Introduction to Functional Analysis

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References: Introductory Functional Analysis with Applications, *Kreyszig*; lectures by someone in MATH301 at the University of Otago, New Zealand.

Contents

1 Metric Spaces

$DEF:$ A metric space is a pair (X, d), where X is a set and we define a *distance function* $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$:

(M1) d is real-valued, finite and nonnegative

(M2) $d(x, y) = 0 \iff x = y$

(M3) $d(x, y) = d(y, x)$ (Symmetry)

(M4) $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle Inequality)

X is called the *underlying set* of (X, d). Its elements are called points. By induction, we can obtain the generalized triangle inequality from (M4):

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n).(1)$$

does the generalized triangle inequality also need to hold on all metric spaces? //// $DEF:$ A subspace $(Y, \tilde{d})$ of (X, d) is obtained by restricting d to $Y \times Y$, denoted as $\tilde{d} = d|_{Y \times Y}$ where $Y \subset X$.

1.1 Some metric spaces

$Ex:$ The *n-dimensional Euclidean space* $\mathbb{R}^n$ is the set of all ordered n-tuples of real numbers

$$x = (\xi_1, \xi_2, \|dots, \xi_n), \qquad y = (\eta_1, \eta_2, \cdots, \eta_n)$$

with d as the *Euclidean metric*:

$$d(x, y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + \cdots + (\xi_n - \eta_n)^2}.$$

The *n-dimensional unitary space* $\mathbb{C}^n$ is the set of all ordered n-tuples of *complex* numbers with metric defined by

$$d(x, y) = \sqrt{|\xi_1 - \eta_1| + |\xi_2 - \eta_2| + \|dots + xi_n - \eta_n|}.$$

$Ex:$ The *2-dimensional Euclidean space* $\mathbb{R}^n$ is the set of all ordered n-tuples of real numbers

$$x = (\xi_1, \xi_2, \cdots, \xi_n), \qquad y = (\eta_1, \eta_2, ..., \eta_n)$$

with d as the *Euclidean metric*:

$$d(x, y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + ... + (\xi_n - \eta_n)^2}.$$

is it l1?? Note that more than one metric can be defined on the same underlying set. Consider the *Manhattan distance* defined on $\mathbb{R}^2$:

$$d(x, y) = |\xi_1 - \eta_1| + |\xi_2 - \eta_2|.$$

This is called the l^1 metric.

The *n-dimensional unitary space* $\mathbb{C}^n$ is the set of all ordered n-tuples of *complex* numbers with metric defined by

$$d(x,y) = \sqrt{|\xi_1 - \eta_1| + |\xi_2 - \eta_2| + ... + |\xi_n - \eta_n|}.$$

****?? Let $\xi_i = a + bi$ and $\eta_i = c + di$. Then

$$|\xi_i - \eta_i| = |a + bi - (c + di)| = |a - c + (b - d)i \coloneqq \sqrt{(a-c)^2 + (b-d)^2}.$$

Ex : The sequence space l^∞ is the set of all bounded sequences of all complex numbers. That is, every point is a complex sequence

$$x = (\xi_1, \xi_2, ...) \qquad \text{briefly} \qquad x = (\xi_j)$$

such that for all j = 1, 2, ... we have

$$|\xi_j| \leq c_x,$$

where c_x is a real number which may depend on x but does not depend on j. The metric is defined by

$$d(x,y) = \sup_{j \in \mathbb{N}} |\xi_j - \eta_j|,$$

where sup denotes the least upper bound.

Ex : The function space C[a, b] is the set of real-valued, well-defined and continuous functions of an independent real variable t on an interval J = [a, b]. The metric is defined by

$$d(x,y) = \max_{t \in J} |x(t) - y(t)|,$$

where *max* denotes the maximum. This metric space is denoted as C[a, b], and called a function space, because every point in C[a, b] is a function.

1.2 Problems

1. Show that $d(x,y) \coloneqq \sqrt{|x-y|}$ is a metric defined on $\mathbb{R}$.

   Solution.
   We verify the four axioms of a metric space.

