$$\begin{cases} P_{4,1} = P\big((4,1)\,|\,(3,1)\big)\times P_{3,1} + P\big((4,1)\,|\,(3,2)\big)\times P_{3,2} = \frac{2}{3}\times\frac{1}{2} + 0\times\frac{1}{2} = \frac{1}{3} \\ P_{4,2} = P\big((4,2)\,|\,(3,1)\big)\times P_{3,1} + P\big((4,2)\,|\,(3,2)\big)\times P_{3,2} = \frac{1}{3}\times\frac{1}{2} + \frac{1}{3}\times\frac{1}{2} = \frac{1}{3} \\ P_{4,3} = P\big((4,3)\,|\,(3,1)\big)\times P_{3,1} + P\big((4,3)\,|\,(3,2)\big)\times P_{3,2} = 0\times\frac{1}{2} + \frac{2}{3}\times\frac{1}{2} = \frac{1}{3} \end{cases}$$

The results indicate that $P_{n,k} = \frac{1}{n-1}$, $\forall k = 1, 2, \cdots, n-1$, and give the hint that the law of total probability can be used in the induction step.

Induction step: given that $P_{n,k} = \frac{1}{n-1}$, $\forall k = 1, 2, \cdots, n-1$, we need to prove $P_{n+1,k} = \frac{1}{(n+1)-1} = \frac{1}{n}$, $\forall k = 1, 2, \cdots, n$. To show it, simply apply the law of total probability:

$$P_{n+1,k} = P\big(miss\,|\,(n,k)\big)P_{n,k} + P\big(score\,|\,(n,k-1)\big)P_{n,k-1}$$
$$= \left(1-\frac{k}{n}\right)\frac{1}{n-1} + \frac{k-1}{n}\frac{1}{n-1} = \frac{1}{n}$$

The equation is also applicable to the $P_{n+1,1}$ and $P_{n+1,n}$, although in these cases $\frac{k-1}{n} = 0$ and $\left(1-\frac{k}{n}\right) = 0$, respectively. So we have $P_{n,k} = \frac{1}{n-1}$, $\forall k = 1, 2, \cdots, n-1$ and $\forall n \geq 2$. Hence, $P_{100,50} = 1/99$.

Cars on road

If the probability of observing at least one car on a highway during any 20-minute time interval is 609/625, then what is the probability of observing at least one car during any 5-minute time interval? Assume that the probability of seeing a car at any moment is uniform (constant) for the entire 20 minutes.

Solution: We can break down the 20-minute interval into a sequence of 4 non-overlapping 5-minute intervals. Because of constant default probability (of observing a car), the probability of observing a car in any 5-minute interval is constant. Let's denote the probability to be p, then the probability that in any 5-minute interval we do not observe a car is $1-p$.

The probability that we do not observe any car in all four of such independent 5-minute intervals is $(1-p)^4 = 1-609/625 = 16/625$, which gives $p = 3/5$.

4.4 Discrete and Continuous Distributions

In this section, we review a variety of distribution functions for random variables that are widely used in quantitative modeling. Although it may not be necessary to memorize the properties of these distributions, having an intuitive understanding of the distributions and having the ability to quickly derive important properties are valuable skills in practice. As usual, let's begin with the theories:

Common function of random variables

Table 4.1 summarizes how the basic properties of discrete and continuous random variables are defined or calculated. These are the basics you should commit to memory.

Random variable (X)	Discrete	Continuous[19]
Cumulative distribution function/cdf	$F(a) = P\{X \le a\}$	$F(a) = \int_{-\infty}^{a} f(x)dx$
Probability mass function /pmf Probability density function /pdf	pmf: $p(x) = P\{X = x\}$	pdf: $f(x) = \frac{d}{dx}F(x)$
Expected value/ $E[X]$	$\sum_{x:p(x)>0} xp(x)$	$\int_{-\infty}^{\infty} xf(x)dx$
Expected value of g(X)/ $E[g(X)]$	$\sum_{x:p(x)>0} g(x)p(x)$	$\int_{-\infty}^{\infty} g(x)f(x)dx$
Variance of X/ $var(X)$	$E[(X-E[X])^2] = E[X^2]-(E[X])^2$	
Standard deviation of X/ $std(X)$	$\sqrt{var(X)}$	

Table 4.1 Basic properties of discrete and continuous random variables

Discrete random variables

Table 4.2 includes some of the most widely-used discrete distributions. Discrete uniform random variable represents the occurrence of a value between number a and b when all values in the set $\{a, a+1, \cdots, b\}$ have equal probability. Binomial random variable represents the number of successes in a sequence of n experiments when each trial is

[19] For continuous random variables, $P(X = x) = 0$, $\forall x \in (-\infty, \infty)$, so $P\{X \le x\} = P\{X < x\}$.

independently a success with probability p. Poisson random variable represents the number of events occurring in a fixed period of time with the expected number of occurrences λt when events occur with a known average rate λ and are independent of the time since the last event. Geometric random variable represents the trial number (n) to get the first success when each trial is independently a success with probability p. Negative Binomial random variable represents the trial number to get to the r-th success when each trial is independently a success with probability p.

Name	Probability mass function (pmf)	$E[X]$	$\text{var}(X)$
Uniform	$P(x)=\frac{1}{b-a+1},\quad x=a,a+1,\cdots,b$	$\frac{b+a}{2}$	$\frac{(b-a+1)^2-1}{12}$
Binomial	$P(x)=\binom{n}{x}p^x(1-p)^{n-x},\quad x=0,1,\cdots,n$	np	$np(1-p)$
Poisson	$P(x)=\frac{e^{-\lambda t}(\lambda t)^x}{x!},\quad x=0,1,\cdots$[20]	λt	λt
Geometric	$P(x)=(1-p)^{x-1}p,\ x=1,2,\cdots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Negative Binomial	$P(x)=\binom{x-1}{r-1}p^r(1-p)^{x-r},\quad x=r,r+1,\cdots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$

Table 4.2 Probability mass function, expected value and variance of discrete random variables

Continuous random variables

Table 4.3 includes some of the commonly encountered continuous distributions. Uniform distribution describes a random variable uniformly distributed over the interval $[a,b]$. Because of the central limit theorem, normal distribution/Gaussian distribution is by far the most popular continuous distribution. Exponential distribution models the arrival time of an event if it has a constant arrival rate λ. Gamma distribution with parameters (α, λ) often arises, in practice, as the distribution of the amount of time one has to wait until a total of n events occur. Beta distributions are used to model events

[20] Here we use the product of arrival rate λ and time t to define the parameter (expected value) since it is the definition used in many Poisson process studies.

that are constrained within a defined interval. By adjusting the shape parameters α and β, it can model different shapes of probability density functions.[21]

Name	Probability density function (pdf)	$E[X]$	$\text{var}(X)$
Uniform	$\frac{1}{b-a},\ a \le x \le b$	$\frac{b+a}{2}$	$\frac{(b-a)^2}{12}$
Normal	$\frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}},\ x \in (-\infty, \infty)$	μ	σ^2
Exponential	$\lambda e^{-\lambda x},\ x \ge 0$	$1/\lambda$	$1/\lambda^2$
Gamma	$\frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)},\ x \ge 0,\ \Gamma(a) = \int_0^\infty e^{-y} y^{a-1}$	α/λ	α/λ^2
Beta	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1},\ 0 < x < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$

Table 4.3 Probability density function, expected value and variance of continuous random variables

Meeting probability

Two bankers each arrive at the station at some random time between 5:00 am and 6:00 am (arrival time for either banker is uniformly distributed). They stay exactly five minutes and then leave. What is the probability they will meet on a given day?

Solution: Assume banker A arrives X minutes after 5:00 am and B arrives Y minutes after 5:00 am. X and Y are independent uniform distribution between 0 and 60. Since both only stay exactly five minutes, as shown in Figure 4.4, A and B meet if and only if $|X-Y| \le 5$.

So the probability that A and B will meet is simply the area of the shadowed region divided by the area of the square (the rest of the region can be combined to a square with size length 55): $\frac{60\times60 - 2\times(1/2\times55\times55)}{60\times60} = \frac{(60+55)\times(60-55)}{60\times60} = \frac{23}{144}$.

[21] For example, beta distribution is widely used in modeling loss given default in risk management. If you are familiar with Bayesian statistics, you will also recognize it as a popular conjugate prior function.

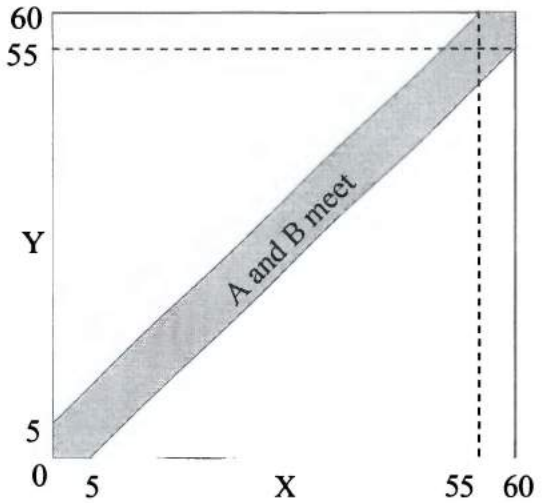


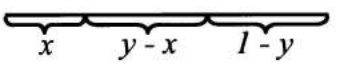
Figure 4.4 Distributions of Banker *A*'s and Banker *B*'s arrival times

Probability of triangle

A stick is cut twice randomly (each cut point follows a uniform distribution on the stick), what is the probability that the 3 segments can form a triangle?[22]

Solution: Without loss of generality, let's assume that the length of the stick is 1. Let's also label the point of the first cut as x and the second cut as y.

If $x < y$, then the three segments are x, y-x and 1-y. The conditions to form a triangle are



$$x+(y-x)>1-y \Rightarrow y>1/2$$
$$x+(1-y)>y-x \Rightarrow y<1/2+x$$
$$(y-x)+(1-y)>x \Rightarrow x<1/2$$

The feasible area is shown in Figure 4.5. The case for $x < y$ is the left gray triangle. Using symmetry, we can see that the case for $x > y$ is the right gray triangle.

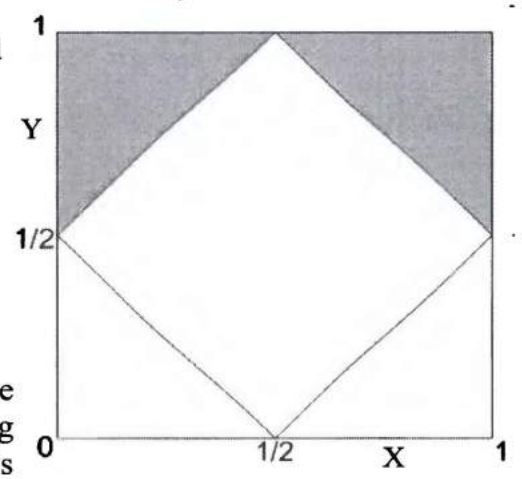


Figure 4.5 Distribution of cuts X and Y

[22] Hint: Let the first cut point be x, the second one be y, use the figure to show the distribution of x and y.

The total shadowed area represents the region where 3 segments can form a triangle, which is 1/4 of the square. So the probability is 1/4.

Property of Poisson process

You are waiting for a bus at a bus station. The buses arrive at the station according to a Poisson process with an average arrival time of 10 minutes ($\lambda = 0.1/\min$). If the buses have been running for a long time and you arrive at the bus station at a random time, what is your expected waiting time? On average, how many minutes ago did the last bus leave?

Solution: Considering the importance of jump-diffusion processes in derivative pricing and the role of Poisson processes in studying jump processes, let's elaborate more on exponential random variables and the Poison process. Exponential distribution is widely used to model the time interval between independent events that happen at a constant average rate (arrival rate) λ: $f(t) = \begin{cases} \lambda e^{-\lambda t} & (t \geq 0) \\ 0 & (t < 0) \end{cases}$. The expected arrival time is $1/\lambda$ and the variance is $1/\lambda^2$. Using integration, we can calculate the cdf of an exponential distribution to be $F(t) = P(\tau \leq t) = 1 - e^{-rt}$ and $P(\tau > t) = e^{-rt}$, where τ is the random variable for arrival time. One unique property of exponential distribution is memorylessness: $P\{\tau > s+t \mid \tau > s\} = P(\tau > t)$.[23] That means if we have waited for s time units, the extra waiting time has the same distribution as the waiting time when we start at time 0.

When the arrivals of a series of events each independently follow an exponential distribution with arrival rate λ, the number of arrivals between time 0 and t can be modeled as a Poisson process $P(N(t) = x) = \frac{e^{-\lambda t}\lambda t^x}{x!}$, $x = 0, 1, \cdots$ [24] The expected number of arrivals is λt and the variance is also λt. Because of the memoryless nature of exponential distribution, the number of arrivals between time s and t is also a Poisson process $P(N(t-s) = x) = \frac{e^{-\lambda(t-s)}\left(\lambda(t-s)\right)^x}{x!}$.

Taking advantage of the memoryless property of exponential distribution, we know that the expected waiting time is $1/\lambda = 10\min$. If you look back in time, the memoryless property stills applies. So on average, the last bus arrived 10 minutes ago as well.

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[23] $P\{\tau > s+t \mid \tau > s\} = e^{-\lambda(s+t)}/e^{-\lambda s} = e^{-\lambda t} = P(x > t)$

[24] More rigorously, $N(t)$ is defined as a right-continuous function.

This is another example that your intuition may misguide you. You may be wondering that if the last bus on average arrived 10 minutes ago and the next bus on average will arrive 10 minutes later, shouldn't the average arrival time be 20 minutes instead of 10? The explanation to the apparent discrepancy is that when you arrive at a random time, you are more likely to arrive in a long time interval between two bus arrivals than in a short one. For example, if one interval between two bus arrivals is 30 minutes and another is 5 minutes, you are more likely to arrive at a time during that 30-minute interval rather than 5-minute interval. In fact, if you arrive at a random time, the expected residual life (the time for the next bus to arrive) is $\frac{E[X^2]}{2E[X]}$ for a general distribution.[25]

Moments of normal distribution

If X follows standard normal distribution ($X \sim N(0, 1)$), what is $E[X^n]$ for n = 1, 2, 3 and 4?

Solution: The first to fourth moments of the standard normal distribution are essentially the mean, the variance, the skewness and the kurtosis. So you probably have remembered that the answers are 0, 1, 0 (no skewness), and 3, respectively.

Standard normal distribution has pdf $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Using simple symmetry we have $E[x^n] = \int_{-\infty}^{\infty} x^n \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 0$ when n is odd. For $n = 2$, integration by parts are often used. To solve $E[X^n]$ for any integer n, an approach using **moment generating functions** may be a better choice. Moment generating functions are defined as

$$M(t) = E[e^{tX}] = \begin{cases} \sum_x e^{tx} p(x), & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x) dx, & \text{if } x \text{ is continuous} \end{cases}$$

Sequentially taking derivative of $M(t)$, we get one frequently-used property of $M(t)$:

$$M'(t) = \frac{d}{dt} E[e^{tX}] = E[Xe^{tX}] \Rightarrow M'(0) = E[X],$$

$$M''(t) = \frac{d}{dt} E[Xe^{tX}] = E[X^2 e^{tX}] \Rightarrow M''(0) = E[X^2],$$

[25] The residual life is explained in Chapter 3 of "***Discrete Stochastic Process***" by Robert G. Gallager.

and $M^n(0) = E[X^n]$, $\forall n \geq 1$ in general.

We can use this property to solve $E[X^n]$ for $X \sim N(0, 1)$. For standard normal distribution $M(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-t)^2/2} dx = e^{t^2/2}$.

($\frac{1}{\sqrt{2\pi}} e^{-(x-t)^2/2}$ is the pdf of normal distribution $X \sim N(t, 1)$, so $\int_{-\infty}^{\infty} f(x)dx = 1$).

Taking derivatives, we have

$M'(t) = te^{t^2/2} \Rightarrow M'(0) = 0$, $M''(t) = e^{t^2/2} + t^2 e^{t^2/2} \Rightarrow M''(0) = e^0 = 1$,

$M^3(t) = te^{t^2/2} + 2te^{t^2/2} + t^3 e^{t^2/2} = 3te^{t^2/2} + t^3 e^{t^2/2} \Rightarrow M^3(0) = 0$,

and $M^4(t) = 3e^{t^2/2} + 3t^2 e^{t^2/2} + 3t^2 e^{t^2/2} + 3t^4 e^{t^2/2} \Rightarrow M^4(0) = 3e^0 = 3$.

4.5 Expected Value, Variance & Covariance

Expected value, variance and covariance are indispensable in estimating returns and risks of any investments. Naturally, they are a popular test subject in interviews as well. The basic knowledge includes the following:

If $E[x_i]$ is finite for all $i = 1, \cdots, n$, then $E[X_1 + \cdots + X_n] = E[X_1] + \cdots + E[X_n]$. The relationship holds whether the x_i's are independent of each other or not.

If X and Y are independent, then $E[g(X)h(Y)] = E[g(x)]E[h(Y)]$.

Covariance: $Cov(X, Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y]$.

Correlation: $\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$

If X and Y are independent, $Cov(X, Y) = 0$ and $\rho(X, Y) = 0$.[26]

General rules of variance and covariance:

$$Cov(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j Cov(X_i, Y_j)$$

$$Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i) + 2 \sum\sum_{i<j} Cov(X_i, X_j)$$

[26] The reverse is not true. $\rho(X, Y) = 0$ only means X and Y are uncorrelated; they may well be dependent.

Conditional expectation and variance

For discrete distribution: $E[g(X)|Y=y]=\sum_x g(x)p_{X|Y}(x|y)=\sum_x g(x)p(X=x|Y=y)$

For continuous distribution: $E[g(X)|Y=y]=\int_{-\infty}^{\infty} g(x)f_{X|Y}(x|y)dx$

Law of total expectation:

$$E[X]=E[E[X|Y]]=\begin{cases}\sum_y E[X|Y=y]p(Y=y), \text{ for discrete } Y\\ \int_{-\infty}^{\infty} E[X|Y=y]f_Y(y)dy, \text{ for continuous } Y\end{cases}$$

Connecting noodles

You have 100 noodles in your soup bowl. Being blindfolded, you are told to take two ends of some noodles (each end on any noodle has the same probability of being chosen) in your bowl and connect them. You continue until there are no free ends. The number of loops formed by the noodles this way is stochastic. Calculate the expected number of circles.

Solution: Again do not be frightened by the large number 100. If you have no clue how to start, let's begin with the simplest case where $n=1$. Surely you have only one choice (to connect both ends of the noodle), so $E[f(1)]=1$. How about 2 noodles? Now you have 4 ends (2×2) and you can connect any two of them. There are $\binom{4}{2}=\frac{4\times 3}{2}=6$ combinations. Among them, 2 combinations will connect both ends of the same noodle together and yield 1 circle and 1 noodle. The other 4 choices will yield a single noodle. So the expected number of circles is

$$E[f(2)]=2/6\times(1+E[f(1)])+4/6\times E[f(1)]=1/3+E[f(1)]=1/3+1.$$

We now move on to 3 noodles with $\binom{6}{2}=\frac{6\times 5}{2}=15$ choices. Among them, 3 choices will yield 1 circle and 2 noodles; the other 12 choices will yield 2 noodles only, so

$$E[f(3)]=3/15\times(1+E[f(2)])+12/15\times E[f(2)]=1/5+E[f(2)]=1/5+1/3+1.$$

See the pattern? For any n noodles, we will have $E[f(n)]=1+1/3+1/5+\cdots+1/(2n-1)$, which can be easily proved by induction. Plug 100 in, we will have the answer.

Actually after the 2-noodle case, you probably have found the key to this question. If you start with n noodles, among $\binom{2n}{2} = n(2n-1)$ possible combinations, we have $\frac{n}{n(2n-1)} = \frac{1}{2n-1}$ probability to yield 1 circle and $n-1$ noodles and $\frac{2n-2}{2n-1}$ probability to yield $n-1$ noodles only, so $E[f(n)] = E[f(n-1)] + \frac{1}{2n-1}$. Working backward, you can get the final solution as well.

Optimal hedge ratio

You just bought one share of stock A and want to hedge it by shorting stock B. How many shares of B should you short to minimize the variance of the hedged position? Assume that the variance of stock A's return is σ_A^2; the variance of B's return is σ_B^2; their correlation coefficient is ρ.

Solution: Suppose that we short h shares of B, the variance of the portfolio return is $\text{var}(r_A - hr_B) = \sigma_A^2 - 2\rho h\sigma_A\sigma_B + h^2\sigma_B^2$

The best hedge ratio should minimize $\text{var}(r_A - hr_B)$. Take the first order partial derivative with respect to h and set it to zero: $\frac{\partial \text{var}}{\partial h} = -2\rho\sigma_A\sigma_B + 2h\sigma_B^2 = 0 \Rightarrow h = \rho\frac{\sigma_A}{\sigma_B}$.

To confirm it's the minimum, we can also check the second-order partial derivative:

$\frac{\partial^2 \text{var}}{\partial h^2} = 2\sigma_B^2 > 0$. So Indeed when $h = \rho\frac{\sigma_A}{\sigma_B}$, the hedge portfolio has the minimum variance.

Dice game

Suppose that you roll a dice. For each roll, you are paid the face value. If a roll gives 4, 5 or 6, you can roll the dice again. Once you get 1, 2 or 3, the game stops. What is the expected payoff of this game?

Solution: This is an example of the law of total expectation. Clearly your payoff will be different depending on the outcome of first roll. Let $E[X]$ be your expected payoff and Y be the outcome of your first throw. You have 1/2 chance to get $Y \in \{1,2,3\}$, in which case the expected value is the expected face value 2, so $E[X \mid Y \in \{1,2,3\}] = 2$; you have

1/2 chance to get $Y \in \{4,5,6\}$, in which case you get expected face value 5 and extra throw(s). The extra throw(s) essentially means you start the game again and have an extra expected value $E[X]$. So we have $E[X \mid Y \in (4,5,6)] = 5 + E[X]$. Apply the law of total expectation, we have $E[X] = E[E[X \mid Y]] = \frac{1}{2} \times 2 + \frac{1}{2} \times (5 + E[X]) \Rightarrow E[X] = 7$.[27]

Card game

What is the expected number of cards that need to be turned over in a regular 52-card deck in order to see the first ace?

Solution: There are 4 aces and 48 other cards. Let's label them as card $1, 2, \cdots, 48$. Let

$$X_i = \begin{cases} 1, & \text{if card } i \text{ is turned over before 4 aces} \\ 0, & \text{otherwise} \end{cases}$$

The total number of cards that need to be turned over in order to see the first ace is $X = 1 + \sum_{i=1}^{48} X_i$, so we have $E[X] = 1 + \sum_{i=1}^{48} E[X_i]$. As shown in the following sequence, each card i is equally likely to be in one of the five regions separated by 4 aces:

1 *A* 2 *A* 3 *A* 4 *A* 5

So the probability that card i appears before all 4 aces is 1/5, and we have $E[X_i] = 1/5$.

Therefore, $E[X] = 1 + \sum_{i=1}^{48} E[X_i] = 1 + 48/5 = 10.6$.

This is just a special case for random ordering of m ordinary cards and n special cards. The expected position of the first special card is $1 + \sum_{i=1}^{m} E[X_i] = 1 + \frac{m}{n+1}$.

Sum of random variables

Assume that X_1, X_2, $\cdots$, and X_n are independent and identically-distributed (IID) random variables with uniform distribution between 0 and 1. What is the probability that $S_n = X_1 + X_2 + \cdots + X_n \le 1$?[28]

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[27] You will also see that the problem can be solved using Wald's equality in Chapter 5.

[28] Hint: start with the simplest case where n =1, 2, and 3. Try to find a general formula and prove it using induction.

Solution: This problem is a rather difficult one. The general principle to start with the simplest cases and try to find a pattern will again help you approach the problem; even though it may not give you the final answer. When $n=1$, $P(S_1 \le 1)$ is 1. As shown in Figure 4.6, when $n=2$, the probability that $X_1+X_2 \le 1$ is just the area under $X_1+X_2 \le 1$ within the square with side length 1 (a triangle). So $P(S_2 \le 1)=1/2$. When $n=3$, the probability becomes the tetrahedron ABCD under the plane $X_1+X_2+X_3 \le 1$ within the cube with side length 1. The volume of tetrahedron ABCD is 1/6.[29] So $P(S_3 \le 1)=1/6$. Now we can guess that the solution is $1/n!$. To prove it, let's again resort to induction. Assume $P(S_n \le 1)=1/n!$. We need to prove that $P(S_{n+1} \le 1)=1/(n+1)!$.

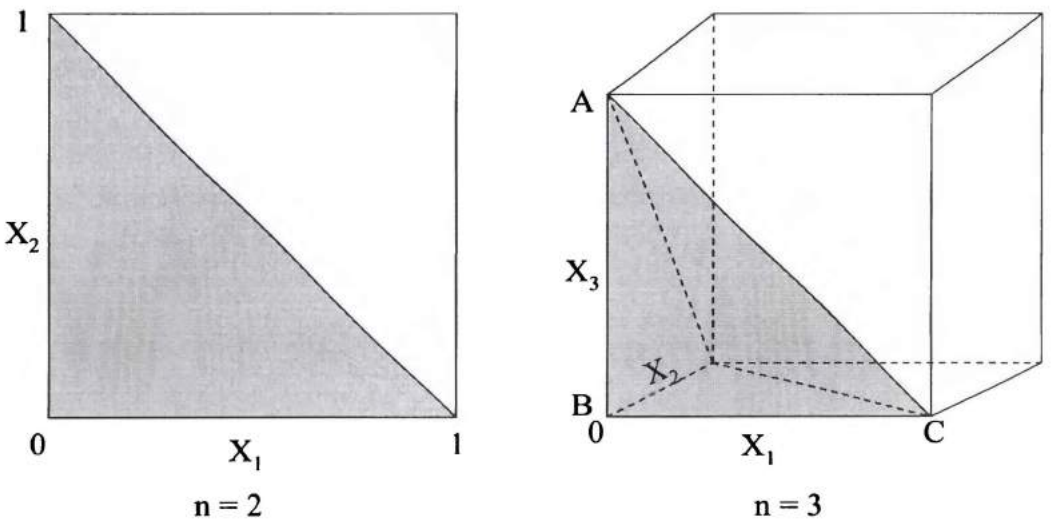

Figure 4.6 Probability that $S_n \le 1$ when n = 2 or n= 3.

Here we can use probability by conditioning. Condition on the value of X_{n+1}, we have $P(S_{n+1} \le 1)=\int_0^1 f(X_{n+1})P(S_n \le 1-X_{n+1})dX_{n+1}$, where $f(X_{n+1})$ is the probability density function of X_{n+1}, so $f(X_{n+1})=1$. But how do we calculate $P(S_n \le 1-X_{n+1})$? The cases of $n=2$ and $n=3$ have provided us with some clue. For $S_n \le 1-X_{n+1}$ instead of $S_n \le 1$, we essentially need to shrink every dimension of the n-dimensional simplex[30] from 1 to

[29] You can derive it by integration: $\int_0^1 A(z)dz = \int_0^1 1/2z^2 dz = 1/6$, where $A(z)$ is the cross-sectional area.

[30] An n-Simplex is the n-dimensional analog of a triangle.

$1-X_{n+1}$. So its volume should be $\frac{(1-X_{n+1})^n}{n!}$ instead of $\frac{1}{n!}$. Plugging in these results, we have $P(S_{n+1} \le 1) = \int_0^1 \frac{(1-X_{n+1})^n}{n!} dX_{n+1} = \frac{1}{n!}\left[-\frac{(1-X_{n+1})^{n+1}}{n+1}\right]_0^1 = \frac{1}{n!} \times \frac{1}{n+1} = \frac{1}{(n+1)!}$.

So the general result is true for $n+1$ as well and we have $P(S_n \le 1) = 1/n!$.

Coupon collection

There are N distinct types of coupons in cereal boxes and each type, independent of prior selections, is equally likely to be in a box.

A. If a child wants to collect a complete set of coupons with at least one of each type, how many coupons (boxes) on average are needed to make such a complete set?

B. If the child has collected n coupons, what is the expected number of distinct coupon types?[31]

Solution: For part *A*, let X_i, $i = 1, 2, \cdots, N$, be the number of additional coupons needed to obtain the i-th type after $(i-1)$ distinct types have been collected. So the total number of coupons needed is $X = X_1 + X_2 + \cdots + X_N = \sum_{i=1}^{N} X_i$.

For any i, $i-1$ distinct types of coupons have already been collected. It follows that a new coupon will be of a different type with probability $1-(i-1)/N = (N-i+1)/N$. Essentially to obtain the i-th distinct type, the random variable X_i follows a geometric distribution with $p = (N-i+1)/N$ and $E[X_i] = N/(N-i+1)$. For example, if $i = 1$, we simply have $X_i = E[X_i] = 1$.

$$\therefore E[X] = \sum_{i=1}^{N} E[X_i] = \sum_{i=1}^{N} \frac{N}{N-i+1} = N\left(\frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{1}\right).$$

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[31] Hint: For part *A*, let X_i be the number of extra coupons collected to get the i-th distinct coupon after $i-1$ types of distinct coupons have been collected. Then the total expected number of coupons to collect all distinct types is $E[X] = \sum_{i=1}^{N} E[X_i]$. For part B, which is the expected probability (P) that the i-th coupon type is not in the n coupons?

For part B, let Y be the number of distinct types of coupons in the set of n coupons. We introduce indicator **random variables** $I_i, i = 1, 2, \cdots, N$, where

$$\begin{cases} I_i = 1, \text{ if at least one coupon of the } i\text{-th type is in the set of } n \text{ coupons} \\ I_i = 0, \text{ otherwise} \end{cases}$$

So we have $Y = I_1 + I_2 + \cdots + I_N = \sum_{i=1}^{N} I_i$

For each collected coupon, the probability that it is not the i-th coupon type is $\frac{N-1}{N}$. Since all n coupons are independent, the probability that none of the n coupons is the i-th coupon type is $P(I_i = 0) = \left(\frac{N-1}{N}\right)^n$ and we have $E[I_i] = P(I_i = 1) = 1 - \left(\frac{N-1}{N}\right)^n$.

$$\therefore E[Y] = \sum_{i=1}^{N} E[I_i] = N - N\left(\frac{N-1}{N}\right)^n .^{32}$$

Joint default probability

If there is a 50% probability that bond A will default next year and a 30% probability that bond B will default. What is the range of probability that at least one bond defaults and what is the range of their correlation?

Solution: The range of probability that at least one bond defaults is easy to find. To have the largest probability, we can assume whenever A defaults, B does not default; whenever B defaults, A does not default. So the maximum probability that at least one bond defaults is $50\% + 30\% = 80\%$. (The result only applies if $P(A) + P(B) \le 1$). For the minimum, we can assume whenever A defaults, B also defaults. So the minimum probability that at least one bond defaults is 50%.

To calculate the corresponding correlation, let I_A and I_B be the indicator for the event that bond A/B defaults next year and ρ_{AB} be their correlation. Then we have $E[I_A] = 0.5$, $E[I_B] = 0.3$, $\text{var}(I_A) = p_A \times (1 - p_A) = 0.25$, $\text{var}(I_B) = 0.21$.

[32] A similar question: if you randomly put 18 balls into 10 boxes, what is the expected number of empty boxes?

$$\begin{aligned}P(A \text{ or } B \text{ defaults}) &= E[I_A]+E[I_B]-E[I_A I_B] \\ &= E[I_A]+E[I_B]-\left(E[I_A]E[I_B]-\text{cov}(I_A,I_B)\right) \\ &= 0.5+0.3-\left(0.5\times 0.3-\rho_{AB}\sigma_A\sigma_B\right) \\ &= 0.65-\sqrt{0.21}/2\rho_{AB}\end{aligned}$$

For the maximum probability, we have $0.65-\sqrt{0.21}/2\rho_{AB}=0.8 \Rightarrow \rho_{AB}=-\sqrt{3/7}$.

