Assignment 2

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2.

		tomorrow will be...		
		sunny	cloudy	rainy
today it's...	sunny	.8	.2	0
	cloudy	.4	.4	.2
	rainy	.2	.6	.2

$\Rightarrow$ Transition matrix $P(x_{t+1} \mid x_t) = \begin{bmatrix} 0.8 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.6 \\ 0 & 0.2 & 0.2 \end{bmatrix}$

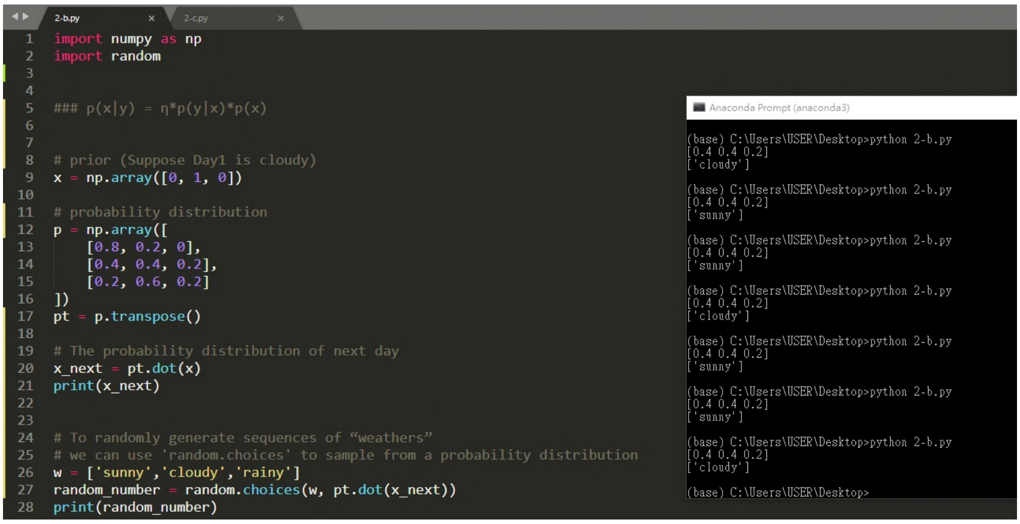
(a)

sunny $\rightarrow$ cloudy $\rightarrow$ cloudy $\rightarrow$ rainy

$0.2 \times 0.4 \times 0.2 = 0.016$ #

(b)

```
import numpy as np
import random

### p(x|y) = η*p(y|x)*p(x)

# prior (Suppose Day1 is cloudy)
x = np.array([0, 1, 0])

# probability distribution
p = np.array([
    [0.8, 0.2, 0],
    [0.4, 0.4, 0.2],
    [0.2, 0.6, 0.2]
])
pt = p.transpose()

# The probability distribution of next day
x_next = pt.dot(x)
print(x_next)


# To randomly generate sequences of “weathers”
# we can use 'random.choices' to sample from a probability distribution
w = ['sunny','cloudy','rainy']
random_number = random.choices(w, pt.dot(x_next))
print(random_number)
```



```
2-b.py    2-c.py

import numpy as np
import random


### p(x|y) = η*p(y|x)*p(x)


# prior (Suppose Day1 is cloudy)
x = np.array([0, 1, 0])

# probability distribution
p = np.array([
    [0.8, 0.2, 0],
    [0.4, 0.4, 0.2],
    [0.2, 0.6, 0.2]
])
pt = p.transpose()

# The probability distribution of next day
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# To randomly generate sequences of “weathers”
# we can use 'random.choices' to sample from a probability distribution
w = ['sunny','cloudy','rainy']
random_number = random.choices(w, pt.dot(x_next))
print(random_number)
```

```
Anaconda Prompt (anaconda3)

(base) C:\Users\USER\Desktop>python 2-b.py
[0.4 0.4 0.2]
['cloudy']

(base) C:\Users\USER\Desktop>python 2-b.py
[0.4 0.4 0.2]
['sunny']

(base) C:\Users\USER\Desktop>python 2-b.py
[0.4 0.4 0.2]
['sunny']

(base) C:\Users\USER\Desktop>python 2-b.py
[0.4 0.4 0.2]
['cloudy']

(base) C:\Users\USER\Desktop>python 2-b.py
[0.4 0.4 0.2]
['sunny']

(base) C:\Users\USER\Desktop>python 2-b.py
[0.4 0.4 0.2]
['sunny']

(base) C:\Users\USER\Desktop>python 2-b.py
[0.4 0.4 0.2]
['cloudy']

(base) C:\Users\USER\Desktop>
```

