Your Name: __________ Group Members: __________ __________

In-class Problem 1 Find the greatest common divisors of the pairs of integers below and write the greatest common divisor as a linear combination of the integers.

(a) $(21, 28)$

Solution: By inspection: $28 - 21 = 7$.

Using the Euclidean Algorithm: $a = 28, b = 21$

$$\begin{aligned} 28 &= 21(1) + 7 & q_1 &= 1, r_1 = 7 & 7 &= 21(1) + 28(-1) \\ 21 &= 7(3) + 0 & q_2 &= 3, r_2 = 0 \end{aligned}$$

so $28 + (-1)21 = 7 = (28, 21)$

(b) $(32, 56)$

Solution: Using the Euclidean Algorithm: $a = 56, b = 32$

$$\begin{aligned} 56 &= 32(1) + 24 & q_1 &= 1, r_1 = 24 & 24 &= 56(1) + 32(-1) \\ 32 &= 24(1) + 8 & q_2 &= 1, r_2 = 8 & 8 &= 32(1) + 24(-1) = 32(1) + (56(1) + 32(-1))(-1) = 32(2) + 56(-1) \\ 32 &= 8(4) + 0 & q_3 &= 4, r_3 = 0. \end{aligned}$$

so $56(-1) + 32(2) = 8 = (56, 32)$

(c) $(0, 113)$

Solution: Since $0 = 113(0)$, $(0, 113) = 113 = 0(0) = 113(1)$.

(d) $(78, 708)$

Solution: Using the Euclidean Algorithm: $a = 708, b = 78$

$$\begin{aligned} 708 &= 78(9) + 6 & q_1 &= 9, r_1 = 6 & 6 &= 708(1) + 78(-9) \\ 78 &= 6(13) + 0 & q_2 &= 13, r_2 = 0. \end{aligned}$$

so $708(1) + 78(-6) = 6 = (78, 708)$

In-class Problem 2 Let p be prime.

(a) If $(a, b) = p$, what are the possible values of (a^2, b)? Of (a^3, b)? Of (a^2, b^3)?

Learning outcomes:
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Solution: If $(a,b) = p$, then there exist $j,k \in \mathbb{Z}$ such that $a = pj, b = pk$, and $p \nmid j$ or $p \nmid k$ (otherwise $(a,b) = p^2$).

$$a^2 = p^2 j^2, \quad a^3 = p^3 j^3, \quad b^3 = p^3 k^3$$

Then (a^2, b) is p if $p \nmid k$ or p^2 if $p \mid k$; and (a^3, b) is p if $p \nmid k$, p^2 if $p \mid k$ and $p^2 \nmid k$, or p^3 if $p^2 \mid k$.

If $p \mid j$, then $p \nmid k$ and $(a^2, b^3) = p^3$. If $p \nmid j$, then $(a^2, b^3) = p^2$.

(b) If $(a,b) = p$ and $(b, p^3) = p^2$, find (ab, p^4) and $(a+b, p^4)$.

Solution: There exists $j,k \in \mathbb{Z}$ such that $a = pj, b = p^2k$, and $p \nmid k, p \nmid k$. Then $ab = p^3jk$ and $a+b = pj + p^2k = p(j+pk)$. Thus, $(ab, p^4) = p^3$ and $(a+b, p^4) = p$.