April 5, 2024

Access GeoGebra at https://www.geogebra.org/m/tuf7y6sh.

Two stills from the GeoGebra interactive are in Figure 1 and Figure 2.

In-class Problem 1 *The steps below outline the proof in the general case, when $p = 7$ and $q = 5$. This case is in Figure 1.*

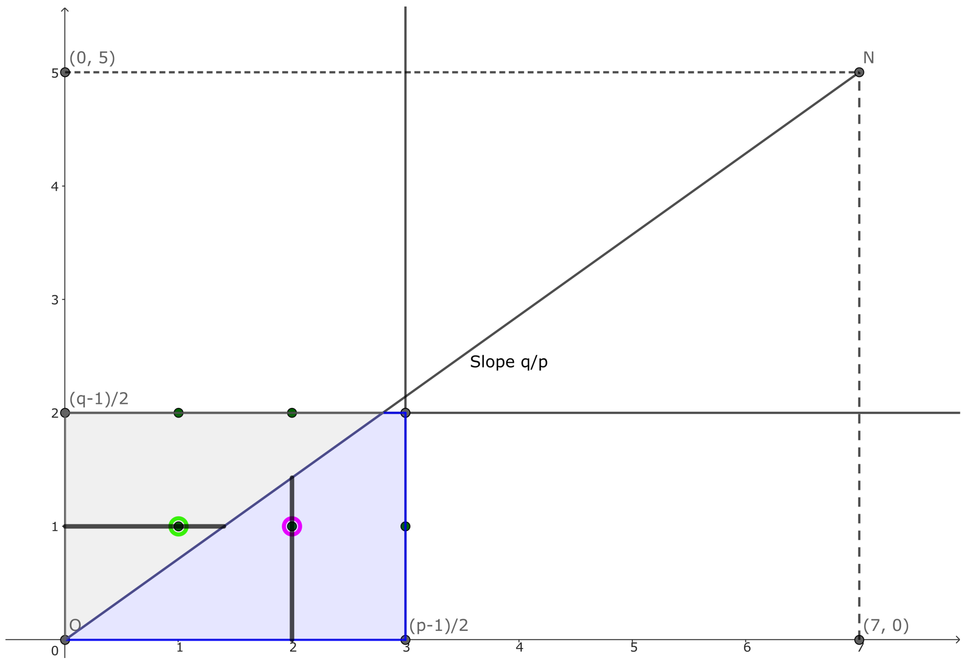

Figure 1: The lattice for the $p = 7, q = 5$ problem, with the $j = 1$ and $k = 2$ cases highlighted

(a) *The line segment between the origin and $(7, 5)$ has slope ______. Since $p = 7$ and $q = 5$ are distinct primes, there are no lattice points on line segment except the endpoints.*

(b) *First, we will count the number of points N_1 where $\frac{5-1}{2} \geq y > \frac{5}{7}x > 0$. This triangle is grey in the GeoGebra. We will count how many lattice points on each horizontal lines $j = 1, 2$. Let's just check the numbers we should get:*

- *When $j = 1$, there are ______ lattice points.*
- *When $j = 2$, there are ______ lattice points.*

For each j, we are counting positive integers $x <$ ______j. Which is,

Thus, the total number of lattice points in this triangle, N_1, is

Learning outcomes:
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(c) *Next we will count the rest of the lattice points in the rectangle, the blue region in the GeoGebra. We willl call this number N_2.*

The region is bounded by $0 < x \leq \frac{7-1}{2}$, $0 < y < \frac{5}{7}x$, and $y \leq \frac{5-1}{2}$. Now, the point A where $y = \frac{5}{7}x$ intersects $y = \frac{5-1}{2}$ is between two consecutive lattice points, with coordinates Similarly, the point B where $y = \frac{5}{7}x$ intersects $x = \frac{7-1}{2}$ is between two consecutive lattice points, with coordinates Thus, the only lattice point in the triangle A, B and $\left(\frac{7-1}{2}, \frac{5-1}{2}\right)$ is $\left(\frac{7-1}{2}, \frac{5-1}{2}\right)$. Therefore, there are also N_2 lattice points in the triangle with vertices $(0,0)$, $\left(\frac{7-1}{2}, 0\right)$, $\left(\frac{7-1}{2}, \frac{5-1}{2}\right)$.