(c)

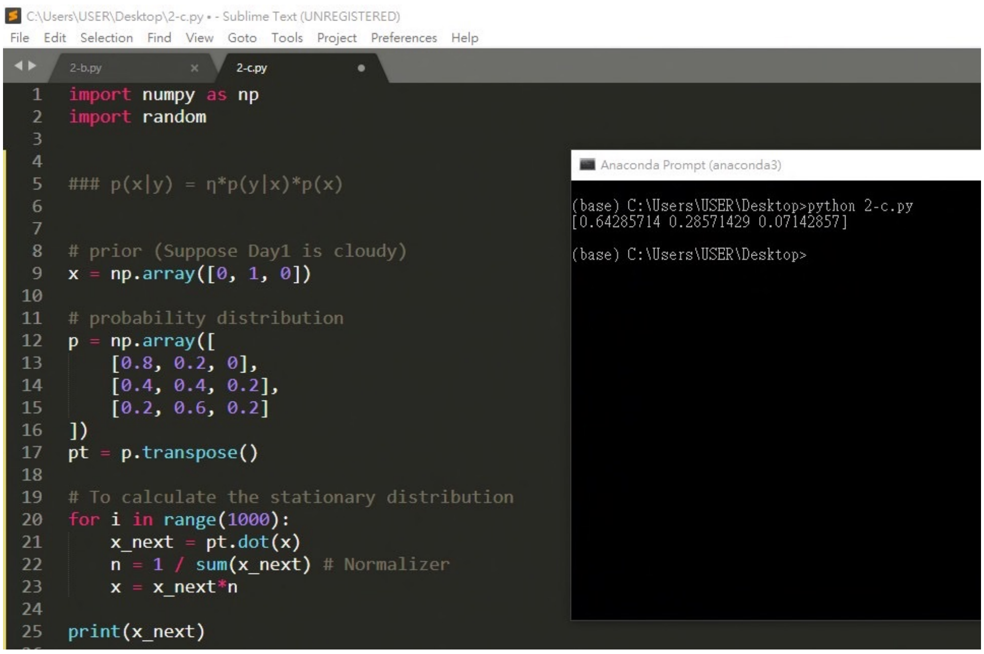
```
import numpy as np
import random

# prior (Suppose Day1 is cloudy)
x = np.array([0, 1, 0])

# probability distribution
p = np.array([
    [0.8, 0.2, 0],
    [0.4, 0.4, 0.2],
    [0.2, 0.6, 0.2]
])
pt = p.transpose()

# To calculate the stationary distribution
for i in range(1000):
    x_next = pt.dot(x)
    n = 1 / sum(x_next) # Normalizer
    x = x_next*n

print(x_next)
```



```
C:\Users\USER\Desktop\2-c.py • - Sublime Text (UNREGISTERED)
File  Edit  Selection  Find  View  Goto  Tools  Project  Preferences  Help
2-b.py    2-c.py

import numpy as np
import random


### p(x|y) = η*p(y|x)*p(x)


# prior (Suppose Day1 is cloudy)
x = np.array([0, 1, 0])

# probability distribution
p = np.array([
    [0.8, 0.2, 0],
    [0.4, 0.4, 0.2],
    [0.2, 0.6, 0.2]
])
pt = p.transpose()

# To calculate the stationary distribution
for i in range(1000):
    x_next = pt.dot(x)
    n = 1 / sum(x_next) # Normalizer
    x = x_next*n

print(x_next)
```

```
Anaconda Prompt (anaconda3)

(base) C:\Users\USER\Desktop>python 2-c.py
[0.64285714 0.28571429 0.07142857]

(base) C:\Users\USER\Desktop>
```

(d)

