

## April 17–Continued fractions calculations and theorems

*We find the continued fraction expansions of some rational and irrational numbers, and prove some theorems about errors of estimates.*

Before diving into irrational numbers, let's take one more look at the Euclidean algorithm to find the continued fraction expansion of  $\frac{a}{b}$  for integers  $a > b > 0$ . First, we let  $r_0 = a$  and  $r_1 = b$ .

$$a = r_0 = r_1 a_0 + r_2 \quad 0 \leq r_2 < r_1 = b$$

If  $0 = r_2$ , we stop. Otherwise,

$$b = r_1 = r_2 a_1 + r_3 \quad 0 \leq r_3 < r_2$$

Continuing until  $0 = r_{n+1} < r_n < r_{n-1} < r_{n-2} < \dots < r_1 = b < r_0 = a$ . We know that  $r_k = a_{k+1} r_{k+1} + r_{k+2}$  for  $k \leq n-1$  and  $(a, b) = r_n$

Then

$$\begin{aligned} \frac{a}{b} &= \frac{r_0}{r_1} = a_0 + T_1, & a_0 &= \left\lfloor \frac{r_0}{r_1} \right\rfloor, T_1 = \frac{r_2}{r_1} \\ \frac{1}{T_1} &= a_1 + T_2, & a_1 &= \left\lfloor \frac{r_1}{r_2} \right\rfloor, T_2 = \frac{r_3}{r_2} \\ &\vdots & & \\ \frac{1}{T_{n-2}} &= a_{n-1} + T_{n-1}, & a_{n-1} &= \left\lfloor \frac{r_{n-2}}{r_{n-1}} \right\rfloor, T_{n-1} = \frac{r_n}{r_{n-1}} \\ \frac{1}{T_{n-1}} &= a_n + 0 & a_n &= \left\lfloor \frac{r_{n-1}}{r_n} \right\rfloor \end{aligned}$$

and  $\frac{a}{b} = [a_0; a_1, a_2, \dots, a_n]$ .

**Definition 1.** Let  $x = [a_0; a_1, a_2, \dots]$ . We call the rational approximations

$\frac{p_i}{q_i} = [a_0; a_1, a_2, \dots, a_i]$  are called convergents, where  $(p_i, q_i) = 1$ .

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Learning outcomes:

Author(s):

**Example 1.** Determine the continued fraction expansion and convergents for  $\frac{36}{13}$ .

$$\frac{36}{13} = \boxed{2} + \frac{1}{\boxed{\frac{13}{10}}} = \boxed{2} + \frac{1}{\boxed{1} + \frac{1}{\boxed{\frac{10}{3}}}} = \boxed{2} + \frac{1}{\boxed{1} + \frac{1}{\boxed{3} + \frac{1}{\boxed{3}}}}$$

$$\frac{p_0}{q_0} = \frac{2}{1}, \frac{p_1}{q_1} = \frac{\boxed{3}}{\boxed{1}}, \frac{p_2}{q_2} = \frac{\boxed{11}}{\boxed{4}}, \frac{p_3}{q_3} = \frac{\boxed{36}}{\boxed{13}}$$

Here is a plot of the convergents



**Example 2.** Determine the continued fraction expansion and convergents for  $\frac{5}{14}$ .

$$\frac{5}{14} = \boxed{0} + \frac{1}{\boxed{\frac{14}{5}}} = \boxed{0} + \frac{1}{\boxed{2} + \frac{1}{\boxed{\frac{5}{4}}}} = \boxed{0} + \frac{1}{\boxed{2} + \frac{1}{\boxed{1} + \frac{1}{\boxed{4}}}}$$

$$\frac{p_0}{q_0} = \frac{0}{1}, \frac{p_1}{q_1} = \frac{\boxed{1}}{\boxed{2}}, \frac{p_2}{q_2} = \frac{\boxed{1}}{\boxed{3}}, \frac{p_3}{q_3} = \frac{\boxed{5}}{\boxed{14}}$$

Here is a plot of the convergents

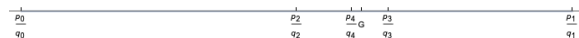


**Example 3.** Now we do our first example of an irrational number, the golden ration  $G = \varphi = \frac{1 + \sqrt{5}}{2}$ . Now,  $G$  is a root of  $\boxed{x^2 - x - 1}$ , so  $G = 1 + \frac{1}{G}$  (check for yourself). Substituting in for  $G$ , we get  $G = 1 + \frac{1}{1 + \frac{1}{G}} = [1; \overline{1}]$  where the  $\overline{\phantom{x}}$  indicates repeating digits, as with decimal expansions.

