Introduction to modular arithmetic

Learning Objectives. By the end of class, students will be able to:

- Prove that congruence modulo m is an equivalence relation on $\mathbb{Z}$.
- Define a complete residue system.
- Practice using modular arithmetic. .

Reading Strayer, Section 2.1 through Example 1.

Turn in The book concludes the section with a caution about division. It states that $6a \equiv 6b \pmod 3$ for all integers a and b. Explain why this is true.

Solution: Since $3 \mid 6a - 6b = 3(2a - 2b)$, $6a \equiv 6b \pmod 3$ for all integers a and b.

Definition (divisibility definition of $a \equiv b \pmod m$)**.** Let $a, b, m \in \mathbb{Z}$ with $m > 0$. We say that a is *congruent to b modulo m* and write $a \equiv b \pmod m$ if $m \mid b - a$, and m is said to be the *modulus of the congruence.* The notation $a \not\equiv b \pmod m$ means a is not congruent to b modulo m, or a is *incongruent to b modulo m.*

Definition (remainder definition of $a \equiv b \pmod m$)**.** Let $a, b, m \in \mathbb{Z}$ with $m > 0$. We say that a is congruent to b modulo m if a and b have the same remainder when divided by m.

Be careful with this idea and negative values. Make sure you understand why $-2 \equiv 1 \pmod 3$ or $-10 \equiv 4 \pmod 7$.

Proposition 1 (Definitions of congruence modulo m are equivalent)**.** *These two definitions are equivalent. That is, for $a, b, m \in \mathbb{Z}$ with $m > 0$, $m \mid b - a$ if and only if a and b have the same remainder when divided by m.*

Proof Let $a, b, m \in \mathbb{Z}$ with $m > 0$. By the **??**, there exists $q_1, q_2, r_1, r_2 \in \mathbb{Z}$ such that

$$\begin{aligned} &aq_1m + r_1, 0 \le r_1 < m, \text{ and} \\ &bq_2m + r_2, 0 \le r_2 < m. \end{aligned}$$

If $m \mid b-a$, then by definition, there exists $k \in \mathbb{Z}$ such that $mk = b-a$. Thus, $mk = q_2m + r_2 - q_1m - r_1$. Rearranging, we get $m(k - q_2 + q_1) = r_2 - r_1$ and $m \mid r_2 - r_1$. Since $0 \le r_1 < m, 0 \le r_2 < m$, we have $-m < r_2 - r_1 < m$. Thus, $r_2 - r_1 = 0$, so a and b have the same remainder when divided by m.

In the other direction, if $r_1 = r_2$, then $a - b = q_1m - q_2m = m(q_1 - q_2)$. Thus, $m \mid a - b$. ■

Example 1. *We will eventually find a function that generates all integers solutions to the equation $a^2 + b^2 = c^2$ (this can be done with only divisibility, so feel free to try for yourself after class).*

Modular arithmetic allows us to say a few things about solutions.

First, let's look at $\pmod 2$**.** *Note that* $0^2 \equiv 0 \pmod 2$ *and* $1^2 \equiv 1 \pmod 2$.

- **Case 1:** $c^2 \equiv 0 \pmod 2$ *In this case,* $c \equiv 0 \pmod 2$ *and either* $1^2 + 1^2 \equiv 0 \pmod 2$ *or* $0^2 + 0^2 \equiv 0 \pmod 2$. *So, we know* $a \equiv b \pmod 2$. (Note: $\pmod 4$ will eliminate the $a \equiv b \equiv 1 \pmod 2$ case)
- **Case 2:** $c^2 \equiv 1 \pmod 2$ *In this case,* $c \equiv 1 \pmod 2$ *and either* $0^2 + 1^2 \equiv 1 \pmod 2$. *So, we know* $a \not\equiv b \pmod 2$.

Learning outcomes:
Author(s): Claire Merriman

Let's start with $\pmod 3$**.** *Note that* $0^2 \equiv 0 \pmod 3$*,* $1^2 \equiv 1 \pmod 3$*, and* $2^2 \equiv 1 \pmod 3$*.*

Case 1: $c^2 \equiv 0 \pmod 3$**.** *In this case,* $c \equiv 0 \pmod 3$ *and* $0^2 + 0^2 \equiv 0 \pmod 3$*. So, we know* $a \equiv b \equiv c \equiv 0 \pmod 3$*.*

Case 2: $c^2 \equiv 1 \pmod 3$**.** *In this case,* c *could be* 1 *or* 2 *modulo* 3*. We also know* $0^2 + 1^2 \equiv 1 \pmod 3$*, so* $a \not\equiv b \pmod 3$*.*

Case 3: $c^2 \equiv 2 \pmod 3$ *has no solutions.*

So at least one of a, b, c *is even, and at least one is divisible by* 3*.*

We can use the idea of congruences to simplify divisibility arguments, as well as nonlinear Diophantine equations.