Division algorithm, divisibility

Learning Objectives. By the end of class, students will be able to:

- Prove facts about divisibility
- Prove basic mathematical statements using definitions and direct proof
- Use truth tables to understand compound propositions
- Prove statements by contradiction
- Use the greatest integer function .

Reading Read Ernst Chapter 1 and Section 2.1. Also read Strayer Introduction and Section 1.1 through the proof of Proposition 1.2 (that is, pages 1-5).

Turn in: From Ernst

Problem 1 *For $n, m \in \mathbb{Z}$, how are the following mathematical expressions similar and how are they different? In particular, is each one a sentence or simply a noun?*

(a) $n \mid m$

(b) $\dfrac{m}{n}$

(c) $^{m}/_{n}$

Solution: *The first means "n divides m," which is a relationship between n and m. This is a sentence. The other two are nouns, that is, the rational number $\dfrac{m}{n}$.*

Problem 2 *Let $a, b, n, m \in \mathbb{Z}$. Determine whether each of the following statements is true or false. If a statement is true, prove it. If a statement is false, provide a counterexample.*

(a) *If $a \mid n$, then $a \mid mn$*

Solution: *Let $a \mid n$. Then by **??**, there exists $k \in \mathbb{Z}$ such that $ak = n$. Multiplying both sides of the equation by m gives*

$$a(km) = mn,$$

*so $a \mid mn$ by definition of **??**.*

(b) *If 6 divides n, then 2 divides n and 3 divides n.*

Solution: *Let $6 \mid n$. Then by definition of **??**, there exists $k \in \mathbb{Z}$ such that $6k = n$. By factoring out 6, we see that $2(3k) = 3(2k) = n$, so $2 \mid n$ and $3 \mid n$.*

Learning outcomes:

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(c) *If ab divides n, then a divides n and b divides n.*

Solution: *Let $ab \mid n$. Then by definition of* **??**, *there exists $k \in \mathbb{Z}$ such that $abk = n$. Thus, we see that $a(bk) = b(ak) = n$, so $a \mid n$ and $b \mid n$.*

Problem 3 *Determine whether the converse of each of Corollary 2.9, Theorem 2.10, and Theorem 2.11 is true. That is, for $a, n, m \in \mathbb{Z}$, determine whether each of the following statements is true or false. If a statement is true, prove it. If a statement is false, provide a counterexample.*

(a) *If a divides n^2, then a divides n. (Converse of Corollary 2.9)*

Solution: *False; $4 \mid 4$ but $4 \nmid 2$.*

(b) *If a divides $-n$, then a divides n. (Converse of Theorem 2.10)*

Solution: *True. If $a \mid -n$, then by definition of* **??**, *there exists $k \in \mathbb{Z}$ such that $ak = -n$. Multiplying both sides by -1 gives*

$$-ak = a(-k) = n.$$

Therefore, $a \mid n$.

(c) *If a divides $m + n$, then a divides m and a divides n. (Converse of Theorem 2.11)*

Solution: *False; $3 \mid 2 + 1$ but $3 \nmid 2$ and $3 \nmid 1$.*

Logic, proof by contradiction, and biconditionals

We will begin by working through Ernst Section 2.2 through Example 2.21. Discuss Problem 2.17 as a class, and note that Problem 2.19 is on Homework 1.

In-class Problem 4 *Construct a truth table for $A \Rightarrow B, \neg(A \Rightarrow B)$ and $A \wedge \neg B$*

Solution:

A	B	$A \Rightarrow B$	$\neg(A \Rightarrow B)$	$A \wedge \neg B$
T	T	T	F	F
T	F	F	T	T
F	T	T	F	F
F	F	T	F	F

This is the basis for *proof by contradiction.* We assume both A and $\neg B$, and proceed until we get a contradiction. That is, A and $\neg B$ cannot both be true.

Definition (Proof by contradiction)**.** Let A and B be propositions. To prove A implies B by contradiction, first assume the B is false. Then work through logical steps until you conclude $\neg A \wedge A$.

First, let's define a *lemma.* A lemma is a minor result whose sole purpose is to help in proving a theorem, although some famous named lemmas have become important results in their own right.

Definition (greatest integer (floor) function)**.** Let $x \in \mathbb{R}$. The *greatest integer function of* x*,* denoted $[x]$ or $\lfloor x \rfloor$, is the greatest integer less than or equal to x.

Lemma (Strayer, Lemma 1.3)**.** *Let* $x \in \mathbb{R}$*. Then* $x - 1 < [x] \le x$*.*

Proof *By the definition of the greatest integer (floor) function,* $[x] \le x$*.*

To prove that $x - 1 < [x]$*, we proceed by contradiction. Assume that* $x - 1 \ge [x]$ *(the negation of* $x - 1 < [x]$*). Then,* $x \ge [x] + 1$*. This contradicts the assumption that* $[x]$ *is the greatest integer* less than or equal to x*. Thus,* $x - 1 < [x]$. ■