For $x_t \to$ probability distribution on day t, $A \to$ transition matrix

$\Rightarrow x_t = A x_{t-1} \Rightarrow x_t = A^n x_0 \quad \forall t \in N$

$\because A$ is diagonalizable $\therefore A = PDP^{-1}$

$$\det(A - \lambda I) = 0 \Rightarrow \det\left(\begin{bmatrix} 0.8-\lambda & 0.4 & 0.2 \\ 0.2 & 0.4-\lambda & 0.6 \\ 0 & 0.2 & 0.2-\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow (\lambda-1)\left(\lambda + \frac{\sqrt{2}-1}{5}\right)\left(\lambda - \frac{\sqrt{2}+1}{5}\right) = 0$$

$$\Rightarrow \lambda_1 = 1,\ \lambda_2 = \frac{-\sqrt{2}+1}{5},\ \lambda_3 = \frac{\sqrt{2}+1}{5}$$

$$(A - \lambda I) \cdot V = 0 \Rightarrow V_1 = \begin{bmatrix} 9 \\ 4 \\ 1 \end{bmatrix},\ V_2 = \begin{bmatrix} \sqrt{2}-1 \\ -\sqrt{2} \\ 1 \end{bmatrix},\ V_3 = \begin{bmatrix} -\sqrt{2}-1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 9 & \sqrt{2}-1 & -\sqrt{2}-1 \\ 4 & -\sqrt{2} & \sqrt{2} \\ 1 & 1 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} \frac{1}{14} & \frac{1}{14} & \frac{1}{14} \\ \frac{2\sqrt{2}-1}{28} & \frac{-5\sqrt{2}-1}{28} & \frac{2\sqrt{2}+13}{28} \\ \frac{-2\sqrt{2}-1}{28} & \frac{5\sqrt{2}-1}{28} & \frac{-2\sqrt{2}+13}{28} \end{bmatrix},\quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1+\sqrt{2}}{5} & 0 \\ 0 & 0 & \frac{1-\sqrt{2}}{5} \end{bmatrix}$$

$$\because \lim_{n\to\infty} D^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \lim_{n\to\infty} A^n = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^{-1} \fallingdotseq \begin{bmatrix} 0.643 \\ 0.286 \\ 0.071 \end{bmatrix}$$

#

(e)

$$H_p(x) = -\sum_x p(x)\log_2 p(x)$$

For transition probability matrix $P = \{P_{ij}\}$, stationary state vector $\mu = \{\mu_i\}$

$$\Rightarrow H_p(x) = -\sum_{ij} \mu_i P_{ij} \log_2 P_{ij}$$

$$P = \begin{bmatrix} 0.8 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.6 \\ 0 & 0.2 & 0.2 \end{bmatrix}, \quad \mu = \begin{bmatrix} 0.643 \\ 0.286 \\ 0.071 \end{bmatrix}$$

$$\begin{aligned} H_p(x) = -\Big[& (0.643 \times 0.8 \times \log_2 0.8) + (0.643 \times 0.4 \times \log_2 0.4) + (0.643 \times 0.2 \times \log_2 0.2) \\ & + (0.286 \times 0.2 \times \log_2 0.2) + (0.286 \times 0.4 \times \log_2 0.4) + (0.286 \times 0.6 \times \log_2 0.6) \\ & + (0.071 \times 0 \times \log_2 0) + (0.071 \times 0.2 \times \log_2 0.2) + (0.071 \times 0.2 \times \log_2 0.2)\Big] \\ = & -(-0.166 - 0.34 - 0.299 - 0.133 - 0.151 - 0.126 + 0 - 0.033 - 0.033) \\ = & \ 1.281 \end{aligned}$$

#

(f)

$$p(x_{t-1} \mid x_t) = \eta\, p(x_t \mid x_{t-1})\, p(x_{t-1})$$

Use stationary distribution $(0.643, 0.286, 0.071)^T$ as prior $p(x_{t-1})$

$$\begin{aligned} p(x_{t-1} = s \mid x_t = s) &= \eta_s P(x_t = s \mid x_{t-1} = s)\ p(x_{t-1} = s) \\ &= \eta_s \times 0.8 \times 0.643 = \eta_s 0.5144 \end{aligned}$$

$$\begin{aligned} p(x_{t-1} = c \mid x_t = s) &= \eta_s P(x_t = s \mid x_{t-1} = c)\ p(x_{t-1} = c) \\ &= \eta_s \times 0.4 \times 0.286 = \eta_s 0.1144 \end{aligned}$$

$$\begin{aligned} p(x_{t-1} = r \mid x_t = s) &= \eta_s P(x_t = s \mid x_{t-1} = r)\ p(x_{t-1} = r) \\ &= \eta_s \times 0.2 \times 0.071 = \eta_s 0.0142 \end{aligned}$$