For the minimum probability, we have $0.65-\sqrt{0.21}/2\rho_{AB}=0.5 \Rightarrow \rho_{AB}=\sqrt{3/7}$.

In this problem, do not start with $P(A \text{ or } B \text{ defaults})=0.65-\sqrt{0.21}/2\rho_{AB}$ and try to set $\rho_{AB}=\pm 1$ to calculate the maximum and minimum probability since the correlation cannot be ± 1. The range of correlation is restricted to $\left[-\sqrt{3/7}, \sqrt{3/7}\right]$.

4.6 Order Statistics

Let X be a random variable with cumulative distribution function $F_X(x)$. We can derive the distribution function for the minimum $Y_n=\min(X_1,X_2,\cdots,X_n)$ and for the maximum $Z_n=\max(X_1,X_2,\cdots,X_n)$ of n IID random variables with cdf $F_X(x)$ as

$$P(Y_n \geq x)=(P(X \geq x))^n \Rightarrow 1-F_{Y_n}(x)=(1-F_X(x))^n \Rightarrow f_{Y_n}(x)=nf_X(x)(1-F_X(x))^{n-1}$$
$$P(Z_n \leq x)=(P(X \leq x))^n \Rightarrow F_{Z_n}(x)=(F_X(x))^n \Rightarrow f_{Z_n}(x)=nf_X(x)(F_X(x))^{n-1}$$

Expected value of max and min

Let $X_1,X_2,\cdots,X_n$ be IID random variables with uniform distribution between 0 and 1. What are the cumulative distribution function, the probability density function and expected value of $Z_n=\max(X_1,X_2,\cdots,X_n)$? What are the cumulative distribution function, the probability density function and expected value of $Y_n=\min(X_1,X_2,\cdots,X_n)$?

Solution: This is a direct test of textbook knowledge. For uniform distribution on [0,1], $F_X(x)=x$ and $f_X(x)=1$. Applying $F_X(x)$ and $f_X(x)$ to $Z_n=\max(X_1,X_2,\cdots,X_n)$ we have

$$P(Z_n \leq x)=(P(X \leq x))^n \Rightarrow F_{Z_n}(x)=(F_X(x))^n=x^n$$
$$\Rightarrow f_{Z_n}(x)=nf_X(x)(F_X(x))^{n-1}=nx^{n-1}$$

and $E[Z_n] = \int_0^1 x f_{Z_n}(x)dx = \int_0^1 nx^n dx = \frac{n}{n+1}\left[x^{n+1}\right]_0^1 = \frac{n}{n+1}$.

Applying $F_X(x)$ and $f_X(x)$ to $Y_n = \min(X_1, X_2, \cdots, X_n)$ we have

$$P(Y_n \geq x) = (P(X \geq x))^n \Rightarrow F_{Y_n}(x) = 1-(1-F_X(x))^n = 1-(1-x)^n$$
$$\Rightarrow f_{Y_n}(x) = nf_X(x)(1-F_X(x))^{n-1} = n(1-x)^{n-1}$$

and $E[Y_n] = \int_0^1 nx(1-x)^{n-1}dx = \int_0^1 n(1-y)y^{n-1}dx = \left[y^n\right]_0^1 - \frac{n}{n+1}\left[y^{n+1}\right]_0^1 = \frac{1}{n+1}$.

Correlation of max and min

Let X_1 and X_2 be IID random variables with uniform distribution between 0 and 1, $Y = \min(X_1, X_2)$ and $Z = \max(X_1, X_2)$. What is the probability of $Y \geq y$ given that $Z \leq z$ for any $y, z \in [0, 1]$? What is the correlation of Y and Z?

Solution: This problem is another demonstration that a figure is worth a thousand words. As shown in Figure 4.7, the probability that $Z \leq z$ is simply the square with side length z. So $P(Z \leq z) = z^2$. Since $Z = \max(X_1, X_2)$ and $Y = \min(X_1, X_2)$, we must have $Y \leq Z$ for any pair of X_1 and X_2. So if $y > z$, $P(Y \geq y \mid Z \leq z) = 0$. For $y \leq z$, that X_1 and X_2 satisfies $Y \geq y$ and $Z \leq z$ is the square with vertices $(y, y), (z, y), (z, z)$, and (y, z), which has an area $(z-y)^2$. So $P(Y \geq y \cap Z \leq z) = (z-y)^2$. Hence

$$P(Y \geq y \mid Z \leq z) = \begin{cases} (z-y)^2 / z^2, & \text{if } 0 \leq z \leq 1 \text{ and } 0 \leq y \leq z \\ 0, & \text{otherwise} \end{cases}.$$

Now let's move on to calculate the correlation of Y and Z.

$$\text{corr}(Y, Z) = \frac{\text{cov}(Y, Z)}{\text{std}(Y) \times \text{std}(Z)} = \frac{E[YZ] - E[Y]E[Z]}{\sqrt{E[Y^2] - E[Y]^2} \times \sqrt{E[Z^2] - E[Z]^2}}$$

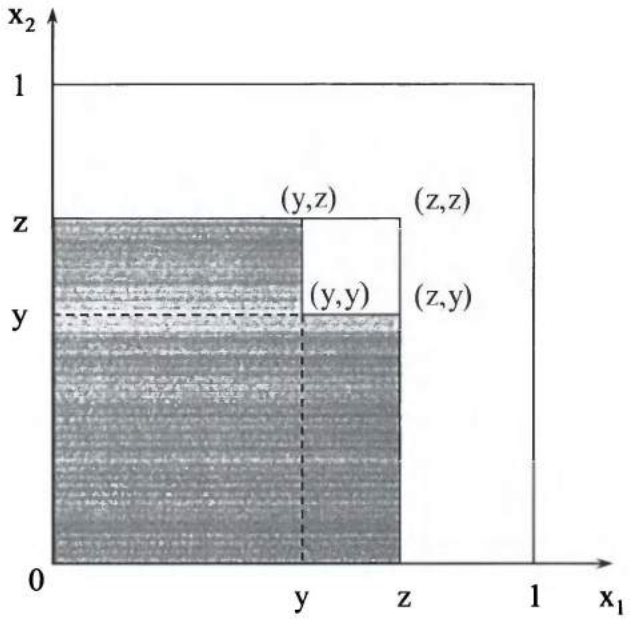

Figure 4.7 Distribution of X_1, X_2, their maximum and minimum.

Using previous problem's conclusions, we have $E[Y]=\frac{1}{2+1}=\frac{1}{3}$, $E[Z]=\frac{2}{2+1}=\frac{2}{3}$. From the pdfs of Y and Z, $f_{Y_n}(x)=n(1-x)^{n-1}=2(1-x)$ and $f_Z(z)=nz^{n-1}=2z$, we can also get $E[Y_n^2]=\int_0^1 2(1-y)y^2dy=\frac{2}{3}-\frac{2}{4}=\frac{1}{6}$ and $E[Z_n^2]=\int_0^1 2z^3dz=\frac{2}{4}$, which give us the variances: $\text{var}(Y)=E[Y^2]-E[Y]^2=\frac{1}{6}-\left(\frac{1}{3}\right)^2=\frac{1}{18}$ and $\text{var}(Z)=\frac{2}{4}-\left(\frac{2}{3}\right)^2=\frac{1}{18}$.[33]

To calculate $E[YZ]$, we can use $E[YZ]=\int_0^1\int_0^z yzf(y,z)dydz$. To solve this equation, we need $f(y,z)$. Let's again go back to Figure 4.7. From the figure we can see that when $0\le z\le 1$ *and* $0\le y\le z$, $F(y,z)$ is the shadowed area with probability

$$F(y,z)=P(Y\le y\cap Z\le z)=P(Z\le z)-P(Y\ge y\cap Z\le z)=z^2-(z-y)^2=2zy-y^2$$

$$\therefore f(y,z)=\frac{\partial}{\partial y\partial z}F(y,z)=2 \text{ and } E[YZ]=\int_0^1\int_0^z 2yzdydz=\int_0^0 z[y^2]_0^z dz=\int_0^1 z^3dz=\frac{1}{4}.$$

[33] You may have noticed that $\text{var}(Y)=\text{var}(Z)$ and wonder whether it is a coincidence for $n=2$. It is actually true for all integer n. You may want to think about why that is true without resorting to calculation. Hint: $\text{var}(x)=\text{var}(1-x)$ for any random variable x.

An alternative and simpler approach to calculate $E[YZ]$ is again to take advantage of symmetry. Notice that no matter $x_1 \le x_2$ or $x_1 > x_2$, we always have $yz = x_1 x_2$ ($z = \max(x_1, x_2)$ and $y = \min(x_1, x_2)$).

$$\therefore E[YZ] = \int_0^1 \int_0^1 x_1 x_2 dx_1 dx_2 = E[X_1]E[X_2] = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

Hence $\text{cov}(Y,Z) = E[YZ] - E[Y]E[Z] = \frac{1}{36}$ and $\text{corr}(Y,Z) = \frac{\text{cov}(Y,Z)}{\sqrt{\text{var}(Y)} \times \sqrt{\text{var}(Z)}} = \frac{1}{2}$.

Sanity check: That Y and Z have positive autocorrelation make sense since when Y becomes large, Z tends to become large as well ($Z \ge Y$).

Random ants

500 ants are randomly put on a 1-foot string (independent uniform distribution for each ant between 0 and 1). Each ant randomly moves toward one end of the string (equal probability to the left or right) at constant speed of 1 foot/minute until it falls off at one end of the string. Also assume that the size of the ant is infinitely small. When two ants collide head-on, they both immediately change directions and keep on moving at 1 foot/min. What is the expected time for all ants to fall off the string?[34]

Solution: This problem is often perceived to be a difficult one. The following components contribute to the complexity of the problem: The ants are randomly located; each ant can go either direction; an ant needs to change direction when it meets another ant. To solve the problem, let's tackle these components.

When two ants collide head-on, both immediately change directions. What does it mean? The following diagram illustrates the key point:

Before collision: $\xrightarrow{A} \quad \xleftarrow{B}$; After collision: $\xleftarrow{A} \quad \xrightarrow{B}$; switch label: $\xleftarrow{B} \quad \xrightarrow{A}$

When an ant A collides with another ant B, both switch direction. But if we exchange the ants' labels, it's like that the collision never happens. A continues to move to the right and B moves to the left. Since the labels are randomly assigned anyway, collisions make no difference to the result. So we can assume that when two ants meet, each just keeps on going in its original direction. What about the random direction that each ant chooses? Once the collision is removed, we can use symmetry to argue that it makes no difference which direction that an ant goes either. That means if an ant is put at the x-th foot, the

[34] Hint: If we switch the label of two ants that collide with each other, it's like that the collision never happened.

expected value for it to fall off is just x min. If it goes in the other direction, simply set x to $1-x$. So the original problem is equivalent to the following:

What is the expected value of the maximum of 500 IID random variables with uniform distribution between 0 and 1?

Clearly the answer is $\frac{499}{500}$ min, which is the expected time for all ants to fall off the string.

Chapter 5 Stochastic Process and Stochastic Calculus

In this chapter, we cover a few topics—Markov chain, random walk and martingale, dynamic programming—that are often not included in introductory probability courses. Unlike basic probability theory, these tools may not be considered to be standard requirements for quantitative researchers/analysts. But a good understanding of these topics can simplify your answers to many interview problems and give you an edge in the interview process. Besides, once you learn the basics, you'll find many interview problems turning into fun-to-solve math puzzles.

5.1 Markov Chain

A Markov chain is a sequence of random variables $X_0, X_1, \cdots, X_n, \cdots$ with the Markov property that given the present state, the future states and the past states are independent:

$P\{X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0\} = p_{ij} = P\{X_{n+1} = j \mid X_n = i\}$ for all n, i_0, $\cdots$, i_{n-1}, i, and j, where $i, j \in \{1, 2, \cdots, M\}$ represent the state space $S = \{s_1, s_2, ..., s_M\}$ of X.

In other words, once the current state is known, past history has no bearing on the future. For a homogenous Markov chain, the transition probability from state i to state j does not depend on n.[1] A Markov chain with M states can be completely described by an $M \times M$ transition matrix P and the initial probabilities $P(X_0)$.

Transition matrix: $P = \{p_{ij}\} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1M} \\ p_{21} & p_{22} & \cdots & p_{2M} \\ \vdots & \vdots & \vdots & \vdots \\ p_{M1} & p_{M2} & \cdots & p_{MM} \end{bmatrix}$, where p_{ij} is the transition probability from state i to state j.

Initial probabilities: $P(X_0) = \big(P(X_0 = 1), P(X_0 = 2), \cdots, P(X_0 = M)\big)$, $\sum_{i=1}^{M} P(X_0 = i) = 1$.

The probability of a path: $P(X_1 = i_1, X_2 = i_2 \cdots, X_n = i_n \mid X_0 = i_0) = p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{n-1} i_n}$

Transition graph: A transition graph is often used to express the transition matrix graphically. The transition graph is more intuitive than the matrix, and it emphasizes

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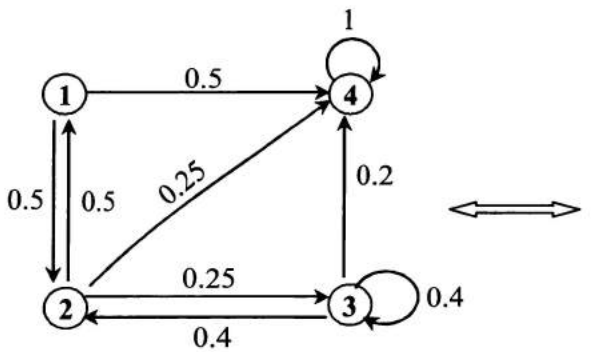
[1] In this chapter, we only consider finite-state homogenous Markov chains (i.e., transition probabilities do not change over time).

possible and impossible transitions. Figure 5.1 shows the transition graph and the transition matrix of a Markov chain with four states:



$$P = \begin{array}{c} i = \\ \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.25 & 0.25 \\ 0 & 0.4 & 0.4 & 0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Figure 5.1 Transition graph and transition matrix of the Play

Classification of states

State j is **accessible** from state i if there is a directed path in the transition graph from i to j ($\exists n$ such that $P_{ij}^{(n)} > 0$). Let $T_{ij} = \min(n : X_n = j \mid X_0 = i)$, then $P(T_{ij} < \infty) > 0$) if and only if state j is accessible from state i. States i and j **communicate** if i is accessible from j and j is accessible from i. In Figure 5.1, state 3 and 1 communicate. State 4 is accessible form state 1, but they do not communicate since state 1 is not accessible from state 4.

We say that state i is **recurrent** if for every state j that is accessible from i, i is also accessible from j ($\forall j,\ P(T_{ij} < \infty) > 0 \Rightarrow P(T_{ij} < \infty) = 1$). A state is called **transient** if it is not recurrent ($\exists j,\ P(T_{ij} < \infty) > 0$ and $P(T_{ij} < \infty) < 1$). In Figure 5.1, only state 4 is recurrent. States 1, 2 and 3 are all transient since 4 is accessible from 1/2/3, but 1/2/3 are not accessible from 4.

Absorbing Markov chains: A state i is called absorbing if it is impossible to leave this state ($p_{ii} = 1, p_{ij} = 0, \forall j \neq i$). A Markov chain is absorbing if it has at least one absorbing state and if from every state it is possible to go to an absorbing state. In Figure 5.1, state 4 is an absorbing state. The corresponding Markov chain is an absorbing Markov chain.

Equations for absorption probability: The probability to reach a specific absorbing state s, $a_1, \cdots, a_M$, are unique solutions to equations $a_s = 1$, $a_i = 0$ for all absorbing state(s) $i \neq s$, and $a_i = \sum_{j-1}^{M} a_j p_{ij}$ for all transient states i. These equations can be easily

derived using the law of total probability by conditioning the absorption probabilities on the next state.

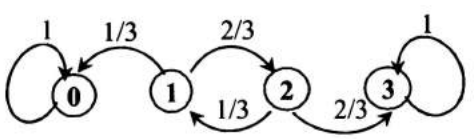
Equations for the expected time to absorption: The expected times to absorption, $\mu_1, \cdots, \mu_M$, are unique solutions to the equations $\mu_i = 0$ for all absorbing state(s) i and $\mu_i = 1 + \sum_{j=1}^{m} p_{ij}\mu_j$ for all transient states i. These equations can be easily derived using the law of total expectation by conditioning the expected times to absorption on the next state. The number 1 is added since it takes one step to reach the next state.

Gambler's ruin problem

Player M has \$1 and player N has \$2. Each game gives the winner \$1 from the other. As a better player, M wins 2/3 of the games. They play until one of them is bankrupt. What is the probability that M wins?

Solution: The most difficult part of Markov chain problems often lies in how to choose the right state space and define the transition probabilities P_{ij}'s, $\forall i, j$. This problem has fairly straightforward states. You can define the state space as the combination of the money that player M has (\$$m$) and the money that player N has (\$$n$): $\{(m,n)\} = \{(3,0),(2,1),(1,2),(0,3)\}$. (Neither m nor n can be negative since the whole game stops when one of them goes bankrupt.) Since the sum of the dollars of both players is always \$3, we can actually simplify the state space using only m: $\{m\} = \{0,1,2,3\}$.

The transition graph and the corresponding transition matrix are shown in Figure 5.2.



$$P = \{P_{ij}\} = \begin{bmatrix} p_{0,0} & p_{0,1} & p_{0,2} & p_{0,3} \\ p_{1,0} & p_{1,1} & p_{1,2} & p_{1,3} \\ p_{2,0} & p_{2,1} & p_{2,2} & p_{2,3} \\ p_{2,0} & p_{2,1} & p_{2,2} & p_{2,3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 5.2 Transition matrix and transition graph for Gambler's ruin problem

The initial state is $X_0 = 1$ (M has \$1 at the beginning). At state 1, the next state is 0 (M loses a game) with probability 1/3 and 2 (M wins a game) with probability 2/3. So $p_{1,0} = 1/3$ and $p_{1,2} = 2/3$. Similarly we can get $p_{2,1} = 1/3$ and $p_{2,3} = 2/3$. Both state 3 (M wins the whole game) and state 0 (M loses the whole game) are absorbing states.

To calculate the probability that M reaches absorbing state 3, we can apply absorption probability equations:

$a_3 = 1, a_0 = 0$, and $a_1 = \sum_{j=0}^{3} p_{1,j} a_j$, $a_2 = \sum_{j=0}^{3} p_{2,j} a_j$

Plugging in the transition probabilities using either the transition graph or transition matrix, we have $\left.\begin{aligned} a_1 &= 1/3 \times 0 + 2/3 \times a_2 \\ a_2 &= 1/3 \times a_1 + 2/3 \times 1 \end{aligned}\right\} \Rightarrow \begin{cases} a_1 = 4/7 \\ a_2 = 6/7 \end{cases}$

So, starting from \$1, player M has $4/7$ probability of winning.

Dice question

Two players bet on roll(s) of the total of two standard six-face dice. Player A bets that a sum of 12 will occur first. Player B bets that two consecutive 7s will occur first. The players keep rolling the dice and record the sums until one player wins. What is the probability that A will win?

Solution: Many of the simple Markov chain problems can be solved using pure conditional probability argument. It is not surprising considering that Markov chain is defined as conditional probability:

$$P\{X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0\} = p_{ij} = P\{X_{n+1} = j \mid X_n = i\}.$$

So let's first solve the problem using conditional probability arguments. Let $P(A)$ be the probability that A wins. Conditioning $P(A)$ on the first throw's sum F, which has three possible outcomes $F = 12$, $F = 7$ and $F \notin \{7,12\}$, we have

$$P(A) = P(A \mid F = 12)P(F = 12) + P(A \mid F = 7)P(F = 7) + P(A \mid F \notin \{7,12\})P(F \notin \{7,12\})$$

Then we tackle each component on the right hand side. Using simple permutation, we can easily see that $P(F = 12) = 1/36$, $P(F = 7) = 6/36$, $P(F \notin \{7,12\}) = 29/36$. Also it is obvious that $P(A \mid F = 12) = 1$ and $P(A \mid F \notin \{7,12\}) = P(A)$. (The game essentially starts over again.) To calculate $P(A \mid F = 7)$, we need to further condition on the second throw's total, which again has three possible outcomes: $E = 12$, $E = 7$, and $E \notin \{7,12\}$.

$$\begin{aligned} P(A \mid F = 7) &= P(A \mid F = 7, E = 12)P(E = 12 \mid F = 7) + P(A \mid F = 7, E = 7)P(E = 7 \mid F = 7) \\ &\quad + P(A \mid F = 7, E \notin \{7,12\})P(E \notin \{7,12\} \mid F = 7) \\ &= P(A \mid F = 7, E = 12) \times 1/36 + P(A \mid F = 7, E = 7) \times 6/36 \\ &\quad + P(A \mid F = 7, E \notin \{7,12\}) \times 29/36 \\ &= 1 \times 1/36 + 0 \times 6/36 + P(A) \times 29/36 = 1/36 + 29/36 P(A) \end{aligned}$$

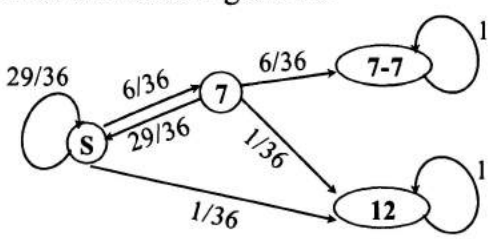
Here the second equation relies on the independence between the second and the first rolls. If $F = 7$ and $E = 12$, A wins; if $F = 7$ and $E = 7$, A loses; if $F = 7$ and

$E \notin \{7,12\}$, the game essentially starts over again. Now we have all the necessarily information for $P(A)$. Plugging it into the original equation, we have

$$P(A) = P(A \mid F = 12)P(F = 12) + P(A \mid F = 7)P(F = 7) + P(A \mid F \notin \{7,12\})P(F \notin \{7,12\})$$
$$= 1 \times 1/36 + 6/36 \times (1/36 + 29/36P(A)) + 29/36P(A)$$

Solving the equation, we get $P(A) = 7/13$.

This approach, although logically solid, is not intuitively appealing. Now let's try a Markov chain approach. Again the key part is to choose the right state space and define the transition probabilities. It is apparent that we have two absorbing states, 12 (*A* wins) and 7-7 (*B* wins), at least two transient states, *S* (starting state) and 7 (one 7 occurs, yet no 12 or 7-7 occurred). Do we need any other states? Theoretically, you can have other states. In fact, you can use all combination of the outcomes of one roll and two consecutive rolls as states to construct a transition matrix and you will get the same final result. Nevertheless, we want to consolidate as many equivalent states as possible. As we just discussed in the conditional probability approach, if no 12 has occurred and the most recent roll did not yield 7, we essentially go back to the initial starting state *S*. So all we need are states *S*, 7, 7-7 and 12. The transition graph and probability to reach state 12 are shown in Figure 5.3.



Probability to absorption state 12

$$\left.\begin{array}{c} a_{12} = 1,\ a_{7\text{-}7} = 0 \\ a_S = 1/36 \times 1 + 6/36 \times a_7 + 29/36 \times a_S \\ a_7 = 1/36 \times 1 + 6/36 \times 0 + 29/36 \times a_S \end{array}\right\} \Rightarrow a_S = 7/13$$

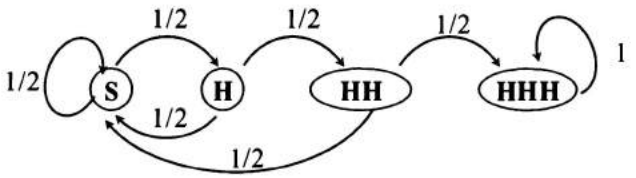
Figure 5.3 Transition graph and probability to absorption for dice rolls

Here the transition probability is again derived from conditional probability arguments. Yet the transition graph makes the process crystal clear.

Coin triplets

Part A. If you keep on tossing a fair coin, what is the expected number of tosses such that you can have *HHH* (heads heads heads) in a row? What is the expected number of tosses to have *THH* (tails heads heads) in a row?

Solution: The most difficult part of Markov chain is, again, to choose the right state space. For the *HHH* sequence, the state space is straightforward. We only need four states: *S* (for the starting state when no coin is tossed or whenever a *T* turns up before *HHH*), *H*, *HH*, and *HHH*. The transition graph is



At state S, after a coin toss, the state will stay at S when the toss gives a T. If the toss gives an H, the state becomes H. At state H, it has 1/2 probability goes back to state S if the next toss is T; otherwise, it goes to state HH. At state HH, it also has 1/2 probability goes back to state S if the next toss is T; otherwise, it reaches the absorbing state HHH.

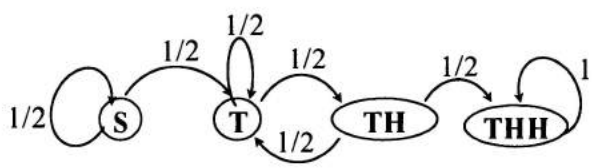
So we have the following transition probabilities: $P_{S,S}=\frac{1}{2}$, $P_{S,H}=\frac{1}{2}$, $P_{H,S}=\frac{1}{2}$, $P_{H,HH}=\frac{1}{2}$, $P_{HH,S}=\frac{1}{2}$, $P_{HH,HHH}=\frac{1}{2}$, and $P_{HHH,HHH}=1$.

We are interested in the expected number of tosses to get HHH, which is the expected time to absorption starting from state S. Applying the standard equations for the expected time to absorption, we have

$$\left.\begin{aligned}\mu_S &= 1+\tfrac{1}{2}\mu_S+\tfrac{1}{2}\mu_H\\ \mu_H &= 1+\tfrac{1}{2}\mu_S+\tfrac{1}{2}\mu_{HH}\\ \mu_{HH} &= 1+\tfrac{1}{2}\mu_S+\tfrac{1}{2}\mu_{HHH}\\ \mu_{HHH} &= 0\end{aligned}\right\}\Rightarrow\left\{\begin{aligned}\mu_S &= 14\\ \mu_H &= 12\\ \mu_{HH} &= 8\\ \mu_{HHH} &= 0\end{aligned}\right.$$

So from the starting state, the expected number of tosses to get HHH is 14.

Similarly for expected time to reach THH, we can construct the following transition graph and estimate the corresponding expected time to absorption:



$$\left.\begin{aligned}\mu_S &= 1+\tfrac{1}{2}\mu_S+\tfrac{1}{2}\mu_T\\ \mu_T &= 1+\tfrac{1}{2}\mu_T+\tfrac{1}{2}\mu_{TH}\\ \mu_{TH} &= 1+\tfrac{1}{2}\mu_T+\tfrac{1}{2}\mu_{THH}\\ \mu_{THH} &= 0\end{aligned}\right\}\Rightarrow\left\{\begin{aligned}\mu_S &= 8\\ \mu_T &= 4\\ \mu_{TH} &= 2\\ \mu_{THH} &= 0\end{aligned}\right.$$

So from the starting state S, the expected number of tosses to get THH is 8.

Part B. Keep flipping a fair coin until either HHH or THH occurs in the sequence. What is the probability that you get an HHH subsequence before THH?[2]

[2] Hint: This problem does not require the drawing of a Markov chain. Just think about the relationship between an HHH pattern and a THH pattern. How can we get an HHH sequence before a THH sequence?

Solution: Let's try a standard Markov chain approach. Again the focus is on choosing the right state space. In this case, we begin with starting state *S*. We only need ordered subsequences of either *HHH* or *THH*. After one coin is flipped, we have either state *T* or *H*. After two flips, we have states *TH* and *HH*. We do not need *TT* (which is equivalent to T for this problem) or *HT* (which is also equivalent to *T* as well). For three coin sequences, we only need *THH* and *HHH* states, which are both absorbing states. Using these states, we can build the following transition graph:

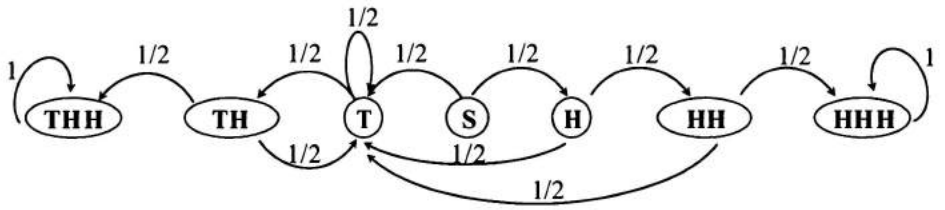


Figure 5.4 Transition graph of coin tosses to reach *HHH* or *THH*

We want to get the probability to reach absorbing state *HHH* from the starting state S. Applying the equations for absorption probability, we have

$$\left.\begin{array}{l} a_{HHH}=1,\ a_{THH}=0 \\ a_S=\frac{1}{2}a_T+\frac{1}{2}a_H \\ a_T=\frac{1}{2}a_T+\frac{1}{2}a_{TH},\ a_H=\frac{1}{2}a_T+\frac{1}{2}a_{HH} \\ a_{TH}=\frac{1}{2}a_T+\frac{1}{2}a_{THH}, a_{HH}=\frac{1}{2}a_T+\frac{1}{2}a_{HHH} \end{array}\right\} \Rightarrow \left\{\begin{array}{c} a_T=0, a_{TH}=0 \\ a_S=\frac{1}{8} \\ a_H=\frac{1}{4} \\ a_{HH}=\frac{1}{2} \end{array}\right.$$

So the probability that we end up with the *HHH* pattern is 1/8.

This problem actually has a special feature that renders the calculation unnecessary. You may have noticed that $a_T=0$. Once a tail occurs, we will always get *THH* before *HHH*. The reason is that the last two coins in *THH* is *HH*, which is the first two coins in sequence *HHH*. In fact, the only way that the sequence reaches state *HHH* before *THH* is that we get three consecutive *H*s in the beginning. Otherwise, we always have a *T* before the first *HH* sequence and always end in *THH* first. So if we don't start the coin flipping sequence with *HHH*, which has a probability of 1/8, we will always have *THH* before *HHH*.

Part C. (Difficult) Let's add more fun to the triplet game. Instead of fixed triplets for the two players, the new game allows both to choose their own triplets. Player 1 chooses a triplet first and announces it; then player 2 chooses a different triplet. The players again toss the coins until one of the two triplet sequences appears. The player whose chosen triplet appears first wins the game.

If both player 1 and player 2 are perfectly rational and both want to maximize their probability of winning, would you go first (as player 1)? If you go second, what is your probability of winning?[3]

Solution: A common misconception is that there is always a best sequence that beats other sequences. This misconception is often founded on a wrong assumption that these sequences are transitive: if sequence A has a higher probability occurring before sequence B and sequence B has a higher probability occurring before sequence C, then sequence A has a higher probability occurring before sequence C. In reality, such transitivity does not exist for this game. No matter what sequence player 1 chooses, player 2 can always choose another sequence with more than 1/2 probability of winning. The key, as we have indicated in Part B, is to choose the last two coins of the sequence as the first two coins of player 1's sequence. We can compile the following table for each pair of sequences:

2's winning Probability		Player 1							
		HHH	**THH**	**HTH**	**HHT**	**TTH**	**THT**	**HTT**	**TTT**
Player 2	**HHH**	/	1/8	2/5	1/2	3/10	5/12	2/5	1/2
	THH	**7/8**	/	1/2	**3/4**	1/3	1/2	1/2	3/5
	HTH	3/5	1/2	/	1/3	3/8	1/2	1/2	7/12
	HHT	1/2	1/4	**2/3**	/	1/2	5/8	**2/3**	7/10
	TTH	7/10	**2/3**	5/8	1/2	/	**2/3**	1/4	1/2
	THT	7/12	1/2	1/2	3/8	1/3	/	1/2	3/5
	HTT	3/5	1/2	1/2	1/3	**3/4**	1/2	/	**7/8**
	TTT	1/2	2/5	5/12	3/10	1/2	2/5	1/8	/

Table 5.1 Player 2's winning probability with different coin sequence pairs

As shown in Table 5.1 (you can confirm the results yourself), no matter what player 1's choices are, player 2 can always choose a sequence to have better odds of winning. The best sequences that player 2 can choose in response to 1's choices are highlighted in bold. In order to maximize his odds of winning, player 1 should choose among HTH, HTT, THH and THT. Even in these cases, player 2 has 2/3 probability of winning.

