Geometric Lemma for Quadratic reciprocity

Learning Objectives. By the end of class, students will be able to:

- Count lattice point in a rectangle with side lengths $\dfrac{p-1}{2}$ and $\dfrac{q-1}{2}$ two different ways
- Use the two counting methods to prove quadratic reciprocity .

Read: None

Theorem 1 (Quadratic Reciprocity)**.** *Let p and a be odd primes with $p \neq q$. Then*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}.$$

Definition 1. *A* lattice point *is a point $(x, y) \in \mathbb{R}^2$ where $x, y \in \mathbb{Z}$. We can write this as $(x, y) \in \mathbb{Z}^2$.*

We will use two different methods to count the number of lattice points in the rectangle with vertices $(0,0)$, $\left(\frac{p-1}{2}\right)$, $\left(\frac{p-1}{2}, \frac{q-1}{2}\right)$, $(0, \frac{q-}{2}$ other than the axes. The easy method is multiplication: there are $\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)$ lattice points. The other method involves counting the lattice points (in the same rectangle) with $y > \frac{q}{p}$, call this number N_1, and those with $y < \frac{q}{p}$, call this number N_2. Then there are a total of $N_1 + N_2$ lattice points.

This changes the statement of quadratic reciprocity to

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{N_1}(-1)^{N_2}$$

Access GeoGebra at https://www.geogebra.org/m/tuf7y6sh.

Two stills from the GeoGebra interactive are in Figure 1 and Figure 2.

Geogebra link: `https://tube.geogebra.org/m/tuf7y6sh`

In-class Problem 1 The steps below outline the proof in the general case, when $p = 7$ and $q = 5$. This case is in Figure 1. Move the sliders to $p = 7$ and $q = 5$.

(a) The line segment between the origin and $(7, 5)$ has slope $\boxed{\frac{5}{7}}$. Since $p = 7$ and $q = 5$ are distinct primes, there are no lattice points on line segment except the endpoints.

(b) First, we will count the number of points N_1 where $\frac{5-1}{2} \geq y > \frac{5}{7}x > 0$. This triangle is grey in the GeoGebra. We will count how many lattice points on each horizontal lines $j = 1, 2$. Let's just check the numbers we should get:

- When $j = 1$, there are $\boxed{1}$ lattice points.
- When $j = 2$, there are $\boxed{2}$ lattice points.

For each j, we are counting positive integers $x < \boxed{\frac{7}{5}}j$. Which is,

Learning outcomes:
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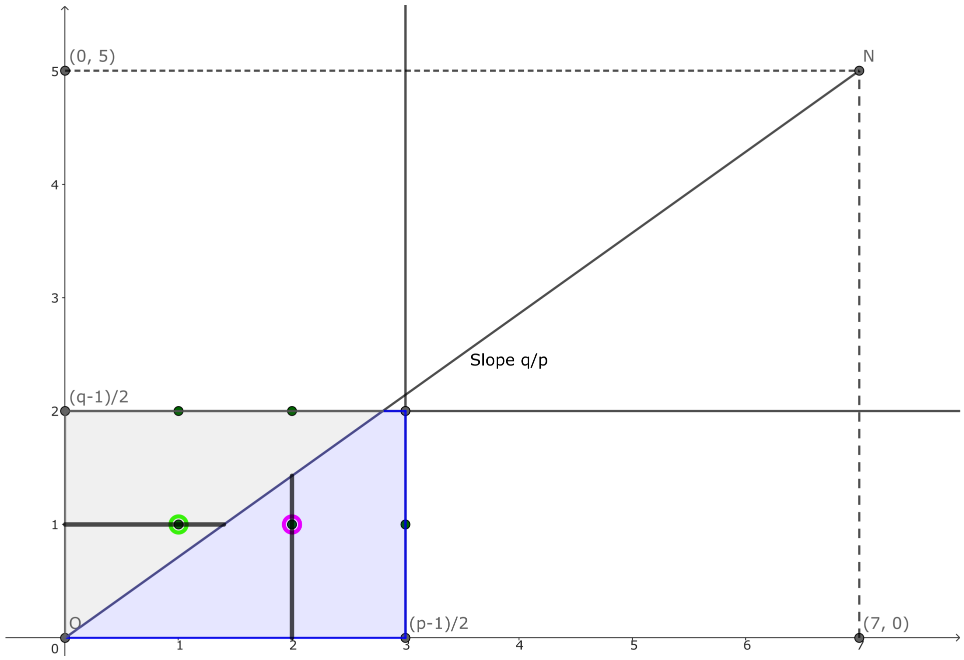

Figure 1: The lattice for the $p = 7, q = 5$ problem, with the $j = 1$ and $k = 2$ cases highlighted

Multiple Choice:

(i) $\left\lfloor \frac{7j}{5} \right\rfloor$ ✓ .

