KUREPA TREES, CONTINUOUS IMAGES, AND PERFECT SET PROPERTIES

CHRIS LAMBIE-HANSON AND ŠÁRKA STEJSKALOVÁ

Abstract. Building upon work of Lücke and Schlicht, we study (higher) Kurepa trees through the lens of higher descriptive set theory, focusing in particular on various perfect set properties and representations of sets of branches through trees as continuous images of function spaces. Answering a question of Lücke and Schlicht, we prove that it is consistent with CH that there exist ω_2-Kurepa trees and yet, for every ω_2-Kurepa tree $T \subseteq {}^{<\omega_2}\omega_2$, the set $[T] \subseteq {}^{\omega_2}\omega_2$ of cofinal branches through T is not a continuous image of ${}^{\omega_2}\omega_2$. We also produce models indicating that the existence of Kurepa trees is not necessary to produce closed subsets of ${}^{\omega_1}\omega_1$ failing to satisfy strong perfect set properties, and prove a number of consistency results regarding *full* and *superthin* trees.

1. Introduction

Two fundamental features of closed subsets of the classical Baire space ${}^{\omega}\omega$ are the following:

(1) every nonempty closed subset of ${}^{\omega}\omega$ is a continuous image (and, in fact, a retract) of ${}^{\omega}\omega$;
(2) every closed subset of ${}^{\omega}\omega$ has the perfect set property.

When passing to the higher Baire spaces of the form ${}^{\kappa}\kappa$ for regular uncountable cardinals κ, both of these properties at least consistently fail. Regarding property (1), for an arbitrary regular uncountable cardinal κ, it is proven in [7, Proposition 1.4] that there exists a nonempty closed subset of ${}^{\kappa}\kappa$ that is not a retract of ${}^{\kappa}\kappa$. If we assume moreover that $\kappa^{<\kappa} = \kappa$, which is a standard assumption in the study of higher Baire spaces, then it is proven in [7, Theorem 1.5] that there exists a nonempty closed subset of ${}^{\kappa}\kappa$ that is not a continuous image of ${}^{\kappa}\kappa$.

When discussing higher analogues of the perfect set property, we need to fix an appropriate notion of perfectness. Unlike in the classical Baire space, where there is a single, unambiguous notion of "perfectness", in the higher Baire space ${}^{\kappa}\kappa$ there exist various degrees of perfectness, leading to a corresponding spectrum of perfect set properties. These will be discussed in more detail below in Section 3, but for now let us consider the strongest and arguably most natural version. We say that a *κ-perfect* subset of ${}^{\kappa}\kappa$ is the set of all cofinal branches through some cofinally-splitting ($<\kappa$)-closed subtree of ${}^{<\kappa}\kappa$ (see Section 2 for a precise definition). A set

2020 *Mathematics Subject Classification.* 03E05, 03E35, 03E47.

Key words and phrases. Kurepa trees, higher Baire space, perfect set properties, continuous images, full trees.

Both authors were supported by the Czech Academy of Sciences (RVO 67985840) and the GAČR project 23-04683S.

$X \subseteq {}^\kappa\kappa$ has the *κ-perfect set property*, denoted $\mathsf{PSP}_\kappa(X)$, if either $|X| \leq \kappa$ or X contains a κ-perfect subset.

As noted in [6, §7], if κ is a regular uncountable cardinal and every closed subset of ${}^\kappa\kappa$ has the perfect set property, then κ^+ is inaccessible in L. On the other hand, by an argument of Schlicht (cf. [6, §9]), if κ is a regular uncountable cardinal, $\nu > \kappa$ is inaccessible, and G is generic for the Lévy collapse $\mathrm{Coll}(\kappa, {<}\nu)$, then, in $V[G]$ $\mathsf{PSP}_\kappa(X)$ holds for every closed $X \subseteq {}^\kappa\kappa$ (in fact, for every Σ^1_1 set $X \subseteq {}^\kappa\kappa$). Thus, the κ-perfect set property for closed subsets of ${}^\kappa\kappa$ for some regular uncountable κ is equiconsistent with the existence of an inaccessible cardinal.

A starring role in the story of the preceding two paragraphs is played by *Kurepa trees*. Indeed, suppose that κ is a regular uncountable cardinal that is not strong limit, e.g., a successor cardinal. If $T \subseteq {}^{<\kappa}\kappa$ is a κ-Kurepa tree[1] then the set $[T]$ of all cofinal branches through T is a closed subset of ${}^\kappa\kappa$ that fails to satisfy the κ-perfect set property (see [6] for a proof of this; the idea of using Kurepa trees as a counterexample for the κ-perfect set property goes back at least to [18] and [9]). The fact that $\mathsf{PSP}_\kappa(X)$ holding for all closed $X \subseteq {}^\kappa\kappa$ implies that κ^+ is inaccessible in L then follows from an argument of Solovay (cf. [5, §4]).

It is thus natural to also consider Kurepa trees in the context of item (1) above, leading to the following general question: for a fixed regular uncountable cardinal κ, given a κ-Kurepa tree T, under what circumstances is $[T]$ a continuous image of, or even a retract of, the space ${}^\kappa\kappa$. This and related questions were considered by Lücke and Schlicht in [8], where they obtained a number of interesting results, including the following:

- If there is $\mu < \kappa$ such that $\mu^\omega \geq \kappa$, and $T \subseteq {}^{<\kappa}\kappa$ is a κ-Kurepa tree with $|[T]| > \kappa^{<\kappa}$, then $[T]$ is not a continuous image of ${}^\kappa\kappa$. In particular, assuming instances of GCH, the question has an easy negative answer if κ is a successor of a cardinal of countable cofinality, e.g., ω_1.
- If $V = L$, then, for every regular uncountable cardinal κ that is not the successor of a cardinal of countable cofinality, there exists a κ-Kurepa tree $T \subseteq {}^{<\kappa}\kappa$ such that $[T]$ is a retract of ${}^\kappa\kappa$.
- If $\mu < \kappa$ are regular uncountable cardinals, with κ inaccessible, then there is a forcing extension in which (1) $\kappa = \mu^+$, (2) there exists a κ-Kurepa tree $T \subseteq {}^{<\kappa}\kappa$ such that $[T]$ is a retract of ${}^\kappa\kappa$; and (3) there exists a κ-Kurepa tree $S \subseteq T$ such that $[S]$ is not a continuous image of ${}^\kappa\kappa$.
- If there is a κ-Kurepa tree, then there is a κ-Kurepa tree $T \subseteq {}^{<\kappa}\kappa$ such that $[T]$ is not a retract of ${}^\kappa\kappa$.

The work of Lücke and Schlicht left open a number of avenues for further research. One question, explicitly asked as [8, Question 6.2], is whether CH together with the existence of an ω_2-Kurepa tree implies the existence of an $\aleph_2$-Kurepa tree $S \subseteq {}^{<\omega_2}\omega_2$ such that $[S]$ is a continuous image of ${}^{\omega_2}\omega_2$. We provide a negative answer to this question (see Theorem B below).

We also undertake further explorations of topics connecting Kurepa trees, (variations on) the perfect set property, and continuous images of higher Baire spaces. Here we make use of a hierarchy of generalizations of perfectness introduced by Väänänen [18] in terms of the existence of winning strategies for certain two-player games. In particular, given a regular cardinal κ and an ordinal δ with

[1] See Section 2 for a precise definition.

$\omega \leq \delta \leq \kappa$, Väänänen defines a notion of *δ-perfectness* for subsets of ${}^{\kappa}\kappa$. These notions strengthen as δ increases, with ω-perfectness being classical perfectness (i.e., closed and having no isolated points), and κ-perfectness being as above.

Recall the classical Cantor-Bendixson theorem: If $E \subseteq {}^{\omega}\omega$ is closed, then there is a countable set $X \subseteq E$ such that $E \setminus X$ is perfect. This generalizes straightforwardly to higher Baire spaces: if $E \subseteq {}^{\kappa}\kappa$ is closed, then there is $X \subseteq E$ such that $|X| \leq \kappa^{<\kappa}$ and $E \setminus X$ is perfect. We prove in Section 3 that, if E is a continuous image of ${}^{\kappa}\kappa$, then we can obtain a stronger conclusion.[2]

In the other direction, recall that κ-Kurepa trees yield examples of closed subsets of ${}^{\kappa}\kappa$ of cardinality greater than κ that do not contain κ-perfect subsets; in fact, if $\kappa = \mu^+$ and $T \subseteq {}^{<\kappa}\kappa$ is a κ-Kurepa tree, then $[T]$ does not contain a $(\mu+1)$-perfect subset. In Section 4, we prove results showing that Kurepa trees are not necessary for the existence of such sets by producing models in which GCH holds, the Kurepa Hypothesis fails (i.e., there are no ω_1-Kurepa trees), but there is a closed subset of ${}^{\omega_1}\omega_1$ of cardinality ω_2 with no $(\omega+1)$-perfect subset. In fact, we prove the following stronger theorem:[3]

Theorem A. *Suppose that there is an inaccessible cardinal. Then there is a forcing extension in which*

(1) GCH *holds;*
(2) the Kurepa Hypothesis fails;
(3) there is a weak Kurepa tree that does not contain a copy of ${}^{<\omega+1}2$.

We actually produce two models witnessing the conclusion of Theorem A. In the first, every weak Kurepa tree contains an Aronszajn (or even Suslin) subtree. In the second, there is a weak Kurepa tree that contains neither a copy of ${}^{<\omega+1}2$ nor an Aronszajn subtree.

In Section 5, we provide a negative answer to the aforementioned question of Lücke and Schlicht. In addition, we show that, in the model we construct, ${}^{\omega_2}\omega_2$ satisfies the strongest possible analogue of the Cantor-Bendixson theorem compatible with the existence of an ω_2-Kurepa tree (recall that, if $T \subseteq {}^{<\omega_2}\omega_2$ is an ω_2-Kurepa tree, then $[T]$ cannot contain an (ω_1+1)-perfect subset). In particular, we prove the following theorem.

Theorem B. *If there exists an inaccessible cardinal, then there is a forcing extension in which* GCH *holds, there exists an ω_2-Kurepa tree, and, for every ω_2-Kurepa tree $S \subseteq {}^{<\omega_2}\omega_2$, $[S]$ is not a continuous image of ${}^{\omega_2}\omega_2$. In addition, in the forcing extension, for every closed subset $E \subseteq {}^{\omega_2}\omega_2$, there is $X \subseteq E$ with $|X| \leq \omega_2$ such that $E \setminus X$ is ω_1-perfect.*

In Section 6, we prove some new results about full trees. A tree T is *full* if, for every limit ordinal β below the height of T, there is at most one branch through $T \restriction \beta$ that is not continued at level β (see Section 6 for a precise definition). Kunen asked whether there could consistently exist a full κ-Suslin tree for some regular uncountable cardinal κ (cf. [10]). This was answered positively by Shelah in [15] for inaccessible κ and recently by Rinot, Yadai, and You in [12] for successors

[2] For more on stronger versions of the Cantor-Bendixson theorem for higher Baire spaces, see [18].

[3] Recall that a *weak Kurepa tree* is a tree of height and size ω_1 with at least ω_2-many cofinal branches.

of regular uncountable cardinals. Here, we are interested in full, splitting trees that may contain some cofinal branches. For example, in Section 6, we prove the following theorem about full trees of height ω_1:

Theorem C. *(1) If $\diamondsuit$ holds, then, for every cardinal $\nu \in \omega \cup \{\omega, \omega_1, 2^{\omega_1}\}$, there is a normal, full, splitting tree $T \subseteq {}^{<\omega_1}\omega_1$ with exactly ν-many cofinal branches.*

(2) CH does not suffice for the conclusion of clause (1). In particular, CH is compatible with the assertion that every full, splitting tree of height ω_1 contains a copy of ${}^{<\omega_1}2$, and hence has 2^{ω_1}-many cofinal branches.

We then move up a cardinal to prove the consistency of the existence of a normal, splitting, full, superthin ω_2-Kurepa tree. We refer the reader to Section 6 for the definition of *superthin*; we only note that superthin Kurepa trees play a central role in the investigations of Lücke and Schlicht in [8]. In particular, they prove there that if there exists a superthin κ-Kurepa tree, then there is a superthin κ-Kurepa tree $T \subseteq {}^{<\kappa}\kappa$ such that $[T]$ is a retract of ${}^{\kappa}\kappa$. In Section 7, we prove the following result, producing a model having ω_2-Kurepa trees with very different behavior from that produced in Theorem B:

Theorem D. *Suppose that GCH holds and κ is the successor of a regular uncountable cardinal. Then there is a cofinality-preserving forcing extension in which*

(1) GCH holds;
(2) there exists a κ-Kurepa tree;
(3) every κ-Kurepa tree contains a normal superthin κ-Kurepa subtree.

Finally, in Section 8, we record some closing remarks and a few questions that remain open.

1.1. **Notation and conventions.** Our notation is for the most part standard. We refer the reader to [4] for undefined notions and notations in set theory. We let Card denote the class of all cardinals. $(\mathsf{G})\mathsf{CH}$ denotes the (generalized) continuum hypothesis. Given a well-ordered set X, we denote its order type by $\mathrm{otp}(X)$. If X and Y are sets, $f : X \to Y$, and $A \subseteq X$, then both $f[A]$ and $f``A$ denote the pointwise image of A, i.e., $\{f(a) \mid a \in A\}$.

Given ordinals α and β, we let ${}^{\alpha}\beta$ denote the set of all functions $f : \alpha \to \beta$, and we let ${}^{<\alpha}\beta$ denote $\bigcup_{\eta<\alpha} {}^{\eta}\beta$. We use ${}^{\leq\alpha}\beta$ and ${}^{<(\alpha+1)}\beta$ interchangeably. If $\sigma, \tau \in {}^{<\alpha}\beta$, then we let $\sigma \sqsubseteq \tau$ denote the assertion that σ is an *initial segment* of τ, i.e., $\mathrm{dom}(\sigma) \leq \mathrm{dom}(\tau)$ and $\sigma = \tau \restriction \mathrm{dom}(\sigma)$. Given a nonempty set $A \subseteq {}^{\leq\alpha}\beta$, we let $\bigwedge A \in {}^{\leq\alpha}\beta$ denote the (unique) $\sqsubseteq$-maximal element of ${}^{\leq\alpha}\beta$ that is an initial segment of every element of A, i.e.,

$$\bigwedge A = \bigcup \{\sigma \in {}^{\leq\alpha}\beta \mid \forall \tau \in A \ (\sigma \sqsubseteq \tau)\}.$$

If σ and τ are two functions whose domains are ordinals α and β, respectively, then $\sigma ^\frown \tau$ denotes the concatenation of σ and τ, i.e., the function ρ with domain $\alpha + \beta$ such that $\rho(\eta) = \sigma(\eta)$ for all $\eta < \alpha$ and $\rho(\alpha + \xi) = \tau(\xi)$ for all $\xi < \beta$. We will sometimes think of functions with ordinal domains as sequences, e.g., a sequence of the form $\langle\gamma\rangle$ will be thought of as a function σ with domain 1 and $\sigma(0) = \gamma$. We will sometimes write, e.g., $\sigma ^\frown \gamma$ instead of $\sigma ^\frown \langle\gamma\rangle$ when there is no risk of confusion.

We make use of various standard forcing notions throughout the paper. In particular, if κ is a regular infinite cardinal, then $\mathrm{Add}(\kappa, 1)$ is the forcing to add a single Cohen subset of κ. If, in addition, $\mu > \kappa$ is a cardinal, then $\mathrm{Coll}(\kappa, {<}\mu)$ is the Lévy collapse that collapses every cardinal in the interval (κ, μ) to have cardinality κ. We assume that every forcing notion $\mathbb{P}$ has a maximum element, which we denote $1_{\mathbb{P}}$. If we define a forcing notion $\mathbb{P}$ that does not have a maximum element, then we implicitly add $\emptyset$ as $1_{\mathbb{P}}$.

2. Combinatorial and topological preliminaries

In this section, we introduce some of the basic notions forming the subject matter of this paper, particularly regarding trees, higher Baire spaces, and forcing.

Definition 2.1. A *tree* is a partial order $(T, \leq_T)$ such that, for every $t \in T$, the set $\mathrm{pred}_T(t) := \{s \in T \mid s <_T t\}$ is well-ordered by $\leq_T$. We will often abuse notation and simply refer to the tree by T, without explicitly mentioning the tree order $\leq_T$. Given $t \in T$, the *height* of t in T is $\mathrm{ht}_T(t) := \mathrm{otp}(\mathrm{pred}_T(t))$. Given an ordinal α, the α^{th} *level* of T, denoted T_α, is $\{t \in T \mid \mathrm{ht}_T(t) = \alpha\}$. The *height* of T, denoted $\mathrm{ht}(T)$, is the least ordinal β such that $T_\beta = \emptyset$. A *subtree* of T is a $\leq_T$-downward closed subset of T, with the tree order inherited from $\leq_T$.

We say that a tree T is *normal* if it satisfies the following two conditions:

- for all $\alpha < \beta < \mathrm{ht}(T)$ and all $s \in T_\alpha$, there is $t \in T_\beta$ such that $s <_T t$; and
- for all limit ordinals $\alpha < \mathrm{ht}(T)$ and all $s, t \in T_\alpha$, if $\mathrm{pred}_T(s) = \mathrm{pred}_T(t)$, then $s = t$.

A *branch* through T is a maximal linearly ordered subset of T. We say that a branch b through T is a *cofinal branch* through T if $b \cap T_\alpha \neq \emptyset$ for all $\alpha < \mathrm{ht}(T)$. The set of all cofinal branches through T is denoted by $[T]$.

Given two nodes $s, t \in T$, we write $s \perp t$ to denote the assertion that s and t are $\leq_T$-incomparable. An *antichain* of T is a set $A \subseteq T$ such that $s \perp t$ for all distinct $s, t \in A$.

Definition 2.2. Suppose that κ is an infinite cardinal. A tree T is a *κ-tree* if $\mathrm{ht}(T) = \kappa$ and $|T_\alpha| < \kappa$ for every $\alpha < \kappa$. A *κ-Aronszajn tree* is a κ-tree with no cofinal branches, and a *κ-Suslin tree* is a κ-tree with no cofinal branches and no antichains of cardinality κ.

A *κ-Kurepa tree* is a κ-tree T with at least κ^+-many cofinal branches. A *weak κ-Kurepa tree* is a tree T of height and size κ with at least κ^+-many cofinal branches.

The *κ-Kurepa Hypothesis* (KH_κ) is the assertion that there exists a κ-Kurepa tree. The *weak κ-Kurepa Hypothesis* (wKH_κ) is the assertion that there exists a weak Kurepa tree. If the parameter κ is omitted in any of the above, then it should be understood that $\kappa = \omega_1$, e.g., a *Suslin tree* is an ω_1-Suslin tree, a *Kurepa tree* is an ω_1-Kurepa tree, and KH is KH_{ω_1}.

Definition 2.3. Let T be a normal tree. For $s, t \in T$, the meet $s \wedge t$ is the unique node $r \leq s, t$ such that there is no $r' > r$ with $r' \leq s, t$.

For ordinals α and β, we will naturally interpret ${}^{<\alpha}\beta$ as a tree by setting, for all $\sigma, \tau \in {}^{<\alpha}\beta$, $\sigma \leq \tau$ if and only if σ is an initial segment of τ (denoted $\sigma \sqsubseteq \tau$). When we say that a tree T is a subtree of ${}^{<\alpha}\beta$, we will implicitly assume that T has height α. In particular, in this context, when α is a limit ordinal, the set $[T]$ of cofinal branches through T can and will be identified with the set

$$\{b \in {}^{\alpha}\beta \mid \forall \eta < \alpha \ (b \restriction \eta \in T)\}.$$

If κ and λ are cardinals, with κ regular, then we implicitly interpret ${}^{\kappa}\lambda$ as a topological space, where each copy of λ is given the discrete topology and the product is given the $(<\kappa)$-box topology. In other words, the topology on ${}^{\kappa}\lambda$ is generated by all basic open sets of the form

$$N_\sigma = \{x \in {}^{\kappa}\lambda \mid \sigma \sqsubseteq x\},$$

where $\sigma \in {}^{<\kappa}\lambda$. We will be most interested in the space ${}^{\kappa}\kappa$ for a regular uncountable cardinal κ; this space is typically called the *higher Baire space at* κ.

Fix for now cardinals κ and λ, with κ regular. Given a nonempty set $X \subseteq {}^{\kappa}\lambda$, let

$$T(X) := \{x \restriction \alpha \mid x \in X \text{ and } \alpha < \kappa\}.$$

Note that $T(X)$ is a subtree of ${}^{<\kappa}\lambda$.

In the other direction, given a tree $T \subseteq {}^{<\kappa}\lambda$, it is a well-known and easily verified fact that $[T]$ is a closed subset of ${}^{\kappa}\lambda$ and, moreover, all closed subsets of ${}^{\kappa}\lambda$ are of this form:

Fact 2.4. *Suppose that κ and λ are cardinals, with κ regular. Then a subset $X \subseteq {}^{\kappa}\lambda$ is closed if and only if it is of the form $[T]$ for some tree $T \subseteq {}^{<\kappa}\lambda$.*

We now recall a number of relevant definitions from [13].

Definition 2.5. Suppose that T is a subtree of ${}^{<\kappa}\lambda$.

(1) T is *normal* if for all $\sigma \in T$ and all $\alpha < \kappa$, there is $\tau \in T$ such that $\sigma \sqsubseteq \tau$ and $\mathrm{dom}(\tau) \geq \alpha$.
(2) T is *splitting* if, for all $\sigma \in T$, if $|\sigma| + 1 < \mathrm{ht}(T)$, then there are distinct $i, j < \lambda$ such that $\sigma^\frown i, \sigma^\frown j \in T$. It is *infinitely splitting* if, for all such $\sigma \in T$, there are infinitely many $i < \lambda$ such that $\sigma^\frown i \in T$.
(3) T is *cofinally splitting* if, for every $\sigma \in T$, there are $\tau_0, \tau_1 \in T$ such that $\sigma \sqsubseteq \tau_0$, $\sigma \sqsubseteq \tau_1$, and $\tau_0 \perp \tau_1$.
(4) Given a cardinal $\mu \leq \kappa$, T is $<\mu$-closed if, for every $\eta < \mu$ and every increasing sequence $\langle \sigma_\xi \mid \xi < \eta \rangle$ from T, we have $\bigcup\{\sigma_\xi \mid \xi < \eta\} \in T$.
(5) T is *κ-perfect* if it is normal, cofinally splitting, and $<\kappa$-closed.
(6) A nonempty closed subset $X \subseteq {}^{\kappa}\lambda$ is said to be κ-perfect if $T(X)$ is κ-perfect.

Definition 2.6. Suppose that S and T are trees and $\iota : S \to T$.

(1) ι is *strict order preserving* if, for all $\sigma, \tau \in S$, if $\sigma < \tau$, then $\iota(\sigma) < \iota(\tau)$.
(2) ι is *$\perp$-preserving* if, for all $\sigma, \tau \in S$, if $\sigma \perp \tau$, then $\iota(\sigma) \perp \iota(\tau)$.
(3) If ι is both strict order preserving and $\perp$-preserving, then we call ι an *isomorphic embedding* of S into T. We say that T *contains a copy of* S if there is an isomorphic embedding $\iota : S \to T$; in this situation, the image $\iota[S]$ will be referred to as *a copy of S in T*.

Remark 2.7. We note that, in the above definition, the maps $\iota : S \to T$ need not preserve levels, i.e., there may be $\sigma \in S$ such that $\mathrm{ht}_S(\sigma) \neq \mathrm{ht}_T(\iota(\sigma))$. In particular, a copy of S in T need not be a subtree of T, as it need not be downward closed. Also note that S and T need not have the same height, e.g., it could be the case that S is a subtree of ${}^{<\alpha}\beta$ and T is a subtree of ${}^{<\gamma}\delta$ for cardinals $\alpha \neq \gamma$ and $\beta \neq \delta$. Note, however, that if $\iota : S \to T$ is strict order preserving, then we will have $\mathrm{ht}_S(s) \leq \mathrm{ht}_T(\iota(s))$ for all $s \in S$.

Definition 2.8. Let κ and λ be cardinals, with κ regular and $\lambda \leq \kappa$, and suppose that $X \subseteq {}^\kappa\lambda$. We say that X has the *κ-perfect set property*, denoted $\mathsf{PSP}_\kappa(X)$, if either $|X| \leq \kappa$ or X contains a κ-perfect subset.

Note that, in the classical Baire space ${}^\omega\omega$, a nonempty closed subset of ${}^\omega\omega$ is perfect in the classical sense (i.e., is closed with no isolated points) if and only if it is ω-perfect in the sense of Definition 2.5(6), and a set $X \subseteq {}^\omega\omega$ has the classical perfect set property if and only if it satisfies $\mathsf{PSP}_\omega(X)$ as in Definition 2.8.

The following well-known proposition is easily proven.

Proposition 2.9. *Let κ and λ be cardinals, with κ regular and $\lambda \leq \kappa$, and suppose that $X \subseteq {}^\kappa\lambda$ is closed. Then $\mathsf{PSP}_\kappa(X)$ holds if and only if $|X| \leq \kappa$ or $T(X)$ contains a copy of ${}^{<\kappa}2$.*

Some of our results discuss situations in which a particular topological space is or is not a continuous image (or retract) of another. Let us recall the relevant definitions here.

Definition 2.10. Suppose that X and Y are topological spaces.

(1) We say that X is a *continuous image* of Y if there is a continuous surjection $g : Y \to X$.
(2) Suppose moreover that X is a subspace of Y. Then we say that X is a *retract* of Y if there is a continuous surjection $g : Y \to X$ such that $g \restriction X$ is the identity map.

We will most often invoke Definition 2.10 in the situation in which κ and λ are cardinals, with κ infinite and regular, $Y = {}^\kappa\lambda$ (with the $(< \kappa)$-box topology), and $X = [T]$ for some subtree $T \subseteq {}^{<\kappa}\lambda$, where $[T]$ is given the subspace topology inherited from Y.

We end this section with some preliminaries on forcing. We first recall the definition of *strategic closure*.

Definition 2.11. Let $\mathbb{P}$ be a partial order and let β be an ordinal.

(1) $\Game_\beta(\mathbb{P})$ is the two-player game in which Players I and II alternate playing conditions from $\mathbb{P}$ to attempt to construct a $\leq_{\mathbb{P}}$-decreasing sequence $\langle p_\alpha \mid \alpha < \beta\rangle$. Player I plays at odd stages, and Player II plays at even stages (including limit stages). Player II is required to play $p_0 = 1_{\mathbb{P}}$. If, during the course of play, a limit ordinal $\alpha < \beta$ is reached such that $\langle p_\eta \mid \eta < \alpha\rangle$ has no lower bound in $\mathbb{P}$, then Player I wins. Otherwise, Player II wins.
(2) $\mathbb{P}$ is said to be *β-strategically closed* if Player II has a winning strategy in $\Game_\beta(\mathbb{P})$.

We now recall the following lemma, due to Silver. The lemma is usually stated with the hypothesis that $\mathbb{R}$ is τ^+-closed (cf. [17, Lemma 4]), but the standard proof is easily seen to work under the weaker assumption that $\mathbb{R}$ is $(\tau + 1)$-strategically closed, so we leave it to the reader.

Lemma 2.12. *Suppose that $\tau < \nu$ are infinite regular cardinals, with $2^\tau \geq \nu$. Suppose that T is a ν-tree and $\mathbb{R}$ is a forcing poset that is $(\tau + 1)$-strategically closed. Then forcing with $\mathbb{R}$ cannot add a branch of length ν through T, i.e., every cofinal branch through T in $V^{\mathbb{R}}$ is in V.*

3. Väänänen's game

The following game was introduced by Väänänen in [18] in order to generalize the notion of perfectness.

Definition 3.1. Suppose that κ is a regular cardinal, E is a subset of ${}^{\kappa}\kappa$, $x_0 \in {}^{\kappa}\kappa$, and $\delta \leq \kappa$ is an infinite ordinal. Then the two-player game $G_\kappa(E, x_0, \delta)$ is defined as follows. The game consists of rounds indexed by ordinals ξ with $1 \leq \xi < \delta$. In round ξ, Player I first plays an ordinal $\alpha_\xi < \kappa$, and then Player II plays an element $x_\xi \in E$. The plays must satisfy the following requirements:

- $\langle \alpha_\xi \mid 1 \leq \xi < \delta \rangle$ is an increasing, continuous sequence of ordinals;
- for all $0 \leq \eta < \xi < \delta$, we have
  - $x_\eta \neq x_\xi$;
  - $x_\eta \restriction \alpha_{\eta+1} = x_\xi \restriction \alpha_{\eta+1}$.

Player II wins if they can successfully play x_ξ for all $1 \leq \xi < \delta$; otherwise, Player I wins.

Definition 3.2. Suppose that κ is a regular cardinal, E is a subset of ${}^{\kappa}\kappa$, and $\delta \leq \kappa$ is an infinite ordinal. Then we say that E is *δ-perfect* if it is closed and, for every $x_0 \in E$, Player II has a winning strategy in $G_\kappa(E, x_0, \delta)$.

Note that $G_\kappa(E, x_0, \delta)$ becomes harder for Player II to win as δ increases; thus, if $\omega \leq \delta_0 \leq \delta_1 \leq \kappa$ and $E \subseteq {}^{\kappa}\kappa$ is δ_1-perfect, then it is also δ_0-perfect.

Remark 3.3. Given a regular uncountable cardinal κ, there is a slight discrepancy between the notion of κ-perfect isolated in Definition 3.2 and the notion of κ-perfect isolated in Definition 2.5(6), which we will call *strongly κ-perfect* when we need to distinguish it from the notion in Definition 3.2. For example, as noted in [18], the set of all $x \in {}^{\kappa}\kappa$ such that $x(\alpha) = 0$ for only finitely many α is κ-perfect but not strongly κ-perfect. However, it is readily verified that every strongly κ-perfect subset of ${}^{\kappa}\kappa$ is κ-perfect and, conversely, every κ-perfect subset of ${}^{\kappa}\kappa$ contains a strongly κ-perfect subset. In particular, the κ-perfect set property is equivalent when defined with either notion. Hence, for the purposes of this paper, it will not be necessarily to distinguish between the two.

In light of the above remark, the following definition generalizes Definition 2.8 in the setting of ${}^{\kappa}\kappa$.

Definition 3.4. Suppose that κ is a regular infinite cardinal, $\delta \leq \kappa$ is an infinite ordinal, and $X \subseteq {}^{\kappa}\kappa$. We say that X has the *δ-perfect set property*, denoted $\mathsf{PSP}_\delta(X)$, if either $|X| \leq \kappa$ or X contains a δ-perfect subset.

The following proposition is readily verified; we leave the proof to the reader.

