Condensed mathematics, extremally disconnected spaces, and forcing

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I. Condensed mathematics

Condensed mathematics

Condensed mathematics is a framework developed by Dustin Clausen and Peter Scholze in order to apply algebraic tools to the study of algebraic structures carrying topologies.

Classical categories of such structures are generally badly behaved from an algebraic viewpoint. For example, in the category TopAb of topological abelian groups, the map

$$\mathbb{R}_{\text{discrete}} \hookrightarrow \mathbb{R}$$

is not an isomorphism, yet it does not have a nontrivial kernel or cokernel witnessing this. In particular, TopAb is not an *abelian category*. We fix this by embedding these categories into richer ones.

Definition

A *condensed set/abelian group/...* is a contravariant functor $T : \mathrm{CHaus} \to \mathrm{Set}/\mathrm{Ab}/\dots$ such that

1. $T(\emptyset) = *$;
2. $T(S_0 \sqcup S_1) = T(S_0) \times T(S_1)$;
3. for any surjection $S' \twoheadrightarrow S$ in CHaus with fiber product $S' \times_S S'$ and its projections π_0 and π_1 to S', the map

   $$T(S) \to \{x \in T(S') \mid T(\pi_0)(x) = T(\pi_1)(x) \in T(S' \times_S S')\}$$

   is a bijection.

More concisely, a condensed [–] is a [–]-valued sheaf on the pro-étale site of a point. Note that the above definition has some cardinality issues; as formulated, the condensed objects T are proper classes. The definition can be fixed by standard methods.

Underlying sets

Given a condensed set/abelian group/... T, we call $T(*)$ the *underlying set/abelian group/...* of T. $T(*)$ can be endowed with a topology, namely the quotient topology induced by

$$\bigsqcup_{S \to T} S \to T(*).$$

More precisely, given any $S \in \mathrm{CHaus}$, every $x \in T(S)$ induces a map g_x from S to $T(*)$ as follows. For every $s \in S$, consider the map $f_s : * \to S$ that takes the point to s, and let

$$g_x(s) = T(f_s)(x).$$

The topology placed on $T(*)$ is the finest that makes all of these maps g_x continuous. This topology is compactly generated (i.e. $T(*)$ is a k-space).

Embedding classical categories

Given a topological space X, one can defined a condensed set $\underline{\mathsf{X}}$ by setting $\underline{\mathsf{X}}(S) = \mathrm{Cont}(S, X)$ for all $S \in \mathsf{CHaus}$. If X is compactly generated, then $\underline{\mathsf{X}}(*) \cong X$ (in Top). Moreover, when restricted to compactly generated spaces, this embedding is *fully faithful*, i.e., bijective on Hom-sets.

Similarly, given a topological abelian group A, one gets a condensed abelian group $\underline{\mathsf{A}}$ by setting $\underline{\mathsf{A}}(S) = \mathrm{Cont}(S, A)$ for all $S \in \mathsf{CHaus}$. This embedding is fully faithful restricted to the category of locally compact abelian groups.

Condensed abelian groups

Recall that a sequence of morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is *exact at B* if $\ker(g) = \mathrm{im}(f)$. A *short exact sequence* is a sequence $0 \to A \to B \to C \to 0$ that is exact at A, B, and C.

The category CondAb of condensed abelian groups is an abelian category (with very nice properties). For example, consider our example of $\mathbb{R}_{\mathrm{discrete}} \hookrightarrow \mathbb{R}$. In CondAb, this fits into a short exact sequence

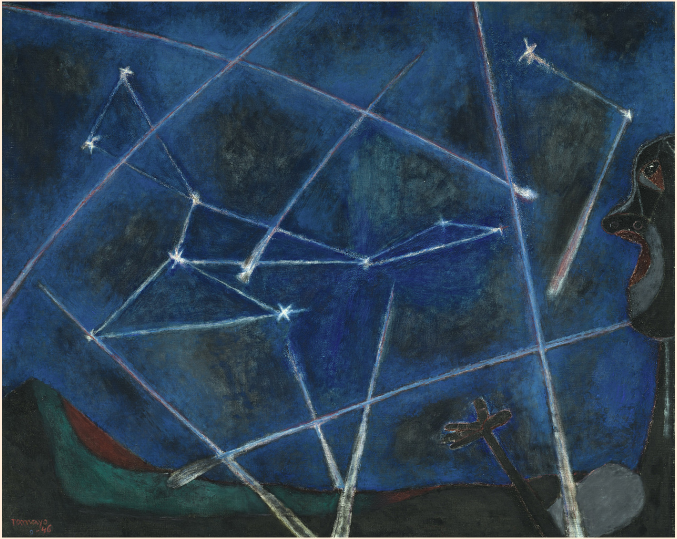
$$0 \to \underline{\mathbb{R}_{\mathrm{discrete}}} \to \underline{\mathbb{R}} \to Q \to 0,$$

where $Q \in \mathrm{CondAb}$ is defined by letting, for all $S \in \mathrm{CHaus}$,

$$Q(S) = \frac{\mathrm{Cont}(S, \mathbb{R})}{\mathrm{LocConst}(S, \mathbb{R})}.$$

Note that $Q(*) = 0$.

