Equivalence Relations

Learning Objectives. By the end of class, students will be able to:

- Prove a given set is an equivalence relation .

Instructor Notes: **Reading** Strayer, Appendix B

Turn in Let R be the equivalence relation on $\mathbb{R}$ defined by

$$[a] = \{b \in \mathbb{R} : \sin(a) = \sin(b) \text{ and } \cos(a) = \cos(b)\}.$$

Prove that R is an equivalence relation on $\mathbb{R}$. Describe the equivalence classes on $\mathbb{R}$

Solution: Since $\sin(a) = \sin(a)$ and $\cos(a) = \cos(a)$, the relation R is reflexive.

If $\sin(a) = \sin(b)$ and $\cos(a) = \cos(b)$, the $\sin(b) = \sin(a)$ and $\cos(b) = \cos(a)$, so the relation is symmetric.

If $\sin(a) = \sin(b)$ and $\cos(a) = \cos(b)$, $\sin(b) = \sin(c)$ and $\cos(b) = \cos(c)$, then $\sin(a) = \sin(c)$ and $\cos(a) = \cos(c)$ is transitive.

Note that $\sin(a) = \sin(b)$ if $b = a + 2\pi k$ or $b = -a + \pi + 2\pi k$ for some $k \in \mathbb{Z}$, and $\cos(a) = \cos(b)$ if $b = a + 2\pi k$ or $b = -a + 2\pi k$ for some $k \in \mathbb{Z}$. These conditions are both true with $b = a + 2\pi k$. Thus, for $a \in [0, 2\pi)$,

$$[a] = \{\dots, a - 4\pi, a - 2\pi, a, a + 2\pi, a + 4\pi, \dots\}.$$

In-class Problem 1 Prove that

$$[a] = \{b \in \mathbb{Z} : 3 \mid (a - b)\}$$

is an equivalence relation on $\mathbb{Z}$.

Proof Let $a, b \in \mathbb{Z}$. We must show that the relation is reflexive, symmetric, and transitive.

To show the relation is reflexive, we must show $a \in \{b \in \mathbb{Z} : 3 \mid (a - b)\}$. Since $\boxed{3 \mid a - a = 0}$, $a \in \{b \in \mathbb{Z} : 3 \mid (a - b)\}$.

To show the relation is symmetric, we must show that if $x \in \{b \in \mathbb{Z} : 3 \mid (a - b)\}$, then $a \in \{b \in \mathbb{Z} : 3 \mid (x - b)\}$. If $x \in \{b \in \mathbb{Z} : 3 \mid (a - b)\}$, then $\boxed{\text{there exists } k \in \mathbb{Z} \text{ such that } 3k = a - x}$. Therefore, $\boxed{-3k = b - a}$ and $a \in \{b \in \mathbb{Z} : 3 \mid (x - b)\}$.

To show the relation is transitive, we must show that if $x \in \{b \in \mathbb{Z} : 3 \mid (a - b)\}$ and $y \in \{b \in \mathbb{Z} : 3 \mid (x - b)\}$, then $x \in \{b \in \mathbb{Z} : 3 \mid (a - b)\}$. If $x \in \{b \in \mathbb{Z} : 3 \mid (a - b)\}$, then $\boxed{\text{there exists } k \in \mathbb{Z} \text{ such that } 3k = a - x}$. Similarly, if $y \in \{b \in \mathbb{Z} : 3 \mid (x - b)\}$, then $\boxed{\text{there exists } m \in \mathbb{Z} \text{ such that } 3m = x - y}$. Therefore, $\boxed{3(m + k) = a - k}$ and $y \in \{b \in \mathbb{Z} : 3 \mid (a - b)\}$. $\boxed{\text{Since the relation is reflexive, symmetric, and transitive, it is an equivalence relation.}}$ ■

Learning outcomes:
Author(s): Claire Merriman