April 20–Farey sequence

We explore the Farey sequence of rational numbers.

There are several equivalent ways of defining successive Farey sequences. We will show that these are equivalent, as well as prove some properties of these sequences.

Definition 1. The *level n Farey fractions between* 0 *and* 1 is the set of reduced form fractions ordered from smallest to largest with

$$S_n = \left\{ \frac{p}{q} : 0 \le \frac{p}{q} \le 1, (p,q) = 1, 1 \le q \le n \right\}.$$

So that every term in the sequence is a fraction, we write $0 = \frac{0}{1}$ and $1 = \frac{1}{1}$. The first four Farey sequences are

$$\begin{aligned} S_1 &= \left\{ \frac{0}{1}, \frac{1}{1} \right\} \\ S_2 &= \left\{ \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \right\} \\ S_3 &= \left\{ \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right\} \\ S_4 &= \left\{ \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right\} \end{aligned}$$

It is clear that each sequence S_n is ordered from smallest to largest and that each $(p,q) = 1$ (since these facts are both part of the definition). However, these facts are less obvious for the next definition, f_n:

Definition 2. Construct a table using the following rules: In the first row, write $\frac{0}{1}$ and $\frac{1}{1}$.

For the n^{th} row, copy the $(n-1)^{st}$ row. For each $\frac{a}{b}$ and $\frac{c}{d}$ in the$(n-1)^{st}$ row, insert $\frac{a+c}{b+d}$ between $\frac{a}{b}$ and $\frac{c}{d}$ if $b+d \le n$. The first four rows are

Learning outcomes:
Author(s):

f_1	$\frac{0}{1}$						$\frac{1}{1}$
f_2	$\frac{0}{1}$			$\frac{1}{2}$			$\frac{1}{1}$
f_3	$\frac{0}{1}$		$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$		$\frac{1}{1}$
f_4	$\frac{0}{1}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{1}{1}$

We call the sequence in the n^{th} row of the table f_n.

It is clear that for each $\dfrac{a}{b}$ and $\dfrac{c}{d}$ in the$(n-1)^{st}$ row, we insert $\dfrac{a+c}{b+d}$ between $\dfrac{a}{b}$ and $\dfrac{c}{d}$ if $b+d \leq n$ (since this is the definition). This unusual (but easier) way of adding fractions gives what we call *median convergents.*

We are going to prove that $f_n = S_n$ for every n; that the n^{th} row of the table consists of all $\dfrac{p}{q}$ with $0 \leq \dfrac{p}{q} \leq 1, (p,q) = 1, 1 \leq q \leq n$; and that the fractions in each row are ordered from smallest to largest. From there, we get that for each $\dfrac{a}{b}$ and $\dfrac{c}{d}$ in the F_{n-1}, we insert $\dfrac{a+c}{b+d}$ between $\dfrac{a}{b}$ and $\dfrac{c}{d}$ if $b+d \leq n$

In both definitions, if $\dfrac{a}{b}$ and $\dfrac{c}{d}$ are next to each other in S_n (or f_n), then we say they are *consecutive fractions in S_n (or f_n).*

Theorem 1 (Textbook Theorem 6.1)**.** If $\dfrac{a}{b}$ and $\dfrac{c}{d}$ are consecutive fractions in f_n with $\dfrac{c}{d}$ to the left of $\dfrac{a}{b}$, then $ad - bc = 1$.

Proof We use induction. For $n = 1$, there are only two fractions, and $1(1) - 0(1) = 1$.

Assume that this fact holds for the $(n-1)^{st}$ row. That is, $ad - bc = 1$ for all $\dfrac{a}{b}$ and $\dfrac{c}{d}$ consecutive Farey fractions in F_{n-1} with $\dfrac{c}{d} < \dfrac{a}{b}$. For each pair, there are two options for the n^{th} row: either $b + d > n$, so $\dfrac{a}{b}$ and $\dfrac{c}{d}$ are still consecutive Farey fractions or $b + d \leq n$, so $\dfrac{a}{b}, \dfrac{a+c}{b+d}$ and $\dfrac{c}{d}$ are consecutive Farey fractions. If $b + d > n$, we are done. If $b + d \leq n$, then $\dfrac{c}{d}, \dfrac{a+c}{b+d}$, and $\dfrac{a}{b}$ are consecutive fractions, and we need to check both new pairs.

$$\begin{aligned}
(a+c)d - (b+d)c &= ad + cd - bc - cd \\
&= ad - bc = 1 \quad \text{(by induction hypothesis)} \\
a(b+d) - b(a+c) &= ab + ad - ba - cb \\
&= ad - bc = 1 \quad \text{(by induction hypothesis)}
\end{aligned}$$

■

Corollary 1 (Textbook Corollary 6.2)**.** Every $\frac{p}{a}$ in the table is in reduced form, ie, $(p, q) = 1$

Corollary 2 (Textbook Corollary 6.3)**.** The fractions in each row are ordered from smallest to largest.

Theorem 2 (Textbook Theorem 6.4)**.** If $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive fractions in any row, then for all f_n that contain $\frac{a+c}{b+d}$, $\frac{a+c}{b+d}$ has the smallest denominator of any rational number between $\frac{a}{b}$ and $\frac{c}{d}$ and is the unique rational number between $\frac{a}{b}$ and $\frac{c}{d}$ with denominator $b+d$.