Proposition 3.5. *Suppose that κ is a regular cardinal, $\delta \leq \kappa$ is an infinite ordinal, and $T \subseteq {}^{<\kappa}\kappa$ is a tree. If $[T]$ is δ-perfect, then T contains a copy of ${}^{<\delta}2$.* □

Note that a set $E \subseteq {}^{\kappa}\kappa$ is ω-perfect if and only if it is perfect in the classical sense. Unlike the case with ${}^{\omega}\omega$, it is not necessarily the case that perfect sets have full cardinality. For example, if κ is regular and uncountable and

$$E = \{x \in {}^{\kappa}\kappa \mid |\{\alpha < \kappa \mid x(\alpha) \neq 0\}| < \aleph_0\},$$

then E is readily seen to be a perfect subset of ${}^{\kappa}\kappa$. However, $|E| = \kappa < 2^\kappa$. Nonetheless, we do recover a version of the Cantor-Bendixson theorem at higher κ:

Proposition 3.6. *Suppose that $E \subseteq {}^{\kappa}\kappa$ is closed. Then there is $X \subseteq E$ such that $|X| \leq \kappa^{<\kappa}$ and $E \setminus X$ is perfect.*

Proof. Let $\Sigma = \{\sigma \in {}^{<\kappa}\kappa \mid |E \cap N_\sigma| \leq \kappa^{<\kappa}\}$, and let $X = \bigcup\{E \cap N_\sigma \mid \sigma \in \Sigma\}$. Then X is as desired. □

Corollary 3.7. *If κ is an infinite regular cardinal such that $\kappa^{<\kappa} = \kappa$, then every closed set $E \subseteq {}^{\kappa}\kappa$ satisfies the ω-perfect set property.*

As noted in the introduction, in ${}^{\omega}\omega$, every nonempty closed set is a continuous image of the entire space. For uncountable κ, this is no longer the case. However, if we know that a closed set E is a continuous image of ${}^{\kappa}\kappa$, then we can slightly improve upon the conclusion of the preceding proposition. The proof of the following theorem is a variation on that of [8, Theorem 1.1]

Theorem 3.8. *Suppose that κ is an uncountable regular cardinal and $E \subseteq {}^{\kappa}\kappa$ is a closed set that is a continuous image of ${}^{\kappa}\kappa$. Then there is $X \subseteq E$ with $|X| \leq \kappa^{<\kappa}$ such that $E \setminus X$ is closed and, for every $x_0 \in E \setminus X$, Player II has a winning strategy in $G_\kappa(E, x_0, \omega + 1)$.*

Proof. Let $f : {}^{\kappa}\kappa \to E$ be a continuous surjection. Let X be as in the proof of Proposition 3.6. Let X_0 be the set of $x \in E$ such that, for every $y \in f^{-1}\{x\}$, there is $\alpha < \kappa$ such that $|f[N_{y\restriction\alpha}]| \leq \kappa^{<\kappa}$.

Claim 3.9. $|X_0| \leq \kappa^{<\kappa}$.

Proof. Let
$$\Sigma = \{\sigma \in {}^{<\kappa}\kappa \mid |f[N_\sigma]| \leq \kappa^{<\kappa}\},$$
and let $Y = \bigcup\{f[N_\sigma] \mid \sigma \in \Sigma\}$. Then $|Y| \leq \kappa^{<\kappa}$ and $X_0 \subseteq Y$. □

Claim 3.10. *$X \subseteq X_0$, and $E \setminus X$ is the closure of $E \setminus X_0$ in ${}^{\kappa}\kappa$.*

Proof. The facts that $X \subseteq X_0$ and $E \setminus X$ is closed follow immediately from the definitions. To show that $E \setminus X$ is the closure of $E \setminus X_0$, it suffices to show that, for every $x \in E \setminus X$ and every open set U with $x \in U$, there is $y \in U \cap (E \setminus X_0)$. Thus, fix such x and U. Without loss of generality, we may assume that $U = N_{x\restriction\alpha}$ for some $\alpha < \kappa$. Since $x \notin X$, we know that $|U \cap E| > \kappa^{<\kappa}$. Thus, by Claim 3.9, we know that $U \cap (E \setminus X_0) \neq \emptyset$. □

It remains to show that Player II has a winning strategy in $G_\kappa(E, x_0, \omega + 1)$ for all $x_0 \in E \setminus X$. We first establish the following claim.

Claim 3.11. *Suppose that $U \subseteq {}^{\kappa}\kappa$ is an open set such that $|f[U]| > \kappa^{<\kappa}$. Then there is $y \in U$ such that $|f[N_{y\restriction\alpha}]| > \kappa^{<\kappa}$ for all $\alpha < \kappa$.*

Proof. If not, then, for every $y \in U$, there is $\alpha_y < \kappa$ such that $|f[N_{y\restriction\alpha_y}]| \leq \kappa^{<\kappa}$. Then we have $|f[U]| \leq |\bigcup\{f[N_{y\restriction\alpha_y}] \mid y \in U\}| \leq \kappa^{<\kappa}$, which is a contradiction. □

Claim 3.12. *Suppose that $x \in E \setminus X$ and $\alpha < \kappa$. Then there is $x' \in E \setminus X_0$ such that $x \neq x'$ and $x' \restriction \alpha = x \restriction \alpha$.*

Proof. If $x \in X_0$ then we can choose any x' in the set $(E \setminus X_0) \cap N_{x\restriction\alpha}$, which is nonempty by Claim 3.10. Thus, we can assume that $x \notin X_0$. Choose $y \in {}^{\kappa}\kappa$ such that $f(y) = x$ and $|f[N_{y\restriction\beta}]| > \kappa^{<\kappa}$ for all $\beta < \kappa$. By the continuity of f, we can find $\beta < \kappa$ such that $f[N_{y\restriction\beta}] \subseteq N_{x\restriction\alpha}$. Let $U = N_{y\restriction\beta} \setminus f^{-1}\{x\}$. Then U is an open set such that $|f[U]| > \kappa^{<\kappa}$. Therefore, by Claim 3.11 we can find $y' \in U$ such that $|f[N_{y'\restriction\gamma}]| > \kappa^{<\kappa}$ for all $\gamma < \kappa$. Then $x' = f(y')$ is as desired. □

Fix $x_0 \in E\setminus X$. We will describe a winning strategy for Player II in $G_\kappa(E, x_0, \omega+1)$. In the course of the game, as the players play the sequences $\langle \alpha_n \mid 1 \leq n \leq \omega\rangle$ and $\langle x_n \mid 1 \leq n \leq \omega\rangle$, Player II will also construct a sequence $\langle y_n \mid 1 \leq n < \omega\rangle$ of elements of ${}^\kappa\kappa$ and increasing sequences $\langle \beta_n \mid 1 \leq n < \omega\rangle$ and $\langle \gamma_n \mid 1 \leq n < \omega\rangle$ of ordinals below κ such that, for all $1 \leq n < \omega$, we have

- $f(y_n) = x_n \in E \setminus X_0$;
- $|f[N_{y_n\restriction\gamma}]| > \kappa^{<\kappa}$ for all $\gamma < \kappa$;
- $f[N_{y_n\restriction\beta_n}] \subseteq N_{x_n\restriction\beta_n}$;
- $\alpha_n \leq \beta_n$;
- $x_n \restriction \beta_n \neq x_{n-1} \restriction \beta_n$;
- $\max\{\alpha_{n+1}, \beta_n\} \leq \gamma_n < \beta_{n+1}$;
- $f[N_{y_n\restriction\gamma_n}] \subseteq N_{x_n\restriction\alpha_{n+1}}$;
- $y_{n+1} \restriction \gamma_n = y_n \restriction \gamma_n$.

We begin by describing Player II's first play. After Player I plays an ordinal α_1, apply Claim 3.12 to find $x_1 \in E \setminus X_0$ such that $x_1 \neq x_0$ and $x_1 \restriction \alpha_1 = x_0 \restriction \alpha_0$. Then fix $y_1 \in {}^\kappa\kappa$ such that $f(y_1) = x_1$ and $|f[N_{y_1\restriction\gamma}]| > \kappa^{<\kappa}$ for all $\gamma < \kappa$. Finally, using the continuity of f and the fact that κ has uncountable cofinality fix β_1 with $\alpha_1 \leq \beta_1 < \kappa$ such that $f[N_{y_1\restriction\beta_1}] \subseteq N_{x_1\restriction\beta_1}$ and $x_1 \restriction \beta_1 \neq x_0 \restriction \beta_0$.

Now suppose that $1 \leq n < \omega$ and $\langle x_0, \alpha_1, x_1, \ldots, \alpha_{n+1}\rangle$ is a partial play of the game, with Player II playing so far according to the strategy that we are about to describe, and that Player II has also specified $\langle (y_i, \beta_i) \mid 1 \leq i \leq n\rangle$. We specify how to choose x_{n+1}, as well as γ_n, β_{n+1}, and y_{n+1}. First, choose $\gamma_n \geq \max\{\alpha_{n+1}, \beta_n\}$ such that $f[N_{y_n\restriction\gamma_n}] \subseteq N_{x_n\restriction\alpha_{n+1}}$, and let

$$U = N_{y_n\restriction\gamma_n} \setminus \bigcup\{f^{-1}\{x_i\} \mid i \leq n\}.$$

Then U is an open set such that $|f[U]| > \kappa^{<\kappa}$, so, by Claim 3.11, we can choose $y_{n+1} \in U$ such that $|f[N_{y_{n+1}\restriction\gamma}]| > \kappa^{<\kappa}$ for all $\gamma < \kappa$. Let $x_{n+1} = f(y_{n+1})$, and choose $\beta_{n+1} > \gamma_n$ such that

- $f[N_{y_{n+1}\restriction\beta_{n+1}}] \subseteq N_{x_{n+1}\restriction\beta_{n+1}}$; and
- $x_{n+1} \restriction \beta_{n+1} \neq x_n \restriction \beta_{n+1}$.

It is readily verified that this satisfies all of the requirements of the construction, and we can move on to the next round of the game.

This completely describes Player II's strategy at rounds indexed by natural numbers. It remains to show that, if they play according to this strategy, then they guarantee that they will be able to play in round ω. To this end, suppose that $\langle (\alpha_n, x_n) \mid 1 \leq n < \omega\rangle$ is an initial segment of the game of length ω, with Player II playing according to the described strategy. Suppose that $\langle (\beta_n, y_n, \gamma_n) \mid 1 \leq n < \omega\rangle$ are the auxiliary objects specified by this strategy.

Now let $\beta_\omega = \sup\{\beta_n \mid n < \omega\} = \sup\{\gamma_n \mid n < \omega\} \geq \alpha_\omega = \sup\{\alpha_n \mid n < \omega\}$. We know that, for all $1 \leq n < \omega$, we have $y_{n+1} \restriction \gamma_n = y_n \restriction \gamma_n$. We can therefore find a $y_\omega \in {}^\kappa\kappa$ such that, for all $1 \leq n < \omega$, we have $y_\omega \restriction \gamma_n = y_n \restriction \gamma_n$. Let $x_\omega = f(y_\omega)$. Then $x_\omega \in E$ and, moreover, for all $1 \leq n < \omega$, we have

$$f[N_{y_\omega\restriction\gamma_n}] = f[N_{y_n\restriction\gamma_n}] \subseteq N_{x_n\restriction\alpha_{n+1}}.$$

Therefore, for all $n < \omega$ we have $x_\omega \restriction \alpha_{n+1} = x_n \restriction \alpha_{n+1}$. By a similar argument, using the fact that $y_\omega \restriction \beta_n = y_n \restriction \beta_n$ for all $1 \leq n < \omega$, we know that $x_\omega \restriction \beta_n = x_n \restriction \beta_n \neq x_{n-1} \restriction \beta_n$, and hence $x_\omega \notin \{x_n \mid n < \omega\}$. Thus, x_ω is a valid play for

Player II in round ω, completing our description of a winning strategy for Player II in $G(E, x_0, \omega+1)$. $\square$

For some observations about this theorem and some remaining open questions, see Section 8 below.

4. Kurepa trees and the perfect set property

As mentioned in the introduction, κ-Kurepa trees often provide natural examples of closed subsets of ${}^{\kappa}\kappa$ that fail to have various perfect set properties. Indeed, suppose that κ is a regular uncountable cardinal that is not strongly inaccessible and $T \subseteq {}^{\kappa}\kappa$ is a κ-Kurepa tree. Let $\lambda < \kappa$ be the least cardinal such that $2^{\lambda} \geq \kappa$. Then we claim that $\mathsf{PSP}_{\lambda+1}([T])$ fails. To see this, first note that, since T is a κ-Kurepa tree, we have $|[T]| > \kappa$. On the other hand, $[T]$ cannot contain a $(\lambda+1)$-perfect subset, for, if it did, then by Proposition 3.5 we could find an isomorphic embedding $\iota : {}^{<\lambda+1}2 \to T$. Since $2^{<\lambda} < \kappa$, we can find $\alpha < \kappa$ such that $\iota[{}^{<\lambda}2] \subseteq T_{<\alpha}$. Then $\{\iota(x) \restriction \alpha \mid x \in {}^{\lambda}2\}$ is a subset of $T_{\leq\alpha}$ of cardinality $2^{\lambda} \geq \kappa$, contradicting the fact that T is a κ-tree.

In this section, we show that κ-Kurepa trees are not needed to produce closed subsets of ${}^{\kappa}\kappa$ failing various perfect set properties. For concreteness and readability, we will focus on the important special case $\kappa = \omega_1$, but the techniques can readily be generalized to produce similar results at other cardinals.

Starting in a model with an inaccessible cardinal, we will produce two models in which GCH holds, the Kurepa Hypothesis fails, and there is a closed subset of ${}^{\omega_1}\omega_1$ that fails to have the $(\omega+1)$-perfect set property.[4] In both models, the closed subset of ${}^{\omega_1}\omega_1$ failing to have the $(\omega+1)$-perfect set property will be of the form $[T]$ for some weak Kurepa tree $T \subseteq {}^{<\omega_1}\omega_1$. The fact that $\mathsf{PSP}_{\omega+1}([T])$ fails will be implied by the fact that, in both case, T will fail to contain a copy of ${}^{<\omega+1}2$. In the first model, constructed in Theorem 4.5, every weak Kurepa tree will contain an Aronszajn subtree. In the second, constructed in Theorem 4.16, there will be a weak Kurepa tree T satisfying $\neg\mathsf{PSP}_{\omega+1}([T])$ which will not contain any Aronszajn subtree. Note that either theorem individually will establish Theorem A from the introduction.

We first need a few preliminary results. We first observe that it suffices to consider isomorphic embeddings of ${}^{<\omega+1}2$ which preserve meets and are continuous at limits:

Lemma 4.1. *Let T be a normal tree, and let f be an isomorphic embedding from ${}^{<\omega+1}2$ to T.*[5] *Then there is an isomorphic embedding g which moreover preserves meets and is continuous at limits, i.e.:*

(i) *(preserves meets)*[6] *for all $a, b \in {}^{<\omega+1}2$, $g(a \wedge b) = g(a) \wedge g(b)$;*
(ii) *(continuous at limits) for all $x \in {}^{\omega}2$, $g(x) = \sup\{g(x \restriction n) \mid n < \omega\}$.*

Proof. We first define g which preserves meets on ${}^{<\omega}2$, and then argue that we can extend g to the whole tree ${}^{<\omega+1}2$. For $a \in {}^{<\omega}2$, define

$$g(a) = f(a^\frown 0) \wedge f(a^\frown 1).$$

[4] Note that this is sharp, by Corollary 3.7

[5] For the purposes of the proof of Theorem 4.4, think of this claim as being applied in $V[G]$.

[6] Note that if g is an isomorphic embedding then g preserves meets for all comparable nodes in ${}^{<\omega+1}2$.

First note that $f(a) \leq g(a)$ for all $a \in {}^{<\omega}2$; from this it is easy to see that g is $\perp$-preserving because f is $\perp$-preserving. Moreover for all $a \subset b \in {}^{<\omega}2$ we have $g(a) < f(b)$ since $g(a) < f(b \restriction (\mathrm{dom}(a)+1)) \leq f(b)$. Therefore for $a \subseteq b \in {}^{<\omega}2$ we have $g(a) \leq f(b) \leq g(b)$, so g preserves $\leq$. To see that g preserves meets, it is enough to verify that g preserves meets for incomparable nodes in ${}^{<\omega}2$, since we already verified that g is an isomorphic embedding from ${}^{<\omega}2$ to T. Assume that $a, b \in {}^{<\omega}2$ are incomparable. Then $g(a) \wedge g(b) = f(a) \wedge f(b) = g(a \wedge b)$: the first equality holds since $f(a) \leq g(a)$ and $f(b) \leq g(b)$ and the second holds by the definition of g (observe that for all $a, a', b, b' \in T$, with a, b incomparable, $a \leq a', b \leq b'$ implies $a \wedge b = a' \wedge b'$).

We extend the definition of g to $x \in {}^{\omega}2$ by letting $g(x) = \sup\{g(x \restriction n) \mid n < \omega\}$. This makes g continuous at limits by definition provided we argue that the suprema exist: This is true because $g(x \restriction n) \leq f(x)$ for all $n < \omega$, and by normality of T, there is a unique node in T on the supremum of the levels of $g(x \restriction n)$, and this node is $\leq f(x)$. Finally notice that if g is an isomorphic embedding from ${}^{<\omega+1}2$ into T which preserves meets on ${}^{<\omega}2$, then it also preserves meets for all $x, y \in {}^{\omega}2$, so g is as required. □

Remark 4.2. Note that if T is a normal tree and f is an isomorphic embedding from ${}^{<\omega+1}2$ to T which is continuous at limits, then f is determined by its restriction to ${}^{<\omega}2$. If T has height with cofinality at least ω_1, then this further implies that the copy of $2^{<\omega+1}$ given by f is bounded in the height of T.

Now, we will prove that no ω_1-distributive forcing can add a copy of ${}^{<\omega+1}2$ to a normal tree which does not contain such a copy in the ground model.

Lemma 4.3. *Let T be a normal tree which does not contain a copy of $({}^{<\omega+1}2)^V$ in ground model and let $\mathbb{P}$ be an ω_1-distributive forcing. Then T does not contain a copy of $({}^{<\omega+1}2)^{V[P]} = ({}^{<\omega+1}2)^V$ in $V[\mathbb{P}]$.*

Proof. This follows from Lemma 4.1 which implies that we can assume that a copy of ${}^{<\omega+1}2$ is determined by its restriction to ${}^{<\omega}2$. Let us give some details. Let F be $\mathbb{P}$-generic over V and let f be an isomorphic embedding from $({}^{<\omega+1}2)^{V[F]}$ to T in $V[F]$. Then since ${}^{<\omega}2$ has size ω and P is ω_1-distributive, $f \restriction {}^{<\omega}2$ is in the ground model and we can extend it to an isomorphic embedding g from $({}^{<\omega+1}2)^V$ by defining $g(x) = \sup\{x \restriction n \mid n < \omega\}$ for $x \in {}^{\omega}2$ and $g(a) = f(a)$ for $a \in {}^{<\omega}2$. Note that $g(x)$ is well defined for all $x \in {}^{\omega}2$ since $f(x)$ is above $\{x \restriction n \mid n < \omega\}$ in $V[F]$ and T is normal. The mapping g determines a copy of $({}^{<\omega+1}2)^V$ in the ground model, which is a contradiction. □

Theorem 4.4. *Assume that G is $\mathbb{Q} = \mathrm{Add}(\omega, 1)$-generic over V. Suppose T is an arbitrary normal tree in V. Then T does not contain a copy of $({}^{<\omega+1}2)^{V[G]}$ in $V[G]$.*

Proof. We will proceed by contradiction and will assume that, in $V[G]$, there is an isomorphic embedding from $({}^{<\omega+1}2)^{V[G]}$ into T. Let us work in $V[G]$. Let T be a normal tree such that $T \in V$. Assume for a contradiction that $f : ({}^{<\omega+1}2)^{V[G]} \to T$ is an isomorphic embedding; by Lemma 4.1, we can assume that f preserve meets and is continuous at limits.

Let $p \in G$ force this property about some name $\dot{f}$ for f. Let us work in V now. For each $x \in ({}^{\omega}2)^V$, there is a condition $q_x \leq p$ in $\mathbb{Q}$ which decides the value of

$\dot{f}(x)$. Since the forcing is countable, the Baire category theorem implies that there is some $q \in \mathbb{Q}$ such that

$$Y = \{x \in ({}^{\omega}2)^V \mid q_x = q\}$$

is not nowhere dense (we say it is *somewhere dense*), i.e.

$$\exists a^* \in {}^{<\omega}2 \; \forall a \in {}^{<\omega}2 \; (a^* \subseteq a \to \exists x \in Y \; x \upharpoonright |a| = a),$$

and hence

$$\exists a^* \in {}^{<\omega}2 \; \forall a \in {}^{<\omega}2 \; (a^* \subseteq a \to \exists x \in ({}^{\omega}2)^V \; x \upharpoonright |a| = a \text{ and } q \text{ decides } \dot{f}(x)). \tag{1}$$

We will now show that the fact that $\dot{f}$ is forced to be continuous allows us to prove that

$$q \text{ decides } \dot{f}[N^V_{a^*}],$$

where $N^V_{a^*}$ denotes the set of all $x \in ({}^{\omega}2)^V$ which extend a^*.

Consider the subset $C = \{x \wedge y \mid x, y \in Y, \; x \neq y\}$ of ${}^{<\omega}2$. This set is in V, since Y is in V. More importantly, q decides the value of $\dot{f}(a)$ for all $a \in C$ since q decides $\dot{f}(x)$ for all $x \in Y$ and $\dot{f}$ is forced to preserve meets. Since Y satisfies (1), it holds that

$$a^* \uparrow = \{a \in {}^{<\omega}2 \mid a^* \subseteq a\}$$

is a subset of C. Since q decides $\dot{f}(a)$ for all $a \in C$, q decides the values of $\dot{f}(a)$ for all $a \supseteq a^*$. Since $\dot{f}$ is forced to be continuous at limits, q also decides the values of $\dot{f}(x)$ for all $x \in ({}^{\omega}2)^V$ such that $a^* \subseteq x$.

Now we finish the proof by arguing that we can read off in V the Cohen subset $c = \bigcup G$ added by $\mathbb{Q}$, which gives the desired contradiction. In $V[G]$, c can be used to define a cofinal branch c' through $a^* \uparrow$ as follows: Identify c with a function from ω to 2 and define $c' = (a^*)^\frown c$; i.e. let $n = \mathrm{dom}(a^*)$, $c'(k) = a^*(k)$ for $k < n$ and $c'(k) = c(k - n)$ for $k \geq n$. Recall that f is an isomorphic embedding from $({}^{<\omega+1}2)^{V[G]}$ into T; therefore there is $t^* \in T$ such that $f(c') = t^*$. Since t^* is in T, t^* is in V. Define $d : \omega \to 2$ in V inductively as follows:

(a) $d(0) = 0$, provided $q \Vdash \dot{f}((a^*)^\frown\langle 0\rangle) \leq t^*$; otherwise let $d(0) = 1$.
(b) If $d \upharpoonright k$ is defined, set $d(k) = 0$, provided $q \Vdash \dot{f}((a^*)^\frown d \upharpoonright k^\frown\langle 0\rangle) \leq t^*$; otherwise let $d(k) = 1$.

Now $q \Vdash \dot{f}((a^*)^\frown d) \leq t^*$ since $\dot{f}$ is forced to be an isomorphic embedding. We claim that $q \Vdash d = \dot{c}$: Since $q \Vdash \dot{f}((a^*)^\frown d) \leq t^*$ and $q \Vdash \dot{f}(\dot{c}') = t^*$ and $\dot{f}$ is forced to be an isomorphic embedding, $q \Vdash (a^*)^\frown d = \dot{c}'$ and hence by the definition of c', $q \Vdash d = \dot{c}$. This yields the desired contradiction. □

We now prove the first of the two main theorems of this section.

Theorem 4.5. *Suppose that there is an inaccessible cardinal κ. Then there is a forcing extension in which*

(1) $\kappa = \omega_2$*;*
(2) GCH*;*
(3) $\neg\mathsf{KH}$*;*
(4) every weak Kurepa tree contains an Aronszajn subtree, moreover if we assume that $\diamondsuit$ holds in V, every weak Kurepa tree contains a Suslin subtree.
(5) there is a weak Kurepa tree that does not contain a copy of ${}^{<\omega+1}2$ of the generic extension.

Proof. Assume that GCH holds. Let $\mathbb{P} = \mathrm{Coll}(\omega_1, {<}\kappa)$ and let $\mathbb{Q} = \mathrm{Add}(\omega, 1)$. We claim that the generic extension by $\mathbb{P} \times \mathbb{Q}$ is the desired forcing extension in which (1)–(5) hold. **Item (1)** and **item (2)** are clear, and **item (3)** follows by standard arguments for the tree property as in [17] or [3].

The proof of item (4). To prove item (4), it suffices to show that every weak Kurepa tree in a generic extension by $\mathbb{P}\times\mathbb{Q}$ contains a copy of $({}^{<\omega_1}2)^V$. To see this, let $G \times H$ be $\mathbb{P}\times\mathbb{Q}$-generic over V. In V, we can construct a special ω_1-Aronszajn tree as a subtree of $({}^{<\omega_1}2)^V$ and this tree is preserved in all forcing extensions which preserve ω_1. In particular, it is still a special ω_1-Aronszajn tree in $V[G][H]$. Moreover, note that $\mathbb{P}\times\mathbb{Q}$ does not add cofinal branches to any ω_1-tree in V since $\mathbb{P}$ is ω_1-closed in V and $\mathbb{Q}$ is ω_1-Knaster in $V[G]$. It follows that all (not only special) ω_1-Aronszajn subtrees of $({}^{<\omega_1}2)^V$ remain Aronszajn in $V[G][H]$. In particular, if S is an ω_1-Suslin subtree of $({}^{<\omega_1}2)^V$ in V, it is still Aronszajn in $V[G][H]$. In fact it is still a Suslin tree in $V[G][H]$: by Easton's Lemma it is Suslin in $V[G]$ since $\mathbb{P}$ is ω_1-closed and it is still Suslin in $V[G][H]$ since $\mathbb{Q}$ is ω_1-Knaster and S is ccc in $V[G]$.

Let us now proceed to show that every weak Kurepa tree in a generic extension by $\mathbb{P}\times\mathbb{Q}$ contains a copy of $({}^{<\omega_1}2)^V$. If $\dot{T}$ is a $\mathbb{P}\times\mathbb{Q}$-name for a weak Kurepa tree then, since $\mathbb{P}\times\mathbb{Q}$ is κ-cc, $\dot{T}$ is a $\mathbb{P}_\theta\times\mathbb{Q}$-name for some regular cardinal $\theta<\kappa$, where we denote $\mathbb{P}_\theta = \mathrm{Coll}(\omega_1, {<}\theta)$. Let G_θ be $\mathbb{P}_\theta$-generic over V and H be $\mathbb{Q}$-generic over $V[G_\theta]$. We will work in $V[G_\theta]$ and we will show that T contains a copy of $({}^{<\omega_1}2)^{V[G_\theta]}$ in $V[G_\theta][H]$. Note that $({}^{<\omega_1}2)^V = ({}^{<\omega_1}2)^{V[G_\theta]}$ since $\mathbb{P}_\theta$ is ω_1-closed.

Note that, in $V[G_\theta][H]$, we have $2^{\omega_1}<\kappa$, and hence T has fewer than κ-many cofinal branches in $V[G_\theta][H]$. Since $\dot{T}$ is a $\mathbb{P}\times\mathbb{Q}$-name for a weak Kurepa tree, it is forced to have κ-many branches in the extension by $\mathbb{P}\times\mathbb{Q}\cong\mathbb{P}_\theta\times\mathbb{P}^\theta\times\mathbb{Q}\cong\mathbb{P}_\theta * (\mathbb{Q}\times\mathbb{P}^\theta)$, where $\mathbb{P}^\theta = \mathrm{Coll}(\omega_1, [\theta, {<}\kappa))^{V[G_\theta]}$. Therefore, we can fix in $V[G_\theta]$ conditions $q^*\in\mathbb{Q}$ and $p^*\in\mathbb{P}^\theta$ and a $\mathbb{Q}\times\mathbb{P}^\theta$-name $\dot{b}$ for a cofinal branch through $\dot{T}$ such that

$$(q^*, p^*) \Vdash \dot{b} \notin V[G_\theta][\dot{H}]. \tag{2}$$

Without loss of generality, we can assume that the underlying set of $\dot{T}$ is forced to be a subset of $\omega_1\times\omega_1$ and that if $(\alpha,\gamma)\in\dot{T}$, then $(\alpha,\gamma)\in\dot{T}_\gamma$. In $V[G_\theta]$, we will build by induction on ω_1 the following objects:

- a labeled tree $\mathcal{T} = \{p_s \mid s \in {}^{<\omega_1}2\}$ of conditions in $\mathbb{P}^\theta$, all of them extending p^* from (2);
- a labeled tree $\{\gamma_s \mid s\in{}^{<\omega_1}2\}$ of ordinals below ω_1;
- a labeled tree $\{A_s \mid s \in {}^{<\omega_1}2\}$ of maximal antichains of conditions in $\mathbb{Q}$ below q^* from (2);

such that the following hold for each $s\in{}^{<\omega_1}2$:

(a) $p_t \le p_s$ for each $s\subseteq t$ in ${}^{<\omega_1}2$;
(b) $\gamma_s<\gamma_t$ for each $s\subset t$ in ${}^{<\omega_1}2$;
(c) for each $q\in A_s$, the conditions $(q, p_{s^\frown 0})$ and $(q, p_{s^\frown 1})$ decide $\dot{b}$ up to γ_s differently; i.e., there are $\gamma\le\gamma_s$ and $\tau_{q,p_{s^\frown 0}} \neq \tau_{q,p_{s^\frown 1}}$ both forced by q to be in $\dot{T}_\gamma$ such that $(q,p_{s^\frown 0}) \Vdash \dot{b}(\gamma) = \tau_{q,p_{s^\frown 0}}$ and $(q,p_{s^\frown 1})\Vdash \dot{b}(\gamma) = \tau_{q,p_{s^\frown 1}}$.

The construction of $\mathcal{T}$ uses the standard method of diagonalizing over antichains in $\mathbb{Q}$ while taking lower bounds in $\mathbb{P}^\theta$, using the ω_1-closure of $\mathbb{P}^\theta$, but we will give details to make the argument self-contained.

Set $p_\emptyset = p^*$. First assume that α is a limit ordinal and for every $\beta < \alpha$ and every $s \in {}^{\beta}2$ the conditions p_s have been constructed. For $s \in {}^{\alpha}2$ let p_s be a lower bound of $\langle p_{s\restriction\beta} \,|\, \beta < \alpha\rangle$. Note that A_s and γ_s for $s \in {}^{\alpha}2$ will be constructed in the successor stage.

Now, assume that α is a successor ordinal $\alpha = \beta+1$ and for every $s \in {}^{\beta}2$, p_s has been constructed, and for every $s \in {}^{<\beta}2$, A_s and γ_s have been constructed. Given $s \in {}^{\beta}2$, we describe the construction of $p_{s^\frown 0}$, $p_{s^\frown 1}$, A_s and γ_s.

Claim 4.6. *For all $q \le q^*$ in $\mathbb{Q}$, all $r^0, r^1 \in \mathbb{P}^\theta$ with $r^0, r^1 \le p^*$, and all $\gamma' < \omega_1$, there are $\gamma' < \gamma < \omega_1$, $(q', p^0) \le (q, r^0)$ and $(q', p^1) \le (q, r^1)$ such that (q', p^0) and (q', p^1) decide $\dot{b}(\gamma)$ differently.*

Proof. Let $q \in \mathbb{Q}$, $r^0, r^1 \in \mathbb{P}^\theta$ and $\gamma' < \omega_1$ be given. Since $\dot{b}$ is forced by (p^*, q^*) to be a cofinal branch through $\dot{T}$ that is not in $V[G_\theta][\dot{H}]$, there are $(\bar{q}, \bar{p}^0) \le (q, r^0)$, $(\bar{q}, \bar{p}^1) \le (q, r^0)$ and $\gamma > \gamma'$ such that $(\bar{q}, \bar{p}^0)$ and $(\bar{q}, \bar{p}^1)$ decide $\dot{b}(\gamma)$ differently; i.e. there are $\tau^0 \ne \tau^1$, both forced by $\bar{q}$ to be in $\dot{T}_\gamma$, such that $(\bar{q}, \bar{p}^0) \Vdash \dot{b}(\gamma) = \tau^0$ and $(\bar{q}, \bar{p}^1) \Vdash \dot{b}(\gamma) = \tau^1$. Now, consider the condition $(\bar{q}, r^1)$: since $\dot{b}$ is a $\mathbb{P}^\theta \times \mathbb{Q}$ name for a cofinal branch through $\dot{T}$, there is an extension $(q', p^1) \le (\bar{q}, r^1)$ which decides $\dot{b}(\gamma)$. Since $\tau^0 \ne \tau^1$, (q', p^1) cannot decide $\dot{b}(\gamma)$ as being equal to both of them. Let p^0 be $\bar{p}^i$, for some $i < 2$, such that (q', p^1) and $(q', \bar{p}^i)$ disagree on $\dot{b}(\gamma)$. Then q', p^0, p^1 and γ' are as required. □

We use the previous claim to inductively construct in δ-many stages for some $\delta < \omega_1$ a maximal antichain $A_s = \{q_i \in \mathbb{Q} \,|\, i < \delta\}$ below q^*, an increasing sequence of ordinals $\langle \gamma_i < \omega_1 \,|\, i < \delta\rangle$ whose supremum will be γ_s, and decreasing sequences $\langle p_i^0 \in \mathbb{P}^\theta \,|\, i < \delta\rangle$ and $\langle p_i^1 \in \mathbb{P}^\theta \,|\, i < \delta\rangle$ with lower bounds $p_{s^\frown 0}$ and $p_{s^\frown 1}$, respectively.