[3] This problem is a difficult one. Interested reader may find the following paper helpful: "Waiting Time and Expected Waiting Time-Paradoxical Situations" by V. C. Hombas, *The American Statistician*, Vol. 51, No. 2 (May, 1997), pp. 130-133. In this section, we will only discuss the intuition.

Color balls

A box contains n balls of n different colors. Each time, you randomly select a pair of balls, repaint the first to match the second, and put the pair back into the box. What is the expected number of steps until all balls in the box are of the same color? (Very difficult)

Solution: Let N_n be the number of steps needed to make all balls the same color, and let F_i, $i=1, 2, \cdots, n$, be the event that all balls have color i in the end. Applying the law of total expectation, we have

$$E[N_n] = E[N_n \mid F_1]P[F_1] + E[N_n \mid F_2]P[F_2] + \cdots + E[N_n \mid F_n]P[F_n].$$

Since all the colors are symmetric (i.e., they should have equivalent properties), we have $P[F_1] = P[F_2] = \cdots = P[F_n] = 1/n$ and $E[N_n] = E[N_n \mid F_1] = E[N_n \mid F_2] = E[N_n \mid F_n]$. That means we can assume that all the balls have color 1 in the end and use $E[N_n \mid F_1]$ to represent $E[N_n]$.

So how do we calculate $E[N_n \mid F_1]$? Not surprisingly, use a Markov chain. Since we only consider event F_1, color 1 is different from other colors and colors $2, \cdots, n$ become equivalent. In other words, any pairs of balls that have no color 1 ball involved are equivalent and any pairs with a color 1 ball and a ball of another color are equivalent if the order is the same as well. So we only need to use the number of balls that have color 1 as the states. Figure 5.5 shows the transition graph.

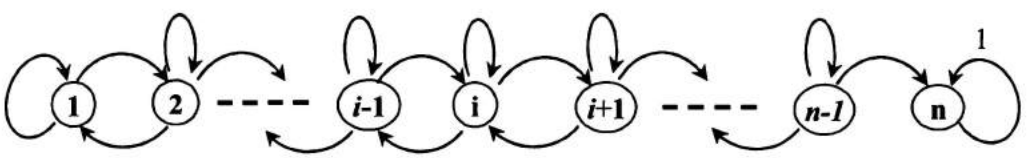


Figure 5.5 Transition graph for all n balls to become color 1

State n is the only absorbing state. Notice that there is no state 0, otherwise it will never reach F_1. In fact, all the transition probability is conditioned on F_1 as well, which makes the transition probability $p_{i,i+1} \mid F_1$ higher than the unconditional probability $p_{i,i+1}$ and $p_{i,i-1} \mid F_1$ lower than $p_{i,i-1}$. For example, $p_{1,0} \mid F_1 = 0$ and $p_{1,0} = 1/n$. (Without conditioning, each ball is likely to be the second ball, so color 1 has $1/n$ probability of being the second ball.) Using the conditional transition probability, the problem essentially becomes expected time to absorption with system equations:

$$E[N_i \mid F_1] = 1 + E[N_{i-1} \mid F_1] \times P_{i,i-1} \mid F_1 + E[N_i \mid F_1] \times P_{i,i} \mid F_1 + E[N_{i+1} \mid F_1] \times P_{i,i+1} \mid F_1.$$

To calculate $P_{i,i-1} \mid F_1$, let's rewrite the probability as $P(x_{k+1} = i-1 \mid x_k = i,\ F_1)$, $\forall k = 0, 1, \ldots$, to make the derivation step clearer:

$$P(x_{k+1} = i-1 \mid x_k = i, F_1) = \frac{P(x_k = i, x_{k+1} = i-1, F_1)}{P(x_k = i, F_1)}$$

$$= \frac{P(F_1 \mid x_{k+1} = i-1, x_k = i) \times P(x_{k+1} = i-1 \mid x_k = i) \times P(x_k = i)}{P(F_1 \mid x_k = i) \times P(x_k = i)}$$

$$= \frac{P(F_1 \mid x_{k+1} = i-1) \times P(x_{k+1} = i-1 \mid x_k = i)}{P(F_1 \mid x_k = i)}$$

$$= \frac{\frac{i-1}{n} \times \frac{i(n-i)}{n(n-1)}}{i/n} = \frac{(n-i) \times (i-1)}{n(n-1)}$$

The first equation is simply the definition of conditional probability; the second equation is the application of Bayes' theorem; the third equation applies the Markov property. To derive $P(F_1 \mid x_k = i) = i/n$, we again need to use symmetry. We have shown that if all the balls have different colors, then we have $P[F_1] = P[F_2] = \cdots = P[F_n] = 1/n$. What is the probability of ending in a given color, labeled as c, if i of the balls are of color c? It is simply i/n. To see that, we can label the color of each of the i balls of color c as c_j, $j = 1, \cdots, i$ (even though they are in fact the same color). Now it's obvious that all balls will end with color c_j with probability $1/n$. The probability for c is the sum of probabilities of c_j's, which gives the result i/n.

Similarly we have $P(F_1 \mid x_{k+1} = i-1) = (i-1)/n$. For $P(x_{k+1} = i-1 \mid x_k = i)$, we use a basic counting method. There are $n(n-1)$ possible permutations to choose 2 balls out of n balls. In order for one color 1 ball to change color, the second ball must be color 1, which has i choices; the first ball needs to be another color, which has $(n-i)$ choices. So $P(x_{k+1} = i-1 \mid x_k = i) = \frac{i(n-i)}{n(n-1)}$.

Applying the same principles, we can get

$$P(x_{k+1} = i \mid x_k = i, F_1) = \frac{(n-i) \times 2i}{n(n-1)},\quad P(x_{k+1} = i+1 \mid x_k = i, F_1) = \frac{(n-i) \times (i+1)}{n(n-1)}.$$

Plugging into $E[N_i \mid F_1]$ and simplifying $E[N_i \mid F_1]$ as Z_i, we have

$$(n-i) \times 2i \times Z_i = n(n-1) + (n-i)(i+1)Z_{i+1} + (n-i)(i-1)Z_{i-1}.$$

Using these recursive system equations and the boundary condition $Z_n = 0$, we can get $Z_1 = (n-1)^2$.[4]

5.2 Martingale and Random walk

Random walk: The process $\{S_n; n \geq 1\}$ is called a random walk if $\{X_i; i \geq 1\}$ are IID (identical and independently distributed) random variables and $S_n = X_1 + \cdots X_n$, where $n = 1, 2, \cdots$ The term comes from the fact that we can think of S_n as the position at time n for a walker who makes successive random steps $X_1, X_2, \cdots$

If X_i takes values 1 and -1 with probabilities p and $1 - p$ respectively, S_n is called a **simple random walk** with parameter p. Furthermore, if $p = \frac{1}{2}$, the process S_n is a **symmetric random walk**. For symmetric random walk, it's easy to show that $E[S_n] = 0$ and $\text{var}(S_n) = E[S_n^2] - E[S_n]^2 = E[S_n^2] = n$.[5]

Symmetric random walk is the process that is most often tested in quantitative interviews. The interview questions on random walk often revolve around finding the first n for which S_n reaches a defined threshold α, or the probability that S_n reaches α for any given value of n.

Martingale: a martingale $\{Z_n; n \geq 1\}$ is a stochastic process with the properties that $E[|Z_n|] < \infty$ for all n and $E[Z_{n+1} | Z_n = z_n, Z_{n-1} = z_{n-1}, \cdots, Z_1 = z_1] = z_n$. The property of a martingale can be extended to $E[Z_m; m > n | Z_n = z_n, Z_{n-1} = z_{n-1}, \cdots, Z_1 = z_1] = z_n$, which means the conditional expected value of future Z_m is the current value Z_n.[6]

A symmetric random walk is a martingale. From the definition of the symmetric random walk we have $S_{n+1} = \begin{cases} S_n + 1 & \text{with probability } 1/2 \\ S_n - 1 & \text{with probability } 1/2 \end{cases}$, so $E[S_{n+1} | S_n = s_n, \cdots, S_1 = s_1] = s_n$.

Since $E[S_{n+1}^2 - (n+1)] = \frac{1}{2}[(S_n + 1)^2 + (S_n - 1)^2] - (n+1) = S_n^2 - n$, $S_n^2 - n$ is a martingale as well.

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[4] Even this step is not straightforward. You need to plug in the i's and try a few cases starting with $i = n - 1$. The pattern will emerge and you can see that all the terms containing $Z_{n-1}, Z_{n-2}, \cdots, Z_2$ cancel out.

[5] Induction again can be used for its proof. $Var(S_1) = Var(Z_1) = 1$. Induction step: If $Var(S_n) = n$, then we have $Var(S_{n+1}) = Var(S_n + x_{n+1}) = Var(S_n) + Var(x_{n+1}) = n + 1$ since x_{n+1} is independent of S_n.

[6] Do not confuse a martingale process with a Markov process. A martingale does not need to be a Markov process; a Markov process does not need to be a martingale process, either.

Stopping rule: For an experiment with a set of IID random variables $X_1, X_2, \cdots$, a stopping rule for $\{X_i; i \geq 1\}$ is a positive integer-value random variable N (stopping time) such that for each $n > 1$, the event $\{N \leq n\}$ is independent of $X_{n+1}, X_{n+2}, \cdots$. Basically it says that whether to stop at n depends only on $X_1, X_2, \cdots, X_n$ (i.e., no look ahead).

Wald's Equality: Let N be a stopping rule for IID random variables $X_1, X_2, \cdots$ and let $S_N = X_1 + X_2 + \cdots + X_N$, then $E[S_N] = E[X]E[N]$.

Since it is an important—yet relatively little known—theorem, let's briefly review its proof. Let I_n be the indicator function of the event $\{N \geq n\}$. So S_N can be written as

$$S_N = \sum_{n=1}^{\infty} X_n I_n, \text{ where } I_n = 1 \text{ if } N \geq n \text{ and } I_n = 0 \text{ if } N \leq n-1.$$

From the definition of stopping rules, we know that I_n is independent of X_n, X_{n+1}, $\cdots$ (it only depends on X_1, X_2, $\cdots$, X_{n-1}). So $E[X_n I_n] = E[X_n]E[I_n] = E[X]E[I_n]$ and

$$E[S_N] = E\left[\sum_{n=1}^{\infty} X_n I_n\right] = \sum_{n=1}^{\infty} E[X_n I_n] = \sum_{n=1}^{\infty} E[X]E[I_n] = E[X]\sum_{n=1}^{\infty} E[I_n] = E[X]E[N].^{7}$$

A martingale stopped at a stopping time is a martingale.

Drunk man

A drunk man is at the 17th meter of a 100-meter-long bridge. He has a 50% probability of staggering forward or backward one meter each step. What is the probability that he will make it to the end of the bridge (the 100th meter) before the beginning (the 0th meter)? What is the expected number of steps he takes to reach either the beginning or the end of the bridge?

Solution: The probability part of the problem—often appearing in different disguises—is among the most popular martingale problems asked by quantitative interviewers. Interestingly, few people use a clear-cut martingale argument. Most candidates either use Markov chain with two absorbing states or treat it as a special version of the gambler's ruin problem with $p = 1/2$. These approaches yield the correct results in the end, yet a martingale argument is not only simpler but also illustrates the insight behind the problem.

[7] For detailed proof and applications of Wald's Equality, please refer to the book ***Discrete Stochastic Processes*** by Robert G. Gallager.

Let's set the current position (the 17th meter) to 0; then the problem becomes a symmetric random walk that stops at either 83 or -17. We also know that both S_n and $S_n^2 - n$ are martingales. Since a martingale stopped at a stopping time is a martingale, S_N and $S_N^2 - N$ (where $S_N = X_1 + X_2 + \cdots + X_N$ with N being the stopping time) are martingales as well. Let p_α be the probability that it stops at $\alpha = 83$, p_β be the probability it stops at $-\beta = -17$ ($p_\beta = 1 - p_\alpha$), and N be the stopping time. Then we have

$$\left.\begin{aligned} E[S_N] &= p_\alpha \times 83 - (1-p_\alpha)\times 17 = S_0 = 0 \\ E[S_N^2 - N] &= E[p_\alpha \times 83^2 + (1-p_\alpha)\times 17^2] - E[N] = S_0^2 - 0 = 0 \end{aligned}\right\} \Rightarrow \begin{cases} p_\alpha = 0.17 \\ E[N] = 1441 \end{cases}$$

Hence, the probability that he will make it to the end of the bridge (the 100th meter) before reaching the beginning is 0.17, and the expected number of steps he takes to reach either the beginning or the end of the bridge is 1441.

We can easily extend the solution to a general case: a symmetric random walk starting from 0 that stops at either α ($\alpha > 0$) or $-\beta$ ($\beta > 0$). The probability that it stops at α instead of $-\beta$ is $p_\alpha = \beta/(\alpha + \beta)$. The expected stopping time to reach either α or $-\beta$ is $E[N] = \alpha\beta$.

Dice game

Suppose that you roll a dice. For each roll, you are paid the face value. If a roll gives 4, 5 or 6, you can roll the dice again. If you get 1, 2 or 3, the game stops. What is the expected payoff of this game?

Solution: In Chapter 4, we used the law of total expectation to solve the problem. A simpler approach—requiring more knowledge—is to apply Wald's Equality since the problem has clear stopping rules. For each roll, the process has 1/2 probability of stopping. So the stopping time N follows a geometric distribution with $p = 1/2$ and we have $E[N] = 1/p = 2$. For each roll, the expected face value is $E[X] = 7/2$. The total expected payoff is $E[S_N] = E[X]E[N] = 7/2 \times 2 = 7$.

Ticket line

At a theater ticket office, $2n$ people are waiting to buy tickets. n of them have only \$5 bills and the other n people have only \$10 bills. The ticket seller has no change to start

with. If each person buys one \$5 ticket, what is the probability that all people will be able to buy their tickets without having to change positions?

Solution: This problem is often considered to be a difficult one. Although many can correctly formulate the problem, few can solve the problem using the reflection principle.[8] This problem is one of the many cases where a broad knowledge makes a difference.

Assign +1 to the n people with \$5 bills and -1 to the n people with \$10 bills. Consider the process as a walk. Let (a,b) represent that after a steps, the walk ends at b. So we start at $(0,0)$ and reaches $(2n,0)$ after $2n$ steps. For these $2n$ steps, we need to choose n steps as +1, so there are $\binom{2n}{n}=\frac{2n!}{n!n!}$ possible paths. We are interested in the paths that have the property $b\geq 0,\ \forall 0<a<2n$ steps. It's easier to calculate the number of complement paths that reach $b=-1,\ \exists 0<a<2n$. As shown in Figure 5.6, if we reflect the path across the line $y=-1$ after a path first reaches -1, for every path that reaches $(2n,0)$ at step $2n$, we have one corresponding reflected path that reaches $(2n,-2)$ at step $2n$. For a path to reach $(2n,-2)$, there are $(n-1)$ steps of $+1$ and $(n+1)$ steps of -1. So there are $\binom{2n}{n-1}=\frac{2n!}{(n-1)!(n+1)!}$ such paths. The number of paths that have the property $b=-1,\ \exists 0<a<2n$, given that the path reaches $(2n,0)$ is also $\binom{2n}{n-1}$ and the number of paths that have the property $b\geq 0,\ \forall 0<a<2n$ is

$$\binom{2n}{n}-\binom{2n}{n-1}=\binom{2n}{n}-\frac{n}{n+1}\binom{2n}{n}=\frac{1}{n+1}\binom{2n}{n}.$$

Hence, the probability that all people will be able to buy their tickets without having to change positions is $1/(n+1)$.

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[8] Consider a random walk starting at a, $S_0=a$, and reaching b in n steps: $S_n=b$. Denote $N_n(a,b)$ as the number of possible paths from $(0,a)$ to (n,b) and $N_n^0(a,b)$ as the number possible paths from $(0,a)$ to (n,b) that at some step k ($k>0$,), $S_k=0$; in other words, $N_n^0(a,b)$ are the paths that contain $(k,0)$, $\exists 0<k<n$. **The reflection principle** says that if $a,\ b>0$, then $N_n^0(a,b)=N_n(-a,b)$. The proof is intuitive: for each path $(0,a)$ to $(k,0)$, there is a one-to-one corresponding path from $(0,-a)$ to $(k,0)$.

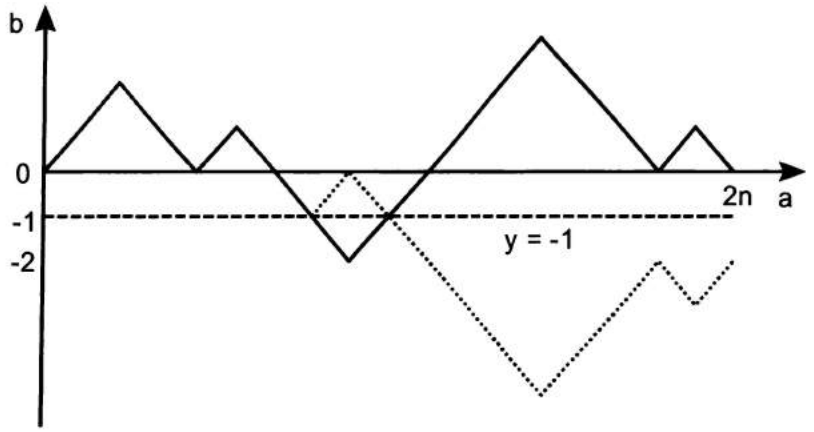
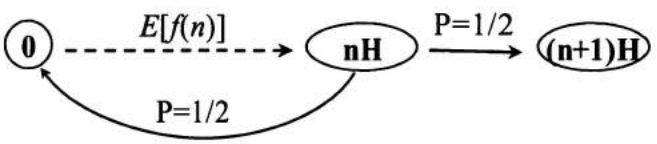


Figure 5.6 Reflected paths: the dashed line is the reflection of the solid line after it reaches -1

Coin sequence

Assume that you have a fair coin. What is the expected number of coin tosses to get n heads in a row?

Solution: Let $E[f(n)]$ be the expected number of coin tosses to get n heads in a row. In the Markov chain section, we discussed the case where $n=3$ (to get the pattern *HHH*). For any integer n, we can consider an induction approach. Using the Markov chain approach, we can easy get that $E[f(1)]=2$, $E[f(2)]=6$ and $E[f(3)]=14$. A natural guess for the general formula is that $E[f(n)]=2^{n+1}-2$. As always, let's prove the formula using induction. We have shown the formula is true for $n=1,2,3$. So we only need to prove that if $E[f(n)]=2^{n+1}-2$, $E[f(n+1)]=2^{n+2}-2$. The following diagram shows how to prove that the equation holds for $E[f(n+1)]$:



The state before $(n+1)$ heads in a row (denoted as $(n+1)H$) must be n heads in a row (denoted as nH). It takes an expected $E[f(n)]=2^{n+1}-2$ tosses to reach nH. Conditioned on state nH, there is a 1/2 probability it will go to $(n+1)H$ (the new toss yields H) and the process stops. There is also a 1/2 probability that it will go to the

starting state *0* (the new toss yields *T*) and we need another expected $E[f(n+1)]$ tosses to reach $(n+1)H$. So we have

$$E[f(n+1)] = E[F(n)] + \tfrac{1}{2}\times 1 + \tfrac{1}{2}\times E[f(n+1)]$$
$$\Rightarrow E[f(n+1)] = 2\times E[F(n)] + 2 = 2^{n+2} - 2$$

General Martingale approach: Let's use $HH\cdots H_n$ to explain a general approach for the expected time to get any coin sequence by exploring the stopping times of martingales.[9] Imagine a gambler has \$1 to bet on a sequence of *n* heads ($HH\cdots H_n$) in a fair game with the following rule: Bets are placed on up to *n* consecutive games (tosses) and each time the gambler bets all his money (unless he goes bankrupt). For example, if *H* appears at the first game, he will have \$2 and he will put all \$2 into the second game. He stops playing either when he loses a game or when he wins *n* games in a roll, in which case he collects $\$2^n$ (with probability $1/2^n$). Now let's imagine, instead of one gambler, before each toss a new gambler joins the game and bets on the same sequence of *n* heads with a bankroll of \$1 as well. After the *i*-th game, *i* gamblers have participated in the game and the total amount of money they have put in the game should be $\$i$. Since each game is fair, the expected value of their total bankroll is $\$i$ as well. In other words, if we denote x_i as the amount of money all the participating gamblers have after the *i*-th game, then $(x_i - i)$ is a martingale.

Now, let's add a stopping rule: the whole game will stop if one of the gamblers becomes the first to get *n* heads in a roll. A martingale stopped at a stopping time is a martingale. So we still have $E[(x_i - i)] = 0$. If the sequence stops after the *i*-th toss ($i \geq n$), the $(i-n+1)$-th player is the (first) player who gets *n* heads in a roll with payoff 2^n. So all the $(i-n)$ players before him went bankrupt; the $(i-n+2)$-th player gets $(n-1)$ heads in a roll with payoff 2^{n-1}; ...; the *i*-th player gets one head with payoff 2. So the total payoff is fixed and $x_i = 2^n + 2^{n-1} + \cdots + 2^1 = 2^{n+1} - 2$.

Hence, $E[(x_i - i)] = 2^{n+1} - 2 - E[i] = 0 \Rightarrow E[i] = 2^{n+1} - 2$.

This approach can be applied to any coin sequences—as well as dice sequences or any sequences with arbitrary number of elements. For example, let's consider the sequence *HHTTHH*. We can again use a stopped martingale process for sequence *HHTTHH*. The gamblers join the game one by one before each toss to bet on the same sequence *HHTTHH* until one gambler becomes the first to get the sequence *HHTTHH*. If the sequence stops after the *i*-th toss, the $(i-5)th$ gambler gets the *HHTTHH* with payoff

[9] If you prefer more details about the approach, please refer to "A Martingale Approach to the Study of Occurrence of Sequence Patterns in Repeated Experiments" by Shuo-Yen Robert Li, *The Annals of Probability*, Vol. 8, No. 6 (Dec., 1980), pp. 1171-1176.

2^6. All the $(i-6)$ players before him went bankrupt; the $(i-4)th$ player loses in the second toss (HT); the $(i-3)th$ player and the $(i-2)th$ player lose in the first toss (T); the $(i-1)th$ player gets sequence HH with payoff 2^2 and the i-th player gets H with payoff 2.

Hence, $E[(x_i - i)] = 2^6 + 2^2 + 2^1 - E[i] = 0 \Rightarrow E[i] = 70$.

5.3 Dynamic Programming

Dynamic Programming refers to a collection of general methods developed to solve sequential, or multi-stage, decision problems.[10] It is an extremely versatile tool with applications in fields such as finance, supply chain management and airline scheduling. Although theoretically simple, mastering dynamic programming algorithms requires extensive mathematical prerequisites and rigorous logic. As a result, it is often perceived to be one of the most difficult graduate level courses.

Fortunately, the dynamic programming problems you are likely to encounter in interviews—although you often may not recognize them as such—are rudimentary problems. So in this section we will focus on the basic logic used in dynamic programming and apply it to several interview problems. Hopefully the solutions to these examples will convey the gist and the power of dynamic programming.

A discrete-time dynamic programming model includes two inherent components:

1. The underlying discrete-time dynamic system

A dynamic programming problem can always be divided into stages with a decision required at each stage. Each stage has a number of states associated with it. The decision at one stage transforms the current state into a state in the next stage (at some stages and states, the decision may be trivial if there is only one choice).

Assume that the problem has $N+1$ stages (time periods). Following the convention, we label these stages as $0, 1, \cdots, N-1, N$. At any stage k, $0 \le k \le N-1$, the state transition can be expressed as $x_{k+1} = f(x_k, u_k, w_k)$, where x_k is the state of system at stage k;[11] u_k is the decision selected at stage k; w_k is a random parameter (also called disturbance).

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[10] This section barely scratches the surface of dynamic programming. For up-to-date dynamic programming topics, I'd recommend the book ***Dynamic Programming and Optimal Control*** by Professor Dimitri P. Bertsekas.

[11] In general, x_k can incorporate all past relevant information. In our discussion, we only consider the present information by assuming Markov property.

Basically the state of next stage x_{k+1} is determined as a function of the current state x_k, current decision u_k (the choice we make at stage k from the available options) and the random variable w_k (the probability distribution of w_k often depends on x_k and u_k).

2. A cost (or profit) function that is additive over time.

Except for the last stage (N), which has a cost/profit $g_N(x_N)$ depending only on x_N, the costs at all other stages $g_k(x_k, u_k, w_k)$ can depend on x_k, u_k, and w_k. So the total cost/profit is $g_N(x_N) + \sum_{k=i}^{N-1} g_k(x_k, u_k, w_k)\}$.

The goal of optimization is to select strategies/policies for the decision sequences $\pi^* = \{u_0^*, \cdots, u_{N-1}^*\}$ that minimize expected cost (or maximize expected profit):

$$J_{\pi^*}(x_0) = \min_{\pi} E\{g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k, w_k)\}.$$

Dynamic programming (DP) algorithm

The dynamic programming algorithm relies on an idea called the **Principle of Optimality**: If $\pi^* = \{u_0^*, \cdots, u_{N-1}^*\}$ is the optimal policy for the original dynamic programming problem, then the tail policy $\pi_i^* = \{u_i^*, \cdots, u_{N-1}^*\}$ must be optimal for the tail subproblem $E\{g_N(x_N) + \sum_{k-i}^{N-1} g_k(x_k, u_k, w_k)\}$.

DP algorithm: To solve the basic problem $J_{\pi^*}(x_0) = \min_{\pi} E\{g_N(x_N) + \sum_{k-0}^{N-1} g_k(x_k, u_k, w_k)\}$, start with $J_N(x_N) = g_N(x_N)$, and go backwards minimizing cost-to-go function $J_k(x_k)$: $J_k(x_k) = \min_{u_k \subset U_k(x_k)} E_{w_k}\{g_k(x_k, u_k, w_k) + J_{k+1}(f(x_k, u_k, w_k))\}, k = 0, \cdots, N-1$. Then the $J_0(x_0)$ generated from this algorithm is the expected optimal cost.

Although the algorithm looks complicated, the intuition is straightforward. For dynamic programming problems, we should start with optimal policy for every possible state of the final stage (which has the highest amount of information and least amount of uncertainty) first and then work backward towards earlier stages by applying the tail policies and cost-to-go functions until you reach the initial stage.

Now let's use several examples to show how the DP algorithm is applied.

Dice game

You can roll a 6-side dice up to 3 times. After the first or the second roll, if you get a number x, you can decide either to get x dollars or to choose to continue rolling. But once you decide to continue, you forgo the number you just rolled. If you get to the third roll, you'll just get x dollars if the third number is x and the game stops. What is the game worth and what is your strategy?

Solution: This is a simple dynamic programming strategy game. As all dynamic programming questions, the key is to start with the final stage and work backwards. For this question, it is the stage where you have forgone the first two rolls. It becomes a simple dice game with one roll. Face values 1, 2, 3, 4, 5, and 6 each have a 1/6 probability and your expected payoff is \$3.5.

Now let's go back one step. Imagine that you are at the point after the second roll, for which you can choose either to have a third roll with an expected payoff of \$3.5 or keep the current face value. Surely you will keep the face value if it is larger than 3.5; in other words, when you get 4, 5 or 6, you stop rolling. When you get 1, 2 or 3, you keep rolling. So your expected payoff before the second roll is $3/6\times 3.5+1/6\times(4+5+6)=\$4.25$.

Now let's go back one step further. Imagine that you are at the point after the first roll, for which you can choose either to have a second roll with expected payoff \$4.25 (when face value is 1, 2, 3 or 4) or keep the current face value. Surely you will keep the face value if it is larger than 4.25; In other words, when you get 5 or 6, you stop rolling. So your expected payoff before the first roll is $4/6\times 4.25+1/6\times(5+6)=\$14/3$.

This backward approach—called tail policy in dynamic programming—gives us the strategy and also the expected value of the game at the initial stage, $\$14/3$.

World series

The Boston Red Sox and the Colorado Rockies are playing in the World Series finals. In case you are not familiar with the World Series, there are a maximum of 7 games and the first team that wins 4 games claims the championship. You have \$100 dollars to place a double-or-nothing bet on the Red Sox.

Unfortunately, you can only bet on each individual game, not the series as a whole. How much should you bet on each game so that if the Red Sox wins the whole series, you win exactly \$100, and if Red Sox loses, you lose exactly \$100?

Solution: Let (i,j) represents the state that the Red Sox has won i games and the Rockies has won j games, and let $f(i,j)$ be our net payoff, which can be negative when we lose money, at state (i,j). From the rules of the game, we know that there may be between 4 and 7 games in total. We need to decide on a strategy so that whenever the

series is over, our final net payoff is either +100—when Red Sox wins the championship—or −100—when Red Sox loses. In other words, the state space of the final stage includes $\{(4,0), (4,1), (4,2), (4,3)\}$ with payoff $f(i,j)=100$ and $\{(0,4), (1,4), (2,4), (3,4)\}$ with payoff $f(i,j)=-100$. As all dynamic programming questions, the key is to start with the final stage and work backwards—even though in this case the number of stages is not fixed. For each state (i, j), if we bet $\$y$ on the Red Sox for the next game, we will have $(f(i, j)+y)$ if the Red Sox wins and the state goes to $(i+1, j)$, or $(f(i,j)-y)$ if the Red Sox loses and the state goes to $(i, j+1)$. So clearly we have

$$\left.\begin{aligned} f(i+1, j) &= f(i, j)+y \\ f(i, j+1) &= f(i, j)-y \end{aligned}\right\} \Rightarrow \begin{cases} f(i, j) = \big(f(i+1, j)+f(i, j+1)\big)/2 \\ y = \big(f(i+1, j)-f(i, j+1)\big)/2 \end{cases}.$$

For example, we have $f(3, 3)=\dfrac{f(4, 3)+f(3, 4)}{2}=\dfrac{100-100}{2}=0$. Let's set up a table with the columns representing i and the rows representing j. Now we have all the information to fill in $f(4, 0)$, $f(4, 1)$, $f(4, 3)$, $f(4, 2)$, $f(0, 4)$, $f(1, 4)$, $f(2, 4)$, $f(3, 4)$, as well as $f(3,3)$. Similarly we can also fill in all $f(i,j)$ for the states where $i=3$ or $j=3$ as shown in Figure 5.7. Going further backward, we can fill in the net payoffs at every possible state. Using equation $y=\big(f(i+1, j)-f(i, j+1)\big)/2$, we can also calculate the bet we need to place at each state, which is essentially our strategy.

If you are not accustomed to the table format, Figure 5.8 redraws it as a binomial tree, a format you should be familiar with. If you consider that the boundary conditions are $f(4, 0)$, $f(4, 1)$, $f(4, 3)$, $f(4, 2)$, $f(0, 4)$, $f(1, 4)$, $f(2, 4)$, and $f(3, 4)$, the underlying asset either increases by 1 or decrease by 1 after each step, and there is no interest, then the problem becomes a simple binomial tree problem and the bet we place each time is the delta in dynamic hedging. In fact, both European options and American options can be solved numerically using dynamic programming approaches.

