DISJOINT TYPE GRAPHS WITH NO SHORT ODD CYCLES

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Abstract. In this note, we provide a proof of a technical result of Erdős and Hajnal about the existence of disjoint type graphs with no odd cycles. We also prove that this result is sharp in a certain sense.

The purpose of this note is to provide a proof of a result of Erdös and Hajnal about the existence of disjoint type graphs with no short odd cycles. As far as we know, a proof of this result has never been published, though forms of it are stated in a number of publications (cf. [2, Theorem 7.4] and [3, Lemma 1.1(d)]). If κ is an uncountable cardinal, then graphs of this form provide, again as far as we know, the only known ZFC examples of graphs with size and chromatic number κ and arbitrarily high odd girth.

Before we state and prove the main result, we need some definitions and conventions. First, if n is a positive integer, we will sometimes think of elements of $[\mathrm{Ord}]^n$ as strictly increasing sequences of length n. So, for instance, if $a \in [\mathrm{Ord}]^n$ and $i < n$, then $a(i)$ is the unique element $\alpha \in a$ such that $|a \cap \alpha| = i$. All graphs considered here will be simple undirected graphs. If G is a graph, then $V(G)$ denotes its vertex set and $E(G)$ denotes its edge set.

Definition 1. Let n be a positive integer. A *disjoint type of width* n is a function $t : 2n \to 2$ such that

$$|t^{-1}(0)| = |t^{-1}(1)| = n.$$

If $a, b \in [\mathrm{Ord}]^n$ are disjoint and $a \cup b$ is enumerated in increasing order as $\{\alpha_i \mid i < 2n\}$, then we say that the *type of* a *and* b is t, denoted $\mathrm{tp}(a, b) = t$, if

$$a = \{\alpha_i \mid i \in t^{-1}(0)\}$$

and

$$b = \{\alpha_i \mid i \in t^{-1}(1)\}.$$

Let $\hat{t}$ denote the disjoint type of width n denoted by letting $\hat{t}(i) = 1 - t(i)$ for all $i < 2n$. It is evident that, if $a, b \in [\mathrm{Ord}]^n$ are disjoint and $\mathrm{tp}(a, b) = t$, then $\mathrm{tp}(b, a) = \hat{t}$.

A type t of width n will sometimes be represented by a binary string of length $2n$ in the obvious way. We will particularly be interested in the following family of types.

Definition 2. Let $1 \leq s < n < \omega$. Then t^n_s is the disjoint type of width n whose binary sequence representation consists of s copies of '0', followed by $n - s$ copies of '01', followed by s copies of '1'. More formally, t^n_s is defined by letting, for all

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$i < 2n$,

$$t_s^n(i) = \begin{cases} 0 & \text{if } i < s \\ 0 & \text{if } s \leq i < 2n - s \text{ and } i - s \text{ is even} \\ 1 & \text{if } s \leq i < 2n - s \text{ and } i - s \text{ is odd} \\ 1 & \text{if } i \geq 2n - s. \end{cases}$$

For example, $t_2^5 = 0001010111$.

Definition 3. Suppose that n is a positive integer, β is an ordinal, and t is a disjoint type of width n. The graph $G(\beta, t)$ is defined as follows. Its vertex set is $V(G(\beta, t)) = [\beta]^n$. Given $a, b \in [\beta]^n$, we put the edge $\{a, b\}$ into $E(G(\beta, t))$ if and only if a and b are disjoint and $\mathrm{tp}(a, b) \in \{t, \hat{t}\}$.

Before we get to our main result, we need a basic lemma. Given a function f from a natural number to $\mathbb{Z}$, let $\max(f)$ and $\min(f)$ denote the maximum and minimum values attained by f, respectively.

Lemma 4. *Suppose that k is a positive integer and $f : k \to \mathbb{Z}$ is a function such that*

- *$f(0) = 0$ and*
- *$|f(i + 1) - f(i)| = 1$ for all $i < k$.*

Then $\max(f) - \min(f) < k$.

