Equivalence Relations

In-class Problem 1 *Prove that*

$$[a] = \{b \in \mathbb{Z} : 3 \mid (a-b)\}$$

is an equivalence relation on $\mathbb{Z}$.

Proof Let $a, b \in \mathbb{Z}$. We must show that the relation is reflexive, symmetric, and transitive.

To show the relation is reflexive, we must show $a \in \{b \in \mathbb{Z} : 3 \mid (a-b)\}$. Since $\boxed{3 \mid a - a = 0}$, $a \in \{b \in \mathbb{Z} : 3 \mid (a-b)\}$.

To show the relation is symmetric, we must show that if $x \in \{b \in \mathbb{Z} : 3 \mid (a-b)\}$, then $a \in \{b \in \mathbb{Z} : 3 \mid (x-b)\}$. If $x \in \{b \in \mathbb{Z} : 3 \mid (a-b)\}$, then $\boxed{\text{there exists } k \in \mathbb{Z} \text{ such that } 3k = a - x}$. Therefore, $\boxed{-3k = b - a}$ and $a \in \{b \in \mathbb{Z} : 3 \mid (x-b)\}$.

To show the relation is transitive, we must show that if $x \in \{b \in \mathbb{Z} : 3 \mid (a-b)\}$ and $y \in \{b \in \mathbb{Z} : 3 \mid (x-b)\}$, then $x \in \{b \in \mathbb{Z} : 3 \mid (a-b)\}$. If $x \in \{b \in \mathbb{Z} : 3 \mid (a-b)\}$, then $\boxed{\text{there exists } k \in \mathbb{Z} \text{ such that } 3k = a - x}$. Similarly, if $y \in \{b \in \mathbb{Z} : 3 \mid (x-b)\}$, then $\boxed{\text{there exists } m \in \mathbb{Z} \text{ such that } 3m = x - y}$. Therefore, $\boxed{3(m+k) = a - k}$ and $y \in \{b \in \mathbb{Z} : 3 \mid (a-b)\}$. $\boxed{\text{Since the relation is reflexive, symmetric, and transitive, it is an equivalence relation.}}$ ■

Learning outcomes:
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