Tests of Uniformity on Compact Homogeneous Polish Spaces

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Abstract. We consider a compact Polish space $(\mathscr{X}, \mathrm{d})$ that is homogeneous in the sense that it can be equipped with a uniform probability measure μ_0: a measure that puts the same mass to all balls sharing the same radius, for every radius. Given a sample of points on $\mathscr{X}$, we introduce and study two different distance-based families of statistical tests. The first ones test that the sample is uniformly spread on $\mathscr{X}$, at least as well as an independently identically distributed (i.i.d.) sample from μ_0, the second ones test that the sample is i.i.d. from μ_0. The tests statistics are based on empirical distance-to-measure signatures, as introduced in a paper by Brécheteau (2019) for two-sample tests of isomorphism between metric measure spaces. These tests are consistent. We illustrate their performance on different examples including compact homogeneous Riemanian manifolds with no boundary: the circle, the sphere, the Grassmannian.

Keywords. Statistical tests, uniformity, Wasserstein distance

1 Introduction

1.1 Compact Homogeneous Spaces and First Examples

The notion of a *uniform* probability measure on a compact Polish space $(\mathscr{X}, \mathrm{d})$ first appeared in Loomis (1945). It refers to a Borel probability measure $\mu_0 \in \mathscr{P}(\mathscr{X})$ that satisfies

$$\forall x, y \in \mathscr{X},\ \forall \epsilon > 0,\ \mu_0(\mathrm{B}(x,\epsilon)) = \mu_0(\mathrm{B}(y,\epsilon)). \tag{1}$$

The space $(\mathscr{X}, \mathrm{d})$ is called *homogeneous* if a uniform probability measure μ_0 exists.

According to a result by Christensen (1970), for a compact metric space $(\mathscr{X}, \mathrm{d})$, such a measure μ_0, if it exists, is unique.

Classic examples of compact homogeneous spaces include the torus, the circle, the sphere, the Grassmannian, and the Bolza surface in the Poincaré disk. Many statistical tests for uniformity exist for the circle and the sphere, as summarized in García-Portugués et al. (2018). There are also such tests on the Grassmannian (see Chikuse (2003)), but to our knowledge, none exist for the Bolza surface.

1.2 Comparing to the Uniform Measure

The Wasserstein distance allows for the comparison of two probability measures and metrizes weak convergence. For $p \in [1, \infty)$, the Wasserstein distance of order p, $\mathcal{W}_{p,\mathrm{d}}$ (or simply $\mathcal{W}_p$), is defined for all measures $\mu, \nu \in \mathscr{P}(\mathscr{X})$ as

$$\mathcal{W}_{p,\mathrm{d}}^p(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathscr{X} \times \mathscr{X}} \mathrm{d}^p(x, y) \mathrm{d}\pi(x, y), \tag{2}$$

where $\Pi(\mu, \nu)$ is the set of probability measures $\pi \in \mathscr{P}(\mathscr{X} \times \mathscr{X})$ with first marginal μ and second marginal ν.

To test whether a sample $X_1, \ldots, X_n$ has been generated as an independent sample from a given law μ, Hallin et al. (2021) compare the empirical distribution associated with the sample, $\boldsymbol{\mu}_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$, to the law μ using $\mathcal{W}_p$.

Unlike Hallin et al. (2021), we use DTM-signatures to test whether a sample is i.i.d. from the law μ_0. These signatures have the advantage of being faster to compute (using nearest-neighbor distances). DTM-signatures are probability measures on $\mathbb{R}$, denoted $s_h(\mu)$ (resp. $s_h(\boldsymbol{\mu}_n)$), associated with a measure μ (resp. an empirical measure $\boldsymbol{\mu}_n$), and they have the property of being stable concerning the Wasserstein distance. Furthermore, it can be shown that these signatures discriminate the uniform law in the sense that

$$\forall h \in [0, 1], s_h(\mu) = s_h(\mu_0) \Leftrightarrow \mu = \mu_0. \tag{3}$$