(ii) $\left\lfloor \frac{5j}{7} \right\rfloor$.

Thus, the total number of lattice points in this triangle, N_1, is

Multiple Choice:

(i) $N_1 = \sum_{j=1}^{2} \left\lfloor \frac{7j}{5} \right\rfloor$ ✓

(ii) $N_1 = \sum_{j=1}^{2} \left\lfloor \frac{5j}{7} \right\rfloor$

(iii) $N_1 = \sum_{j=1}^{3} \left\lfloor \frac{7j}{5} \right\rfloor$

(iv) $N_1 = \sum_{j=1}^{3} \left\lfloor \frac{5j}{7} \right\rfloor$

(c) Next we will count the rest of the lattice points in the rectangle, the blue region in the GeoGebra. We willl call this number N_2.

The region is bounded by $0 < x \leq \frac{7-1}{2}$, $0 < y < \frac{5}{7}x$, and $y \leq \frac{5-1}{2}$. Now, the point A where $y = \frac{5}{7}x$ intersects $y = \frac{5-1}{2}$ is between two consecutive lattice points, with coordinates $x = \boxed{2}, y = \boxed{2}$ and $x = \boxed{3}, y = \boxed{2}$.

Similarly, the point B where $y = \frac{5}{7}x$ intersects $x = \frac{7-1}{2}$ is between two consecutive lattice points, with coordinates $x = \boxed{3}, y = \boxed{2}$ and $x = \boxed{3}, y = \boxed{3}$. Thus, the only lattice point in the triangle A, B and $\left(\frac{7-1}{2}, \frac{5-1}{2}\right)$ is $\left(\frac{7-1}{2}, \frac{5-1}{2}\right)$. Therefore, there are also N_2 lattice points in the triangle with vertices $(0,0)$, $\left(\frac{7-1}{2}, 0\right)$, $\left(\frac{7-1}{2}, \frac{5-1}{2}\right)$.

(d) We use the same method as N_1 to find N_2. We will count how many lattice points on each vertical lines $k = 1, 2, 3$. Let's just check the numbers we should get:

- When $k = 1$, there are $\boxed{0}$ lattice points.
- When $k = 2$, there are $\boxed{1}$ lattice points.
- When $k = 3$, there are $\boxed{2}$ lattice points.

For each k, we are counting positive integers $y < \boxed{\frac{5}{7}}k$. Which is,

Multiple Choice:

(i) $\left\lfloor \frac{7j}{5} \right\rfloor$.

(ii) $\left\lfloor \frac{5j}{7} \right\rfloor$ ✓ .

Thus, the total number of lattice points in this triangle is

Multiple Choice:

(i) $N_2 = \sum_{k=1}^{2} \left\lfloor \frac{7k}{5} \right\rfloor$

(ii) $N_2 = \sum_{k=1}^{2} \left\lfloor \frac{5k}{7} \right\rfloor$

(iii) $N_2 = \sum_{k=1}^{3} \left\lfloor \frac{7k}{5} \right\rfloor$

(iv) $N_2 = \sum_{k=1}^{3} \left\lfloor \frac{5k}{7} \right\rfloor$ ✓

Thus, the total number of lattice points is $N_1 + N_2 = \boxed{(3)(2)}$.

In-class Problem 2 The steps below outline the proof in the general case, when $p = 23$ and $q = 13$. Move the sliders to $p = 23$ and $q = 13$.

(a) The line segment between the origin and $(23, 13)$ has slope $\boxed{\frac{13}{23}}$. Since $p = 23$ and $q = 13$ are distinct primes, there are no lattice points on line segment except the endpoints.

(b) First, we will count the number of points N_1 where $\frac{13-1}{2} \geq y > \frac{13}{23}x > 0$. This triangle is grey in the GeoGebra. We will count how many lattice points on each horizontal lines $j = 1, 2, \ldots, \boxed{6}$. Let's just check one case, we should get:

- When $j = 3$, as in Figure 2, there are $\boxed{5}$ lattice points.

For each j, we are counting positive integers $x < \boxed{\frac{23}{13}}j$. Which is,

Multiple Choice:

