April 17–Continued fractions calculations and theorems

We find the continued fraction expansions of some rational and irrational numbers, and prove some theorems about errors of estimates.

Before diving into irrational numbers, let's take one more look at the Euclidean algorithm to find the continued fraction expansion of $\frac{a}{b}$ for integers $a > b > 0$. First, we let $r_0 = a$ and $r_1 = b$.

$$a = r_0 = r_1 a_0 + r_2 \quad 0 \le r_2 < r_1 = b$$

If $0 = r_2$, we stop. Otherwise,

$$b = r_1 = r_2 a_1 + r_3 \quad 0 \le r_3 < r_2$$

Continuing until $0 = r_{n+1} < r_n < r_{n-1} < r_{n-2} < \cdots < r_1 = b < r_0 = a$. We know that $r_k = a_{k+1} r_{k+1} + r_{k+2}$ for $k \le n-1$ and $(a,b) = r_n$

Then

$$\begin{aligned}
\frac{a}{b} = \frac{r_0}{r_1} &= a_0 + T_1, & a_0 &= \left\lfloor \frac{r_0}{r_1} \right\rfloor, T_1 = \frac{r_2}{r_1} \\
\frac{1}{T_1} &= a_1 + T_2, & a_1 &= \left\lfloor \frac{r_1}{r_2} \right\rfloor, T_2 = \frac{r_3}{r_2} \\
&\vdots \\
\frac{1}{T_{n-2}} &= a_{n-1} + T_{n-1}, & a_{n-1} &= \left\lfloor \frac{r_{n-2}}{r_{n-1}} \right\rfloor, T_{n-1} = \frac{r_n}{r_{n-1}} \\
\frac{1}{T_{n-1}} &= a_n + 0 & a_n &= \left\lfloor \frac{r_{n-1}}{r_n} \right\rfloor
\end{aligned}$$

and $\frac{a}{b} = [a_0; a_1, a_2, \ldots, a_n]$.

Definition 1. Let $x = [a_0; a_1, a_2, \ldots]$. We call the rational approximations $\frac{p_i}{q_i} = [a_0; a_1, a_2, \ldots, a_i]$ are called *convergents*, where $(p_i, q_i) = 1$.

Learning outcomes:
Author(s):

Example 1. Determine the continued fraction expansion and convergents for $\dfrac{36}{13}$.

$$\frac{36}{13} = 2 + \frac{1}{\frac{13}{10}} = 2 + \cfrac{1}{1 + \cfrac{1}{\frac{10}{3}}} = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{3}}}$$

$$\frac{p_0}{q_0} = \frac{2}{1}, \frac{p_1}{q_1} = \frac{3}{1}, \frac{p_2}{q_2} = \frac{11}{4}, \frac{p_3}{q_3} = \frac{36}{13}$$

Here is a plot of the convergents

Example 2. Determine the continued fraction expansion and convergents for $\dfrac{5}{14}$

$$\frac{5}{14} = 0 + \frac{1}{\frac{14}{5}} = 0 + \cfrac{1}{2 + \cfrac{1}{\frac{5}{4}}} = 0 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{4}}}$$

$$\frac{p_0}{q_0} = \frac{0}{1}, \frac{p_1}{q_1} = \frac{1}{2}, \frac{p_2}{q_2} = \frac{1}{3}, \frac{p_3}{q_3} = \frac{5}{14}$$

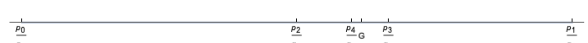
Here is a plot of the convergents

Example 3. Now we do our first example of an irrational number, the golden ration $G = \varphi = \dfrac{1+\sqrt{5}}{2}$. Now, G is a root of $x^2 - x - 1$, so $G = 1 + \dfrac{1}{G}$ (check for yourself). Substituting in for G, we get $G = 1 + \cfrac{1}{1 + \cfrac{1}{G}} = [1; \overline{1}]$ where the $\bar{}$ indicates repeating digits, as with decimal expansions.