		Red Sox				
	wins	0	1	2	3	4
Colorado Rockies	0					100
	1					100
	2					100
	3				0	100
	4	-100	-100	-100	-100	

		Red Sox				
	wins	0	1	2	3	4
Colorado Rockies	0				87.5	100
	1				75	100
	2				50	100
	3	-87.5	-75	-50	0	100
	4	-100	-100	-100	-100	

		Red Sox				
	wins	0	1	2	3	4
Colorado Rockies	0	0	31.25	62.5	87.5	100
	1	-31.3	0	37.5	75	100
	2	-62.5	-37.5	0	50	100
	3	-87.5	-75	-50	0	100
	4	-100	-100	-100	-100	

		Red Sox				
	bets	0	1	2	3	4
Colorado Rockies	0	31.25	31.25	25	12.5	
	1	31.25	37.5	37.5	25	
	2	25	37.5	50	50	
	3	12.5	25	50	100	
	4					

Figure 5.7 Payoffs and bets at different states

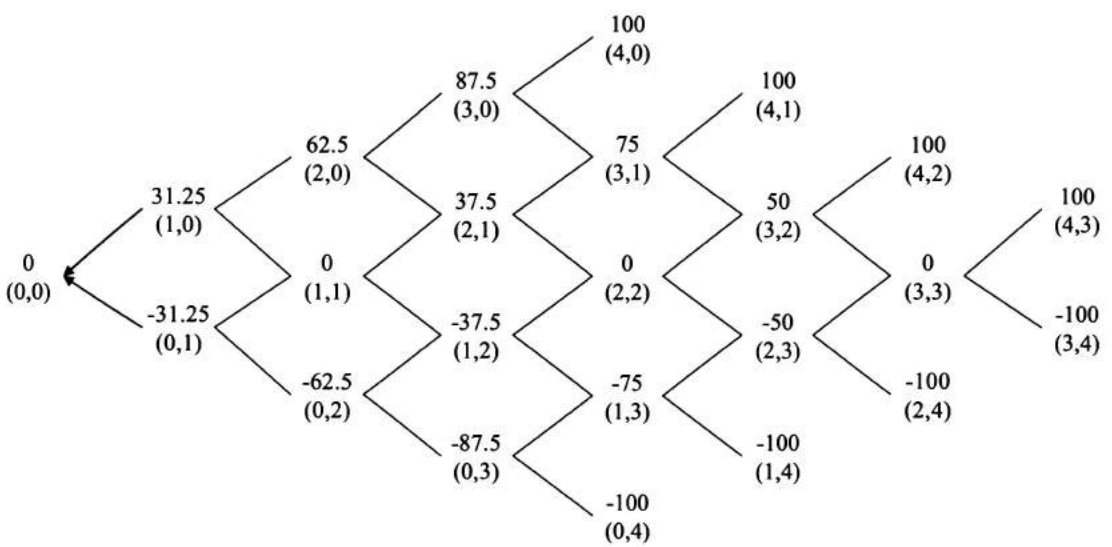

Figure 5.8 Payoff at different states expressed in a binomial tree

Dynamic dice game

A casino comes up with a fancy dice game. It allows you to roll a dice as many times as you want unless a 6 appears. After each roll, if 1 appears, you will win \$1; if 2 appears, you will win \$2; ...; if 5 appears, you win \$5; but if 6 appears all the moneys you have won in the game is lost and the game stops. After each roll, if the dice number is 1-5, you can decide whether to keep the money or keep on rolling. How much are you willing to pay to play the game (if you are risk neutral)?[12]

Solution: Assuming that we have accumulated n dollars, the decision to have another roll or not depends on the expected profit versus expected loss. If we decide to have an extra roll, our expected payoff will become

$$\frac{1}{6}(n+1)+\frac{1}{6}(n+2)+\frac{1}{6}(n+3)+\frac{1}{6}(n+4)+\frac{1}{6}(n+5)+\frac{1}{6}\times 0=\frac{5}{6}n+2.5.$$

We have another roll if the expected payoff $\frac{5}{6}n+2.5>n$, which means that we should keep rolling if the money is no more than \$14. Considering that we will stop rolling when $n\geq 15$, the maximum payoff of the game is \$19 (the dice rolls a 5 after reaching the state $n=14$). We then have the following: $f(19)=19$, $f(18)=18$, $f(17)=17$, $f(16)=16$, and $f(15)=15$. When $n\leq 14$, we will keep on rolling, so $E[f(n)\,|\,n\leq 14]=\frac{1}{6}\sum_{i=1}^{5}E[f(n+i)]$. Using this equation, we can calculate the value for $E[f(n)]$ recursively for all $n=14, 13, \cdots, 0$. The results are summarized in Table 5.2. Since $E[f(0)]=6.15$, we are willing to pay at most \$6.15 for the game.

n	19	18	17	16	15	14	13	12	11	10
E[$f(n)$]	19.00	18.00	17.00	16.00	15.00	14.17	13.36	12.59	11.85	11.16
n	9	8	7	6	5	4	3	2	1	0
E[$f(n)$]	10.52	9.91	9.34	8.80	8.29	7.81	7.36	6.93	6.53	6.15

Table 5.2 Expected payoff of the game when the player has accumulated n dollars

[12] Hint: If you decide to have another roll, the expected amount you have after the roll should be higher than the amount before the roll. As the number of dollars increases, you risk losing more money if a 6 appears. So when the amount of dollar reaches a certain number, you should stop rolling.

Dynamic card game

A casino offers yet another card game with the standard 52 cards (26 red, 26 black). The cards are thoroughly shuffled and the dealer draws cards one by one. (Drawn cards are not returned to the deck.) You can ask the dealer to stop at any time you like. For each red card drawn, you win \$1; for each black card drawn, you lose \$1. What is the optimal stopping rule in terms of maximizing expected payoff and how much are you willing to pay for this game?

Solution: It is another problem perceived to be difficult by many interviewees. Yet it is a simple dynamic programming problem. Let (b, r) represent the number of black and red cards left in the deck, respectively. By symmetry, we have

$$red\ cards\ drawn - black\ cards\ drawn = black\ cards\ left - red\ cards\ left = b - r$$

At each (b, r), we face the decision whether to stop or keep on playing. If we ask the dealer to stop at (b, r), the payoff is $b-r$. If we keep on going, there is $\frac{b}{b+r}$ probability that the next card will be black—in which case the state changes to $(b-1, r)$—and $\frac{r}{b+r}$ probability that the next card will be red—in which case the state changes to $(b, r-1)$. We will stop if and only if the expected payoff of drawing more cards is less than $b-r$. That also gives us the system equation:

$$E[f(b,r)] = \max\left(b-r, \frac{b}{b+r}E[f(b-1,r)] + \frac{r}{b+r}[f(b,r-1)]\right).^{13}$$

As shown in Figure 5.9 (next page), using the boundary conditions $f(0, r) = 0$, $f(b, 0) = b$, $\forall b, r = 0, 1, \cdots, 26$, and the system equation for $E[f(b, r)]$, we can recursively calculate $E[f(b, r)]$ for all pairs of b and r.

The expected payoff at the beginning of the game is $E[f(26, 26)] = \$2.62$.

[13] You probably have recognized this system equation as the one for American options. Essentially you decide whether you want to exercise the option at state (b, r).

f(b,r)	Number of Black Cards Left																										
Number of Red Cards Left	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
1	0	0.50	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	0	0.33	0.67	1.20	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
3	0	0.25	0.50	0.85	1.34	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
4	0	0.20	0.40	0.66	1.00	1.44	2.07	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
5	0	0.17	0.33	0.54	0.79	1.12	1.55	2.15	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
6	0	0.14	0.29	0.45	0.66	0.91	1.23	1.66	2.23	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
7	0	0.13	0.25	0.39	0.56	0.76	1.01	1.34	1.75	2.30	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
8	0	0.11	0.22	0.35	0.49	0.66	0.86	1.11	1.43	1.84	2.36	3.05	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
9	0	0.10	0.20	0.31	0.43	0.58	0.75	0.95	1.21	1.52	1.92	2.43	3.10	4	5	6	7	8	9	10	11	12	13	14	15	16	17
10	0	0.09	0.18	0.28	0.39	0.52	0.66	0.83	1.04	1.30	1.61	2.00	2.50	3.15	4	5	6	7	8	9	10	11	12	13	14	15	16
11	0	0.08	0.17	0.26	0.35	0.46	0.59	0.74	0.91	1.12	1.38	1.69	2.08	2.57	3.20	4	5	6	7	8	9	10	11	12	13	14	15
12	0	0.08	0.15	0.24	0.32	0.42	0.54	0.66	0.81	0.99	1.20	1.46	1.77	2.15	2.63	3.24	4	5	6	7	8	9	10	11	12	13	14
13	0	0.07	0.14	0.22	0.30	0.39	0.49	0.60	0.73	0.89	1.06	1.28	1.53	1.84	2.22	2.70	3.28	4.03	5	6	7	8	9	10	11	12	13
14	0	0.07	0.13	0.20	0.28	0.36	0.45	0.55	0.67	0.80	0.95	1.13	1.35	1.60	1.91	2.29	2.75	3.33	4.06	5	6	7	8	9	10	11	12
15	0	0.06	0.13	0.19	0.26	0.33	0.42	0.51	0.61	0.73	0.86	1.02	1.20	1.42	1.67	1.98	2.36	2.81	3.38	4.09	5	6	7	8	9	10	11
16	0	0.06	0.12	0.18	0.24	0.31	0.39	0.47	0.57	0.67	0.79	0.93	1.08	1.27	1.48	1.74	2.05	2.42	2.87	3.43	4.13	5	6	7	8	9	10
17	0	0.06	0.11	0.17	0.23	0.29	0.36	0.44	0.53	0.62	0.73	0.85	0.99	1.15	1.33	1.55	1.81	2.11	2.48	2.93	3.48	4.16	5	6	7	8	9
18	0	0.05	0.11	0.16	0.22	0.28	0.34	0.41	0.49	0.58	0.67	0.78	0.90	1.04	1.21	1.39	1.61	1.87	2.17	2.54	2.99	3.53	4.19	5	6	7	8
19	0	0.05	0.10	0.15	0.20	0.26	0.32	0.39	0.46	0.54	0.63	0.73	0.84	0.96	1.10	1.26	1.45	1.67	1.93	2.24	2.60	3.04	3.57	4.22	5.01	6	7
20	0	0.05	0.10	0.14	0.19	0.25	0.31	0.37	0.43	0.51	0.59	0.68	0.78	0.89	1.01	1.16	1.32	1.51	1.73	1.99	2.30	2.66	3.09	3.62	4.25	5.03	6
21	0	0.05	0.09	0.14	0.19	0.24	0.29	0.35	0.41	0.48	0.55	0.63	0.72	0.83	0.94	1.07	1.21	1.38	1.57	1.79	2.05	2.35	2.72	3.15	3.66	4.28	5.05
22	0	0.04	0.09	0.13	0.18	0.23	0.28	0.33	0.39	0.45	0.52	0.60	0.68	0.77	0.87	0.99	1.12	1.26	1.43	1.62	1.85	2.11	2.41	2.77	3.20	3.71	4.32
23	0	0.04	0.08	0.13	0.17	0.22	0.26	0.32	0.37	0.43	0.49	0.56	0.64	0.72	0.82	0.92	1.04	1.17	1.32	1.48	1.68	1.90	2.16	2.47	2.82	3.25	3.75
24	0	0.04	0.08	0.12	0.16	0.21	0.25	0.30	0.35	0.41	0.47	0.53	0.60	0.68	0.77	0.86	0.97	1.08	1.22	1.37	1.54	1.73	1.96	2.22	2.52	2.88	3.30
25	0	0.04	0.08	0.12	0.16	0.20	0.24	0.29	0.34	0.39	0.45	0.51	0.57	0.64	0.72	0.81	0.90	1.01	1.13	1.26	1.42	1.59	1.78	2.01	2.27	2.57	2.93
26	0	0.04	0.07	0.11	0.15	0.19	0.23	0.28	0.32	0.37	0.43	0.48	0.54	0.61	0.68	0.76	0.85	0.95	1.06	1.18	1.31	1.46	1.64	1.83	2.06	2.32	**2.62**

Figure 5.9 Expected payoffs at different states (b, r)

5.4 Brownian Motion and Stochastic Calculus

In this section, we briefly go over some problems for stochastic calculus, the counterpart of stochastic processes in continuous space. Since the basic definitions and theorems of Brownian motion and stochastic calculus are directly used as interview problems, we'll simply integrate them into the problems instead of starting with an overview of definitions and theorems.

Brownian motion

A. Define and enumerate some properties of a Brownian motion?[1]

Solution: This is the most basic Brownian motion question. Interestingly, part of the definition, such as $W(0)=0$, and some properties are so obvious that we often fail to recite all the details.

A continuous stochastic process $W(t),\ t\geq 0$, is a Brownian motion if

- $W(0)=0$;
- The increments of the process $W(t_1)-W(0), W(t_2)-W(t_1), \cdots, W(t_n)-W(t_{n-1})$, $\forall 0\leq t_1\leq t_2\leq\cdots\leq t_n$ are independent;
- Each of these increments is normally distributed with distribution $W(t_{i+1})-W(t_i)\sim N(0, t_{i+1}-t_i)$.

Some of the important properties of Brownian motion are the following: continuous (no jumps); $E\left[W(t)\right]=0$; $E\left[W(t)^2\right]=t$; $W(t)\sim N(0,t)$; martingale property $E\left[W(t+s)\,|\,W(t)\right]=W(t)$; $\text{cov}\left(W(s),W(t)\right)=s,\ \forall 0<s<t$; and Markov property (in continuous space).

There are two other important martingales related to Brownian motion that are valuable tools in many applications.

- $Y(t)=W(t)^2-t$ is a martingale.
- $Z(t)=\exp\left\{\lambda W(t)-\frac{1}{2}\lambda^2 t\right\}$, where λ is any constant and $W(t)$ is a Brownian motion, is a martingale. (Exponential martingale).

---

[1] A Brownian motion is often denoted as B_t. Alternatively it is denoted as $W(t)$ since it is a Wiener process. In this section, we use both notations interchangeably so that you get familiar with both.

We'll show a proof of the first martingale using Ito's lemma in the next section. A sketch for the exponential martingale is the following:[2]

$$\begin{aligned} E[Z(t+s)] &= E\left[\exp\left\{\lambda\left(W(t)+W(s)\right)-\tfrac{1}{2}\lambda^2(t+s)\right\}\right] \\ &= \exp\left\{\lambda W(t)-\tfrac{1}{2}\lambda^2 t\right\}\exp\left\{-\tfrac{1}{2}\lambda^2 s\right\}E\left[\exp\left\{\lambda W(s)\right\}\right] \\ &= Z_t \exp\left\{-\tfrac{1}{2}\lambda^2 s\right\}\exp\left\{\tfrac{1}{2}\lambda^2 s\right\} = Z_t \end{aligned}$$

B. What is the correlation of a Brownian motion and its square?

Solution: The solution to this problem is surprisingly simple. At time t, $B_t \sim N(0,t)$, by symmetry, $E[B_t]=0$ and $E[B_t^3]=0$. Applying the equation for covariance $Cov(X,Y)=E[XY]-E[X]E[Y]$, we have $Cov(B_t,B_t^2)=E[B_t^3]-E[B_t]E[B_t^2]=0-0=0$. So the correlation of a Brownian motion and its square is 0, too.

C. Let B_t be a Brownian motion. What is the probability that $B_1>0$ and $B_2<0$?

Solution: A standard solution takes advantage of the fact that $B_1 \sim N(0,1)$, and B_2-B_1 is independent of B_1, which is again a normal distribution: $B_2-B_1 \sim N(0,1)$. If $B_1=x>0$, then for $B_2<0$, we must have $B_2-B_1<-x$.

$$\begin{aligned} P(B_1>0,B_2<0) &= P(B_1>0,B_2-B_1<-B_1) \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy = \int_0^\infty\int_{-\infty}^{x}\frac{1}{2\pi}e^{-(x^2+y^2)/2}dxdy \\ &= \int_0^\infty\int_{3/2\pi}^{7/4\pi}\frac{1}{2\pi}e^{-r^2/2}rdrd\theta = \frac{7/4\pi-3/2\pi}{2\pi}\left[-e^{-r^2/2}\right]_0^\infty = \frac{1}{8} \end{aligned}$$

But do we really need the integration step? If we fully take advantage of the facts that B_1 and B_2-B_1 are two IID $N(0,1)$, the answer is no. Using conditional probability and independence, we can reformulate the equation as

$$\begin{aligned} P(B_1>0,B_2<0) &= P(B_1>0)P(B_2-B_1<0)P(|B_2-B_1|>|B_1|) \\ &= 1/2\times 1/2\times 1/2 = 1/8 \end{aligned}$$

---

[2] $W(s)\sim N(0,s)$. So $E\left[\exp\left\{\lambda W(s)\right\}\right]$ is the moment generating function of normal random variable $N(0,s)$.

This approach is better demonstrated in Figure 5.10. When we have $B_1 > 0$ and $B_2 - B_1 < -B_1$, which accounts for 1/8 of the density volume. (All 8 regions separated by $x = 0$, $y = 0$, $y = x$, and $y = -x$ have the same density volume by symmetry.)

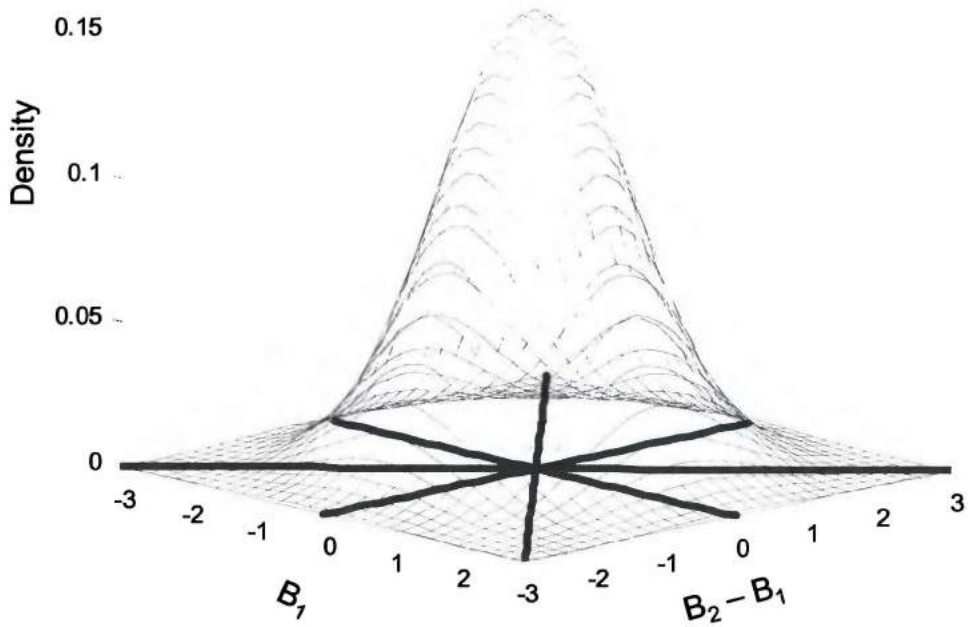

Figure 5.10 Probability density graph of (B_1, B_2-B_1)

Stopping time/ first passage time

A. What is the mean of the stopping time for a Brownian motion to reach either -1 or 1?

Solution: As we have discussed, $B_t^2 - t$ is martingale. It can be proved by applying Ito's lemma:

$$d(B_t^2 - t) = \frac{\partial(B_t^2 - t)}{\partial B_t} dB_t + \frac{\partial(B_t^2 - t)}{\partial t} dt + \frac{1}{2} \frac{\partial^2 (B_t^2 - t)}{\partial B_t^2} dt = 2B_t dB_t - dt + dt = 2B_t dB_t .$$

So $d(B_t^2 - t)$ has no drift term and is a martingale. Let $T = \min\{t;\ B_t = 1 \text{ or } -1\}$. At continuous time and space, the following property still applies: A martingale stopped at

a stopping time is a martingale! So $B_T^2 - T$ is a martingale and $E\left[B_T^2 - T\right] = B_0^2 - 0 = 0$. The probability that B_t hits 1 or -1 is 1, so $B_T^2 = 1 \Rightarrow E[T] = E\left[B_T^2\right] = 1$.

B. Let $W(t)$ be a standard Wiener process and τ_x ($x > 0$) be the first passage time to level x ($\tau_x = \min\{t;\, W(t) = x\}$). What is the probability density function of τ_x and the expected value of τ_x?

Solution: This is a textbook problem that is elegantly solved using the reflection principle, so we will simply summarize the explanation. For any Wiener process paths that reach x before t ($\tau_x \le t$), they have equal probability ending above x or below x at time t, $P(\tau_x \le t, W(t) \ge x) = P(\tau_x \le t, W(t) \le x)$. The explanation lies in the reflection principle. As shown in Figure 5.11, for each path that reaches x before t and is at a level y above x at time t, we can switch the sign of any move starting from τ_x and the reflected path will end at $2x - y$ that is below x at time t. For a standard Wiener process (Brownian motion), both paths have equal probability.

$$P(\tau_x \le t) = P(\tau_x \le t, W(t) \ge x) + P(\tau_x \le t, W(t) \le x) = 2P(\tau_x \le t, W(t) \ge x)$$
$$= 2P(W(t) \ge x) = 2\int_x^\infty \frac{1}{\sqrt{2\pi t}} e^{-w^2/2t} dw$$

Let $v = \frac{w}{\sqrt{t}}$, we have $e^{-w^2/2t} = e^{-v^2/2}$ and $dv = \frac{dw}{\sqrt{t}}$.

$$\therefore\ P(\tau_x \le t) = 2\int_m^\infty \frac{1}{\sqrt{2\pi t}} e^{-w^2/2t} dw = 2\int_{x/\sqrt{t}}^\infty \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv = 2 - 2N(x/\sqrt{t}).^3$$

Take the derivative with respect to t, we have

$$f_{\tau_x}(t) = \frac{dP\{\tau_x \le t\}}{dt} = \frac{dP\{\tau_x \le t\}}{d(x/\sqrt{t})} \frac{d(x/\sqrt{t})}{dt} = 2N'(x/\sqrt{t}) \times \frac{x}{2} t^{-3/2} \Rightarrow \frac{xe^{-x^2/2t}}{t\sqrt{2\pi t}},\ \forall x > 0.$$

From part A, it's easy to show that the expected stopping time to reach either α ($\alpha > 0$) or $-\beta$ ($\beta > 0$) is again $E[N] = \alpha\beta$. The expected first passage time to level x is

---

[3] If we define $M(t) = \max_{0 \le s \le t} W(s)$, then $P(\tau \le t)$ if and only if $M(t) \ge x$. Taking the derivative of $P(\tau_x \le t)$ with respect to x, we can derive the probability density function of $M(t)$.

essentially the expected stopping time to reach either x or $-\infty$ and $E[\tau_x] = x \times \infty = \infty$. Although we have $P(\tau_x \le \infty) = 2 - 2N(x/\sqrt{\infty}) = 1$, the expected value of τ_x is ∞!

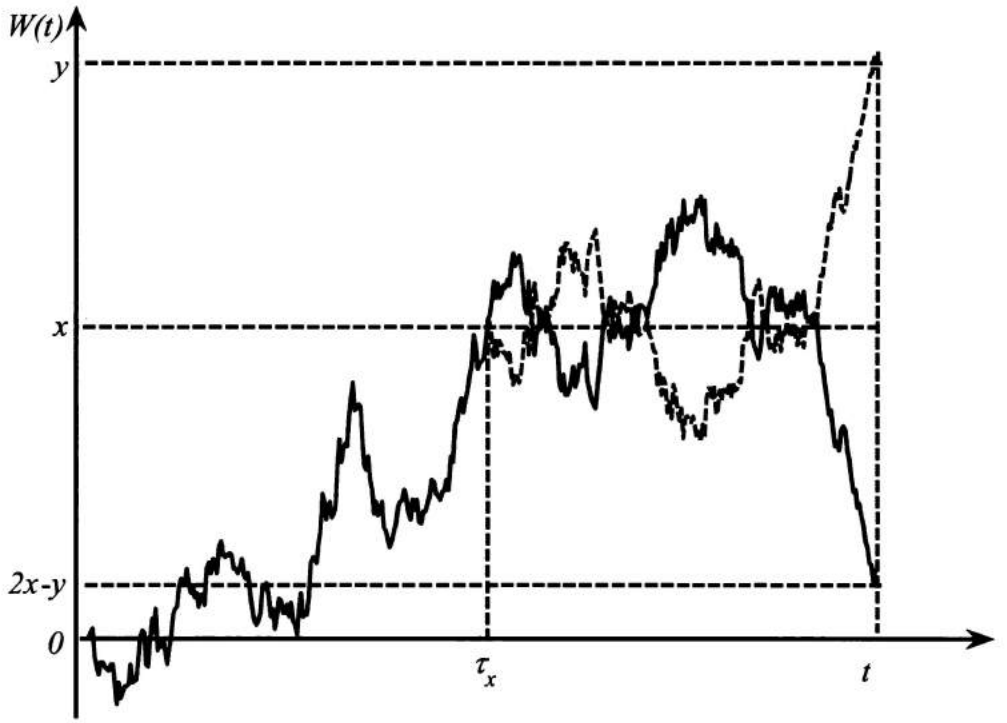

Figure 5.11 Sample path of a standard Weiner process and its reflected path

C. Suppose that X is a Brownian motion with no drift, i.e. $dX(t) = dW(t)$. If X starts at 0, what is the probability that X hits 3 before hitting -5? What if X has drift m, i.e. $dX(t) = mdt + dW(t)$?

Solution: A Brownian motion is a martingale. Let p_3 be the probability that the Brownian motion hits 3 before -5. Since a martingale stopped at a stopping time is a martingale, we have $3P_3 + (-5)(1 - P_3) = 0 \Rightarrow P_3 = 5/8$. Similar to random walk, if we have stopping boundaries ($\alpha > 0$) and $-\beta$ ($\beta > 0$), the probability that it stops at α instead of $-\beta$ is $p_\alpha = \beta/(\alpha + \beta)$. The expected stopping time to reach either α or $-\beta$ is again $E[N] = \alpha\beta$.

When X has drift m, the process is no longer a martingale. Let $P(t, x)$ be the probability that the process hits 3 before hitting -5 when $X = x$ at time t. Although X is no longer a

martingale process, it is still a Markov process. So $P(t,x) = P(x)$ is actually independent of t. Applying the Feynman-Kac equation[4], we have

$mP_x(x) + 1/2P_{xx}(x) = 0$ for $-5 < x < 3$.

We also have boundary conditions that $P(3) = 1$ and $P(-5) = 0$.

$mP_x(x) + 1/2P_{xx}(x) = 0$ is a homogeneous linear differential equation with two real roots: $r_1 = 0$ and $r_2 = -2m$. So the general solution is $P(x) = c_1 e^{0x} + c_2 e^{-2mx} = c_1 + c_2 e^{-2mx}$. Applying the boundary conditions, we have

$$\begin{cases} c_1 + c_2 e^{-6m} = 1 \\ c_1 + c_2 e^{10m} = 0 \end{cases} \Rightarrow \begin{cases} c_1 = -e^{10m}/(e^{-6m} - e^{10m}) \\ c_2 = 1/(e^{-6m} - e^{10m}) \end{cases} \Rightarrow P(0) = c_1 + c_2 = \frac{e^{10m} - 1}{e^{10m} - e^{-6m}}$$

A different and simpler approach takes advantage of the exponential martingale: $Z(t) = \exp\{\lambda W(t) - \frac{1}{2}\lambda^2 t\}$. Since $W(t) = X(t) - mt$, $X(t) - mt$ is a Brownian motion as well. Applying the exponential martingale, we have $E\left[\exp\left(\lambda(X - mt) - \frac{1}{2}\lambda^2 t\right)\right] = 1$ for any constant λ. To remove the terms including time t, we can set $\lambda = -2m$ and the equation becomes $E\left[\exp(-2mX)\right] = 1$. Since a martingale stopped at a stopping time is a martingale, we have $P_3 \exp(-2m \times 3) + (1 - P_3)\exp(-2m \times -5) = 1 \Rightarrow \frac{e^{10m} - 1}{e^{10m} - e^{-6m}}$.

D. Suppose that X is a generalized Weiner process $dX = dt + dW(t)$, where $W(t)$ is a Brownian motion. What is the probability that X ever reaches -1?

Solution: To solve this problem, we again can use the equation $E\left[\exp(-2mX)\right] = 1$ from the previous problem with $m = 1$. It may not be obvious since we only have one apparent boundary, -1. To apply the stopping time, we also need a corresponding positive boundary. To address this problem, we can simply use $+\infty$ as the positive boundary and the equation becomes

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[4] Let X be an Ito process given by equation $dX(t) = \beta(t,X)dt + \gamma(t,X)dW$ and $f(x)$ be a function of X. Define function $V(t,x) = E[f(X_T) \mid X_t = x]$, then $V(t,x)$ is a martingale process that satisfies the partial differential equation $\frac{\partial V}{\partial t} + \beta(t,x)\frac{\partial V}{\partial S} + \frac{1}{2}\gamma^2(t,x)\frac{\partial^2 V}{\partial S^2} = 0$ and terminal condition $V(T,x) = f(x)$ for all x.

$$P_{-1}\exp(-2\times -1)+(1-P_{-1})\exp(-2\times +\infty)=P_{-1}e^{2}=1\Rightarrow P_{-1}=e^{-2}.$$

Ito's lemma

Ito's lemma is the stochastic counterpart of the chain rule in ordinary calculus. Let $X(t)$ be an Ito process satisfying $dX(t)=\beta(t,X)dt+\gamma(t,X)dW(t)$, and $f(X(t),t)$ be a twice-differentiable function of $X(t)$ and t. Then $f(X(t),t)$ is an Ito process satisfying

$$df=\left(\frac{\partial f}{\partial t}+\beta(t,X)\frac{\partial f}{\partial x}+\frac{1}{2}\gamma^{2}(t,X)\frac{\partial^{2}f}{\partial x^{2}}\right)dt+\gamma(t,X)\frac{\partial f}{\partial x}dW(t).$$

$$\text{Drift rate}=\frac{\partial f}{\partial t}+\beta(t,X)\frac{\partial f}{\partial x}+\frac{1}{2}\gamma^{2}(t,X)\frac{\partial^{2}f}{\partial x^{2}}$$

A. Let B_t be a Brownian motion and $Z_t=\sqrt{t}B_t$. What is the mean and variance of Z_t? Is Z_t a martingale process?

Solution: As a Brownian motion, $B_t\sim N(0,t)$, which is symmetric about 0. Since $\sqrt{t}$ is a constant at t, $Z_t=\sqrt{t}B_t$ is symmetric about 0 and has mean 0 and variance $t\times\text{var}(B_t)=t^{2}$. More exactly, $Z_t\sim N(0,t^{2})$.

Although Z_t has unconditional expected value 0, it is not a martingale. Applying Ito's lemma to $Z_t=\sqrt{t}B_t$, we have $dZ_t=\frac{\partial Z_t}{\partial B_t}dB_t+\frac{\partial Z_t}{\partial t}dt+\frac{1}{2}\times\frac{\partial^{2}Z_t}{\partial B_t^{2}}dt=\frac{1}{2}t^{-1/2}B_t dt+\sqrt{t}dB_t$. For all the cases that $B_t\neq 0$, which has probability 1, the drift term $\frac{1}{2}t^{-1/2}B_t dt$ is not zero.[5] Hence, the process $Z_t=\sqrt{t}B_t$ is not a martingale process.

B. Let $W(t)$ be a Brownian motion. Is $W(t)^3$ a martingale process?

[5] A generalized Wiener process $dx=a(x,t)dt+b(x,t)dW(t)$ is a martingale process if and only if the drift term has coefficient $a(x,t)=0$.

Solution: Applying Ito's lemma to $f\left(W(t), t\right) = W(t)^3$, we have $\frac{\partial f}{\partial W(t)} = 3W(t)^2$, $\frac{\partial f}{\partial t} = 0$, $\frac{\partial^2 f}{\partial W(t)^2} = 6W(t)$, and $df\left(W(t), t\right) = 3W(t)dt + 3W(t)^2 dW(t)$. So again for the cases $W(t) \neq 0$, which has probability 1, the drift term is not zero. Hence, $W(t)^3$ is not a martingale process.

Chapter 6 Finance

It used to be common for candidates with no finance knowledge to get hired into quantitative finance positions. Although this still happens for candidates with specialized knowledge that is in high demand, it's more likely that you are required, or at least expected, to have a basic grasp of topics in finance. So you should expect to answer some finance questions and be judged on your answers.