Proof. The proof is by induction on k. If $k = 1$, then $\max(f) = \min(f) = f(0) = 0$. Suppose that $k > 0$ and we have proven the lemma for $k - 1$. Fix $f : k \to \mathbb{Z}$, and let $f^- = f \restriction (k - 1)$. If $f(k) - f(k - 1) = 1$, then we have $\max(f) \leq max(f^-) + 1$ and $\min(f) = \min(f^-)$, so, applying the induction hypothesis to f^-, we obtain

$$\max(f) - \min(f) \leq 1 + \big(\max(f^-) - \min(f^-)\big) < 1 + (k - 1) = k.$$

If $f(k) - f(k-1) = -1$, then we have $\max(f) = \max(f^-)$ and $\min(f) \geq \min(f^-) - 1$, so

$$\max(f) - \min(f) \leq \big(\max(f) - \min(f^-)\big) + 1 < (k - 1) + 1 = k.$$

□

We are now ready for the main result of this note. The proof is rather technical; we recommend that the reader first draw some pictures to convince themselves of the truth of the theorem in the special case $s = 1$, $n = 3$ (this pair does not satisfy $n > 2s^2 + 3s + 1$, but the conclusion of the theorem still holds). This will help the reader to get a feel for the problem and motivate the calculations in the proof. We also note that the lower bound of $2s^2 + 3s + 1$ is probably not optimal and can likely be improved with a more careful analysis. Since a precise lower bound for n is not necessary for our desired applications (cf. [5]), the primary interest of the result for us is the fact that such a lower bound exists at all.

Theorem 5. *Suppose that s and n are positive integers with $n > 2s^2 + 3s + 1$, and suppose that β is an ordinal. Then the graph $G(\beta, t_s^n)$ has no odd cycles of length $2s + 1$ or shorter.*

Proof. Let $t = t_s^n$, $V = [\beta]^n$, $E = E(G(\beta, t))$, and $G = G(\beta, t) = (V, E)$. We begin by making some preliminary observations. If $\{a, b\} \in E$, then either $\mathrm{tp}(a, b) = t$ or $\mathrm{tp}(a, b) = \hat{t}$. If $\mathrm{tp}(a, b) = t$, then, for all i with $s < i < n$, we have

$$b(i - s - 1) < a(i) < b(i - s).$$

If $\mathrm{tp}(a,b) = \hat{t}$, then, for all $i < n - s - 1$, we have

$$b(i+s) < a(i) < b(i+s+1).$$

Suppose that k is a positive integer and $P = \langle a_0, \ldots, a_k \rangle$ is a path of length k in G. For $j \leq k$, let

$$U_j(P) = \big\{ i < j \mid \mathrm{tp}(a_i, a_{i+1}) = t \big\},$$

and

$$D_j(P) = \big\{ i < j \mid \mathrm{tp}(a_i, a_{i+1}) = \hat{t} \big\}.$$

Intuitively, $U_j(P)$ is the set of steps "up" in the path among the first j steps, and $D(P)$ is the set of steps "down" among the first j steps. Then set $u_j(P) = |U_j(P)|$ and $d_j(p) = |D_j(P)|$; note that $u_j(P) + d_j(P) = j$ for all $j \leq k$.

Claim 6. *Suppose that $1 \leq k \leq 2s+1$ and $P = \langle a_0, \ldots, a_k \rangle$ is a path in G. Let $u = u_k(P)$ and $d = d_k(P)$. Then there is $i < n$ such that*

$$a_k\big(i - u(s+1) + ds\big) < a_0\big(i\big) < a_k\big(i - us + d(s+1)\big).$$

Remark 7. Implicit in the statement of the claim is the assertion that

$$0 \leq i - u(s+1) + ds < i - us + d(s+1) < n,$$

the truth of which will follow readily from the proof.

Proof of Claim 6. Define a function $f : k+1 \to \mathbb{Z}$ by letting $f(j) = u_j(P) - d_j(P)$ for every $j \leq k$. Then f satisfies the hypotheses of Lemma 4, so, letting $M = \max(f)$ and $m = \min(f)$, we have $M - m \leq k \leq 2s+1$.

Let $i = M(s+1)$. Note that

$$M(s+1) \leq (2s+1)(s+1) = 2s^2 + 3s + 1,$$

so we certainly have $i < n$.

Subclaim 8. *For every $0 < j \leq k$, we have*

$$a_j\big(i - sf(j) - u_j(P)\big) < a_0\big(i\big) < a_j\big(i - sf(j) + d_j(P)\big).$$

Remark 9. Implicit in the statement of this subclaim is the assertion that, for each $0 < j \leq k$, we have

$$0 \leq i - sf(j) - u_j(P) < i - sf(j) + d_j(P) < n.$$

This will follow readily from the proof.

