

# Introduction to Functional Analysis

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December 2024

References: Introductory Functional Analysis with Applications, *Kreyszig*; lectures by someone in MATH301 at the University of Otago, New Zealand.

## Contents

<b>1</b>	<b>Metric Spaces</b>	<b>2</b>
1.1	Some metric spaces . . . . .	2
1.2	Problems . . . . .	3
1.3	More metric spaces . . . . .	4

# 1 Metric Spaces

**DEF :** A metric space is a pair  $(X, d)$ , where  $X$  is a set and we define a *distance function*  $d : X \times X \rightarrow \mathbb{R}$  such that for all  $x, y, z \in X$ :

(M1)  $d$  is real-valued, finite and nonnegative

(M2)  $d(x, y) = 0 \iff x = y$

(M3)  $d(x, y) = d(y, x)$  (Symmetry)

(M4)  $d(x, y) \leq d(x, z) + d(z, y)$  (Triangle Inequality)

$X$  is called the *underlying set* of  $(X, d)$ . Its elements are called points. By induction, we can obtain the generalized triangle inequality from (M4):

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n). (1)$$

does the generalized triangle inequality also need to hold on all metric spaces? //// **DEF :** A subspace  $(Y, \tilde{d})$  of  $(X, d)$  is obtained by restricting  $d$  to  $Y \times Y$ , denoted as  $\tilde{d} = d|_{Y \times Y}$  where  $Y \subset X$ .

## 1.1 Some metric spaces

**Ex :** The  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  is the set of all ordered  $n$ -tuples of real numbers

$$x = (\xi_1, \xi_2, \dots, \xi_n), \quad y = (\eta_1, \eta_2, \dots, \eta_n)$$

with  $d$  as the *Euclidean metric*:

$$d(x, y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + \cdots + (\xi_n - \eta_n)^2}.$$

The  $n$ -dimensional unitary space  $\mathbb{C}^n$  is the set of all ordered  $n$ -tuples of *complex* numbers with metric defined by

$$d(x, y) = \sqrt{|\xi_1 - \eta_1|^2 + |\xi_2 - \eta_2|^2 + \dots + |\xi_n - \eta_n|^2}.$$

**Ex :** The 2-dimensional Euclidean space  $\mathbb{R}^2$  is the set of all ordered  $n$ -tuples of real numbers

$$x = (\xi_1, \xi_2, \dots, \xi_n), \quad y = (\eta_1, \eta_2, \dots, \eta_n)$$

with  $d$  as the *Euclidean metric*:

$$d(x, y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + \dots + (\xi_n - \eta_n)^2}.$$

is it l1?? Note that more than one metric can be defined on the same underlying set. Consider the *Manhattan distance* defined on  $\mathbb{R}^2$ :

$$d(x, y) = |\xi_1 - \eta_1| + |\xi_2 - \eta_2|.$$

This is called the  $l^1$  metric.

The  $n$ -dimensional unitary space  $\mathbb{C}^n$  is the set of all ordered  $n$ -tuples of *complex* numbers with metric defined by

$$d(x, y) = \sqrt{|\xi_1 - \eta_1| + |\xi_2 - \eta_2| + \dots + |\xi_n - \eta_n|}.$$

\*\*\*\*?? Let  $\xi_i = a + bi$  and  $\eta_i = c + di$ . Then

$$|\xi_i - \eta_i| = |a + bi - (c + di)| = |a - c + (b - d)i| := \sqrt{(a - c)^2 + (b - d)^2}.$$

**Ex :** The sequence space  $l^\infty$  is the set of all bounded sequences of all complex numbers. That is, every point is a complex sequence

$$x = (\xi_1, \xi_2, \dots) \quad \text{briefly} \quad x = (\xi_j)$$

such that for all  $j = 1, 2, \dots$  we have

$$|\xi_j| \leq c_x,$$

where  $c_x$  is a real number which may depend on  $x$  but does not depend on  $j$ . The metric is defined by

$$d(x, y) = \sup_{j \in \mathbb{N}} |\xi_j - \eta_j|,$$

where  $\sup$  denotes the least upper bound.

**Ex :** The function space  $C[a, b]$  is the set of real-valued, well-defined and continuous functions of an independent real variable  $t$  on an interval  $J = [a, b]$ . The metric is defined by

$$d(x, y) = \max_{t \in J} |x(t) - y(t)|,$$

where  $\max$  denotes the maximum. This metric space is denoted as  $C[a, b]$ , and called a function space, because every point in  $C[a, b]$  is a function.