Since every digit is 1, this continued fraction expansion converges very slowly. Calculate a few of the convergents:

$$\frac{p_0}{q_0} = \frac{1}{1}, \frac{p_1}{q_1} = \frac{\boxed{2}}{\boxed{1}}, \frac{p_2}{q_2} = \frac{\boxed{3}}{\boxed{2}}, \frac{p_3}{q_3} = \frac{\boxed{5}}{\boxed{3}}, \frac{p_4}{q_4} = \frac{\boxed{8}}{\boxed{5}}$$

Here is a plot of the convergents



Looking at all three plots, we can start to see that for  $x \in \mathbb{R}, x > 0$ ,  $\frac{p_0}{q_0} < x < \frac{p_1}{q_1}$  and  $\frac{p_0}{q_0} < \frac{p_2}{q_2} < x < \frac{p_3}{q_3} < \frac{p_1}{q_1}$  (contrast this to the decimal expansion where  $d_0 \leq d_0.d_1 \leq d_0.d_1d_2 \leq d_0.d_1d_2d_3 \leq \dots$ ). In order to prove that this pattern continues (and holds for all positive real numbers), we need to prove a few more facts about  $p_i$  and  $q_i$ .

**Theorem 1.** Let  $p_{-1} = 1, q_{-1} = 0, p_0 = a_0, q_0 = 1$  (see that this matches with  $p_0, q_0$  above). Then

$$p_n = a_n p_{n-1} + p_{n-2} \quad (1)$$

$$q_n = a_n q_{n-1} + q_{n-2} \quad (2)$$

for  $n \geq 1$ .

**Proof** We start by checking  $\frac{p_1}{q_1} = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1} = \frac{p_0 a_1 + p_{-1}}{q_0 a_1 + q_{-1}}$ .

Next, we check  $k = 2$ ,  $\frac{p_2}{q_2} = a_0 + \frac{1}{a_1 + \frac{1}{a_2}} = a_0 + \frac{a_2}{a_2 a_1 + 1} = \frac{a_0(a_2 a_1 + 1) + a_2}{a_2 a_1 + 1} =$

$$\frac{a_2 p_1 + p_0}{a_2 q_1 + q_0}.$$

Now we proceed by induction. Assume that these recurrence relations hold for  $n \leq k$ . Then

$$\frac{p_k}{q_k} = [a_0; a_1, a_2, \dots, a_k] = \frac{a_k p_{k-1} + p_{k-2}}{a_k q_{k-1} + q_{k-2}}.$$

By definition,

$$\begin{aligned} \frac{p_{k+1}}{q_{k+1}} &= [a_0; a_1, a_2, \dots, a_k, a_{k+1}] = [a_0; a_1, a_2, \dots, a_k + \frac{1}{a_{k+1}}] \\ &= \frac{\left(a_k + \frac{1}{a_{k+1}}\right) p_{k-1} + p_{k-2}}{\left(a_k + \frac{1}{a_{k+1}}\right) q_{k-1} + q_{k-2}} \text{ (by induction hypothesis*)} \\ &= \frac{a_{k+1} (a_k p_{k-1} + p_{k-2}) + p_{k-1}}{a_{k+1} (a_k q_{k-1} + q_{k-2}) + q_{k-1}} \\ &= \frac{a_{k+1} p_k + p_{k-1}}{a_{k+1} q_k + q_{k-1}} \text{ (by induction hypothesis)} \end{aligned}$$

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\*Often the recurrence relations (1) and (2) are proven using matrix multiplication instead of justifying that it is ok to substitute  $a_k + \frac{1}{a_{k+1}}$  for  $a_k$ . The matrix definition also allows us to prove the next theorem by taking determinants. Instead, we will use induction and the recurrence relations.

**Theorem 2.** For all  $n \geq 1$ ,  $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$ , which is equivalent to  $\frac{p_n}{q_n} = \frac{(-1)^{n-1}}{q_n q_{n-1}} + \frac{p_{n-1}}{q_{n-1}}$ .

**Proof** The second equation is the first equation rewritten.

We start with  $i = 1$ , so from the previous theorem  $p_1q_0 - p_0q_1 = (a_0a_1 + 1)1 - a_0a_1 = 1 = (-1)^0$ .