DTM-signatures are defined in Brécheteau (2019) as

$$s_h(\mu) = \mathrm{d}_{\mu,h|\#\mu}$$

for a parameter $h \in [0, 1]$, where $\mathrm{d}_{\mu,h}$ is the distance-to-measure (DTM) function defined in Chazal et al. (2010), and $\mathrm{d}_{\mu,h|\#\mu}$ is the pushforward measure of μ by $\mathrm{d}_{\mu,h}$. Thus, the DTM-signature with parameter h is the distribution of $\mathrm{d}_{\mu,h}(X)$, where X follows the distribution μ. The DTM is defined as

$$\mathrm{d}_{\mu,h} : x \in \mathscr{X} \mapsto \mathcal{W}_2(\delta_x, \mathrm{Sub}_h(\mu)) \in \mathbb{R}_+,$$

where $\mathrm{Sub}_h(\mu) = \{\nu \in \mathscr{P}(\mathscr{X}) \mid h\nu \leq \mu\}$ is the set of sub-measures of μ with mass h. We recall that

$$s_h\left(\frac{1}{n}\sum_{j=1}^n \delta_{X_j}\right) = \frac{1}{n}\sum_{j=1}^n \delta_{\sqrt{\frac{1}{nh}\sum_{i=1}^{nh} \mathrm{d}^2(X^{(i)}(X_j), X_j)}} \in \mathscr{P}(\mathbb{R}),$$

where $X^{(i)}(x)$ denotes the i-th nearest neighbor of x in $\{X_1, \ldots, X_n\}$. This corresponds to the empirical distribution of the average distances of the sample points to their nh nearest neighbors in the sample.

As the sample size n tends to infinity, the signature converges to

$$s_h(\mu_0) = \delta_{\mathrm{d}_h},$$

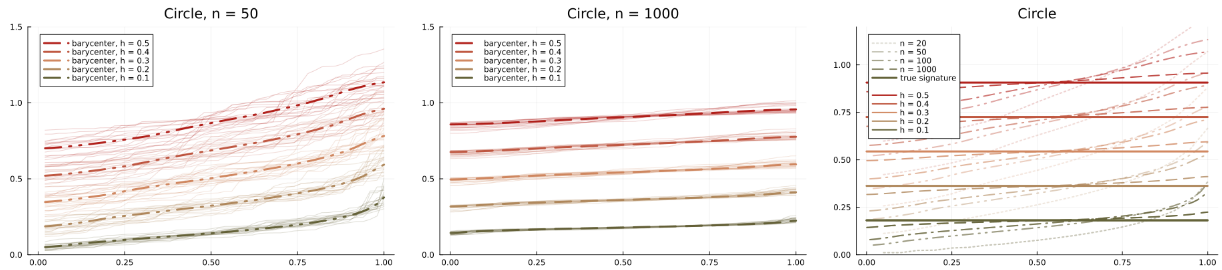

Figure 1: Barycenter signatures for the uniform distribution on the circle, and for samples of size 50 (left), 1000 (middle), from 20 to 1000 together with the limit signature (*true signature*, right).

where d_h is the constant value of the DTM of μ_0, which depends on h. It is also possible to compute the barycenter of the DTM-signature as the law of the quantile function

$$F^{-} : x \in \mathbb{R} \mapsto \mathbb{E}_{\mu_0}\left[F^{-}_{s_h\left(\frac{1}{n}\sum_{j=1}^{n}\delta_{X_j}\right)}(x)\right],$$

which is given by the expectation of the quantile functions of the random signatures $s_h\left(\frac{1}{n}\sum_{j=1}^{n}\delta_{X_j}\right)$ for iid n-samples from the distribution μ_0. We denote this barycenter signature as $\bar{s}_h(\hat{\mathbb{P}}_{0,n})$. The barycenter and limiting signatures for the circle are illustrated in Figure 1.