Since every digit is 1, this continued fraction expansion converges very slowly. Calculate a few of the convergents:

$$\frac{p_0}{q_0} = \frac{1}{1}, \frac{p_1}{q_1} = \frac{2}{1}, \frac{p_2}{q_2} = \frac{3}{2}, \frac{p_3}{q_3} = \frac{5}{3}, \frac{p_4}{q_4} = \frac{8}{5}$$

Here is a plot of the convergents



Looking at all three plots, we can start to see that for $x \in \mathbb{R}, x > 0$, $\dfrac{p_0}{q_0} < x < \dfrac{p_1}{q_1}$ and $\dfrac{p_0}{q_0} < \dfrac{p_2}{q_2} < x < \dfrac{p_3}{q_3} < \dfrac{p_1}{q_1}$ (contrast this to the decimal expansion where $d_0 \leq d_0.d_1 \leq d_0.d_1d_2 \leq d_0.d_1d_2d_3 \leq \cdots$). In order to prove that this pattern continues (and holds for all positive real numbers), we need to prove a few more facts about p_i and q_i.

Theorem 1. Let $p_{-1} = 1, q_{-1} = 0, p_0 = a_0, q_0 = 1$ (see that this matches with p_0, q_0 above). Then

$$p_n = a_n p_{n-1} + p_{n-2} \tag{1}$$
$$q_n = a_n q_{n-1} + q_{n-2} \tag{2}$$

for $n \geq 1$.

Proof We start by checking $\frac{p_1}{q_1} = a_0 + \frac{1}{a_1} = \frac{a_0a_1+1}{a_1} = \frac{p_0a_1+p_{-1}}{q_0a_1+q_{-1}}$.

Next, we check $k=2$, $\frac{p_2}{q_2} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}} = a_0 + \frac{a_2}{a_2a_1+1} = \frac{a_0(a_2a_1+1)+a_2}{a_2a_1+1} = \frac{a_2p_1+p_0}{a_2q_1+q_0}$.

Now we proceed by induction. Assume that these recurrence relations hold for $n \leq k$. Then

$$\frac{p_k}{q_k} = [a_0; a_1, a_2, \ldots, a_k] = \frac{a_kp_{k-1}+p_{k-2}}{a_kq_{k-1}+q_{k-2}}.$$

By definition,

$$\begin{aligned}
\frac{p_{k+1}}{q_{k+1}} &= [a_0; a_1, a_2, \ldots, a_k, a_{k+1}] = [a_0; a_1, a_2, \ldots, a_k + \frac{1}{a_{k+1}}] \\
&= \frac{\left(a_k + \frac{1}{a_{k+1}}\right)p_{k-1} + p_{k-2}}{\left(a_k + \frac{1}{a_{k+1}}\right)q_{k-1} + q_{k-2}} \text{(by induction hypothesis*)} \\
&= \frac{a_{k+1}\left(a_kp_{k-1}+p_{k-2}\right)+p_{k-1}}{a_{k+1}\left(a_kq_{k-1}+q_{k-2}\right)+q_{k-1}} \\
&= \frac{a_{k+1}p_k+p_{k-1}}{a_{k+1}q_k+q_{k-1}} \text{(by induction hypothesis)}
\end{aligned}$$

■

*Often the recurrence relations (**??**) and (**??**) are proven using matrix multiplication instead of justifying that it is ok to substitute $a_k + \frac{1}{a_{k+1}}$ for a_k. The matrix definition also allows us to prove the next theorem by taking determinants. Instead, we will use induction and the recurrence relations.

Theorem 2. For all $n \geq 1$, $p_nq_{n-1} - p_{n-1}q_n = (-1)^{n-1}$, which is equivalent to $\frac{p_n}{q_n} = \frac{(-1)^{n-1}}{q_nq_{n-1}} + \frac{p_{n-1}}{p_{n-1}}$.

Proof The second equation is the first equation rewritten.

We start with $i = 1$, so from the previous theorem $p_1 q_0 - p_0 q_1 = (a_0 a_1 + 1)1 - a_0 a_1 = 1 = (-1)^0$.