Now we proceed by induction. Assume that these recurrence relations hold for  $n \leq k$ . Then  $p_kq_{k-1} - p_{k-1}q_k = (-1)^{k-1}$ . Substituting in the recurrence relations (1) and (2), we get

$$\begin{aligned} p_{k+1}q_k - p_kq_{k+1} &= (a_{k+1}p_k + p_{k-1})q_k - p_k(a_{k+1}q_k + q_{k-1}) \\ &= p_{k-1}q_k - p_kq_{k-1} \\ &= -(p_kq_{k-1} - p_{k-1}q_k) \\ &= -(-1)^{k-1} = (-1)^k \end{aligned}$$

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Notice that  $\frac{p_n}{q_n} = \frac{(-1)^{n-1}}{q_nq_{n-1}} + \frac{p_{n-1}}{p_{n-1}}$  is still a recurrence relation. Expanding this out, we get

$$\frac{p_n}{q_n} = a_0 + \frac{1}{q_0q_1} - \frac{1}{q_1q_2} + \cdots + \frac{(-1)^{n-1}}{q_nq_{n-1}} = a_0 + \sum_{k=1}^n \frac{(-1)^{k-1}}{q_{k-1}q_k}.$$

Thus  $x = \lim_{n \rightarrow \infty} \frac{p_n}{q_n} = a_0 + \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{(-1)^{k-1}}{q_{k-1}q_k} = a_0 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{q_{k-1}q_k}$  which converges by the alternating series test. This shows that the continued fraction expansion of  $x$  really does converge to  $x$  for all  $x \in \mathbb{R}, x > 0$ .

**Theorem 3.** For  $x \in \mathbb{R}, x > 0$ ,  $\frac{p_0}{q_0} < \frac{p_2}{q_2} < \cdots < x < \cdots < \frac{p_3}{q_3} < \frac{p_1}{q_1}$ .

**Proof** From (2), we find that  $q_0 < q_1 < q_2 < \cdots$ , since we have a recurrence relation that add positive integers to get a larger positive integer. Thus  $\frac{1}{q_{n+1}q_n} < \frac{1}{q_nq_{n-1}}$ .

For  $n = 2k$ , we have that

$$\begin{aligned} \frac{p_{2k}}{q_{2k}} &= \frac{(-1)^{2k-1}}{q_{2k}q_{2k-1}} + \frac{p_{2k-1}}{p_{2k-1}} = \frac{-1}{q_{2k}q_{2k-1}} + \frac{(-1)^{2k-2}}{q_{2k-1}q_{2k-2}} + \frac{p_{2k-2}}{p_{2k-2}} \\ &= \frac{-1}{q_{2k}q_{2k-1}} + \frac{1}{q_{2k-1}q_{2k-2}} + \frac{p_{2k-2}}{p_{2k-2}}, \end{aligned}$$

and  $\frac{-1}{q_{2k}q_{2k-1}} + \frac{1}{q_{2k-1}q_{2k-2}} > 0$ , so  $\frac{p_0}{q_0} < \frac{p_2}{q_2} < \cdots < \frac{p_{2k}}{q_{2k}} < \cdots$ . We also have that  $\lim_{k \rightarrow \infty} \frac{p_{2k}}{q_{2k}} = x$  (from analysis)  $\frac{p_0}{q_0} < \frac{p_2}{q_2} < \cdots < \frac{p_{2k}}{q_{2k}} < \cdots < x$ .

For  $n = 2k + 1$ , we have that

$$\begin{aligned}\frac{p_{2k+1}}{q_{2k+1}} &= \frac{(-1)^{2k}}{q_{2k+1}q_{2k}} + \frac{p_{2k}}{p_{2k}} = \frac{1}{q_{2k+1}q_{2k}} + \frac{(-1)^{2k-1}}{q_{2k}q_{2k-1}} + \frac{p_{2k-1}}{p_{2k-1}} \\ &= \frac{1}{q_{2k+1}q_{2k}} + \frac{-1}{q_{2k}q_{2k-1}} + \frac{p_{2k-1}}{p_{2k-1}},\end{aligned}$$

and  $\frac{1}{q_{2k+1}q_{2k}} + \frac{-1}{q_{2k}q_{2k-1}} < 0$ , so  $\dots < \frac{p_{2k-1}}{q_{2k-1}} < \frac{p_{2k+1}}{1_{2k+1}} \dots < \frac{p_3}{q_3} < \frac{p_1}{q_1}$ . We also have that  $\lim_{k \rightarrow \infty} \frac{p_{2k+1}}{q_{2k+1}} = x$  (from analysis)  $x < \dots < \frac{p_{2k-1}}{q_{2k-1}} < \frac{p_{2k+1}}{1_{2k+1}} \dots < \frac{p_3}{q_3} < \frac{p_1}{q_1}$ . ■