Geometry Pythagorean Triples

Project on geometry and Pythagorean Triples.

Exploration 1 ***Instructor Notes:***

Definition 1. A *rational point* is a point (x, y) whose coordinates x and y are both rational numbers.

Use rational points on the unit circle and trigonometry to derive the formula for generating Pythagorean triples.

Problem 1.1 *Let (x, y) be a rational point on the unit circle. That is, there exists $a, b, c \in \mathbb{Z}$ such that* $x = \frac{a}{c}$ and $y = \frac{b}{c}$.

Explain why we can write x and y with the same denominator.

Rubric. 2 points.

Problem 1.2 *Let (x, y) be any point on the unit circle, ie $x^2 + y^2 = 1$. Consider the line segment between $(0, 0)$ and (x, y). As long as (x, y) is not $(-1, 0)$, the slope of this line is*

$$t = \frac{y}{1+x} = \tan\theta$$

where θ is the angle between the line segment and the x-axis.

(a) *Show that*

$$x = \frac{1-t^2}{1+t^2}, y = \frac{2t}{1+t^2}$$

either using algebra or trig identities.

(b) *Show that (x, y) is a rational point not equal to $(-1, 0)$ if and only if t is rational.*

(c) *Let $t = \frac{m}{n}$. Prove that if $x, y > 0$, then $m > n$, $(m, n) = 1$, exactly one of m, n is even, and all nontrivial Pythagorean triples have the form $a = k(m^2 - n^2), b = k(2mn), c = k(m^+n^2)$, for some $k \in \mathbb{Z}$.*

Hint: *Use that* $k = \frac{c}{m^2+n^2}$.

Rubric. 4 points if individual, 3 if presenting as a pair.

Learning outcomes:
Author(s): Claire Merriman
$x = -1$ and $y = 0$ is the limit as $t \to \infty$

Exploration 2 *The following problems are from* Number Theory: A Lively Introduction with Proofs, Applications, and Stories *by Erica Flapan, Tim Marks, and James Pommersheim15. Some Nonlinear Diophantine Equations,15.5 A Geometric look at the Equation $x^4 + y^4 = z^2$ [?].*

Theorem 1 (Fermat's Right Triangle Theorem)**.** The does not exist a right triangle with rational side lengths and area a perfect square.

Problem 2.1 *Let x, y, z be positive integers be the side lengths of a triangle with hypotenuse z and the area of the triangle is a perfect square. Prove that if z is the smallest such integer, then (x, y, z) is a primitive Pythagorean triple.*

Rubric. 4 points if individual, 3 if presenting as a pair.

Problem 2.2 *Let x, y, z as in the previous problem. Then there exists $m, n \in \mathbb{Z}$ with $m > n > 0, (m, n) = 1$ and exactly one of m and n even such that $x = m^2 - n^2,\ y = 2mn,$ and $z = m^2 + n^2$. Furthermore, $m,\ n,\ m+n$ and $m-n$ are perfect squares.*

(a) *Prove that if $c, d \in \mathbb{Z}$ such that $c^2 = m + n$, and $d^2 = m - n$, then exactly one of $c + d$ and $c - d$ is divisible by 4.*

(b) *If $4 \mid c + d$, prove that $\frac{b^2}{4} = \frac{c+d}{4}\frac{c-d}{2}$, where the two factors on the right side of the equation are relatively prime.*

(c) *Show that there exists $s, t \in \mathbb{Z}$ such that $\frac{c+d}{4} = s^2$ and $\frac{c-d}{2} = t^2$*

(d) *Show that $(2s^2, t^2, a)$ is a Pythagorean triple and finish the proof of the right triangle theorem for this case.*

(e) *Repeat parts (b)-(d) for $4 \mid c - d$.*

Rubric. Part (a) 4 points if individual, 3 if presenting as a pair. Parts (b)-(d) 3 points per case if individual, 2 points per case if presenting as a pair.

Problem total: 10 points if individual, 7 points if pair.

Exploration 3 *(If presenting as a pair)*

Definition 2. A positive integer n is a *congruent number* if there exists a right triangle whose sides are all rational numbers and whose area is n.

Problem 3.1 *Show that the following are congruent numbers:*

- 6.
- 5.

Rubric. 2 points.

Problem 3.2 *Prove that there is no right triangle with* integer *sides whose area is* 5. *There is no right triangle with* integer *sides whose area is* 1.

Rubric. 3 points.