II. Extremally disconnected spaces



Extremally disconnected spaces

Recall that a topological space X is *extremally disconnected* if the closure of every open subset of X is open. Let ED denote the class of extremally disconnected compact Hausdorff spaces.

Theorem (Gleason)

The elements of ED *are precisely the projective objects in* CHaus.

Typical examples of ED spaces are the Čech-Stone compactifications βX, where X is discrete. In fact, every ED space is a retract of such a space.

Condensed objects revisited

Definition

A *condensed set/abelian group/...* is a contravariant functor $T : \mathsf{ED} \to \mathsf{Set}/\mathsf{Ab}/\ldots$ such that

1. $T(\emptyset) = *$;
2. $T(S_0 \sqcup S_1) = T(S_0) \times T(S_1)$.

Remark

Two observations are in order here:

1. *Replacing* CHaus *by* ED *involves no loss of information. The fact that every element of* CHaus *admits a surjection from an element of* ED *implies that a condensed object is determined by its restriction to* ED.
2. *Requirement (3) in the previous definition is not needed anymore; it automatically holds when restricted to* ED *spaces.*

Condensed projective objects

A category $\mathcal{C}$ is said to have *enough projective objects* if, for every $X \in \mathcal{C}$, there is a projective object $P \in \mathcal{C}$ and an epimorphism (surjection) from P to X.

Cond has enough projective objects; the compact projective objects are precisely the condensed sets $\underline{S}$, where $S \in \mathrm{ED}$.

CondAb also has enough projective objects; the compact projective objects are precisely the condensed abelian groups $\mathbb{Z}[\underline{S}]$ for $S \in \mathrm{ED}$.

Here, for $T \in \mathrm{Cond}$, $\mathbb{Z}[T] \in \mathrm{CondAb}$ is defined by letting $\mathbb{Z}[T](S)$ be the free group on $T(S)$ for all $S \in \mathrm{ED}$.

Condensed projective objects

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Here, for $T \in \mathrm{Cond}$, $\mathbb{Z}[T] \in \mathrm{CondAb}$ is the sheafification of the functor that sends each $S \in \mathrm{ED}$ to the free group on $T(S)$.

Products of ED spaces

Theorem (folklore)

If S_0 and S_1 are infinite ED *spaces, then* $S_0 \times S_1 \notin \mathrm{ED}$

As a result, in Cond and CondAb, the class of projective objects is not closed under taking products.

The *projective dimension* of an object X of an abelian category with enough projective objects is the least $n < \omega$ (if it exists) such that there exists an exact sequence

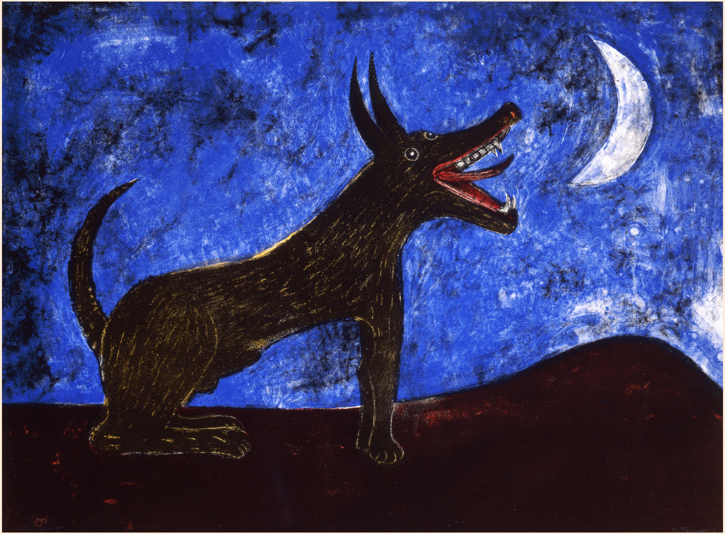
$$0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to X \to 0,$$

where each P_i is projective (and ∞ otherwise).

Question (Clausen–Scholze)

Suppose $S_0, S_1 \in \mathrm{ED}$. *Does* $\mathbb{Z}[\underline{S_0 \times S_1}]$ *have finite projective dimension in* CondAb*?*

III. The Whitehead problem



Whitehead groups

Definition

An abelian group A is *Whitehead* if, for every surjective group homomorphism $\pi : B \to A$ with $\ker(\pi) \cong \mathbb{Z}$, there is a homomorphism $\sigma : A \to B$ such that $\pi \circ \sigma = \mathrm{id}_A$.

Equivalently, every short exact sequence of the form

$$0 \to \mathbb{Z} \to B \to A \to 0$$

splits. Equivalently, $\mathrm{Ext}^1(A, \mathbb{Z}) = 0$.

Whitehead's problem

Fact

A group A is free if and only if $\operatorname{Ext}^1(A, C) = 0$ for every group C. Equivalently, for every surjective homomorphism $\pi : B \to A$, there is a homomorphism $\sigma : A \to B$ such that $\pi \circ \sigma = \mathrm{id}_A$.