Let us initialize the construction and define the required objects for $i = 0$. First set $\gamma'_s = \sup\{\gamma_{s\restriction\beta'} \,|\, \beta' < \beta\}$ (in case $\beta = 0$, take $\gamma'_s = 0$). By Claim 4.6 there are $\gamma'_s < \gamma_0 < \omega_1$ and $(q_0, p_0^0), (q_0, p_0^1) \le (q^*, p_s)$ such that (q_0, p_0^0) and (q_0, p_0^1) decide $\dot{b}(\gamma_0)$ differently. The condition q_0, p_0^0, p_0^1 and the ordinal γ_0 are as required.

Now assume that $0 < i < \omega_1$ and for all $j < i$ we already have q_j, p_j^0, p_j^1 and γ_j.

If there is $q \in \mathbb{Q}$ below q^* such that q is incompatible with all q_j for $j < i$, let us fix such q. If i is a limit ordinal, fix some lower bounds r^0 and r^1 of $\langle p_j^0 \in \mathbb{P}^\theta \,|\, j < i\rangle$ and $\langle p_j^1 \in \mathbb{P}^\theta \,|\, j < i\rangle$, respectively, and a supremum γ' of $\langle \gamma_j < \omega_1 \,|\, j < i\rangle$. If i is a successor of j, set $r^0 = p_j^0$, $r^1 = p_j^1$ and $\gamma' = \gamma_j$. By Claim 4.6, there are $\gamma' < \gamma_i < \omega_1$ and $(q_i, p_i^0) \le (q, r^0)$, $(q_i, p_i^1) \le (q, r^1)$ such that (q_i, p_i^0) and (q_i, p_i^1) decide $\dot{b}(\gamma_i)$ differently. The conditions q_i, p_i^0, p_i^1 and the ordinal γ_i are as required.

If there is no $q \in \mathbb{Q}$ below q^* such that q is incompatible with q_j for all $j < i$, we stop the construction and set $\delta = i$ and $A_s = \{q_j \in \mathbb{Q} \,|\, j < \delta\}$. If i is a limit ordinal, we set $p_{s^\frown 0}$ and $p_{s^\frown 1}$ to be lower bounds of $\langle p_j^0 \in \mathbb{P}^\theta \,|\, j < \delta\rangle$ and $\langle p_j^1 \in \mathbb{P}^\theta \,|\, j < \delta\rangle$, respectively, and γ_s to be a supremum of $\langle \gamma_j < \omega_1 \,|\, j < \delta\rangle$. If i is a successor of j, then we set $p_{s^\frown 0} = p_j^0$, $p_{s^\frown 1} = p_j^1$ and $\gamma_s = \gamma_j$. Note that the construction will end after countably many steps since $\mathbb{Q}$ is ccc, and hence $\delta < \omega_1$.

This ends the construction of the labeled tree $\mathcal{T}$ and the related objects.

In $V[G_\theta][H]$, we define a copy of $({}^{<\omega_1}2)^{V[G_\theta]}$ in $T = \dot{T}^{V[G_\theta][H]}$ using the tree $\mathcal{T}$. This copy is given by an embedding $h : ({}^{<\omega_1}2)^{V[G_\theta]} \to T$ which maps sequences $s^\frown 0$ and $s^\frown 1$ to the nodes $\tau_{q,p_{s^\frown 0}}$ and $\tau_{q,p_{s^\frown 1}}$, respectively where q is the unique element of $H \cap A_s$ (see item (c) in the properties of $\mathcal{T}$ for definitions). The definition of h extends continuously to limit levels: if $s \in ({}^{\gamma}2)^{V[G_\theta]}$ for a limit ordinal $\gamma < \omega_1$, then let $h(s)$ be the supremum of $\{h(s \restriction \alpha) \mid \alpha < \gamma\}$. This supremum exists in T by normality and the fact that, for instance, $h(s^\frown 0)$ is above $h(s \restriction \alpha)$ for all $\alpha < \delta$. Since the q's are chosen from H, the forcing statements from item (c)

$$(q, p_{s^\frown 0}) \Vdash \dot{b}(\gamma) = \tau_{q,p_{s^\frown 0}} \text{ and } (q, p_{s^\frown 1}) \Vdash \dot{b}(\gamma) = \tau_{q,p_{s^\frown 1}}$$

respect the tree T, i.e. the embedding h preserves the strict ordering and the incompatibility of nodes between the trees $({}^{<\omega_1}2, \subseteq)^{V[G_\theta]}$ and $(T, <_T)$. It follows that T contains a copy of $({}^{<\omega_1}2)^{V[G_\theta]} = ({}^{<\omega_1}2)^V$ as required.

The proof of item (5). Item (5) is a consequence of Theorem 4.4: In $V[G]$ there are normal weak Kurepa trees, e.g., $({}^{<\omega_1}2)^V$. Such a tree remains a weak Kurepa tree in $V[G][H]$ and, by Theorem 4.4, cannot contain a copy of $({}^{<\omega+1}2)^{V[G][H]}$. $\square$

We now turn to proving the second main theorem of this section, which will produce a model similar to that of Theorem 4.5, except we will obtain a weak Kurepa tree which does not contain a copy of ${}^{<\omega+1}2$ of the generic extension and does not contain an Aronszajn subtree. First we define a forcing which adds a weak Kurepa tree with these properties and establish some basic facts about the forcing. The definition of the forcing is motivated by the standard forcing for adding an ω_1-Kurepa tree due to Stewart [16].

Definition 4.7. Let λ be an uncountable cardinal. We define a poset $\mathbb{K}_\lambda$ which adds a tree with size and height ω_1 with λ-many cofinal branches. Conditions $q \in \mathbb{K}_\lambda$ are pairs (T_q, f_q) such that

- there is $\eta_q < \omega_1$ such that T_q is a normal, infinitely splitting subtree of ${}^{<\eta_q+1}\omega_1$ that does not contain a copy of ${}^{<\omega+1}2$;
- f_q is a countable partial function from λ to $T_q \cap {}^{\eta_q}\omega_1$.

If $q_0, q_1 \in \mathbb{K}_\lambda$, then $q_1 \leq q_0$ if and only if

- $\eta_{q_1} \geq \eta_{q_0}$;
- $T_{q_1} \cap {}^{<\eta_{q_0}+1}\omega_1 = T_{q_0}$;
- $\mathrm{dom}(f_{q_1}) \supseteq \mathrm{dom}(f_{q_0})$;
- for all $\alpha \in \mathrm{dom}(f_{q_0})$, $f_{q_1}(\alpha) \supseteq f_{q_0}(\alpha)$.

We also include the pair $(\emptyset, \emptyset)$ in $\mathbb{K}_\lambda$ as $1_{\mathbb{K}_\lambda}$.

It is easy to verify that $\mathbb{K}_\lambda$ is separative. Note that $\mathbb{K}_\lambda$ is not ω_2-cc: in fact, it collapses 2^{ω_1} to ω_1 since we can code subsets of ω_1 in the ground model into the levels of the generic tree added by $\mathbb{K}_\lambda$. Therefore if $\lambda \leq 2^{\omega_1}$, then the generic tree added by $\mathbb{K}_\lambda$ is not a weak Kurepa tree.

Lemma 4.8. *Let λ and μ be uncountable cardinals such that $\mu > 2^{\omega_1}$ is regular and $\mu > \gamma^\omega$ for all $\gamma < \mu$. Then $\mathbb{K}_\lambda$ is μ-Knaster. In particular $\mathbb{K}_\lambda$ is $(2^{\omega_1})^+$-Knaster.*

Proof. Let a set of conditions $\{q_\alpha = (T_\alpha, f_\alpha) \in \mathbb{K}_\lambda \,|\, \alpha < \mu\}$ be given. Since $\mu > 2^{\omega_1}$ is regular and there are only 2^{ω_1}-many possibilities for T_α's, there is a tree $T \subseteq {}^{<\eta+1}\omega_1$ of countable height $\eta + 1$ and $I \subseteq \mu$ of size μ such that $T_\alpha = T$ for all $\alpha \in I$.

Since $\gamma^\omega < \mu$ for all $\gamma < \mu$, there is $I' \subseteq I$ of size μ such that the set $\{\mathrm{dom}(f_\alpha) \,|\, \alpha \in I'\}$ forms a Δ-system with root $a \subseteq \lambda$. Since a is at most countable, there are at most $2^\omega < \mu$ many functions from a to T_η and therefore there is a countable f from a to T_η and $J \subseteq I'$ of size μ such $f = f_\alpha \cap f_\beta$ for all $\alpha \neq \beta \in J$. Then all conditions in $\{q_\alpha \,|\, \alpha \in J\}$ are compatible. $\square$

Lemma 4.9. *Let λ be an uncountable cardinal. If* CH *holds, then $\mathbb{K}_\lambda$ is ω_1-closed.*

Proof. Let $\langle q_n \mid n < \omega\rangle$ be a decreasing sequence from $\mathbb{K}_\lambda$. To avoid trivialities, assume that the sequence $\langle \eta_{q_n} \mid n < \omega\rangle$ is strictly increasing. Let $\eta := \sup\{\eta_{q_n} \mid n < \omega\}$. We will construct a lower bound q for $\langle q_n \mid n < \omega\rangle$ such that $\eta_q = \eta$. Let $T = \bigcup_{n<\omega} T_{q_n}$. To define T_q, we simply need to decide which cofinal branches through T should continue.

We first note that there may be countably many branches that we are obliged to extend because of the functions $\{f_{q_n} \mid n < \omega\}$. Namely, let $a = \bigcup_{n<\omega} \mathrm{dom}(f_{q_n})$ and, for each $\alpha \in a$, let

$$b_\alpha := \bigcup\{f_{q_n}(\alpha) \mid n < \omega \wedge \alpha \in \mathrm{dom}(f_{q_n})\}.$$

Then each b_α is a cofinal branch through T, and we are obliged to put b_α in T_q.

We may need to extend additional branches through T in order to ensure that T_q is normal; i.e., for each $\sigma \in T$, we need to ensure that there is $\tau \in T_q \cap {}^\eta\omega_1$ with $\sigma \subseteq \tau$. However, when doing so, we must be careful not to add a copy of ${}^{<\omega+1}2$ to T_q. We will do so through the use of the following bookkeeping device.

Let $\langle \iota_\xi \mid \xi < \omega_1\rangle$ enumerate all isomorphic embeddings ι from ${}^{<\omega}2$ to T such that, for every $x \in {}^\omega 2$, the union $\bigcup\{\iota(x \restriction n) \mid n < \omega\}$ is a cofinal branch through T. Note that this is possible, due to the fact that CH holds and $|T| \leq \omega_1$. For each $\xi < \omega_1$, let

$$[\iota_\xi] := \left\{ \bigcup\{\iota_\xi(x \restriction n) \mid n < \omega\} \;\middle|\; x \in {}^\omega 2\right\}.$$

Note that $[\iota_\xi] \subseteq [T]$, and $|[\iota_\xi]| = 2^\omega = \omega_1$. Let us enumerate T as $\langle \sigma_\xi \mid \xi < \omega_1\rangle$, with repetitions if T is countable. We now recursively construct disjoint subsets $\{c_\xi \mid \xi < \omega_1\}$ and $\{d_\xi \mid \xi < \omega_1\}$ of $[T]$. Suppose that $\xi < \omega_1$ and we have constructed c_ζ and d_ζ for all $\zeta < \xi$. First, choose $c_\xi \in [T]$ such that

- $\sigma_\xi \subseteq c_\xi$;
- $c_\xi \notin \{d_\zeta \mid \zeta < \xi\}$.

Note that this is possible to do: T is normal and splitting, and $\mathrm{cf}(\gamma) = \omega$, so there are 2^ω many elements of $[T]$ extending σ_ξ. Next, choose $d_\xi \in [T]$ such that

- $d_\xi \in [\iota_\xi]$;
- $d_\xi \notin \{b_\alpha \mid \alpha \in a\} \cup \{c_\zeta \mid \zeta \leq \xi\}$.

At the end of the construction, set $T_q := T \cup \{b_\alpha \mid \alpha \in a\} \cup \{c_\xi \mid \xi < \omega_1\}$. Let f_q be such that $\mathrm{dom}(f_q) = a$ and, for all $\alpha \in a$, $f_q(a) = b_\alpha$.

We claim that q is a condition in $\mathbb{K}_\lambda$ and it is a lower bound of $\langle q_n \mid n < \omega\rangle$. The only nontrivial thing to verify is the fact that T_q does not contain a copy of ${}^{<\omega+1}2$. Assume for a contradiction that T_q contains a copy of ${}^{<\omega+1}2$. Since, for every $n < \omega$, we know that T_{q_n} does not contain a copy of ${}^{<\omega+1}2$, there must be a $\xi < \omega_1$ such that $[\iota_\xi] \subseteq (T_q)_\eta = \{b_\alpha \,|\, \alpha \in a\} \cup \{c_\zeta \,|\, \zeta < \omega_1\}$. However, this means that $d_\xi \in \{b_\alpha \,|\, \alpha \in a\} \cup \{c_\zeta \,|\, \zeta < \omega_1\}$ which contradicts our choice of d_ξ. □

Lemma 4.10. *Let $\lambda < \kappa$ be uncountable cardinals. Then there is a projection from $\mathbb{K}_\kappa$ to $\mathbb{K}_\lambda$.*

Proof. We define π from $\mathbb{K}_\kappa$ to $\mathbb{K}_\lambda$ by letting $\pi(T_q, f_q) = (T_q, f_q \restriction \lambda)$ for all $q \in \mathbb{K}_\kappa$. It is routine to verify that π is order-preserving and that $\pi(1_{\mathbb{K}_\kappa}) = 1_{\mathbb{K}_\lambda}$.

Let $q \in \mathbb{K}_\kappa$ and $r \in \mathbb{K}_\lambda$ be such that $r \leq \pi(q) = (T_q, f_q \restriction \lambda)$. Define $r' \leq q$ by first letting $T_{r'} = T_r$ (note that T_r is an end-extension of T_q). Let $\mathrm{dom}(f_{r'}) = \mathrm{dom}(f_q) \cup \mathrm{dom}(f_r)$ and define

– $f_{r'}(\alpha) = f_r(\alpha)$ for every $\alpha \in \mathrm{dom}(f_r)$ and
– $f_{r'}(\alpha) = \tau$, where $\tau \in (T_{r'})_{\eta_r}$ with $\tau \supseteq f_q(\alpha)$, for every $\alpha \in \mathrm{dom}(f_q) \setminus \mathrm{dom}(f_r)$.

It is easy to check that $\pi(r') = r$. Therefore π is a projection. $\square$

Assume that $\lambda < \kappa$ are uncountable cardinals. Let H be a $\mathbb{K}_\lambda$-generic filter, $T_H = \bigcup\{T_q \mid q \in H\}$, and let $\mathbb{K}_\kappa/H = \{r \in \mathbb{K}_\kappa \mid \pi(r) \in H\}$ be the quotient given by H and the projection π. Then $\mathbb{K}_\kappa$ is forcing equivalent to a two step iteration $\mathbb{K}_\lambda * \mathbb{K}_\kappa/H$. It is easy to see that in $V[H]$, $\mathbb{K}_\kappa/H$ is forcing equivalent to the forcing notion $\mathbb{K}_{\lambda,\kappa}$, where conditions in $\mathbb{K}_{\lambda,\kappa}$ are pairs $r = (\eta_r, f_r)$ such that f_r is a countable partial function from $\kappa \setminus \lambda$ to $(T_H)_{\eta_r}$ and $r \leq q$ if and only if $\eta_r \geq \eta_q$, $\mathrm{dom}(f_q) \subseteq \mathrm{dom}(f_r)$, and $f_q(\alpha) \subseteq f_r(\alpha)$ for all $\alpha \in \mathrm{dom}(f_q)$.

Lemma 4.11. *Let $\lambda < \kappa$ be uncountable cardinals and H be a $\mathbb{K}_\lambda$-generic filter. If* CH *holds in V, then $\mathbb{K}_\kappa/H$ is ω_1-distributive in $V[H]$.*

Proof. The forcing $\mathbb{K}_\kappa$ is forcing equivalent to a two step iteration $\mathbb{K}_\lambda * \mathbb{K}_\kappa/\dot{H}$. Since $\mathbb{K}_\kappa$ is ω_1-closed by Lemma 4.9, $\mathbb{K}_\kappa/H$ cannot add new countable sequences of ordinals over $V[H]$, hence it is ω_1-distributive in $V[H]$. $\square$

We fix the following notation: for $r \in \mathbb{K}_\kappa$ and $\lambda < \kappa$, let $r \restriction \lambda$ denote $(T_r, f_r \restriction \lambda)$, i.e. $r \restriction \lambda = \pi(r)$ for the projection π defined in Lemma 4.10.

Lemma 4.12. *Let $\lambda < \kappa$ be uncountable cardinals and $\dot{H}$ be a canonical $\mathbb{K}_\lambda$-name for the $\mathbb{K}_\lambda$-generic filter. Let $q \in \mathbb{K}_\lambda$ and $r \in \mathbb{K}_\kappa$. Then the following are equivalent.*

(i) $q \Vdash r \in \mathbb{K}_\kappa/\dot{H}$;
(ii) $q \leq r \restriction \lambda$.

Proof. To see that (i) implies (ii), note that if $q \not\leq r \restriction \lambda$, then by the separativity of $\mathbb{K}_\lambda$ there is $q' \leq q$ which is incompatible with $r \restriction \lambda$, and hence q does not force r into $\mathbb{K}_\kappa/\dot{H}$. The other direction is clear, since $q \leq r \restriction \lambda$ means that $q \leq \pi(r)$. $\square$

Lemma 4.13. *Assume* GCH *and let $\lambda > \omega_2$ be a cardinal. Let H be a $\mathbb{K}_\lambda$-generic filter over V. Then the generic tree $T_H = \bigcup\{T_q \mid q \in H\}$ is a weak Kurepa tree with λ-many cofinal branches which does not contain a copy of $({}^{<\omega+1}2)^{V[H]}$.*

Proof. Since $\mathbb{K}_\lambda$ is ω_1-closed, ω_1 is preserved by $\mathbb{K}_\lambda$ and the generic tree $T_H = \bigcup\{T_q \mid q \in H\}$ is thus a tree with height and size ω_1. By a standard density argument T_H has λ-many cofinal branches in $V[H]$. Since GCH holds in the ground model, $\mathbb{K}_\lambda$ is ω_3-Knaster by Lemma 4.8, hence all cardinals greater than ω_2 are preserved (recall that 2^{ω_1} is always collapsed); in particular λ is preserved. Since $\lambda > \omega_2$ in the ground model, $\lambda > \omega_1$ in $V[H]$; therefore the generic tree T_H is a weak Kurepa tree in $V[H]$.

Assume for a contradiction that there is a copy of $({}^{<\omega+1}2)^{V[H]}$ in T_H. By Lemma 4.1 and Remark 4.2, there is one which is bounded in the height of T_H and therefore there is a condition $q \in H$ such that T_q contains a copy of $({}^{<\omega+1}2)^{V[H]}$. Since T_q is a normal tree in the ground model and $\mathbb{K}_\lambda$ is ω_1-closed by Lemma 4.9, T_q contains a copy of $({}^{<\omega+1}2)^{V}$ in V by Lemma 4.3; this contradicts the definition of the forcing $\mathbb{K}_\lambda$. $\square$

We show now that the generic tree added by $\mathbb{K}_\lambda$ does not contain an Aronszajn subtree and that this property is preserved by all ω_1-closed forcings.

Lemma 4.14. *Assume* CH*. Let $\lambda > \omega_1$ be a cardinal and let H be a $\mathbb{K}_\lambda$-generic filter over V. Assume that $\mathbb{P}$ is an ω_1-closed forcing in $V[H]$. Then the generic tree $T_H = \bigcup\{T_q \mid q \in H\}$ does not contain an Aronszajn subtree in the generic extension of $V[H]$ by $\mathbb{P}$. In particular T_H does not contain an Aronszajn subtree in $V[H]$.*

Proof. We will prove this lemma in V using names. The statement is formulated with generic filters for easier reading. Let $\dot{T}_H$ denote a canonical $\mathbb{K}_\lambda$-name for the generic tree added by $\mathbb{K}_\lambda$. Assume for a contradiction that $\dot{S}$ is a $\mathbb{K}_\lambda * \dot{\mathbb{P}}$-name for an Aronszajn subtree of $\dot{T}_H$ and $(q, \dot{p}) \in \mathbb{K}_\lambda * \dot{\mathbb{P}}$ is a condition which forces this.

We begin with some easy observations. For every $r \in \mathbb{K}_\lambda$ and all $\sigma, \tau \in {}^{<\omega_1}\omega_1$, it holds that $r \Vdash \sigma \leq_{\dot{T}_H} \tau$ if and only if $\sigma \subseteq \tau \in T_r$. Moreover, if $(r, \dot{p}') \leq (q, \dot{p})$ decides that level $\dot{S}_\gamma = x$ for some $\gamma < \omega_1$, then $\mathrm{ht}(T_r) \geq \gamma + 1$ and $x \subseteq (T_r)_\gamma$.

Claim 4.15. *Let $(r, \dot{p}') \leq (q, \dot{p})$ decide that level $\dot{S}_\gamma = x$ for some $\gamma < \omega_1$. Then there is a condition $(r^*, \dot{p}^*) = ((T_{r^*}, f_{r^*}), \dot{p}^*)$ stronger than $(r, \dot{p}')$ such that the following hold:*

(i) for each $\sigma \in x$ there is $\alpha \in \mathrm{dom}(f_{r^*})$ such that $f_{r^*}(\alpha) \supseteq \sigma$;
(ii) for each $\alpha \in \mathrm{dom}(f_{r^*})$, $(r^*, \dot{p}^*) \Vdash f_{r^*}(\alpha) \notin \dot{S}$.

Proof. Note that we can assume that $(r, \dot{p}')$ is such that for each $\sigma \in x$ there is $\alpha \in \mathrm{dom}(f_r)$ with $f_r(\alpha) \supseteq \sigma$; in particular $(r, \dot{p}')$ forces that σ is in some cofinal branch of $\dot{T}_H$. If this is not the case, then we can extend f_r appropriately using that T_r is normal. We now build by induction on ω a decreasing sequence of conditions $\langle (r_n, \dot{p}_n) \in \mathbb{K}_\lambda * \dot{\mathbb{P}} \mid n < \omega \rangle$ such that the desired $(r^*, \dot{p}^*)$ will be a lower bound of this sequence.

Begin by letting $(r_0, \dot{p}_0) = (r, \dot{p}')$. Now fix $n < \omega$ and suppose that we have constructed $(r_n, \dot{p}_n)$. Note that, for each $\alpha < \omega_1$, $\bigcup\{f_s(\alpha) \mid s \in \dot{H}$ and $\alpha \in \mathrm{dom}(f_s)\}$ is forced to be a cofinal branch through $\dot{T}_H$, where $\dot{H}$ is a name for the $\mathbb{K}_\lambda$-generic filter. Therefore, since $\dot{S}$ is forced by $(r_n, \dot{p}_n)$ to be an Aronszajn subtree of $\dot{T}_H$, and since $\mathrm{dom}(f_{r_n})$ is countable, there is $(r_{n+1}, \dot{p}_{n+1}) \leq (r_n, \dot{p}_n)$ such that, for all $\alpha \in \mathrm{dom}(f_{r_n})$, we have $(r_{n+1}, \dot{p}_{n+1}) \Vdash f_{r_{n+1}}(\alpha) \notin \dot{S}$. Let $(q^*, \dot{p}^*)$ be a lower bound of $\langle (r_n * \dot{p}_n) \in \mathbb{K}_\lambda * \dot{\mathbb{P}} \mid n < \omega \rangle$ such that $\mathrm{dom}(f_{q^*}) = \bigcup\{\mathrm{dom}(f_{r_n}) \mid n < \omega\}$. Then it is easy to see that $(q^*, \dot{p}^*)$ satisfy condition (i) and (ii) above. $\square$

Now, we build a decreasing sequence of conditions $\langle (r^*_n, \dot{p}^*_n) \in \mathbb{K}_\lambda * \dot{\mathbb{P}} \mid n < \omega \rangle$ such that each condition will satisfy an instance of Claim 4.15. Let $(r_0, \dot{p}_0) \leq (q, \dot{p})$ decide that level $\dot{S}_{\gamma_0} = x_{\gamma_0}$ for some $\gamma_0 < \omega_1$. Then take $(r^*_0, \dot{p}^*_0) \leq (r_0, \dot{p}_0)$ to be a condition which satisfies Claim 4.15 for γ_0. Note that $\mathrm{ht}(T_{r^*_0}) \geq \gamma_0$. Let $(r^*_n, \dot{p}^*_n)$ be constructed for some $n < \omega$. Let $(r_{n+1}, \dot{p}_{n+1}) \leq (r^*_n, \dot{p}^*_n)$ be a condition which decides $\dot{S}_{\gamma_{n+1}} = x_{\gamma_{n+1}}$ for some $\gamma_{n+1} > \mathrm{ht}(T_{r^*_n})$, and then take $(r^*_{n+1}, \dot{p}^*_{n+1}) \leq (r_{n+1}, \dot{p}_{n+1})$ to be a condition which satisfies Claim 4.15 for γ_{n+1}. Note that $\mathrm{ht}(T_{r^*_{n+1}}) \geq \gamma_{n+1}$.

It now follows from the construction that $T^\gamma = \bigcup_{n<\omega} T_{r^*_n}$ is a subtree of ${}^{<\gamma}\omega_1$ of limit height γ, where $\gamma = \sup\{\gamma_n \mid n < \omega\}$. Let

$$S^\gamma = \bigcup\{\sigma \in {}^{<\gamma}\omega_1 \mid \exists n < \omega \; (r^*_n, \dot{p}^*_n) \Vdash \sigma \in \dot{S}\}.$$

Note that S^γ is a subtree of T^γ with countable levels and height γ in V, and that any lower bound for $\langle (r^*_n, \dot{p}^*_n) \mid n < \omega \rangle$ will force that $\dot{S} \cap {}^{<\gamma}\omega_1 = S^\gamma$.

Let $a = \bigcup_{n<\omega} \mathrm{dom}(f_{r^*_n})$. Then, for each $\alpha \in a$

$$b_\alpha = \bigcup\{f_{r^*_n}(\alpha) \mid n < \omega \wedge \alpha \in \mathrm{dom}(f_{r^*_n})\}.$$

is a cofinal branch through T^γ. However, b_α is not a cofinal branch through S^γ since there is $n < \omega$ such that $(r^*_n, \dot{p}^*_n) \Vdash f_{r^*_n}(\alpha) \notin \dot{S}$. Hence we are not obligated to put any cofinal branch through S^γ into a lower bound of $\langle r^*_n \in \mathbb{K}_\lambda \,|\, n < \omega\rangle$.

We claim that we can find a lower bound r^* of $\langle r^*_n \,|\, n < \omega\rangle$ such that T_{r^*} has height $\gamma + 1$ and for each $\tau \in T_{r^*}$ at level γ, $\mathrm{pred}_{T_{r^*}}(\tau)$ is not a cofinal branch through S^γ. Consider $r = (T_r, f_r)$, where $T_r = T^\gamma \cup \{b_\alpha \,|\, \alpha \in a\}$, $\mathrm{dom}(f_r) = a$ and for each $\alpha \in a$, $f_r(\alpha) = b_\alpha$. Each lower bound of $\langle r^*_n \in \mathbb{K}_\lambda \,|\, n < \omega\rangle$ has to extend r, but r itself does not have to be a condition in $\mathbb{K}_\lambda$, since T_r does not have to be normal. However for each $\sigma \in S^\gamma$ there is $\alpha \in a$ such that $\sigma \subseteq b_\alpha$, hence to extend T_r to a normal tree we only need to extend nodes in $T^\gamma \setminus S^\gamma$. As in the proof that $\mathbb{K}_\lambda$ is ω_1-closed, extend T_r to a normal tree $T^* = T_r \cup \{c_\tau \mid \tau \in T^\gamma \setminus S^\gamma\}$ by carefully picking cofinal branches c_τ in T^γ for each $\tau \in T^\gamma \setminus S^\gamma$ to ensure that T^* does not contain a copy of ${}^{<\omega+1}2$. Since we add only nodes to level γ above $T^\gamma \setminus S^\gamma$, we did not add any cofinal branch through S^γ into T^*. Therefore every cofinal branch b through S^γ is vanishing in T^*; i.e. there is no node on level γ above b.

Let $r^* = (T^*, f_r)$. Then r^* is a condition in $\mathbb{K}_\lambda$ which is a lower bound of $\langle r^*_n \in \mathbb{K}_\lambda \,|\, n < \omega\rangle$. Consider $(r^*, \dot{p}^*)$, where $\dot{p}^*$ is chosen so that $(r^*, \dot{p}^*)$ is a lower bound of $\langle (r^*_n, \dot{p}^*_n) \in \mathbb{K}_\lambda * \dot{\mathbb{P}} \,|\, n < \omega\rangle$. Then the condition $(r^*, \dot{p}^*)$ forces that $\dot{S} \cap (\dot{T}_H)_\gamma = \emptyset$, which is a contradiction since $(q, \dot{p})$ forces that $\dot{S}$ is an Aronszajn subtree of $\dot{T}_H$ and in particular unbounded in $\dot{T}_H$. □

We are now ready to prove the following variation on Theorem 4.5.

Theorem 4.16. *Suppose that there is an inaccessible cardinal κ. Then there is a forcing extension in which*

(1) $\kappa = \omega_2$;
(2) GCH;
(3) $\neg\mathsf{KH}$;
(4) there is a weak Kurepa tree $T_H \subseteq {}^{<\omega_1}\omega_1$ which does not contain a copy of ${}^{<\omega+1}2$ of the generic extension, nor does it contain an Aronszajn subtree.

