Introduction to Functional Analysis

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References: Introductory Functional Analysis with Applications, *Kreyszig*; lectures by someone in MATH301 at the University of Otago, New Zealand.

Contents

1 Background

Let us begin by recalling some properties of the real numbers $\mathbb{R}$:

1. Any non-empty, bounded subset S $\subset \mathbb{R}$ has a least upper bound $\sup(S)$, which is a real number c such that $x \leq c$ for all $x \in S$, and such that if c' is any other number with this property then $c' \geq c$;
2. Similarly, any non-empty, bounded subset S $\subset \mathbb{R}$ has a greatest lower bound $\inf(S)$, which is a real number c such that $c \leq x$ for all $x \in S$, and such that if c' is any other number with this property then $c \geq c'$;
3. Thm : (Bolzano-Weierstrass) Any bounded sequence of reals has a convergent subsequence;

   The proof of the theorem relies on the Monotone Subsequence Theorem (Every Sequence has a Monotone Subsequence) and the Monotone Convergence Theorem (A monotone sequence (a_n) converges $\iff$ it is bounded).
4. DEF : A sequence $\{a_n\}_{n=1}^{\infty}$ is called a *Cauchy sequence* if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $m, n \geq \mathbb{N}$, then $|a_m - a_n| < \epsilon$.
   Thm : Any Cauchy sequence of real numbers converges.
   The proof relies on the fact that every Cauchy sequences is bounded and hence, by B-W, has a convergent subsequence.

Recall the analytic definition of continuity:

DEF : If $f : A \to \mathbb{R}$ is a function, we say that f is *continuous at* $a \in A$ if, for any $\epsilon > 0$, we can find a $\delta > 0$ such that if $|x - a| < \delta$ then $|f(\mathrm{x}) - f(\mathrm{a})| < \epsilon$.

and the topological/neighborhood definition of continuity:

DEF : For all $V_\varepsilon(f(c))$, there exists $V_\delta(c)$ such that $x \in V_\delta(c) \cap A \implies f(x) \in V_\varepsilon(f(c))$.

2 Metric Spaces

DEF : A metric space is a pair (X, d), where X is a set and d is a *distance function* $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$:

(M1) d is real-valued, finite and nonnegative

(M2) $d(x, y) = 0 \iff x = y$

(M3) $d(x, y) = d(y, x)$ (Symmetry)

(M4) $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle Inequality)

X is called the *underlying set* of (X, d). Its elements are called points. By induction, we can obtain the generalized triangle inequality from (M4):

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n). \tag{1}$$

Lemma : Let x, y, z be points in a metric space. Then we have

$$|d(x,y) - d(x,z)| \leq d(y,z). \tag{2}$$

Proof : We need to show that $d(x,y) - d(x,z) \leq d(y,z)$ and $d(x,z) - d(x,y) \leq d(y,z)$. These are both instances of the triangle inequality. ■

DEF : A subspace $(Y, \tilde{d})$ of (X, d) is obtained by restricting d to $Y \times Y$, denoted as $\tilde{d} = d|_{Y \times Y}$ where $Y \subset X$.

2.1 Some metric spaces

Ex : The *n-dimensional Euclidean space* $\mathbb{R}^n$ is the set of all ordered n-tuples of real numbers

$$x = (\xi_1, \xi_2, \cdots, \xi_n), \qquad y = (\eta_1, \eta_2, \cdots, \eta_n).$$

There are several possible metrics on $\mathbb{R}^n$:

$$\begin{aligned} d_1(x,y) &= \sum_{i=1}^{n} (|\xi_i - \eta_i|); \\ d_2(x,y) &= \Big((\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + \cdots + (\xi_n - \eta_n)^2\Big)^{1/2}; \\ d_\infty(x,y) &= \max_{i \in \{1,2,\cdots,n\}} |\xi_i - \eta_i|. \end{aligned}$$

These are called the ℓ^1-, ℓ^2- (or Euclidean) and ℓ^∞-distances, respectively.

The *n-dimensional unitary space* $\mathbb{C}^n$ is the set of all ordered n-tuples of *complex* numbers with metric defined by

$$d(x,y) = \sqrt{|\xi_1 - \eta_1| + |\xi_2 - \eta_2| + ... + |\xi_n - \eta_n|}.$$

****?? Let $\xi_i = a + bi$ and $\eta_i = c + di$. Then

$$|\xi_i - \eta_i| = |a + bi - (c + di)| = |a - c + (b - d)i := \sqrt{(a-c)^2 + (b-d)^2}.$$

Ex : The sequence space l^∞ is the set of all bounded sequences of all complex numbers. That is, every point is a complex sequence

$$x = (\xi_1, \xi_2, ...) \qquad \text{briefly} \qquad x = (\xi_j)$$

such that for all $j = 1, 2, ...$ we have

$$|\xi_j| \leq c_x,$$

where c_x is a real number which may depend on x but does not depend on j. The metric is defined by

$$d(x,y) = \sup_{j \in \mathbb{N}} |\xi_j - \eta_j|,$$

where sup denotes the least upper bound.