Proof of Subclaim 8. We proceed by induction on j. We begin by proving the subclaim for $j = 1$. Suppose first that $\mathrm{tp}(a_0, a_1) = t$, so $f(1) = 1$, $u_1(P) = 1$, and $d_1(P) = 0$. Then $M \geq 1$, so $i \geq s+1$. Therefore, since $\mathrm{tp}(a_0, a_1) = t$, the preliminary observations at the beginning of the proof of the theorem imply that

$$a_1(i - s - 1) < a_0(i) < a_1(i - s),$$

as desired.

If, on the other hand, $\mathrm{tp}(a_0, a_1) = \hat{t}$, and hence $f(1) = -1$, $u_1(P) = 0$, and $d_1(P) = 1$, then $m \leq -1$. Therefore, we have $M \leq 2s$, so $i = M(s+1) \leq 2s^2 + 2s < n - s - 1$. Therefore, since $\mathrm{tp}(a_0, a_1) = \hat{t}$, the preliminary observations at the beginning of the proof imply that

$$a_1(i+s) < a_0(i) < a_1(i+s+1),$$

as desired.

Now suppose that $0 < j < k$ and we have established that

$$a_j\big(i - sf(j) - u_j(P)\big) < a_0\big(i\big) < a_j\big(i - sf(j) + d_j(P)\big).$$

We will prove the corresponding statement for $j+1$. Suppose to begin that $\mathrm{tp}(a_j, a_{j+1}) = t$, so $f(j+1) = f(j)+1$, $u_{j+1}(P) = u_j(P)+1$, and $d_{j+1}(P) = d_j(P)$. In this case, it follows that $f(j) \leq (M-1)$ and $u_j(P) \leq (M-1)$. In particular, we have

$$i - sf(j) - u_j(P) \geq M(s+1) - s(M-1) - (M-1) = s+1 > s.$$

Therefore, by the preliminary observations, we have

$$\begin{aligned} a_{j+1}\big(i - sf(j) - u_j(P) - s - 1\big) &< a_j\big(i - sf(j) - u_j(P)\big) \\ a_{j+1}\big(i - sf(j+1) - u_{j+1}(P)\big) &< a_j\big(i - sf(j) - u_j(P)\big) \end{aligned}$$

and

$$\begin{aligned} a_j\big(i - sf(j) + d_j(P)\big) &< a_{j+1}\big(i - sf(j) + d_j(P) - s\big) \\ a_j\big(i - sf(j) + d_j(P)\big) &< a_{j+1}\big(i - sf(j+1) + d_{j+1}(P)\big). \end{aligned}$$

Combining these inequalities with the inductive hypothesis yields

$$a_{j+1}\big(i - sf(j+1) - u_{j+1}(P)\big) < a_0(i) < a_{j+1}\big(i - sf(j+1) + d_{j+1}(P)\big),$$

as desired.

On the other hand, suppose that $\mathrm{tp}(a_j, a_{j+1}) = \hat{t}$, so $f(j+1) = f(j) - 1$, $u_{j+1}(P) = u_j(P)$, and $d_{j+1}(P) = d_j(P) + 1$. In this case, it follows that $f(j) \geq (m+1)$ and $d_j(P) \leq -(m+1)$. In particular, we have

$$i - sf(j) + d_j(P) \leq i - s(m+1) - (m+1) = i - (m+1)(s+1).$$

We know that $M - m \leq 2s+1$, so $m+1 \geq M - 2s$. As a result, the above inequality becomes

$$i - sf(j) + d_j(P) \leq M(s+1) - (M-2s)(s+1) = 2s^2 + 2s < n - s - 1.$$

Therefore, by the preliminary observations, we have

$$\begin{aligned} a_{j+1}\big(i - sf(j) - u_j(P) + s\big) &< a_j\big(i - sf(j) - u_j(P)\big) \\ a_{j+1}\big(i - sf(j+1) - u_{j+1}(P)\big) &< a_j\big(i - sf(j) - u_j(P)\big) \end{aligned}$$

and

$$\begin{aligned} a_j\big(i - sf(j) + d_j(P)\big) &< a_{j+1}\big(i - sf(j) + d_j(P) + s + 1\big) \\ a_j\big(i - sf(j) + d_j(P)\big) &< a_{j+1}\big(i - sf(j+1) + d_{j+1}(P)\big). \end{aligned}$$

Combining these inequalities with the inductive hypothesis yields

$$a_{j+1}\big(i - sf(j+1) - u_{j+1}(P)\big) < a_0\big(i\big) < a_{j+1}\big(i - sf(j+1) + d_{j+1}(P)\big),$$

as desired, finishing the proof of the subclaim. □

Since $f(k) = u_k(P) - d_k(P)$, we have

$$i - u(s+1) + ds = i - sf(k) - u_k(P) \quad \text{and} \quad i - us + d(s+1) = i - sf(k) + d_k(P).$$