(d) *We use the same method as N_1 to find N_2. We will count how many lattice points on each vertical lines $k = 1, 2, 3$. Let's just check the numbers we should get:*

- *When $k = 1$, there are ______ lattice points.*
- *When $k = 2$, there are ______ lattice points.*
- *When $k = 3$, there are ______ lattice points.*

For each k, we are counting positive integers $y <$ ______k. Which is,

Thus, the total number of lattice points in this triangle is

Thus, the total number of lattice points is $N_1 + N_2 =$ ____________.

In-class Problem 2 *The steps below outline the proof in the general case, when $p = 23$ and $q = 13$.*

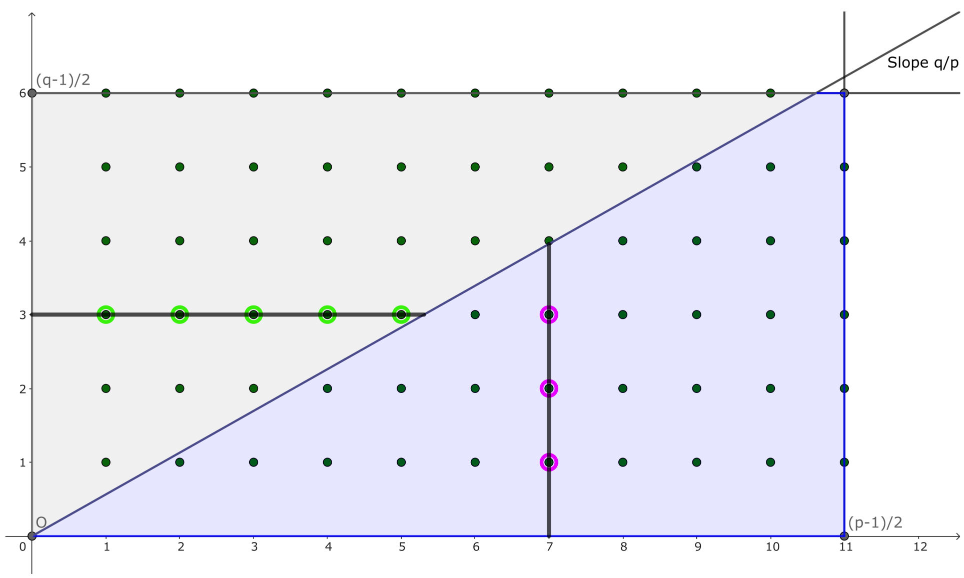

Figure 2: The lattice for the $p = 23, q = 13$, with the $j = 3$ and $k = 7$ cases highlighted

(a) *The line segment between the origin and $(23, 13)$ has slope ______. Since $p = 23$ and $q = 13$ are distinct primes, there are no lattice points on line segment except the endpoints.*

(b) *First, we will count the number of points N_1 where $\frac{13-1}{2} \geq y > \frac{13}{23}x > 0$. This triangle is grey in the GeoGebra. We will count how many lattice points on each horizontal lines $j = 1, 2, \ldots,$ ______. Let's just check one case, we should get:*

- *When $j = 3$, as in Figure 2, there are ______ lattice points.*

For each j, we are counting positive integers $x <$ ______ j. Which is,

Thus, the total number of lattice points in this triangle is

(c) Next we will count the rest of the lattice points in the rectangle, the blue region in the GeoGebra. We willl call this number N_2.

The region is bounded by $0 < x \leq \frac{23-1}{2}$, $0 < y < \frac{13}{23}x$, and $y \leq \frac{13-1}{2}$. Now, the point A where $y = \frac{13}{23}x$ intersects $y = \frac{13-1}{2}$ is between two consecutive lattice points, with coordinates Similarly, the point B where $y = \frac{13}{23}x$ intersects $x = \frac{23-1}{2}$ is between two consecutive lattice points, with coordinates Thus, the only lattice point in the triangle A, B and $\left(\frac{23-1}{2}, \frac{13-1}{2}\right)$ is $\left(\frac{23-1}{2}, \frac{13-1}{2}\right)$. Therefore, there are also N_2 lattice points in the triangle with vertices $(0,0)$, $\left(\frac{23-1}{2}, 0\right)$, $\left(\frac{23-1}{2}, \frac{13-1}{2}\right)$.

(d) We use the same method as N_1 to find N_2. We will count how many lattice points on each vertical lines $k = 1, 2, \ldots,$ ______. Let's just check the numbers we should get:

- When $k = 7$, as in Figure 2, there are ______ lattice points.

For each k, we are counting positive integers $y <$ ______ k. Which is,

Thus, the total number of lattice points in this triangle is

Thus, the total number of lattice points is $N_1 + N_2 =$ ____________.