# Sums and Differences of Squares

Reading and Turn in: This fill-in-the-blank version of the proof of Sum of Three Squares

**In-class Problem 1 (Chapter 6, Exercise 34)** Prove that a positive integer can be written as the difference of two squares of integers if and only if it is not of the form $4n+2$ for some $n \in \mathbb{Z}$.

**Solution:** ($\Rightarrow$) We will show that if a positive integer can be written as the difference of two squares of integers, then it is not of the form $4n+2$ for some $n \in \mathbb{Z}$.

(Proof)

($\Leftarrow$) We will show that any positive integer not of the form $4n+2$ for some $n \in \mathbb{Z}$ can be written as the difference of two squares of integers.

First, we will show that if $a$ and $b$ are positive integers that can be written as the difference of two squares of integers, then so can $ab$.

(Proof)

Now we will show that every odd prime can be written as the difference of two squares of integers. Let $p$ be an odd prime. Then $p = x^2 - y^2 = (x-y)(x+y)$ when $x = \frac{p+1}{2}$ and $y = \frac{p-1}{2}$. Therefore every odd number can be written as the difference of two squares since

It remains to show that every positive integer of the form $4n$ for some $n \in \mathbb{Z}$ can be written as the difference of two squares of integers. Why is this the only remaining case?

Similar to the odd prime case, $4n = x^2 - y^2 = (x-y)(x+y)$ when $x = n+1$ and $y = n-1$ or when $x = \frac{n}{2} - 2$ and $y = \frac{n}{2} + 2$ (solutions are not unique).

If there is time

**In-class Problem 2 (Chapter 6, Exercise 14d)** Let $x, y, z$ be a primitive Pythagorean triple with $y$ even. Prove that $x + y \equiv x - y \equiv 1, 7 \pmod 8$.

***Hint:*** *First show that* $x + y \equiv x - y \pmod 8$.

**Solution:** Let $x, y, z$ be a primitive Pythagorean triple with $y$ even. Then from **??**, there exist positive integers $m, n$ with $m > n, (m, n) = 1$ and exactly one of $m, n$ even such that

$$\begin{aligned} x &= m^2 - n^2 \\ y &= 2mn \\ z &= m^2 + n^2. \end{aligned}$$

Thus, $4 \mid y$ and $y \equiv -y \pmod 8$. Adding $x$ to both sides of the congruence gives $x + y \equiv x - y \pmod 8$.

Since exactly one of $m, n$ even, $x \equiv 0-1, 4-1, 1-0, 1-4 \pmod 8$. When $x \equiv \pm 1 \pmod 8$, $4 \mid mn$ and thus $y = 2mn \equiv 0 \pmod 8$. Thus, $x + y \equiv x - y \equiv \pm 1 \pmod 8$. When $x \equiv \pm 3 \pmod 8$, $4 \nmid mn$ and thus $y = 2mn \equiv 4 \pmod 8$. Thus, $x + y \equiv x - y \equiv \pm 1 \pmod 8$.

Learning outcomes:

Author(s): Claire Merriman