March 30–Proof of quadratic reciprocity

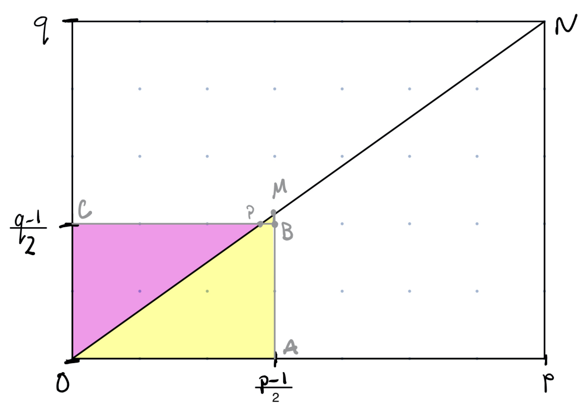
We finally prove quadratic reciprocity!

Theorem 1 (Restatement of quadratic reciprocity)**.** Let p and a be odd primes with $p \neq q$. Then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}.$$

Definition 1. A *lattice point* is a point $(x, y) \in \mathbb{R}^2$ where $x, y \in \mathbb{Z}$. We can write this as $(x, y) \in \mathbb{Z}^2$.

Proof Without loss of generality, assume that $p > q$. We draw the rectangle $O = (0, 0)$, $A = \left(\frac{p-1}{2}, 0\right)$, $B = \left(\frac{p-1}{2}, \frac{q-1}{2}\right)$, and $C = \left(0, \frac{q-1}{2}\right)$, like in the graphic below:



The participation assignment is to count the lattice points in the rectangle $OABC$ outlined in grey, including those on the lines AB and BC, but not those on OA or OC.

In order to count these lattice points another way, we are going to show that there are N_1 lattice points in the triangle OPC not including OC(pink) and N_2 lattice points in in OAM not including OA (yellow), thus the total number of lattice points is $N_1 + N_2$. We will find that $N_1 = \sum_{j=1}^{\frac{q-1}{2}} \left\lfloor \frac{jp}{q} \right\rfloor$ and $N_2 = \sum_{j=1}^{\frac{-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor$.

Learning outcomes:
Author(s):

Thus, by the previous lemma, $\left\lfloor \frac{p}{q} \right\rfloor = (-1)^{N_1}$ and $\left\lfloor \frac{q}{p} \right\rfloor = (-1)^{N_2}$, which will let us finish the proof.

We will do an examples first:

Example 1. We look at the example above with $p = 7$ and $q = 5$.

a) The line ON has slope $\frac{5}{7}$. Since p and q are distinct primes, there are no lattice points on ON except the endpoints.

b) The x-coordinate of M is 3, y-coordinate of M is $\frac{15}{7}$.

c) The y-coordinate of M lies between two consecutive integers 2 and 3.

Thus, the triangle PMB has no lattice points except possibly those on PB. We can then count the number of lattice points in $OABC$ by adding the number of lattice points in OCP to those in OAM.

To find N_1, the number of lattice points in OPC, not including those on OC, we count how many lattice points on the line $y = j$ are to the left of ON for $j = 1, 2, \ldots, \frac{q-1}{2}$ (in our case, this is only $j = 1, 2$.) Another was of saying this is for each j, we want the number of nonnegative integers less than

Multiple Choice:

(a) $\frac{7j}{5}$ ✓

(b) $\frac{5j}{7}$

Thus, we have for each j, there are

Multiple Choice:

(a) $\left\lfloor \frac{7j}{5} \right\rfloor$ ✓

(b) $\left\lfloor \frac{5j}{7} \right\rfloor$

lattice points in OPC. Then $N_1 =$

Multiple Choice:

(a) $\sum_{j=1}^{2} \left\lfloor \frac{7j}{5} \right\rfloor$ ✓

(b) $\displaystyle\sum_{j=1}^{2} \left\lfloor \frac{5j}{7} \right\rfloor$

To find N_2, we use a similar counting method on OAM. Now, we count the lattice points on $x = j$ for $j = 1, 2, \ldots, \frac{p-1}{2}$. Thus, for each j, we want the number of nonnegative integers less than

Multiple Choice:

(a) $\frac{7j}{5}$

(b) $\frac{5j}{7}$ ✓

Thus, we have for each j, there are

Multiple Choice:

(a) $\left\lfloor \frac{7j}{5} \right\rfloor$

(b) $\left\lfloor \frac{5j}{7} \right\rfloor$ ✓

l attice points in OPC. Then $N_2 =$

Multiple Choice:

(a) $\displaystyle\sum_{j=1}^{3} \left\lfloor \frac{7j}{5} \right\rfloor$ ✓

(b) $\displaystyle\sum_{j=1}^{3} \left\lfloor \frac{5j}{7} \right\rfloor$

Now we generalize this idea to any odd primes p and q with $p > q$.

a) The line ON has slope $\frac{q}{p}$. Since p and q are distinct primes, there are no lattice points on ON except the endpoints.

b) The x-coordinate of M is $\frac{p-1}{2}$, y-coordinate of M is $\frac{(p-1)}{2}\frac{q}{p} = \frac{q}{2} - \frac{q}{2p}$.

c) The y-coordinate of M lies between two consecutive integers $\frac{q-1}{2}$ and $\frac{q+1}{2}$, since

$$\frac{q-1}{2} = \frac{q}{2} - \frac{1}{2} < \frac{q}{2} - \frac{q}{2p} < \frac{q}{2} < \frac{q+1}{2}$$

Thus, the triangle PMB has no lattice points except possibly those on PB. We can then count the number of lattice points in $OABC$ by adding the number of lattice points in OCP to those in OAM.

To find N_1, the number of lattice points in OPC, not including those on OC, we count how many lattice points on the line $y = j$ are to the left of ON for $j = 1, 2, \ldots, \frac{q-1}{2}$. Another was of saying this is for each j, we want the number of nonnegative integers less than

Multiple Choice:

(a) $\frac{jp}{q}$ ✓

(b) $\frac{jq}{p}$

Thus, we have for each j, there are

Multiple Choice:

(a) $\left\lfloor \frac{jp}{q} \right\rfloor$ ✓

(b) $\left\lfloor \frac{jq}{p} \right\rfloor$

lattice points in OPC. Then $N_1 =$

Multiple Choice:

(a) $\sum_{j=1}^{2} \left\lfloor \frac{jp}{q} \right\rfloor$ ✓

(b) $\sum_{j=1}^{2} \left\lfloor \frac{jq}{p} \right\rfloor$

To find N_2, we use a similar counting method on OAM. Now, we count the lattice points on $x = j$ for $j = 1, 2, \ldots, \frac{p-1}{2}$. Thus, for each j, we want the number of nonnegative integers less than

Multiple Choice:

(a) $\dfrac{jp}{q}$

(b) $\dfrac{jq}{p}$ ✓

Thus, we have for each j, there are

Multiple Choice:

(a) $\left\lfloor \dfrac{jp}{q} \right\rfloor$

(b) $\left\lfloor \dfrac{jq}{p} \right\rfloor$ ✓

lattice points in OPC. Then $N_2 =$

Multiple Choice:

(a) $\displaystyle\sum_{j=1}^{2} \left\lfloor \frac{jp}{q} \right\rfloor$

(b) $\displaystyle\sum_{j=1}^{2} \left\lfloor \frac{jq}{p} \right\rfloor$ ✓

From the previous Lemma, $\left(\dfrac{p}{q}\right) = (-1)^{N_1}$ and $\left(\dfrac{q}{p}\right) = (-1)^{N_2}$. Thus,

$$\begin{aligned} \left(\frac{p}{q}\right)\left(\frac{p}{q}\right) &= (-1)^{N_1}(-1)^{N_2} \\ &= (-1)^{N_1+N_2} \\ &= (-1)^{\frac{p-1}{2}\frac{q-1}{2}} \end{aligned}$$

with the result from the participation assignment. ■

Quadratic reciprocity means that determining all quadratic residues (perfect squares) modulo an odd prime is a finite problem. In terms of Legendre symbol, this is finding all a where $\left(\dfrac{a}{p}\right) = 1$ for a given p. For example, when $p = 11$, we can check all positive integers a. However, what about the reverse? Quadratic reciprocity allows us to find all odd primes p where $\left(\dfrac{11}{p}\right) = 1$, even though there are infinitely many odd primes. This idea is the last homework problem.