Decimal expansions

We take a number theoretic view of decimal (ie, regular) expansion of numbers.

We are going to look at our regular, decimal expansions of numbers from a number theory perspective in order to study something familiar as an analogy for continued fractions. We start with some familiar definitions.

Definition 1. *Let $\alpha \in \mathbb{R}$. Then α is a* rational number *(or $\alpha \in \mathbb{Q}$) if $\alpha = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$. Otherwise α is irrational.*

Example 1. (a) $0.5 = \frac{5}{10} = \frac{1}{2}$

(b) 0.666..... *where the 6s repeat forever is rational since* $0.666\cdots = \boxed{\frac{2}{3}}$. *We will actually prove that the decimal expansion of a rational number either repeats or terminates.*

(c) *The real number $\sqrt{2}$ is irrational. This is our first proof.*

(d) *The real constants π and e are irrational. We will prove that e is irrational in the homework. The proof that π is irrational is much harder.*

(e) *The real numbers $2^{\sqrt{2}}, e^{\pi}$, and πe are irrational. These were not proven until 1929.*

(f) *We still do not know if $\pi^{\sqrt{2}}, \pi^{e}$ or 2^{e} are rational or irrational.*

Theorem 1. $\sqrt{2} \notin \mathbb{Q}$.

Proof In order to get a contradiction, assume that $\sqrt{2} \in \mathbb{Q}$. Then $\sqrt{2} \in \frac{a}{b}$ for some $a, b \in \mathbb{Z}$, with $b \neq 0$. Without loss of generality, assume $(a, b) = 1$. By squaring both sides, we get $2 = \frac{a^2}{b^2}$, so $2b^2 = a^2$. Thus, $2 \mid a^2$ and $2 \mid a$. Thus, there is some integer c where $a = 2c$. Then $2b^2 = 4c^2$, so $b^2 = 2c^2$. Now we get that $2 \mid b$. Thus $2 \mid a$ and $2 \mid b$, which contradicts $(a, b) = 1$. So $\sqrt{2} \notin \mathbb{Q}$. ■

Proof by contradiction is a useful technique for proving a number is irrational.

Theorem 2. *Let $\alpha, \beta \in \mathbb{Q}$. Then $\alpha \pm \beta, \alpha\beta \in \mathbb{Q}$, and if $\beta \neq 0$, then $\frac{\alpha}{\beta} \in \mathbb{Q}$.*

Learning outcomes:
Author(s):

Proof The participation assignment covers $\alpha+\beta, \alpha\beta$. Replacing β with $-\beta$ gives $\alpha-\beta$. If $\beta \neq 0$, then there exists $a, b, c, d \in \mathbb{Z}$ where $\alpha = \dfrac{a}{b}$ and $\beta = \dfrac{c}{d}$ where none of b, c, d are zero.

$$\frac{\alpha}{\beta} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}.$$

■

The analogous statement for irrational numbers does not hold. For example $\sqrt{2}\sqrt{2} = 2$. The participation assignment is to find an example that does not work for addition.

Theorem 3. *Let $\alpha \in \mathbb{R}$ be the root of the polynomial*

$$f(x) = x^n + c_{n-1}x^{n-1} + \cdot + c_1 x + c_0$$

where $c_i \in \mathbb{Z}$ and $c_0 \neq 0$. Then $\alpha \in \mathbb{Z}$ or α is irrational.

Proof Assume that $\alpha \in \mathbb{Q}$. We must show that $\alpha \in \mathbb{Z}$. Now $\alpha = \dfrac{a}{b}$ for some integers a and $b \neq 0$. Without loss of generality, $(a, b) = 1$. Then for $f(\alpha) = 0$ implies

$$\left(\frac{a}{b}\right)^n + c_{n-1}\left(\frac{a}{b}\right)^{n-1} + c_{n-2}\left(\frac{a}{b}\right)^{n-2} + \cdots + c_1\left(\frac{a}{b}\right) + c_0 = 0.$$

■

Multiplying both sides by b^n, we get

$$a^n + c_{n-1}a^{n-1}b + c_{n-2}a^{n-2}b^2 + \cdots + c_1 ab^{n-1} + c_0 b^n = 0.$$

Then

$$a^n = b(-c_{n-1}a^{n-1} - c_{n-2}a^{n-2}b - \cdots - c_1 ab^{n-2} - c_0 b^{n-1}.$$

Thus, $b \mid a^n$. Since $(a, b) = 1$, we have that $b = \pm 1$. Then $\alpha = \dfrac{a}{\pm 1} = \pm a \in \mathbb{Z}$.