Besides classic textbooks,[1] there are a few interview books in the market to help you prepare for finance interviews.[2] If you want to get prepared for general finance problems, you may want to read a finance interview book to get a feel for what types of questions are asked. The focus of this chapter is more on the intuitions and mathematics behind derivative pricing instead of basic finance knowledge. Derivative problems are popular choices in quantitative interviews—even for divisions that are not directly related to derivative markets—because these problems are complex enough to test your understanding of quantitative finance.

6.1. Option Pricing

Let's begin with some notations that we will use in the following sections.

T: maturity date; t: the current time; $\tau = T - t$: time to maturity; S: stock price at time t; r: continuous risk-free interest rate; y: continuous dividend yield; σ: annualized asset volatility; c: price of a European call; p: price of a European put; C: price of an American call; P: price of an American put; D: present value, at t, of future dividends; K: strike price; PV: present value at t.

Price direction of options

How do vanilla European/American option prices change when S, K, τ, σ, r, or D changes?

Solution: The payoff of a call is $\max(S-K,0)$ and the payoff of a put is $\max(K-S,0)$. A European option can only be exercised at the expiration time, while an American option can be exercised at any time before maturity. Intuitively we can figure out that the price of a European/American call should decrease when the strike price increases

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[1] For basic finance theory and financial market knowledge, I recommend ***Investments*** by Zvi Bodie, Alex Kane and Alan J. Marcus. For derivatives, ***Options, Futures and Other Derivatives*** by John C. Hull is a classic. If you want to gain a deeper understanding of stochastic calculus and derivative pricing, I'd recommend ***Stochastic Calculus for Finance*** (Volumes I and II) by Steven E. Shreve.

[2] For example, ***Vault Guide to Finance Interviews*** and ***Vault Guide to Advanced and Quantitative Finance Interviews***.

since a call with a higher strike has no higher—and sometimes lower—payoff than a call with a lower strike. Using similar analyses, we summarize the effect of changing market conditions on an option's value in Table 6.1.

The impact of time to maturity on the price of a European call/put is uncertain. If there is a large dividend payoff between two different maturity dates, a European call with shorter maturity that expires before the ex-dividend date may be worth more than a call with longer maturity. For deep in-the-money European puts, the one with shorter maturity is worth more since it can be exercised earlier (time value of the money).

Variable	European call	European put	American call	American Put
Stock price ↑	↑	↓	↑	↓
Strike price ↑	↓	↑	↓	↑
Time to maturity ↑	?	?	↑	↑
Volatility ↑	↑	↑	↑	↑
Risk-free rate ↑	↑	↓	↑	↓
Dividends ↑	↓	↑	↓	↑

Table 6.1 Impact of S, K, τ, σ, r, and D on option prices

↑: increase; ↓: decrease; ?: increase or decrease

It is also worth noting that Table 6.1 assumes that only one factor changes value while all others stay the same, which in practice may not be realistic since some of the factors are related. For example, a large decrease in interest rate often triggers a stock market rally and increases the stock price, which has an opposite effect on option value.

Put-call parity

Put-call parity: $c + K^{-r\tau} = p + S - D$, where the European call option and the European put option have the same underlying security, the same maturity T and the same strike price K. Since $p \geq 0$, we can also derive boundaries for c, $S - D - Ke^{-r\tau} \leq c \leq S$, from the put-call parity.

For American options, the equality no longer holds and it becomes two inequalities: $S - D - K \leq C - P \leq S - K^{-r\tau}$.

Can you write down the put-call parity for European options on non-dividend paying stocks and prove it?

Solution: The put-call parity for European options on non-dividend paying stocks is $c + K^{-r\tau} = p + S$. We can treat the left side of the equation as portfolio *A*—a call and a zero-coupon bond with face value *K*—and the right side as portfolio *B*—a put and the underlying stock, which is a protective put. Portfolio *A* has payoff $\max(S_T - K, 0) + K = \max(S_T, K)$ at maturity *T*; portfolio *B* has payoff $\max(K - S_T, 0) + S_T = \max(S_T, K)$ at *T*. Since both portfolios have the same payoff at *T* and no payoff between *t* and *T*, the no-arbitrage argument[3] dictates that they must have the same value at *t*. Hence, $c + K^{-r\tau} = p + S$.

If we rearrange the put-call parity equation into $c - p = S - K^{-r\tau}$, it will give us different insight. The portfolio on the left side of the equation—long a call and short a put—has the payoff $\max(S_T - K, 0) - \max(K - S_T, 0) = S_T - K$, which is the payoff of a forward with delivery price *K*. A forward with delivery price *K* has present value $S - K^{-r\tau}$. So we again have the put-call parity $c - p = S - K^{-r\tau}$. This expression shows that when the strike price $K = S^{r\tau}$ (forward price), a call has the same value as put; when $K < S^{r\tau}$, a call has higher value; and when $K > S^{r\tau}$, a put has higher value.

American v.s. European options

A. Since American options can be exercised at any time before maturity, they are often more valuable than European options with the same characteristics. But when the stock pays no dividend, the theoretical price for an American call and European call should be the same since it is never optimal to exercise the American call. Why should you never exercise an American call on a non-dividend paying stock before maturity?

Solution: There are a number of solutions to this popular problem. We present three arguments for the conclusion.

Argument 1. If you exercises the call option, you will only get the intrinsic value of the call $S - K$. The price of the American/European call also includes time value, which is positive for a call on a non-dividend paying stock. So the investor is better off selling the option than exercising it before maturity.

In fact, if we rearrange the put-call parity for European options, we have $c = S - K^{-r\tau} + p = (S - K) + (K - K^{-r\tau}) + p$. The value of a European call on a non-dividend paying stock includes three components: the first component is the intrinsic value $S - K$; the second component is the time value of the strike (if you exercise now,

[3] A set of transactions is an arbitrage opportunity if the initial investment ≤ 0; payoff ≥ 0; and at least one of the inequalities is strict.

you pay K now instead of K at the maturity date, which is lower in present value); and the third component is the value of the put, which is often considered to be a protection against falling stock price. Clearly the second and the third components are both positive. So the European call should be worth more than its intrinsic value. Considering that the corresponding American call is worth at least as much as the European call, it is worth more than its intrinsic value as well. As a result, it is not optimal to exercise the American call before maturity.

Argument 2. Let's compare two different strategies. In strategy 1, we exercise the call option[4] at time t ($t < T$) and receive cash $S-K$. Alternatively, we can keep the call, short the underlying stock and lend K dollars with interest rate r (the cash proceedings from the short sale, S, is larger than K). At the maturity date T, we exercise the call if it's in the money, close the short position and close the lending. Table 6.2 shows the cash flow of such a strategy:

It clearly shows that at time t, we have the same cash flow as exercising the call, $S-K$. But at time T, we always have positive cash flow as well. So this strategy is clearly better than exercising the call at time t. By keeping the call alive, the extra benefit can be realized at maturity.

Cash flow	t	T	
		$S_T \le K$	$S_T > K$
Call option	0	0	$S_T - K$
Short Stock	S	$-S_T$	$-S_T$
Lend K at t	$-K$	$Ke^{r\tau}$	$Ke^{r\tau}$
Total	$S-K$	$Ke^{r\tau} - S_T > 0$	$Ke^{r\tau} - K > 0$

Table 6.2 Payoff of an alternative strategy without exercising the call

Argument 3. Let's use a mathematical argument relying on risk-neutral pricing and **Jensen's inequality**—if $f(X)$ is a convex function,[5] then $E[f(X)] \ge f(E[X])$. From Figure 6.1, it's obvious that the payoff (if exercised when $S > K$) of a call option $C(S) = (S-K)^+$ is a convex function of stock price with property

$$C(\lambda S_1 + (1-\lambda)S_2) \le \lambda C(S_1) + (1-\lambda)C(S_2), \quad 0 < \lambda < 1.$$

[4] We assume $S > K$ in our discussion. Otherwise, the call surely should not be exercised.

[5] A function $f(X)$ is convex if and only if $f\left(\lambda x + (1-\lambda)y\right) \le \lambda f(x) + (1-\lambda)f(y)$, $0 < \lambda < 1$. If $f''(x) > 0, \forall x$, then $f(X)$ is convex.

Let $S_1 = S$ and $S_2 = 0$, then $C(\lambda S) \le \lambda C(S) + (1-\lambda)C(0) = \lambda C(S)$ since $C(0) = 0$.

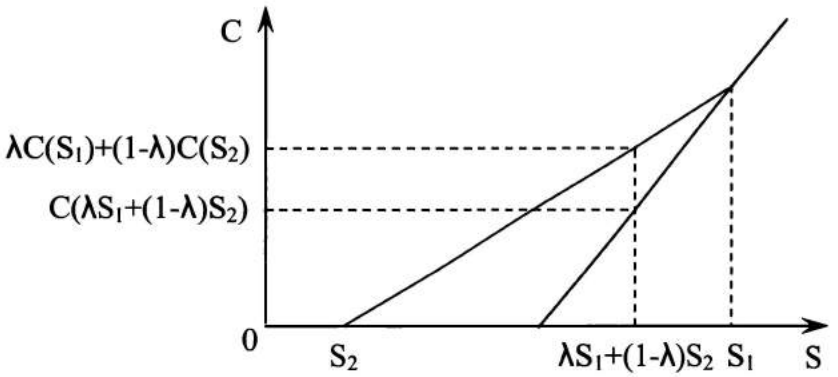

Figure 6.1 Payoff of a European call option

If the option is exercised at time t, the payoff at t is $C(S_t - K)$. If it is not exercised until maturity, the discounted expected payoff (to t) is $\tilde{E}\left[e^{-r\tau}C(S_T)\right]$ under risk-neutral measure. Under risk-neutral probabilities, we also have $\tilde{E}[S_T] = S_t e^{r\tau}$.

So $\tilde{E}\left[e^{-r\tau}C(S_T)\right] = e^{-r\tau}\tilde{E}\left[C(S_T)\right] \ge e^{-r\tau}C\left(\tilde{E}\left[S_T\right]\right) = e^{-r\tau}C\left(e^{r\tau}S_t\right)$,

where the inequality is from Jensen's inequality.

Let $S = e^{r\tau}S_t$ and $\lambda = e^{-r\tau}$, we have $C(\lambda S) = C(S_t) \le e^{-r\tau}C\left(e^{r\tau}s_t\right) \le e^{-r\tau}\tilde{E}\left[C(S_T)\right]$.

Since the discounted payoff $e^{-r\tau}\tilde{E}\left[C(S_T)\right]$ is no less than $C(S_t)$ for any $t \le T$ under the risk neutral measure, it is never optimal to exercise the option before expiration.

I should point out that the payoff of a put is also a convex function of the stock price. But it is often optimal to exercise an American put on a non-dividend paying stock. The difference is that $P(0) = K$, so it does not have the property that $P(\lambda S) \le \lambda P(S)$. In fact, $P(\lambda S) \ge \lambda P(S)$. So the argument for American calls does not apply to American puts.

Similar analysis can also show that early exercise of an American call option for dividend-paying stocks is never optimal except possibly for the time right before an ex-dividend date.

B. A European put option on a non-dividend paying stock with strike price \$80 is currently priced at \$8 and a put option on the same stock with strike price \$90 is priced at \$9. Is there an arbitrage opportunity existing in these two options?

Solution: In the last problem, we mentioned that the payoff of a put is a convex function in **stock price**. The price of a put option as a function of the **strike price** is a convex function as well. Since a put option with strike 0 is worthless, we always have $P(0)+\lambda P(K)=\lambda P(K)>P(\lambda K)$.

For this specific problem, we should have $8/9\times P(90)=8/9\times 9=8>P(80)$. Since the put option with strike price \$80 is currently price at 8, it is overpriced and we should short it. The overall arbitrage portfolio is to short 9 units of put with K = \$80 and long 8 units of put with K = 90. At time 0, the initial cash flow is 0. At the maturity date, we have three possible scenarios:

$S_T \geq 90$, payoff $= 0$ (No put is exercised.)

$90 > S_T \geq 80$, payoff $= 8\times(90-S_T)>0$ (Puts with $K=90$ are exercised.)

$S_T < 80$, payoff $= 8\times(90-S_T)-9\times(80-S_T)=S_T>0$ (All puts are exercised.)

The final payoff ≥ 0 with positive probability that payoff > 0. So it is clearly an arbitrage opportunity.

Black-Scholes-Merton differential equation

Can you write down the Black-Scholes-Merton differential equation and briefly explain how to derive it?

Solution: If the evolution of the stock price is a geometric Brownian motion, $dS=\mu S dt+\sigma S dW(t)$, and the derivative $V=V(S,t)$ is a function of S and t, then applying Ito's lemma yields:

$$dV=(\frac{\partial V}{\partial t}+\mu S\frac{\partial V}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2})dt+\sigma S\frac{\partial V}{\partial S}dW(t),$$ where $W(t)$ is a Brownian motion.

The Black-Scholes-Merton differential equation is a partial differential equation that should be satisfied by V: $\frac{\partial V}{\partial t}+rS\frac{\partial V}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}=rV$.

To derive the Black-Scholes-Merton differential equation, we build a portfolio with two components: long one unit of the derivative and short $\frac{\partial V}{\partial S}$ unit of the underlying stock.

Then the portfolio has value $\Pi=V-\frac{\partial V}{\partial S}S$ and the change of Π follows equation

$$
\begin{aligned}
d\Pi &= dV - \frac{\partial V}{\partial S} dS \\
&= (\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt + \sigma S \frac{\partial V}{\partial S} dW(t) - \frac{\partial V}{\partial S}(\mu S dt + \sigma S dW(t)) \\
&= (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt
\end{aligned}
$$

It is apparent that this portfolio is risk-free since it has no diffusion term. It should have risk-free rate of return as well: $d\Pi = r(V - \frac{\partial V}{\partial S} S)dt$. Combining these results we have

$$
(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt = r(V - \frac{\partial V}{\partial S} S)dt \Rightarrow \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV,
$$

which is the Black-Scholes-Merton differential equation.

The Black-Scholes-Merton differential equation is a special case of the discounted Feynman-Kac theorem. The discounted Feynman-Kac theorem builds the bridge between stochastic differential equations and partial differential equations and applies to all Ito processes in general:

Let X be an Ito process given by equation $dX(t) = \beta(t, X)dt + \gamma(t, X)dW(t)$ and $f(x)$ be a function of X. Define function $V(t,x) = E[e^{-r(T-t)} f(X_T) \mid X_t = x]$, then $V(t,x)$ is a martingale process that satisfies the partial differential equation

$$
\frac{\partial V}{\partial t} + \beta(t,x)\frac{\partial V}{\partial x} + \frac{1}{2}\gamma^2(t,x)\frac{\partial^2 V}{\partial x^2} = rV(t,x)
$$

and boundary condition $V(T,x) = f(x)$ for all x.

Under risk-neutral measure, $dS = rSdt + \sigma S dW(t)$. Let $S = X$, $\beta(t,X) = rS$ and $\gamma(t,X) = \sigma S$, then the discounted Feynman-Kac equation becomes the Black-Scholes-Merton differential equation $\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$.

Black-Scholes formula

The Black-Scholes formula for European calls and puts with continuous dividend yield y is:

$c = Se^{-y\tau}N(d_1) - Ke^{-r\tau}N(d_2)$ and $p = Ke^{-r\tau}N(-d_2) - Se^{-y\tau}N(-d_1)$,

$$d_1 = \frac{\ln(Se^{-y\tau}/K)+(r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}} = \frac{\ln(S/K)+(r-y+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$

where

$$d_2 = \frac{\ln(S/K)+(r-y-\sigma^2/2)\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau}$$

$N(x)$ is the cdf of the standard normal distribution and $N'(x)$ is the pdf of the standard normal distribution: $N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$ and $N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

If the underlying asset is a futures contract, then yield $y = r$. If the underlying asset is a foreign currency, then yield $y = r_f$, where r_f is the foreign risk-free interest rate.

A. What are the assumptions behind the Black-Scholes formula?

Solution: The original Black-Scholes formula for European calls and puts consists of the equations $c = SN(d_1) - Ke^{-r\tau}N(d_2)$ and $p = Ke^{-r\tau}N(-d_2) - SN(-d_1)$, which require the following assumptions:

1. The stock pays no dividends.
2. The risk-free interest rate is constant and known.
3. The stock price follows a geometric Brownian motion with constant drift μ and volatility σ: $dS = \mu Sdt + \sigma SdW(t)$.
4. There are no transaction costs or taxes; the proceeds of short selling can be fully invested.
5. All securities are perfectly divisible.
6. There are no risk-free arbitrage opportunities.

B. How can you derive the Black-Scholes formula for a European call on a non-dividend paying stock using risk-neutral probability measure?

Solution: The Black-Scholes formula for a European call on a non-dividend paying stock is

$c = SN(d_1) - Ke^{-r\tau}N(d_2)$, where $d_1 = \frac{\ln(S/K)+(r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}$ and $d_2 = d_1 - \sigma\sqrt{\tau}$.

Under the risk-neutral probability measure, the drift of stock price becomes the risk-free interest rate $r(t)$: $dS = r(t)Sdt + \sigma SdW(t)$. Risk-neutral measure allows the option to be priced as the discounted value of its expected payoff with the risk-free interest rate: $V(t) = E\left[e^{-\int_t^T r(u)du} V(T) \middle| S(t) \right]$, $0 \le t \le T$, where $V(T)$ is the payoff at maturity T.

When r is constant, the formula can be further simplified as $V(t) = e^{-r\tau} E\left[V(T) \middle| S(t) \right]$. Under risk-neutral probabilities, $dS = rSdt + \sigma SdW(t)$. Applying Ito's lemma, we get

$$d\left(\ln(S)\right) = (r - \sigma^2/2)dt + \sigma dW(t) \Rightarrow \ln S_T \sim N\left(\ln S + (r - \sigma^2/2)\tau, \sigma^2\tau\right).$$

So $S_T = Se^{(r-\sigma^2/2)\tau + \sigma\sqrt{\tau}\varepsilon}$, where $\varepsilon \sim N(0, 1)$. For a European option, we have

$$V(T) = \begin{cases} Se^{(r-\sigma^2/2)\tau+\sigma\sqrt{\tau}\varepsilon} - K, & \text{if } Se^{(r-\sigma^2/2)\tau+\sigma\sqrt{\tau}\varepsilon} > K \\ 0, & \text{otherwise} \end{cases}$$

$$Se^{(r-\sigma^2/2)\tau+\sigma\sqrt{\tau}\varepsilon} > K \Rightarrow \varepsilon > \frac{\ln(K/S) - (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}} = -d_2 \text{ and}$$

$$E\left[V(T) \mid S\right] = E\left[\max(S_T - K, 0) \mid S\right] = \int_{-d_2}^{\infty} \left(Se^{(r-\sigma^2/2)\tau+\sigma\sqrt{\tau}\varepsilon} - K \right) \frac{1}{\sqrt{2\pi}} e^{-\varepsilon^2/2} d\varepsilon$$

$$= Se^{r\tau} \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\left(\varepsilon - \sqrt{\tau}\sigma\right)^2/2} d\varepsilon - K \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\varepsilon^2/2} d\varepsilon$$

Let $\tilde{\varepsilon} = \varepsilon - \sigma\sqrt{\tau}$, then $d\varepsilon = d\tilde{\varepsilon}$, $\varepsilon = -d_2 \Rightarrow \tilde{\varepsilon} = -d_2 - \sigma\sqrt{\tau} = -d_1$ and we have

$$Se^{r\tau} \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\left(\varepsilon - \sqrt{\tau}\sigma\right)^2/2} d\varepsilon = Se^{r\tau} \int_{-d_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\tilde{\varepsilon}^2/2} d\tilde{\varepsilon} = Se^{r\tau} N\left(d_1\right),$$

$$K \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\varepsilon^2/2} d\varepsilon = K\left(1 - N(-d_2)\right) = KN\left(d_2\right)$$

$$\therefore E\left[V(T)\right] = Se^{r\tau} N\left(d_1\right) - KN(d_2) \text{ and } V(t) = e^{-r\tau} E\left[V(T)\right] = SN\left(d_1\right) - Ke^{-r\tau} N(d_2)$$

From the derivation process, it is also obvious that $1 - N(-d_2) = N(d_2)$ is the risk-neutral probability that the call option finishes in the money.

C. How do you derive the Black-Scholes formula for a European call option on a non-dividend paying stock by solving the Black-Scholes-Merton differential equation?

Solution: You can skip this problem if you don't have background in partial differential equations (PDE). One approach to solving the problem is to convert the Black-Scholes-Merton differential equation to a heat equation and then apply the boundary conditions to the heat equation to derive the Black-Scholes formula.

Let $y = \ln S$ ($S = e^y$) and $\tilde{\tau} = T - t$, then $\frac{\partial V}{\partial t} = -\frac{\partial V}{\partial \tilde{\tau}}$, $\frac{\partial V}{\partial S} = \frac{\partial V}{\partial y}\frac{dy}{dS} = \frac{1}{S}\frac{\partial V}{\partial y}$ and

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial V}{\partial S}\left(\frac{\partial V}{\partial S}\right) = \frac{\partial V}{\partial S}\left(\frac{1}{S}\frac{\partial V}{\partial y}\right) = \frac{-1}{S^2}\frac{\partial V}{\partial y} + \frac{1}{S}\frac{\partial V}{\partial S}\left(\frac{\partial V}{\partial y}\right) = \frac{-1}{S^2}\frac{\partial V}{\partial y} + \frac{1}{S^2}\frac{\partial^2 V}{\partial y^2}.$$ [6]

The Black-Scholes-Merton differential equation $\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rV = 0$ can be converted to $-\frac{\partial V}{\partial \tilde{\tau}} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial V}{\partial y} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial y^2} - rV = 0.$

Let $u = e^{r\tilde{\tau}}V$, the equation becomes $-\frac{\partial u}{\partial \tilde{\tau}} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial y} + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial y^2} = 0$.

Finally, let $x = y + \left(r - \frac{1}{2}\sigma^2\right)\tilde{\tau} = \ln S + \left(r - \frac{1}{2}\sigma^2\right)\tilde{\tau}$ and $\tau = \tilde{\tau}$, then $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial \tilde{\tau}} = \frac{\partial u}{\partial \tau} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial x}$, which transforms the equation to

$$-\frac{\partial u}{\partial \tau} - \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial x} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2}$$

So the original equation becomes a heat/diffusion equation $\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2}$. For heat equation $\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2}$, where $u = u(x,\tau)$ is a function of time τ and space variable x, with boundary condition $u(x,0) = u_0(x)$, the solution is

$$u(x,\tau) = \frac{1}{\sqrt{2\pi\tau\sigma}}\int_{-\infty}^{\infty} u_0(\psi)\exp\left(-\frac{(x-\psi)^2}{2\sigma^2\tau}\right)d\psi.$$ [7]

---

[6] The log is taken to convert the geometric Brownian motion to an arithmetic Brownian motion; $\tau = T - t$ is used to convert the equation from a backward equation to a forward equation with initial condition at $\tau = 0$ (the boundary condition at $t = T \Rightarrow \tau = 0$).

For European calls, the boundary condition is $u_0(S_T) = \max(S_T - K, 0)$.

$S = \exp\left(x - (r - 0.5\sigma^2)\tau\right)$. When $x = \psi$ and $\tau = 0$, $S_T = e^{\psi}$.

$$u(S,\tau) = u(x,\tau) = \frac{1}{\sqrt{2\pi\tau}\sigma}\int_{-\infty}^{\infty}\max\left(e^{\psi} - K, 0\right)\exp\left(-\frac{(x-\psi)^2}{2\sigma^2\tau}\right)d\psi$$

$$= \frac{1}{\sqrt{2\pi\tau}\sigma}\int_{\ln K}^{\infty}(e^{\psi} - K)\exp\left(-\frac{(x-\psi)^2}{2\sigma^2\tau}\right)d\psi$$

Let $\varepsilon = \frac{\psi - x}{\sigma\sqrt{\tau}}$, then $d\varepsilon = \frac{d\psi}{\sigma\sqrt{\tau}}$, $e^{\psi} = e^{x + \varepsilon\sigma\sqrt{\tau}}$, $\exp\left(-\frac{(x-\psi)^2}{2\sigma^2\tau}\right) = e^{-\varepsilon^2/2}$ and when

$\psi = \ln K$, $\varepsilon = \frac{\ln(K/S) - (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}} = -d_2$

$$\therefore\ u(S,\tau) = \int_{-d_2}^{\infty}\left(Se^{(r-\sigma^2/2)\tau + \sigma\sqrt{\tau}\varepsilon} - K\right)\frac{1}{\sqrt{2\pi}}e^{-\varepsilon^2/2}d\varepsilon$$

Now, it's clear that the equation for $u(S,\tau)$ is exactly the same as the equation for $E[V(T)|S]$ in question *B*. Hence, we have $V(S,t) = e^{-r\tau}u(S,\tau) = SN(d_1) - Ke^{-r\tau}N(d_2)$ as well.

D. Assume zero interest rate and a stock with current price at \$1 that pays no dividend. When the price hits level \$H ($H > 1$) for the first time you can exercise the option and receive \$1. What is this option worth to you today?

Solution: First let's use a brute-force approach to solve the problem by assuming that the stock price follows a geometric Brownian motion under risk-neutral measure: $dS = rSdt + \sigma SdW(t)$. Since $r = 0$, $dS = \sigma SdW(t) \Rightarrow d(\ln S) = -\frac{1}{2}\sigma^2 dt + \sigma dW(t)$. When $t = 0$, we have $S_0 = 1 \Rightarrow \ln(S_0) = 0$.

---

[7] The **fundamental solution** to heat equation $\frac{\partial u}{\partial \tau} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$ with initial condition $u_0(\psi) = f(\psi)$ is $u(x,t) = \int_{\psi=-\infty}^{\infty} p(x_t = x | x_0 = \psi) f(\psi) d\psi$, where $p(x_t = x \mid x_0 = \psi) = \frac{1}{\sqrt{2\pi t}}\exp\left\{-(x-\psi)^2/2t\right\}$.

For detailed discussion about heat equation, please refer to *The Mathematics of Financial Derivatives* by Paul Wilmott, Sam Howison, and Jeff Dewynne.

Hence, $\ln S = -\frac{1}{2}\sigma^2 t + \sigma W(t) \Rightarrow \frac{\ln S + \frac{1}{2}\sigma^2 t}{\sigma} = W(t)$ is a Brownian motion.

Whenever S reaches \$$H$, the payoff is \$1. Because the interest rate is 0, the discounted payoff is also \$1 under risk-neutral measure. So the value of the option is the probability that S ever reaches \$$H$, which is equivalent to the probability that $\ln S$ ever reaches $\ln H$. Again we can apply the exponential martingale $Z(t) = \exp\left\{\lambda W(t) - \frac{1}{2}\lambda^2 t\right\}$ as we did in Chapter 5: $E\left[Z(t)\right] = E\left[\exp\left\{\lambda \frac{\ln S + \frac{1}{2}\sigma^2 t}{\sigma} - \frac{1}{2}\lambda^2 t\right\}\right] = 1$.

To remove the terms including time t, we can set $\lambda = \sigma$ and the equation becomes $E\left[\exp(\ln S)\right] = 1$. The Let P be the probability that $\ln S$ ever reaches $\ln H$ (using $-\infty$ as the negative boundary for stopping time), we have

$P\exp(\ln H) + (1-P)\exp(-\infty) = P \times H = 1 \Rightarrow P = 1/H$.

So the probability that S ever reaches \$$H$ is $1/H$ and the price of the option should be \$$1/H$. Notice that S is a martingale under the risk-neutral measure;[8] but $\ln S$ has a negative drift. The reason is that $\ln S$ follows a (symmetrical) normal distribution, but S itself follows a lognormal distribution, which is positively skewed. As $T \to \infty$, although the expected value of S_T is 1, the probability that $S_T \geq 1$ actually approaches 0.

It is simpler to use a no-arbitrage argument to derive the price. In order to pay \$1 when the stock price hits \$$H$, we need to buy $1/H$ shares of the stock (at \$$1/H$). So the option should be worth no more than \$$1/H$. Yet if the option price C is less than \$$1/H$ ($C < 1/H \Rightarrow CH < 1$), we can buy an option by borrowing C shares of the stock. The initial investment is 0. Once the stock price hits \$$H$, we will excise the option and return the stock by buying C shares at price \$$H$, which gives payoff $1 - CH > 0$. That means we have no initial investment, yet we have possible positive future payoff, which is contradictory to the no arbitrage argument. So the price cannot be less than \$$1/H$. Hence, the price is exactly \$$1/H$.

E. Assume a non-dividend paying stock follows a geometric Brownian motion. What is the value of a contract that at maturity T pays the inverse of the stock price observed at the maturity?

---

[8] Once we recognize that S is a martingale under the risk neutral measure, we do not need the assumption that S follows a geometric Brownian motion. S has two boundaries for stopping: 0 and H. The boundary conditions are $f(0) = 0$ and $f(H) = 1$. Using the martingale, the probability that it will ever reaches H is $P \times H + (1-P) \times 0 = S_0 = 1 \Rightarrow P = 1/H$.

Solution: Under risk-neutral measure $dS = rSdt + \sigma SdW(t)$. Apply Ito's lemma to $V = \frac{1}{S}$:

$$dV = \left(\frac{\partial V}{\partial S}rS + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial V}{\partial S}\sigma SdW(t)$$
$$= \left(-\frac{1}{S^2}rS + 0 + \frac{1}{2}\frac{2}{S^3}\sigma^2 S^2\right)dt - \frac{1}{S^2}\sigma SdW(t) = (-r+\sigma^2)Vdt - \sigma VdW(t)$$

So V follows a geometric Brownian motion as well and we can apply Ito's lemma to $\ln V$:

$$d(\ln V) = \left(\frac{V}{V}(-r+\sigma^2) + 0 - \frac{1}{2}\frac{V^2}{V^2}\sigma^2\right)dt + \frac{V}{V}\sigma dW(t) = \left(-r + \frac{1}{2}\sigma^2\right)dt - \sigma dW(t).$$

Hence, $\ln(V_T) \sim \ln(V_t) + N\left((-r+\frac{1}{2}\sigma^2)\tau, \sigma^2\tau\right)$ and $E[V_T] = E[e^{\ln V_T}] = \frac{1}{S_t}e^{-r\tau+\sigma^2\tau}$.

Discounting the payoff by $e^{-r\tau}$, we have $V = e^{-r\tau}E[V_T] = \frac{1}{S_t}e^{-2r\tau+\sigma^2\tau}$.

6.2. The Greeks

All Greeks are first-order or second-order partial derivatives of the option price with respect to different underlying factors, which are used to measure the risks—as well as potential returns—of the financial derivative. The following Greeks for a derivative f are routinely used by financial institutions:

Delta: $\Delta = \frac{\partial f}{\partial S}$; **Gamma:** $\Gamma = \frac{\partial^2 f}{\partial S^2}$; **Theta:** $\Theta = \frac{\partial f}{\partial t}$; **Vega:** $\upsilon = \frac{\partial f}{\partial \sigma}$; **Rho:** $\rho = \frac{\partial f}{\partial r}$

Delta

For a European call with dividend yield y: $\Delta = e^{-y\tau}N(d_1)$

For a European put with dividend yield y: $\Delta = -e^{-y\tau}[1 - N(d_1)]$

A. What is the delta of a European call option on a non-dividend paying stock? How do you derive the delta?

Solution: The delta of a European call on a non-dividend paying stock has a clean expression: $\Delta = N(d_1)$. For the derivation, though, many make the mistake by treating

$N(d_1)$ and $N(d_2)$ as constants in the call pricing formula $c = SN(d_1) - Ke^{-r\tau}N(d_2)$ and simply taking the partial derivative on S to yield $N(d_1)$. The derivation step is actually more complex than that since both $N(d_1)$ and $N(d_2)$ are functions of S through d_1 and d_2. So the correct partial derivative is $\frac{\partial c}{\partial S} = N(d_1) + S \times \frac{\partial}{\partial S} N(d_1) - Ke^{-r\tau} \frac{\partial}{\partial S} N(d_2)$.

