Divisibility practice

Learning Objectives. By the end of class, students will be able to:

- Prove facts about divisibility
- Prove basic mathematical statements using definitions and direct proof
- Use truth tables to understand compound propositions
- Prove statements by contradiction
- Use the greatest integer function .

Instructor Notes: **Reading** Read Ernst Chapter 1 and Section 2.1. Also read Strayer Introduction and Section 1.1 through the proof of Proposition 1.2 (that is, pages 1-5).

Turn in: From Ernst: Problem 2.6 and 2.8

Divisibility practice

Proposition 1. *Let $a, b \in \mathbb{Z}$. If $a \mid b$ and $b \mid c$, then $a \mid c$.*

Since this is the first result in the chapter, the only tool we have is the definition of "$a \mid b$".

Proof Since $a \mid b$ and $b \mid c$, there exist $d, e \in \mathbb{Z}$ such that $b = ae$ and $c = bf$. Combining these, we see

$$c = bf = (ae)f = a(ef),$$

so $a \mid c$. ■

This means that division is *transitive*.

Proposition 2. *Let $a, b, c, m, n \in \mathbb{Z}$. If $c \mid a$ and $c \mid b$ then $c \mid ma + nb$.*

Proof *Let $a, b, c, m, n \in \mathbb{Z}$ such that $c \mid a$ and $c \mid b$. Then by definition of divisibility, there exists $j, k \in \mathbb{Z}$ such that $cj = a$ and $ck = b$. Thus,*

$$ma + nb = m(cj) + n(ck) = c(mj + nk).$$

Therefore, $c \mid ma + nb$ by definition. ■

Definition. The expression $ma + nb$ in Proposition 1.2 is called *an (integral) linear combination of a and b.*

Proposition 1.2 says that an integer dividing each of two integers also divides any integral linear combination of those integers. This fact will be extremely valuable in establishing theoretical results. But first, let's get some more practice with proof writing

Break into three groups. Using the proofs of Propositions 1.1 and 1.2 as examples, prove the following facts. Each group will prove one part.

In-class Problem 1 *Prove or disprove the following statements.*

(a) *If a, b, c, and d are integers such that if $a \mid b$ and $c \mid d$, then $a + c \mid b + d$.*

Learning outcomes:
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(b) *If a, b, c, and d are integers such that if $a \mid b$ and $c \mid d$, then $ac \mid bd$.*

(c) *If a, b, and c are integers such that if $a \nmid b$ and $b \nmid c$, then $a \nmid c$.*

Solution: Problem on Homework 1.

Logic, proof by contradiction, and biconditionals

We will begin by working through Ernst Section 2.2 through Example 2.21. Discuss Problem 2.17 as a class. Problem 2.17 is also provided below:

In-class Problem 2 *Determine whether each of the following is a proposition. Explain your reasoning.*

- *All cars are red.*
- *Every person whose name begins with J has the name Joe.*
- $x^2 = 4$.
- *There exists a real number x such that $x^2 = 4$.*
- *For all real numbers x, $x^2 = 4$.*
- $\sqrt{2}$ *is an irrational number.*
- p *is prime.*
- *Is it raining?*
- *It will rain tomorrow.*
- *Led Zeppelin is the best band of all time.*

In-class Problem 3 *Construct a truth table for $A \Rightarrow B, \neg(A \Rightarrow B)$ and $A \wedge \neg B$*

Solution:

A	B	$A \Rightarrow B$	$\neg(A \Rightarrow B)$	$A \wedge \neg B$
T	T	T	F	F
T	F	F	T	T
F	T	T	F	F
F	F	T	F	F

This is the basis for *proof by contradiction.* We assume both A and $\neg B$, and proceed until we get a contradiction. That is, A and $\neg B$ cannot both be true.

Definition (Proof by contradiction)**.** Let A and B be propositions. To prove A implies B by contradiction, first assume the B is false. Then work through logical steps until you conclude $\neg A \wedge A$.

All definitions are ‘biconditionals but we normally only write the “if.”

We say that two definitions are *equivalent* if definition A is true if and only if definition B is true.