Decimal expansions

Project on decimal expansions.

In this project, you will classify decimal (and duadecimal) expansions are finite, periodic, or aperiodic.

Read Strayer Section 7.1.
Rubric. Introducing topic, definitions, and any necessary results: 4 points

Exploration 1 *We say that the decimal expansion of a real number is* finite *if it terminates after a finite number of digits. If the pattern of digits eventually repeats, such as* $\frac{1}{3} = 0.33333\cdots = 0.\overline{3}$ *or* $\frac{1}{28} = 0.03571428571428\cdots = 0.03\overline{571428}$, *we say the decimal is* periodic. *We could also say that finite decimal expansions are periodic, where the periodic part is* $\overline{0}$. *Otherwise, we say the decimal is* aperiodic.

Problem 1.1 (a) *Find the decimal expansions of* $\frac{3}{2}, \frac{2}{3}, \frac{7}{25}, \frac{2}{7}, \frac{3}{20}, \frac{2}{15}$ *using WolframAlpha or another resource or method that will tell you if the decimal is periodic. These particular examples should be fine with a regular calculator.* You do not need to present these answers, they are to help with the next part.

(b) *Complete and prove the statement:*
Conjecture 1. Let $a, b \in \mathbb{Z}$ with $b \neq 0$ and $(a, b) = 1$. Then $\frac{a}{b}$ has a finite decimal expansion if and only if b has no prime factors other than $2, 5$.

Rubric. 4 points if individual, 3 points if pair.

Problem 1.2 *Prove that if a real number has terminating decimal expansion, then it must be rational. Prove that if a real number has a periodic decimal expansion, then it must be rational.*

Hint: *Generalize the idea in Example 4 and the proof of Proposition 7.4.*
Rubric. 6 points if individual, 4 points if pair.

Exploration 2
Definition 1. Let $x \in \mathbb{R}$ with $0 < x < 1$. Then the *base 12* expansion of x is

$$\sum_{n=1}^{\infty} \frac{d_n}{12^n} = 0.d_1 d_2 \ldots, \quad d_i \in \{1, 2, \ldots, 11, 12\}.$$

Learning outcomes:
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If there exist a positive integer r and N such that $d_n = d_{n+r}$ for all $n \geq N$, then x is *periodic* and $d_N d_{N+1} \ldots d_{N+r}$ is the periodic part of the base 12 expansion. The base 12 expansion of x is finite if the periodic part is 0.

If m is a positive integer, then the base 12 expansion of m is

$$\sum_{k=1}^{n} a_k 12^k, \quad a_i \in \{1, 2, \ldots, 11, 12\}.$$

Combining these definitions gives the base 12 expansion of any positive real number (the only reason to separate the definitions is dealing with the indices of the summation)

Example 1. (a) To find the base 12 expansion of $\frac{3}{2}$, first write $\frac{3}{2} = 1 + \frac{1}{2}$. Since $1 \in \{1, 2, \ldots, 12\}$ the base 12 expansion of 1 is still 1. Then we find the base 12 expansion of $\frac{1}{2}$:

$$\begin{aligned} \frac{1}{2} &= \frac{a_1}{12} + \frac{a_2}{12^2} + \cdots \\ 6(1) &= a_1 + \frac{a_2}{12} + \cdots \end{aligned}$$

Then we can make $a_1 = 6$ and the rest of the $a_i = 0$. So the base 12 expansion of $\frac{3}{2}$ is 1.6

(b) To find the base 12 expansion of $\frac{1}{10}$,

$$\begin{aligned} \frac{1}{11} &= \frac{a_1}{12} + \frac{a_2}{12^2} + \cdots \\ 12\left(\frac{1}{11}\right) &= a_1 + \frac{a_2}{12} + \cdots \\ 1 + \frac{1}{11} &= a_1 + \frac{a_2}{12} + \cdots \end{aligned}$$

So $a_1 = 1$, and we are back to where we started, finding the base 12 expansion of $\frac{1}{11}$. Thus $\frac{1}{11} = 0.\overline{1}$.

(c) Often a is used to represent 10 and b is used to represent 11. The base 12 expansion of

$$\frac{10}{11} = b\frac{1}{11} = b\left(\frac{1}{12} + \frac{1}{12^2} + \cdots + \right) = 0.\overline{b}.$$

Problem 2.1 *Adjust the technique from Example* **??** *or check with* WolframAlpha *to find the base 12 expansions of the fractions from Problem 1.1*??. *Present some of these in class: you do not need to present the work, only the expansion.*
Rubric. 2 points.

Problem 2.2 *Complete and prove the statement:*

Conjecture 2. Let $a, b \in \mathbb{Z}$ with $b \neq 0$ and $(a, b) = 1$. Then $\frac{a}{b}$ has a finite base 12 expansion if and only if b has no prime factors other than $2, 3$.

Rubric. 4 points if individual, 3 points if pair.

Problem 2.3 *(If presenting as a pair) Prove that if a real number has terminating base 12 expansion, then it must be rational.*

Hint: *Generalize the idea in Example 4 and the proof of Proposition 7.4.*

Rubric. 4 points.