Proof. Let κ be an inaccessible cardinal, and assume that GCH holds. Let $\mathbb{P} = \mathrm{Coll}(\omega_1, <\kappa)$ and let $G \times H$ be $\mathbb{P} \times \mathbb{K}_\kappa$-generic over V. $\mathbb{P} \times \mathbb{K}_\kappa$ is ω_1-closed and κ-Knaster and hence preserves ω_1 and all cardinals greater or equal κ. Since $\mathbb{P}$ collapses cardinals between ω_1 and κ to ω_1, κ is ω_2 in $V[G][H]$. Since $\mathbb{P} \times \mathbb{K}_\kappa$ is ω_1-closed and has cardinality κ, it follows that GCH holds in $V[G][H]$. By Lemma 4.13, $T_H = \bigcup\{T_q \,|\, q \in H\}$ is a weak Kurepa tree with κ-many cofinal branches which does not contain a copy of ${}^{<\omega+1}2$ in $V[H]$. Note that T_H is still a weak Kurepa tree in $V[H][G]$ since $\mathbb{P}$ preserves ω_1 and κ over $V[H]$. Moreover, since $\mathbb{P}$ is ω_1-closed in $V[H]$, T_H does not contain any Aronszajn subtree in $V[H][G] = V[G][H]$ by Lemma 4.14 and T_H does not contain a copy of ${}^{<\omega+1}2$ in $V[H][G]$ by Lemma 4.3.[7]

To finish the proof we need to show that there are no Kurepa trees in $V[G][H]$. Assume that S is an ω_1-tree in $V[G][H]$. Since S has size ω_1 and $\mathbb{P} \times \mathbb{K}_\kappa$ is κ-Knaster, there is a nice $\mathbb{P} \times \mathbb{K}_\kappa$-name $\dot{S}$ of size less than κ for S. Since $\dot{S}$ has size less than κ, $\dot{S}$ is a $\mathbb{P}_\theta \times \mathbb{K}_\theta$-name for some regular cardinal $\theta < \kappa$, where $\mathbb{P}_\theta = \mathrm{Coll}(\omega_1, <\theta)$. Let G_θ denote the $\mathbb{P}_\theta$-generic over V determined by G; i.e. $G_\theta = \{p \restriction \theta \,|\, p \in G\}$, and let H_θ denote the $\mathbb{K}_\theta$-generic filter over $V[G_\theta]$ determined by H and π, where

[7]Since $\mathbb{P}$ and $\mathbb{K}_\kappa$ are ω_1-closed, it holds that $({}^{<\omega+1}2)^V = ({}^{<\omega+1}2)^{V[G]} = ({}^{<\omega+1}2)^{V[G][H]}$; therefore we do not need to specify from which model we take ${}^{<\omega+1}2$.

π is the projection from $\mathbb{K}_\kappa$ to $\mathbb{K}_\theta$ from Lemma 4.10. The ω_1-tree S is an element of $V[G_\theta][H_\theta]$ and has at most $(2^{\omega_1})^{V[G_\theta][H_\theta]}$-many cofinal branches here, which is less than κ, since κ is still inaccessible in $V[G_\theta][H_\theta]$. We show that the quotient forcing $\mathbb{P}^\theta \times \mathbb{K}_\kappa/H_\theta$, where $\mathbb{P}^\theta$ denotes $\mathrm{Coll}(\omega_1, [\theta, {<}\kappa))$, cannot add a cofinal branch to S over $V[G_\theta][H_\theta]$. It will follow that S has ${<}\kappa = \omega_2$-many cofinal branches in $V[G][H]$, i.e., it is not a Kurepa tree. Since S was an arbitrary ω_1-tree, this will conclude the proof of Theorem 4.16. The proof that there are no Kurepa trees in $V[G][H]$ is relatively long; for easier reading, it is divided into several Claims (Claims 4.18 to 4.21).

First we observe that we can express the generic extension $V[G][H]$ as a forcing extension given by $(\mathbb{P}_\theta \times \mathbb{K}_\theta) * (\mathbb{K}_\kappa/\dot{H}_\theta \times \mathbb{P}^\theta)$: Since $\mathbb{P}$ is forcing equivalent to $\mathbb{P}_\theta \times \mathbb{P}^\theta$, we can view G as a filter $G_\theta \times g$ which is $\mathbb{P}_\theta \times \mathbb{P}^\theta$-generic over V. Similarly, since $\mathbb{K}_\kappa$ is forcing equivalent to $\mathbb{K}_\theta * \mathbb{K}_\kappa/\dot{H}_\theta$, we can view H as a filter $H_\theta * h$ which is generic over $V[G_\theta][g]$. Therefore $V[G][H]$ is equal to the generic extension $V[G_\theta][g][H_\theta][h]$. Moreover, since $\mathbb{P}^\theta$ and $\mathbb{K}_\theta$ live in $V[G_\theta]$ and H_θ is generic over $V[G_\theta][g]$, $V[G_\theta][g][H_\theta][h] = V[G_\theta][H_\theta][g][h]$. Similarly, we can exchange g and h since both $\mathbb{P}^\theta$ and $\mathbb{K}_\kappa/H_\theta$ live in $V[G_\theta][H_\theta]$ and h is generic over $V[G_\theta][H_\theta][g]$. It follows that $V[G_\theta][H_\theta][g][h] = V[G_\theta][H_\theta][h][g]$.

Since $\mathbb{P}^\theta$ and $\mathbb{P}_\theta \times \mathbb{K}_\kappa$ are both ω_1-closed in V, $\mathbb{P}^\theta$ is still ω_1-closed in $V[G_\theta][H_\theta][h]$. By Lemma 2.12, ω_1-closed forcing cannot add new cofinal branches to an ω_1-tree and hence any cofinal branch through S is already in $V[G_\theta][H_\theta][h]$.

To show that $\mathbb{K}_\kappa/H_\theta$ cannot add a cofinal branch to S over $V[G_\theta][H_\theta]$, we will work in $V[G_\theta]$. (Note that $\mathbb{K}_\kappa/H_\theta$ is only ω_1-distributive in $V[G_\theta][H_\theta]$ and therefore we cannot simply apply Fact 2.12 in $V[G_\theta][H_\theta]$.) We will show that if there is a $\mathbb{K}_\theta * (\mathbb{K}_\kappa/\dot{H}_\theta)$-name $\dot{b}$ and a condition $(q^*, r^*) \in \mathbb{K}_\theta * (\mathbb{K}_\kappa/\dot{H}_\theta)$ which forces that $\dot{b}$ is a cofinal branch through $\dot{S}$ that is not in $V[G_\theta][\dot{H}_\theta]$, then there is a condition $q \leq q^*$ which forces that $\dot{S}$ has an uncountable level. This is a contradiction, since we can assume that (q^*, r^*) is in $H_\theta * h$ and that q^* forces that $\dot{S}$ is an ω_1-tree.

The proof is similar to Silver's argument for ω_1-closed forcings: we build a tree $\mathcal{T}$ of conditions in $\mathbb{K}_\kappa$ labeled by ${}^{<\omega}2$, but since we are working with the quotient $\mathbb{K}_\kappa/\dot{H}_\theta$ and we work in $V[G_\theta]$ and not in $V[G_\theta][H_\theta]$, we will guide this construction by a decreasing sequence of length ω of conditions in $\mathbb{K}_\theta$, which will force conditions from $\mathcal{T}$ into the quotient $\mathbb{K}_\kappa/\dot{H}_\theta$.

We make a natural identification and view $(\dot{S}, <_{\dot{S}})$ as a name for a tree with underlying set ω_1.

Work in $V[G_\theta]$. Assume that $\dot{b}$ is a $\mathbb{K}_\theta * (\mathbb{K}_\kappa/\dot{H}_\theta)$-name and $(q^*, r^*) \in \mathbb{K}_\theta * (\mathbb{K}_\kappa/\dot{H}_\theta)$ forces that $\dot{b}$ is a cofinal branch through $\dot{S}$ that is not in $V[G_\theta][\dot{H}_\theta]$. We will build by induction on ω the following objects:

- a decreasing sequence $\langle q_n \mid n < \omega \rangle$ of conditions in $\mathbb{K}_\theta$ with $q_0 \leq q^*$;
- a labeled tree $\mathcal{T} = \{r_s \mid s \in {}^{<\omega}2\}$ of conditions in $\mathbb{K}_\kappa$, with $r_s \leq r^*$ for all $s \in {}^{<\omega}2$;
- a strictly increasing sequence $\langle \gamma_n \mid n < \omega \rangle$ of ordinals below ω_1,

such that the following hold for all $n < \omega$ and all $s \in {}^{<\omega}2$ of length n:

(a) $q_n = r_s \restriction \theta$; in particular $q_n \Vdash r_s \in \mathbb{K}_\kappa/\dot{H}_\theta$;
(b) the conditions $(q_{n+1}, r_{s^\frown 0})$ and $(q_{n+1}, r_{s^\frown 1})$ decide $\dot{b}$ up to γ_{n+1} differently; i.e., there are $\delta \leq \gamma_{n+1}$ and $\tau_{s^\frown 0} \neq \tau_{s^\frown 1}$ forced to be in $\dot{S}_\delta$ such that $(q_n, r_{s^\frown 0}) \Vdash \dot{b}(\delta) = \tau_{s^\frown 0}$ and $(q_n, r_{s^\frown 1}) \Vdash \dot{b}(\delta) = \tau_{s^\frown 1}$;

(c) $(q_{|t|}, r_t) \leq (q_{|s|}, r_s)$ for all $s \subseteq t$ in ${}^{<\omega}2$;
(d) $\mathrm{rng}(f_{r_s} \restriction [\theta, \kappa)) \cap \mathrm{rng}(f_{r_t} \restriction [\theta, \kappa)) = \emptyset$ for distinct s, t in ${}^{<\omega}2$ of the same length.

Definition 4.17. Let us call a system above, satisfying conditions (a)–(d), a *labeled system* for $\dot{b}$. We say just a labeled system if $\dot{b}$ is clear from the context.

A labeled system will be constructed below, using Claims 4.18 and 4.19. First we prove an auxiliary claim which will be useful for the construction.

Claim 4.18. *For every $(q, r^0), (q, r^1) \leq (q^*, r^*) \in \mathbb{K}_\theta * (\mathbb{K}_\kappa/\dot{H}_\theta)$ and $\gamma' < \omega_1$ there are $\gamma' < \gamma < \omega_1$, $(q', p^0) \leq (q, r^0)$ and $(q', p^1) \leq (q, r^1)$ such that (q', p^0) and (q', p^1) decide $\dot{b}(\gamma)$ differently and the condition q' extends both $p^0 \restriction \theta$ and $p^1 \restriction \theta$.*

Proof. Let $(q, r^0), (q, r^1) \leq (q^*, r^*)$ and $\gamma' < \omega_1$ be given. Fix for the moment a $\mathbb{K}_\theta$-generic F over $V[G_\theta]$ such that $q \in F$ and work in $V[G_\theta][F]$. Since $\dot{b}$ is forced by (q^*, r^*) to be a new cofinal branch through $\dot{S}$ and $q^* \in F$, there are $\bar{p}^0 \leq r^0$, $\bar{p}^1 \leq r^0$ and $\gamma > \gamma'$ such that $\bar{p}^0$ and $\bar{p}^1$ are both in $\mathbb{K}_\kappa/F$ and decide $\dot{b}(\gamma)$ differently; i.e., there are $\tau^0 \neq \tau^1$ in $\dot{S}_\gamma$ such that $\bar{p}^0 \Vdash \dot{b}(\gamma) = \tau^0$ and $\bar{p}^1 \Vdash \dot{b}(\gamma) = \tau^1$.

Since $\bar{p}^0$ and $\bar{p}^1$ are in $\mathbb{K}_\kappa/F$, $\bar{p}^0 \restriction \theta$ and $\bar{p}^1 \restriction \theta$ are in F and hence they are compatible. Let $\bar{q}$ be a common extension of $\bar{p}^0 \restriction \theta$, $\bar{p}^1 \restriction \theta$ and q such that, for each $i < 2$, we have $(\bar{q}, \bar{p}^i) \Vdash \dot{b}(\gamma) = \tau^i$.

Now, we return back to working in $V[G_\theta]$. Since $\bar{q}$ extends both $\bar{p}^0 \restriction \theta$ and $\bar{p}^1 \restriction \theta$, it forces both of them into the quotient $\mathbb{K}_\kappa/\dot{H}_\theta$. It follows $(\bar{q}, \bar{p}^0)$ and $(\bar{q}, \bar{p}^1)$ are conditions in $\mathbb{K}_\theta * (\mathbb{K}_\kappa/\dot{H}_\theta)$ which extend (q, r^0) and decide $\dot{b}(\gamma)$ differently.

Now, consider the condition $(\bar{q}, r^1)$. Since $\dot{b}$ is a $\mathbb{K}_\theta * (\mathbb{K}_\kappa/\dot{H}_\theta)$ name for a cofinal branch through $\dot{S}$, there is an extension $(q', p^1) \leq (\bar{q}, r^1)$ which decides $\dot{b}(\gamma)$. Since $\tau^0 \neq \tau^1$, (q', p^1) cannot decide $\dot{b}(\gamma)$ as being equal to both of them. Let p^0 be $\bar{p}^i$, for $i < 2$, such that (q', p^1) and $(q', \bar{p}^i)$ disagree on $\dot{b}(\gamma)$. Then q', p^0, p^1 and γ are as required. Note that $q' \leq p^1 \restriction \theta$ since q' forces p^1 into $\mathbb{K}_\kappa/\dot{H}_\theta$ and $\mathbb{K}_\theta$ is separative. $\square$

Now, we are ready to construct our labeled system. The construction is by induction on ω.

Let γ_0 be an arbitrary ordinal below ω_1 and let $(q_0, r_\emptyset)$ be (q^*, r^*). By Lemma 4.12, $q^* \leq r^* \restriction \theta$ since q^* forces r^* into $\mathbb{K}_\kappa/\dot{H}_\theta$ and $\mathbb{K}_\theta$ is separative. If $q^* \neq r^* \restriction \theta$, we can extend r^* appropriately to ensure the condition (a); for more details, see Claim 4.19 below in the successor step of the construction.

Now fix $n < \omega$ and assume that we have constructed γ_n, q_n and r_s for all $s \in {}^n2$. Let $\langle s_i \mid i < 2^n \rangle$ enumerate ${}^n2$. We describe how to construct γ_{n+1}, q_{n+1}, and r_s for $s \in {}^{n+1}2$.

We proceed by induction on $2^n = m$. Let us start with s_0. By Claim 4.18 there are $q^0 \leq q_n$, $r'_{s_0 ^\frown 0}$,$r'_{s_0 ^\frown 1} \leq r_{s_0}$ and $\gamma^0 \geq \gamma_n$ such that $(q^0, r'_{s_0 ^\frown 0})$, $(q^0, r'_{s_0 ^\frown 1})$ decide $\dot{b}(\gamma^0)$ differently and q^0 extends both $r'_{s_0 ^\frown 0} \restriction \theta$ and $r'_{s_0 ^\frown 1} \restriction \theta$. Now fix $1 \leq i < m$, and suppose that γ^{i-1} and q^{i-1} have been constructed. By Claim 4.18 there are $q^i \leq q^{i-1}$, $r'_{s_i ^\frown 0}$, $r'_{s_i ^\frown 1} \leq r_{s_i}$ and $\gamma^i \geq \gamma^{i-1}$ such that $(q^i, r'_{s_i ^\frown 0})$, $(q^i, r'_{s_i ^\frown 1})$ decide $\dot{b}(\gamma^i)$ differently and q^i extends both $r'_{s_i ^\frown 0} \restriction \theta$ and $r'_{s_i ^\frown 1} \restriction \theta$.

Let q'_{n+1} be q^{m-1} and γ_{n+1} be γ^{m-1}. It follows by the construction that the objects q'_{n+1}, γ_{n+1}, $r'_{s ^\frown j}$, for $j < 2$ and all $s \in {}^n2$ satisfy the desired conditions (b) and (c).

However, we have only ensured that q'_{n+1} extends $r'_s \restriction \theta$ for $s \in {}^{n+1}2$, but not that they are equal as is required in condition (a), and we have not ensured condition (d) either. To ensure conditions (a) and (d), we define appropriate extensions of q'_{n+1} and r'_s for all $s \in {}^{n+1}2$.

Claim 4.19. *The objects constructed above can be extended to satisfy conditions (a)–(d) of a labeled system in Definition 4.17.*

Proof. Since q'_{n+1} extends $r'_s \restriction \theta$ for all $s \in {}^{n+1}2$, $T_{q'_{n+1}}$ is an end-extension of $T_{r'_s}$ for all $s \in {}^{n+1}2$. Let $\eta+1$ be the height of $T_{q'_{n+1}}$ and let T' be a one-level extension of $T_{q'_{n+1}}$ such that for every node τ in $T_{q'_{n+1}}$ on level η we add at least countably many new nodes above τ into T'. The height of T' is $\eta+2$, and T' is normal and infinitely splitting since $T_{q'_{n+1}}$ is normal and infinitely splitting. Let f' be a function such that $\mathrm{dom}(f') = \mathrm{dom}(f_{q'_{n+1}})$ and for each $\alpha \in \mathrm{dom}(f')$ let $f'(\alpha) \supseteq f_{q'_{n+1}}(\alpha)$ be some node of T' on level $\eta+1$. Then $q_{n+1} = (T', f')$ is a condition in $\mathbb{K}_\theta$ and it extends q'_{n+1}.

Now, we define extensions of r'_s for $s \in {}^{n+1}2$. For $s \in {}^{n+1}2$, let f_s be a function such that $\mathrm{dom}(f_s) = \mathrm{dom}(f_{r'_s}) \cap [\theta, \kappa)$ and for each $\alpha \in \mathrm{dom}(f_s)$, let $f_s(\alpha) \supseteq f_{r'_s}(\alpha)$ be some node of T' on level $\eta+1$. Moreover, ensure that $\mathrm{rng}(f_s) \cap \mathrm{rng}(f_t) = \emptyset$ for all $s \neq t \in {}^{n+1}2$. Note that we can find such functions since T' is infinitely splitting and the sets $\mathrm{dom}(f_{r'_s})$ for $s \in {}^{n+1}2$ are at most countable. Set $r_s = (T', f' \cup f_s)$ for $s \in {}^{n+1}2$. Clearly, r_s are conditions in $\mathbb{K}_\kappa$ for all $s \in {}^{n+1}2$ such that q_{n+1} is equal to $r_s \restriction \theta$, hence condition (a) holds. Condition (d) follows from the definition of f_s for $s \in {}^{n+1}2$.

Since q_{n+1} extends q'_{n+1} and r_s extends r'_s for all $s \in {}^{n+1}2$, conditions (b) and (c) are still satisfied for q_{n+1}, γ_{n+1}, $r_{s^\frown j}$, for $j < 2$ and all $s \in {}^{n}2$. $\square$

This completes our construction of a labeled system. Let γ be the supremum of $\langle \gamma_n \,|\, n < \omega \rangle$. To finish the proof of the whole Theorem 4.16, we would like to find a lower bound q of the sequence $\langle q_n \,|\, n < \omega \rangle$ and a lower bound r_x of sequences $\langle r_{x \restriction n} \,|\, n < \omega \rangle$ for all $x \in {}^\omega 2$ such that q forces r_x into the quotient $\mathbb{K}_\kappa / \dot{H}_\theta$ for every x, thus ensuring that every (q, r_x) is a condition in $\mathbb{K}_\theta * (\mathbb{K}_\kappa / \dot{H}_\theta)$. If this is the case, then q forces that level γ of $\dot{S}$ has size 2^ω, and hence q forces that $\dot{S}$ is not an ω_1-tree, finishing the proof.

In the rest of the proof we will construct such conditions. Before we start, note that we do not need to ensure that q forces r_x into the quotient for *all* $x \in {}^\omega 2$; it is enough to have this for uncountably many $x \in {}^\omega 2$.

Let $T^* = \bigcup_{n<\omega} T_{q_n}$ and $a = \bigcup_{n<\omega} \mathrm{dom}(f_{q_n})$. If the height of T^* is a successor ordinal, then the construction of the appropriate lower bounds is analogous but simpler than the one we state below and we leave it as an exercise for the reader. Let us therefore assume that T^* is a normal tree with a limit height. Note that T^* does not contain a copy of ${}^{<\omega+1}2$. Let $\eta < \omega_1$ denote the height of T^*. By condition (a), $T^* = \bigcup_{n<\omega} T_{r_{x\restriction n}}$ and $a = \bigcup_{n<\omega} \mathrm{dom}(f_{r_{x\restriction n}}) \cap \theta$ for all $x \in {}^\omega 2$.

Note that if we want to ensure that a lower bound of $\langle q_n \,|\, n < \omega \rangle$ forces a lower bound of $\langle r_{x\restriction n} \,|\, n < \omega \rangle$, for some $x \in {}^\omega 2$, into the quotient, we are obliged not only to extend all cofinal branches through T^* which are given by functions $\{ f_{q_n} \,|\, n < \omega \}$ (recall the proof that $\mathbb{K}_\kappa$ is ω_1-closed), but also to extend all cofinal branches given by $\{ f_{r_{x\restriction n}} \,|\, n < \omega \}$, otherwise it can happen that the lower bounds of $\langle q_n \,|\, n < \omega \rangle$ and $\langle r_{x\restriction n} \,|\, n < \omega \rangle$ for some $x \in {}^\omega 2$ will be incompatible. Therefore if we want to ensure that a lower bound of $\langle q_n \,|\, n < \omega\rangle$ forces lower bounds of $\langle r_{x\restriction n} \,|\, n < \omega\rangle$ for uncountably many $x \in {}^{\omega}2$ into the quotient, we are obliged to extend uncountably many cofinal branches of T^*. This can be done because the trees in the conditions can be wide, but we need to make sure that while doing it, we do not add a copy of ${}^{<\omega+1}2$. To argue that T_q does not contain a copy of ${}^{<\omega+1}2$, we use following claim which says that if we add only countably many cofinal branches to a tree which does not contain a copy of ${}^{<\omega+1}2$, then we do not add a copy of ${}^{<\omega+1}2$.

Claim 4.20. *Let T be a tree with countable height $\eta+1$, where η is limit. Let D be a countable set of cofinal branches of $T \restriction \eta$. If T does not contain a copy of ${}^{<\omega+1}2$, then T' also does not contain a copy of ${}^{<\omega+1}2$, where $T' = T \cup \{d \,|\, d \in D\}$.*

Proof. This is a straightforward consequence of the following well-known fact: If P is a perfect subset of ${}^{\omega}2$ and $Q \subseteq P$ is countable, then $P \setminus Q$ contains a perfect set. □

To build suitable lower bounds, we will proceed similarly as in the proof of Lemma 4.9. First apply the construction in Lemma 4.9 to find q' which is a lower bound of $\langle q_n \,|\, n < \omega\rangle$ such that $T_{q'}$ has height $\eta + 1$ and $\mathrm{dom}(f_{q'}) = a$. Note that each node $\tau \in T_{q'}$ on level η determines a cofinal branch through T^* which we denote b_τ. Note also that $T_{q'} = T^* \cup \{b_\tau \,|\, \tau \in (T_{q'})_\eta\}$.

For each $x \in {}^{\omega}2$, let $a_x = \bigcup_{n<\omega} \mathrm{dom}(f_{r_{x\restriction n}}) \setminus \theta$. In order to ensure that q is compatible with a lower bound of $\langle r_{x\restriction n} \,|\, n < \omega\rangle$, we are obliged to extend cofinal branches given by $\{r_{x\restriction n} \,|\, n < \omega\}$. For $\alpha \in a_x$, let

$$d^x_\alpha := \bigcup\{f_{r_{x\restriction n}}(\alpha) \,|\, n < \omega \wedge \alpha \in \mathrm{dom}(f_{r_{x\restriction n}})\} \tag{3}$$

and let $D_x = \{d^x_\alpha \,|\, \alpha \in a_x\}$. Note that $D_x \cap D_y = \emptyset$ for all $x \neq y \in {}^{\omega}2$ by condition (d). Moreover, let as define f_x to be a function with domain a_x such that for each $\alpha \in a_x$, $f_x(\alpha) = d^x_\alpha$.

We will use the following bookkeeping device to extend cofinal branches in uncountably many D_x (but not in all D_x because this might add a copy of ${}^{<\omega+1}2$).

Let $\langle \iota_\xi \,|\, \xi < \omega_1\rangle$ enumerate all isomorphic embeddings ι from ${}^{<\omega}2$ to T^* such that, for every $x \in {}^{\omega}2$, the function $\bigcup\{\iota(x \restriction n) \,|\, n < \omega\}$ is a cofinal branch through T^*. Note that this is possible, due to the fact that CH holds and $|T^*| \leq \omega_1$. For each $\xi < \omega_1$, let

$$[\iota_\xi] := \left\{\bigcup\{\iota_\xi(x \restriction n) \mid n < \omega\} \;\middle|\; x \in {}^{\omega}2\right\}. \tag{4}$$

Note that $[\iota_\xi] \subseteq [T^*]$, and $|[\iota_\xi]| = 2^\omega = \omega_1$.

We now recursively build two disjoint sequences $\langle x_\xi \,|\, \xi < \omega_1\rangle$ and $\langle y_\xi \,|\, \xi < \omega_1\rangle$ of functions in ${}^{\omega}2$. The first sequence picks functions x in ${}^{\omega}2$ for which we will extend the corresponding cofinal branches D_x. The second sequence picks functions y in ${}^{\omega}2$ for which we will forbid the extension of corresponding cofinal branches in D_y to ensure that T_q does not contain a copy of ${}^{<\omega+1}2$. Note that we need here that the D_x's are pairwise disjoint.

Suppose that we have constructed x_ζ and y_ζ for all $\zeta < \xi$, where $\xi < \omega_1$. First, set x_ξ to be any function such that $x_\xi \notin \{x_\zeta \,|\, \zeta < \xi\} \cup \{y_\zeta \,|\, \zeta < \xi\}$. Next, if there is $y \notin \{x_\zeta \,|\, \zeta \leq \xi\} \cup \{y_\zeta \,|\, \zeta < \xi\}$ such that

$$([\iota_\xi] \setminus \{b_\tau \,|\, \tau \in (T_{q'})_\eta\}) \cap D_y \neq \emptyset, \tag{5}$$

then let y_ξ be any such y. If there is no such y, then let y_ξ be any y which is not in $\{x_\zeta \,|\, \zeta \leq \xi\} \cup \{y_\zeta \,|\, \zeta < \xi\}$.

At the end of the construction, set $T_q = T_{q'} \cup \bigcup_{\xi<\omega_1} D_{x_\xi}$ and $f_q = f'_q$. Set $T_{r_{x_\xi}} = T_q$ and $f_{r_{x_\xi}} = f_{q'} \cup f_{x_\xi}$ for every $\xi < \omega_1$.

Claim 4.21. *The following hold:*

(i) q is a condition in $\mathbb{K}_\theta$;
(ii) r_{x_ξ} is a condition in $\mathbb{K}_\kappa$ for all $\xi < \omega_1$;
(iii) q forces r_{x_ξ} into the quotient $\mathbb{K}_\kappa/\dot{H}_\theta$, for all $\xi < \omega_1$.

Proof. Note that T_q is a normal tree since $T_{q'}$ is normal and $T_q = T_{q'} \cup \bigcup_{\xi<\omega_1} D_{x_\xi}$. The only nontrivial thing to check is that T_q does not contain a copy of ${}^{<\omega+1}2$. Assume for contradiction that T_q contains a copy of ${}^{<\omega+1}2$; hence there is $\xi < \omega_1$ such that $[\iota_\xi] \subseteq (T_q)_\eta$. Let I be the set of all $\zeta < \omega_1$ such that $(D_{x_\zeta} \setminus (T_{q'})_\eta) \cap [\iota_\xi] \neq \emptyset$; in particular, $[\iota_\xi] \subseteq (T_{q'})_\eta \cup (\bigcup_{\zeta\in I} D_{x_\zeta})$. Note that I cannot be countable by Claim 4.20, since $T_{q'}$ does not contain a copy of ${}^{<\omega+1}2$. Hence I is unbounded in ω_1 which means that there is $\zeta > \xi$ such that $(D_{x_\zeta} \setminus (T_{q'})_\eta) \cap [\iota_\xi] \neq \emptyset$. Therefore in step ξ of the induction, condition (5) was satisfied and we chose y_ξ such that the intersection $(D_{y_\xi} \setminus (T_{q'})_\eta) \cap [\iota_\xi]$ is nonempty. This is a contradiction since D_{y_ξ} should be disjoint from D_{x_ζ} for all $\zeta \in I$, and we did not add any cofinal branches from $D_{y_\xi} \setminus (T_{q'})_\eta$ into T_q. This shows that q is a condition in $\mathbb{K}_\theta$ and r_{x_ξ} are conditions in $\mathbb{K}_\kappa$ for all $\xi < \omega_1$.

By the definition of r_{x_ξ}, $r_{x_\xi} \restriction \theta = q$ and hence q forces r_{x_ξ} into the quotient $\mathbb{K}_\kappa/\dot{H}_\theta$ for every $\xi < \omega_1$. □

It is straightforward to check that q forces $\dot{S}_\gamma$ to be uncountable: Let F be any $\mathbb{K}_\theta$-generic which contains q. Then r_{x_ξ} is a condition in the quotient $\mathbb{K}_\theta/F$ for every $\xi < \omega_1$. Let $r^*_{x_\xi}$ be an extension of r_{x_ξ} which decides $\dot{b}(\gamma) = \tau_\xi$. Then for every $\xi \neq \zeta < \omega_1$, $\tau_\xi \neq \tau_\zeta$ since (q, r_{x_ξ}) and (q, r_{x_ζ}) decide $\dot{b}$ differently below γ. This concludes the proof of Theorem 4.16. □

5. Kurepa trees and continuous images

In this section, we prove Theorem B, answering a question of Lücke and Schlicht from [8]. In particular, we will construct a model of ZFC in which GCH holds, there exist ω_2-Kurepa trees, and, for every ω_2-Kurepa tree $S \subseteq {}^{<\omega_2}\omega_2$, $[S]$ is not a continuous image of ${}^{\omega_2}\omega_2$.

We will also prove that, in the model that we construct, closed subsets of ${}^{\omega_2}\omega_2$ satisfy the strongest possible perfect set property compatible with the existence of ω_2-Kurepa trees. Note that, by the discussion at the beginning of Section 4, if $S \subseteq {}^{<\omega_2}\omega_2$ is a Kurepa tree, then $\mathsf{PSP}_{\omega_1+1}([S])$ necessarily fails. In the model we construct, this failure will be sharp: clause (5) in the statement of Theorem 5.1 below will imply that, for every closed $E \subseteq {}^{\omega_2}\omega_2$, we have $\mathsf{PSP}_{\omega_1}(E)$.

In what follows, when referring to trees, we will sometimes write *countably closed* in place of *$(<\omega_1)$-closed*.

Theorem 5.1. *Suppose that there is an inaccessible cardinal κ. Then there is a forcing extension in which*

(1) $\kappa = \omega_3$;
(2) GCH holds;

(3) *there is an ω_2-Kurepa tree;*
(4) *for every ω_2-Kurepa tree $S \subseteq {}^{<\omega_2}\omega_2$, $[S]$ is not a continuous image of ${}^{\omega_2}\omega_2$;*
(5) *for every closed subset $E \subseteq {}^{\omega_2}\omega_2$, there is $X \subseteq E$ with $|X| \leq \omega_2$ such that $E \setminus X$ is ω_1-perfect.*

Proof. We can assume that GCH holds in the ground model. Let $\mathbb{P} := \mathrm{Coll}(\omega_2, <\kappa)$, and let $\mathbb{Q}$ be the forcing to add a countably closed ω_2-Kurepa tree with κ-many cofinal branches. More precisely, $\mathbb{Q}$ consists of all pairs $q = (T_q, f_q)$ such that

- there is an $\eta_q < \omega_2$ such that T_q is a normal, splitting, countably closed subtree of ${}^{<\eta_q+1}\omega_2$;
- for all $\xi \leq \eta_q$, $|T_q \cap {}^{\xi}\omega_2| \leq \omega_1$;
- f_q is a partial function of size at most ω_1 from κ to $T_q \cap {}^{\eta_q}\omega_2$.

