Farey Fractions

Project on Farey Fractions.

The following problems are based on *The Theory of Number* by Ivan Niven, Herbert S. Zuckerman, and Hugh L. Montgomery [**?**].

The subject of *Diophantine Approximation* concerns approximating real numbers with rational numbers. One way to do this is with *Farey fractions*–which we can generate by adding "wrong." However, there is a much simpler way to generate Farey fractions. One goal is to show these two methods give the same sequences.

Exploration 1

Definition 1. The N^{th} *Farey sequence* is the list of all fractions, written from smallest to largest, between 0 and 1 where the denominator is less than or equal to N when written as a reduced fraction. We write this sequence as $\mathcal{F}_N$.

The first three Farey sequences are:

$$\mathcal{F}_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\},$$
$$\mathcal{F}_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\},$$
$$\mathcal{F}_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$$

Problem 1.1 *Prove that for a positive integer $N \geq 2$, the number of elements of $\mathcal{F}_N$ with denominator N is equal to $\phi(N)$.*

Rubric. 5 points if individual, 4 points if pair.

Exploration 2 *Another way to generate the Farey sequence is using a table. In the first row, write $\frac{0}{1}$ and $\frac{1}{1}$.*

To form the second row, copy the first row. Then insert $\frac{0+1}{1+1}$ between $\frac{0}{1}$ and $\frac{1}{1}$.

To form the n^{th} row, copy the $(n-1)^{st}$ row. Then for each $\frac{a}{b}, \frac{c}{d}$ in the $(n-1)$ row, if $b+d \leq n$, insert $\frac{a+c}{b+d}$ between $\frac{a}{b}$ and $\frac{c}{d}$.

Learning outcomes:
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$\frac{0}{1}$												$\frac{1}{1}$
$\frac{0}{1}$						$\frac{1}{2}$						$\frac{1}{1}$
$\frac{0}{1}$				$\frac{1}{3}$		$\frac{1}{2}$		$\frac{2}{3}$				$\frac{1}{1}$
$\frac{0}{1}$			$\frac{1}{4}$	$\frac{1}{3}$		$\frac{1}{2}$		$\frac{2}{3}$	$\frac{3}{4}$			$\frac{1}{1}$
$\frac{0}{1}$		$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$		$\frac{1}{1}$
$\frac{0}{1}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{1}{1}$

Lemma 1 (Niven-Zukerman-Montgomery, Theorem 6.1)**.** If $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive fractions in the n^{th} row of the table, with $\frac{a}{b}$ to the left of $\frac{c}{d}$, then $bc - ad = 1$.

This lemma matches with the definition of a Farey pair *from Strayer Chapter 7, Student Project 2.*

Problem 2.1 *Prove this lemma.*
Rubric. 5 points if individual, 4 points if pair.

Problem 2.2 *(If presenting as a pair) Strayer Chapter 7, Student Project 2.*
Rubric. Parts (a)-(c): 4 points. Part (d): 4 points.

Problem 2.3 *(If presenting as an individual) Prove*
Corollary 1 (Niven-Zukerman-Montgomery, Corollary 6.2)**.** Every $\frac{a}{b}$ in the table is in reduced form.
Corollary 2 (Niven-Zukerman-Montgomery, Corollary 6.3)**.** The fractions in each row are listed in order from smallest to largest.
Rubric. 5 points.

Problem 2.4 *Prove that if $0 \le m \le n$ and $(m, n) = 1$, then fraction $\frac{m}{n}$ is in the n^{th} row of the table. Therefore, algorithmic definition of the n^{th} Farey sequence equivalent to the definition: The N^{th}* Farey sequence *is the list of all fractions, written from smallest to largest, between 0 and 1 where the denominator is less than or equal to N when written as a reduced fraction.*
Rubric. 5 points if individual, 4 points if pair.