MAT-255, Homework #1 Rubrics

**Problem 1** *Prove or disprove the following statements.*

(a) *If $a, b, c$, and $d$ are integers such that $a \mid b$ and $c \mid d$, then $a + c \mid b + d$.*

(b) *If $a, b, c$, and $d$ are integers such that $a \mid b$ and $c \mid d$, then $ac \mid bd$.*

(c) *If $a, b$, and $c$ are integers such that $a \nmid b$ and $b \nmid c$, then $a \nmid c$.enumerate*

***Solution:*** (a) *False: Let $a = b = c = 1$ and $d = 2$. Then $a+b = 2$ and $c+d = 3$. Therefore, $a+b \nmid c+d$.*

(b) *Let $a, b, c$, and $d$ be integers such that $a \mid b$ and $c \mid d$. Then there exists $m, n \in \mathbb{Z}$ such that $a = bm$ and $c = dn$, by definition of divides. Multiplying these two equations gives*

$$ac = (bm)(dn) = (bd)(mn).$$

*Therefore, $ac \mid bd$.*

(c) *False: Let $a = 2, b = 3$, and $c = 4$. Then $a \nmid b, b \nmid c$ and $a \mid c$.*

***Rubric:***

***0 points*** *Work does not contain enough of the relevant concepts to provide feedback.*

***1 points*** ***Does not demonstrate understanding*** *Contains a reasonable attempt to prove each part, but does not meet the criteria for two points.*

***2 points*** ***Needs revisions*** *Two parts are mathematically correct, with the third containing major mathematical errors. Or proof has some significant gaps or excess information.*

***3 points*** ***Demonstrates understanding*** *Two parts are mathematically correct, with minor errors in the third part. Or does not follow the homework guide for mathematical writing.*

***4 points*** ***Exemplary*** *All three parts are mathematically correct and follows the homework guide for mathematical writing.*

**Problem 2** (a) *Let $n \in \mathbb{Z}$. Prove that $3 \mid n^3 - n$.*

(b) *Let $n \in \mathbb{Z}$. Prove that $5 \mid n^5 - n$.*

(c) *Let $n \in \mathbb{Z}$. Is it true that $4 \mid n^4 - n$? Provide a proof or counter example.enumerate*

***Proof*** *There are several ways to do these problems. I used slightly different methods for parts (a) and (b).*

(a) *Let $n \in \mathbb{Z}$. Then $n^3 - n =$. By the Division Algorithm, there exists unique $q, r \in \mathbb{Z}$ such that $n = 3q + r$ and $0 \le r < 3$. Thus, $n^3 - n = 27q^3 + 27q^2r + 9qr^2 + r^3 - 3q - r$*

*If $r = 0$, then $n^3 - n = 27q^3 - 3q = 3(9q^3 - q)$, so$3 \mid n^3 - n$.*

*If $r = 1$, then $n^3 - n = 27q^3 + 27q^2 + 6q = 3(9q^3 + 9q^2 + 2q)$, so$3 \mid n^3 - n$.*

*If $r = 2$, then $n^3 - n = 27q^3 + 54q^2 + 15q + 6 = 3(9q^3 + 18q^2 + 5q + 2)$, so$3 \mid n^3 - n$.*

*Therefore, $3 \mid n^3 - n$ in all possible cases.*

(b) *Let $n \in \mathbb{Z}$. Then $n^5 - n = n(n-1)(n+1)(n^2+1)$. By the Division Algorithm, there exists unique $q, r \in \mathbb{Z}$ such that $n = 5q + r$ and $0 \le r < 5$.*

*If $r = 0$, then $5 \mid n$. If $r = 1$, then $5 \mid (n-1) = 5q$. If $r = 4$, then $5 \mid (n+1) = 5q$.*

*If $r = 2$, then $n = 5q + 2$ and $n^2 + 1 = 25q^2 + 20q + 4 + 1 = 5(5q^2 + 4q + 1)$. Therefore, $5 \mid (n^2 + 1)$. If $r = 3$, then $n = 5q + 3$ and $n^2 + 1 = 25q^2 + 30q + 9 + 1 = 5(5q^2 + 6q + 2)$. Therefore, $5 \mid (n^2 + 1)$. Thus, by transitivity of division, $5 \mid n^5 - n$.*