If $q_0, q_1 \in \mathbb{Q}$, then $q_1 \leq q_0$ if

- $\eta_{q_1} \geq \eta_{q_0}$;
- $T_{q_1} \cap {}^{<\eta_{q_0}+1}\omega_2 = T_{q_0}$;
- $\mathrm{dom}(f_{q_1}) \supseteq \mathrm{dom}(f_{q_0})$;
- for all $\alpha \in \mathrm{dom}(f_{q_0})$, $f_{q_1}(\alpha) \sqsupseteq f_{q_0}(\alpha)$.

We also include $(\emptyset, \emptyset)$ in $\mathbb{Q}$ as $1_{\mathbb{Q}}$. By standard arguments, $\mathbb{P} \times \mathbb{Q}$ is ω_2-closed and has the κ-cc.

For a cardinal $\delta < \kappa$, let $\mathbb{P}_\delta := \mathrm{Coll}(\omega_2, <\delta)$, and let $\mathbb{Q}_\delta$ be the set of $q \in \mathbb{Q}$ for which $\mathrm{dom}(q) \subseteq \delta$. It is routine to verify that, for all $\delta < \kappa$, the map $\pi_\delta : \mathbb{P} \times \mathbb{Q} \to \mathbb{P}_\delta \times \mathbb{Q}_\delta$ defined by letting $\pi_\delta(p, q) = (p \restriction \delta, (T_q, f_q \restriction \delta))$ is a projection.

Claim 5.2. *Let $\delta < \kappa$ be a cardinal. Then, in $V^{\mathbb{P}_\delta \times \mathbb{Q}_\delta}$, the quotient forcing $\mathbb{R}_\delta := (\mathbb{P} \times \mathbb{Q})/(\mathbb{P}_\delta \times \mathbb{Q}_\delta)$ is countably closed.*

Proof. Let $G_\delta \times H_\delta$ be $\mathbb{P}_\delta \times \mathbb{Q}_\delta$-generic over V, and move to $V[G_\delta \times H_\delta]$. Let $\langle (p_n, q_n) \mid n < \omega \rangle$ be be a decreasing sequence from $\mathbb{P} \times \mathbb{Q}$ such that, for all $n < \omega$, we have $\pi_\delta(p_n, q_n) \in G_\delta \times H_\delta$. Note that this sequence is in fact in V, by the distributivity of $\mathbb{P} \times \mathbb{Q}$.

Let $p = \bigcup\{p_n \mid n < \omega\}$. It is evident that $p \in \mathbb{P}$ and $p \restriction \delta \in G_\delta$. We next define a condition $q \in \mathbb{Q}$. If the sequence $\langle T_{q_n} \mid n < \omega \rangle$ is eventually constant, then, by removing an initial segment of the sequence, we can assume it is constant. Let T_q be this constant value, and let $f_q = \bigcup\{f_{q_n} \mid n < \omega\}$. Then (p, q) is a lower bound for $\langle (p_n, q_n) \mid n < \omega \rangle$ and $\pi_\delta(p, q) \in G_\delta \times H_\delta$.

So suppose that $\langle T_{q_n} \mid n < \omega \rangle$ is not eventually constant. Let $T^* = \bigcup\{T_{q_n} \mid n < \omega\}$. Then T^* is a normal subtree of ${}^{<\eta}\omega_2$ for some $\eta < \omega_2$ of countable cofinality. Moreover, by CH, we have $|[T^*]| \leq \omega_1$. Let $T_q := T^* \cup [T^*]$. By construction and the fact that each T_{q_n} is countably closed, it follows that T_q is a normal countably closed subtree of ${}^{<\eta+1}\omega_2$. Since the trees in the first coordinate of conditions in $\mathbb{Q}$ are required to be countably closed, we must have $(T_q, \emptyset) \in H_\delta$. Let $D = \bigcup\{\mathrm{dom}(f_{q_n}) \mid n < \omega\}$. Define $f_q : D \to B$ by letting

$$f_q(\alpha) = \bigcup\{f_{q_n}(\alpha) \mid n < \omega \ \wedge \ \alpha \in \mathrm{dom}(f_{q_n})\}$$

for all $\alpha \in D$. Then (p, q) is a lower bound for $\langle (p_n, q_n) \mid n < \omega \rangle$ and $\pi_\delta(p, q) \in G_\delta \times H_\delta$. $\square$

Let $G \times H$ be $\mathbb{P} \times \mathbb{Q}$-generic over V and, for all cardinals $\delta < \kappa$, let $G_\delta \times H_\delta$ be the $\mathbb{P}_\delta \times \mathbb{Q}_\delta$-generic filter induced by $G \times H$. We claim that $V[G \times H]$ is the desired forcing extension. We have $(\omega_2)^V = (\omega_2)^{V[G\times H]}$, and $\kappa = (\omega_3)^{V[G\times H]}$; moreover, GCH holds in $V[G\times H]$. Let $T_H = \bigcup\{T_q \mid q \in H\}$ and, for each $\alpha < \kappa$, let

$$b_\alpha = \bigcup\{f_q(\alpha) \mid q \in H \ \wedge\ \alpha \in \mathrm{dom}(f_q)\}.$$

Then T_H is a normal, countably closed subtree of ${}^{<\omega_2}\omega_2$, all of whose levels have size at most ω_1. Moreover, for all $\alpha < \kappa$, b_α is a cofinal branch through T_H and, by genericity, for all $\alpha < \beta < \kappa$, we have $b_\alpha \neq b_\beta$. Therefore, T_H is an ω_2-Kurepa tree in $V[G\times H]$.

To verify clause (4) in the statement of the theorem, we must show that, in $V[G\times H]$, there is no ω_2-Kurepa tree $S \subseteq {}^{<\omega_2}\omega_2$ such that $[S]$ is a continuous image of ${}^{\omega_2}\omega_2$. Suppose for the sake of contradiction that S is such a tree, and let $g : {}^{\omega_2}\omega_2 \to [S]$ be a continuous surjection. Define a function $h : {}^{<\omega_2}\omega_2 \to S \cup [S]$ by letting $h(\sigma) = \bigwedge\{g(x) \mid x \in N_\sigma\}$ for all $\sigma \in {}^{<\omega_2}\omega_2$, and let

$$W = \{\sigma \in {}^{<\omega_2}\omega_2 \mid \mathrm{dom}(h(\sigma)) \geq \mathrm{dom}(\sigma)\}.$$

Note that h is monotone, i.e., for all $\sigma, \tau \in {}^{<\omega_2}\omega_2$, if $\sigma \sqsubseteq \tau$, then $h(\sigma) \sqsubseteq h(\tau)$. It follows that W is $(<\omega_2)$-closed, i.e., if $\eta < \omega_2$ and $\langle \sigma_\xi \mid \xi < \eta\rangle$ is a $\sqsubseteq$-increasing sequence of elements of W, then $\bigcup\{\sigma_\xi \mid \xi < \eta\} \in W$.

Claim 5.3. *For all $x \in {}^{\omega_2}\omega_2$, $C_x := \{\alpha < \omega_2 \mid x \restriction \alpha \in W\}$ is a club in ω_2.*

Proof. Fix $x \in {}^{\omega_2}\omega_2$. The fact that C_x is closed follows immediately from the fact that h is monotone. To see that it is unbounded, fix an $\alpha_0 < \omega_2$. We will find $\alpha \in C_x \setminus \alpha_0$. Starting with α_0, recursively define an increasing sequence $\langle \alpha_n \mid n < \omega\rangle$ of ordinals below ω_2 such that, for all $n < \omega$ and all $y \in N_{x\restriction\alpha_{n+1}}$, we have $g(y) \restriction \alpha_n = g(x) \restriction \alpha_n$. It is straightforward to build such a sequence due to the continuity of g.

Let $\alpha = \sup\{\alpha_n \mid n < \omega\}$. For all $y \in N_{x\restriction\alpha}$ and all $n < \omega$, we have $g(y) \restriction \alpha_n = g(x) \restriction \alpha_n$; therefore, $g(y) \restriction \alpha = g(x) \restriction \alpha$. In particular, $h(x \restriction \alpha) \sqsupseteq g(x) \restriction \alpha$, so $\alpha \in C_x$. $\square$

Note that S, h, and W are objects of size ω_2, so, by the chain condition of $\mathbb{P}\times\mathbb{Q}$, we can fix a $\delta < \kappa$ so that $S, h, W \in V[G_\delta \times H_\delta]$ and, in that model, the quotient forcing $\mathbb{R}_\delta$ forces them to have the relevant properties isolated above. In particular, letting $\dot{g}$ be an $\mathbb{R}_\delta$-name for g, it is forced by every condition in $\mathbb{R}_\delta$ that S is an ω_2-Kurepa tree, $\dot{g}$ is a continuous surjection from ${}^{\omega_2}\omega_2$ to $[S]$, and h and W are as defined above from S and $\dot{g}$.

Work now in $V[G_\delta \times H_\delta]$. In this model, $2^{\omega_2} < \kappa$; therefore, forcing with $\mathbb{R}_\delta$ must add new cofinal branches to S. Let $\dot{b}$ be an $\mathbb{R}_\delta$-name for a cofinal branch through S such that $\Vdash_{\mathbb{R}_\delta} \dot{b} \notin V[G_\delta \times H_\delta]$. Let $\dot{x}$ be an $\mathbb{R}_\delta$-name for an element of ${}^{\omega_2}\omega_2$ such that $\Vdash_{\mathbb{R}_\delta} \dot{g}(\dot{x}) = \dot{b}$.

Now, by recursion on $\mathrm{dom}(\sigma)$, we construct a labeled tree $\langle (r_\sigma, b_\sigma, x_\sigma) \mid \sigma \in {}^{<\omega_1}2\rangle$ of elements of $\mathbb{R}_\delta \times S \times {}^{<\omega_2}\omega_2$. We will arrange so that the following requirements are satisfied.

1. For all $\sigma \in {}^{<\omega_1}2$, we have
   (a) $r_\sigma \Vdash_{\mathbb{R}_\delta}$ "$\dot{b} \sqsupseteq b_\sigma$ and $\dot{x} \sqsupseteq x_\sigma$";
   (b) $x_\sigma \in W$;
   (c) $h(x_\sigma) \sqsupseteq b_\sigma$.
2. For all $\sigma \sqsubseteq \tau \in {}^{<\omega_1}2$, we have $r_\tau \leq r_\sigma$, $b_\tau \sqsupseteq b_\sigma$, and $x_\tau \sqsupseteq x_\sigma$.
3. For all $\sigma \in {}^{<\omega_1}2$, $b_{\sigma^\frown\langle 0\rangle}$ and $b_{\sigma^\frown\langle 1\rangle}$ are incomparable in S.

Begin by letting $r_\emptyset = (\emptyset, \emptyset)$ and $b_\emptyset = x_\emptyset = \emptyset$. Next, suppose that $\sigma \in {}^{<\omega_1}2$ is of limit length, and suppose that $(r_{\sigma\restriction\eta}, b_{\sigma\restriction\eta}, x_{\sigma\restriction\eta})$ has been defined for all $\eta < \mathrm{dom}(\sigma)$. By Lemma 5.2, $\mathbb{R}_\delta$ is countably closed, so we can let r_σ be a lower bound of $\langle r_{\sigma\restriction\eta} \mid \eta < \mathrm{dom}(\sigma)\rangle$. Let $b_\sigma = \bigcup\{b_{\sigma\restriction\eta} \mid \eta < \mathrm{dom}(\sigma)\}$ and $x_\sigma = \bigcup\{x_{\sigma\restriction\eta} \mid \eta < \mathrm{dom}(\sigma)\}$. For all $\eta < \mathrm{dom}(\sigma)$, we have $r_\sigma \Vdash_{\mathbb{R}_\delta}$ "$\dot{b} \sqsupseteq b_{\sigma\restriction\eta}$ and $\dot{x} \sqsupseteq x_{\sigma\restriction\eta}$". As a result, we get $r_\sigma \Vdash_{\mathbb{R}_\delta}$ "$\dot{b} \sqsupseteq b_\sigma$ and $\dot{x} \sqsupseteq x_\sigma$". In particular, it follows that b_σ is in fact in S. The fact that $x_\sigma \in W$ and $h(x_\sigma) \sqsupseteq b_\sigma$ follows from the closure of W and the monotonicity of h. We have therefore satisfied all of the requirements of the construction.

Finally, suppose that $\sigma \in {}^{<\omega_1}2$ and we have defined $(r_\sigma, b_\sigma, x_\sigma)$. We describe how to define $(r_{\sigma^\frown\langle i\rangle}, b_{\sigma^\frown\langle i\rangle}, x_{\sigma^\frown\langle i\rangle})$ for $i < 2$. First, since $\Vdash_{\mathbb{R}_\delta} \dot{b} \notin V[G_\delta \times H_\delta]$, we can find $r_i^* \leq r_\sigma$ and $b_i^* \sqsupseteq b_\sigma$ for $i < 2$ such that b_0^* and b_1^* are incomparable in S and, for $i < 2$, we have $r_i^* \Vdash_{\mathbb{R}_\delta} \dot{b} \sqsupseteq b_i^*$.

By Claim 5.3, there are forced to be unboundedly many $\xi < \omega_2$ such that $\dot{x} \restriction \xi \in W$. Therefore, for each $i < 2$, we can find $r_{\sigma^\frown\langle i\rangle} \leq r_i^*$ and $x_{\sigma^\frown\langle i\rangle} \in W$ such that

- $x_{\sigma^\frown\langle i\rangle} \sqsupseteq x_\sigma$;
- $\mathrm{dom}(x_{\sigma^\frown\langle i\rangle}) \geq \mathrm{dom}(b_i^*)$;
- $r_{\sigma^\frown\langle i\rangle} \Vdash_{\mathbb{R}_\delta} \dot{x} \sqsupseteq x_{\sigma^\frown\langle i\rangle}$.

Finally, for $i < 2$, let $b_{\sigma^\frown\langle i\rangle} = h(x_{\sigma^\frown\langle i\rangle})$. Note first that, since

$$r_{\sigma^\frown\langle i\rangle} \Vdash \dot{g}(\dot{x}) = \dot{b} \wedge \dot{x} \sqsupseteq x_{\sigma^\frown\langle i\rangle},$$

it follows that $r_{\sigma^\frown\langle i\rangle} \Vdash \dot{b} \sqsupseteq b_{\sigma^\frown\langle i\rangle}$. Next observe that $b_{\sigma^\frown\langle i\rangle}$ must be in S, since if it were in $[S]$, then we would have $r_{\sigma^\frown\langle i\rangle} \Vdash \dot{b} = b_{\sigma^\frown\langle i\rangle}$, contradicting the fact that $\dot{b}$ is forced not to be in $V[G_\delta \times H_\delta]$. We have therefore satisfied requirements (1)–(3) above, and we can continue with the construction.

At the end of the construction, since CH holds, $\{b_\sigma \mid \sigma \in {}^{<\omega_1}2\}$ is a subset of S of size ω_1. Therefore, we can fix a $\xi < \omega_2$ such that $b_\sigma \in {}^{<\xi}\omega_2$ for all $\sigma \in {}^{<\omega_1}2$. For each $\nu \in {}^{\omega_1}2$, let $b_\nu = \bigcup\{b_{\nu\restriction\eta} \mid \eta < \omega_1\}$.

Claim 5.4. *For all $\nu \in {}^{\omega_1}2$, $b_\nu \in S$.*

Proof. Fix $\nu \in {}^{\omega_1}2$. Let $x_\nu = \bigcup\{x_{\nu\restriction\eta} \mid \eta < \omega_1\}$. By monotonicity of h, we have $h(x_\nu) \sqsupseteq h(x_{\nu\restriction\eta}) \sqsupseteq b_{\nu\restriction\eta}$ for all $\eta < \omega_1$. Therefore, by definition of b_ν, it follows that $h(x_\nu) \sqsupseteq b_\nu$. Since h maps into $S \cup [S]$, we get $b_\nu \in S$, as desired. □

Claim 5.5. *For all distinct $\nu_0, \nu_1 \in {}^{\omega_1}2$, we have $b_{\nu_0} \neq b_{\nu_1}$.*

Proof. Fix distinct $\nu_0, \nu_1 \in {}^{\omega_1}2$. Let $\eta < \omega_1$ be least such that $\nu_0(\eta) \neq \nu_1(\eta)$. Without loss of generality, assume that $\nu_i(\eta) = i$ for $i < 2$. Let $\sigma = \nu_0 \restriction \eta$. Then, for $i < 2$, we have $b_{\nu_i} \sqsupseteq b_{\sigma^\frown\langle i\rangle}$. Since $b_{\sigma^\frown\langle 0\rangle}$ and $b_{\sigma^\frown\langle 1\rangle}$ are incomparable in S, b_{ν_0} and b_{ν_1} are incomparable, as well; in particular, they are not equal. □

It follows that $\{b_\nu \mid \nu \in {}^{\omega_1}2\}$ is a collection of ω_2-many elements of $S_{\leq\xi}$. This contradicts the fact that S is an ω_2-tree, thus completing the verification of clause (4).

We finally verify clause (5) in the statement of the theorem. To this end, fix a closed set $E \subseteq {}^{\omega_2}\omega_2$ in $V[G \times H]$. Let $\Sigma = \{\sigma \in {}^{<\omega_2}\omega_2 \mid |E \cap N_\sigma| \leq \omega_2\}$. Equivalently, by the chain condition of $\mathbb{P} \times \mathbb{Q}$, Σ is the set of all $\sigma \in {}^{<\omega_2}\omega_2$ for which there exists $\delta < \kappa$ such that $E \cap N_\sigma \subseteq V[G_\delta \times H_\delta]$. Let $X = \bigcup\{E \cap N_\sigma \mid \sigma \in \Sigma\}$.

Then $|X| \leq \omega_2$ and, since E is closed and $\bigcup\{N_\sigma \mid \sigma \in \Sigma\}$ is open, it follows that $E \setminus X$ is closed. It remains to show that, for all $x_0 \in E \setminus X$, Player II has a winning strategy in $G_{\omega_2}(E \setminus X, x_0, \omega_1)$. To this end, fix an arbitrary $x_0 \in E \setminus X$.

Recall that $T(E) = \{x \restriction \eta \mid x \in E \text{ and } \eta < \omega_2\}$ is a subtree of ${}^{<\omega_2}\omega_2$; since E is closed, it follows that $E = [T(E)]$. Moreover, $T(E) \setminus \Sigma$ is a subtree of ${}^{<\omega_2}\omega_2$, and $E \setminus X = [T(E) \setminus \Sigma]$. In V, let $\dot{E}$ be a $\mathbb{P} \times \mathbb{Q}$-name for E and $\dot{X}$ be a $\mathbb{P} \times \mathbb{Q}$-name for X. By the chain condition of $\mathbb{P} \times \mathbb{Q}$, we can find a nonzero cardinal $\gamma < \kappa$ such that

- $x_0, T(E), \Sigma, X \in V[G_\gamma \times H_\gamma]$;
- interpreting $\dot{E}$ in $V[G_\gamma \times H_\gamma]$ as an $\mathbb{R}_\gamma$-name, the empty condition in $\mathbb{R}_\gamma$ forces all of the following statements:
  - $\dot{E} = [T(E)]$;
  - $\dot{E} \setminus X = [T(E) \setminus \Sigma]$;
  - $\forall \sigma \in T(E) \setminus \Sigma \ (|\dot{E} \cap N_\sigma| = \kappa)$.

To describe a winning strategy for Player II in $G_{\omega_2}(E \setminus X, x_0, \omega_1)$, we look more carefully at the quotient forcing $\mathbb{R}_\gamma$ in $V[G_\gamma \times H_\gamma]$. Note that T_H, the subtree of ${}^{<\omega_2}\omega_2$ added by H, is in $V[G_\gamma \times H_\gamma]$. It is not hard to show that $\mathbb{R}_\gamma$ is forcing equivalent to $\mathbb{S} \times \mathbb{T}$, where $\mathbb{S} = \mathrm{Coll}(\omega_2, [\gamma, \kappa))$ and $\mathbb{T}$ is the set of all pairs (η, f) such that $\eta < \omega_2$ and f is a partial function of size ω_1 from $\kappa \setminus \gamma$ to $T_H \cap {}^\eta\omega_2$. Given two elements $(\eta, f), (\xi, g) \in \mathbb{T}$, we have $(\xi, g) \leq_{\mathbb{T}} (\eta, f)$ if and only if $\xi \geq \eta$, $\mathrm{dom}(g) \supseteq \mathrm{dom}(f)$, and, for all $\alpha \in \mathrm{dom}(f)$, we have $g(\alpha) \sqsupseteq f(\alpha)$.

Recall that elements of $\mathrm{Coll}(\omega_2, [\gamma, \kappa))$ are all functions s such that $\mathrm{dom}(s)$ is a subset of $\mathrm{Card} \cap [\gamma, \kappa)$ of size $\leq \omega_1$ and, for each $\nu \in \mathrm{dom}(s)$, $s(\nu)$ is a partial function from ω_2 to ν of size $\leq \omega_1$. Given $I \subseteq \kappa \setminus \gamma$, let $\mathbb{S}_I$ be the set of all $s \in \mathbb{S}$ such that $\mathrm{dom}(s) \subseteq I$, and let $\mathbb{T}_I$ be the set of all $(\eta, f) \in \mathbb{T}$ such that $\mathrm{dom}(f) \subseteq I$.

Note that, if $I_0, I_1 \subseteq \kappa \setminus \gamma$ and $\pi : I_0 \to I_1$ is a bijection, then $\mathbb{T}_{I_0} \cong \mathbb{T}_{I_1}$ via the map $(\eta, f) \mapsto (\eta, \hat{\pi}(f))$, where $\mathrm{dom}(\hat{\pi}(f)) = \pi[\mathrm{dom}(f)]$ and, for all $\alpha \in \mathrm{dom}(f)$, we let $\hat{\pi}(f)(\alpha) = f(\alpha)$. If, moreover, π is a bijection between $\mathrm{Card} \cap I_0$ and $\mathrm{Card} \cap I_1$ such that $\pi(\nu) \leq \nu$ for all $\nu \in \mathrm{Card} \cap I_0$, then π introduces a projection $\tilde{\pi} : \mathbb{S}_{I_0} \to \mathbb{S}_{I_1}$ defined as follows: for each $s \in \mathbb{S}_{I_0}$, let $\tilde{\pi}(s) \in \mathbb{S}_{I_1}$ be such that

- $\mathrm{dom}(\tilde{\pi}(s)) = \pi[\mathrm{dom}(s)]$;
- for all $\nu \in \mathrm{dom}(s)$, we have $\mathrm{dom}(\tilde{\pi}(s)(\pi(\nu))) = \mathrm{dom}(s(\nu))$;
- for all $\nu \in \mathrm{dom}(s)$ and all $\eta \in \mathrm{dom}(s(\nu))$,

$$\tilde{\pi}(s)(\pi(\nu))(\eta) = \begin{cases} s(\nu)(\eta) & \text{if } s(\nu)(\eta) < \pi(\nu) \\ 0 & \text{otherwise.} \end{cases}$$

Suppose in particular that $\delta \in (\gamma, \kappa)$ is a cardinal, say $\delta = \gamma^{+\varepsilon}$ for some $\varepsilon < \kappa$. Then we can partition the interval $[\gamma, \kappa)$ into intervals $\{I_\zeta \mid \zeta < \kappa\}$, where $I_\zeta = [\gamma^{+\varepsilon \cdot \zeta}, \gamma^{+\varepsilon \cdot (\zeta+1)})$ for all $\zeta < \kappa$. The preceding discussion then implies that

- $\mathbb{R}_\gamma$ is forcing equivalent to the $<\omega_2$-support product of $\langle \mathbb{S}_{I_\zeta} \times \mathbb{T}_{I_\zeta} \mid \zeta < \kappa \rangle$;
- for all $\zeta < \kappa$, there is a projection from $\mathbb{S}_{I_\zeta} \times \mathbb{T}_{I_\zeta}$ to $\mathbb{S}_{I_0} \times \mathbb{T}_{I_0}$.

In particular, forcing with $\mathbb{R}_\gamma$ adds κ-many pairwise mutually $(\mathbb{S}_{I_0} \times \mathbb{T}_{I_0})$-generic filters.

We are now ready to describe Player II's winning strategy in $G_{\omega_2}(E \setminus X, x_0, \omega_1)$. Some aspects of the strategy can only be precisely specified after Player I makes their first move, so suppose that, in round 1 of the game, Player I plays the ordinal $\alpha_1 < \omega_2$. Let $\sigma_1 = x_0 \restriction \alpha_1$. Since $\sigma_1 \in T(E) \setminus \Sigma$, moving to $V[G_\gamma \times H_\gamma]$, we can find a nice $\mathbb{R}_\gamma$-name $\dot{y}$ that is forced to be an element of $\dot{E} \cap N_{\sigma_1} \setminus V[G_\gamma \times H_\gamma]$. By the chain condition of $\mathbb{R}_\gamma$, we can find a limit ordinal ε such that, letting $I_0 = [\gamma, \gamma^{+\varepsilon})$, $\dot{y}$ is an $\mathbb{S}_{I_0} \times \mathbb{T}_{I_0}$-name. Let $\mathbb{U}$ denote the $<\omega_2$-support product of κ-many copies of $\mathbb{S}_{I_0} \times \mathbb{T}_{I_0}$; for $\zeta < \kappa$, let $\mathbb{U}(\zeta)$ denote its ζ^{th} factor. By the preceding discussion, there is a projection from $\mathbb{R}_\gamma$ to $\mathbb{U}$, so we can view $\mathbb{R}_\gamma$ as a two-step iteration of the form $\mathbb{U} * \dot{\mathbb{W}}$. For each $\zeta < \kappa$, let $\dot{y}_\zeta$ denote the name for the interpretation of $\dot{y}$ with respect to the generic filter for $\mathbb{U}(\zeta)$. By the product lemma, for all $\zeta_0 < \zeta_1 < \kappa$, $\dot{y}_{\zeta_0}$ and $\dot{y}_{\zeta_1}$ are forced to be distinct elements of $N_{\sigma_1} \cap (\dot{E} \setminus X)$.

Let $K * L$ be $\mathbb{U} * \dot{\mathbb{W}}$-generic over $V[G_\gamma \times H_\gamma]$ such that $V[G \times H] = V[G_\gamma \times H_\gamma][K * L]$. For $\zeta < \kappa$, let K_ζ denote the $\mathbb{S}_{I_0} \times \mathbb{T}_{I_0}$-generic filter induced by the ζ^{th} factor of $\mathbb{U}$. In the course of the run of $G_{\omega_2}(E \setminus X, x_0, \omega_1)$ in which Player II plays according to the strategy we will specify here, producing the play, $\langle (\alpha_i, x_i) \mid 1 \leq i < \omega_1 \rangle$, Player II will construct a strictly increasing sequence $\langle \zeta_i \mid 1 \leq i < \omega_1 \rangle$ of ordinals below κ and a decreasing sequence $\langle p_i \mid 1 \leq i < \omega_1 \rangle$ of conditions in $\mathbb{S}_{I_0} \times \mathbb{T}_{I_0}$ satisfying the following requirements for all $1 \leq i < \omega_1$:

(1) x_i is the interpretation of $\dot{y}_{\zeta_i}$ in $V[G_\gamma \times H_\gamma][K * L]$;
(2) $p_i, p_{i+1} \in K_{\zeta_i}$;
(3) p_i decides the value of $\dot{y} \restriction \alpha_i$.

We first describe Player II's first move. Let $\zeta_1 = 0$, let their play x_1 be the interpretation of $\dot{y}_0$ in $V[G_\gamma \times H_\gamma][K * L]$, and, let $p_1 = 1_{\mathbb{S}_{I_0} \times \mathbb{T}_{I_0}}$.

Now suppose that $1 \leq j < \omega_1$ and, in our run of $G_{\omega_2}(E \setminus X, x_0, \omega_1)$, the players have played $\langle \alpha_i \mid i \leq j \rangle$ and $\langle x_i \mid i < j \rangle$, with Player II playing according to their winning strategy and also specifying $\langle (p_i, \zeta_i) \mid i < j \rangle$.

Suppose first that j is a successor ordinal, say $j = j_0 + 1$. First, choose $p_j \leq p_{j_0}$ such that $p_j \in K_{\zeta_{j_0}}$ and p_j decides the value of $\dot{y} \restriction \alpha_j$. Note that, in $V[G_\gamma \times H_\gamma]$, the set

$$\{f \in \mathbb{U} \mid \exists \zeta \in (\zeta_{j_0}, \kappa) \ (f(\zeta) \leq p_j)\}$$

is a dense open subset of $\mathbb{U}$. Therefore, by genericity, Player II can choose $\zeta_j \in (\zeta_{j_0}, \kappa)$ such that $p_j \in K_{\zeta_j}$. Then Player II plays the interpretation of $\dot{y}_{\zeta_j}$ as x_j.

Next, suppose that j is a limit ordinal. In this case, we have $\alpha_j = \sup\{\alpha_i \mid i < j\}$. Since $\mathbb{S}_{I_0} \times \mathbb{T}_{I_0}$ is countably closed, we can fix a lower bound p_j for $\langle p_i \mid i < j \rangle$. As in the previous paragraph, by genericity we can choose a $\zeta_j < \kappa$ such that $p_j \in K_{\zeta_j}$ and $\zeta_j > \zeta_i$ for all $i < j$. Then Player II plays the interpretation of $\dot{y}_{\zeta_j}$ as x_j.

It is readily verified that this describes a winning strategy for Player II in $G_{\omega_2}(E \setminus X, x_0, \omega_1)$, thus completing the proof that $E \setminus X$ is ω_1-perfect in $V[G \times H]$. □

6. Full trees

Recall that, given a tree T and an ordinal $\beta < \mathrm{ht}(T)$, we let $[T_{<\beta}]$ denote the set of cofinal branches through $T_{<\beta}$, i.e., the set of all elements $b \in \prod_{\alpha<\beta} T_\alpha$ such that the range of b is linearly ordered by $<_T$. Note that we can identify each element of T_β with an element of $[T_{<\beta}]$, namely the branch given by its predecessors; if T is normal, then this identification is injective. With a slight abuse of notation, then, we let $[T_{<\beta}] \setminus T_\beta$ denote the set of *vanishing* branches through T of length β, i.e., the set of $b \in [T_{<\beta}]$ such that the range of b does not have an upper bound in T_β.

Definition 6.1. A tree T is *full* if, for every limit ordinal $\beta < \mathrm{ht}(T)$, there is at most one vanishing branch through T of length β.

There has been some research in recent years into the existence of full κ-Suslin trees. For example, in [15], Shelah establishes the consistency of the existence of full κ-Suslin trees for a Mahlo cardinal κ, answering a question of Jech (cf. [10]). In [12], Rinot, Yadai, and You prove the consistency of the existence of full κ-Suslin trees at accessible cardinals κ; for example, they can consistently exist at all successors of regular uncountable cardinals.

Here, we are interested in full, splitting trees that may contain some cofinal branches. We begin by investigating full trees of height and size ω_1. We first show that, under $\diamondsuit$, we have considerable control over the number of cofinal branches through such trees, establishing Theorem C(1).

Theorem 6.2. *Suppose that $\diamondsuit$ holds. Then, for every cardinal $\nu \in \omega \cup \{\omega, \omega_1, 2^{\omega_1}\}$, there is a normal, full, splitting tree $T \subseteq {}^{<\omega_1}\omega_1$ such that $|[T]| = \nu$.*

Proof. If $\nu = 2^{\omega_1}$, we can simply let $T = {}^{<\omega_1}\omega_1$. For concreteness, we will prove the theorem in case $\nu = \omega_1$; the proof for smaller values of ν is similar but easier.