Proof From the definition, $\frac{a+c}{b+d}$ is the first fraction to appear between $\frac{a}{b}$ and $\frac{c}{d}$ in row $b+d$. Thus, $\frac{c}{d} < \frac{a+c}{b+d} < \frac{a}{b}$ by Corollary 6.3. Now consider any $\frac{x}{y}$ with $\frac{c}{d} < \frac{x}{y} < \frac{a}{b}$. Then

$$\begin{aligned}\frac{a}{b} - \frac{c}{d} &= \left(\frac{a}{b} - \frac{x}{y}\right) + \left(\frac{x}{y} - \frac{c}{y}\right)\\ &= \frac{ay-bx}{by} + \frac{xd-cd}{dy} \geq \frac{1}{by} + \frac{1}{dy} = \frac{b+d}{bdy}\end{aligned}$$

and thus

$$\frac{b+d}{bdy} \leq \frac{ad-bd}{bd} = \frac{1}{bd}$$

and $y \geq b+d$. If $y > b+d$, then $\frac{x}{y}$ does not have the least denominator of rational numbers between $\frac{a}{b}$ and $\frac{c}{d}$. If $y = b+d$, then $\frac{ay-bx}{by} + \frac{xd-cy}{dy} = \frac{1}{by} + \frac{1}{dy}$ and $ay - bx = xd - cy = 1$. By solving, we find $x = a+c$ and $y = b+d$, so $\frac{x}{y} = \frac{a+c}{b+d}$. ∎

Theorem 3 (Textbook Theorem 6.5)**.** If $0 \leq x \leq y, (x, y) = 1$, then the fraction $\frac{x}{y}$ appears in the y^{th} row of the table and all future rows.

Proof The theorem is true by definition for $y = 1$. Assume that the theorem is true for $y = y_0 - 1$, with $y_0 > 1$. Then if $y = y_0 - 1$, then the fraction $\frac{x}{y}$ cannot be in the $(y-1)^{st}$ row by definition. Then $\frac{c}{d} < \frac{x}{y} < \frac{a}{b}$ for some $\frac{a}{b}, \frac{c}{d}$ in the $(y-1)^{st}$ row. Since $\frac{c}{d} < \frac{a+c}{b+d} < \frac{a}{b}$, and $\frac{a}{b}, \frac{c}{d}$ are consecutive fractions, $\frac{a+c}{b+d}$ is not in the $(y-1)^{st}$ row. Thus $b+d > y-1$ by the induction hypothesis.

By Theorem 6.4, $y \geq b+d$, so $y = b+d$. By the uniqueness part of Theorem 6.4, $x = a+c$. Thus, $\dfrac{x}{y}$ appears in the y^{th} row of the table and all future rows. ■

Corollary 3 (Textbook Corollary 6.6)**.** The n^{th} row consists of all reduce fractions with $\dfrac{a}{b}$ such that $0 \leq \dfrac{a}{b} \leq 1$ and $0 < b \leq n$. The fractions are listed from smallest to largest.

Now we have the definition of the Farey sequences which contain the level n Farey fractions between 0 and 1.

Definition 3. The n^{th} Farey sequence

$$F_n = \left\{ \frac{p}{q} : (p,q) = 1, 1 \leq q \leq n \right\}.$$

So that every term in the sequence is a fraction, we write $n = \dfrac{n}{1}$. The first four Farey sequences are

$$F_1 = \left\{ \ldots, -\frac{1}{1}, \frac{0}{1}, \frac{1}{1}, \ldots \right\}$$
$$F_2 = \left\{ \ldots, -\frac{1}{1}, -\frac{1}{2}, \frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \ldots \right\}$$
$$F_3 = \left\{ \ldots, -\frac{1}{1}, -\frac{2}{3}, -\frac{1}{2}, -\frac{1}{3}, \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \ldots \right\}$$
$$F_4 = \left\{ \ldots, -\frac{1}{1}, -\frac{3}{4}, -\frac{2}{3}, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}, \ldots \right\}$$

Depending on the source, Farey sequences may or may not be restricted to the interval $[0,1]$. However, if you look at the sets $f_n = S_n$, they aren't really sequences since they are finite.

Remark 1. If $\dfrac{p}{q} = [a_0; a_1, a_2, \ldots, a_n]$ with $(p,q) = 1$, then the fractions on either side of $\dfrac{p}{q}$ in the q^{th} Farey sequence are $[a_0; a_1, a_2, \ldots, a_n - 1]$ and $[a_0; a_1, a_2, \ldots, a_{n-1}]$. For example, $\dfrac{2}{9}$ has continued fraction expansion $[0;4,2]$ and fractions on either side of $\dfrac{2}{9}$ in the 9^{th} Farey sequence are $\dfrac{1}{5} = [0;4,1]$ (smaller) and $\dfrac{1}{4} = [0;4]$ (larger).

The following great image from Wikipedia highlights successive Farey fractions connected by semicircles. F_1 in brown, F_2 in red, F_3 in yellow, etc (the arc from

0 to $\dfrac{1}{n}$ tells you the color of the n^{th} Farey sequence). If you view the image on Wikipedia, hovering over a curve to highlights it and the terms in that Farey sequence (this seems to require a desktop browser) `https://upload.wikimedia.org/wikipedia/commons/9/91/Farey_diagram_horizontal_arc_9.svg`