

Introduction to Functional Analysis

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References: Introductory Functional Analysis with Applications, *Kreyszig*; lectures by someone in MATH301 at the University of Otago, New Zealand.

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1 Background

Let us begin by recalling some properties of the real numbers \mathbb{R} :

1. Any non-empty, bounded subset $S \subset \mathbb{R}$ has a least upper bound $\sup(S)$, which is a real number c such that $x \leq c$ for all $x \in S$, and such that if c' is any other number with this property then $c' \geq c$;
2. Similarly, any non-empty, bounded subset $S \subset \mathbb{R}$ has a greatest lower bound $\inf(S)$, which is a real number c such that $c \leq x$ for all $x \in S$, and such that if c' is any other number with this property then $c \geq c'$;
3. **Thm :** (Bolzano-Weierstrass) Any bounded sequence of reals has a convergent subsequence; The proof of the theorem relies on the Monotone Subsequence Theorem (Every Sequence has a Monotone Subsequence) and the Monotone Convergence Theorem (A monotone sequence (a_n) converges \iff it is bounded).
4. **DEF :** A sequence $\{a_n\}_{n=1}^\infty$ is called a *Cauchy sequence* if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $m, n \geq N$, then $|a_m - a_n| < \epsilon$.
Thm : Any Cauchy sequence of real numbers converges.
The proof relies on the fact that every Cauchy sequence is bounded and hence, by B-W, has a convergent subsequence.

Recall the analytic definition of continuity:

DEF : If $f : A \rightarrow \mathbb{R}$ is a function, we say that f is *continuous at* $a \in A$ if, for any $\epsilon > 0$, we can find a $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$.

and the topological/neighborhood definition of continuity:

DEF : For all $V_\epsilon(f(c))$, there exists $V_\delta(c)$ such that $x \in V_\delta(c) \cap A \implies f(x) \in V_\epsilon(f(c))$.

2 Metric Spaces

DEF : A metric space is a pair (X, d) , where X is a set and d is a *distance function* $d : X \times X \rightarrow \mathbb{R}$ such that for all $x, y, z \in X$:

(M1) d is real-valued, finite and nonnegative

(M2) $d(x, y) = 0 \iff x = y$

(M3) $d(x, y) = d(y, x)$ (Symmetry)

(M4) $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle Inequality)

X is called the *underlying set* of (X, d) . Its elements are called points. By induction, we can obtain the generalized triangle inequality from (M4):

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n). \quad (1)$$

Lemma : Let x, y, z be points in a metric space. Then we have

$$|d(x, y) - d(x, z)| \leq d(y, z). \quad (2)$$

Proof : We need to show that $d(x, y) - d(x, z) \leq d(y, z)$ and $d(x, z) - d(x, y) \leq d(y, z)$. These are both instances of the triangle inequality. ■

DEF : A subspace (Y, \tilde{d}) of (X, d) is obtained by restricting d to $Y \times Y$, denoted as $\tilde{d} = d|_{Y \times Y}$ where $Y \subset X$.

2.1 Some metric spaces

Ex : The n -dimensional Euclidean space \mathbb{R}^n is the set of all ordered n -tuples of real numbers

$$x = (\xi_1, \xi_2, \dots, \xi_n), \quad y = (\eta_1, \eta_2, \dots, \eta_n).$$

There are several possible metrics on \mathbb{R}^n :

$$\begin{aligned} d_1(x, y) &= \sum_{i=1}^n (|\xi_i - \eta_i|); \\ d_2(x, y) &= \left((\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + \dots + (\xi_n - \eta_n)^2 \right)^{1/2}; \\ d_\infty(x, y) &= \max_{i \in \{1, 2, \dots, n\}} |\xi_i - \eta_i|. \end{aligned}$$

These are called the ℓ^1 -, ℓ^2 - (or Euclidean) and ℓ^∞ -distances, respectively.

The n -dimensional unitary space \mathbb{C}^n is the set of all ordered n -tuples of complex numbers with metric defined by

$$d(x, y) = \sqrt{|\xi_1 - \eta_1| + |\xi_2 - \eta_2| + \dots + |\xi_n - \eta_n|}.$$

****?? Let $\xi_i = a + bi$ and $\eta_i = c + di$. Then

$$|\xi_i - \eta_i| = |a + bi - (c + di)| = |a - c + (b - d)i| := \sqrt{(a - c)^2 + (b - d)^2}.$$

Ex : The sequence space l^∞ is the set of all bounded sequences of all complex numbers. That is, every point is a complex sequence

$$x = (\xi_1, \xi_2, \dots) \quad \text{briefly} \quad x = (\xi_j)$$

such that for all $j = 1, 2, \dots$ we have

$$|\xi_j| \leq c_x,$$

where c_x is a real number which may depend on x but does not depend on j . The metric is defined by

$$d(x, y) = \sup_{j \in \mathbb{N}} |\xi_j - \eta_j|,$$

where sup denotes the least upper bound.