We will construct a tree T by recursively specifying its α^{th} level $T_\alpha = T \cap {}^\alpha\omega_1$ for $\alpha < \omega_1$. When specifying T_α, we will also specify an injective function $f_\alpha : \alpha \to T_\alpha$ with the requirement that, for all $\eta < \alpha < \beta < \omega_1$, we have $f_\alpha(\eta) \subseteq f_\beta(\eta)$. The idea is that, at the end of the construction, for each $\eta < \omega_1$, $b_\eta := \bigcup\{f_\alpha(\eta) \mid \eta < \alpha < \omega_1\}$ will be a cofinal branch through T, and we will arrange so that *every* cofinal branch through T is equal to b_η for some $\eta < \omega_1$.

Since $\diamondsuit$ holds, we can fix a sequence $\langle a_\alpha \mid \alpha < \omega_1 \rangle$ such that

- for all $\alpha < \omega_1$, $a_\alpha : \alpha \to \alpha$;
- for all $b : \omega_1 \to \omega_1$, there are stationarily many $\alpha < \omega_1$ for which $b \restriction \alpha = a_\alpha$.

We now describe the construction of T. We must set $T_0 = f_0 = \emptyset$. Given T_α and f_α, first form $T_{\alpha+1}$ by splitting maximally, i.e., $T_{\alpha+1} = \{\sigma^\frown i \mid \sigma \in T_\alpha,\ i < \omega_1\}$. For each $\eta < \alpha$, let $f_{\alpha+1}(\eta) = f_\alpha(\eta)^\frown 0$, and let $f_{\alpha+1}(\alpha)$ be any element of $T_{\alpha+1}$ not equal to $f_{\alpha+1}(\eta)$ for some $\eta < \alpha$.

Now suppose that $\beta < \omega_1$ is a limit ordinal and $T_{<\beta}$, together with $\langle f_\alpha \mid \alpha < \beta \}$, has been specified. For each $\eta < \beta$, let $b^\beta_\eta := \bigcup\{f_\alpha(\eta) \mid \eta < \alpha < \beta\}$. By construction, each b^β_η is in $[T_{<\beta}]$. Now consider the function a_β given by our $\diamondsuit$-sequence. There are two cases to consider.

Case 1: a_β is a branch through $T_{<\beta}$ and, for all $\eta < \beta$, we have $a_\beta \neq b^\beta_\eta$. In this case, let $T_\beta = [T_{<\beta}] \setminus \{a_\beta\}$. For all $\eta < \beta$, let $f_\beta(\eta) = b^\beta_\eta$.

Case 2: otherwise. In this case, let $T_\beta = [T_{<\beta}]$ and, again, let $f_\beta(\eta) = b^\beta_\eta$ for all $\eta < \beta$.

This completes the construction of T and $\langle f_\alpha \mid \alpha < \omega_1 \rangle$. It is easily verified that T is a normal, full, splitting tree and $\langle b_\eta \mid \eta < \omega_1 \rangle$ is an injective sequence of cofinal branches through T. It remains to show that every cofinal branch through T is equal to b_η for some $\eta < \omega_1$.

To this end, fix $b \in {}^{\omega_1}\omega_1$ such that, for all $\eta < \omega_1$, we have $b \neq b_\eta$. We will find $\beta < \omega_1$ such that $b \restriction \beta \notin T$. First, let C be the set of limit ordinals $\beta < \omega_1$ such that

- $b[\beta] \subseteq \beta$;
- for all $\eta < \beta$, we have $b_\eta \restriction \beta \neq b \restriction \beta$.

Then C is a club in ω_1, so we can fix $\beta \in C$ such that $b \restriction \beta = a_\beta$. Now consider stage β of the construction of T. If $a_\beta \notin [T_{<\beta}]$, then we are done, since this immediately implies that $b \restriction \beta \notin T$. If a_β *is* a branch through $T_{<\beta}$, then, for all $\eta < \beta$, we have

$a_\beta \neq b_\eta \restriction \beta = b_\eta^\beta$. We are therefore in Case 1 of the construction, and hence we have $b \restriction \beta = a_\beta \notin T$, as desired. $\square$

CH is a necessary condition for the existence of full, splitting trees of height ω_1 with few cofinal branches as it is easily seen that, if CH fails, then every full, splitting tree of height ω_1 has at least $2^{\aleph_0}$-many cofinal branches. However, we now show that CH is not a sufficient condition for this, i.e., the hypothesis of $\diamondsuit$ in Theorem 6.2 cannot be weakened to CH. Recall that a forcing notion $\mathbb{P}$ is *totally proper* if it is proper and adds no reals. We will need an iterable strengthening of total properness known as *complete properness*, introduced in [11] (cf. also [2]). Let us recall the relevant definitions, beginning with α-properness.

Definition 6.3. Suppose that $\mathbb{Q}$ is a forcing notion and θ is a sufficiently large regular cardinal. A countable elementary submodel M of $H(\theta)$ is said to be *suitable for* $\mathbb{Q}$ if $\mathbb{Q}, \mathscr{P}(\mathbb{Q}) \in M$.

If M is suitable for $\mathbb{Q}$ and $q \in \mathbb{Q}$, then we say that q is $(M, \mathbb{Q})$-generic if, for every dense open subset D of $\mathbb{Q}$ with $D \in M$ and every $r \leq_{\mathbb{Q}} q$, r is compatible with an element of $D \cap M$. If $G \subseteq \mathbb{Q} \cap M$ is a filter, then we say that G is $(M, \mathbb{Q})$-generic if $G \cap D \neq \emptyset$ for every dense open subset D of $\mathbb{Q}$ with $D \in M$. A condition $q \in \mathbb{Q}$ is *totally* $(M, \mathbb{Q})$*-generic* if the set $\{p \in M \cap \mathbb{Q} \mid q \leq p\}$ is an $(M, \mathbb{Q})$-generic filter.

Definition 6.4. Suppose that $\mathbb{Q}$ is a forcing notion and $\alpha < \omega_1$. A *suitable α-tower for* $\mathbb{Q}$ is a continuous, $\subseteq$-increasing sequence $\langle M_\eta \mid \eta < \alpha \rangle$ of countable elementary submodels of $H(\theta)$ for some sufficiently large regular cardinal θ such that

- M_0 is suitable for $\mathbb{Q}$;
- for all ξ with $\xi + 1 < \alpha$, we have $\langle M_\eta \mid \eta \leq \xi \rangle \in M_{\xi+1}$.

We say that $\mathbb{Q}$ is *α-proper* if, for every suitable α-tower $\langle M_\eta \mid \eta < \alpha \rangle$ for $\mathbb{Q}$ and every $q_0 \in M_0 \cap \mathbb{Q}$, there is $q \leq_{\mathbb{Q}} q_0$ such that q is $(M_\eta, \mathbb{Q})$-generic for all $\eta < \alpha$. We say that $\mathbb{Q}$ is *totally α-proper* if we can additionally require that q is totally $(M_\eta, \mathbb{Q})$-generic for all $\eta < \alpha$.

We now turn to Moore's notion of *complete properness*.

Definition 6.5. If M and N are sets, then the notation $M \to N$ denotes the existence of an elementary embedding $\varepsilon : (M, \in) \to (N, \in)$ such that $\varepsilon \in N$ and $N \models$ "M is countable", i.e., N contains an injection of M into ω. If $X \in M$ and $M \to N$, as witnessed by ε, then we will let X^N denote $\varepsilon(X)$. If $X \subseteq M$ is not an element of M, then X^N denotes the pointwise image $\varepsilon[X]$.

Remark 6.6. The above definition, as well as those below, involves a slight abuse of notation, as the role of the elementary embedding ε is notationally suppressed. In practice, though, this will not cause any confusion, as we will always specify the elementary embeddings under consideration. In what follows, we will refer to instances of $M \to N$ as "arrows". So, for instance, if we have a fixed set M, then the quantification "for every arrow $M \to N\ldots$" should be interpreted as "for every set N and every elementary embedding $\varepsilon : (M, \in) \to (N, \in)$ witnessing $M \to N\ldots$".

Definition 6.7. Suppose that $\mathbb{Q}$ is a forcing notion, M is suitable for $\mathbb{Q}$, and $M \to N$. Then a filter $G \subseteq \mathbb{Q} \cap M$ is $\overrightarrow{MN}$*-prebounded* if, for every arrow $N \to P$ such that $G \in P$, we have $P \models$ "G^P has a lower bound in $\mathbb{Q}^P$", where G^P and $\mathbb{Q}^P$ are defined as in Definition 6.5 via the composition $M \to N \to P$.

Definition 6.8. Suppose that $\mathbb{Q}$ is a forcing notion. We say that $\mathbb{Q}$ is *completely proper* if whenever M is suitable for $\mathbb{Q}$, $q \in \mathbb{Q} \cap M$, and $M \to N_i$ for $i < 2$, there is an $(M, \mathbb{Q})$-generic filter $G \subseteq \mathbb{Q} \cap M$ that is $\overrightarrow{MN_i}$-prebounded for all $i < 2$ with $q \in G$.

Moore proved in [11, Lemma 4.11] that completely proper forcings are 2-complete with respect to some completeness system $\mathbb{D}$ (cf. [14, §V.5]). Combined with Shelah's iteration theorem concerning such forcings ([14, Theorem VIII.4.5]), this yields the following theorem (cf. [2, Main Theorem]).

Theorem 6.9. *Suppose that $\langle \mathbb{P}_\eta, \dot{\mathbb{Q}}_\xi \mid \eta \leq \delta,\ \xi < \delta \rangle$ is a countable support iteration of totally proper forcing notions such that, for all $\eta < \delta$, we have*

$$\Vdash_{\mathbb{P}_\eta} \text{“}\dot{\mathbb{Q}}_\eta \textit{ is completely proper and totally } \alpha\textit{-proper for all } \alpha < \omega_1\text{”}.$$

Then $\mathbb{P}_\delta$ is totally proper.

We are now ready to prove that the assumption of $\diamondsuit$ cannot be weakened to CH in Theorem 6.2. In fact, we will prove that it is consistent with CH that every full, splitting tree of height ω_1 contains a copy of ${}^{<\omega_1}2$, thus establishing Theorem C(2).

Theorem 6.10. *It is consistent that CH holds and every full, splitting tree of height ω_1 contains a copy of ${}^{<\omega_1}2$.*

Proof. We first note that it is enough to consider trees of height and *size* ω_1. To see this, suppose that CH holds and T is a tree of height ω_1. Let θ be a sufficiently large regular cardinal and let $M \prec H(\theta)$ be such that

- $|M| = \omega_1$;
- ${}^{\omega}M \subseteq M$;
- $T \in M$.

Then $T \cap M$ is a full, splitting subtree of T of height and size ω_1; if $T \cap M$ contains a copy of ${}^{<\omega_1}2$ then, *a fortiori*, so does T.

Suppose that GCH holds. Fix for now a full, splitting tree T of height and size ω_1. Without loss of generality, we can assume that T is a subtree of ${}^{<\omega_1}\omega_1$. We will describe a totally proper forcing $\mathbb{P}(T)$ of size ω_1 that adds a copy of ${}^{<\omega_1}2$ to T and then show that this forcing can be iterated without adding reals.

Given a subtree $S \subseteq {}^{<\omega_1}\omega_1$, let ∂S be the set of $\sigma \in {}^{<\omega_1}\omega_1$ such that

- $\mathrm{dom}(\sigma)$ is a limit ordinal;
- $\sigma \notin S$;
- for all $\eta < \mathrm{dom}(\sigma)$, $\sigma \restriction \eta \in S$.

Conditions of $\mathbb{P}(T)$ are all pairs of the form $p = (S_p, f_p)$ such that

(1) S_p is a countable subtree of ${}^{<\omega_1}2$;
(2) $f_p : S_p \to T$ is an isomorphic embedding;
(3) for all $\sigma \in \partial S_p$, we have $\bigcup\{f_p(\sigma \restriction \eta) \mid \eta < \mathrm{dom}(\sigma)\} \in T$.

If $p, q \in \mathbb{P}(T)$, then $q \leq p$ if and only if $S_q \supseteq S_p$ and $f_q \restriction S_p = f_p$.

Since CH holds, $\mathbb{P}(T)$ is of size ω_1. Given $p = (S_p, f_p) \in \mathbb{P}(T)$, let T_p denote the $\leq_T$-downward closure of f_p“S_p. Condition (3) above can then be expressed as the assertion that $\partial T_p \subseteq T$.

Let $\mathcal{L}$ be the set of $\beta \in \lim(\omega_1)$ such that $[T_{<\beta}] \setminus T_\beta \neq \emptyset$ and, for each $\beta \in \mathcal{L}$, let b_β be the unique element of $[T_{<\beta}] \setminus T_\beta$. When constructing elements of $\mathbb{P}(T)$, we need to be careful to avoid the branches b_β for $\beta \in \mathcal{L}$. The following lemmas show that this can be done.

Lemma 6.11. *Suppose that $\beta \in \lim(\omega_1)$ and $b \in {}^\beta\omega_1$. Suppose moreover that*

- $p = (S_p, f_p) \in \mathbb{P}(T)$;
- $\sigma \in {}^{<\omega_1}2$ *is such that* $\sigma, \sigma^\frown 0, \sigma^\frown 1 \in S_p$;
- $f_p(\sigma) \sqsubseteq b$ *but, for all* $i < 2$, *we have* $f_p(\sigma^\frown i) \perp b$.

Let $\xi < \beta$ be such that $f_p(\sigma^\frown i) \perp (b \restriction \xi)$ for all $i < 2$. Then, for every $q \leq p$ and every $\tau \in S_q$, it is not the case that $f_q(\tau) \sqsupseteq b \restriction \xi$.

Proof. Note that $f_p(\sigma) = b \restriction \eta$ for some $\eta < \xi$. Fix $q \leq p$ and $\tau \in S_q$. There are three cases to consider.

First, if $\tau \sqsubseteq \sigma$, then $f_q(\tau) = f_p(\tau) \sqsubseteq b \restriction \eta$, so $f_q(\tau) \not\sqsupseteq b \restriction \xi$.

Second, if $\sigma \sqsubseteq \tau$, then there is $i < 2$ such that $\sigma^\frown \langle i \rangle \sqsubseteq \tau$, and hence $f_p(\sigma^\frown \langle i \rangle) \sqsubseteq f_q(\tau)$. Since $f_p(\sigma^\frown \langle i \rangle) \perp b \restriction \xi$, it follows that $f_q(\tau) \perp b \restriction \xi$.

The remaining case is that in which $\tau \perp \sigma$. In this case, $f_q(\tau) \perp f_p(\sigma) = b \restriction \eta$. □

Lemma 6.12. *Suppose that M is suitable for $\mathbb{P}(T)$, $p \in M \cap \mathbb{P}(T)$, $\beta = M \cap \omega_1$, and $\beta \in \mathcal{L}$. Let*

$$\xi := \sup\{\eta < \beta \mid \exists \sigma \in S_p\ [f_p(\sigma) = b_\beta \restriction \eta]\},$$

and assume that $b_\beta \restriction \xi \in M$. Then there is $q \leq p$ with $q \in M$ and $\sigma \in S_q$ such that

- $f_q(\sigma) \sqsubseteq b_\beta$;
- *for all* $i < 2$, $\sigma^\frown i \in S_q$ *and* $f_q(\sigma^\frown i) \perp b_\beta$.

Proof. We may assume that p itself does not satisfy the conclusion of the theorem. Since $p \in \mathbb{P}(T)$ and $b_\beta \notin T$, we must have $\xi < \beta$. Assume first that there is $\sigma \in S_p$ such that $f_p(\sigma) = b_\beta \restriction \xi$. By the definition of ξ, if $i < 2$ and $\sigma^\frown i \in S_p$, then $f_p(\sigma^\frown i) \perp b_\beta$. We will define a condition $q \leq p$ in M such that $S_q = S_p \cup \{\sigma^\frown 0, \sigma^\frown 1\}$. It suffices to define $f_q(\sigma^\frown i)$ for $i < 2$. Using the fact that T is splitting, we know that there are two distinct nodes $t_0, t_1 \in T_{\xi+2} \cap M$ such that, for all $i < 2$, we have $b_\beta \restriction \xi \sqsubseteq t_i$ but $b_\beta \restriction (\xi + 2) \perp t_i$. We can also choose t_0 and t_1 so that, if $i < 2$ and $\sigma^\frown i \in S_p$, then $f_p(\sigma^\frown i) \perp t_{1-i}$. For $i < 2$, if $\sigma^\frown i \notin S_p$, set $f_q(\sigma^\frown i) = t_i$. It is readily verified that $q \leq p$ is as desired.

Now suppose that there is no $\sigma \in S_p$ such that $f_p(\sigma) = b_\beta \restriction \xi$. By our assumptions about p, there must be an increasing sequence of ordinals $\langle \eta_n \mid n < \omega \rangle$ and a $\sqsubseteq$-increasing sequence $\langle \sigma_n \mid n < \omega \rangle$ from S_p such that

- $\sup\{\eta_n \mid n < \omega\} = \xi$;
- $\bigcup\{\sigma_n \mid n < \omega\} \notin S_p$; and
- for all $n < \omega$, $f_p(\sigma_n) = b_\beta \restriction \eta_n$.

Let $\sigma := \bigcup\{\sigma_n \mid n < \omega\}$. Then $\sigma \in M$, since it is definable in M as the unique element τ of ∂S_p such that $\bigcup\{f_p(\tau \restriction \eta) \mid \eta < \mathrm{dom}(\tau)\} = b_\beta \restriction \xi$. We will define a condition $q \leq p$ in M such that $S_q = S_p \cup \{\sigma, \sigma^\frown 0, \sigma^\frown 1\}$. Work entirely in M. First, set $f_q(\sigma) = b_\beta \restriction \xi$. Then define $f_q(\sigma^\frown i)$ for $i < 2$ exactly as in the previous case. It is again readily verified that $q \leq p$ is as desired. □

Lemma 6.13. *$\mathbb{P}(T)$ is totally α-proper for all $\alpha < \omega_1$.*

Proof. The proof is by induction on α. We will in fact prove the following stronger statement, which will be used in the inductive step:

> For every suitable $(\alpha + 1)$-tower $\langle M_\eta \mid \eta \leq \alpha \rangle$ and every $p_0 \in M_0 \cap \mathbb{P}(T)$, there is $p \leq p_0$ such that

- p is totally $(M_\eta, \mathbb{P}(T))$-generic for all $\eta \leq \alpha$;
- $S_p, f_p \subseteq M_\alpha$.

Fix $\beta < \omega_1$ and suppose that we have established the induction hypothesis for all $\alpha < \beta$. Let $\langle M_\eta \mid \eta \leq \beta \rangle$ be a suitable $(\beta+1)$-tower for $\mathbb{P}(T)$, and fix $p_0 \in M_0 \cap \mathbb{P}(T)$. We will assume that β is a limit ordinal, and hence $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$; the case in which β is a successor is easier and proven similarly. Note also that if a condition p is totally $(M_\alpha, \mathbb{P}(T))$-generic for all $\alpha < \beta$, then it is also totally $(M_\beta, \mathbb{P}(T))$-generic.

Let $\delta := M_\beta \cap \omega_1$. Assume that $\delta \in \mathcal{L}$; the case in which $\delta \notin \mathcal{L}$ is similar and easier. There are now two cases to consider; we will deal with them in parallel:

- **Case 1:** There is $\xi^* < \delta$ such that $b_\delta \restriction \xi^* \notin M_\beta$;
- **Case 2:** For every $\xi < \delta$, $b_\delta \restriction \xi \in M_\beta$.

If we are in Case 2, let

$$\xi := \sup\{\eta < \beta \mid \exists \sigma \in S_{p_0} \ [f_{p_0}(\sigma) = b_\beta \restriction \eta]\}.$$

There are now two subcases to consider:

- **Case 2a:** $b_\delta \restriction \xi \in M_0$;
- **Case 2b:** $b_\delta \restriction \xi \notin M_0$.

If we are in Case 2a, begin by applying Lemma 6.12 to find a $p_0' \leq p_0$ and a $\sigma_0 \in S_{p_0'}$ such that

- $p_0' \in M_0$;
- $f_{p_0'}(\sigma_0) \sqsubseteq b_\delta$;
- for all $i < 2$, $\sigma_0{}^\frown i \in S_{p_0'}$ and $f_{p_0'}(\sigma_0{}^\frown i) \perp b_\delta$.

If we are in Case 1 or 2b, let $p_0' = p_0$ and leave σ_0 undefined for now.

Let $\langle \alpha_n \mid n < \omega \rangle$ be a strictly increasing sequence of ordinals that is cofinal in β. If we are in Case 2b, additionally choose α_0 so that $\alpha_0 + 1$ is the least ordinal $\varepsilon < \beta$ such that $b_\delta \restriction \xi \in M_\varepsilon$ (note that this ε must be a successor ordinal by the continuity of $\langle M_\eta \mid \eta < \beta \rangle$).

We will build a decreasing sequence $\langle p_n \mid n < \omega \rangle$ of conditions in $\mathbb{P}(T)$ such that, for all $n < \omega$, we have

- $p_{n+1} \in M_{\alpha_n+1}$;
- p_{n+1} is totally $(M_\eta, \mathbb{P}(T))$-generic for all $\eta \leq \alpha_n$.

We will also arrange so that $\langle p_n \mid n < \omega \rangle$ will have a lower bound

$$p_\infty = \left(\bigcup_{n<\omega} S_{p_n}, \bigcup_{n<\omega} f_{p_n} \right)$$

in $\mathbb{P}(T)$. This lower bound will then be the desired condition that is totally $(M_\eta, \mathbb{P}(T))$-generic for all $\eta \leq \beta$. Note that, by construction, we will have $S_{p_\infty} \subseteq M_\beta$.

Let $\langle \gamma_n \mid 0 < n < \omega \rangle$ enumerate $\mathcal{L} \cap \delta$ (with repetitions, if necessary) in such a way that $\gamma_n \in M_{\alpha_n+1}$ for all $0 < n < \omega$. We will arrange so that, for all $0 < n < \omega$, there is $\sigma_n \in S_{p_n}$ such that

- $f_{p_n}(\sigma_n) \sqsubseteq b_{\gamma_n}$;
- for all $i < 2$, $\sigma_n{}^\frown i \in S_{p_n}$ and $f_{p_n}(\sigma_n{}^\frown i) \perp b_{\gamma_n}$.

We now describe the rest of the construction. The first step will be different from the others due to the need to take care of Case 2b. We begin by applying the inductive hypothesis in M_{α_0+1} to find $p_1' \leq p_0'$ such that

- $p_1' \in M_{\alpha_0+1}$ and $S_{p_1'}, f_{p_1'} \subseteq M_{\alpha_0}$;
- p_1' is totally $(M_\eta, \mathbb{P}(T))$-generic for all $\eta \leq \alpha_0$.

Suppose first that we are in Case 2b. By our choice of α_0 and the fact that $S_{p_1'} \subseteq M_{\alpha_0}$, we know that $b_\delta \restriction \xi \notin T_{p_1'}$. Thus, we still have

$$\xi = \sup\{\eta < \beta \mid \exists \sigma \in S_{p_1'} \ [f_{p_1'}(\sigma) = b_\beta \restriction \eta]\}.$$

We can therefore apply Lemma 6.12 to find $p_1^* \leq p_1'$ and $\sigma_0 \in S_{p_1^*}$ such that

- $p_1^* \in M_{\alpha_0+1}$;
- $f_{p_1^*}(\sigma_0) \sqsubseteq b_\delta$;
- for all $i < 2$, $\sigma_0{}^\frown\langle i\rangle \in S_{p_1^*}$ and $f_{p_1^*}(\sigma_0{}^\frown\langle i\rangle) \perp b_\delta$.

If we are in Case 1 or 2a, let $p_1^* = p_1'$; in Case 1, leave σ_0 permanently undefined. Now, regardless of the case we are in, apply Lemma 6.12 once again to find $p_1 \leq p_1^*$ and $\sigma_1 \in S_{p_1}$ such that

- $p_1 \in M_{\alpha_0+1}$;
- $f_{p_1}(\sigma_1) \sqsubseteq b_{\gamma_1}$;
- for all $i < 2$, $\sigma_1{}^\frown i \in S_{p_1}$ and $f_{p_1}(\sigma_1{}^\frown i) \perp b_{\gamma_1}$.

The construction is now uniform across all cases. Suppose that $0 < n < \omega$ and we have constructed p_n. Apply the inductive hypothesis to the tower $\langle M_\eta \mid \alpha_{n-1} < \eta \leq \alpha_n\rangle$ inside M_{α_n+1} to find $p_{n+1}' \leq p_n$ that is totally $(M_\eta, \mathbb{P}(T))$-generic for all $\eta \in (\alpha_{n-1}, \alpha_n]$. Since $p_{n+1}' \leq p_n$ and p_n is totally $(M_\eta, \mathbb{P}(T))$-generic for all $\eta \leq \alpha_{n-1}$, it follows that p_{n+1}' is in fact totally $(M_\eta, \mathbb{P}(T))$-generic for all $\eta \leq \alpha_n$. Now apply Lemma 6.12 to find $p_{n+1} \leq p_{n+1}'$ and $\sigma_{n+1} \in S_{p_{n+1}}$ such that

- $p_{n+1} \in M_{\alpha_n+1}$;
- $f_{p_{n+1}}(\sigma_{n+1}) \sqsubseteq b_{\gamma_{n+1}}$;
- for all $i < 2$, $\sigma_{n+1}{}^\frown i \in S_{p_{n+1}}$ and $f_{p_{n+1}}(\sigma_{n+1}{}^\frown i) \perp b_{\gamma_{n+1}}$.

This completes the construction. It remains to verify that $p_\infty \in \mathbb{P}(T)$. Since $S_{p_\infty}, f_{p_\infty} \subseteq M_\beta$, the only way that p_∞ could fail to be in $\mathbb{P}(T)$ is if there is $\gamma \in \mathcal{L} \cap (\delta + 1)$ and $\sigma \in \partial S_\infty$ such that $\bigcup\{f_{p_\infty}(\sigma \restriction \eta) \mid \eta < \mathrm{dom}(\sigma)\} = b_\gamma$. Suppose there are such a γ and σ.

Assume first that $\gamma = \delta$. If we are in Case 1, then there is $\xi^* < \delta$ such that $b_\delta \restriction \xi^* \notin M_\beta$. Since $S_{p_\infty}, f_{p_\infty} \subseteq M_\beta$, there can be no $\eta < \mathrm{dom}(\sigma)$ such that $f_{p_\infty}(\sigma \restriction \eta) \sqsupseteq b_\delta \restriction \xi^*$, which is a contradiction. If we are in Case 2, then recall our choice of σ_0, and let $\xi' < \delta$ be such that $f_{p_1}(\sigma_0{}^\frown i) \perp b_\delta \restriction \xi'$ for all $i < 2$. Then, by Lemma 6.11, there is no $\eta < \mathrm{dom}(\sigma)$ such that $f_{p_\infty}(\sigma \restriction \eta) \sqsupseteq b_\delta \restriction \xi'$, which is again a contradiction.

Finally, assume that $\gamma \in \mathcal{L} \cap \delta$. Then there is $0 < n < \omega$ such that $\gamma = \gamma_n$. Let $\xi' < \gamma$ be such that $f_{p_n}(\sigma_n{}^\frown i) \perp b_\gamma \restriction \xi'$ for all $i < 2$. Then, again by Lemma 6.11, there is no $\eta < \mathrm{dom}(\sigma)$ such that $f_{p_\infty}(\sigma \restriction \eta) \sqsupseteq b_\gamma \restriction \xi'$, yielding the final contradiction and completing the proof. □

Lemma 6.14. *$\mathbb{P}(T)$ is completely proper.*

Proof. Suppose that M is suitable for $\mathbb{Q}$ and we are given $M \to N_i$ for $i < 2$. Let $\varepsilon_i : (M, \in) \to (N_i, \in)$ witness this. Let $\delta := M \cap \omega_1$. Note that $\delta \in N_0 \cap N_1$. By elementarity, for $i < 2$, the restriction $\varepsilon_i \restriction \delta$ must be the identity map on δ, and hence $\pi \restriction (T \cap M)$ is the identity map on $T \cap M$. Without loss of generality, assume that $\delta \in \mathcal{L}^{N_0} \cap \mathcal{L}^{N_1}$; the other cases are similar but easier. For each $i < 2$, let b_δ^i be such that

$$N_i \models \text{“}b^i_\delta \text{ is the unique element of } [T^{N_i}_{<\delta}] \setminus T^{N_i}_\delta\text{”}.$$

Let $p \in M \cap \mathbb{P}(T)$ be arbitrary. We will now build a decreasing sequence $\langle p_n \mid n < \omega\rangle$ from $M \cap \mathbb{P}(T)$ below p such that the upward closure of $\{p_n \mid n < \omega\}$ in $M \cap \mathbb{P}(T)$ is an $(M, \mathbb{P}(T))$-generic filter. Begin by letting $p_0 = p$. Now define p_1 as follows. If there is $\xi < \delta$ such that $b^0_\delta \restriction \xi \notin M$, then simply let $p_1 = p_0$. Otherwise, apply Lemma 6.12 to find $p_1 \leq p_0$ and $\sigma_1 \in S_{p_1}$ such that

- $p_1 \in M$;
- $f_{p_1}(\sigma_1) \sqsubseteq b^0_\delta$;
- for all $j < 2$, $\sigma_1{}^\frown j \in S_{p_1}$ and $f_{p_1}(\sigma_1{}^\frown j) \perp b^0_\delta$.

Next, find $p_2 \leq p_1$ in the same way, but replacing b^0_δ by b^1_δ. Finally, let $\langle D_n \mid n < \omega\rangle$ enumerate all dense open subsets of $\mathbb{P}(T)$ that are elements of M, and recursively define a decreasing sequence $\langle p_n \mid 2 < n < \omega\rangle$ from $M \cap \mathbb{P}(T)$ such that $p_n \in D_{n-3}$ for all $n < \omega$. Notice that, for all $\gamma \in \mathcal{L} \cap \delta$, the set D^*_γ of all $q \in \mathbb{P}(T)$ for which there is $\sigma \in S_q$ such that

- $f_q(\sigma) \sqsubseteq b_\gamma$; and
- for all $j < 2$ $\sigma{}^\frown j \in S_q$ and $f_q(\sigma{}^\frown j) \perp b_\gamma$

is a dense open subset of $\mathbb{P}(T)$ that is in M; there is therefore $n < \omega$ such that $p_n \in D^*_\gamma$.

Let G be the upward closure of $\{p_n \mid n < \omega\}$ in $M \cap \mathbb{P}(T)$. By construction, it is clear that G is an $(M, \mathbb{P}(T))$-generic filter. It remains to show that it is $\overrightarrow{MN_i}$-prebounded for all $i < 2$. To this end, fix $i < 2$ and an arrow $N_i \to P$, witnessed by a map $\epsilon : (N_i, \in) \to (P, \in)$, such that $G \in P$. Note that, for each $n < \omega$, we have $\epsilon \circ \varepsilon_i(p_n) = p_n$, and hence $G^P = G$. We will therefore be done if we show that

$$p_\infty = \left(\bigcup_{n<\omega} S_{p_n}, \bigcup_{n<\omega} f_{p_n}\right) = \left(\bigcup_{q\in G} S_q, \bigcup_{q\in G} f_q\right)$$

is in $\mathbb{P}(T)^P$.

First note that $p_\infty \in P$, since $G \in P$. Also, the range of f_{p_∞} is contained in $T^P_{<\delta} = T_{<\delta}$. Therefore, the only way that p_∞ could fail to be in $\mathbb{P}(T)^P$ is if there is $\gamma \in \mathcal{L}^P \cap (\delta + 1)$ and $\sigma \in \partial S_\infty$ such that $\bigcup\{f_{p_\infty}(\sigma \restriction \eta) \mid \eta < \mathrm{dom}(\sigma)\} = b^P_\gamma$, where b^P_γ is such that

$$P \models \text{“}b^P_\gamma \text{ is the unique element of } [T^P_{<\gamma}] \setminus T^P_\gamma\text{”}.$$

Suppose that there are such a γ and σ.