(c) *False: Let $n = 2$. Then $n^4 - n = 14$ and $4 \nmid 14$.*

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*The hint in the back of the book says to use induction for part (a). That is not necessary and likely harder.*

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***0 points*** *Work does not contain enough of the relevant concepts to provide feedback.*

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***2 points*** ***Needs revisions*** *Two parts are mathematically correct with some missing details, with the third containing major mathematical errors. Or proof has some significant gaps or excess information.*

***3 points*** ***Demonstrates understanding*** *Two parts are mathematically correct, with minor errors. Or does not follow the homework guide for mathematical writing.*

***4 points*** ***Exemplary*** *All three parts are mathematically correct and follows the homework guide for mathematical writing.*

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**Problem 3** *Use the definition of even and odd from Strayer and not Ernst.*

(a) *Let $n \in \mathbb{Z}$. Prove that $n$ is an even integer if and only if $n = 2m$ with $m \in \mathbb{Z}$.*

(b) *Let $n \in \mathbb{Z}$. Prove that $n$ is an odd integer if and only if $n = 2m + 1$ with $m \in \mathbb{Z}$.*

(c) *Prove that the sum and product of two even integers are even.*

(d) *Prove that the sum of two odd integers is even and that their product is odd.*

(e) *Prove that the sum of an even integer and an odd integer is odd and that their product is even.*

*Rubric for parts (a) and (b), graded together:*

***Rubric:***

***0 points*** *Work does not contain enough of the relevant concepts to provide feedback.*

***1 points*** ***Does not demonstrate understanding*** *Contains a reasonable attempt to prove each part, but does not meet the criteria for two points.*

***2 points*** ***Needs revisions*** *Proof has some significant gaps or excess information. Or does not prove both directions for both parts.*

***3 points*** ***Demonstrates understanding*** *Contains minor arithmetic, spelling, or grammatical errors. Or uses informal mathematical writing.*

*Exemplory* *Correctly proves both directions of both (a) and (b). Work is easy to follow with formal mathematical writing.*

*Rubric for parts (c),(d), and (e), graded together:*

***Rubric:***

***0 points*** *Work does not contain enough of the relevant concepts to provide feedback.*

***1 points*** ***Does not demonstrate understanding*** *Contains a reasonable attempt to prove each part, but does not meet the criteria for two points.*

***2 points*** ***Needs revisions*** *Proof has some significant gaps or excess information. Or missing the proof for either the sum or product in one part.*

***3 points*** ***Demonstrates understanding*** *Mathematically correct proof for all three parts with minor arithmetic, spelling, or grammatical errors. Or uses informal mathematical writing.*

*Exemplory* *Mathematically correct proof for all three parts. Work is easy to follow with formal mathematical writing.*

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**Problem 4** *Let $A$ represent "6 is an even integer" and $B$ represent "4 divides 6." Express each of the following compound propositions in an ordinary English sentence and then determine its truth value.*

a. $A \wedge B$

b. $A \vee B$

c. $\neg A$

d. $\neg B$

e. $\neg(A \wedge B)$

f. $\neg(A \vee B)$

g. $A \Rightarrow B$

***Solution:*** *(a) "6 is an even integer and 4 divides 6". This is false because 4 does not divide 6.*

*(b) "6 is an even integer or 4 divides 6". This is true because 6 is an even.*

*(c) "6 is not an even number". This is false.*

*(d) "6 is not an even number or 4 does not divide 6". This is true because 4 does not divide 6.*

*(e) "6 is not an even number and 4 does not divide 6". This is false because 6 is an even number.*

*(f) "If 6 is an even integer, then 4 divides 6". This is false because 4 does not divide 6.*

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***Rubric:***

***0 points*** *Work does not contain enough of the relevant concepts to provide feedback.*

***1 points*** ***Does not demonstrate understanding*** *Contains a reasonable attempt to prove each part, but does not meet the criteria for two points.*