Take the partial derivative with respect to S for $N(d_1)$ and $N(d_2)$[9]:

$$\frac{\partial}{\partial S} N(d_1) = N'(d_1) \frac{\partial}{\partial S} d_1 = \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2} \times \frac{1}{S\sigma\sqrt{\tau}} = \frac{1}{S\sigma\sqrt{2\pi\tau}} e^{-d_1^2/2}$$

$$\frac{\partial}{\partial S} N(d_2) = N'(d_2) \frac{\partial}{\partial S} d_2 = \frac{1}{\sqrt{2\pi}} e^{-d_2^2/2} \times \frac{1}{S\sigma\sqrt{\tau}} = \frac{1}{S\sigma\sqrt{2\pi\tau}} e^{-(d_1 - \sigma\sqrt{\tau})^2/2}$$

$$= \frac{1}{S\sigma\sqrt{2\pi\tau}} e^{-d_1^2/2} e^{\sigma\sqrt{\tau} d_1} e^{-\sigma^2\tau/2} = \frac{1}{S\sigma\sqrt{2\pi\tau}} e^{-d_1^2/2} \times \frac{S}{K} e^{r\tau}$$

So we have $\frac{\partial}{\partial S} N(d_2) = \frac{S}{K} e^{r\tau} N(d_1) \Rightarrow S \times \frac{\partial}{\partial S} N(d_1) - Ke^{-r\tau} \frac{\partial}{\partial S} N(d_2) = 0$. Hence, the last two components of $\frac{\partial c}{\partial S}$ cancel out and $\frac{\partial c}{\partial S} = N(d_1)$.

B. What is your estimate of the delta of an at-the-money call on a stock without dividend? What will happen to delta as the at-the-money option approaches maturity date?

Solution: For an at-the-money European call, the stock price equals the strike price. $S = K \Rightarrow d_1 = \frac{(r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} = (\frac{r}{\sigma} + \frac{\sigma}{2})\sqrt{\tau} > 0$ and $\Delta = N(d_1) > 0.5$. As shown in Figure 6.2, all at-the-money call options indeed have $\Delta > 0.5$ and the longer the maturity, the higher the Δ. As $T - t \to 0$, $(\frac{r}{\sigma} + \frac{\sigma}{2})\sqrt{\tau} \to 0 \Rightarrow N(d_1) = N(0) = 0.5$, which is also shown in Figure 6.2 ($T = 10$ days). The same argument is true for calls on stock with continuous dividend rate y if $r > y$.

Figure 6.2 also shows that when S is large ($S >> K$), Δ approaches 1. Furthermore, the shorter the maturity, the faster the delta approaches 1. On the other hand, if S is small ($S << K$), Δ approaches 0 and the shorter the maturity, the faster the delta approaches 0.

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[9] $d_2 = d_1 - \sigma\sqrt{\tau} \Rightarrow N'(d_2) = \frac{S}{K} e^{(r-y)\tau} N'(d_1)$, $\frac{\partial d_2}{\partial S} = \frac{\partial d_1}{\partial S}$

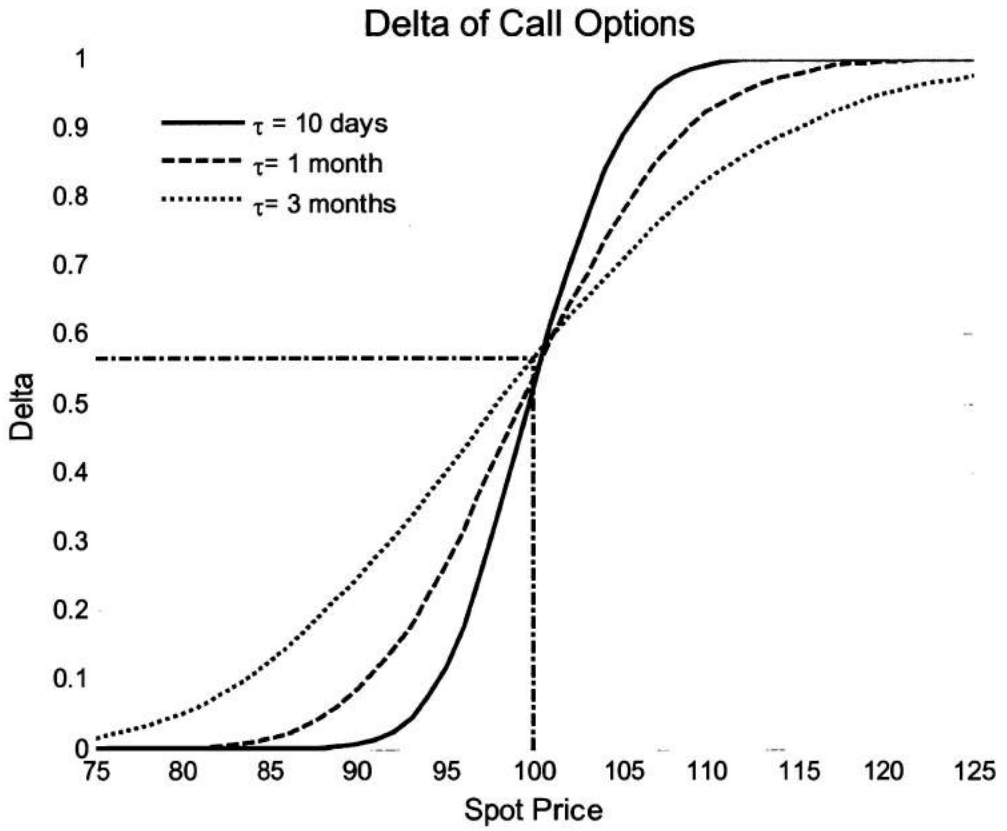

Figure 6.2 Variation of delta of a European call option with respect to S and T. $K = 100$, $r = 0.05$, $\sigma = 0.25$.

C. You just entered a long position for a European call option on GM stock and decide to dynamically hedge the position to eliminate the risk from the fluctuation of GM stock price. How will you hedge the call option? If after your hedge, the price of GM has a sudden increase, how will you rebalance your hedging position?

Solution: Since $d_1 = \dfrac{\ln(S/K)+(r-y+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}$ and $\Delta = e^{-y\tau}N(d_1)$ is a monotonously increasing function of d_1, we have $S\uparrow \Rightarrow d_1\uparrow \Rightarrow \Delta\uparrow$.

One hedging method is delta hedging, for which we short $\Delta = e^{-y\tau}N(d_1)$ shares of stock for each unit of call option to make the portfolio delta-neutral. Since Δ shares of GM stock costs more than one unit of GM option, we also need to invest cash (if the option price exactly follows the Black-Scholes formula, we need to lend $\$Ke^{-r\tau}N(d_2)$ for each

unit of option) in the money market. If there is a sudden increase in S, d_1 increases and Δ increases as well. That means we need to short more stock and lend more cash ($Ke^{-r\tau}N(d_2)$ also increases).

The delta hedge only replicates the value and the slope of the option. To hedge the curvature of the option, we will need to hedge gamma as well.

D. Can you estimate the value of an at-the-money call on a non-dividend paying stock? Assume the interest rate is low and the call has short maturity.

Solution: When $S = K$, we have $c = S\left(N(d_1) - e^{-r\tau}N(d_2)\right)$. In a low-interest environment, $r \approx 0$ and $e^{-r\tau} \approx 1$, so $c \approx S\left(N(d_1) - N(d_2)\right)$.

We also have $N(d_1) - N(d_2) = \int_{d_2}^{d_1} \frac{1}{\sqrt{2\pi}} e^{-1/2x^2} dx$,

where $d_2 = (\frac{r}{\sigma} - \frac{\sigma}{2})\sqrt{\tau}$ and $d_1 = (\frac{r}{\sigma} + \frac{\sigma}{2})\sqrt{\tau}$.

For a small r, a typical σ for stocks (< 40% per year) and a short maturity (< 3 months), both d_2 and d_1 are close to 0. For example, if $r = 0.03$, $\sigma = 0.3$, and $\tau = 1/6$ year, then $d_2 = -0.02$ and $e^{-1/2d_2^2} = 0.98$.

$$\therefore N(d_1) - N(d_2) \approx \frac{1}{\sqrt{2\pi}}(d_1 - d_2) = \frac{\sigma\sqrt{\tau}}{\sqrt{2\pi}} \approx 0.4\sigma\sqrt{T-t} \Rightarrow c \approx 0.4\sigma S\sqrt{\tau}.$$

In practice, this approximation is used by some volatility traders to estimate the implied volatility of an at-the-money option.

(The approximation $e^{-1/2x^2} \approx 1$ causes a small overestimation since $e^{-1/2x^2} < 1$; but the approximation $-e^{-r\tau}K \approx -K$ causes a small underestimation. To some extent, the two opposite effects cancel out and the overall approximation is fairly accurate.)

Gamma

For a European call/put with dividend yield y: $\Gamma = \frac{N'(d_1)e^{-y\tau}}{S_0\sigma\sqrt{\tau}}$

What happens to the gamma of an at-the-money European option when it approaches its maturity?

Solution: From the put-call parity, it is obvious that a call and a put with identical characteristics have the same gamma (since $\Gamma = 0$ for both the cash position and the underlying stock). Taking the partial derivative of the Δ of a call option with respect to S, we have $\Gamma = \frac{N'(d_1)e^{-y\tau}}{S\sigma\sqrt{\tau}}$, where $N'(d_1) = \frac{1}{\sqrt{2\pi}} e^{-1/2d_1^2}$.

So for plain vanilla call and put options, gamma is **always positive**.

Figure 6.3 shows that gamma is high when options are at the money, which is the stock price region that Δ changes rapidly with S. If $S << K$ or $S >> K$ (deep in the money or out of the money), gamma approaches 0 since Δ stays constant at 1 or 0.

The gamma of options with shorter maturities approaches 0 much faster than options with longer maturities as S moves away from K. So for deep in-the-money or deep out-of-the-money options, longer maturity means higher gamma. In contrast, if the stock prices are close to the strike price (at the money) as the maturity nears, the slope of delta for an at-the-money call becomes steeper and steeper. So for options close to the strike price, shorter-term options have higher gammas.

As $\tau \to 0$, an at-the-money call/put has $\Gamma \to \infty$ (Δ becomes a step function). This can be shown from the formula of gamma for a European call/put with no dividend, $\Gamma = \frac{N'(d_1)}{S\sigma\sqrt{\tau}}$:

When $S = K$, $d_1 = \lim_{\tau \to 0}(\frac{r}{\sigma} + \frac{\sigma}{2})\sqrt{\tau} \to 0 \Rightarrow \lim_{\tau \to 0} N'(d_1) \to \frac{1}{\sqrt{2\pi}}$. The numerator is $1/\sqrt{2\pi}$; yet the denominator has a limit $\lim_{\tau \to 0} S\sigma\sqrt{\tau} \to 0$, so $\Gamma \to \infty$. In other words, When $t = T$, delta becomes a step function. This phenomenon makes hedging at-the-money options difficult when $t \to T$ since delta is extremely sensitive to changes in S.

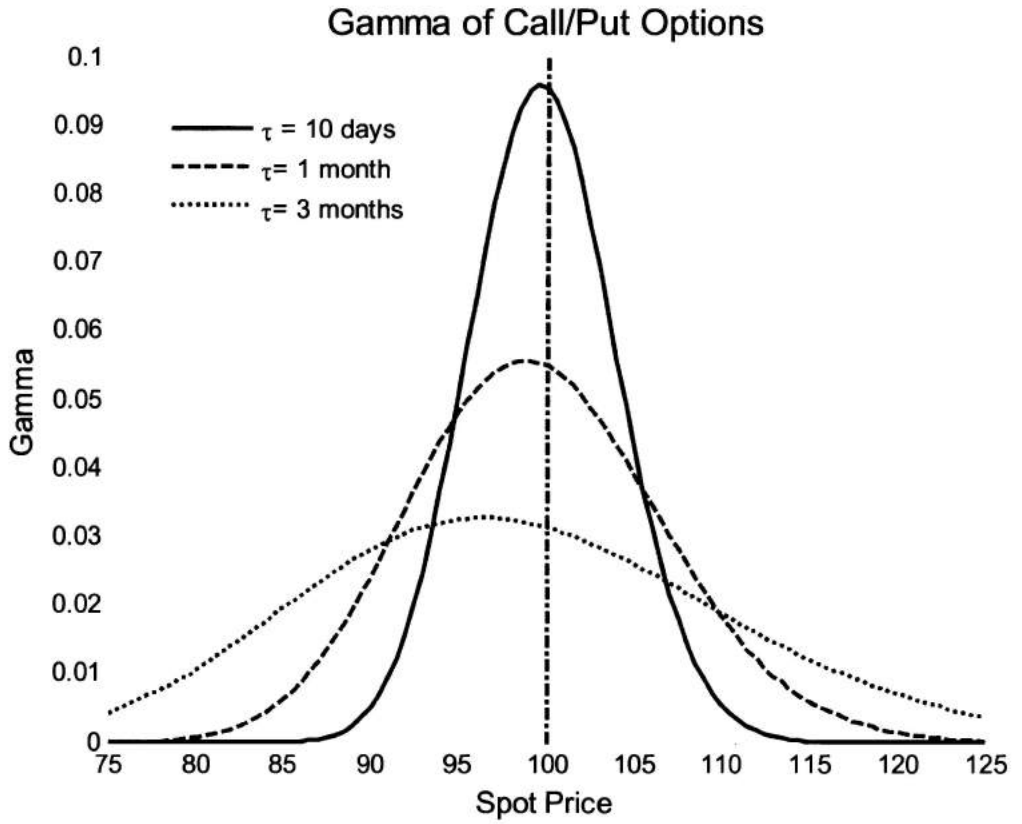

Figure 6.3 Variation of gamma of a European call option with respect to S and T. $K = 100$, $r = 0.05$, $\sigma = 0.25$.

Theta

For a European call option: $\Theta = -\dfrac{SN'(d_1)\sigma e^{-y\tau}}{2\sqrt{\tau}} + ySe^{-y\tau}N(d_1) - rKe^{-r\tau}N(d_2)$

For a European put option: $\Theta = -\dfrac{SN'(d_1)\sigma e^{-y\tau}}{2\sqrt{\tau}} - ySe^{-y\tau}N(-d_1) + rKe^{-r\tau}N(-d_2)$

When there is no dividend, the theta for a European call option is simplified to $\Theta = -\dfrac{SN'(d_1)}{2\sqrt{\tau}} - rKe^{-r\tau}N(d_2)$, which is always negative. As shown in Figure 6.4, when $S << K$, $N(d_2) \approx 0$ and $N'(d_1) \approx 0$. Hence, $\Theta \to 0$. When $S >> K$, $N(d_2) \approx 1$ and

$N'(d_1) \approx 0$. Hence, $\Theta \to -rKe^{-r\tau}$. When $S \approx K$, Θ has large negative value and the smaller the τ, the more negative the Θ.

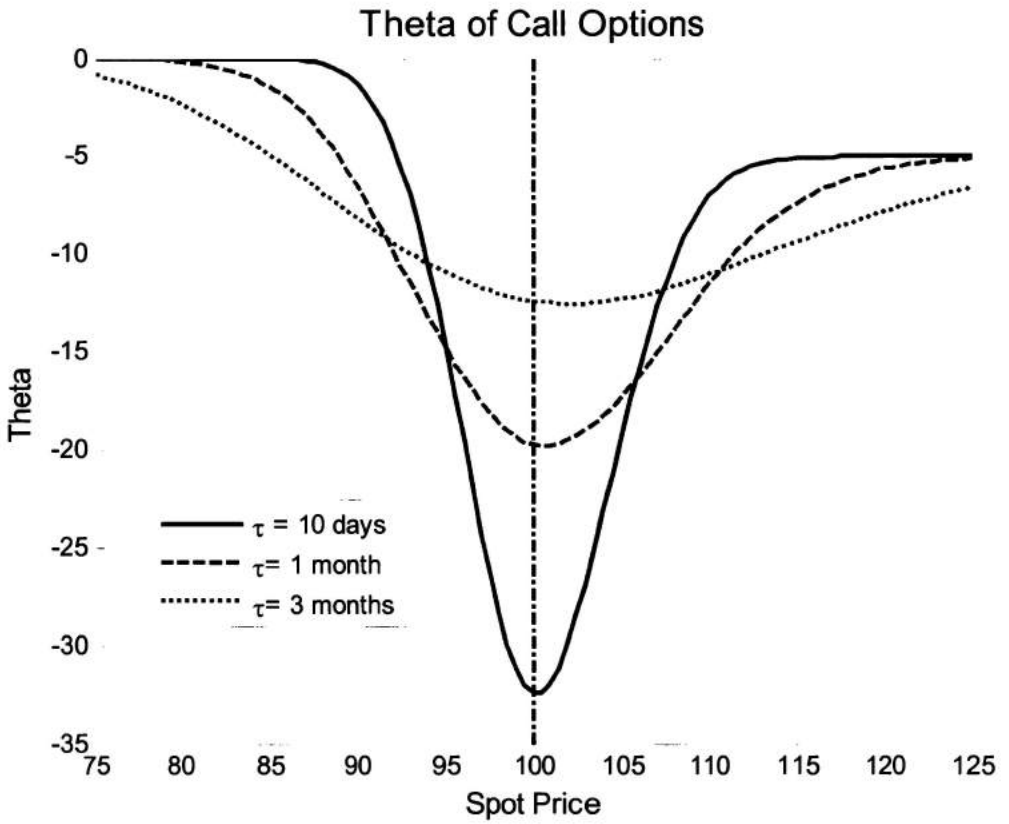

Figure 6.4 Variation of theta of a European call option with respect to S and T. $K = 100$, $\sigma = 0.25$, $r = 0.05$

A. When will a European option have positive theta?

Solution: For American options as well as European calls on non-dividend paying assets, theta is always negative. But for deep in-the-money European puts, their values may increase as t approaches T if all other factors remain the same, so they may have positive theta.

A put option on a non-dividend paying asset has $\Theta = -\frac{SN'(d_1)\sigma}{2\sqrt{\tau}} + rKe^{-r\tau}N(-d_2)$. If the put option is deep in-the-money ($S << K$), then $N'(d_1) \approx 0$ and $N(-d_2) \approx 1$. Hence,

$\Theta \approx rKe^{-r\tau} > 0$. That's also the reason why it can be optimal to exercise a deep in-the-money American put before maturity.

For deep in-the-money European call options with high dividend yield, the theta can be positive as well. If a call option with high dividend yield is deep in-the-money ($S >> K$), $N(d_1) \approx N(d_2) \approx 1$, $N'(d_1) \approx 0$, so the component $ySe^{-y\tau}N(d_1)$ can make Θ positive.

B. You just entered a long position for a call option on GM and hedged the position by shorting GM shares to make the portfolio delta neutral. If there is an immediate increase or decrease in GM's stock price, what will happen to the value of your portfolio? Is it an arbitrage opportunity? Assume that GM does not pay dividends.

Solution: A position in the underlying asset has zero gamma. So the portfolio is delta-neutral and long gamma. Therefore, either an immediate increase or decrease in the GM stock price will increase the portfolio value. The convexity (positive gamma) enhances returns when there is a large move in the stock price in either direction.

Nevertheless, it is not an arbitrage opportunity. It is a trade-off between gamma and theta instead. From the Black-Scholes-Merton differential equation, the portfolio V satisfies the equation $\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = \Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = rV$. For a delta-neutral portfolio, we have $\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = rV$. This indicates that gamma and theta often have opposite signs. For example, when an at-the-money call approaches maturity, gamma is large and positive, so theta is large and negative. Our delta neutral portfolio has positive gamma and negative theta. That means if the price does not move, the passage of time will result in a lower portfolio value unless we rebalance. So the portfolio does not provide an arbitrage opportunity.

Vega

For European options: $\upsilon = \frac{\partial c}{\partial \sigma} = \frac{\partial p}{\partial \sigma} = Se^{-y\tau}\sqrt{\tau}N'(d_1)$

At-the-money options are most sensitive to volatility change, so they have higher vegas than either in-the-money or out-of-the-money options. The vegas of all options decrease as time to expiration becomes shorter ($\sqrt{\tau} \to 0$) since a long-term option is more sensitive to change in volatility.

A. Explain implied volatility and volatility smile. What is the implication of volatility

smile for the Black-Scholes pricing model?

Solution: Implied volatility is the volatility that makes the model option price equal to the market option price. Volatility smile describes the relationship between the implied volatility of the options and the strike prices for a given asset. For currency options, implied volatilities tend to be higher for in-the-money and out-of-the-money options than for at-the-money options. For equity, volatility often decreases as the strike price increases (also called volatility skew). The Black-Scholes model assumes that the asset price follows a lognormal distribution with constant volatility. In reality, volatilities are neither constant nor deterministic. In fact, the volatility is a stochastic process itself. Furthermore, there may be jumps in asset prices.

B. You have to price a European call option either with a constant volatility 30% or by drawing volatility from a random distribution with a mean of 30%. Which option would be more expensive?

Solution: Many would simply argue that stochastic volatility makes the stock price more volatile, so the call price is more valuable when the volatility is drawn from a random distribution. Mathematically, the underlying argument is that the price of a European call option is a convex function of volatility and as a result $c\left(E[\sigma]\right) \le E\left[c(\sigma)\right]$, where σ is the random variable representing volatility and c is the call option price. Is the underlying argument correct? It's correct in most, but not all, cases. If the call price c is always a convex function of σ, then $\frac{\partial^2 c}{\partial \sigma^2} \ge 0$. $\frac{\partial c}{\partial \sigma}$ is the Vega of the option. For a European call option,

$$\upsilon = \frac{\partial c}{\partial \sigma} = S\sqrt{\tau}N'(d_1) = \frac{S\sqrt{\tau}}{\sqrt{2\pi}}\exp\left(-d_1^2/2\right).$$

The secondary partial derivative $\frac{\partial^2 c}{\partial \sigma^2}$ is called Volga. For a European call option,

$$\frac{\partial^2 c}{\partial \sigma^2} = \frac{S\sqrt{\tau}}{\sqrt{2\pi}}\exp\left(-d_1^2/2\right)\frac{d_1 d_2}{\sigma} = \upsilon\frac{d_1 d_2}{\sigma}.$$

υ is always positive. For most out-of-the-money call options, both d_1 and d_2 are negative; for most in-the-money call options, both d_1 and d_2 are positive. So $d_1 d_2 > 0$ in most cases and c is a convex function of σ when $d_1 d_2 > 0$. But theoretically, we can have conditions that $d_1 > 0$ and $d_2 < 0$ and $\frac{\partial^2 c}{\partial \sigma^2} < 0$ when the option is close to being

at-the-money. So the function is not always convex. In those cases, the option with constant volatility may have a higher value.

C. The Black-Scholes formula for non-dividend paying stocks assumes that the stock follows a geometric Brownian motion. Now assume that you don't know the stochastic process followed by the stock price, but you have the European call prices for all (continuous) strike prices K. Can you determine the risk-neutral probability density function of the stock price at time T?

Solution: The payoff a European call at its maturity date is $Max(S_T - K, 0)$. Therefore under risk-neutral measure, we have $c = e^{-r\tau}\int_K^\infty (s-K) f_{S_T}(s)ds$, where $f_{S_T}(s)$ is the probability density function of S_T under the risk-neutral probability measure. Taking the first and second derivatives of c with respect to K,[10] we have

$$\begin{aligned}\frac{\partial c}{\partial K} &= e^{-r\tau}\frac{\partial}{\partial K}\int_K^\infty (s-K) f_{S_T}(s)ds \\ &= e^{-r\tau}\int_K^\infty \frac{\partial (s-K)}{\partial K} f_{S_T}(s)ds - e^{-r\tau}(K-K)\times 1 \\ &= e^{-r\tau}\int_K^\infty -f_{S_T}(s)ds\end{aligned}$$

and $\dfrac{\partial^2 c}{\partial K^2} = \dfrac{\partial}{\partial K}\left(\dfrac{\partial c}{\partial K}\right) = e^{-r\tau}\dfrac{\partial}{\partial K}\int_K^\infty -f_{S_T}(s)ds = e^{-r\tau} f_{S_T}(K)$.

Hence the risk-neutral probability density function is $f_{S_T}(K) = e^{r\tau}\dfrac{\partial^2 c}{\partial K^2}$.

6.3. Option Portfolios and Exotic Options

In addition to the pricing and properties of vanilla European and American options, you may be expected to be familiar with the construction and payoff of basic option-based trading strategies—covered call, protective put, bull/bear spread, butterfly spread, straddle, etc. Furthermore, if you are applying for a derivatives-related position, you

[10] To calculate the derivatives requires the Leibniz integral rule, a formula for differentiating a definite integral whose limits are functions of the differential variable:

$$\frac{\partial}{\partial z}\int_{a(z)}^{b(z)} f(x,z)dx = \int_{a(z)}^{b(z)} \frac{\partial f(x,z)}{\partial z}dx + f(b(z),z)\frac{\partial b}{\partial z} - f(a(z),z)\frac{\partial a}{\partial z}$$

should also have a good understanding of pricing and hedging of some of the common exotic derivatives—binary option, barrier option, Asian option, chooser option, etc.

Bull spread

What are the price boundaries for a bull call spread?

Solution: A bull call spread is a portfolio with two options: long a call c_1 with strike K_1 and short a call c_2 with strike K_2 ($K_1 < K_2$). The cash flow of a bull spread is summarized in table 6.3.

Cash flow	Time 0	Maturity T		
		$S_T \le K_1$	$K_1 < S_T < K_2$	$S_T \ge K_2$
Long c_1	$-c_1$	0	$S_T - K_1$	$S_T - K_1$
Short c_2	c_2	0	0	$-(S_T - K_2)$
Total	$c_2 - c_1 < 0$	0	$S_T - K_1$	$K_2 - K_1$

Table 6.3 Cash flows of a bull call spread.

Since $K_1 < K_2$, the initial cash flow is negative. Considering that the final payoff is bounded by $K_2 - K_1$, the price of the spread, $c_1 - c_2$, is bounded by $e^{-rT}(K_2 - K_1)$. Besides, the payoff is also bounded by $\frac{K_2 - K_1}{K_2} S_T$, so the price is also bounded by $\frac{K_2 - K_1}{K_2} S$.

Straddle

Explain what a straddle is and when you want to purchase a straddle.

Solution: A straddle includes long positions in both a call option and a put option with the same strike price K and maturity date T on the same stock. The payoff of a long straddle is $|S_T - K|$. So a straddle may be used to bet on large stock price moves. In practice, a straddle is also used as a trading strategy for making bets on volatility. If an investor believes that the realized (future) volatility should be much higher than the implied volatility of call and put options, he or she will purchase a straddle. For example,

the value of an at-the-money call or put is almost a linear function of volatility. If the investor purchases an at-the-money straddle, both the call and the put options have the price $c \approx p \approx 0.4\sigma_i S\sqrt{\tau}$, where σ_i is the implied volatility. If the realized volatility $\sigma_r > \sigma_i$, both options are undervalued. When the market prices converge to the prices with the realized volatility, both the call and the put will become more valuable.

Although initially a straddle with an at-the-money call and an at-the-money put ($K = S$) has a delta close to 0, as the stock price moves away from the strike price, the delta is no longer close to 0 and the investor is exposed to stock price movements. So a straddle is not a pure bet on stock volatility. For a pure bet on volatility, it is better to use volatility swaps or variance swaps.[11] For example, a variance swap pays $N \times (\sigma_r^2 - K_{\text{var}})$, where N is the notional value, σ_r^2 is the realized variance and K_{var} is the strike for the variance.

Binary options

What is the price of a binary (cash-or-nothing digital) European call option on a non-dividend paying stock if the stock price follows a geometric Brownian motion? How would you hedge a cash-or-nothing call option and what's the limitation of your hedging strategy?

Solution: A cash-or-nothing call option with strike price K pays \$1 if the asset price is above the strike price at the maturity date, otherwise it pays nothing. The price of the option is $c_B = e^{-r\tau}N(d_2)$ if the underlying asset is a non-dividend paying stock. As we have discussed in the derivation of the Black-Scholes formula, $N(d_2)$ is the probability that a vanilla call option finishes in the money under the risk-neutral measure. So its discounted value is $e^{-r\tau}N(d_2)$.

Theoretically, a cash-or-nothing call option can be hedged using the standard delta hedging strategy. Since $\Delta = \frac{\partial c_B}{\partial S} = e^{-r\tau}N'(d_2)\frac{1}{S\sigma\sqrt{\tau}}$, a long position in a cash-or-nothing call option can be hedged by shorting $e^{-r\tau}N'(d_2)\frac{1}{S\sigma\sqrt{\tau}}$ shares (and a risk-free money market position). Such a hedge works well when the difference between S and K is large and τ is not close to 0. But when the option is approaching maturity T ($\tau \to 0$)

[11] For detailed discussion about volatility swaps, please refer to the paper "More Than You Ever Wanted to Know about Volatility Swaps" by Kresimir Demeterfi, et al. The paper shows that a variance swap can be approximated by a portfolio of straddles with proper weights inversely proportional to $1/k^2$.

and the stock price S is close to K, Δ is extremely volatile[12] and small changes in the stock price cause very large changes in Δ. In these cases, it is practically impossible to hedge a cash-or-nothing call option by delta hedging.

We can also approximate a digital option using a bull spread with two calls. If call options are available for all strike prices and there are no transaction costs, we can long $1/2\varepsilon$ call options with strike price $K-\varepsilon$ and short $1/2\varepsilon$ call options with strike price $K+\varepsilon$. The payoff of the bull spread is the same as the digital call option if $S_T \le K-\varepsilon$ (both have payoff 0) or $S_T \ge K+\varepsilon$ (both have payoff \$1). When $K-\varepsilon < S_T < K+\varepsilon$, their payoffs are different. Nevertheless, if we set $\varepsilon \to 0$, such a strategy will exactly replicate the digital call. So it provides another way of hedging a digital call option. This hedging strategy suffers its own drawback. In practice, not all strike prices are traded in the market. Even if all strike prices were traded in the market, the number of options needed for hedging, $1/2\varepsilon$, will be large in order to keep ε small.

Exchange options

How would you price an exchange call option that pays $\max\left(S_{T,1}-S_{T,2},\,0\right)$ at maturity. Assume that S_1 and S_2 are non-dividend paying stocks and both follow geometric Brownian motions with correlation ρ.

Solution: The solution to this problem uses change of numeraire. Numeraire means a unit of measurement. When we express the price of an asset, we usually use the local currency as the numeraire. But for modeling purposes, it is often easier to use a different asset as the numeraire. The only requirement for a numeraire is that it must always be positive.

The payoff of the exchange option depends on both $S_{T,1}$ (price of S_1 at maturity date T) and $S_{T,2}$ (price of S_2 at T), so it appears that we need two geometric Brownian motions:

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dW_{t,1}$$

$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dW_{t,2}$$

Yet if we use S_1 as the numeraire, we can convert the problem to just one geometric Brownian motion. The final payoff is $\max\left(S_{T,2}-S_{T,1},0\right)=S_{T,1}\max\left(\frac{S_{T,2}}{S_{T,1}}-1,0\right)$. When

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[12] $S \to K$ and $\tau \to 0 \Rightarrow \ln(S/K) \to 0 \Rightarrow d_2 \to (r/\sigma + 0.5\sigma)\sqrt{\tau} \to 0 \Rightarrow \Delta \to \frac{1}{\sqrt{2\pi}}\frac{e^{-rt}}{S\sigma\sqrt{\tau}} \to \infty$.