Note that $\delta < N_i \cap \omega_1$, so $\epsilon \restriction (\delta + 1)$ is the identity map. It follows that $\mathcal{L}^P \cap (\delta + 1) = \mathcal{L}^{N_i} \cap (\delta + 1)$, and, moreover, $\mathcal{L}^P \cap \delta = \mathcal{L} \cap \delta$. Suppose first that $\gamma = \delta$, in which case we have $b^P_\delta = b^i_\delta$. If there is $\xi < \delta$ such that $b^i_\delta \restriction \xi \notin M$, then, since $S_{p_\infty}, f_{p_\infty} \subseteq M$, there can be no $\eta < \mathrm{dom}(\sigma)$ such that $f_{p_\infty}(\sigma \restriction \eta) \sqsupseteq b^i_\delta \restriction \xi$, which is a contradiction. Otherwise, when constructing p_{i+1}, we fixed a $\sigma_{i+1} \in S_{p_{i+1}}$ such that $f_{p_{i+1}}(\sigma_{i+1}) \sqsubseteq b^i_\delta$ and for which there exists some $\xi' < \delta$ such that $f_{p_{i+1}}(\sigma_{i+1}{}^\frown j) \perp b^i_\delta \restriction \xi'$ for all $j < 2$. Then, by Lemma 6.11, there is no $\eta < \mathrm{dom}(\sigma)$ such that $f_{p_\infty}(\sigma \restriction \eta) \sqsupseteq b^i_\delta \restriction \xi'$, which is again a contradiction.

Suppose finally that $\gamma < \delta$, in which case $b^P_\gamma = b_\gamma$. Then there is $n < \omega$ such that $p_n \in D^*_\gamma$, and we reach a contradiction exactly as in the second case in the previous paragraph. Thus, G is in fact $\overrightarrow{MN}_i$-prebounded, and hence $\mathbb{P}(T)$ is completely proper. $\square$

By a standard genericity argument, if G is $\mathbb{P}(T)$-generic over V, then $\bigcup\{f_p \mid p \in G\}$ witnesses that, in $V[G]$, T contains a copy of ${}^{<\omega_1}2$ (which, since $\mathbb{P}(T)$ is totally proper, is the same when calculated in V or in $V[G]$).

We now define a countable support iteration $\langle \mathbb{P}_\eta, \dot{\mathbb{Q}}_\xi \mid \eta \leq \omega_2,\ \xi < \omega_2 \rangle$ such that, for each $\eta < \omega_2$, there is a $\mathbb{P}_\eta$-name $\dot{T}_\eta$ for a full, splitting subtree of ${}^{<\omega_1}\omega_1$ of height and size ω_1 such that

$$\Vdash_{\mathbb{P}_\eta} \dot{\mathbb{Q}}_\eta = \mathbb{P}(\dot{T}_\eta).$$

By Theorem 6.9, $\mathbb{P}_\eta$ will be totally proper for each $\eta \leq \omega_2$, and hence

- CH will hold in $V^{\mathbb{P}_\eta}$; and
- $({}^{<\omega_1}\omega_1)^{V^{\mathbb{P}_\eta}} = ({}^{<\omega_1}\omega_1)^V$.

It follows that, for each $\eta < \omega_2$, we have

$$\Vdash_{\mathbb{P}_\eta} \text{“}\dot{\mathbb{Q}}_\eta \text{ is a proper forcing of size } \omega_1\text{”}.$$

Therefore, by [1, Theorem 2.10], $\mathbb{P}_\eta$ has the ω_2-cc and is of size $\leq 2^{\omega_1} = \omega_2$ for each $\eta \leq \omega_2$. Thus, by a standard bookkeeping argument, we can arrange so that, for every $\mathbb{P}_{\omega_2}$-name $\dot{T}$ for a full, splitting subtree of ${}^{<\omega_1}\omega_1$ of height and size ω_1, there is $\eta < \omega_2$ such that $\Vdash_{\mathbb{P}_{\omega_2}} \dot{T} = \dot{T}_\eta$. By genericity,

$$\Vdash_{\mathbb{P}_{\eta+1}} \text{“}\dot{T}_\eta \text{ contains a copy of } {}^{<\omega_1}2\text{”}.$$

Since ${}^{<\omega_1}2$ is the same when calculated in V or in $V^{\mathbb{P}_\eta}$ for any $\eta \leq \omega_2$, it follows that

$$\Vdash_{\mathbb{P}_{\omega_2}} \text{“}\dot{T} \text{ contains a copy of } {}^{<\omega_1}2\text{”}.$$

Hence, $V^{\mathbb{P}_{\omega_2}}$ is our desired model. $\square$

If κ is a regular uncountable cardinal, then we follow [8] and say that a κ-tree T is *superthin* if $|[T_{<\delta}]| < \kappa$ for all limit ordinals $\delta < \kappa$. Note that cofinally splitting ω_1-trees cannot be superthin, so the notion is primarily of interest for $\kappa > \omega_1$. In [8], Lücke and Schlicht prove that if κ is a regular uncountable cardinal and there exists a superthin κ-Kurepa tree, then there exists a superthin κ-Kurepa tree $T \subseteq {}^{<\kappa}\kappa$ such that $[T]$ is a retract of ${}^{\kappa}\kappa$. They moreover show that such trees consistently exist; for example, if $V = L$, then superthin κ-Kurepa trees exist whenever κ is the successor of a cardinal of uncountable cofinality. Superthinness seems to be in tension with fullness, but we show now that, for regular $\kappa > \omega_1$, we can consistently have full, superthin, κ-Kurepa trees. For concreteness, we focus on the case $\kappa = \omega_2$.

Theorem 6.15. *Suppose that $\diamondsuit + (2^{\omega_1} = \omega_2)$ holds. Then there is a cardinality- and cofinality-preserving poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there is a normal, splitting, full, superthin ω_2-Kurepa tree.*

Proof. The poset $\mathbb{P}$ consists of all pairs $p = (T_p, f_p)$ such that

- there is an $\eta_p < \omega_2$ such that T_p is a normal, splitting, full subtree of ${}^{<\eta_p+1}\omega_2$;
- for all $\xi \leq \eta_p$, we have $|T_p \cap {}^{\xi}\omega_2| \leq \omega_1$;
- for every limit ordinal $\xi \leq \eta_p$, we have $|[(T_p)_{<\xi}]| \leq \omega_1$;
- f_p is a partial function of size ω_1 from ω_3 to $T_p \cap {}^{\eta_q}\omega_2$.

If $p_0, p_1 \in \mathbb{P}$, then $p_1 \leq p_0$ if

- $\eta_{p_1} \geq \eta_{p_1}$;
- $T_{p_1} \cap {}^{<\eta_{p_0}+1}\omega_2 = T_{p_0}$;
- $\mathrm{dom}(f_{p_1}) \supseteq \mathrm{dom}(f_{p_0})$;
- for all $\alpha \in \mathrm{dom}(f_{p_0})$, we have $f_{p_1}(\alpha) \sqsupseteq f_{p_0}(\alpha)$.

By a standard Δ-system argument, and using the assumption that $2^{\omega_1} = \omega_2$, it follows that $\mathbb{P}$ has the ω_3-cc.

Claim 6.16. *$\mathbb{P}$ is (ω_1+1)-strategically closed.*

Proof. We describe a winning strategy for Player II in $\Game_{\omega_1+1}(\mathbb{P})$. Given a (partial) play $\langle p_\alpha \mid \alpha < \gamma\rangle$ of $\Game_{\omega_1+1}(\mathbb{P})$, for all $\alpha < \gamma$, we let T^α, f^α, and η_α denote T_{p_α}, f_{p_α}, and η_{p_α}, respectively. As part of Player II's winning strategy, they also fix, as they play a round $\langle p_\alpha \mid \alpha \leq \omega_1\rangle$, arbitrary surjections $\pi_\alpha : \omega_1 \to T^\alpha_{\eta_\alpha}$ for each ordinal $\alpha < \omega_1$. We will also ensure that, for every even ordinal $\alpha < \omega_1$, the map f^α is a bijection between its domain and $T^\alpha_{\eta_\alpha}$.

Using $\diamondsuit$, fix a sequence $\langle a_\alpha : \alpha \to \alpha \mid \alpha < \omega_1\rangle$ such that, for all $f : \omega_1 \to \omega_1$, the set

$$\{\alpha < \omega_1 \mid f \restriction \alpha = a_\alpha\}$$

is stationary in ω_1.

Suppose now that $\beta \leq \omega_1$ is a nonzero even ordinal and $\langle p_\alpha \mid \alpha < \beta\rangle$ is a partial run of $\Game_{\omega_1+1}(\mathbb{P})$, with Player II playing thus far according to their winning strategy. Suppose also that Player II has fixed surjections $\pi_\alpha : \omega_1 \to T^\alpha_{\eta_\alpha}$ for each α such that there exists an even ordinal β' with $\alpha \leq \beta' < \beta$. We now describe how Player II should select p_β.

Suppose first that $\beta = \alpha + 1$ is a successor ordinal. In this case, simply let $\eta_\beta = \eta_\alpha + 1$ and

$$T^\beta_{\eta_\beta} = \{\sigma^\frown i \mid \sigma \in T^\alpha_{\eta_\alpha} \text{ and } i < \omega_1\}.$$

We now describe how to choose f^β. We are going to let $\mathrm{dom}(f^\beta) = \mathrm{dom}(f^\alpha) \cup e_\beta$ for some (possibly empty) $e_\beta \in [\omega_3]^{\leq\omega_1}$ disjoint from $\mathrm{dom}(f^\alpha)$. First, for all $\gamma \in \mathrm{dom}(f^\alpha)$, let $f^\beta(\gamma) = f^\alpha(\gamma)^\frown i$ for some $i < \omega_1$ in such a way that $f^\beta \restriction \mathrm{dom}(f^\alpha)$ is injective (this is possible, since each element of $T^\alpha_{\eta_\alpha}$ has ω_1-many successors in $T^\beta_{\eta_\beta}$). The resulting function may not be surjective, though, so choose some e_β as above of the appropriate cardinality and define f^β on e_β in such a way that the resulting function is a bijection. Finally, let π_α and π_β be arbitrary surjections from ω_1 onto $T^\alpha_{\eta_\alpha}$ and $T^\beta_{\eta_\beta}$, respectively.

Suppose next that β is a limit ordinal of countable cofinality. Let $\eta_\beta = \sup\{\eta_\alpha \mid \alpha < \beta\}$. Note that, by the specification of Player II's strategy at successor stages, we know that $\eta_\beta > \eta_\alpha$ for all $\alpha < \beta$; in particular, η_β is a limit ordinal of countable cofinality. Let $T^{<\beta} = \bigcup\{T^\alpha \mid \alpha < \beta\}$, so $T^{<\beta}$ is a normal, splitting, full subtree of ${}^{<\eta_\beta}\omega_2$. Let $d_\beta = \bigcup\{\mathrm{dom}(f^\alpha) \mid \alpha < \beta\}$ and, for each $\gamma \in d_\beta$, let

$$b^\beta_\gamma = \bigcup\{f^\alpha(\gamma) \mid \alpha < \beta \text{ and } \gamma \in \mathrm{dom}(f^\alpha)\}.$$

Note that b^β_γ is in $[T^{<\beta}]$. Our $\diamondsuit$ sequence gives us a function $a_\beta : \beta \to \beta$. Consider the subset $A^\beta = \{\pi_\alpha(a_\beta(\alpha)) \mid \alpha < \beta\}$ of $T^{<\beta}$. If A^β is linearly ordered by $\subseteq$, then let $b^\beta_* = \bigcup A^\beta$, and note that $b^\beta_* \in [T^{<\beta}]$. There are now two cases to consider.

Case 1. Suppose first that A^β is linearly ordered and, for every $\gamma \in d_\beta$, we have $b^\beta_* \neq b^\beta_\gamma$. In this case, let $T^\beta_{\eta_\beta} = [T^{<\beta}] \setminus \{b^\beta_*\}$. Note that, since $\mathrm{cf}(\beta) = \omega$, CH holds, and every level of $T^{<\beta}$ has size at most ω_1, we have $|T^\beta_{\eta_\beta}| = \omega_1$. To define f^β, we will let $\mathrm{dom}(f^\beta) = d_\beta \cup e_\beta$ for some $e_\beta \in [\omega_3]^{\leq \omega_1}$ disjoint from d_β. For $\gamma \in d_\beta$, let $f^\beta(\gamma) = b^\beta_\gamma$. Note that $f^\beta \restriction d_\beta$ is injective since, for all even $\alpha < \beta$, f^α is injective. As in the successor case, now choose a set e_β of the appropriate cardinality and define f^β on e_β so that the resulting function is a bijection. Finally, let π_β be an arbitrary surjection from ω_1 onto $T^\beta_{\eta_\beta}$.

Case 2. If we were not in Case 1, i.e., if either A^β is not linearly ordered or $b^\beta_* = b^\beta_\gamma$ for some $\gamma \in d^*$, then proceed exactly as in Case 1, except let $T^\beta_{\eta_\beta} = [T^{<\beta}]$.

Finally, suppose that $\beta = \omega_1$. In this case, we just need to show that $\langle p_\alpha \mid \alpha < \omega_1\rangle$ has a lower bound. As above, let $\eta_{\omega_1} = \sup\{\eta_\alpha \mid \alpha < \omega_1\}$, let $T^{<\omega_1} = \bigcup\{T^\alpha \mid \alpha < \omega_1\}$, and let $d_{\omega_1} = \bigcup\{\mathrm{dom}(f^\alpha) \mid \alpha < \omega_1\}$. For each $\gamma \in d_{\omega_1}$, let

$$b^{\omega_1}_\gamma = \bigcup\{f^\alpha(\gamma) \mid \alpha < \omega_1 \text{ and } \gamma \in \mathrm{dom}(f^\alpha)\}.$$

For all $\gamma \in d_{\omega_1}$, we have $b^{\omega_1}_\gamma \in [T^{<\omega_1}]$.

Subclaim 6.17. *For every $b \in [T^{<\omega_1}]$, there is $\gamma \in d_{\omega_1}$ such that $b = b^{\omega_1}_\gamma$.*

Proof. Suppose for the sake of contradiction that $b \in [T^{<\omega_1}]$ and, for all $\gamma \in d_{\omega_1}$, we have $b \neq b^{\omega_1}_\gamma$. Recall that, for each $\alpha \in \lim(\omega_1)$, the function f^α is bijective; therefore, there is a unique $\gamma_\alpha \in \mathrm{dom}(f^\alpha)$ such that $f^\alpha(\gamma_\alpha) \sqsubseteq b$. By assumption, for each $\alpha \in \lim(\omega_1)$, we can find $\alpha^\dagger \in \lim(\omega_1)$ such that $b \restriction \alpha^\dagger \neq b^{\omega_1}_{\gamma_\alpha} \restriction \alpha^\dagger$. Let

$$C = \{\alpha' \in \lim(\omega_1) \mid \forall \alpha \in \lim(\alpha')\ \alpha^\dagger < \alpha'\}.$$

Then C is a club in ω_1 and, for all $\alpha' \in C$ and all $\gamma \in d_{\alpha'}$, we have $b \restriction \eta_{\alpha'} \neq b^{\alpha'}_\gamma$.

Let $f : \omega_1 \to \omega_1$ be such that, for each $\alpha < \omega_1$, we have $\pi_\alpha(f(\alpha)) = b \restriction \eta_\alpha$. We can then find $\alpha' \in C$ such that $f \restriction \alpha' = a_{\alpha'}$. Now recall the specification of Player II's strategy at stage α' of this run of the game. Unraveling the definitions, we have $b^{\alpha'}_* = b \restriction \eta_{\alpha'}$, and, for every $\gamma \in d_{\alpha'}$, we have $b^{\alpha'}_* \neq b^{\alpha'}_\gamma$. Therefore, Player II played $T^{\alpha'}_{\eta_{\alpha'}} = [T^{<\alpha'}] \setminus \{b^{\alpha'}_*\}$. In particular, $b \restriction \eta_{\alpha'} \notin T^{<\omega_1}$, contradicting the assumption that $b \in [T^{<\omega_1}]$. $\square$

Since $|d_{\omega_1}| = \omega_1$, it follows that $|[T^{<\omega_1}]| = \omega_1$. Define a subtree T^{ω_1} of ${}^{<\eta_{\omega_1}+1}\omega_2$ by letting $T^{\omega_1} \cap {}^{<\eta_{\omega_1}}\omega_2 = T^{<\omega_1}$ and $T^{\omega_1}_{\eta_{\omega_1}} = [T^{<\omega_1}]$. The normality of T^{ω_1} follows from the fact that f^α is surjective onto $T^\alpha_{\eta_\alpha}$ for all even ordinals $\alpha < \omega_1$, and hence each element of $T^{<\omega_1}$ is an initial segment of $b^{\omega_1}_\gamma$ for some $\gamma \in d_{\omega_1}$. Let $\mathrm{dom}(f^{\omega_1}) = d_{\omega_1}$ and, for each $\gamma \in d_{\omega_1}$, let $f^{\omega_1}(\gamma) = b^{\omega_1}_\gamma$. Then $(T^{\omega_1}, f^{\omega_1})$ is a lower bound for $\langle p_\alpha \mid \alpha < \omega_1\rangle$ in $\mathbb{P}$. $\square$

Since $\mathbb{P}$ has the ω_3-cc and is $(\omega_1 + 1)$-strategically closed, it follows that it preserves all cardinalities and cofinalities. Let G be $\mathbb{P}$-generic over V, let $T_G = \bigcup\{T_p \mid p \in G\}$, let $d = \bigcup\{\mathrm{dom}(f_p) \mid p \in G\}$, and define a function f with domain d by letting $f(\gamma) = \bigcup\{f_p(\gamma) \mid p \in G \text{ and } \gamma \in \mathrm{dom}(p)\}$. Then standard genericity arguments, combined with the arguments of the proof of Claim 6.16, show that

- T_G is a normal, splitting, full, superthin ω_2-tree;
- $d = \omega_3$;
- f is an injective function from ω_3 to $[T_G]$.

Thus, T_G is the desired tree as in the statement of the theorem. $\square$

7. Adding superthin subtrees

In this section, we prove Theorem D. Fix for the remainder of the section a regular uncountable cardinal μ such that $\mu^{<\mu} = \mu$, $2^\mu = \mu^+$, and $2^{\mu^+} = \mu^{++}$, and let $\kappa = \mu^+$. Fix for now a closed set $E \subseteq {}^\kappa\kappa$ such that $|E| > \kappa$. Recall that the tree $T(E) \subseteq {}^{<\kappa}\kappa$ is defined to be $\{x \restriction \alpha \mid x \in E \text{ and } \alpha < \kappa\}$.

We introduce a forcing notion $\mathbb{P}(E)$ that will add a superthin κ-Kurepa subtree to $T(E)$. We first thin E out to a subset such that all nonempty neighborhoods are large. Let $\Sigma = \{\sigma \in {}^{<\kappa}\kappa \mid |E \cap N_\sigma| \leq \kappa\}$, and let

$$E' = E \setminus \bigcup\{N_\sigma \mid \sigma \in \Sigma\}.$$

By replacing E with E', assume from now on that E has the property that, for all $\sigma \in {}^{<\kappa}\kappa$, either $E \cap N_\sigma = \emptyset$ or $|E \cap N_\sigma| > \kappa$.

Conditions of $\mathbb{P}(E)$ are all triples of the form $p = (B^p, \gamma^p, t^p)$ such that

- $B^p \in [E]^{\leq\mu}$ is nonempty;
- $\gamma^p < \kappa$ is such that, for all distinct $b, b' \in B^p$, we have $b \restriction \gamma^p \neq b' \restriction \gamma^p$;
- $t^p = \{b \restriction \alpha \mid b \in B^p,\ \alpha \leq \gamma^p\}$;
- t^p looks like an initial segment of a normal superthin subtree of $T(E)$, i.e.,
  - t^p is a normal subtree of ${}^{\leq\gamma^p}\kappa$;
  - all levels of t^p have cardinality at most μ;
  - for all limit ordinals $\gamma \leq \gamma^p$, we have $|[t^p_{<\gamma}]| \leq \mu$.

We note that, for a condition $p \in \mathbb{P}(E)$, t^p is uniquely determined by B^p and γ^p; we include it in the notation for convenience.

Given $p, q \in \mathbb{P}(E)$, we set $q \leq p$ if and only if

- $B^q \supseteq B^p$;
- $\gamma^q \geq \gamma^p$;
- t^q end-extends t^p, i.e., $t^q \cap {}^{\leq\gamma^p}\kappa = t^p$.

We also include $(\emptyset, \emptyset, \emptyset)$ as $1_{\mathbb{P}(E)}$.

Proposition 7.1. *$\mathbb{P}(E)$ is κ^+-Knaster.*

Proof. Let $\langle p_\delta \mid \delta < \kappa^+\rangle$ be a sequence of conditions from $\mathbb{P}(E)$, with each p_δ of the form $(B^\delta, \gamma^\delta, t^\delta)$. Since $\kappa^{<\kappa} = \kappa$, by thinning out our sequence if necessary we can assume that there are fixed $\gamma < \kappa$ and $t \subseteq {}^{\leq\gamma}\kappa$ such that, for all $\delta < \kappa^+$, we have $\gamma^\delta = \gamma$ and $t^\delta = t$.

We claim that, for all $\delta_0 < \delta_1 < \kappa^+$, the conditions p_{δ_0} and p_{δ_1} are compatible. To this end, fix such $\delta_0 < \delta_1$. For each $\sigma \in t_\gamma$ and each $i < 2$, there is a unique $b_{\sigma,i} \in B^{\delta_i}$ such that $b_{\sigma,i} \restriction \gamma = \sigma$. Choose $\gamma' \in [\gamma, \kappa)$ large enough so that, for all $\sigma \in t_\gamma$, either $b_{\sigma,0} = b_{\sigma,i}$ or $b_{\sigma,0} \restriction \gamma' \neq b_{\sigma,1} \restriction \gamma'$. Let $B' = B^{\delta_0} \cup B^{\delta_1}$ and $t' = \{b \restriction \alpha \mid b \in B',\ \alpha \leq \gamma'\}$. Then it is readily verified that $p' = (B', \gamma', t')$ is a common extension of p^{δ_0} and p^{δ_1} in $\mathbb{P}(E)$. For example, to verify that $|[t_{<\beta}]| \leq \mu$ for all limit ordinals $\beta \leq \gamma'$, fix such a β. If $\beta \leq \gamma$, then the desired conclusion follows from the fact that p^{δ_0} and p^{δ_1} are in $\mathbb{P}(E)$. If $\beta \in (\gamma, \gamma']$, then our construction immediately implies that every element of the form $[t_{<\beta}]$ is of the form $b \restriction \beta$ for some $b \in B'$. $\square$

Proposition 7.2. *$\mathbb{P}(E)$ is κ-strategically closed.*

Proof. We describe a winning strategy for Player II in $\Game_\kappa(\mathbb{P}(E))$. The winning strategy is very simple: at every even stage of the game, Player II does the minimal amount of work necessary. More precisely, suppose that $\beta < \kappa$ is an even ordinal and $\langle p_\alpha \mid \alpha < \beta \rangle$ is a partial play of the game, with Player II playing so far according to the strategy being described here. For each $\alpha < \beta$, let $p_\alpha = (B^\alpha, \gamma^\alpha, t^\alpha)$.

If β is a successor ordinal, then Player II can play arbitrarily; for instance, they can simply play $p_\beta = p_{\beta-1}$. If β is a limit ordinal, then they specify p_β by setting $B^\beta = \bigcup\{B^\alpha \mid \alpha < \beta\}$ and $\gamma^\beta = \sup\{\gamma^\alpha \mid \alpha < \beta\}$ (this uniquely determines t^β).

In order to prove that this is a winning strategy, we must only verify that p_β thus defined is indeed a condition in $\mathbb{P}(E)$. Let $\gamma = \gamma^\beta$ and $t = t^\beta$. The only nontrivial condition to verify is the requirement that $|[t_{<\gamma}]| \leq \mu$. If $\mathrm{cf}(\beta) < \mu$, then this follows immediately from the fact that $\mu^{<\mu} = \mu$ and all levels of t have cardinality at most μ.

Thus, assume that $\mathrm{cf}(\beta) = \mu$. We claim that, in this case, every element of $[t_{<\gamma}]$ is of the form $b \restriction \gamma$ for some $b \in B^\beta$. Since $|B^\beta| \leq \mu$, this will suffice. To this end, fix $d \in [t_{<\gamma}]$. Let $\langle \alpha_i \mid i < \mu \rangle$ be a strictly increasing enumeration of a club in β. By construction, for each $i < \mu$, there is a unique $b_i \in B^{\alpha_i}$ such that $d \restriction \gamma^{\alpha_i} = b_i \restriction \gamma^{\alpha_i}$. If $j < \mu$ is a limit ordinal, then the assumption that Player II has played so far according to the described strategy implies that $B^{\alpha_j} = \bigcup\{B^{\alpha_i} \mid i < j\}$ and $\gamma^{\alpha_j} = \sup\{\gamma^{\alpha_i} \mid i < j\}$. Therefore, for each limit ordinal $j < \mu$, we can find $i(j) < j$ such that $b_j \in B^{\alpha_{i(j)}}$. By Fodor's Lemma, we can find a stationary $S \subseteq \mu$ and a fixed $i < \mu$ such that $i(j) = i$ for all $j \in S$. Since distinct elements of B^{α_i} have distinct restrictions to γ^{α_i}, it follows that there must be a fixed $b \in B^{\alpha_i}$ such that $b_j = b$ for all $j \in S$. But then $d \restriction \gamma^{\alpha_j} = b \restriction \gamma^{\alpha_j}$ for all $j \in S$, and hence $d = b \restriction \gamma$. □

By Propositions 7.1 and 7.2, forcing with $\mathbb{P}(E)$ preserves all cardinalities and cofinalities. We now show that it adds a normal, cofinally splitting, superthin κ-Kurepa subtree of $T(E)$.

Theorem 7.3. *Suppose that G is $\mathbb{P}(E)$-generic over V, and let $T_G = \bigcup\{t^p \mid p \in G\}$. Then in $V[G]$, T_G is a normal, cofinally splitting, superthin κ-Kurepa subtree of $T(E)$.*

Proof. In V, let $\dot{G}$ be the canonical $\mathbb{P}(E)$-name for the generic filter, and let $\dot{T}_G$ be a $\mathbb{P}(E)$-name for $\bigcup\{t^p \mid p \in \dot{G}\}$. It is immediate that, for every $\gamma < \kappa$, the set $D_\gamma = \{p \in \mathbb{P}(E) \mid \gamma^p \geq \gamma\}$ is dense in $\mathbb{P}(E)$. As a result, the definition of $\mathbb{P}(E)$ implies that $\dot{T}_G$ is forced to be a superthin normal subtree of $T(E)$.

We now show that $\dot{T}_G$ is forced to be cofinally splitting. To this end, fix a condition $p \in \mathbb{P}(E)$ and a $\sigma \in {}^{<\kappa}\kappa$ such that $p \Vdash \sigma \in \dot{T}_G$. Note that there must be $b \in B^p$ such that $\sigma \sqsubseteq b$; otherwise, if $\gamma \in [\gamma^p, \kappa)$ is such that $\sigma \in {}^{\leq\gamma}\kappa$, then the condition q with $B^q = B^p$ and $\gamma^q = \gamma$ would extend p and force $\sigma \notin \dot{T}_G$. By increasing γ^p if necessary, we can assume that $\sigma \in {}^{\leq\gamma^p}\kappa$. Let $\sigma' \in t^p_{\gamma^p}$ be such that $\sigma \sqsubseteq \sigma'$, and let b be the unique element of B^p such that $\sigma' \sqsubseteq b$. By our assumption about E, we know that $|E \cap N_{\sigma'}| > \kappa$, so we can fix $b' \in E \cap N_{\sigma'}$ such that $b' \neq b$. Define a condition q by letting $B^q = B^p \cup \{b'\}$ and letting $\gamma^q \in (\gamma^p, \kappa)$ be large enough so that $b \restriction \gamma^q \neq b' \restriction \gamma^q$. Then q extends p and forces that $b \restriction \gamma^q$ and $b' \restriction \gamma^q$ are incomparable elements of $\dot{T}_G$ extending σ.

We finally show that $\dot{T}_G$ is forced to have at least κ^+-many cofinal branches. Let $\dot{B}$ be a $\mathbb{P}(E)$-name for $\bigcup\{B^p \mid p \in \dot{G}\}$. It will suffice to show that $\dot{B}$ is forced to have cardinality greater than κ. Suppose for the sake of contradiction that $p \in \mathbb{P}(E)$ and $p \Vdash |\dot{B}| \leq \kappa$. Since $\mathbb{P}(E)$ has the κ^+-c.c., we can find $q \leq p$ and $A \in [E]^\kappa$ such that $q \Vdash \dot{B} \subseteq A$. Let σ be an arbitrary element of $t^q_{\gamma^q}$. Since $|E \cap N_\sigma| > \kappa$, we can find $b \in (E \cap N_\sigma) \setminus (A \cup B^q)$. Precisely as in the previous paragraph, we can find $r \leq q$ such that $b \in B^r$, contradicting the fact that $r \notin A$ and $q \Vdash \dot{B} \subseteq A$. □

We now show that appropriate iterations of forcings of the form $\mathbb{P}(E)$ are well-behaved.

Theorem 7.4. *Suppose that $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j \mid i \leq \varepsilon, j < \varepsilon \rangle$ is a $(\leq\mu)$-support iteration such that, for all $i < \varepsilon$, there is a $\mathbb{P}_i$-name $\dot{E}_i$ such that*

- *$\dot{E}_i$ is forced to be a nonempty closed subset of ${}^\kappa\kappa$ such that, for all $\sigma \in T(\dot{E}_i)$, we have $|\dot{E}_i \cap N_\sigma| > \kappa$;*
- *$\Vdash_{\mathbb{P}_i} \dot{\mathbb{Q}}_i = \mathbb{P}(\dot{E})$.*

Then $\mathbb{P}_\varepsilon$ is κ^+-Knaster and κ-strategically closed.

Proof. Since $\kappa = \mu^+$, the iteration is taken with supports of size $\leq\mu$, and each iterand is forced to be κ-strategically closed, standard arguments imply that $\mathbb{P}_\varepsilon$ is κ-strategically closed (roughly speaking, Player II simply plays according to their winning strategy on each coordinate).

To show that $\mathbb{P}_\varepsilon$ is κ^+-Knaster, we first isolate a well-behaved dense subset of $\mathbb{P}_\varepsilon$. For concreteness, for all $j \leq \varepsilon$, we will think of conditions of $\mathbb{P}_j$ as being functions whose domains are subsets of j of cardinality $\leq\mu$.

Claim 7.5. *For all $j \leq \varepsilon$, let $\mathbb{P}^*_j$ be the set of $p \in \mathbb{P}_j$ such that, for all $i \in \operatorname{dom}(p)$, there are $\gamma^{p,i} < \kappa$, $t^{p,i} \subseteq {}^{\leq\gamma^{p,i}}\kappa$, a collection $B^{p,i}$ of $\mathbb{P}_i$-names, and a bijection $\pi^{p,i} : B^{p,i} \to t^{p,i}_{\gamma^{p,i}}$ such that*

- *$p \restriction i \Vdash_{\mathbb{P}_i} p(i) = (B^{p,i}, \gamma^{p,i}, t^{p,i})$;*
- *for all $\dot{b} \in B^{p,i}$, $p \restriction i \Vdash_{\mathbb{P}_i} \dot{b} \restriction \gamma^{p,i} = \pi^{p,i}(\dot{b})$.*

Then $\mathbb{P}^*_j$ is dense in $\mathbb{P}_j$.