2 Statistical Tests

2.1 Statistical Tests definition

Given the empirical measure $\boldsymbol{\mu}_n = \frac{1}{n}\sum_{i=1}^{n}\delta_{X_i}$ of law $\mathbb{P}_n$ defined on the set of uniform measures supported on n points $\mathscr{P}_n(\mathscr{X}) = \{\frac{1}{n}\sum_{i=1}^{n}\delta_{x_i},\, x_1,\ldots,x_n \in \mathscr{X}\}$, we define two families of statistical tests, $(\boldsymbol{\phi}^{\mathrm{hom}}_{n,h})_{0<h<1}$ and $(\boldsymbol{\phi}^{\mathrm{iid}}_{n,h})_{0<h<1}$, for the null hypothesis

$$\mathrm{H}_0 : \mathbb{P}_n = \hat{\mathbb{P}}_{0,n}$$

against the alternative hypothesis

$$\mathrm{H}_1 : \mathbb{P}_n \neq \hat{\mathbb{P}}_{0,n},$$

where $\hat{\mathbb{P}}_{0,n}$ is the law of $\boldsymbol{\mu}_n = \frac{1}{n}\sum_{i=1}^{n}\delta_{X_i}$ when $X_1,\ldots,X_n$ is a sample of n independent variables with law μ_0. We consider alternatives of the form:

$$\mathrm{H}_1(\mathrm{d},\mu_0,\mathrm{c}) = \{\hat{\mathbb{P}}_{\mathrm{n}} \in \mathscr{P}(\mathscr{P}_{\mathrm{n}}(\mathscr{X})) \mid \mathrm{d}(\mu,\mu_0) > \mathrm{c}\}, \tag{4}$$

where the measure $\mathbb{P}_n = \hat{\mathbb{P}}_n$ is based on a sample of n independent variables $(X_1, X_2, \ldots, X_n)$, each with the same law $\mu \in \mathscr{P}(\mathscr{X})$.

The test statistic with parameter h for the family of uniformity tests is given by:

$$\mathbf{T}_{n,h}^{\mathrm{hom}} = \mathcal{W}_1(s_h(\boldsymbol{\mu}_n), s_h(\mu_0)) \tag{5}$$

and the test

$$\boldsymbol{\phi}_{n,h}^{\mathrm{hom}} = \mathbb{1}_{\mathbf{T}_{n,h}^{\mathrm{hom}} > \mathrm{q}_{1-\alpha,\mathrm{n,h}}^{\mathrm{hom}}}, \tag{6}$$

where $\mathrm{q}_{1-\alpha,\mathrm{n,h}}^{\mathrm{hom}}$ is the $1-\alpha$ quantile of $\mathbf{T}_{n,h}^{\mathrm{hom}}$ under H_0, approximated by Monte Carlo.

The test statistic with parameter h for the family of uniformity and independence tests is given by:

$$\mathbf{T}_{n,h}^{\mathrm{iid}} = \mathcal{W}_2(s_h(\boldsymbol{\mu}_n), \bar{s}_h(\hat{\mathbb{\mu}}_{0,n})) \tag{7}$$

and the test

$$\boldsymbol{\phi}_{n,h}^{\mathrm{iid}} = \mathbb{1}_{\mathbf{T}_{n,h}^{\mathrm{iid}} > \mathrm{q}_{1-\alpha,\mathrm{n,h}}^{\mathrm{iid}}}, \tag{8}$$

where $\mathrm{q}_{1-\alpha,\mathrm{n,h}}^{\mathrm{iid}}$ is the $1-\alpha$ quantile of $\mathbf{T}_{n,h}^{\mathrm{iid}}$ under H_0, approximated by Monte Carlo.

2.2 Consistency of the Tests

The tests $\boldsymbol{\phi}_{n,h}^{\mathrm{hom}}$ and $\boldsymbol{\phi}_{n,h}^{\mathrm{iid}}$ are asymptotically uniformly consistent against alternatives of the type

$$\mathrm{H}_1 \text{“}\mathcal{W}_1(s_h(\mu), s_h(\mu_0)) > \eta\text{”},$$

for $\eta > 0$, in the following sense.