Therefore, the claim follows immediately from Subclaim 8. □

Now suppose for sake of contradiction that G has an odd cycle of length $2s+1$ or shorter. In other words, there is a positive integer $k \leq s$ and a path $C = \langle a_0, \ldots, a_{2k+1} \rangle$ with $a_0 = a_{2k+1}$. Let $u = u_k(C)$ and $d = d_k(C)$. Note that $u + d = 2k + 1$. Apply Claim 6 to find $i < n$ such that

$$a_{2k+1}\big(i - u(s+1) + ds\big) < a_0(i) < a_{2k+1}\big(i - us + d(s+1)\big).$$

Since $a_0 = a_{2k+1}$, this reduces to

$$i - u(s+1) + ds < i < i - us + d(s+1).$$

Cancelling i from all three terms yields

$$ds - u(s+1) < 0 < d(s+1) - us.$$

Since d and u are both non-negative integers, this implies that they are both nonzero. Therefore, the left inequality gives us

$$\frac{d}{u} < \frac{s+1}{s}$$

and the right inequality gives us

$$\frac{s}{s+1} < \frac{d}{u},$$

so we have

$$\frac{s}{s+1} < \frac{d}{u} < \frac{s+1}{s}.$$

In particular, $\frac{d}{u}$ is close to 1. But we know that $d + u = 2k + 1$; the assignments of values to d and u subject to this constraint that put $\frac{d}{u}$ closest to 1 are either $d = k$ and $u = k+1$ or vice versa. But $k \leq s$, so, if $d = k$ and $u = k+1$, then

$$\frac{d}{u} \leq \frac{s}{s+1}$$

and, if $d = k+1$ and $u = k$, then

$$\frac{d}{u} \geq \frac{s+1}{s}.$$

Either possibility gives us a contradiction, so we are done. □

We end this note by making a few further observations about these disjoint type graphs. We first point out a minor error in the literature. In [1, Remark 1], the authors write, using slightly different terminology, that, for any positive integer $n \geq 3$, the graph $G(\beta, t_1^n)$ has no odd cycles of length less than $2\lceil n/2 \rceil$. This is true for $n = 3$ but false for every larger value of n; $G(\beta, t_1^n)$ always has a cycle of length 5, as long as β is large enough to allow room for the cycle. In fact, we have the following general result, showing that Theorem 5 is sharp in a sense.

Proposition 10. *Suppose that* $0 < s < n < \omega$ *and*

$$\beta > (n-1)(2s+3) + (2s+1)(2s+2).$$

Then the graph $G(\beta, t_s^n)$ *has a cycle of length* $2s+3$.

Proof. Let $m = 2s+3$. We will define a path $\langle a_0, a_1, \ldots, a_m \rangle$ in $G(\beta, t^n_s)$ with $a_m = a_0$. First define $a_0 = a_m$ by letting $a_m(i) = im$ for all $i < n$. The definition of each of the remaining elements of the cycle depends on the parity of its index. For j with $0 < j \leq s+1$, define a_{2j-1} by setting

$$a_{2j-1}(i) = (i+s+j)m - (2j-1)$$

for all $i < n$, and define a_{2j} by setting

$$a_{2j}(i) = (i+j)m - 2j$$

for all $i < n$. The following facts are easily verified and left to the reader.

- For all $j \leq s$, $\mathrm{tp}(a_{2j}, a_{2j+1}) = t^n_s$.
- For all $j \leq s$, $\mathrm{tp}(a_{2j+1}, a_{2j+2}) = \hat{t}^n_s$.
- $\mathrm{tp}(a_{2s+2}, a_m) = \hat{t}^n_s$.
- The largest element of any of the vertices in the cycle is

$$a_{2s+1}(n-1) = (n-1)(2s+3) + (2s+1)(2s+2).$$

Therefore, $\langle a_0, a_1, \ldots, a_m \rangle$ forms a cycle of length $2s+3$ in $G(\beta, t^n_s)$. $\square$

We conclude by noting the following result, which is one of the primary reasons for interest in disjoint type graphs. The result is due to Erdős and Hajnal [2]; the special case $t = t^3_1$ is due to Erdős and Rado [4]. A proof of the full result can be found in [1, Theorem 2.1].

Theorem 11. *Suppose that n is a positive integer and t is a disjoint type of width n. For every infinite cardinal κ, the graph $G(\kappa, t)$ has chromatic number κ.*

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