Linear Diophantine Equations

Definition 1. *A* Diophantine equation *is any equation in one or more variables to be solved in the integers.*

Definition 2. *Let $a_1, a_2, \ldots, a_n, b \in \mathbb{Z}$ with $a_1, a_2, \ldots, a_n$ not zero. A Diophantine equation of the form*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

is a linear Diophantine equation in the n variable $x_1, \ldots, x_n$.

The question of whether there are solutions to Diophantine equations becomes harder when there is more than one variable.

Theorem 1. *Let $ax + by = c$ be a linear Diophantine equation in the variables x and y. Let $d = (a, b)$. If $d \nmid c$, then the equation has no solutions; if $d \mid c$, then the equation has infinitely many solutions. Furthermore, if x_0, y_0 is a particular solution of the equation, then all solution are given by $x = x_0 + \frac{b}{d}n$ and $y = y_0 - \frac{a}{d}n$ where $n \in \mathbb{Z}$.*

Proof *Since $d \mid a, d \mid b$, we have that $d \mid \boxed{c}$. So, if $d \nmid c$, then the given linear Diophantine equation has no solutions. Assume that $d \mid c$. Then, there exists $r, s \in \mathbb{Z}$ such that*

$$d = (a, b) = ar + bs.$$

Furthermore, $d \mid c$ implies $c = de$ for some $e \in \mathbb{Z}$. Then

$$c = de = (ar + bs)e = a(re) + b(se).$$

Thus, $x = re$ and $y = se$ are integer solutions.

Let x_0, y_0 be a particular solution to $ax + by = c$ Then, if $n \in \mathbb{Z}, x = x_0 + \frac{b}{d}n$ and $y = y_0 - \frac{a}{d}n$,

$$ax + by = a(x_0 + \frac{b}{d}n) + b(y_0 - \frac{a}{d}n) = ax_0 + \frac{abn}{d} + by_0 - \frac{abn}{d} = c.$$

We now need to show that every solution has this form. Let x and y be any solution to $ax + by = c$. Then

$$(ax + by) - (ax_0 + by_0) = c - c = 0.$$

Rearranging, we get

$$a(x - x_0) = b(y_0 - y).$$

Dividing both sides by d gives

$$\frac{a}{d}(x - x_0) = \frac{b}{d}(y_0 - y).$$

Now $\frac{b}{d} \mid \frac{a}{d}(x - x_0)$ and $(\frac{a}{d}, \frac{b}{d}) = 1$, so $\frac{b}{d} \mid x - x_0$. Thus, $x - x_0 = \frac{b}{d}n$ for some $n \in \mathbb{Z}$. The proof for y is similar. ■

Example 1. *Is $24x + 60y = 15$ is solvable?*

Multiple Choice:

Learning outcomes:
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(a) *Yes*

(b) *No* ✓

Example 2. *Find all solutions to* $803x + 154y = 11$.

Using the Euclidean Algorithm, we find:

$$\begin{aligned} 803 &= 154 * \boxed{5} + \boxed{33} \\ 154 &= \boxed{33} * \boxed{4} + \boxed{22} \\ \boxed{33} &= \boxed{22} * 1 + \boxed{11} \end{aligned}$$

Thus

$$\begin{aligned} (803, 154) &= \boxed{33} - \boxed{22} \\ &= \boxed{33} - (154 - \boxed{33} * \boxed{4}) = \boxed{33} * \boxed{5} - 154 \\ &= (803 - 154 * \boxed{5}) * \boxed{5} - 154 = 803 * \boxed{5} - 154 * \boxed{26} \end{aligned}$$

Thus, all solutions to the Diophantine equation have the form $x = \boxed{5} + \frac{\boxed{154}}{\boxed{11}} n$ *and* $y = \boxed{-26} - \frac{\boxed{803}}{\boxed{11}} n$.

Example 3. *There is a famous riddle about Diophantus: "God gave him his boyhood one-sixth of his life, One twelfth more as youth while whiskers grew rife; And then yet one-seventh ere marriage begun; In five years there came a bouncing new son. Alas, the dear child of master and sage After attaining half the measure of his father's life chill fate took him. After consoling his fate by the science of numbers for four years, he ended his life."*

That is: Diophantus's childhood was $1/6^{th}$ *of his life, adolescence was* $1/12^{th}$ *of his life, after another* $1/7^{th}$ *of his life he married, his son was born 5 years after he married, his son then died at half the age that Diophantus died, and 4 years later Diophantus died.*

The Diophantine equation that let's us solve this riddle is:

$$x = \frac{x}{6} + \frac{x}{12} + \frac{x}{7} + 5 + \frac{x}{2} + 4.$$

Then, Diophantus's childhood was $\boxed{14}$ *years, his adolescence was* $\boxed{7}$ *years, he married when he was* $\boxed{33}$, *his son was born when he was* $\boxed{38}$ *and died* $\boxed{42}$ *years later, then Diophantus died when he was* $\boxed{84}$.