$$\eta_s = (0.5144 + 0.1144 + 0.0142)^{-1} = 1.555$$

$$p(x_{t-1}=s \mid x_t=c) = \eta_c\, p(x_t=c \mid x_{t-1}=s)\; p(x_{t-1}=s)$$
$$= \eta_c \times 0.2 \times 0.643 = \eta_c\, 0.1286$$
$$p(x_{t-1}=c \mid x_t=c) = \eta_c\, p(x_t=c \mid x_{t-1}=c)\; p(x_{t-1}=c)$$
$$= \eta_c \times 0.4 \times 0.286 = \eta_c\, 0.1144$$
$$p(x_{t-1}=r \mid x_t=c) = \eta_c\, p(x_t=c \mid x_{t-1}=r)\; p(x_{t-1}=r)$$
$$= \eta_c \times 0.6 \times 0.071 = \eta_c\, 0.0426$$
$$\eta_c = (0.1286 + 0.1144 + 0.0426)^{-1} = 3.501$$

$$p(x_{t-1}=s \mid x_t=r) = \eta_r\, p(x_t=r \mid x_{t-1}=s)\; p(x_{t-1}=s)$$
$$= \eta_r \times 0 \times 0.643 = 0$$
$$p(x_{t-1}=c \mid x_t=r) = \eta_r\, p(x_t=r \mid x_{t-1}=c)\; p(x_{t-1}=c)$$
$$= \eta_r \times 0.2 \times 0.286 = \eta_r\, 0.0572$$
$$p(x_{t-1}=r \mid x_t=r) = \eta_r\, p(x_t=r \mid x_{t-1}=r)\; p(x_{t-1}=r)$$
$$= \eta_r \times 0.2 \times 0.071 = \eta_r\, 0.0142$$
$$\eta_r = (0.0572 + 0.0142)^{-1} = 14$$

$\therefore$

today \ yesterday	sunny	cloudy	rainy
sunny	0.8	0.18	0.02
cloudy	0.45	0.4	0.15
rainy	0	0.8	0.2

#

(g)

Take seasons as an time-varying state varible s_t of four values (Spring, Summer, Fall, Winter).

By extending the original 3x3 transition matrix to 12x12 (3 weathers x 4 seasons), we can still get all possible transition from $\{x_t, s_t\}$ to $\{x_{t+1}, s_{t+1}\}$

original state variable (weather)

$$\Rightarrow p(x_{t+1}, s_{t+1} \mid x_{1:t}, s_{1:t}) = p(x_{t+1}, s_{t+1} \mid x_t, s_t)$$

$\therefore$ The system wouldn't violate the Markov property. #

3.

		our sensor tells us...		
		sunny	cloudy	rainy
the actual weather is...	sunny	.6	.4	0
	cloudy	.3	.7	0
	rainy	0	0	1

(a)

	①	②	③	④	⑤
sensor z_t	x	c	c	r	s
actual x_t	s	?	?	?	s

$$P(x_5 \mid x_1, z_{2:5}) = \eta\, P(z_5 \mid x_5, x_1, z_{2:4})\, P(x_5 \mid x_1, z_{2:4})$$

$$= \eta\, P(z_5 \mid x_5)\, \underline{P(x_5 \mid x_1, z_{2:4})}$$

$\because P(x) = \sum_y P(x \mid y)\, P(y)$

$$\therefore P(x_5 \mid x_1, z_{2:4}) = \sum_{x_4} P(x_5 \mid x_4, x_1, z_{2:4})\,(x_4 \mid x_1, z_{2:4})$$

$$= \sum_{x_4} P(x_5 \mid x_4)\, P(x_4 \mid x_1, z_{2:4})$$

$\because$ the transition matrix shows that $z_4 = \text{rainy} \rightarrow x_4 = \text{rainy}$

$\therefore$ $P(x_5 \mid x_1, z_{2:4}) = P(x_5 \mid x_4 = r) \cdot 1$

So that

$$P(x_5 = s \mid x_1, z_{2:3}, z_4 = r, z_5 = s)$$

$$= \eta\, P(z_5 = s \mid x_5 = s)\, P(x_5 = s \mid x_4 = r)$$

$$= \frac{P(z_5 = s \mid x_5 = s)\, P(x_5 = s \mid x_4 = r)}{\sum_{x_5'} P(z_5 = s \mid x_5')\, P(x_5' \mid x_4 = r)}$$

$$= \frac{0.6 \times 0.2}{0.6 \times 0.2 + 0.3 \times 0.6 + 0} = 0.4 \quad \#$$

(b)

