Sums of three squares

We prove which integers cannot be written as the sum of four squares.

We finish out the sums of squares section by classifying which integers can be written as the sum of three squares and sum of four squares. These cases are more difficult than the sum of two squares since there is no formula analogous to the April 8 participation assignment.

Theorem 1 (Sum of three squares necessary condition)**.** Let $m, n \in \mathbb{Z}$ with $m, n \geq 0$. If $N = 4^m(8n+7)$, then N can not be written as the sum of 3 squares.

Proof We start by proving the $m = 0$ case. In order to get a contradiction, assume that $N = 8n + 7$ can be written as the sum of three squares. Thus, there exists $x, y, z \in \mathbb{Z}$ such that

$$8n + 7 = x^2 + y^2 + z^2.$$

Now, $8n + 7 \equiv 7 \pmod 8$ and $x^2 + y^2 + z^2 \not\equiv 7 \pmod 8$ (by participation assignment), which gives the contradiction we are looking for.

Now we assume $m > 0$. and again assume $N = 4^m(8n+7)$ can be written as the sum of three squares. As before, there exist $x, y, z \in \mathbb{Z}$ such that

$$4^m(8n+7) = x^2 + y^2 + z^2$$

and x, y, z are even (by participation assignment). So there exists x', y' and z' such that $x = 2x', y = 2y'$, and $z = 2z'$. Substituting into our definition of N, we get

$$4^{m-1}(8n+7) = (x')^2 + (y')^2 + (z')^2.$$

Repeating this process $m - 1$ times, we find $8n + 7$ is expressible as a sum of three squares, a contradiction. Thus, $N = 4^m(8n+7)$ cannot be written as the sum of three squares. ■

Now, the converse is true. Legendre proved this in 1798, but is much harder to prove, due to the lack of formula like the one from April 8 participation assignment. Note that any integer that cannot be written as the sum of three squares cannot be written as the sum of two squares.

Example 1. Determine whether 1584 is expressible as the sum of three squares.

The highest power of 4 that divides 1584 evenly is 16, leaving $99 \equiv 3 \pmod 8$. Thus, 1584 can be written as the sum of three squares:

Learning outcomes:
Author(s):

Multiple Choice:

(a) True ✓

(b) False

(c) Not enough information

Since this also allows us to factor 1584, we also know 1584 can be written as the sum of two squares:

Multiple Choice:

(a) True

(b) False ✓

(c) Not enough information