

Farey Fractions

Project on Farey Fractions.

The following problems are based on *The Theory of Number* by Ivan Niven, Herbert S. Zuckerman, and Hugh L. Montgomery.

The subject of *Diophantine Approximation* concerns approximating real numbers with rational numbers. One way to do this is with *Farey fractions*—which we can generate by adding “wrong.” However, there is a much simpler way to generate Farey fractions. One goal is to show these two methods give the same sequences.

Exploration 1

Definition 1. The N^{th} Farey sequence is the list of all fractions, written from smallest to largest, between 0 and 1 where the denominator is less than or equal to N when written as a reduced fraction. We write this sequence as \mathcal{F}_N .

The first three Farey sequences are:

$$\begin{aligned}\mathcal{F}_1 &= \left\{ \frac{0}{1}, \frac{1}{1} \right\}, \\ \mathcal{F}_2 &= \left\{ \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \right\}, \\ \mathcal{F}_3 &= \left\{ \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right\}\end{aligned}$$

Problem 1.1 Prove that for a positive integer $N \geq 2$, the number of elements of \mathcal{F}_N with denominator N is equal to $\phi(N)$.

Rubric. 5 points if individual, 4 points if pair.

Exploration 2 Another way to generate the Farey sequence is using a table. In the first row, write $\frac{0}{1}$ and $\frac{1}{1}$.

To form the second row, copy the first row. Then insert $\frac{0+1}{1+1}$ between $\frac{0}{1}$ and $\frac{1}{1}$.

To form the n^{th} row, copy the $(n-1)^{\text{st}}$ row. Then for each $\frac{a}{b}, \frac{c}{d}$ in the $(n-1)$ row, if $b+d \leq n$, insert $\frac{a+c}{b+d}$ between $\frac{a}{b}$ and $\frac{c}{d}$.

Learning outcomes:
Author(s): Claire Merriman

$\frac{0}{1}$												$\frac{1}{1}$
$\frac{0}{1}$					$\frac{1}{2}$							$\frac{1}{1}$
$\frac{0}{1}$			$\frac{1}{3}$		$\frac{1}{2}$		$\frac{2}{3}$					$\frac{1}{1}$
$\frac{0}{1}$			$\frac{1}{3}$	$\frac{1}{1}$	$\frac{2}{2}$	$\frac{1}{1}$	$\frac{3}{3}$	$\frac{2}{2}$	$\frac{3}{3}$			$\frac{1}{1}$
$\frac{0}{1}$		$\frac{1}{5}$	$\frac{1}{4}$	$\frac{3}{1}$	$\frac{2}{2}$	$\frac{1}{1}$	$\frac{3}{3}$	$\frac{2}{2}$	$\frac{4}{3}$	$\frac{4}{5}$		$\frac{1}{1}$
$\frac{0}{1}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{3}{1}$	$\frac{2}{2}$	$\frac{1}{1}$	$\frac{3}{3}$	$\frac{2}{2}$	$\frac{4}{3}$	$\frac{5}{4}$	$\frac{5}{6}$	$\frac{1}{1}$
$\frac{0}{1}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{3}{1}$	$\frac{2}{2}$	$\frac{1}{1}$	$\frac{3}{3}$	$\frac{2}{2}$	$\frac{4}{3}$	$\frac{5}{4}$	$\frac{5}{6}$	$\frac{1}{1}$

Lemma 1 (Niven-Zukerman-Montgomery, Theorem 6.1). *If $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive fractions in the n^{th} row of the table, with $\frac{a}{b}$ to the left of $\frac{c}{d}$, then $bc - ad = 1$.*

This lemma matches with the definition of a Farey pair from Strayer Chapter 7, Student Project 2.

Problem 2.1 *Prove this lemma.*

Rubric. 5 points if individual, 4 points if pair.

Problem 2.2 *(If presenting as a pair) Strayer Chapter 7, Student Project 2.*

Rubric. Parts (a)-(c): 4 points. Part (d): 4 points.

Problem 2.3 *(If presenting as an individual) Prove*

Corollary 1 (Niven-Zukerman-Montgomery, Corollary 6.2). *Every $\frac{a}{b}$ in the table is in reduced form.*

Corollary 2 (Niven-Zukerman-Montgomery, Corollary 6.3). *The fractions in each row are listed in order from smallest to largest.*

Rubric. 5 points.

Problem 2.4 *Prove that if $0 \leq m \leq n$ and $(m, n) = 1$, then fraction $\frac{m}{n}$ is in the n^{th} row of the table.*

Therefore, algorithmic definition of the n^{th} Farey sequence equivalent to the definition: The N^{th} Farey sequence is the list of all fractions, written from smallest to largest, between 0 and 1 where the denominator is less than or equal to N when written as a reduced fraction.

Rubric. 5 points if individual, 4 points if pair.