Now we proceed by induction. Assume that these recurrence relations hold for $n \leq k$. Then $p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}$. Substituting in the recurrence relations (**??**) and (**??**), we get

$$\begin{aligned} p_{k+1} q_k - p_k q_{k+1} &= (a_{k+1} p_k + p_{k-1}) q_k - p_k (a_{k+1} q_k + q_{k-1}) \\ &= p_{k-1} q_k - p_k q_{k-1} \\ &= -(p_k q_{k-1} - p_{k-1} q_k) \\ &= -(-1)^{k-1} = (-1)^k \end{aligned}$$

■

Notice that $\frac{p_n}{q_n} = \frac{(-1)^{n-1}}{q_n q_{n-1}} + \frac{p_{n-1}}{p_{n-1}}$ is still a recurrence relation. Expanding this out, we get

$$\frac{p_n}{q_n} = a_0 + \frac{1}{q_0 q_1} - \frac{1}{q_1 q_2} + \cdots + \frac{(-1)^{n-1}}{q_n q_{n-1}} = a_0 + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{q_{k-1} q_k}.$$

Thus $x = \lim_{n\to\infty} \frac{p_n}{q_n} = a_0 + \lim_{n\to\infty} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{q_{k-1} q_k} = a_0 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{q_{k-1} q_k}$ which converges by the alternating series test. This shows that the continued fraction expansion of x really does converge to x for all $x \in \mathbb{R}, x > 0$.

Theorem 3. For $x \in \mathbb{R}, x > 0$, $\frac{p_0}{q_0} < \frac{p_2}{q_2} < \cdots < x < \cdots < \frac{p_3}{q_3} < \frac{p_1}{q_1}$.

Proof From (**??**), we find that $q_0 < q_1 < q_2 < \cdots$, since we have a recurrence relation that add positive integers to get a larger positive integer. Thus $\frac{1}{q_{n+1} q_n} < \frac{1}{q_n q_{n-1}}$.

For $n = 2k$, we have that

$$\begin{aligned} \frac{p_{2k}}{q_{2k}} &= \frac{(-1)^{2k-1}}{q_{2k} q_{2k-1}} + \frac{p_{2k-1}}{p_{2k-1}} = \frac{-1}{q_{2k} q_{2k-1}} + \frac{(-1)^{2k-2}}{q_{2k-1} q_{2k-2}} + \frac{p_{2k-2}}{p_{2k-2}} \\ &= \frac{-1}{q_{2k} q_{2k-1}} + \frac{1}{q_{2k-1} q_{2k-2}} + \frac{p_{2k-2}}{p_{2k-2}}, \end{aligned}$$

and $\frac{-1}{q_{2k} q_{2k-1}} + \frac{1}{q_{2k-1} q_{2k-2}} > 0$, so $\frac{p_0}{q_0} < \frac{p_2}{q_2} < \cdots < \frac{p_{2k}}{q_{2k}} < \cdots$. We also have that $\lim_{k\to\infty} \frac{p_{2k}}{q_{2k}} = x$ (from analysis) $\frac{p_0}{q_0} < \frac{p_2}{q_2} < \cdots < \frac{p_{2k}}{q_{2k}} < \cdots < x$.

For $n = 2k+1$, we have that

$$\begin{aligned}\frac{p_{2k+1}}{q_{2k+1}} &= \frac{(-1)^{2k}}{q_{2k+1}q_{2k}} + \frac{p_{2k}}{p_{2k}} = \frac{1}{q_{2k+1}q_{2k}} + \frac{(-1)^{2k-1}}{q_{2k}q_{2k-1}} + \frac{p_{2k-1}}{p_{2k-1}} \\ &= \frac{1}{q_{2k+1}q_{2k}} + \frac{-1}{q_{2k}q_{2k-1}} + \frac{p_{2k-1}}{p_{2k-1}},\end{aligned}$$

and $\frac{1}{q_{2k+1}q_{2k}} + \frac{-1}{q_{2k}q_{2k-1}} < 0$, so $\cdots < \frac{p_{2k-1}}{q_{2k-1}} < \frac{p_{2k+1}}{1_{2k+1}} \cdots < \frac{p_3}{q_3} < \frac{p_1}{q_1}$. We also have that $\lim_{k\to\infty} \frac{p_{2k+1}}{q_{2k+1}} = x$ (from analysis) $x < \cdots < \frac{p_{2k-1}}{q_{2k-1}} < \frac{p_{2k+1}}{1_{2k+1}} \cdots < \frac{p_3}{q_3} < \frac{p_1}{q_1}$. ■