Example 2. (a) *$\sqrt{3}$ is a root of the polynomial $f(x) = x^2 - 3$. Since $\sqrt{3} \notin \mathbb{Z}$, then $\sqrt{3}$ is irrational.*

(b) *$2+\sqrt{7}$ is a root of the polynomial $f(x) = x^2 - 4x - 3$ and $\sqrt{7} \notin \mathbb{Z}$, $2+\sqrt{7}$ is irrational.*

(c) *$\sqrt[3]{5}$ is a root of the polynomial $f(x) = x^3 - 5$. Since 5 is between the perfect cubes $\boxed{1^3}$ and $\boxed{2^3}$, we have that $\boxed{1} < \sqrt[3]{5} < \boxed{2}$. Thus, $\sqrt[3]{5}$ is not an integer, and thus is irrational.*

Using this theorem involves finding a polynomial where x is a root. Sometimes this is basic algebra, like rewriting $x = 3 + \sqrt{2}$ as $0 = \boxed{x^2 - 6x + 7}$. However, for numbers like π and e, no such polynomial exists.

Every real number has a decimal expansion, which is how we are used to writing numbers.

Definition 2. *Let $\alpha \in \mathbb{R}$ with $0 \leq \alpha < 1$ and let $\sum_{n=1}^{\infty} \frac{a_n}{10^n} = 0.a_1a_2a_3\dots$ be a decimal representation of α. If there exist a positive integer ρ and N such that $a_n = a_{n+\rho}$ for all $n \geq N$, then α is* eventually periodic*; the sequence $a_Na_{N+1}\cdots a_{N+\rho-1}$ with ρ minimal is the* period *of α and ρ is the* period length*. If the smallest such N is 1, then α is* periodic*. An eventually periodic real number*

$$\alpha = 0.a_1a_2a_3\dots a_{N-1}a_Na_{N+1}\cdots a_{N+\rho-1}a_Na_{N+1}\cdots a_{N+\rho-1}a_Na_{N+1}\cdots a_{N+\rho-1}\cdots$$

is written

$$\alpha = 0.a_1a_2a_3\dots a_{N-1}\overline{a_Na_{N+1}\cdots a_{N+\rho-1}}.$$

This is a formalized definition of a repeating decimal.

Example 3. (a) *A decimal representation of $\frac{1}{2}$ is $0.5 = 0.5\overline{0}$, so $\frac{1}{2}$ is eventually periodic with period $\boxed{0}$ and length $\boxed{1}$. Any terminating decimal can be considered periodic with the same period and length.*

(b) *A decimal representation of $\frac{2}{3}$ is $0.\overline{6}$ is eventually periodic with period $\boxed{6}$ and length $\boxed{1}$.*

(c) *A decimal representation of $\sqrt{2}$ to 20 digits is* $1.41421356237309504880\dots$ *which does not* appear *to be eventually periodic, but maybe we have not computed enough digits.*

(d) *A decimal representation of π to 20 digits is* $3.14159263558979323846\dots$ *which does not* appear *to be eventually periodic but maybe we have not computed enough digits.*

You have probably heard that the decimal expansion of a ration number either terminates or repeats. We have formalized the definition of repeats to "eventually periodic," and show that terminating decimals are also eventually periodic. Now we prove that fact.