## 1.2 Problems

1. Show that  $d(x, y) := \sqrt{|x - y|}$  is a metric defined on  $\mathbb{R}$ .

*Solution.*

We verify the four axioms of a metric space.

$$(M1) \quad d(x, y) = \sqrt{|x - y|} \text{ is real-valued, finite and nonnegative}$$

$$(M2) \quad d(x, y) = 0 \iff \sqrt{|x - y|} = 0 \iff |x - y| = 0 \iff x = y$$

$$(M3) \quad d(x, y) = \sqrt{|x - y|} = \sqrt{|y - x|} = d(y, x)$$

$$(M4)$$

$$d(x, z) = \sqrt{|x - z|} \Rightarrow d(x, z)^2 = |x - z|$$

$$d(x, y) + d(y, z) = \sqrt{|x - y|} + \sqrt{|y - z|}$$

$$\Rightarrow (d(x, y) + d(y, z))^2 = (\sqrt{|x - y|} + \sqrt{|y - z|})^2 = |x - y| + |y - z| + 2\sqrt{|x - y||y - z|}$$

And therefore, since  $|x - z| \leq |x - y| + |y - z| \leq |x - y| + |y - z| + 2\sqrt{|x - y||y - z|}$ , we have that  $d(x, z)^2 \leq (d(x, y) + d(y, z))^2$ . Since the square root function is order-preserving, we obtain  $d(x, z) \leq d(x, y) + d(y, z)$ , as required. ■

2. Using the generalized triangle inequality in (1), prove that

$$|d(x, y) - d(z, w)| \leq d(x, z) + d(y, w)$$

*Solution.*

Consider the relationships  $x \rightarrow z \rightarrow w \rightarrow y \rightarrow x$ . Then

$$d(x, y) \leq d(x, z) + d(z, w) + d(w, y)$$

$$d(z, w) \leq d(y, w) + d(x, y) + d(x, z),$$

where the second inequality uses (M3). We rearrange to obtain

$$d(x, y) - d(z, w) \leq d(x, z) + d(w, y)$$

$$d(z, w) - d(x, y) \leq d(y, w) + d(x, z).$$

$$\Rightarrow d(x, y) - d(z, w) \leq d(x, z) + d(w, y) \leq d(x, y) - d(z, w)$$

$$\Rightarrow |d(x, y) - d(z, w)| \leq d(y, w) + d(x, z)$$

■

### 1.3 More metric spaces

We now provide examples of slightly more complex metric spaces.

**DEF :** The sequence space  $s$  is the set of *all* (unbounded or bounded) sequences of complex numbers and the metric  $d$  defined by

$$d(x, y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|},$$

where  $x = (\xi_j)$  and  $y = (\eta_j)$ . We cannot use the metric defined for  $l^\infty$  since unbounded functions could violate the finiteness of (M1).

**Thm :** The sequence space  $s$  is a metric space.

*Proof.* To prove that  $s$  is a metric space, we use the auxiliary function  $f$  defined on  $\mathbb{R}$  by

$$f(t) = \frac{t}{1 + t}.$$

Differentiating using the product rule gives  $f'(t) = 1/(1 + t)^2$ , which is positive, and hence  $f$  is monotone increasing. Hence,  $|a + b| \leq |a| + |b| \Rightarrow f(|a + b|) \leq f(|a| + |b|)$ . Therefore,

$$f(|a + b|) = \frac{|a + b|}{1 + |a + b|} \leq \frac{|a| + |b|}{1 + |a| + |b|}.$$

Let  $z = (\zeta_j)$ , and  $a = \xi_j - \zeta_j$  and  $b = \zeta_j - \eta_j$ . Then  $a + b = \xi_j - \eta_j$ , so

$$\frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} \leq \frac{|\xi_j - \zeta_j|}{1 + |\xi_j - \zeta_j|} + \frac{|\zeta_j - \eta_j|}{1 + |\zeta_j - \eta_j|}.$$

$$\sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} \leq \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \zeta_j|}{1 + |\xi_j - \zeta_j|} + \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\zeta_j - \eta_j|}{1 + |\zeta_j - \eta_j|}.$$

$$d(x, y) \leq d(x, z) + d(z, y).$$

■

hello this is me testing