# Nonlinear Diophantine equations

**Learning Objectives.** By the end of class, students will be able to:

- Define a nonlinear Diophantine equation.

Reading None

**Definition 1.** A Diophantine equation is *nonlinear* if it is not linear.

**Example 1.** (a) The Diophantine equation $x^2 + y^2 = z^2$ is our next section. Solutions are called Pythagorean triples.

(b) Let $n \in \mathbb{Z}$ with $n \geq 3$. The Diophantine equation $x^n + y^n = z^n$ is the subject of the famous Fermat's Last Theorem. We will also prove one case of this.

(c) Let $n \in \mathbb{Z}$. The Diophantine equation $x^2 + y^2 = n$ tells us which integers can be represented as the sum of two squares.

(d) Let $d, n \in \mathbb{Z}$. The Diophantine equation $x^2 - dy^2 = n$ is known as Pell's equation.

Sometimes we can use congruences to show that a particular nonlinear Diophantine equation has no solutions.

**Example 2.** Prove that $3x^2 + 2 = y^2$ is not solvable.

**Solution:** Assume that there is a solution. Then any solution to the Diophantine equation is also a solution to the congruence $3x^2 + 2 \equiv y^2 \mod 3$, which implies $2 \equiv y^2 \mod 3$, which we know is false. Thus there are no integer solutions to $3x^2 + 2 = y^2$.

Note: viewing the same equation modulo 2 says $x^2 \equiv y^2 \mod 2$, which does not give us enough information to prove a solution does not exist–it also is not enough information to conclude a solution exists.

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Learning outcomes:
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