Theorem 1 *For all $\eta > 0$,*

$$\lim_{n\to\infty} \inf_{\mu \in \mathscr{P}(\mathscr{X}),\, \mathcal{W}_1(s_h(\mu), s_h(\mu_0)) > \eta} \mathbb{P}_\mu(\mathbf{T}_{n,h}^{\mathrm{hom}} > \mathrm{q}_{1-\alpha,\mathrm{n,h}}^{\mathrm{hom}}) = 1 \tag{9}$$

and,

$$\lim_{n\to\infty} \inf_{\mu \in \mathscr{P}(\mathscr{X}),\, \mathcal{W}_2(s_h(\mu), s_h(\mu_0)) > \eta} \mathbb{P}_\mu(\mathbf{T}_{n,h}^{\mathrm{iid}} > \mathrm{q}_{1-\alpha,\mathrm{n,h}}^{\mathrm{iid}}) = 1. \tag{10}$$

3 Numerical Illustrations

3.1 Comparison with Uniformity Tests on the Circle, Sphere, and Grassmannian

In Figures 2 and 3, we compare our two families of tests with classical methods implemented in the R package by García-Portugués et al. (2024) for the circle and the sphere. In addition to varying the parameter $h \in [0, 1]$ in our tests, we consider an aggregated version of the tests in which the best parameter h is automatically chosen from the data. The critical value is then recalibrated to ensure that the test maintains a 5% significance level.

3.1 Comparison with Uniformity Tests on the Circle, Sphere, and Grassmannian

Our methods with a parameter h close to 1 detect von Mises–Fisher-type alternatives with a single mode almost as well as the best tests. Conversely, our methods with a parameter h close to 0 detect von Mises–Fisher mixture alternatives, with modes symmetrically distributed on the circle or sphere, at least as well as or even better than the best existing tests. The same phenomenon appears in the Grassmannian, as shown in Figure 4, where the performance of our tests $(\phi^{\mathrm{iid}}_{n,h})_h$ is compared to the `grassmann.utest` function from the R `Riemann` package.

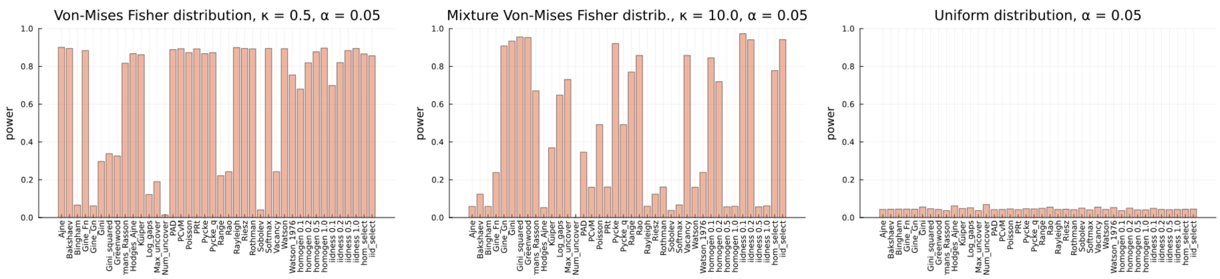

Figure 2: Power on the circle. Alternatives: unimodal, multimodal, uniform distribution

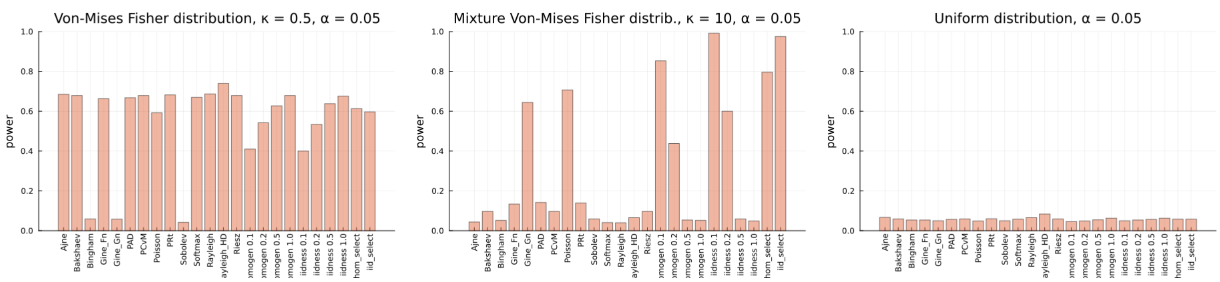

Figure 3: Power on the sphere. Alternatives: unimodal, multimodal, uniform distribution