Ex : The function space C[a, b] is the set of real-valued, well-defined and continuous functions of an independent real variable t on an interval J = [a, b]. The metric is defined by

$$d(x, y) = \max_{t\in J} |x(t) - y(t)|,$$

where *max* denotes the maximum. This metric space is denoted as C[a, b], and called a function space, because every point in C[a, b] is a function.

2.2 Problems

1. Show that $d(x, y) \coloneqq \sqrt{|x - y|}$ is a metric defined on $\mathbb{R}$.

   Solution.
   We verify the four axioms of a metric space.

   (M1) $d(x, y) = \sqrt{|x - y|}$ is real-valued, finite and nonnegative

   (M2) $d(x, y) = 0 \iff \sqrt{|x - y|} = 0 \iff |x - y| = 0 \iff x = y$

   (M3) $d(x, y) = \sqrt{|x - y|} = \sqrt{|y - x|} = d(y, x)$

   (M4)

   $$d(x, z) = \sqrt{|x - z|} \Rightarrow d(x, z)^2 = |x - z|$$

   $$d(x, y) + d(y, z) = \sqrt{|x - y|} + \sqrt{|y - z|}$$

   $$\Rightarrow (d(x, y)+d(y, z))^2 = (\sqrt{|x - y|}+\sqrt{|y - z|})^2 = |x-y|+|y-z|+2\sqrt{|x - y||y - z|}$$

   And therefore, since $|x-z| \le |x-y|+|y-z| \le |x-y|+|y-z|+2\sqrt{|x - y||y - z|}$, we have that $d(x, z)^2 \le (d(x, y) + d(y, z))^2$. Since the square root function is order-preserving, we obtain $d(x, z) \le d(x, y) + d(y, z)$, as required. ■

2. Using the generalized triangle inequality in (1), prove that

   $$|d(x, y) - d(z, w)| \le d(x, z) + d(y, w)$$

   Solution.
   Consider the relationships x $\to z \to w \to y \to x$. Then

   $$d(x, y) \le d(x, z) + d(z, w) + d(w, y)$$

   $$d(z, w) \le d(y, w) + d(x, y) + d(x, z),$$

   where the second inequality uses (M3). We rearrange to obtain

   $$d(x, y) - d(z, w) \le d(x, z) + d(w, y)$$

   $$d(z, w) - d(x, y) \le d(y, w) + d(x, z).$$

   $$\Rightarrow d(x, y) - d(z, w) \le d(x, z) + d(w, y) \le d(x, y) - d(z, w)$$

   $$\Rightarrow |d(x, y) - d(z, w)| \le d(y, w) + d(x, z)$$

   ■

2.3 More metric spaces

We now provide examples of slightly more complex metric spaces.

DEF : The sequence space s is the set of *all* (unbounded or bounded) sequences of complex numbers and the metric d defined by

$$d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|},$$

where $x = (\xi_j)$ and $y = (\eta_j)$. We cannot use the metric defined for l^∞ since unbounded functions could violate the finiteness of (M1).

Thm : The sequence space s is a metric space.

Proof : To prove that s is a metric space, we use the auxiliary function f defined on $\mathbb{R}$ by

$$f(t) = \frac{t}{1+t}.$$

Differentiating using the product rule gives $f'(t) = 1/1 + t^2$, which is positive, and hence f is monotone increasing. Hence, $|a + b| \leq |a| + |b| \Rightarrow f(|a + b|) \leq f(|a| + |b|)$. Therefore,

$$f(|a+b|) = \frac{|a+b|}{1+|a+b|} \leq \frac{|a|+|b|}{1+|a|+|b|}.$$

Let $z = (\zeta_j)$, and $a = \xi_j - \zeta_j$ and $b = \zeta_j - \eta_j$. Then $a + b = \xi_j + \eta_j$, so

$$\frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} \leq \frac{|\xi_j - \zeta_j|}{1 + |\xi_j - \zeta_j|} + \frac{|\zeta_j - \eta_j|}{1 + |\zeta_j - \eta_j|}.$$

$$\sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} \leq \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \zeta_j|}{1 + |\xi_j - \zeta_j|} + \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\zeta_j - \eta_j|}{1 + |\zeta_j - \eta_j|}.$$

$$d(x,y) \leq d(x,z) + d(z,y).$$

■