S_1 and S_2 are geometrical Browian motions, $f = \frac{S_2}{S_1}$ is a geometric Brownian motion as well. One intuitive explanation is that both $\ln S_1$ and $\ln S_2$ follow normal distributions, so $\ln f = \ln S_2 - \ln S_1$ follows a normal distribution as well and f follows a lognormal distribution. More rigorously, we can apply the Ito's lemma to $f = \frac{S_2}{S_1}$:

$$\frac{\partial f}{\partial S_1} = \frac{-S_2}{S_1^2}, \frac{\partial f}{\partial S_2} = \frac{1}{S_1}, \frac{\partial^2 f}{\partial S_1^2} = \frac{2S_2}{S_1^3}, \frac{\partial^2 f}{\partial S_2^2} = 0, \frac{\partial^2 f}{\partial S_1 \partial S_2} = \frac{-1}{S_1^2}$$

$$\begin{aligned} df &= \frac{\partial f}{\partial S_1} dS_1 + \frac{\partial f}{\partial S_2} dS_2 + \frac{1}{2}\frac{\partial^2 f}{\partial S_1^2}(dS_1)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial S_2^2}(dS_2)^2 + \frac{\partial^2 f}{\partial S_1 \partial S_2} dS_1 dS_2 \\ &= -\mu_1 \frac{S_2}{S_1} dt - \sigma_1 \frac{S_2}{S_1} dW_{t,1} + \mu_2 \frac{S_2}{S_1} dt + \sigma_2 \frac{S_2}{S_1} dW_{t,2} + \sigma_1^2 \frac{S_2}{S_1} dt - \rho\sigma_1\sigma_2 \frac{S_2}{S_1} dt \\ &= (\mu_2 - \mu_1 + \sigma_1^2 - \rho\sigma_1\sigma_2) f dt - \sigma_1 f dW_{t,1} + \sigma_2 f dW_{t,2} \\ &= (\mu_2 - \mu_1 + \sigma_1^2 - \rho\sigma_1\sigma_2) f dt + \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2} \times f dW_{t,3} \end{aligned}$$

To make $f = \frac{S_2}{S_1}$ a martingale, set $\mu_2 - \mu_1 + \sigma_1^2 - \rho\sigma_1\sigma_2 = 0$ and we have $\tilde{E}\left[\frac{S_{T,2}}{S_{T,1}}\right] = \frac{S_2}{S_1}$, and $\frac{S_{t,2}}{S_{t,1}}$ is a martingale under the new measure. The value of the exchange option using S_1 as the numeraire is $C_s = \tilde{E}\left[\max\left(\frac{S_{T,2}}{S_{T,1}} - 1, 0\right)\right]$, which is just the value of a call option with underlying asset price $S = \frac{S_2}{S_1}$, strike price $K = 1$, interest rate $r = 0$, and volatility $\sigma_S = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$. So its value is $C_S = \frac{S_2}{S_1} N(d_1) - N(d_2)$, where $d_1 = \frac{\ln(S_2/S_1) + 0.5\sigma_S^2 \tau}{\sigma_S \sqrt{\tau}}$ and $d_2 = d_1 - \sigma\sqrt{\tau}$. The payoff of the exchange option expressed in local currency is $S_1 C_S = S_2 N(d_1) - S_1 N(d_2)$.

6.4. Other Finance Questions

Besides option pricing problems, a variety of other quantitative finance problems are tested in quantitative interviews as well. Many of these problems tend to be position-specific. For example, if you are applying for a risk management job, prepare to answer questions about VaR; for fixed-income jobs, get ready to answer questions about interest rate models. As I explained in Chapter 1, it always helps if you grasp the basic knowledge before the interview. In this section, we use several examples to show some typical interview problems.

Portfolio optimization

You are constructing a simple portfolio using two stocks A and B. Both have the same expected return of 12%. The standard deviation of A's return is 20% and the standard deviation of B's return is 30%; the correlation of their returns is 50%. How will you allocate your investment between these two stocks to minimize the risk of your portfolio?

Solution: Portfolio optimization has always been a crucial topic for investment management firms. Harry Markowitz's mean-variance portfolio theory is by far the most well-known and well-studied portfolio optimization model. The essence of the mean-variance portfolio theory assumes that investors prefer (1) higher expected returns for a given level of standard deviation/variance and (2) lower standard deviations/variances for a given level of expected return. Portfolios that provide the minimum standard deviation for a given expected return are termed efficient portfolios. The expected return and the variance of a portfolio with N assets can be expressed as

$$\mu_p = w_1\mu_1 + w_2\mu_2 + \cdots + w_N\mu_N = w^T\mu$$

$$\text{var}(r_p) = \sum_{i=1}^{N}\sigma_i^2 w_i^2 + \sum_{i\neq j}\sigma_{ij}w_i w_j = w^T\Sigma w$$

where $w_i, \forall i = 1,\cdots,N$, is the weight of the i-th asset in the portfolio; $\mu_i, \forall i = 1,\cdots,N$, is the expected return of the i-th asset; σ_i^2 is the variance of i-th asset's return; $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$ is the covariance of the returns of the i-th and the j-th assets and ρ_{ij} is their correlation; w is an $N\times 1$ column vector of w_i's; μ is an $N\times 1$ column vector of μ_i's; Σ is the covariance matrix of the returns of N assets, an $N\times N$ matrix.

Since the optimal portfolio minimizes the variance of the return for a given level of expected return, the efficient portfolio can be formulated as the following optimization problem:

$$\min_{w} w^T \Sigma w \quad s.t.\ w^T \mu = \mu_p,\ w^T e = 1$$, where e is an $N \times 1$ vector with all elements equal to 1.[13]

For this specific problem, the expected returns are 12% for both stocks. So μ_p is always 12% no matter what w_A and w_B ($w_A + w_B = 1$) are. The variance of the portfolio is

$$\begin{aligned}\text{var}(r_p) &= \sigma_A^2 w_A^2 + \sigma_B^2 w_B^2 + 2\rho_{A,B}\sigma_A\sigma_B w_A w_B \\ &= \sigma_A^2 w_A^2 + \sigma_B^2 (1-w_A)^2 + 2\rho_{A,B}\sigma_A\sigma_B w_A (1-w_A)\end{aligned}$$

Taking the derivative of $\text{var}(r_p)$ with respect to w_A and setting it to zero, we have

$$\frac{\partial \text{var}(r_p)}{\partial w_A} = 2\sigma_A^2 w_A - 2\sigma_B^2 (1-w_A) + 2\rho_{A,B}\sigma_A\sigma_B (1-w_A) - 2\rho_{A,B}\sigma_A\sigma_B w_A = 0$$

$$\Rightarrow w_A = \frac{\sigma_B^2 - \rho_{A,B}\sigma_A\sigma_B}{\sigma_A^2 - 2\rho_{A,B}\sigma_A\sigma_B + \sigma_B^2} = \frac{0.09 - 0.5\times 0.2\times 0.3}{0.04 - 2\times 0.5\times 0.2\times 0.3 + 0.09} = \frac{6}{7}.$$

So we should invest 6/7 of the money in stock A and 1/7 in stock B.

Value at risk

Briefly explain what VaR is. What is the potential drawback of using VaR to measure the risk of derivatives?

Solution: Value at Risk (VaR) and stress test—or more general scenario analysis—are two important aspects of risk management. In the *Financial Risk Manager Handbook,*[14] VaR is defined as the following: VAR is the maximum loss over a target horizon such that there is a low, pre-specified probability that the actual loss will be larger.

Given a confidence level $\alpha \in (0, 1)$, the VaR can be implicitly defined as $\alpha = \int_{-VaR}^{\infty} xf(x)dx$, where x is the dollar profit (loss) and $f(x)$ is its probability density function. In practice, α is often set to 95% or 99%. VaR is an extremely popular choice in financial risk management since it summarizes the risk to a single dollar number.

---

[13] The optimal weights have closed form solution $w^* = \lambda\Sigma^{-1}e + \gamma\Sigma^{-1}\mu$, where $\lambda = \frac{C - \mu_p B}{D}$, $\gamma = \frac{\mu_p A - B}{D}$, $A = e'\Sigma^{-1}e > 0$, $B = e'\Sigma^{-1}\mu$, $C = \mu'\Sigma^{-1}\mu > 0$, $D = AC - B^2$.

[14] ***Financial Risk Manager Handbook*** by Phillippe Jorion is a comprehensive book covering different aspects of risk management. A classic book for VaR is ***Value at Risk***, also by Philippe Jorion.

Mathematically, it is simply the (negative) first or fifth percentile of the profit distribution.

As a percentile-based measure on the profit distribution, VaR does not depend on the shape of the tails before (and after) probability $1-\alpha$, so it does not describe the loss on the left tail. When the profit/loss distribution is far from a normal distribution, as in the cases of many derivatives, the tail portion has a large impact on the risk, and VaR often does not reflect the real risk.[15] For example, let's consider a short position in a credit default swap. The underlying asset is bond A with a \$1M notional value. Further assume that A has a 3% default probability and the loss given default is 100% (no recovery). Clearly we are facing the credit risk of bond A. Yet if we use 95% confidence level, $VaR(A)=0$ since the probability of default is less than 5%.

Furthermore, VaR is not sub-additive and is not a coherent measure of risk, which means that when we combine two positions A and B to form a portfolio C, we do not always have $VaR(C) \le VaR(A)+VaR(B)$. For example, if we add a short position in a credit default swap on bond B with a \$1M notional value. B also has a 3% default probability independent of A and the loss given default is 100%. Again we have $VaR(B)=0$. When A and B form a portfolio C, the probability that at least one bond will default becomes $1-(1-3\%)(1-3\%) \approx 5.9\%$. So $VaR(C)=\$1\text{M} > VaR(A)+VaR(B)$. Lack of sub-additivity directly contradicts the intuitive idea that diversification reduces risk. So it is a theoretical drawback of VaR.

(Sub-additivity is one property of a coherent risk measure. A risk measure $\rho(X)$ is considered coherent if the following conditions holds: $\rho(X+Y) \le \rho(X)+\rho(Y)$; $\rho(aX)=a\rho(X), \forall a>0$; $\rho(X) \le \rho(Y)$, if $X \le Y$; and $\rho(X+k)=\rho(X)-k$ for any constant k. It is defined in *Coherent Measure of Risk* by Artzner, P., et al., Mathematical Finance, 9 (3):203–228. Conditional VaR is a coherent risk measure.)

Duration and convexity

The duration of a bond is defined as $D=-\frac{1}{P}\frac{dP}{dy}$, where P is the price of the bond and y is yield to maturity. The convexity of a bond is defined as $C=\frac{1}{P}\frac{d^2P}{dy^2}$. Applying Taylor's expansion, $\frac{\Delta P}{P} \approx -D\Delta y+\frac{1}{2}C\Delta y^2$. when Δy is small, $\frac{\Delta P}{P} \approx -D\Delta y$.

For a fixed-rate bond with coupon rate c and time-to-maturity T:

[15] Stress test is often used as a complement to VaR by estimating the tail risk.

$T\uparrow \Rightarrow D\uparrow\ c\uparrow \Rightarrow D\downarrow\ y\uparrow \Rightarrow D\downarrow \quad T\uparrow \Rightarrow C\uparrow\ c\uparrow \Rightarrow C\downarrow\ y\uparrow \Rightarrow C\downarrow.$

Another important concept is dollar duration: $\$D = -\frac{dP}{dy} = P \times D$. Many market participants use a concept called DV01: $DV01 = -\frac{dP}{10,000 \times dy}$, which measures the price change when the yield changes by one basis point. For some bond derivatives, such as swaps, dollar duration is especially important. A swap may have value $P = 0$, in which case dollar duration is more meaningful than duration.

When n bonds with values P_i, $i = 1, \cdots, n$, and Durations D_i (convexities C_i) form a portfolio, the duration of the portfolio is the value-weighted average of the durations of the components: $D = \sum_{i=1}^{n} \frac{P_i}{P} D_i$ ($C = \sum_{i=1}^{n} \frac{P_i}{P} C_i$), where $P = \sum_{i=1}^{n} P_i$. The dollar duration of the portfolio is simply the sum of the dollar durations of the components: $\$D = \sum_{i=1}^{n} \$D_i$.

What are the price and duration of an inverse floater with face value \$100 and annual coupon rate $30\% - 3r$ that matures in 5 years? Assume that the coupons are paid semiannually and the current yield curve is flat at 7.5%.

Solution: The key to solving basic fixed-income problems is cash flow replication. To price a fixed-income security with exotic structures, if we can replicate its cash flow using a portfolio of fundamental bond types such as fixed-rate coupon bonds (including zero-coupon bonds) and floating-rate bonds, no-arbitrage arguments give us the following conclusions:
Price of the exotic security = Price of the replicating portfolio
Dollar duration of the exotic security = Dollar duration of the replicating portfolio

To replicate the described inverse floater, we can use a portfolio constructed by shorting 3 floating rate bonds, which is worth \$100 each, and longing 4 fixed-rate bonds with a 7.5% annual coupon rate, which is worth \$100 each as well. The coupon rate of a floating-rate bond is adjusted every 0.5 years payable in arrear: the coupon rate paid at $t + 0.5y$ is determined at t. The cash flows of both positions and the whole portfolio are summarized in the following table. It is apparent that the total cash flows of the portfolio are the same as the described inverse floater. So the price of the inverse float is the price of the replicating portfolio: $P_{inverse} = \$100$.

Cash flow	Year 0	Year 0.5	...	Year 4.5	Year 5
Short 3 floating-rate bonds	300	$-150r_0$	...	$-150r_4$	$-300-150r_{4.5}$
Long 4 bonds with 7.5% coupon rate	-400	15	...	15	400+15
Total	-100	$15-150r_0$	...	$30-300r_0$	$115-150r_{4.5}$

The dollar duration of the inverse floater is the same as the dollar duration of the portfolio as well: $\$D_{inverse} = 4\times\$D_{fixed} - 3\times\$D_{floating}$. Since the yield curve is flat, $r_0 = 7.5\%$ and the floating-rate bond is always worth \$103.75 (after the payment of \$3.75, the price of the floating-rate bond is \$100) at year 0.5, and the dollar duration[16] is

$$\$D_{floating} = -\frac{d\left(103.75/(1+y/2)\right)}{dy} = 0.5\times\frac{103.75}{(1+y/2)^2} = 100\times\frac{0.5}{1+y/2} = 48.19.$$

The price of a fixed-rate bond is $P = \sum_{t=1}^{2T}\frac{c/2}{(1+y/2)^t} + \frac{100}{(1+y/2)^{2T}}$, where T is the maturity of the bond. So the dollar duration of the fixed-rate bond is

$$\$D_{fixed} = -\frac{dP}{dy} = \frac{1}{1+y/2}\left(\sum_{t=1}^{2T}\frac{t}{2}\frac{c/2}{(1+y/2)^t} + \frac{100T}{(1+y/2)^{2T}}\right) = 410.64.$$

So $\$D_{inverse} = 4\times\$D_{fixed} - 3\times\$D_{floating} = 1498$ and the duration of the inverse floater is

$D_{inverse} = \$D_{inverse} / P_{inverse} = 14.98.$

Forward and futures

What's the difference between futures and forwards? If the price of the underlying asset is strongly positively correlated with interest rates, and the interest rates are stochastic, which one has higher price: futures or forwards? Why?

Solution: Futures contracts are exchange-traded standardized contracts; forward contracts are over-the-counter agreements so they are more flexible. Futures contracts are marked-to-market daily; forwards contacts are settled at the end of the contract term.

[16] The initial duration of a floating rate bond is the same as the duration of a six-month zero coupon bond.

If the interest rate is deterministic, futures and forwards have the same theoretical price: $F = Se^{(r+u-y)\tau}$, where u represents all the storage costs and y represents dividend yield for investment assets, convenience yield for commodities and foreign risk-free interest rate for foreign currencies.

The mark-to-market property of futures makes their values differ from forwards when interest rates vary unpredictably (as they do in the real world). As the life of a futures contract increases, the differences between forward and futures contracts may become significant. If the futures price is positively correlated with the interest rate, the increases of the futures price tend to occur the same time when interest rate is high. Because of the mark-to-market feature, the investor who longs the futures has an immediate profit that can be reinvested at a higher rate. The loss tends to occur when the interest rate is low so that it can be financed at a low rate. So a futures contract is more valuable than the forward when its value is positively correlated with interest rates and the futures price should be higher.

Interest rate models

Explain some of the basic interest rate models and their differences.

Solution: In general, interest rate models can be separated into two categories: short-rate models and forward-rate models. The short-rate models describe the evolution of the instantaneous interest rate $R(t)$ as stochastic processes, and the forward rate models (e.g., the one- or two-factor Heath-Jarrow-Morton model) capture the dynamics of the whole forward rate curve. A different classification separates interest rate models into arbitrage-free models and equilibrium models. Arbitrage-free models take the current term structure—constructed from most liquid bonds—and are arbitrage-free with respect to the current market prices of bonds. Equilibrium models, on the other hand, do not necessarily match the current term structure.

Some of the simplest short-rate models are the Vasicek model, the Cox-Ingersoll-Ross model, the Ho-Lee model, and the Hull-White model.

Equilibrium short-rate models

Vasicek model: $dR(t) = a\big(b - R(t)\big)dt + \sigma\, dW(t)$

When $R(t) > b$, the drift rate is negative; when $R(t) < b$, the drift rate is positive. So the Vasicek model has the desirable property of mean-reverting towards long-term average b. But with constant volatility, the interest rate has positive probability of being negative, which is undesirable.

Cox-Ingersoll-Ross model: $dR(t) = a\big(b - R(t)\big)dt + \sigma\sqrt{R(u)}\, dW(t)$

The Cox-Ingersoll-Ross model keeps the mean-reversion property of the Vasicek model. But the diffusion rate $\sigma\sqrt{R(u)}$ addresses the drawback of Vasicek model by guaranteeing that the short rate is positive.

No-arbitrage short-rate models

Ho-Lee model: $dr = \theta(t)dt + \sigma dz$

The Ho-Lee model is the simplest no-arbitrage short-rate model where $\theta(t)$ is a time-dependent drift. $\theta(t)$ is adjusted to make the model match the current rate curve.

Hull-White model: $dR(t) = a\big(b(t) - R(t)\big)dt + \sigma dW(t)$

The Hull-White model has a structure similar to the Vasicek model. The difference is that $b(t)$ is a time-dependent variable in the Hull-White model to make it fit the current term structure.

Chapter 7 Algorithms and Numerical Methods

Although the percentage of time that a quant spends on programming varies with the job function (e.g., quant analyst/researcher versus quant developer) and firm culture, a typical quant generally devotes part of his or her time to implementing models through programming. Therefore, programming skill test is often an inherent part of the quantitative interview.

To a great extent, the programming problems asked in quantitative interviews are similar to those asked in technology interviews. Not surprisingly, many of these problems are platform- or language-specific. Although C++ and Java still dominate the market, we've seen a growing diversification to other programming languages such as Matlab, SAS, S-Plus, and R. Since there are many existing books and websites dedicated to technology interviews, this chapter will not give a comprehensive review of programming problems. Instead, it discusses some algorithm problems and numerical methods that are favorite topics of quantitative interviews.

7.1. Algorithms

In programming, the analysis of algorithm complexity often uses asymptotic analysis that ignores machine-dependent constants and studies the running time $T(n)$ —the number of primitive operations such as addition, multiplication, and comparison—as the number of inputs $n \to \infty$.[1]

Three of the most important notations in algorithm complexity are big-O notation, Ω notation and Θ notation:

$O(g(n)) = \{ f(n)$: there exist positive constants c and n_0 such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0 \}$. It is the asymptotic upper bound of $f(n)$.

$\Omega(g(n)) = \{ f(n)$: there exist positive constants c and n_0 such that $0 \le cg(n) \le f(n)$ for all $n \ge n_0 \}$. It is the asymptotic lower bound of $f(n)$.

$\Theta(g(n)) = \{ f(n)$: there exist positive constants c_1, c_2, and n_0 such that $c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0 \}$. It is the asymptotic tight bound of $f(n)$.

Besides notations, it is also important to explain two concepts in algorithm complexity:

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[1] If you want to review basic algorithms, I highly recommend "***Introduction to Algorithm***" by Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest and Clifford Stein. It covers all the theories discussed in this section and includes many algorithms frequently appearing in interviews.

Worst-case running time $W(n)$: an upper bound on the running time for any n inputs.

Average-case running time $A(n)$: the expected running time if the n inputs are randomly selected.

For many algorithms, $W(n)$ and $A(n)$ have the same $O(g(n))$. But as we will discuss in some problems, they may well be different and their relative importance often depends on the specific problem at hand.

A problem with n inputs can often be split into a subproblems with n/b inputs in each subproblem. This paradigm is commonly called divide-and-conquer. If it takes $f(n)$ primitive operations to divide the problem into subproblems and to merge the solutions of the subproblems, the running time can be expressed as a recurrence equation $T(n) = aT(n/b) + f(n)$, where $a \geq 1$, $b > 1$, and $f(n) \geq 0$.

The **master theorem** is a valuable tool in finding the tight bound for recurrence equation $T(n) = aT(n/b) + f(n)$: If $f(n) = O\left(n^{\log_b a - \varepsilon}\right)$ for some constant $\varepsilon > 0$, $T(n) = \Theta\left(n^{\log_b a}\right)$, since $f(n)$ grows slower than $n^{\log_b a}$. If $f(n) = \Theta\left(n^{\log_b a} \log^k n\right)$ for some $k \geq 0$, $T(n) = \Theta\left(n^{\log_b a} \log^{k+1} n\right)$, since $f(n)$ and $n^{\log_b a}$ grow at similar rates. If $f(n) = \Omega\left(n^{\log_b a + \varepsilon}\right)$ for some constant $\varepsilon > 0$, and $af(n/b) \leq cf(n)$ for some constant $c < 1$, $T(n) = \Theta(f(n))$, since $f(n)$ grows faster than $n^{\log_b a}$.

Let's use binary search to show the application of the master theorem. To find an element in an array, if the numbers in the array are sorted ($a_1 \leq a_2 \leq \cdots \leq a_n$), we can use binary search: The algorithm starts with $a_{\lfloor n/2 \rfloor}$. If $a_{\lfloor n/2 \rfloor} = x$, the search stops. If $a_{\lfloor n/2 \rfloor} > x$, we only need to search $a_1, \cdots, a_{\lfloor n/2-1 \rfloor}$. If $a_{\lfloor n/2 \rfloor} < x$, we only need to search $a_{\lfloor n/2+1 \rfloor}, \cdots, a_n$. Each time we can reduce the number of elements to search by half after making one comparison. So we have $a = 1$, $b = 2$, and $f(n) = 1$. Hence, $f(n) = \Theta\left(n^{\log_2 1} \log^0 n\right)$ and the binary search has complexity $\Theta(\log n)$.

Number swap

How do you swap two integers, i and j, without using additional storage space?

Solution: Comparison and swap are the basic operations for many algorithms. The most common technique for swap uses a temporary variable, which unfortunately is forbidden in this problem since the temporary variable requires additional storage space. A simple

mathematic approach is to store the sum of i and j first, then extract i's value and assign it to j and finally assign j's value to i. The implementation is shown in the following code:[2]

```
void swap(int &i, int &j) {
    i = i + j;  //store the sum of i and j
    j = i - j;  //change j to i's value
    i = i - j;  //change i to j's value
}
```

An alternative solution uses bitwise XOR (^) function by taking advantage of the fact that $x \wedge x = 0$ and $0 \wedge x = x$:

```
void swap(int &i, int &j){
    i = i ^ j;
    j = j ^ i; //j = i ^ (j ^ i) = i
    i = i ^ j; //i = (i ^ j) ^ i = j
}
```

Unique elements

If you are given a sorted array, can you write some code to extract the unique elements from the array? For example, if the array is [1, 1, 3, 3, 3, 5, 5, 5, 9, 9, 9, 9], the unique elements should be [1, 3, 5, 9].

Solution: Let a be an n-element sorted array with elements $a_0 \le a_1 \le \cdots \le a_{n-1}$. Whenever we encounter a new element a_i in the sorted array, its value is different from its previous element ($a_i \ne a_{i-1}$). Using this property we can easily extract the unique elements. One implementation in C++ is shown as the following function:[3]

```
template <class T> vector<T> unique(T a[], int n) {
    vector<T> vec; // vector used to avoid resizing problem
    vec.reserve(n); //reserver to avoid reallocation
    vec.push_back(a[0]);
    for(int i=1; i<n; ++i) {
```

[2] This chapter uses C++ to demonstrate some implementations. For other problems, the algorithms are described using pseudo codes.
The following is a one-line equivalent function for swapping two integers. It is not recommend, though, as it lacks clarity.

```
void swap(int &i, int &j) { i-=j=(i+=j)-j; };
```

[3] I should point out that C++ STL has general algorithms for this basic operation: unique and unique_copy.

```
            if(a[i] != a[i-1])
                  vec.push_back(a[i]);
      }
      return vec;
}
```

Horner's algorithm

Write an algorithm to compute $y = A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots + A_nx^n$.

Solution: A naïve approach calculates each component of the polynomial and adds them up, which takes $O(n^2)$ number of multiplications. We can use Horner's algorithm to reduce the number of multiplications to $O(n)$. The algorithm expresses the original polynomial as $y = \Big(\big(\big((A_nx + A_{n-1})x + A_{n-2}\big)x + \cdots + A_2\big)x + A_1\Big)x + A_0$ and sequentially calculate $B_n = A_n$, $B_{n-1} = B_nx + A_{n-1}$, $\cdots$, $B_0 = B_1x + A_0$. We have $y = B_0$ with at most n multiplications.

Moving average

Given a large array A of length m, can you develop an efficient algorithm to build another array containing the n-element moving average of the original array ($B_1, \cdots, B_{n-1} = NA$, $B_i = \big(A_{i-n+1} + A_{i-n+2} + \cdots + A_i\big)/n$, $\forall i = n, \cdots, m$)?

Solution: When we calculate the moving average of the next n consecutive numbers, we can reuse the previously computed moving average. Just multiply that average by n, subtract the first number in that moving average and then add the new number, and you have the new sum. Dividing the new sum by n yields the new moving average. Here is the pseudo-code for calculating the moving average:

S= A[1] + $\cdots$ + A[n]; B[n] = S/n;

for (i=n+1 to m) { S = S – A[i-n] + A[i]; B[i] = S/n; }

Sorting algorithm

Could you explain three sorting algorithms to sort n distinct values $A_1, \cdots, A_n$ and analyze the complexity of each algorithm?

Solution: Sorting is a fundamental process that is directly or indirectly implemented in many programs. So a variety of sorting algorithms have been developed for different

purposes. Here let's discuss three such algorithms: insertion sort, merge sort and quick sort.

Insertion sort: Insertion sort uses an incremental approach. Assume that we have sorted subarray A[1, ..., *i*-1]. We insert element A_i into the appropriate place in A[1, ..., *i*-1], which yields sorted subarray A[1, ..., *i*]. Starting with $i=1$ and increases i step by step to n, we will have a fully sorted array. For each step, the expected number of comparisons is $i/2$ and the worst-case number of comparisons is i. So we have

$$A(n)=\Theta\left(\sum_{i=1}^{n} i/2\right)=\Theta(n^2) \text{ and } W(n)=\Theta\left(\sum_{i=1}^{n} i\right)=\Theta(n^2).$$

Merge sort: Merge sort uses the divide-and-conquer paradigm. It divides the array into two subarrays each with $n/2$ items and sorts each subarray. Unless the subarray is small enough (with no more than a few elements), the subarray is again divided for sorting. Finally, the sorted subarrays are merged to form a single sorted array.

The algorithm can be expressed as the following pseudocode:

```
mergesort(A, beginindex, endindex)
if beginindex < endindex
  then centerindex ← (beginindex + endindex)/2
       merge1 <- mergesort(A, beginindex, centerindex)
       merge2 <- mergesort(A, centerindex + 1, endindex)
       merge(merge1, merge2)
```

The merge of two sorted arrays with $n/2$ elements each into one array takes $\Theta(n)$ primitive operations. The running time $T(n)$ follows the following recursive function:

$$T(n)=\begin{cases}2T(n/2)+\Theta(n), & \text{if } n>1\\ 1, & \text{if } n=1\end{cases}.$$

Applying the master theorem to $T(n)$ with $a=2, b=2$, and $f(n)=\Theta(n)$, we have $f(n)=\Theta\left(n^{\log_b a}\log^0 n\right)$. So $T(n)=\Theta(n\log n)$. For merge sort, $A(n)$ and $W(n)$ are the same as $T(n)$.

Quicksort: Quicksort is another recursive sorting method. It chooses one of the elements, A_i, from the sequence and compares all other values with it. Those elements smaller than A_i are put in a subarray to the left of A_i; those elements larger than A_i are put in a subarray to the right of A_i. The algorithm is then repeated on both subarrays (and any subarrays from them) until all values are sorted.

In the worst case, quicksort requires the same number of comparisons as the insertion sort. For example, if we always choose the first element in the array (subarray) and compare all other elements with it, the worst case happens when $A_1, \cdots, A_n$ are already sorted. In such cases, one of the subarray is empty and the other has $n-1$ element. Each step only reduces the subarray size by one. Hence, $W(n) = \Theta\left(\sum_{i=1}^{n} i\right) = \Theta(n^2)$.

To estimate the average-case running time, let's assume that the initial ordering is random so that each comparison is likely to be any pair of elements chosen from $A_1, \cdots, A_n$. If we suspect that the original sequence of elements has a certain pattern, we can always randomly permute the sequence first with complexity $\Theta(n)$ as explained in the next problem. Let $\tilde{A}_p$ and $\tilde{A}_q$ be the pth and qth element ($1 \le p < q \le n$) in the final sorted array. There are $q - p + 1$ numbers between $\tilde{A}_p$ and $\tilde{A}_q$. The probability that $\tilde{A}_p$ and $\tilde{A}_q$ is compared is the probability that $\tilde{A}_q$ is compared with $\tilde{A}_p$ before $\tilde{A}_{p+1}, \cdots$, or $\tilde{A}_{q-1}$ is compared with either $\tilde{A}_p$ or $\tilde{A}_q$ (otherwise, $\tilde{A}_p$ and $\tilde{A}_q$ are separated into different subarrays and will not be compared), which happens with probability $P(p,q) = \frac{2}{q-p+1}$ (you can again use the symmetry argument to derive this probablity).

The total expected number of comparison is $A(n) = \sum_{q=2}^{n}\sum_{p=1}^{q-1} P(p,q) = \sum_{q=2}^{n}\sum_{p=1}^{q-1}\left(\frac{2}{q-p+1}\right)$

$= \Theta(n \lg n)$.

Although theoretically quicksort can be slower than merge sort in the worst cases, it is often as fast as, if not faster than, merge sort.

Random permutation

A. If you have a random number generator that can generate random numbers from either discrete or continuous uniform distributions, how do you shuffle a deck of 52 cards so that every permutation is equally likely?

Solution: A simple algorithm to permute *n* elements is random permutation by sorting. It assigns a random number to each card and then sorts the cards in order of their assigned random numbers.[4] By symmetry, every possible order (out of *n*! possible ordered sequences) is equally likely. The complexity is determined by the sorting step, so the

[4] If we use the continuous uniform distribution, theoretically any two random numbers have zero probability of being equal.

running time is $\Theta(n \log n)$. For a small n, such as $n = 52$ in a deck of cards, the complexity $\Theta(n \log n)$ is acceptable. For large n, we may want to use a faster algorithm known as the Knuth shuffle. For n elements $A[1], \cdots, A[n]$, the Knuth shuffle uses the following loop to generate a random permutation:

for (i=1 to n) swap($A[i]$, A[Random(i, n)]),

where Random(i, n) is a random number from the discrete uniform distribution between i and n.

The Knuth shuffle has a complexity of $\Theta(n)$ and an intuitive interpretation. In the first step, each of the n cards has equal probability of being chosen as the first card since the card number is chosen from the discrete uniform distribution between 1 and n; in the second step, each of the remaining $n-1$ cards elements has equal probability of being chosen as the second card; and so on. So naturally each ordered sequence has $1/n!$ probability.

B. You have a file consisting of characters. The characters in the file can be read sequentially, but the length of the file is unknown. How do you pick a character so that every character in the file has equal probability of being chosen?

