Other number systems: $\mathbb{Z}[\sqrt{d}]$

Project on $\mathbb{Z}[\sqrt{d}]$.

Definition 1. *Let $d \in \mathbb{Z}$ such that d is not a perfect square. Define $\mathbb{Z}[\sqrt{d}] = \{a+b\sqrt{d} : a, b \in \mathbb{Z}\}$, pronounced "Z adjoin square-root d" If $d < 0$, then $\mathbb{Z}[\sqrt{d}]$ is a set of complex numbers. We will use the regular definition of addition and multiplication.*

Definition 2. *Let $d \in \mathbb{Z}$ such that d is not a perfect square, and let $z \in \mathbb{Z}[\sqrt{d}]$ where $z = a + b\sqrt{d}$. The* conjugate of z *is $\overline{z} = a - b\sqrt{d}$, and the* norm of z *is $|z| = z\overline{z} = a^2 - db^2$.*

Strayer uses the notation z' for the conjugate.

Exploration 1 Problem 1.1 *Strayer Chapter 1, Exercise 77 explores the case $d = -10$. Then $\mathbb{Z}[\sqrt{d}]$ is not have unique factorization. Do this problem.*

Rubric. Parts (a) and (b): 3 points each if individual, 2 points each if pair. Part (c): 2 points.

Problem total: 8 points if individual, 6 points if pair.

Problem 1.2 *If $x, y \in \mathbb{Z}[\sqrt{d}]$, prove that $|xy| = |x||y|$. If $\frac{x}{y} \in \mathbb{Z}[\sqrt{d}]$, prove that $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$.*

Rubric. 2 points.

Problem 1.3 *(If presenting as a pair)*

(a) *Prove that the norm of any nonzero element (ie, $a \neq 0$ or $b \neq 0$ of $\mathbb{Z}[\sqrt{-5}]$ is positive.*

(b) *When, if ever, does $|a + b\sqrt{-5}| = 1$?*

(c) *When, if ever, does $|a + b\sqrt{-5}| = 13$?*

Rubric. 4 points.

Problem 1.4 *Strayer Chapter 8, Student Project 6.*

Rubric. Part (a): 2 points. Parts (b) and (c): 4 points each if individual, 3 points each if pair.

Problem total: 10 points if individual, 8 points if pair.

Learning outcomes:
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