Induction

Learning Objectives. By the end of class, students will be able to:

- Construct a proof by induction .

The following reading assignment covers the basics on proof by induction:

Read Strayer Appendix A.1: The First Principle of Mathematical Induction or Ernst Section 4.1 and Section 4.2

Turn in Strayer Exercise Set A, Exercise 1a. If n is a positive integer, then

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

In-class Problem 1 *Theorems in Ernst Section 4.1*

Theorem (Ernst Theorem 4.5)**.** *For all $n \in \mathbb{N}$, 3 divides $4^n - 1$.*

Solution: *We proceed by induction. When $n = 1$, $3 \mid 4^n - 1 = 3$. Thus, the statement is true for $n = 1$.*

Now assume $k \geq 1$ and the desired statement is true for $n = k$. Then the induction hypothesis is

$$3 \mid 4^k - 1.$$

By the definition of **??**, *there exists $m \in \mathbb{Z}$ such that $3m = 4^k - 1$. In other words, $3m + 1 = 4^k$. Multiplying both sides by 4 gives $12m + 4 = 4^{k+1}$. Rewriting this equation gives $3(4m + 1) = 4^{k+1} - 1$. Thus, $3 \mid 4^{k+1} - 1$, and the desired statement is true for $n = k + 1$. By the (first) principle of mathematical induction, the statement is true for all positive integers, and the proof is complete.*

Theorem (Ernst Theorem 4.7)**.** *Let $p_1, p_2, \ldots, p_n$ be n distinct points arranged on a circle. Then the number of line segments joining all pairs of points is $\frac{n^2 - n}{2}$.*

Solution: *We proceed by induction. When $n = 1$, there is only one point, so there are no lines connecting pairs of points. Additionally, $\frac{1^2 - 1}{2} = 0$.*

Now assume $k \geq 1$ and the desired statement is true for $n = k$. Then the induction hypothesis is for k distinct points arranged in a circle, the number of line segments joining all pairs of points is $\frac{k^2 - k}{2}$. Adding a $(k+1)^{st}$ point on the circle will add an additional k line segments joining pairs of points, one for each existing point. Note that

$$\frac{k^2 - k}{2} + k = \frac{k^2 + k}{2} = \frac{k^2 + k + k + 1 - (k+1)}{2} = \frac{(k+1)^2 - (k+1)}{2}$$

Learning outcomes:
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Alternately, you could use $n = 2$ for the base case. Then there is one line connecting the only pair of points and $\frac{2^2 - 2}{2} = 1$

In-class Problem 2 . *Use the first principle of mathematical induction to prove each statement.*

(a) *If n is a positive integer, then*

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

(b) *If n is an integer with $n \geq 5$, then*

$$2^n > n^2.$$