Participation assignment for March 23

This assignment looks at arithmetic functions in number theory. We have already seen the examples $\phi(n) = \#\{x : 0 < x < n, (x, n) = 1\}$ and the floor/greatest integer function $\lfloor x \rfloor$ =greatest integer less than x. So far, we have a formula for the value of $\phi(n)$ when n is prime or the product of two distinct primes. We will prove a general formula for the ϕ function and look at some other functions.

Euler ϕ function: number of relatively prime positive integers less than n

Recall from before break that $\phi(p) = p - 1$ if and only if p is prime. We also proved that $\phi(pq) = (p - 1)(q - 1)$ for primes p and q.

On the homework, you will prove that for relatively prime positive integers m and n, $\phi(mn) = \phi(m)\phi(n)$. Since Ximera can really only handle numerical answers, let's prove this is true for a particular example:

Question 1 *Let us prove that $\phi(20) = \phi(4)\phi(5)$. First, note that $\phi(4) = 2$ and $\phi(5) = 4$, so $\phi(20) = 8$.*

(a) *A number a is relatively prime to 20 if and only if a is relatively prime to* 4 *and* 5 *(first blank should be smaller than second blank for the automatic grading to work, both should be relevant to what we are trying to show).*

(b) *We can partition the positive integers less that 20 into*

$$\begin{aligned} 0 \equiv 4 \equiv 8 \equiv 12 \equiv 16 \pmod 4 \\ 1 \equiv 5 \equiv 9 \equiv 13 \equiv 17 \pmod 4 \\ 2 \equiv 6 \equiv 10 \equiv 14 \equiv 18 \pmod 4 \\ 3 \equiv 7 \equiv 11 \equiv 15 \equiv 19 \pmod 4 \end{aligned}$$

For any b in the range $0, 1, 2, 3$, define s_b to be the number of integers a in the range $0, 1, 2, \ldots, 19$ such that $a \equiv b \pmod 4$ and $\gcd(a, 20) = 1$. Thus, $s_0 = 0, s_1 = 4, s_2 = 0$, and $s_3 = 4$.

We can see that when $(b, 4) = 1$, $s_b = \phi(5)$ and when $(b, 4) > 1$, $s_b = 0$.

(c) $\phi(20) = s_0 + s_1 + s_2 + s_3$. *Why?*

Learning outcomes:
Author(s):

All of the positive integers less than or equal to 20 *is in exactly one of the congruence classes above. The* s_i *count how many integers in each congruence class are relatively prime to* 20*. If we add them up, we have counted all positive integers less than or equal to* 20*.*

(d) *We have seen that* $\phi(20) = s_0 + s_1 + s_2 + s_3$*, that when* $(b, 4) = 1$*,* $s_b = \phi(5)$*, and that when* $(b, 4) > 1$*,* $s_b = 0$*. Thus, we can say that* $\phi(20) = 0 + \phi(5) + 0 + \phi(5)$*. To finish the "proof" we show that there are* $\phi(4)$ *integers* b *where* $(b, 4) = 1$*.*

There are 4 *congruence classes modulo 4. Of these,* $2 = \phi(4)$ *have elements that are relatively prime to* 20*. Thus,* $\phi(20) = \phi(4)\phi(5)$*.*

Other common number theory functions

We are going to look at some other functions that show up in analytic number theory.

- $d(n)$ is the number of positive divisors of n. For example, $d(12) = 6$. We introduce the notation $\sum_{d|n}$ as "the sum over the divisors of n," called the *divisor sum*. For the normal sum: $\sum_{i=1}^{n} 1 = n$. Then, $\sum_{d|n} 1 = d(n)$.

- $\sigma(n)$ is the sum of positive divisors of n. For example, $\sigma(12) = 28$. Then $\sum_{d|n} d = \sigma(n)$.

- $\sigma_k(n)$ is sum of the k^{th} powers of positive divisors of n. For example, $\sigma_2(12) = 1^2 + 2^2 + 3^2 + 4^2 + 6^2 + 12^2 = 210$. Generally, $\sum_{d|n} d^k = \sigma_k(n)$.

- $\omega(n)$ is the number of distinct prime divisors of n. For example, $\omega(12) = 2$. We can modify the divisor sum to sum over prime divisors of n, $\sum_{p|n}$. Then, $\sum_{p|n} 1 = \omega(n)$.

- $\Omega(n)$ is the number of primes dividing n counting multiplicity. For example, $\Omega(12) = 3$. Then $\sum_{p^\beta|n} 1 = \Omega(n)$.