Theorem 4. *Let $\alpha \in \mathbb{R}$ with $0 \leq \alpha < 1$. Then $\alpha \in \mathbb{Q}$ if and only if α is eventually periodic.*

Proof ($\Rightarrow$) Assume that $\alpha \in \mathbb{Q}$. The $\alpha = \frac{a}{b}$ for some integers a and $b \neq 0$. Since, $0 \leq \alpha <$, we also have that $0 \leq a < b$. Now divide b into a by using long division; let the resulting decimal representation of α be

$$\sum_{n=1}^{\infty} \frac{q_n}{10^n} = 0.q_1q_2q_3\dots$$

By the division algorithm, the possible remainders when dividing a by b are $0, 1, 2, \dots, b-1$. At each stage of the long-division process, b is being divided by one of these remainders times 10 (ie, $0, 10, 20, \dots, (b-1)10$). The first such remainder is a. Accordingly, let $r_1 = a, r_2, r_3, \dots$ be the sequence of remainders corresponding to the quotients $q_1, q_2, q_3 \dots$ (so that $\frac{a}{b} = 0.q_1q_2q_3\dots$). Since the number of possible remainders is finite, $r_N = r_M$ for some N and M with $N < M$. If $p = M - N$, then $r_n = r_n + p$ for all $n \geq N$, from which $q_n = q_{n+p}$ for all $n \geq N$, and α is eventually periodic.

($\Leftarrow$) Assume that α is eventually periodic. Then there exists positive integers p and N such that $\alpha = 0.a_1a_2a_3\dots a_{N-1}\overline{a_Na_{N+1}\cdots a_{N+\rho-1}}$ Now

$$10^{N-1}\alpha = a_1a_2a_3\dots a_{N-1}.\overline{a_Na_{N+1}\cdots a_{N+\rho-1}}$$

and

$$10^p10^{N-1}\alpha = a_1a_2a_3\dots a_{N-1}a_Na_{N+1}\cdots a_{N+\rho-1}.\overline{a_Na_{N+1}\cdots a_{N+\rho-1}}.$$

Furthermore, $10^p10^{N-1}\alpha - 10^{N-1}\alpha$ is an integer since the identical repeating blocks cancel (leaving $a_1a_2a_3\dots a_{N-1}a_Na_{N+1}\cdots a_{N+\rho-1} - a_1a_2a_3\dots a_{N-1}$). Since $10^p10^{N-1}\alpha - 10^{N-1}\alpha = (10^p-1)10^{N-1}\alpha$, we have that $(10^p-1)10^{N-1}\alpha = m \in \mathbb{Z}$. Then

$$\alpha = \frac{m}{(10^p-1)10^{N-1}}.$$

Since $(10^p-1)10^{N-1}$ is a nonzero integer, then $\alpha \in \mathbb{Q}$ as desired. ■

Homework: parallel this proof for specific numbers.

A very different look at decimal numbers

Here is a very different way of generating decimal expansions using ideas from dynamical systems. The idea is to divide the unit interval $[0, 1)$ into intervals $\left[\frac{i}{10}, \frac{i+1}{10}\right)$ where $i = 0, 1, 2, \dots, 9$. If a number $x \in \left[\frac{i}{10}, \frac{i+1}{10}\right)$, then the first digit of the decimal expansion is i. For example, when $i = 1$, the interval is $\left(\boxed{\frac{1}{10}}, \boxed{\frac{2}{10}}\right)$ and the first digit of every x in the interval is $\boxed{1}$.

To get the second digit, we break each of these intervals into 10 smaller intervals $\left[\frac{i}{10} + \frac{j}{10^2}, \frac{i}{10} + \frac{j+1}{10^2}\right)$, $0 \leq i \leq 9, 0 \leq j \leq 9$. For each $x \in \left[\frac{i}{10} + \frac{j}{10^2}, \frac{i}{10} + \frac{j+1}{10^2}\right)$, $x =$

$0.ij\ldots$. For example, when $i=2, j=3$, the interval is $\left(\boxed{\frac{2}{10}+\frac{3}{100}}, \boxed{\frac{2}{10}+\frac{4}{100}}\right)$ and the first digit of every $x = 0.\boxed{2}\boxed{3}\ldots$.

Determining the rest of the digits involves iterating this process.