Geometry Pythagorean Triples

Project on geometry and Pythagorean Triples.

Definition 1. A *rational point* is a point (x, y) whose coordinates x and y are both rational numbers.

Use rational points on the unit circle and trigonometry to derive the formula for generating Pythagorean triples.

Problem 1.1 *Let (x, y) be a rational point on the unit circle. That is, there exists $a, b, c \in \mathbb{Z}$ such that $x = \dfrac{a}{c}$ and $y = \dfrac{b}{c}$.*

Explain why we can write x and y with the same denominator.
Rubric. 2 points.

Problem 1.2 *Let (x, y) be any point on the unit circle, ie $x^2 + y^2 = 1$. Consider the line segment between $(0, 0)$ and (x, y). As long as (x, y) is not $(-1, 0)$, the slope of this line is*

$$t = \frac{y}{1 + x} = \tan \theta$$

where θ is the angle between the line segment and the x-axis.

(a) *Show that*

$$x = \frac{1 - t^2}{1 + t^2}, y = \frac{2t}{1 + t^2}$$

either using algebra or trig identities.

(b) *Show that (x, y) is a rational point not equal to $(-1, 0)$ if and only if t is rational.*

(c) *Let $t = \dfrac{m}{n}$. Prove that if $x, y > 0$, then $m > n, (m, n) = 1$, exactly one of m, n is even, and all nontrivial Pythagorean triples have the form $a = k(m^2 - n^2), b = k(2mn), c = k(m^+n^2)$, for some $k \in \mathbb{Z}$.*

 Hint: *Use that $k = \dfrac{c}{m^2 + n^2}$.*
Rubric. 4 points if individual, 3 if presenting as a pair.

Exploration 2 *The following problems are from* Number Theory: A Lively Introduction with Proofs, Applications, and Stories *by Erica Flapan, Tim Marks, and James Pommersheim Chapter 15, Some Nonlinear Diophantine Equations,15.5 A Geometric look at the Equation $x^4 + y^4 = z^2$ [?].*

$x = -1$ and $y = 0$ is the limit as $t \to \infty$

Theorem 1 (Fermat's Right Triangle Theorem)**.** The does not exist a right triangle with rational side lengths and area a perfect square.

Problem   2.1   *Let x, y, z be positive integers be the side lengths of a triangle with hypotenuse z and the area of the triangle is a perfect square. Prove that if z is the smallest such integer, then (x, y, z) is a primitive Pythagorean triple.*
Rubric. 4 points if individual, 3 if presenting as a pair.

Problem   2.2   *Let x, y, z as in the previous problem. Then there exists $m, n \in \mathbb{Z}$ with $m > n > 0$, $(m, n) = 1$ and exactly one of m and n even such that $x = m^2 - n^2$, $y = 2mn$, and $z = m^2 + n^2$. Furthermore, m, n, $m + n$ and $m - n$ are perfect squares.*

(a) *Prove that if $c, d \in \mathbb{Z}$ such that $c^2 = m + n$, and $d^2 = m - n$, then exactly one of $c + d$ and $c - d$ is divisible by 4.*

(b) *If $4 \mid c + d$, prove that $\dfrac{b^2}{4} = \dfrac{c + d}{4} \dfrac{c - d}{2}$, where the two factors on the right side of the equation are relatively prime.*

(c) *Show that there exists $s, t \in \mathbb{Z}$ such that $\dfrac{c + d}{4} = s^2$ and $\dfrac{c - d}{2} = t^2$*

(d) *Show that $(2s^2, t^2, a)$ is a Pythagorean triple and finish the proof of the right triangle theorem for this case.*

(e) *Repeat parts ??-?? for $4 \mid c - d$.*
Rubric. Part **??** 4 points if individual, 3 if presenting as a pair. Parts **??-??** 3 points per case if individual, 2 points per case if presenting as a pair.

Problem total: 10 points if individual, 7 points if pair.

Exploration   3   *(If presenting as a pair)*
Definition 2. A positive integer n is a *congruent number* if there exists a right triangle whose sides are all rational numbers and whose area is n.

Problem   3.1   *Show that the following are congruent numbers:*

- 6.

- 5.
Rubric. 2 points.

Problem   3.2   *Prove that there is no right triangle with integer sides whose area is 5. There is no right triangle with integer sides whose area is 1.*
Rubric. 3 points.