Circle of fifths

Project on frequency ratios for musical cords and continued fractions.

Read Strayer Sections 7.2 and 7.3, paying particular attention to the examples. I will use more standard notation than Strayer.

Exploration 1 *Give an introduction to continued fraction expansions, with enough detail for classmates to follow the presented problems.*

Rubric. Present as much as necessarily for classmates to follow: 4 points if individual, 3 points if pair.

Definition 1. *Let x be a positive real number. Then the* continued fraction expansion of x *is*

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} = [a_0; a_1, a_2, \dots]$$

where $a_1, a_2, \dots$ are positive integers and a_0 is a nonnegative integer.

Define the convergents of x *to be the sequence of $\frac{p_n}{q_n}$ where*

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_n}}}}$$

Problem 1.1 *Use induction to show that $p_n = a_n p_{n-1} + p_{n-2}, q_n = a_n q_{n-1} + q_{n-2}$.*

Hint: *Consider $\cfrac{1}{a_n + \cfrac{1}{a_{n+1}}}$ and use $a_n + \frac{1}{a_{n+1}}$ in the induction step.*

Rubric. 4 points if individual, 3 points if pair.

Problem 1.2 *(If presenting as a pair) Chapter 7, Exercise 18*

Rubric. 3 points.

Learning outcomes:
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Exploration 2 *The following problems are from* Number Theory: A Lively Introduction with Proofs, Applications, and Stories *by Erica Flapan, Tim Marks, and James Pommersheim.*

Read the scanned notes on Moodle. Present as much as necessarily for classmates to follow.

Rubric. Present as much as necessarily for classmates to follow: 4 points if individual, 3 points if pair.

Problem 2.1 *An acoustically correct major third has frequency ratio* $5:4$

(a) *Show that there do not exist natural numbers* m *and* n *such that*

$$\left(\frac{5}{4}\right)^m = 2^n.$$

Thus, no number of acoustically correct major thirds that can make up a. whole number of octaves.

(b) *Find the first four convergents of* $\log_2\left(\frac{5}{4}\right)$

(c) *For each of your answer to part (b), find a pair of integers* m *and* n *for which* $\left(\frac{5}{4}\right)^m = 2^n$ *is approximately correct.*

(d) *In the equal-tempered scale, and octave consists of exactly 3 major thirds. To which convergent of* $\log_2\left(\frac{5}{4}\right)$ *does this correspond?*

Rubric. Each part: 2 points.