Before we prove Claim 7.5, we establish the following useful fact.

Claim 7.6. *Fix $j \leq \varepsilon$, let $\beta < \kappa$ be a limit ordinal, and suppose that $\vec{p} = \langle p_\alpha \mid \alpha < \beta \rangle$ is a decreasing sequence of conditions from $\mathbb{P}_j$ such that*

- *for all $\alpha < \beta$, we have $p_\alpha \in \mathbb{P}^*_j$, as witnessed by $\langle (B^{p_\alpha,i}, \gamma^{p_\alpha,i}, t^{p_\alpha,i}, \pi^{p_\alpha,i}) \mid i \in \operatorname{dom}(p_\alpha) \rangle$;*
- *for all $\alpha < \alpha' < \beta$ and all $i \in \operatorname{dom}(p_\alpha)$, we have $B^{p_\alpha,i} \subseteq B^{p_{\alpha'},i}$;*
- *for all limit ordinals $\alpha' < \beta$, we have*
  - $\operatorname{dom}(p_{\alpha'}) = \bigcup\{\operatorname{dom}(p_\alpha) \mid \alpha < \alpha'\}$*;*
  - *for all $i \in \operatorname{dom}(p_{\alpha'})$, we have $\gamma^{p_{\alpha'},i} = \sup\{\gamma^{p_\alpha,i} \mid \alpha < \alpha'$ and $i \in \operatorname{dom}(p_\alpha)\}$ and $B^{p_{\alpha'},i} = \bigcup\{B^{p_\alpha,i} \mid \alpha < \alpha'$ and $i \in \operatorname{dom}(p_\alpha)\}$.*

Then $\vec{p}$ has a lower bound p_β that is in $\mathbb{P}^*_j$, as witnessed by $\langle (B^{p_\beta,i}, \gamma^{p_\beta,i}, t^{p_\beta,i}, \pi^{p_\beta,i}) \mid i \in \operatorname{dom}(p_\beta) \rangle$ satisfying:

- $\operatorname{dom}(p_\beta) = \bigcup\{\operatorname{dom}(p_\alpha) \mid \alpha < \beta\}$*;*
- *for all $i \in \operatorname{dom}(p_\beta)$, we have $\gamma^{p_\beta,i} = \sup\{\gamma^{p_\alpha,i} \mid \alpha < \beta$ and $i \in \operatorname{dom}(p_\alpha)\}$ and $B^{p_\beta,i} = \bigcup\{B^{p_\alpha,i} \mid \alpha < \beta$ and $i \in \operatorname{dom}(p_\alpha)\}$.*

Proof. We define p_β as follows. First, let $\operatorname{dom}(p_\beta) = \bigcup\{\operatorname{dom}(p_\alpha) \mid \alpha < \beta\}$. Then, for each $i \in \operatorname{dom}(p_\beta)$, we let $p_\beta(i)$ be a $\mathbb{P}_i$-name for the triple $(B^{p_\beta,i}, \gamma^{p_\beta,i}, t^{p_\beta,i})$, defined as follows. First, let $B^{p_\beta,i} = \bigcup\{B^{p_\alpha,i} \mid \alpha < \beta \text{ and } i \in B^{p_\alpha}\}$ and let $\gamma^{p_\beta,i} = \sup\{\gamma^{p_\alpha,i} \mid \alpha < \beta \text{ and } i \in \operatorname{dom}(p_\alpha)\}$.

Let $s^{p_\beta,i} = \bigcup\{t^{p_\alpha,i} \mid \alpha < \beta \text{ and } i \in \mathrm{dom}(p_\alpha)\}$ and, for each $\dot{b} \in B^{p_\beta,i}$, let

$$d_{\dot{b}} = \bigcup\{\pi^{p_\alpha,i}(\dot{b}) \mid \alpha < \beta,\ i \in \mathrm{dom}(p_\alpha),\ \text{and } \dot{b} \in B^{p_\alpha,i}\}.$$

Then $d_{\dot{b}}$ is a cofinal branch through $s^{p_\beta,i}$ and, if $\dot{b}$ and $\dot{b}'$ are distinct elements of $B^{p_\beta,i}$, then they both appear in $B^{p_\alpha,i}$ for some $\alpha < \beta$, and the fact that $\pi^{p_\alpha,i}$ is injective implies that $d_{\dot{b}}$ and $d_{\dot{b}'}$ are distinct. We end by setting $t^{p_\beta,i} = s^{p_\beta,i} \cup \{d_{\dot{b}} \mid \dot{b} \in B^{p_\beta,i}\}$ and, for each $\dot{b} \in B^{p_\beta,i}$, setting $\pi^{p_\beta,i}(\dot{b}) = d_{\dot{b}}$.

We claim that p_β thus defined is as desired. The only nontrivial requirement to verify is that, for all $i \in \mathrm{dom}(p_\beta)$, the tree $t^{p_\beta,i}$ looks like an initial segment of a normal superthin subtree of ${}^{<\kappa}\kappa$. This amounts to verifying that $|[s^{p_\beta,i}]| \leq \mu$. This is proven exactly as in the proof of Proposition 7.2, so we leave it to the reader. □

Proof of Claim 7.5. The proof is by induction on j. We will actually establish the following technical strengthening of the density of $\mathbb{P}^*_j$:

> Suppose that $p_{00} \in \mathbb{P}^*_j$, as witnessed by $\langle (B^{p_{00},i}, \gamma^{p_{00},i}, t^{p_{00},i}, \pi^{p_{00},i}) \mid i \in \mathrm{dom}(p_{00})\rangle$, and $p_0 \leq_{\mathbb{P}_j} p_{00}$. Then there is $q \leq p_0$ in $\mathbb{P}^*_j$ witnessed by $\langle (B^{q,i}, \gamma^{q,i}, t^{q,i}, \pi^{q,i}) \mid i \in \mathrm{dom}(q)\rangle$ such that, for all $i \in \mathrm{dom}(p_{00})$, we have $B^{p_{00},i} \subseteq B^{q,i}$.

To see that this does indeed yield the density of $\mathbb{P}^*_j$, note that in the above statement we can let $p_0 \in \mathbb{P}_j$ be arbitrary and take p_{00} to be such that $\mathrm{dom}(p_{00}) = \emptyset$. Then $p_0 \leq p_{00}$, and p_{00} is trivially in $\mathbb{P}^*_j$.

Fix $j \leq \varepsilon$, and suppose that the inductive hypothesis has been established for all $i < j$. Fix $p_0 \leq p_{00}$ as in the statement of the hypothesis; we will find $q \leq p_0$ as desired.

If $j = 0$, then we can take $q = p_0$. Suppose next that j is a successor ordinal, say $j = j_0 + 1$. Assume that $j_0 \in \mathrm{dom}(p_0)$; otherwise, $p \in \mathbb{P}^*_{j_0}$, and we can apply the inductive hypothesis to obtain the desired conclusion. Suppose also that $j_0 \in \mathrm{dom}(p_{00})$; the argument is similar but easier in the other case.

Since $\mathbb{P}_{j_0}$ is κ-strategically closed and hence $({<}\kappa)$-distributive, we can find $p_1 \leq p_0 \restriction j_0$ in $\mathbb{P}_{j_0}$, an ordinal $\gamma^{j_0} < \kappa$, a tree $t^{j_0} \subseteq {}^{\leq\gamma^{j_0}}\kappa$, a set B^{j_0} of $\mathbb{P}_{j_0}$-names, and a bijection $\pi^{j_0} : B^{j_0} \to t^{j_0}_{\gamma^{j_0}}$ such that

- $p_1 \Vdash_{\mathbb{P}_{j_0}} p_0(j_0) = (B^{j_0}, \gamma^{j_0}, t^{j_0})$;
- for all $\dot{b} \in B^{j_0}$, $p_1 \Vdash \dot{b} \restriction \gamma^{j_0} = \pi^{j_0}(\dot{b})$.

Since $p_0 \restriction j_0 \Vdash p_0(j_0) \leq p_{00}(j_0)$, by extending p_1 further if necessary we can assume that, for each $\dot{b} \in B^{p_{00},j_0}$, there is a unique $\dot{b}' \in B^{j_0}$ such that

$$p_1 \Vdash_{\mathbb{P}_{j_0}} \text{“} \Vdash_{\dot{\mathbb{Q}}_{j_0}} \dot{b}' = \dot{b}\text{”}.$$

By replacing each $\dot{b}'$ with $\dot{b}$ in B^{j_0} and redefining π^{j_0} in the obvious way, we may thus assume that $B^{j_0} \supseteq B^{p_{00},j_0}$.

Now apply the inductive hypothesis to find $p_2 \leq_{\mathbb{P}_{j_0}} p_1$ such that $p_2 \in \mathbb{P}^*_{j_0}$ witnessed by $\langle (B^{p_2,i}, \gamma^{p_2,i}, t^{p_2,i}, \pi^{p_2,i}) \mid i \in \mathrm{dom}(p_2)\rangle$ such that, for all $i \in \mathrm{dom}(p_{00}) \cap j_0$, we have $B^{p_{00},i} \subseteq B^{p_2,i}$. Finally, define $q \in \mathbb{P}_j$ by setting $\mathrm{dom}(q) = \mathrm{dom}(p_2) \cup \{j_0\}$, $q \restriction j_0 = p_2$, and $q(j_0) = (B^{j_0}, \gamma^{j_0}, t^{j_0})$. Then q is as desired.

Suppose next that $j > 0$ is a limit ordinal. If $\mathrm{cf}(j) > \mu$, then there is $i < j$ such that $p_0 \in \mathbb{P}_i$, so we can apply the inductive hypothesis to obtain the desired conclusion. Thus, assume that $\mathrm{cf}(j) \leq \mu$. Let $\langle i_\eta \mid \eta < \mathrm{cf}(j)\rangle$ be an increasing enumeration of a club in j. Using the inductive hypothesis and Claim 7.6, recursively construct a sequence $\langle q_\eta \mid \eta < \mathrm{cf}(j)\rangle$ such that, for all $\eta < \mathrm{cf}(j)$, we have:

- $q_\eta \in \mathbb{P}^*_{i_\eta}$, as witnessed by $\langle (B^{q_\eta,i}, \gamma^{q_\eta,i}, t^{q_\eta,i}, \pi^{q_\eta,i}) \mid i \in \mathrm{dom}(q_\eta)\rangle$;
- $q_\eta \leq_{\mathbb{P}_{i_\eta}} p_0 \restriction i_\eta$;
- for all $i \in \mathrm{dom}(p_{00}) \cap i_\eta$, $B^{p_{00},i} \subseteq B^{q_\eta,i}$;
- for all $\xi < \eta$, we have
  - $q_\eta \restriction i_\xi \leq_{\mathbb{P}_{i_\xi}} q_\xi$;
  - for all $i \in \mathrm{dom}(q_\xi)$, $B^{q_\xi,i} \subseteq B^{q_\eta,i}$;
- if η is a limit ordinal, then
  - $\mathrm{dom}(q_\eta) = \bigcup\{\mathrm{dom}(q_\xi) \mid \xi < \eta\}$;
  - for all $i \in \mathrm{dom}(q_\eta)$, we have $\gamma^{q_\eta,i} = \sup\{\gamma^{q_\xi,i} \mid \xi < \eta \text{ and } i \in \mathrm{dom}(q_\xi)\}$ and $B^{q_\eta,i} = \bigcup\{B^{q_\xi,i} \mid \xi < \eta \text{ and } i \in \mathrm{dom}(q_\xi)\}$.

The construction is straightforward, so we leave it to the reader. At the end of the construction, another appeal to Claim 7.6 yields a condition $q \in \mathbb{P}^*_j$, as witnessed by $\langle (B^{q,i}, \gamma^{q,i}, t^{q,i}, \pi^{q,i}) \mid i \in \mathrm{dom}(q)\rangle$ such that

- q is a lower bound of $\langle q_\eta \mid \eta < \mathrm{cf}(j)\rangle$;
- $\mathrm{dom}(q) = \bigcup\{\mathrm{dom}(q_\eta) \mid \eta < \mathrm{cf}(j)\}$;
- for all $i \in \mathrm{dom}(q)$, we have $B^{q,i} = \bigcup\{B^{q_\eta,i} \mid \eta < \mathrm{cf}(j) \text{ and } i \in \mathrm{dom}(q_\eta)\}$.

Then q is as desired, completing the proof of the claim. □

The following will be the key claim in establishing that $\mathbb{P}_\varepsilon$ is κ^+-Knaster.

Claim 7.7. *Suppose that $j \leq \varepsilon$ and, for $\ell < 2$, $p_\ell \in \mathbb{P}^*_j$, as witnessed by*

$$\langle (B^{p_\ell,i}, \gamma^{p_\ell,i}, t^{p_\ell,i}, \pi^{p_\ell,i}) \mid i \in \mathrm{dom}(p_\ell)\rangle.$$

Suppose moreover that, for all $i \in \mathrm{dom}(p_0) \cap \mathrm{dom}(p_1)$, we have $t^{p_0,i} = t^{p_1,i}$. Then $p_0 \parallel p_1$.

Proof. The proof is by induction on j. If $j = 0$, the conclusion is trivial. Suppose next that j is a successor ordinal, say $j = j_0 + 1$. Assume that $j_0 \in \mathrm{dom}(p_0) \cap \mathrm{dom}(p_1)$, as otherwise we can simply apply the induction hypothesis. First, apply the induction hypothesis to find q_0 such that $q_0 \leq_{\mathbb{P}_{j_0}} p_\ell \restriction j_0$ for $\ell < 2$. Next, let $\gamma = \gamma^{p_0,j_0} = \gamma^{p_1,j_0}$ and $t = t^{p_0,j_0} = t^{p_1,j_0}$. For every $u \in t_\gamma$ and $\ell < 2$, there is a unique $\dot{b}_{u,\ell} \in B^{p_\ell,j_0}$ such that $q_0 \Vdash_{\mathbb{P}_{j_0}} \dot{b}_{u,\ell} \restriction \gamma = u$. Since $\mathbb{P}_{j_0}$ is $({<}\kappa)$-distributive, we can find $q_1 \leq_{\mathbb{P}_{j_0}} q_0$ such that, for all $u \in t_\gamma$,

- q_1 decides the statement "$\dot{b}_{u,0} = \dot{b}_{u,1}$";
- if $q_1 \Vdash \dot{b}_{u,0} \neq \dot{b}_{u,1}$, then there is $\gamma_u < \kappa$ such that $q_1 \Vdash \dot{b}_{u,0} \restriction \gamma_u \neq \dot{b}_{u,1} \restriction \gamma_u$.

Let $s_0 = \{u \in t_\gamma \mid q_1 \Vdash \dot{b}_{u,0} \neq \dot{b}_{u,1}\}$ and $s_1 = t_\gamma \setminus s_0$. Let $\gamma^* = \sup\{\gamma_u \mid u \in s_0\}$ (or $\gamma^* = \gamma$ if $s_0 = \emptyset$). Again using the $({<}\kappa)$-distributivity of $\mathbb{P}_{j_0}$, find $q_2 \leq_{\mathbb{P}_{j_0}} q_1$ such that, for all $u \in t_\gamma$ and $\ell < 2$, q_2 decides the value of $\dot{b}_{u,\ell} \restriction \gamma^*$, say as $b^*_{u,\ell} \in {}^{\gamma^*}\kappa$. Let t^* be the downward closure of the set $\{b^*_{u,\ell} \mid u \in t_\gamma,\ \ell < 2\}$ in ${}^{<\kappa}\kappa$, and let

$$B^* = \{\dot{b}_{u,0} \mid u \in t_\gamma\} \cup \{\dot{b}_{u,1} \mid u \in s_0\}.$$

Define a condition $q \in \mathbb{P}_j$ by letting $\mathrm{dom}(q) = \mathrm{dom}(q_2) \cup \{j_0\}$, $q \restriction j_0 = q_2$, and $q(j_0)$ be such that $q_2 \Vdash q(j_0) = (B^*, \gamma^*, t^*)$.

Finally, suppose that j is a limit ordinal. If $\mathrm{cf}(j) > \mu$, then there is $j_0 < j$ such that $p_0, p_1 \in \mathbb{P}_{j_0}$, so we can simply apply the induction hypothesis. Thus, assume that $\mathrm{cf}(j) \leq \mu$. Let $\langle i_\eta \mid \eta < \mathrm{cf}(j)\rangle$ be an increasing enumeration of a club in j. Recursively build a sequence $\langle q_\eta \mid \eta < \mathrm{cf}(j)\rangle$ such that, for all $\eta < \mathrm{cf}(j)$, we have:

- $q_\eta \in \mathbb{P}^*_{i_\eta}$, as witnessed by $\langle (B^{q_\eta,i}, \gamma^{q_\eta,i}, t^{q_\eta,i}, \pi^{q_\eta,i}) \mid i \in \mathrm{dom}(q_\eta)\rangle$;
- $q_\eta \leq_{\mathbb{P}_{i_\eta}} p_0 \restriction i_\eta, p_1 \restriction i_\eta$;
- for all $\xi < \eta$, we have
  - $q_\eta \restriction i_\xi \leq_{\mathbb{P}_{i_\xi}} q_\xi$;
  - for all $i \in \mathrm{dom}(q_\xi)$, $B^{q_\xi,i} \subseteq B^{q_\eta,i}$;
- if η is a limit ordinal, then
  - $\mathrm{dom}(q_\eta) = \bigcup\{\mathrm{dom}(q_\xi) \mid \xi < \eta\}$;
  - for all $i \in \mathrm{dom}(q_\eta)$, we have $\gamma^{q_\eta,i} = \sup\{\gamma^{q_\xi,i} \mid \xi < \eta \text{ and } i \in \mathrm{dom}(q_\xi)\}$ and $B^{q_\eta,i} = \bigcup\{B^{q_\xi,i} \mid \xi < \eta \text{ and } i \in \mathrm{dom}(q_\xi)\}$.

The construction is straightforward using the induction hypothesis and Claim 7.6, so we leave it to the reader. At the end, another appeal to Claim 7.6 yields $q \in \mathbb{P}_j$ that is a lower bound for $\langle q_\eta \mid \eta < \mathrm{cf}(j)\rangle$. This condition q extends both p_0 and p_1, thus establishing the claim. □

We are now finally ready to prove that $\mathbb{P}_\varepsilon$ is κ^+-Knaster. To this end, suppose that $\langle p_\eta \mid \eta < \kappa^+\rangle$ is a sequence of conditions in $\mathbb{P}_\varepsilon$. By extending the conditions if necessary, we may assume that each p_η is in $\mathbb{P}^*_\varepsilon$, as witnessed by $\langle (B^{\eta,i}, \gamma^{\eta,i}, t^{\eta,i}, \pi^{\eta,i}) \mid i \in \mathrm{dom}(p_\eta)\rangle$. Since $2^\mu = \kappa$, we can find an unbounded $A \subseteq \kappa^+$ such that

- $\langle \mathrm{dom}(p_\eta) \mid \eta \in A\rangle$ forms a Δ-system, with root r;
- there is a sequence $\langle t^i \mid i \in r\rangle$ such that, for all $\eta \in A$ and all $i \in r$, we have $t^{\eta,i} = t^i$.

Then, by Claim 7.7, $\langle p_\eta \mid \eta \in A\rangle$ is a sequence of pairwise compatible conditions. □

We can now easily prove the main consistency result of this section. To make its statement self-contained, we recall the standing cardinal assumptions of this section at the start of the theorem.

Theorem 7.8. *Suppose that μ is a regular uncountable cardinal such that $\mu^{<\mu} = \mu$, $2^\mu = \mu^+$, and $2^{\mu^+} = \mu^{++}$, and let $\kappa = \mu^+$. Then there is a forcing extension in which all cardinalities and cofinalities are preserved and in which*

1. *there exists a κ-Kurepa tree;*
2. *every κ-Kurepa tree contains a normal superthin κ-Kurepa subtree;*
3. *the cardinal arithmetic of the ground model is preserved; in particular, if* GCH *holds in the ground model, then it continues to hold in the extension.*

Proof. We can assume that there is a κ-Kurepa tree in V; if not, then first force with one of the standard forcings to add a κ-Kurepa tree with κ^+-many cofinal branches (e.g., the analogue of the forcing $\mathbb{Q}$ from the proof of Theorem 5.1).

We will force with a $(\leq\mu)$-support iteration $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j \mid i \leq \kappa^+,\ j < \kappa^+\rangle$ such that, for all $i < \kappa^+$, $\dot{\mathbb{Q}}_i$ is forced by $\mathbb{P}_i$ to be either trivial forcing or of the form $\mathbb{P}([\dot{T}_i])$, where $\dot{T}_i$ is a $\mathbb{P}_i$-name for a κ-Kurepa tree such that

$$\Vdash_{\mathbb{P}_i} \forall \sigma \in \dot{T}_i\ |[\dot{T}_i] \cap N_\sigma| = \kappa^+.$$

By Theorem 7.4, $\mathbb{P} = \mathbb{P}_{\kappa^+}$ will be κ^+-Knaster and κ-strategically closed. It will therefore preserve all cardinalities and cofinalities, and hence will preserve the κ-Kurepa tree that exists in the ground model. Moreover, since the length of the iteration is κ^+, it will not change the value of 2^ν for any cardinal ν. Moreover, for every $i < \kappa^+$, the quotient forcing $\mathbb{P}/\mathbb{P}_i$ is equivalent to an iteration of the same shape as $\mathbb{P}$ and therefore shares the same properties.

We will recursively specify $\mathbb{P}$ by concurrently specifying a sequence $\langle \dot{T}_i \mid i < \kappa^+ \rangle$ such that

- for all $i < \kappa^+$, $\dot{T}_i$ is a $\mathbb{P}_i$-name for a subtree of ${}^{<\kappa}\kappa$;
- it is forced by $\mathbb{P}_i$ that:
  - if $\dot{T}_i$ is not a κ-Kurepa tree, then $\dot{\mathbb{Q}}_i$ is trivial forcing;
  - if $\dot{T}_i$ is a κ-Kurepa tree, then $\dot{Q}_i$ is $\mathbb{P}([\dot{T}_i^*])$, where $\dot{T}_i^*$ is the $\mathbb{P}_i$-name for the set of $\sigma \in \dot{T}_i$ that are contained in κ^+-many distinct cofinal branches through $\dot{T}_i$ (note that, in this case, $\dot{T}_i^*$ is forced to be a κ-Kurepa tree itself).

By the cardinal arithmetic assumptions and the chain condition of $\mathbb{P}$, there will only be κ^+-many nice $\mathbb{P}$-names for subtrees of ${}^{<\kappa}\kappa$. Moreover, also by the arithmetic and chain condition, each such name $\dot{T}$ will in fact be a $\mathbb{P}_i$-name for some $i < \kappa^+$. Therefore, by a standard bookkeeping argument, we can construct the sequence $\langle \dot{T}_i \mid i < \kappa^+ \rangle$ in such a way that, in $V^{\mathbb{P}}$, for every subtree T of ${}^{<\kappa}\kappa$, there is $i < \kappa^+$ such that T equals the interpretation of $\dot{T}_i$ in $V^{\mathbb{P}}$.

Let G be $\mathbb{P}$-generic over V. For all $i < \kappa^+$, let G_i be the $\mathbb{P}_i$-generic filter induced by G. We claim that $V[G]$ is the desired forcing extension. We have already argued that requirements (1) and (3) in the statement of the theorem hold. It remains to verify (2). To this end, let $T \in V[G]$ be a κ-Kurepa tree. We may assume that it is a subtree of ${}^{<\kappa}\kappa$. By our construction of $\mathbb{P}$, we can find $i < \kappa$ such that T equals the interpretation of $\dot{T}_i$. In particular, $T \in V[G_i]$. Moreover, since the quotient forcing $\mathbb{P}/\mathbb{P}_i$ is κ-strategically closed, Lemma 2.12 implies that all of the cofinal branches through T in $V[G]$ are already in $V[G_i]$. In particular, T is a κ-Kurepa tree in $V[G_i]$. Therefore, $\mathbb{Q}_i$ is $\mathbb{P}([T^*])$, where T^* is the set of all $\sigma \in T$ such that κ^+-many cofinal branches through T contain σ. Forcing with $\mathbb{Q}_i$ therefore adds a normal superthin κ-Kurepa subtree to T^*, and hence to T. Since T was arbitrary, this completes the proof of the theorem. $\square$

8. Open questions

In this final section, we present some closing remarks and survey some of the remaining open questions. We first make the following observations about Theorem 3.8.

- In Theorem 3.8, the assumption that E is a continuous image of ${}^{\kappa}\kappa$ is slightly more than is needed. As the proof makes clear, it is enough to assume that E is the continuous image of $[T]$ for some normal, splitting, $({<}\omega_1)$-closed tree $T \subseteq {}^{<\kappa}\kappa$. Note that, for such a tree T, the set $[T]$ is necessarily $(\omega+1)$-perfect.
- The conclusion of Theorem 3.8 falls slightly short of the assertion that $E \setminus X$ is $(\omega+1)$-perfect, since it only gives a winning strategy for games $G_\kappa(E, x_0, \omega+1)$ rather than $G_\kappa(E \setminus X, x_0, \omega+1)$.

In light of these observations, the following questions are natural.

Question 8.1. *Suppose that κ is a regular uncountable cardinal and $E \subseteq {}^{\kappa}\kappa$ is a closed set.*

(1) *Suppose that E is a continuous image of ${}^{\kappa}\kappa$. Must there be $X \subseteq E$ such that $|X| \leq \kappa^{<\kappa}$ and $E \setminus X$ is $(\omega+1)$-perfect?*
(2) *Suppose that there is an $(\omega+1)$-perfect set $D \subseteq {}^{\kappa}\kappa$ such that E is a continuous image of D. Must there be $X \subseteq E$ such that $|X| \leq \kappa^{<\kappa}$ and, for every $x_0 \in E \setminus X$, Player II has a winning strategy in $G_\kappa(E, x_0, \omega+1)$? Must there be $X \subseteq E$ such that $|X| \leq \kappa^{<\kappa}$ and $E \setminus X$ is $(\omega+1)$-perfect?*

We now turn to some questions arising from Theorem 5.1. The first concerns whether an inaccessible cardinal is necessary to obtain the conclusion of the theorem.

Question 8.2. *Suppose that* GCH *holds, there is an ω_2-Kurepa tree and, for every ω_2-Kurepa tree $S \subseteq {}^{<\omega_2}\omega_2$, $[S]$ is not a continuous image of ${}^{\omega_2}\omega_2$. Must $(\omega_3)^V$ be inaccessible in L?*

We also ask whether the opposite extreme to that obtained in Theorem 5.1 can hold.

Question 8.3. *Is it consistent with* GCH *that there are ω_2-Kurepa trees and, for every ω_2-Kurepa tree $T \subseteq {}^{<\omega_2}\omega_2$, $[T]$* is *a continuous image of ${}^{\omega_2}\omega_2$?*

Note that the strengthening of the previous question asking for $[T]$ to be a *retract* of ${}^{\omega_2}\omega_2$ for every ω_2-Kurepa tree $T \subseteq {}^{<\omega_2}\omega_2$ has a negative answer by [8, Theorem 1.12].

In all of the known consistent examples of ω_2-Kurepa trees $T \subseteq {}^{<\omega_2}\omega_2$ such that $[T]$ is a continuous image of ${}^{\omega_2}\omega_2$, there exist $x \in [T]$ and $\sigma \in T$ such that $\sigma \sqsubseteq x$ and $|[T] \cap N_\sigma| \leq \omega_2$. We conjecture that this is in necessarily the case.

Conjecture 8.4. *Suppose that* GCH *holds, and let $T \subseteq {}^{<\omega_2}\omega_2$ be an ω_2-Kurepa tree such that, for every $\sigma \in T$, we have $|[T] \cap N_\sigma| > \omega_2$. Then $[T]$ is not a continuous image of ${}^{\omega_2}\omega_2$*

Note that, if GCH holds and $T \subseteq {}^{<\omega_2}\omega_2$ is an ω_2-Kurepa tree, then $T' := \{\sigma \in T \mid |[T] \cap N_\sigma| > \omega_2\}$ is an ω_2-Kurepa tree satisfying the hypothesis of Conjecture 8.4. Therefore, a positive resolution of Conjecture 8.4 would entail a negative answer to Question 8.3.

References

[1] Uri Abraham. Proper forcing. In *Handbook of set theory. Vols. 1, 2, 3*, pages 333–394. Springer, Dordrecht, 2010.

[2] Todd Eisworth, Justin Tatch Moore, and David Milovich. Iterated forcing and the continuum hypothesis. In *Appalachian set theory 2006–2012*, volume 406 of *London Math. Soc. Lecture Note Ser.*, pages 207–244. Cambridge Univ. Press, Cambridge, 2013.

[3] Radek Honzik and Šárka Stejskalová. The tree property and the continuum function below $\aleph_\omega$. *MLQ Math. Log. Q.*, 64(1-2):89–102, 2018.

[4] Thomas Jech. *Set theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, millennium edition, 2003.

[5] Thomas J. Jech. Trees. *J. Symbolic Logic*, 36:1–14, 1971.

[6] Philipp Lücke. Σ^1_1-definability at uncountable regular cardinals. *J. Symbolic Logic*, 77(3):1011–1046, 2012.

[7] Philipp Lücke and Philipp Schlicht. Continuous images of closed sets in generalized Baire spaces. *Israel J. Math.*, 209(1):421–461, 2015.

[8] Philipp Lücke and Philipp Schlicht. Descriptive properties of higher Kurepa trees. *arXiv preprint arXiv:2010.11597*, 2020.

[9] Alan Mekler and Jouko Väänänen. Trees and Π^1_1-subsets of ${}^{\omega_1}\omega_1$. *J. Symbolic Logic*, 58(3):1052–1070, 1993.

[10] Arnold W. Miller. Arnie Miller's problem list. In *Set theory of the reals (Ramat Gan, 1991)*, volume 6 of *Israel Math. Conf. Proc.*, pages 645–654. Bar-Ilan Univ., Ramat Gan, 1993.

[11] Justin Tatch Moore. ω_1 and $-\omega_1$ may be the only minimal uncountable linear orders. *Michigan Math. J.*, 55(2):437–457, 2007.

[12] Assaf Rinot, Shira Yadai, and Zhixing You. Full Souslin trees at small cardinals. *J. Lond. Math. Soc. (2)*, 110(1):Paper No. e12957, 26, 2024.

[13] Philipp Schlicht and Dorottya Sziráki. The open dihypergraph dichotomy for generalized Baire spaces and its applications. *arXiv preprint arXiv:2301.13274*, 2023.

[14] Saharon Shelah. *Proper and improper forcing*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, second edition, 1998.

[15] Saharon Shelah. On full Suslin trees. *Colloq. Math.*, 79(1):1–7, 1999.

[16] D. H. Stewart. *The consistency of the Kurepa hypothesis with the axioms of set theory*. Master's thesis, University of Example, 1966.

[17] Spencer Unger. Fragility and indestructibility of the tree property. *Arch. Math. Logic*, 51(5-6):635–645, 2012.

[18] Jouko Väänänen. A Cantor-Bendixson theorem for the space $\omega_1^{\omega_1}$. *Fund. Math.*, 137(3):187–199, 1991.

(Lambie-Hanson) Institute of Mathematics, Czech Academy of Sciences, Žitná 25, Prague 1, 115 67, Czech Republic

Email address: lambiehanson@math.cas.cz

URL: https://users.math.cas.cz/~lambiehanson/

(Stejskalová) Charles University, Department of Logic, Celetná 20, Prague 1, 116 42, Czech Republic

Email address: sarka.stejskalova@ff.cuni.cz

URL: logika.ff.cuni.cz/sarka

Institute of Mathematics, Czech Academy of Sciences, Žitná 25, Prague 1, 115 67, Czech Republic