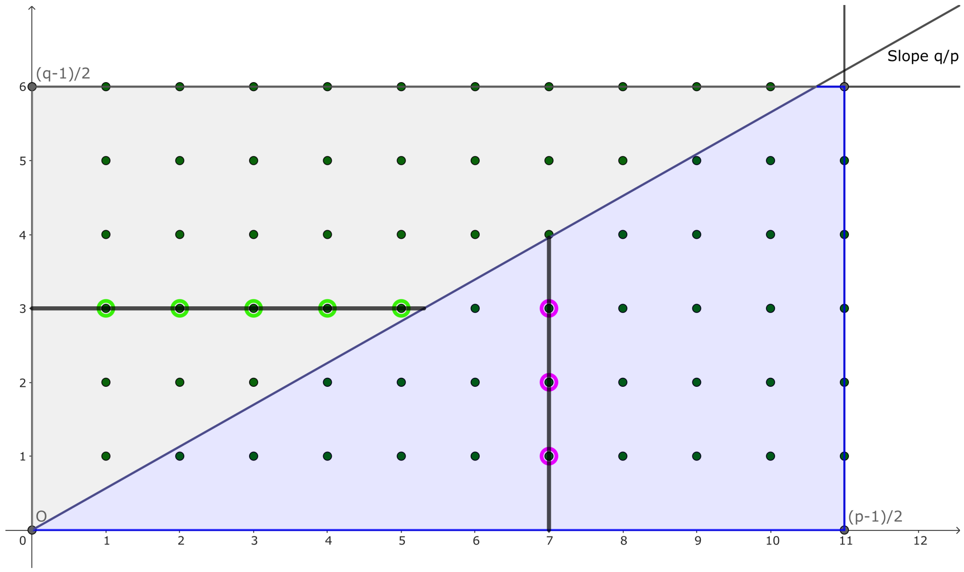

Figure 2: The lattice for the $p = 23, q = 13$, with the $j = 3$ and $k = 7$ cases highlighted

(i) $\left\lfloor \dfrac{23j}{13} \right\rfloor$ ✓ .

(ii) $\left\lfloor \dfrac{13j}{23} \right\rfloor$.

Thus, the total number of lattice points in this triangle is

Multiple Choice:

(i) $N_1 = \sum_{j=1}^{6} \left\lfloor \dfrac{23j}{13} \right\rfloor$ ✓

(ii) $N_1 = \sum_{j=1}^{6} \left\lfloor \dfrac{13j}{23} \right\rfloor$

(iii) $N_1 = \sum_{j=1}^{11} \left\lfloor \dfrac{23j}{13} \right\rfloor$

(iv) $N_1 = \sum_{j=1}^{11} \left\lfloor \dfrac{13j}{23} \right\rfloor$

(c) Next we will count the rest of the lattice points in the rectangle, the blue region in the GeoGebra. We willl call this number N_2.

The region is bounded by $0 < x \leq \frac{23-1}{2}$, $0 < y < \frac{13}{23}x$, and $y \leq \frac{13-1}{2}$. Now, the point A where $y = \frac{13}{23}x$ intersects $y = \frac{13-1}{2}$ is between two consecutive lattice points, with coordinates $x = \boxed{10}, y = \boxed{6}$ and $x = \boxed{11}, y = \boxed{6}$. Similarly, the point B where $y = \frac{13}{23}x$ intersects $x = \frac{23-1}{2}$ is between two consecutive lattice points, with coordinates $x = \boxed{11}, y = \boxed{6}$ and $x = \boxed{11}, y = \boxed{7}$. Thus, the only lattice point in the triangle A, B and $\left(\frac{23-1}{2}, \frac{13-1}{2}\right)$ is $\left(\frac{23-1}{2}, \frac{13-1}{2}\right)$. Therefore, there are also N_2 lattice points in the triangle with vertices $(0, 0), \left(\frac{23-1}{2}, 0\right), \left(\frac{23-1}{2}, \frac{13-1}{2}\right)$.

(d) We use the same method as N_1 to find N_2. We will count how many lattice points on each vertical lines $k = 1, 2, \ldots, \boxed{11}$. Let's just check the numbers we should get:

- When $k = 7$, as in Figure 2, there are $\boxed{3}$ lattice points.

For each k, we are counting positive integers $y < \boxed{\frac{13}{23}}k$. Which is,

Multiple Choice:

(i) $\left\lfloor \dfrac{23j}{13} \right\rfloor$.

(ii) $\left\lfloor \dfrac{13j}{23} \right\rfloor$ ✓ .

Thus, the total number of lattice points in this triangle is

Multiple Choice:

(i) $N_2 = \sum_{k=1}^{6} \left\lfloor \dfrac{23k}{13} \right\rfloor$

(ii) $N_2 = \sum_{k=1}^{6} \left\lfloor \dfrac{13k}{23} \right\rfloor$

(iii) $N_2 = \sum_{k=1}^{11} \left\lfloor \dfrac{23k}{13} \right\rfloor$

(iv) $N_2 = \sum_{k=1}^{11} \left\lfloor \dfrac{13k}{23} \right\rfloor$ ✓

Thus, the total number of lattice points is $N_1 + N_2 = \boxed{(11)(6)}$.

