Primes

Learning Objectives. By the end of class, students will be able to:

- Every integer greater than 1 has a prime divisor.
- Prove that there are infinitely many prime numbers.

Instructor Notes: **Read** Strayer, Section 1.2

Turn in

- The proof method for Euclid's infinitude of primes is an important method. Summarize this method in your own words.

  Solution: Summaries will vary

- Identify any other new proof methods in this section

  Solution: Proof by construction may be new to some students. Students also identified:

  – Introducing a variable to aid in proof

  – Without loss of generality

- Exercise 22. Prove that 2 is the only even prime number.

  Solution: Assume that there exists another even prime number, call it p. Then there exists $2 \mid p$ by the definition of even, but that implies that $p = 2$ by the definition of prime. Thus, 2 is the only even prime number.

Definition (prime and composite)**.** An integer $p > 1$ is *prime* if the only positive divisors of p are 1 and itself. An integer n which is not prime is *composite*.

Why is 1 not prime?

Lemma 1 (Lemma 1.5)**.** *Every integer greater than 1 has a prime divisor.*

We will not go over this proof in class.

Proof Assume by contradiction that there exists $n \in \mathbb{Z}$ greater than 1 with no prime divisor. By the **??**, we may assume n is the least such integer. By definition, $n \mid n$, so n is not prime. Thus, n is composite and there exists $a, b \in \mathbb{Z}$ such that $n = ab$ and $1 < a < n$, $1 < b < n$. Since $a < n$, then it has a prime divisor p. But since $p \mid a$ and $p \mid n$, $p \mid n$. This contradicts our assumption, so no such integer exists. ■

Theorem (Euclid's Infinitude of Primes)**.** *(Theorem 1.6) There are infinitely many prime numbers.*

Learning outcomes:
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Proof Assume by way of contradiction, that there are only finitely many prime numbers, so $p_1, p_2, \ldots, p_n$. Consider the number $N = p_1 p_2 \cdots p_n + 1$. Now N has a prime divisor, say, p, by Lemma 1.5. So $p = p_i$ for some i, $i = 1, 2, \ldots, n$. Then $p \mid N - p_1 p_2 \ldots p_n$, which implies that $p \mid 1$, a contradiction. Hence, there are infinitely many prime numbers. ■

One famous open problem is the Twin Primes Conjecture. A *conjecture* is a proposition you (or in this case, the mathematical community) believe to be true, but have not proven.

Conjecture 1 (Twin Prime Conjecture)**.** *There are infinitely many prime number p for which $p + 2$ is also prime number.*

Another important fact is there are arbitrarily large sequences of composite numbers. Put another way, there are arbitrarily large gaps in the primes. Another important proof method, which is a *constructive proof*:

Theorem 1. *For any positive integer n, there are at least n consecutive positive integers.*

Proof Given the positive integer n, consider the n consecutive positive integers

$$(n+1)! + 2, (n+1)! + 3, \ldots, (n+1)! + n + 1.$$

Let i be a positive integer such that $2 \leq i \leq n + 1$. Since $i \mid (n+1)!$ and $i \mid i$, we have

$$i \mid (n+1)! + i, \quad 2 \leq i \leq n + 1$$

by linear combination (**??**). So each of the n consecutive positive integers is composite. ■

In-class Problem 1 *Let n be a positive integer with $n \neq 1$. Prove that if $n^2 + 1$ is prime, then $n^2 + 1$ can be written in the form $4k + 1$ with $k \in \mathbb{Z}$.*

Solution: *Assume that n is a positive integer, $n \neq 1$, and $n^2 + 1$ is prime. If n is odd, then n^2 is odd, which would imply $n^2 + 1 = 2$, the only even prime. However, $n \neq 1$ by assumption. Thus, n is even.*

By definition of even, there exists $j \in \mathbb{Z}$ such that $n = 2k$ and $n^2 = 4j^2$. Thus, $n^2 + 1 = 4k + 1$ when $k = j^2$.

In-class Problem 2 *Prove or disprove the following conjecture, which is similar to Twin Prime Conjecture:*

Conjecture 2. *There are infinitely many prime number p for which $p + 2$ and $p + 4$ are also prime numbers.*