Solution: Let's start with picking the first character. If there is a second character, we keep the first character with probability 1/2 and replace the pick with the second character with probability 1/2. If there is a third character, we keep the pick (from the first two characters) with probability 2/3 and replace the pick with the third character with probability 1/3. The same process is continued until the final character. In other words, let C_n be the character that we pick after we have scanned n characters and the $(n+1)th$ character exists, the probability of keeping the pick is $\frac{n}{n+1}$ and the probability of switching to the $(n+1)th$ character is $\frac{1}{n+1}$. Using simple induction, we can easily prove that each character has $1/m$ probability of being chosen if there are m characters.

Search algorithm

A. Develop an algorithm to find both the minimum and the maximum of n numbers using no more than $3n/2$ comparisons.

Solution: For an unsorted array of n numbers, it takes $n-1$ comparisons to identify either the minimum or the maximum of the array. However, it takes at most $3n/2$ comparisons to identify both the minimum and the maximum. If we separate the elements to $n/2$ pairs, compare the elements in each pair and put the smaller one in group

A and the larger one in group B. This step takes $n/2$ comparisons. Since the minimum of the whole array must be in group A and the maximum must be in group B, we only need to find the minimum in A and the maximum in B, either of which takes $n/2-1$ comparisons. So the total number of comparisons is at most 3n/2.[5]

B. You are given an array of numbers. From the beginning of the array to some position, all elements are zero; after that position, all elements are nonzero. If you don't know the size of the array, how do you find the position of the first nonzero element?

Solution: We can start with the 1st element; if it is zero, we check the 2nd element; if the 2nd element is zero, we check the 4th element... The process is repeated until the ith step when the $2^i th$ element is nonzero. Then we check the $\frac{2^i + 2^{i-1}}{2} th$ element. If it is zero, the search range is limited to the elements between the $\frac{2^i + 2^{i-1}}{2} th$ element and the $2^i th$ element; otherwise the search range is limited to the elements between the $2^{i-1} th$ element and the $\frac{2^i + 2^{i-1}}{2} th$ element... Each time, we cut the range by half. This method is basically a binary search. If the first nonzero element is at position n, the algorithm complexity is $\Theta(\log n)$.

C. You have a square grid of numbers. The numbers in each row increase from left to right. The numbers in each column increase from top to bottom. Design an algorithm to find a given number from the grid. What is the complexity of your algorithm?

Solution: Let A be an $n \times n$ matrix representing the grid of numbers and x be the number we want to find in the grid. Begin the search with the last column from top to bottom: $A_{1,n}, \cdots, A_{n,n}$. If the number is found, then stop the search. If $A_{n,n} < x$, x is not in the grid and the search stops as well. If $A_{i,n} < x < A_{i+1,n}$, then we know that all the numbers in rows $1, \cdots, i$ are less than x and are eliminated as well.[6] Then we search the $(i+1)th$ row from right to left. If the number is found in the $(i+1)th$ row, the search stops. If $A_{1,i+1} > x$, x is not in the grid since all the number in rows $i+1$ and above are larger than x. If $A_{i+1,j+1} > x > A_{i+1,j}$, we eliminate all the numbers in columns $j+1, \cdots, n$. Then we can search along column from $A_{i+1,j}$ towards $A_{n,j}$ until we find x (or x does not exist in

[5] Slight adjustment needs to be made if n is odd, but the upper bound $3n/2$ still applies.

[6] i can be 0, which means $x < A_{1,n}$, in which case we can search the first row from right to left.

the grid) or a k that makes $A_{k,j} < x < A_{k+1,j}$ and then we search left along the row $k+1$ from $A_{k+1,j}$ towards $A_{k+1,1}$... Using this algorithm, the search takes at most $2n$ steps. So its complexity is $O(n)$.

Fibonacci numbers

Consider the following C++ program for producing Fibonacci numbers:

```
int Fibonacci(int n)
{
    if (n <= 0)
        return 0;
    else if (n==1)
        return 1;
    else
        return Fibonacci(n-1)+Fibonacci(n-2);
}
```

If for some large n, it takes 100 seconds to compute Fibonacci(n), how long will it take to compute Fibonacci(n+1), to the nearest second? Is this algorithm efficient? How would you calculate Fibonacci numbers?

Solution: This C++ function uses a rather inefficient recursive method to calculate Fibonacci numbers. Fibonacci numbers are defined as the following recurrence:

$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2},\ \forall n \geq 2$

F_n has closed-formed solution $F_n = \dfrac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}}$, which can be easily proven using induction. From the function, it is clear that

$T(0) = 1,\ T(1) = 1,\ T(n) = T(n-1) + T(n-2) + 1.$

So the running time is a proportional to a sequence of Fibonacci numbers as well. For a large n, $(1-\sqrt{5})^n \to 0$, so $\dfrac{T(n+1)}{T(n)} \approx \dfrac{\sqrt{5}+1}{2}$. If it takes 100 seconds to compute Fibonacci(n), the time to compute Fibonacci(n+1) is $T(n+1) \approx \dfrac{\sqrt{5}+1}{2} T(n) \approx 162$ seconds.[7]

[7] $\phi = \dfrac{\sqrt{5}+1}{2}$ is called the golden ratio.

The recursive algorithm has exponential complexity $\Theta\left(\left(\frac{\sqrt{5}+1}{2}\right)^n\right)$, which is surely inefficient. The reason is that it fails to effectively use the information from Fibonacci numbers with smaller n in the Fibonacci number sequence. If we compute $F_0, F_1, \cdots, F_n$ in sequence using the definition, the running time has complexity $\Theta(n)$.

An algorithm called recursive squaring can further reduce the complexity to $\Theta(\log n)$. Since $\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix}$ and $\begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, we can show that $\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$ using induction. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, we can again apply the divide-and-conquer paradigm to calculate A^n: $A^n = \begin{cases} A^{n/2} \times A^{n/2}, & \text{if } n \text{ is even} \\ A^{(n-1)/2} \times A^{(n-1)/2} \times A, & \text{if } n \text{ is odd} \end{cases}$. The multiplication of two 2×2 matrices has complexity $\Theta(1)$. So $T(n) = T(n/2) + \Theta(1)$. Applying the master theorem, we have $T(n) = \Theta(\log n)$.

Maximum contiguous subarray

Suppose you have a one-dimensional array A with length n that contains both positive and negative numbers. Design an algorithm to find the maximum sum of any contiguous subarray $A[i, j]$ of A: $V(i, j) = \sum_{x=i}^{j} A[x]$, $1 \le i \le j \le n$.

Solution: Almost all trading systems need such an algorithm to calculate maximum run-up or maximum drawdown of either real trading books or simulated strategies. Therefore this is a favorite algorithm question of interviewers, especially interviewers at hedge funds and trading desks.

The most apparent algorithm is an $O(n^2)$ algorithm that sequentially calculates the $V(i, j)$'s from scratch using the following equations:

$V(i,i) = A[i]$ when $j = i$ and $V(i, j) = \sum_{x-i}^{j} A[x] = V(i, j-1) + A[j]$ when $j > i$.

As the $V(i, j)$'s are calculated, we also keep track of the maximum of $V(i, j)$ as well as the corresponding subarray indices i and j.

A more efficient approach uses the divide-and-conquer paradigm. Let's define $T(i)=\sum_{x=1}^{i} A[x]$ and $T(0)=0$, then $V(i,j)=T(j)-T(i-1), \forall 1 \le i \le j \le n$. Clearly for any fixed j, when $T(i-1)$ is minimized, $V(i,j)$ is maximized. So the maximum subarray ending at j is $V_{max}=T(j)-T_{min}$ where $T_{min}=\min(T(1),\cdots,T(j-1))$. If we keep track of and update V_{max} and T_{min} as j increases, we can develop the following $O(n)$ algorithm:

$T = A[1]; V_{max} = A[1]; T_{min} = \min(0,T)$

For $j = 2$ to n

{ $T = T + A[j]$;

If $T - T_{min} > V_{max}$ then $V_{max} = T - T_{min}$;

If $T < T_{min}$, then $T_{min} = T$;

}

Return V_{max};

The following is a corresponding C++ function that returns V_{max} and indices i and j given an array and its length:

```
double maxSubarray(double A[], int len, int &i, int &j)
{
     double T=A[0], Vmax=A[0];
     double Tmin = min(0.0, T);
     for(int k=1; k<len; ++k)
     {
          T+=A[k];
          if (T-Tmin > Vmax) {Vmax=T-Tmin; j=k;}
          if (T<Tmin) {Tmin = T; i = (k+1<j)? (k+1):j;}
     }
     return Vmax;
}
```

Applying it to the following array A,

```
double A[]={1.0,2.0,-5.0,4.0,-3.0, 2.0, 6.0, -5.0, -1.0};
int i = 0, j =0;
```

```
double Vmax = maxSubarray(A, sizeof(a)/sizeof(A[1]), i, j);
```

will give $V_{max} = 9$, $i = 3$ and $j = 6$. So the subarray is [4.0, – 3.0, 2.0, 6.0].

7.2. The Power of Two

There are only 10 kinds of people in the world—those who know binary, and those who don't. If you happen to get this joke, you probably know that computers operate using the binary (base-2) number system. Instead of decimal digits 0-9, each bit (binary digit) has only two possible values: 0 and 1. Binary representation of numbers gives some interesting properties that are widely explored in practice and makes it an interesting topic to test in interviews.

Power of 2

How do you determine whether an integer is a power of 2?

Solution: Any integer $x = 2^n$ ($n \geq 0$) has a single bit (the $(n+1)th$ bit from the right) set to 1. For example, 8 ($= 2^3$) is expressed as $0\cdots01000$. It is also easy to see that $2^n - 1$ has all the n bits from the right set to 1. For example, 7 is expressed as $0\cdots00111$. So 2^n and $2^n - 1$ do not share any common bits. As a result, $x \;\&\; (x-1) == 0$, where & is a bitwise AND operator, is a simple way to identify whether the integer x is a power of 2.

Multiplication by 7

Give a fast way to multiply an integer by 7 without using the multiplication (*) operator?

Solution: (x << 3) – x, where << is the bit-shift left operator. x << 3 is equivalent to x*8. Hence (x << 3) – x is x*7.[8]

Probability simulation

You are given a fair coin. Can you design a simple game using the fair coin so that your probability of winning is p, $0 < p < 1$?[9]

[8] The result could be wrong if << causes an overflow.

[9] Hint: Computer stores binary numbers instead of decimal ones; each digit in a binary number can be simulated using a fair coin.

Solution: The key to this problem is to realize that $p \in (0,1)$ can also be expressed as a binary number and each digit of the binary number can be simulated using a fair coin. First, we can express the probability p as binary number:

$$p = 0.p_1 p_2 \cdots p_n = p_1 2^{-1} + p_2 2^{-2} + \cdots + p_n 2^{-n},\ p_i \in \{0,1\},\ \forall i = 1, 2, \cdots, n.$$

Then, we can start tossing the fair coin, and count heads as 1 and tails as 0. Let $s_i \in \{0,1\}$ be the result of the i-th toss starting from $i = 1$. After each toss, we compare p_i with s_i. If $s_i < p_i$, we win and the coin tossing stops. If $s_i > p_i$, we lose and the coin tossing stops. If $s_i = p_i$, we continue to toss more coins. Some p values (e.g., 1/3) are infinite series when expressed as a binary number ($n \to \infty$). In these cases, the probability to reach $s_i \neq p_i$ is 1 as i increases. If the sequence is finite, (e.g., 1/4=0.01) and we reach the final stage with $s_n = p_n$, we lose (e.g., for 1/4, only the sequence 00 will be classified as a win; all other three sequences 01, 10 and 11 are classified as a loss). Such a simulation will give us probability p of winning.

Poisonous wine

You've got 1000 bottles of wines for a birthday party. Twenty hours before the party, the winery sent you an urgent message that one bottle of wine was poisoned. You happen to have 10 lab mice that can be used to test whether a bottle of wine is poisonous. The poison is so strong that any amount will kill a mouse in exactly 18 hours. But before the death on the 18th hour, there are no other symptoms. Is there a sure way that you can find the poisoned bottle using the 10 mice before the party?

Solution: If the mice can be tested sequentially to eliminate half of the bottles each time, the problem becomes a simple binary search problem. Ten mice can identify the poisonous bottle in up to 1024 bottles of wines. Unfortunately, since the symptom won't show up until 18 hours later and we only have 20 hours, we cannot sequentially test the mice. Nevertheless, the binary search idea still applies. All integers between 1 and 1000 can be expressed in 10-bit binary format. For example, bottle 1000 can be labeled as 1111101000 since $1000 = 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^3$.

Now let mouse 1 take a sip from every bottle that has a 1 in the first bit (the lowest bit on the right); let mouse 2 take a sip from every bottle with a 1 in the second bit; ...; and, finally, let mouse 10 take a sip from every bottle with a 1 in the 10th bit (the highest bit). Eighteen hours later, if we line up the mice from the highest to the lowest bit and treat a live mouse as 0 and a dead mouse as 1, we can easily back track the label of the poisonous bottle. For example, if the 6th, 7th, and 9th mice are dead and all others are alive, the line-up gives the sequence 0101100000 and the label for the poisonous bottle is $2^8 + 2^6 + 2^5 = 352$.

7.3 Numerical Methods

The prices of many financial instruments do not have closed-form analytical solutions. The valuation of these financial instruments relies on a variety of numerical methods. In this section, we discuss the application of Monte Carlo simulation and finite difference methods.

Monte Carlo simulation

Monte Carlo simulation is a method for iteratively evaluating a deterministic model using random numbers with appropriate probabilities as inputs. For derivative pricing, it simulates a large number of price paths of the underlying assets with probability corresponding to the underlying stochastic process (usually under risk-neutral measure), calculates the discounted payoff of the derivative for each path, and averages the discounted payoffs to yield the derivative price. The validity of Monte Carlo simulation relies on the law of large numbers.

Monte-Carlo simulation can be used to estimate derivative prices if the payoffs only depend on the final values of the underlying assets, and it can be adapted to estimate prices if the payoffs are path-dependent as well. Nevertheless, it cannot be directly applied to American options or any other derivatives with early exercise options.

A. Explain how you can use Monte Carlo simulation to price a European call option?

Solution: If we assume that stock price follows a geometric Brownian motion, we can simulate possible stock price paths. We can split the time between t and T into N equally-spaced time steps.[10] So $\Delta t = \frac{T-t}{N}$ and $t_i = t + \Delta t \times i$, for $i = 0,1,2,\cdots,N$. We then simulate the stock price paths under risk-neutral probability using equation $S_i = S_{i-1} e^{(r-\sigma^2/2)(\Delta t)+\sigma\sqrt{\Delta t}\varepsilon_i}$, where ε_i's are IID random variables from standard normal distribution. Let's say that we simulate M paths and each one yields a stock price $S_{T,k}$, where $k = 1,2,\cdots,M$, at maturity date T.

[10] For European options, we can simply set N=1. But for more general options, especially the path-dependent ones, we want to have small time steps and therefore N should be large.

The estimated price of the European call is the present value of the expected payoff, which can be calculated as $C = e^{-r(T-t)} \frac{\sum_{k=1}^{M} \max(S_{T,k} - K, 0)}{M}$.

B. How do you generate random variables that follow $N(\mu, \sigma^2)$ (normal distribution with mean μ and variance σ^2) if your computer can only generate random variables that follow continuous uniform distribution between 0 and 1?

Solution: This is a great question to test the basic knowledge of random number generation, the foundation of Monte Carlo simulation. The solution to this question can be dissected to two steps:

1. Generate random variable of $x \sim N(0,1)$ from uniform random number generator using inverse transform method and rejection method.
2. Scale x to $\mu + \sigma x$ to generate the final random variables that follow $N(\mu, \sigma^2)$.

The second step is straightforward; the first step deserves some explanations. A popular approach to generating random variables is the inverse transform method: For any continuous random variable X with cumulative density function F ($U = F(X)$), the random variable X can be defined as the inverse function of U: $X = F^{-1}(U)$, $0 \le U \le 1$. It is obvious that $X = F^{-1}(U)$ is a one-to-one function with $0 \le U \le 1$. So any continuous random variable can be generated using the following process:

- Generate a random number u from the standard uniform distribution.
- Compute the value x such that $u = F(x)$ as the random number from the distribution described by F.

For this model to work, $F^{-1}(U)$ must be computable. For standard normal distribution, $U = F(X) = \int_{-\infty}^{X} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$. The inverse function has no analytical solution. Theoretically, we can come up with the one-to-one mapping of X to U as the numeric solution of ordinary differential equation $F'(x) = f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ using numerical integration method such as the Euler method.[11] Yet this approach is less efficient than the rejection method:

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[11] To integrate $y = F(x)$ with first derivative $y' = f(x)$ and a known initial value $y_0 = F(x_0)$, the Euler method chooses a small step size h (h can be positive or negative) to sequentially approximate y values:

Some random variables have pdf $f(x)$, but no analytical solution for $F^{-1}(U)$. In these cases, we can use a random variable with pdf $g(y)$ and $Y = G^{-1}(U)$ to help generate random variables with pdf $f(x)$. Assume that M is a constant such that $\frac{f(y)}{g(y)} \le M, \ \forall y$.

We can implement the following acceptance-rejection method:

- Sampling step: Generate random variable y from $g(y)$ and a random variable v from standard uniform distribution $[0,1]$.
- Acceptance/rejection step: If $v \le \frac{f(y)}{Mg(y)}$, accept $x = y$; otherwise, repeat the sampling step.[12]

An exponential random variable ($g(x) = \lambda e^{-\lambda x}$) with $\lambda = 1$ has cdf $u = G(x) = 1 - e^{-x}$. So the inverse function has analytical solution $x = -\log(1-u)$ and a random variable with exponential distribution can be conveniently simulated. For standard normal distribution, $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$,

$$\frac{f(x)}{g(x)} = \sqrt{\frac{2}{\pi}} e^{x - x^2/2} < \sqrt{\frac{2}{\pi}} e^{-(x-1)^2/2 + 1/2} \le \sqrt{\frac{2}{\pi}} e^{1/2} \approx 1.32, \ \forall 0 < x < \infty$$

So we can choose $M = 1.32$ and use the acceptance-rejection method to generate $x \sim N(0,1)$ random variables and scale them to $N(\mu, \sigma^2)$ random variables.

C. Can you explain a few variance reduction techniques to improve the efficiency of Monte Carlo simulation?

Solution: Monte Carlo simulation, in its basic form, is the mean of IID random variables $Y_1, Y_2, \cdots, Y_M$: $\overline{Y} = \frac{1}{M} \sum_{i=1}^{M} Y_i$. Since the expected value of each Y_i is unbiased, the estimator $\overline{Y}$ is unbiased as well. If $Var(Y) = \sigma$ and we generate IID Y_i, then $Var(\overline{Y}) = \sigma / \sqrt{M}$, where M is the number of simulations. Not surprisingly, Monte Carlo

---

$F(x_0 + h) = F(x_0) + f(x_0) \times h, \ F(x_0 + 2h) = F(x_0 + h) + f(x_0 + h) \times h, \cdots$. The initial value of the cdf of a standard normal can be $F(0) = 0.5$.

[12] $P(X \le x) \propto \int_{-\infty}^{x} g(y) \frac{f(y)}{Mg(y)} dy = M \int_{-\infty}^{x} f(y) dy \Rightarrow F(x) = \frac{P(X \le x)}{P(X < \infty)} = \int_{-\infty}^{x} f(y) dy$

simulation is computationally intensive if σ is large. Thousands or even millions of simulations are often required to get the desired accuracy. Depending on the specific problems, a variety of methods have been applied to reduce variance.

Antithetic variable: For each series of ε_i's, calculate its corresponding payoff $Y(\varepsilon_1,\cdots,\varepsilon_N)$. Then reverse the sign of all ε_i's and calculate the corresponding payoff $Y(-\varepsilon_1,\cdots,-\varepsilon_N)$. When $Y(\varepsilon_1,\cdots,\varepsilon_N)$ and $Y(-\varepsilon_1,\cdots,-\varepsilon_N)$ are negatively correlated, the variance is reduced.

Moment matching: Specific samples of the random variable may not match the population distribution well. We can draw a large set of samples first and then rescale the samples to make the samples' moments (mean and variance are the most commonly used) match the desired population moments.

Control variate: If we want to price a derivative X and there is a closely related derivative Y that has an analytical solution, we can generate a series of random numbers and use the same random sequences to price both X and Y to yield $\hat{X}$ and $\hat{Y}$. Then X can be estimated as $\hat{X}+(Y-\hat{Y})$. Essentially we use $(Y-\hat{Y})$ to correct the estimation error of $\hat{X}$.

Importance sampling: To estimate the expected value of $h(x)$ from distribution $f(x)$, instead of drawing x from distribution $f(x)$, we can draw x from distribution $g(x)$ and use Monte Carlo simulation to estimate expected value of $\frac{h(x)f(x)}{g(x)}$:

$$E_{f(x)}[h(x)]=\int h(x)f(x)dx=\int\frac{h(x)f(x)}{g(x)}g(x)dx=E_{g(x)}\left[\frac{h(x)f(x)}{g(x)}\right].$$ [13]

If $\frac{h(x)f(x)}{g(x)}$ has a smaller variance than $h(x)$, then importance sampling can result in a more efficient estimator. This method is better explained using a deep out-of-the-money option as an example. If we directly use risk-neutral $f(S_T)$ as the distribution, most of the simulated paths will yield $h(S_T)=0$ and as a result the estimation variance will be large. If we introduce a distribution $g(S_T)$ that has much wider span (fatter tail for S_T), more simulated paths will have positive $h(S_T)$. The scaling factor $\frac{f(x)}{g(x)}$ will keep the estimator unbiased, but the approach will have lower variance.

[13] Importance sampling is essentially a variance reduction method using a change of measure.

Low-discrepancy sequence: Instead of using random samples, we can generate a deterministic sequence of "random variable" that represents the distribution. Such low-discrepancy sequences may make the convergence rate $1/M$.

D. If there is no closed-form pricing formula for an option, how would you estimate its delta and gamma?

Solution: As we have discussed in problem *A*, the prices of options with or without closed-form pricing formulas can be derived using Monte Carlo simulation. The same methods can also be used to estimate delta and gamma by slightly changing the current underlying price from S to $S \pm \delta S$, where δS is a small positive value. Run Monte Carlo simulation for all three starting prices $S - \delta S$, S and $S + \delta S$, we will get their corresponding option prices $f(S - \delta S)$, $f(S)$ and $f(S + \delta S)$.

Estimated delta: $\Delta = \dfrac{\delta f}{\delta S} = \dfrac{f(S + \delta S) - f(S - \delta S)}{2\delta S}$

Estimated gamma: $\Gamma = \dfrac{\big(f(S + \delta S) - f(S)\big) - \big(f(S) - f(S - \delta S)\big)}{\delta S^2}$

To reduce variance, it's often better to use the same random number sequences to estimate $f(S - \delta S)$, $f(S)$ and $f(S + \delta S)$.[14]

E. How do you use Monte Carlo simulation to estimate π?

Solution: Estimation of π is a classic example of Monte Carlo simulation. One standard method to estimate π is to randomly select points in the unit square (x and y are independent uniform random variables between 0 and 1) and determine the ratio of points that are within the circle $x^2 + y^2 \le 1$. For simplicity, we focus on the first quadrant. As shown in Figure 7.1, any points within the circle satisfy the equation $x_i^2 + y_i^2 \le 1$. The percentage of the points within the circle is proportional to its area: $\hat{p} = \dfrac{\text{Number of } (x_i, y_i) \text{ within } x_i^2 + y_i^2 \le 1}{\text{Number of } (x_i, y_i) \text{ within the square}} = \dfrac{1/4\pi}{1 \times 1} = \dfrac{1}{4}\pi \Rightarrow \hat{\pi} = 4\hat{p}$.

So we generate a large number of independent (x, y) points, estimate the ratio of the points within the circle to the points in the square, and multiply the ratio by 4 to yield an estimation of π. Figure 7.1 uses only 1000 points for illustration. With today's

[14] The method may not work well if the payoff function is not continuous.

computing power, we can easily generate millions of (x, y) pairs to estimate π with good precision. 1,000 simulations with 1,000,000 (x, y) points each using Matlab took less than 1 minute on a laptop and gave an average estimation of π as 3.1416 with standard deviation 0.0015.

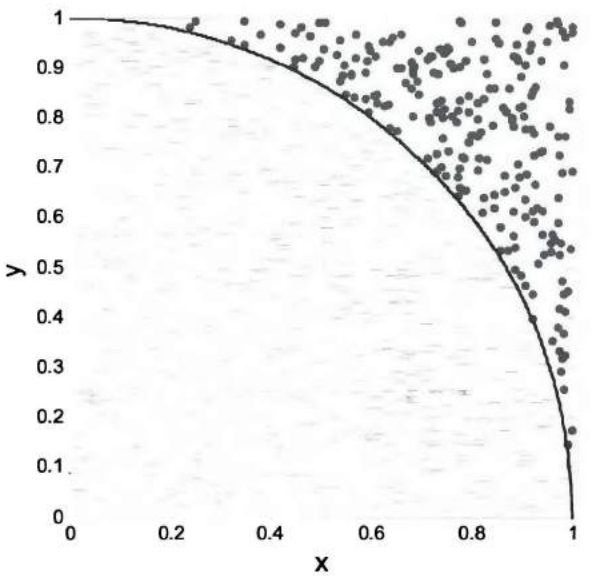

Figure 7.1 A Monte Carlo simulation method to estimate π

Finite difference method

The finite difference method is another popular numerical technique for derivative pricing. It numerically solves a differential equation to estimate the price of a derivative by discretizing the time and the price of the underlying security. We can convert the Black-Scholes-Merton equation, a second order nonlinear partial differential equation, to a heat diffusion equation (as we did in Chapter 6). This new equation, expressed as a function of τ (time to maturity) and x (a function of the price of the underlying security), is a general differential equation for derivatives. The difference between various derivatives lies in the boundary conditions. By building a grid of x and τ and using the boundary conditions, we can recursively calculate u at every x and τ using finite difference methods.

A. Can you briefly explain finite difference methods?

Solution: There are several version of finite difference methods used in practice. Let's briefly go over the explicit difference method, the implicit difference method and the

Crank-Nicolson method. As shown in Figure 7.2, if we divide the range of τ, $[0, T]$, into N discrete intervals with increment $\Delta\tau = T/N$ and divide the range of x, $[x_0, x_J]$, into J discrete intervals with increment $\Delta x = (x_J - x_0)/J$, the time τ and the space of x can be expressed as a grid of τ_n, $n = 1, \cdots, N$ and x_j, $j = 1, \cdots, J$.

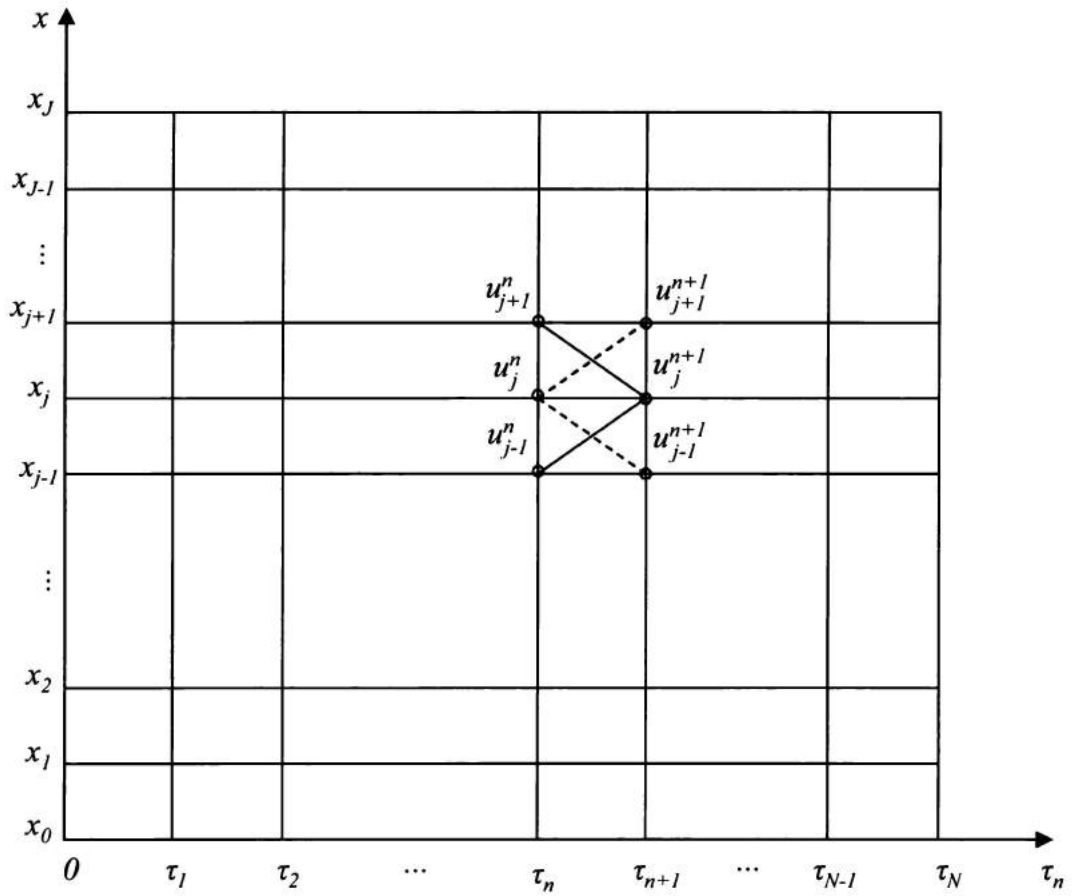

Figure 7.2 Grid of τ and x for finite different methods

The **explicit difference method** uses the forward difference at time τ_n and the second-order central difference at x_j: $\dfrac{\partial u}{\partial \tau} \approx \dfrac{u_j^{n+1} - u_j^n}{\Delta\tau} = \dfrac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} \approx \dfrac{\partial^2 u}{\partial x^2}$.

Rearranging terms, we can express u_j^{n+1} as a linear combination of u_{j+1}^n, u_j^n and u_{j-1}^n: $u_j^{n+1} = \alpha u_{j-1}^n + (1-2\alpha)u_j^n + \alpha u_{j+1}^n$, where $\alpha = \Delta t/(\Delta x)^2$. Besides, we often have boundary conditions u_j^0, u_0^n, and u_J^n for all $n = 1, \cdots, N$; $j = 0, \cdots, J$. Combining the boundary

conditions and equation $u_j^{n+1} = \alpha u_{j-1}^n + (1-2\alpha)u_j^n + \alpha u_{j+1}^n$, we can estimate all u_j^n's on the grid.

The **implicit difference method** uses the backward difference at time t_{n+1} and the second-order central difference at x_j: $\dfrac{\partial u}{\partial \tau} \approx \dfrac{u_j^{n+1} - u_j^n}{\Delta \tau} = \dfrac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\Delta x)^2} \approx \dfrac{\partial^2 u}{\partial x^2}$.

The Crank-Nicolson method uses the central difference at time $(t_n + t_{n+1})/2$ and the second-order central difference at x_j:

$$\frac{\partial u}{\partial \tau} \approx \frac{u_j^{n+1} - u_j^n}{\Delta \tau} = \frac{1}{2}\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} + \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\Delta x)^2} \right) \approx \frac{\partial^2 u}{\partial x^2}.$$

B. If you are solving a parabolic partial differential equation using the explicit finite difference method, is it worse to have too many steps in the time dimension or too many steps in the space dimension?

Solution: The equation for u_j^{n+1} in the explicit finite difference method is $u_j^{n+1} = \alpha u_{j-1}^n + (1-2\alpha)u_j^n + \alpha u_{j+1}^n$, where $\alpha = \Delta t/(\Delta x)^2$. For the explicit finite difference method to be stable, we need to have $1-2\alpha > 0 \Rightarrow \Delta t/(\Delta x)^2 < 1/2$. So a small Δt (i.e., many time steps) is desirable, but a small Δx (too many space steps) may make $\Delta t/(\Delta x)^2 > 1/2$ and the results unstable. In that sense, it is worse to have too many steps in the space dimension. In contrast, the implicit difference method is always stable and convergent.

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