***2 points*** ***Needs revisions*** *Three or four parts are mathematically and grammatically correct sentences.*

***3 points*** ***Demonstrates understanding*** *Five or six parts are mathematically and grammatically correct sentences.*

***4 points*** ***Exemplary*** *All parts are mathematically and grammatically correct sentences.*

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**Problem 5** *Let $A$ and $B$ represent the statements from Problem 2.19. Express each of the following in an ordinary English sentence.*

*(a) The converse of $A \Rightarrow B$*

*(b) The contrapositive of $A \Rightarrow B$*

***Solution:*** *Let $A$ represent "6 is an even integer" and $B$ represent "4 divides 6."*

*(a) "If 4 divides 6, then 6 is an even integer." Since 4 does not divide 6, any conclusion is vacuously true.*

*(b) "If 4 does not divides 6, then 6 is not an even integer." Since 6=2(3), this statement is false.*

**Rubric:**

**0 points** *Work does not contain enough of the relevant concepts to provide feedback.*

**1 points** **Does not demonstrate understanding** *Contains a reasonable attempt to prove each part, but does not meet the criteria for two points.*

**2 points** **Needs revisions** *One part is not mathematically correct. Or both parts are missing the proof or counterexample.*

**3 points** **Demonstrates understanding** *Both parts correctly translate the statements into grammaticallly correct sentences. One part contains a mathematically correct proof or counterexample. Or does not follow the homework guide for mathematical writing.*

**4 points** **Exemplary** *Both parts correctly translate the statements into grammaticallly correct sentences with a mathematically correct proof or counterexample.*

**Problem 6** *For each of the following equation, find what real numbers $x$ make the statement true. Prove your statement.*

(a) $\lfloor x \rfloor + \lfloor x \rfloor = \lfloor 2x \rfloor$

(b) $\lfloor x+3 \rfloor = 3 + \lfloor x \rfloor$

(c) $\lfloor x+3 \rfloor = 3 + x$

**Solution:** *All of these statements could be stated as "if and only if." The proofs below also work for both directions of the biconditional.*

(a) *If the decimal part of $x$ is less than 0.5, then $\lfloor x \rfloor + \lfloor x \rfloor = \lfloor 2x \rfloor$ . In otherwords, if $x - \frac{1}{2} < \lfloor x \rfloor \leq x$, then $\lfloor x \rfloor + \lfloor x \rfloor = \lfloor 2x \rfloor$ .*

***Proof*** *We will use the characterization $x - \frac{1}{2} < \lfloor x \rfloor \leq x$. Then we add this inequality to itself to find*

$$x - \frac{1}{2} + x - \frac{1}{2} < \lfloor x \rfloor + \lfloor x \rfloor \leq x + x$$
$$2x - 1 < \lfloor x \rfloor + \lfloor x \rfloor \leq 2x.$$

*By Strayer, Lemma 1.3, $2x - 1 < \lfloor 2x \rfloor \leq 2x$. Since the characterizations are the same, it must be that $\lfloor x \rfloor + \lfloor x \rfloor = \lfloor 2x \rfloor$.* ■

(b) *If $x \in \mathbb{R}$, then $\lfloor x+3 \rfloor = 3 + \lfloor x \rfloor$.*

***Proof*** *By Strayer, Lemma 1.3, for all* $x \in \mathbb{R}$, $x - 1 + 3 < 3 + \lfloor x \rfloor \leq x + 3$ *and* $x + 2 < \lfloor x + 3 \rfloor \leq x + 3$. *That is,* $3 + \lfloor x \rfloor = \lfloor x + 3 \rfloor$. ■

(c) *If* $x \in \mathbb{Z}$, *then* $\lfloor x + 3 \rfloor = 3 + x$

***Proof*** *Since* $\lfloor x + 3 \rfloor \in \mathbb{Z}$, $\lfloor x + 3 \rfloor = 3 + x$ *requires* $x \in \mathbb{Z}$. *By part (b),* $\lfloor x + 3 \rfloor = 3 + \lfloor x \rfloor$ *for all* $x \in \mathbb{Z}$. ■

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