Ex : The function space $C[a, b]$ is the set of real-valued, well-defined and continuous functions of an independent real variable t on an interval $J = [a, b]$. The metric is defined by

$$d(x, y) = \max_{t \in J} |x(t) - y(t)|,$$

where \max denotes the maximum. This metric space is denoted as $C[a, b]$, and called a function space, because every point in $C[a, b]$ is a function.

2.2 Problems

1. Show that $d(x, y) := \sqrt{|x - y|}$ is a metric defined on \mathbb{R} .

Solution.

We verify the four axioms of a metric space.

$$(M1) \quad d(x, y) = \sqrt{|x - y|} \text{ is real-valued, finite and nonnegative}$$

$$(M2) \quad d(x, y) = 0 \iff \sqrt{|x - y|} = 0 \iff |x - y| = 0 \iff x = y$$

$$(M3) \quad d(x, y) = \sqrt{|x - y|} = \sqrt{|y - x|} = d(y, x)$$

$$(M4)$$

$$d(x, z) = \sqrt{|x - z|} \Rightarrow d(x, z)^2 = |x - z|$$

$$d(x, y) + d(y, z) = \sqrt{|x - y|} + \sqrt{|y - z|}$$

$$\Rightarrow (d(x, y) + d(y, z))^2 = (\sqrt{|x - y|} + \sqrt{|y - z|})^2 = |x - y| + |y - z| + 2\sqrt{|x - y||y - z|}$$

And therefore, since $|x - z| \leq |x - y| + |y - z| \leq |x - y| + |y - z| + 2\sqrt{|x - y||y - z|}$, we have that $d(x, z)^2 \leq (d(x, y) + d(y, z))^2$. Since the square root function is order-preserving, we obtain $d(x, z) \leq d(x, y) + d(y, z)$, as required. ■

2. Using the generalized triangle inequality in (1), prove that

$$|d(x, y) - d(z, w)| \leq d(x, z) + d(y, w)$$

Solution.

Consider the relationships $x \rightarrow z \rightarrow w \rightarrow y \rightarrow x$. Then

$$d(x, y) \leq d(x, z) + d(z, w) + d(w, y)$$

$$d(z, w) \leq d(y, w) + d(x, y) + d(x, z),$$

where the second inequality uses (M3). We rearrange to obtain

$$d(x, y) - d(z, w) \leq d(x, z) + d(w, y)$$

$$d(z, w) - d(x, y) \leq d(y, w) + d(x, z).$$

$$\Rightarrow d(x, y) - d(z, w) \leq d(x, z) + d(w, y) \leq d(x, y) - d(z, w)$$

$$\Rightarrow |d(x, y) - d(z, w)| \leq d(y, w) + d(x, z)$$

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2.3 More metric spaces

We now provide examples of slightly more complex metric spaces.

DEF : The sequence space s is the set of *all* (unbounded or bounded) sequences of complex numbers and the metric d defined by

$$d(x, y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|},$$

where $x = (\xi_j)$ and $y = (\eta_j)$. We cannot use the metric defined for l^∞ since unbounded functions could violate the finiteness of (M1).

Thm : The sequence space s is a metric space.

Proof : To prove that s is a metric space, we use the auxiliary function f defined on \mathbb{R} by

$$f(t) = \frac{t}{1+t}.$$

Differentiating using the product rule gives $f'(t) = 1/(1+t)^2$, which is positive, and hence f is monotone increasing. Hence, $|a+b| \leq |a|+|b| \Rightarrow f(|a+b|) \leq f(|a|+|b|)$. Therefore,

$$f(|a+b|) = \frac{|a+b|}{1+|a+b|} \leq \frac{|a|+|b|}{1+|a|+|b|}.$$

Let $z = (\zeta_j)$, and $a = \xi_j - \zeta_j$ and $b = \zeta_j - \eta_j$. Then $a+b = \xi_j - \eta_j$, so

$$\begin{aligned} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} &\leq \frac{|\xi_j - \zeta_j|}{1 + |\xi_j - \zeta_j|} + \frac{|\zeta_j - \eta_j|}{1 + |\zeta_j - \eta_j|}. \\ \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} &\leq \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \zeta_j|}{1 + |\xi_j - \zeta_j|} + \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\zeta_j - \eta_j|}{1 + |\zeta_j - \eta_j|}. \\ d(x, y) &\leq d(x, z) + d(z, y). \end{aligned}$$

■

hello this is me testing ok now i'm testing again i am testing once again teojfhuighiguhifb hello
I am editing my notes once again edit