	①	②	③	④
sensor z_t	X	s	s	r
actual x_t	s			

Day 2 :

① $P(x_2 = s \mid z_2 = s) = \eta\, P(z_2 = s \mid x_2 = s)\, P(x_2 = s)$

$= \eta \times 0.6 \times 0.8 = 0.48\eta$

$P(x_2 = c \mid z_2 = s) = \eta\, P(z_2 = s \mid x_2 = c)\, P(x_2 = c)$

$= \eta \times 0.3 \times 0.2 = 0.06\eta$

$P(x_2 = r \mid z_2 = s) = \eta\, P(z_2 = s \mid x_2 = r)\, P(x_2 = r)$

$= 0$

$\eta = (0.48 + 0.06)^{-1} = 1.85$

$\therefore$ The probability of the weather is

$s \to 0.48 \times 1.85 = 0.889$, $c \to 0.06 \times 1.85 = 0.111$, $r \to 0$ #

② $P(x_2 \mid x_1, z_{2:4}) = \eta P(x_2 \mid x_1) P(z_{2:4} \mid x_2)$

$= \eta P(x_2 \mid x_1) P(z_2 \mid z_{3:4}, x_2) P(z_{3:4} \mid x_2)$

$= \eta P(x_2 \mid x_1) P(z_2 \mid x_2) \sum_{x_3} P(x_3, z_{3:4} \mid x_2)$

$= \eta P(x_2 \mid x_1) P(z_2 \mid x_2) \sum_{x_3} P(x_3 \mid x_2) P(z_3 \mid x_3) P(z_4 \mid x_3)$

$= \eta P(x_2 \mid x_1) P(z_2 \mid x_2) \sum_{x_3} P(x_3 \mid x_2) P(z_3 \mid x_3) \sum_{x_4} P(x_4, z_4 \mid x_3)$

$= \eta P(x_2 \mid x_1) P(z_2 \mid x_2) \sum_{x_3} P(x_3 \mid x_2) P(z_3 \mid x_3) \sum_{x_4} P(x_4 \mid x_3) P(z_4 \mid x_4)$

$= \eta \begin{bmatrix} 0.00576 \\ 0.00144 \\ 0 \end{bmatrix} \Rightarrow \eta = (0.00576 + 0.00144)^{-1} = 138.9$

$\therefore$ The probability of the weather is

$s \to 0.00576 \times 138.9 = 0.8$, $c \to 0.00144 \times 138.9 = 0.2$, $r \to 0$ #

Day 3 :

① $P(x_3 \mid x_1, z_{2:3}) = \eta P(z_3 \mid x_3, x_1, z_2) P(x_3 \mid x_1, z_2)$

$= \eta P(z_3 \mid x_3) \sum_{x_2} P(x_3 \mid x_2) P(x_2 \mid x_1, z_2)$

$= \eta \begin{bmatrix} 4.08 \\ 0.6 \\ 0 \end{bmatrix} \Rightarrow \eta = (4.08 + 0.6)^{-1} = 0.214$

$\therefore$ The probability of the weather is

$s \to 4.08 \times 0.214 = 0.87$, $c \to 0.6 \times 0.214 = 0.13$, $r \to 0$ #

② $P(x_3 \mid x_1, z_{2:4}) = \eta P(x_3 \mid x_1, z_{2:3}) P(z_4 \mid x_3, x_1, z_{2:3})$

$= \eta P(x_3 \mid x_2, x_3) P(z_4 \mid x_3)$

$$= \eta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ The probability of the weather is $s \to 0, c \to 1, r \to 0$ #

Day 4 :

$$P(x_4 \mid x_1, z_{2:4}) = P(x_4 \mid x_3, z_4) = \eta \, P(z_4 \mid x_4) \, P(x_4 \mid x_3)$$

$$= \eta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ The probability of the weather is $s \to 0, c \to 0, r \to 1$ #

(c)

According to the result of (b), the most likely sequence of weather is <u>sunny, cloudy, rainy</u> $\Rightarrow 0.8 \times 1 \times 1 = 80\%$ #