   (M1) $d(x,y) = \sqrt{|x-y|}$ is real-valued, finite and nonnegative
   (M2) $d(x,y) = 0 \iff \sqrt{|x-y|} = 0 \iff |x-y| = 0 \iff x = y$
   (M3) $d(x,y) = \sqrt{|x-y|} = \sqrt{|y-x|} = d(y,x)$
   (M4)

   $$d(x,z) = \sqrt{|x-z|} \Rightarrow d(x,z)^2 = |x-z|$$

   $$d(x,y) + d(y,z) = \sqrt{|x-y|} + \sqrt{|y-z|}$$

   $$\Rightarrow (d(x,y) + d(y,z))^2 = (\sqrt{|x-y|} + \sqrt{|y-z|})^2 = |x-y| + |y-z| + 2\sqrt{|x-y||y-z|}$$

   And therefore, since $|x-z| \leq |x-y| + |y-z| \leq |x-y| + |y-z| + 2\sqrt{|x-y||y-z|}$, we have that $d(x,z)^2 \leq (d(x,y) + d(y,z))^2$. Since the square root function is order-preserving, we obtain $d(x,z) \leq d(x,y) + d(y,z)$, as required. ■

2. Using the generalized triangle inequality in (1), prove that

$$|d(x,y) - d(z,w)| \leq d(x,z) + d(y,w)$$

Solution.
Consider the relationships x $\rightarrow z \rightarrow w \rightarrow y \rightarrow x$. Then

$$d(x,y) \leq d(x,z) + d(z,w) + d(w,y)$$

$$d(z,w) \leq d(y,w) + d(x,y) + d(x,z),$$

where the second inequality uses (M3). We rearrange to obtain

$$d(x,y) - d(z,w) \leq d(x,z) + d(w,y)$$

$$d(z,w) - d(x,y) \leq d(y,w) + d(x,z).$$

$$\Rightarrow d(x,y) - d(z,w) \leq d(x,z) + d(w,y) \leq d(x,y) - d(z,w)$$

$$\Rightarrow |d(x,y) - d(z,w)| \leq d(y,w) + d(x,z)$$

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1.3 More metric spaces

We now provide examples of slightly more complex metric spaces.

$DEF:$ The sequence space s is the set of *all* (unbounded or bounded) sequences of complex numbers and the metric d defined by

$$d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|},$$

where $x = (\xi_j)$ and $y = (\eta_j)$. We cannot use the metric defined for l^∞ since unbounded functions could violate the finiteness of (M1).
$Thm:$ The sequence space s is a metric space.
Proof. To prove that s is a metric space, we use the auxiliary function f defined on $\mathbb{R}$ by

$$f(t) = \frac{t}{1+t}.$$

Differentiating using the product rule gives $f'(t) = 1/1 + t^2$, which is positive, and hence f is monotone increasing. Hence, $|a+b| \leq |a| + |b| \Rightarrow f(|a+b|) \leq f(|a| + |b|)$. Therefore,

$$f(|a+b|) = \frac{|a+b|}{1+|a+b|} \leq \frac{|a|+|b|}{1+|a|+|b|}.$$

Let $z = (\zeta_j)$, and $a = \xi_j - \zeta_j$ and $b = \zeta_j - \eta_j$. Then $a + b = \xi_j + \eta_j$, so

$$\frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} \leq \frac{|\xi_j - \zeta_j|}{1 + |\xi_j - \zeta_j|} + \frac{|\zeta_j - \eta_j|}{1 + |\zeta_j - \eta_j|}.$$

$$\sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} \leq \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \zeta_j|}{1 + |\xi_j - \zeta_j|} + \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\zeta_j - \eta_j|}{1 + |\zeta_j - \eta_j|}.$$

$$d(x,y) \leq d(x,z) + d(z,y).$$

■

hello this is me testing ok now i"m testing again i am testing once again teojfhuighiguhifb hello I am editing my notes once again edit