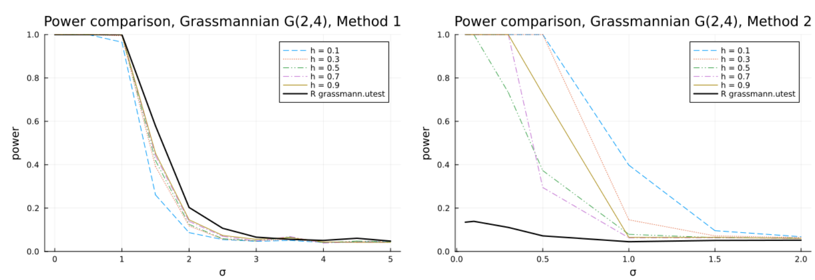

Figure 4: Power on the Grassmannian $\mathbb{G}(2,4)$. Method 1: The first two eigenvectors of the covariance matrix of 50-sample points $(s_i + X_i)_{1\le i\le 50}$, where $(s_i)_{1\le i\le 50} \in \mathbb{R}^4$ are taken from the R dataset `iris`, and $(X_i)_{1\le i\le 50}$ are 50 i.i.d. samples from a normal distribution $\mathcal{N}(0,\sigma)$. Method 2: A mixture of six normal distributions in $\mathbb{G}(2,4)$ (defined using the exponential and logarithm maps), centered at (e_1,e_2), (e_1,e_3), (e_1,e_4), (e_2,e_3), (e_2,e_4), (e_3,e_4).

3.2 Comparison of Power for Non-i.i.d. Samples

The two families of tests behave differently when the data $(X_1, \ldots, X_n)$ are not independently generated. We consider points from a regular grid on the sphere, with von Mises noise added and an increasing concentration parameter κ. (For $\kappa = 0$, the sample is an i.i.d. sample from μ_0 on the sphere. As $\kappa \to +\infty$, the sample forms a regular grid.)

According to Figure 5, the uniformity tests $\phi_{n,h}^{\text{hom}}$ never reject H_0 for the grid. Indeed, the points are at least as uniformly distributed on the sphere as an i.i.d. uniform sample would be. Conversely, the uniformity and independence tests $\phi_{n,h}^{\text{iid}}$ systematically reject H_0 since the points are not independent.

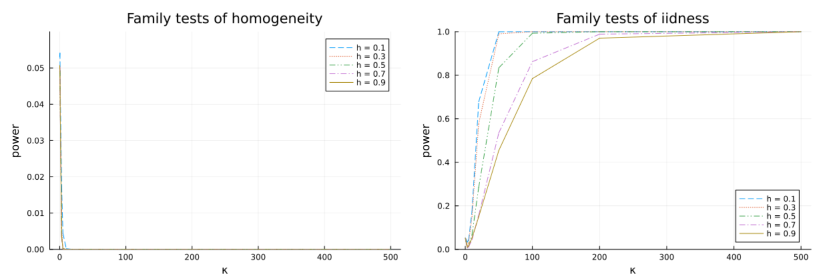

Figure 5: Comparison of the power of the two families of tests

4 Conclusion and Perspectives

For large h values, the tests compare favorably to the best tests in the literature for detecting unimodal alternatives. Meanwhile, for small h values, the tests outperform existing methods for detecting symmetric multimodal alternatives. The uniformity and independence tests could, for example, be used to assess deterministic methods for generating independent samples from a uniform distribution.

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