

# Sums of squares

Reading None

The first result will prove which primes can be written as the sum of two squares. Note  $1^2 + 1^2 = 2$ , and if  $a$  is a positive integer such that  $a \equiv 3 \pmod{4}$ , then  $a$  cannot be written as the sum of two squares.

**Proposition 1.** Let  $m, n \in \mathbb{Z}$  with  $m, n > 0$ . If  $m$  and  $n$  can be written as the sums of two squares of integers, then  $mn$  can be written as the sum of two squares of integers.

**Proof** Let  $m, n \in \mathbb{Z}$  with  $m, n > 0$  and assume that there exists  $a, b, c, d \in \mathbb{Z}$  such that  $m = a^2 + b^2$  and  $n = c^2 + d^2$ . Then

$$\begin{aligned} mn &= (a^2 + b^2)(c^2 + d^2) = a^2c^2 + b^2c^2 + a^2d^2 + b^2d^2 \\ &= a^2c^2 + 2abcd + b^2d^2 + a^2d^2 - 2abcd + b^2c^2 \\ &= (ac + bd)^2 + (ad - bc)^2. \end{aligned}$$

■

We need two lemmas to prove

**Theorem 1.** Let  $n \in \mathbb{Z}$  with  $n > 0$ . Then  $n$  is expressible as the sum of two squares if and only if every prime factor congruent to 3 modulo 4 occurs to an even power in the prime factorization of  $n$ .

**Lemma 1.** If  $p$  is a prime such that  $p \equiv 1 \pmod{4}$ , then there are integers  $x, y$  such that  $x^2 + y^2 = kp$  for some  $k \in \mathbb{Z}$  with  $0 < k < p$ .

**Proof** Since  $p \equiv 1 \pmod{4}$ , we have that  $\left(\frac{-1}{p}\right) = 1$ . Thus, there exists  $x \in \mathbb{Z}$  with  $0 < x \leq \frac{p-1}{2}$  such that  $x^2 \equiv -1 \pmod{p}$ . Then,  $p \mid x^2 + 1$ , and we have that  $x^2 + 1 = kp$  for some  $k \in \mathbb{Z}$ . Thus, we found  $x$  and  $y = 1$ . Since  $x^2 + 1$  and  $p$  are positive, so is  $k$ . Also,

$$kp = x^2 + y^2 < \left(\frac{p}{2}\right)^2 + 1 < p^2$$

implies  $k < p$ .

■

**Proposition 2.** A prime  $p$  can be written as the sum of two squares if and only if  $p = 2$  or  $p \equiv 1 \pmod{4}$ .

**Proof** If  $p \equiv 3 \pmod{4}$ , then  $p$  cannot be written as the sum of two squares. Since the squares modulo 4 are 0 and 1, the integers that can be written as a sum of two squares are congruent to  $0^2 + 0^2 \equiv 0 \pmod{4}$ ,  $1^2 + 0^2 \equiv 1 \pmod{4}$  or  $1^2 + 1^2 \equiv 2 \pmod{4}$ , so no integer that is congruent to 3 modulo 4 can be written as the sum of two squares. Thus, if  $p$  can be written as the sum of two squares,  $p \not\equiv 3 \pmod{4}$ . That is,  $p = 2$  or  $p \equiv 1 \pmod{4}$ .

We will prove the other direction with two cases. When  $p = 2$ , then  $1^2 + 1^2 = 2$ . It remains to show that every prime  $p \equiv 1 \pmod{4}$  can be written as the sum of two squares.

Let  $p \equiv 1 \pmod{4}$ , and let  $m$  be the least integer such that there exists  $x, y \in \mathbb{Z}$  with  $x^2 + y^2 = mp$  and  $0 < m < p$  as in the previous theorem. We show that  $m = 1$ . Assume, by way of contradiction, that  $m > 1$ . Let  $a, b \in \mathbb{Z}$  such that

$$a \equiv x \pmod{m}, \quad \frac{-m}{2} < a \leq \frac{m}{2}$$

and

$$b \equiv y \pmod{m}, \quad \frac{-m}{2} < b \leq \frac{m}{2}.$$

Then

$$a^2 + b^2 \equiv x^2 + y^2 = mp \equiv 0 \pmod{m},$$

and so there exists  $k \in \mathbb{Z}$  with  $k > 0$  such that  $a^2 + b^2 = km$ . (Why?)

Now,

$$(a^2 + b^2)(x^2 + y^2) = (km)(mp) = km^2p.$$

By ??,  $(a^2 + b^2)(x^2 + y^2) = (ax + by)^2 + (ay - bx)^2$ , so  $(ax + by)^2 + (ay - bx)^2 = km^2p$ . Since  $a \equiv x \pmod{m}$  and  $b \equiv y \pmod{m}$ ,

$$ax + by \equiv x^2 + y^2 \equiv 0 \pmod{m}$$

and

$$ay - bx \equiv xy - yx \equiv 0 \pmod{m}$$

so  $\frac{ax + by}{m}, \frac{ay - bx}{m} \in \mathbb{Z}$  and

$$\left(\frac{ax + by}{m}\right)^2 + \left(\frac{ay - bx}{m}\right)^2 = \frac{km^2p}{m^2} = kp.$$

Now,  $\frac{-m}{2} < a \leq \frac{m}{2}$  and  $\frac{-m}{2} < b \leq \frac{m}{2}$  imply that  $a^2 \leq \frac{m^2}{4}$  and  $b^2 \leq \frac{m^2}{4}$ . Thus,  $km = a^2 + b^2 \leq \frac{m^2}{2}$ . Thus,  $k \leq \frac{m}{2} < m$ , but this contradicts that  $m$  is the smallest such integer.

Thus,  $m = 1$  and  $p$  can be written as the sum of two squares of integers. ■