Now we will use without proof that

Lemma 1. *Let p be an odd prime number and let $a \in \mathbb{Z}$ with $p \nmid a$ and a odd. If*

$$N = \sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{ja}{p} \right\rfloor,$$

then

$$\left(\frac{a}{p}\right) = (-1)^N.$$

Proof of ?? Let p and q be distinct odd primes. Then from ,

$$\left(\frac{p}{q}\right) = (-1)^{N_1}, \quad \left(\frac{q}{p}\right) = (-1)^{N_2}, \qquad \text{where} \quad N_1 = \sum_{j=1}^{\frac{q-1}{2}} \left\lfloor \frac{p}{q} j \right\rfloor \quad \text{and} \quad N_2 = \sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{q}{p} j \right\rfloor.$$

Thus, $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{N_1+N_2}$. It remains to show that $N_1 + N_2 = \left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)$.

Without loss of generality, assume that $p > q$. We draw the rectangle $(0,0)$, $\left(\frac{p-1}{2}, 0\right)$, $\left(\frac{p-1}{2}, \frac{q-1}{2}\right)$, and $\left(0, \frac{q-1}{2}\right)$, as in the GeoGebra example. Then there are $\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)$ lattice points in this rectangle, excluding the axes.

The line segment between the origin and (p, q) has slope $\boxed{\frac{q}{p}}$. Since p and q are distinct primes, there are no lattice points on line segment except the endpoints.

For each j, we are counting positive integers $x < \boxed{\frac{p}{q}}j$. Which is,

Multiple Choice:

(a) $\left\lfloor \dfrac{pj}{q} \right\rfloor$ ✓ .

(b) $\left\lfloor \dfrac{qj}{p} \right\rfloor$.

Thus, the total number of lattice points in this triangle is

Multiple Choice:

(a) $N_1 = \displaystyle\sum_{j=1}^{(q-1)/2} \left\lfloor \frac{pj}{q} \right\rfloor$ ✓

(b) $N_1 = \displaystyle\sum_{j=1}^{(q-1)/2} \left\lfloor \frac{qj}{p} \right\rfloor$

(c) $N_1 = \displaystyle\sum_{j=1}^{(p-1)/2} \left\lfloor \frac{pj}{q} \right\rfloor$

(d) $N_1 = \displaystyle\sum_{j=1}^{(p-1)/2} \left\lfloor \frac{qj}{p} \right\rfloor$

Next we will count the rest of the lattice points in the rectangle, the blue region in the GeoGebra. We willl call this number N_2. The region is bounded by $0 < x \leq \frac{p-1}{2}$, $0 < y < \frac{q}{p}x$, and $y \leq \frac{q-1}{2}$. Now, the point A where $y = \frac{q}{p}x$ intersects $y = \frac{q-1}{2}$ is between two consecutive lattice points, with coordinates $x = \boxed{10}$, $y = \boxed{(q-1)/2}$ and $x = \boxed{(p-1)/2}$, $y = \boxed{(q-1)/2}$. Similarly, the point B where $y = \frac{q}{p}x$ intersects $x = \frac{p-1}{2}$ is between two consecutive lattice points, with coordinates $x = \boxed{(p-1)/2}$, $y = \boxed{(q-1)/2}$ and $x = \boxed{(p-1)/2}$, $y = \boxed{7}$. Thus, the only lattice point in the triangle A, B and $\left(\frac{p-1}{2}, \frac{q-1}{2}\right)$ is $\left(\frac{p-1}{2}, \frac{q-1}{2}\right)$. Therefore, there are also N_2 lattice points in the triangle with vertices $(0,0)$, $\left(\frac{p-1}{2}, 0\right)$, $\left(\frac{p-1}{2}, \frac{q-1}{2}\right)$.

We use the same method as N_1 to find N_2. We will count how many lattice points on each vertical lines $k = 1, 2, \ldots, \boxed{\dfrac{p-1}{2}}$. For each k, we are counting positive integers $y < \boxed{\frac{q}{p}}k$. Which is,

Multiple Choice:

(a) $\left\lfloor \dfrac{pj}{q} \right\rfloor$.

(b) $\left\lfloor \dfrac{qj}{p} \right\rfloor$ ✓ .

Thus, the total number of lattice points in this triangle is

Multiple Choice:

(a) $N_2 = \sum_{k=1}^{(q-1)/2} \left\lfloor \frac{pk}{q} \right\rfloor$

(b) $N_2 = \sum_{k=1}^{(q-1)/2} \left\lfloor \frac{qk}{p} \right\rfloor$

(c) $N_2 = \sum_{k=1}^{(p-1)/2} \left\lfloor \frac{pk}{q} \right\rfloor$

(d) $N_2 = \sum_{k=1}^{(p-1)/2} \left\lfloor \frac{qk}{p} \right\rfloor$ ✓

Thus, the total number of lattice points is $N_1 + N_2 = \sum_{k=1}^{(q-1)/2} \left\lfloor \frac{pk}{q} \right\rfloor + \sum_{k=1}^{(p-1)/2} \left\lfloor \frac{qk}{p} \right\rfloor = \left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)$ ■