A free group is therefore manifestly Whitehead. Whitehead's problem, first posed in the 1940s/50s, asks whether the converse holds.

Question (Whitehead)

Is every Whitehead group free?

A solution

Shortly after the problem was posed, Stein proved that all *countable* Whitehead groups are free. Twenty years later, Shelah resolved the full question in a way that was very surprising at the time.

Theorem (Shelah)

Whitehead's problem is independent of ZFC*.*

1. *If $V = L$, then every Whitehead group is free.*
2. *If $\mathrm{MA}_{\aleph_1}$ holds, then there is a nonfree Whitehead group of cardinality $\aleph_1$.*

Internal Hom

For $T_0, T_1 \in \mathsf{CondAb}$, $\mathrm{Hom}(T_0, T_1)$ is an abelian group. CondAb also has a tensor product and an *internal Hom functor*, $\underline{\mathrm{Hom}}(\cdot,\cdot)$, which takes values in CondAb. It satisfies the adjunction

$$\mathrm{Hom}(T_0, \underline{\mathrm{Hom}}(T_1, T_2)) \cong \mathrm{Hom}(T_0 \otimes T_1, T_2).$$

If A and G are locally compact topological abelian groups, then

$$\underline{\mathrm{Hom}}(\underline{A}, \underline{G}) \cong \underline{\mathrm{Hom}(A, G)},$$

where $\mathrm{Hom}(A, G)$ is given the compact-open topology (if A and G are both discrete, then this is just the product topology). Thus, for $S \in \mathsf{CHaus}$, we have $\underline{\mathrm{Hom}}(\underline{A}, \underline{G})(S) \cong \mathrm{Cont}(S, \mathrm{Hom}(A, G))$.

Internal Ext

$\underline{\mathrm{Hom}}(\cdot,\cdot)$ has a first derived functor, $\underline{\mathrm{Ext}}^1(\cdot,\cdot)$. For $A, B \in \mathsf{Ab}$, $\underline{\mathrm{Ext}}^1(\underline{A},\underline{B})(*) = \mathrm{Ext}(A,B)$. When Whitehead's problem is formulated in terms of $\underline{\mathrm{Ext}}^1$ (applied to abelian groups with the discrete topology), it turns out that it is *not* independent of ZFC.

Theorem (Clausen–Scholze)

Suppose that A is an abelian group and $\underline{\mathrm{Ext}}^1(\underline{A},\underline{\mathbb{Z}}) = 0$. Then A is free.

Clausen and Scholze's original proof relies heavily on deep structural facts about the category of condensed abelian groups (and the subcategory of *solid abelian groups*). It is also very inexplicit, e.g., given a nonfree group A, it does not identify a space S for which $\underline{\mathrm{Ext}}^1(\underline{A},\underline{\mathbb{Z}})(S) \neq 0$. This motivated us to find a more explicit, combinatorial proof.

Free resolutions

Given any group A, we can form a short exact sequence

$$0 \to K \to F \to A \to 0$$

where K is a subgroup of F and F is free (hence K is also free). It turns out that A is Whitehead if and only if the induced map

$$\mathrm{Hom}(F, \mathbb{Z}) \to \mathrm{Hom}(K, \mathbb{Z})$$

is surjective, i.e., iff every element of $\mathrm{Hom}(K, \mathbb{Z})$ extends to an element of $\mathrm{Hom}(F, \mathbb{Z})$.

One half of Shelah's theorem can be formulated as follows: given sufficiently many instances of $\diamondsuit$, if A is not free, then there is an element of $\mathrm{Hom}(K, \mathbb{Z})$ that does not extend to an element of $\mathrm{Hom}(F, \mathbb{Z})$.

Similarly, A is Whitehead in the condensed sense iff the map

$$\underline{\mathrm{Hom}}(\underline{F}, \underline{\mathbb{Z}})(S) \to \underline{\mathrm{Hom}}(\underline{K}, \underline{\mathbb{Z}})(S)$$

is surjective for all $S \in \mathsf{CHaus}$.

Recall that $\underline{\mathrm{Hom}}(\underline{F}, \underline{\mathbb{Z}})(S) = \mathrm{Cont}(S, \mathrm{Hom}(F, \mathbb{Z}))$, where $\mathrm{Hom}(F, \mathbb{Z})$ is endowed with the compact-open topology.

So to show that A is not Whitehead (in the condensed sense), it suffices to find an $S \in \mathsf{CHaus}$ and a continuous map $\varphi : S \to \mathrm{Hom}(K, \mathbb{Z})$ such that there is no continuous map $\psi : S \to \mathrm{Hom}(F, \mathbb{Z})$ for which $\psi(s)$ extends $\varphi(s)$ for all $s \in S$.

Theorem (Bergfalk–LH–Šaroch)

If A is not free, then there is such a map for $S = 2^\kappa$ (with the product topology), where κ is the least cardinality of a nonfree subgroup of A. I.e., $\underline{\mathrm{Ext}}^1(\underline{A}, \underline{\mathbb{Z}})(2^\kappa) \neq 0$.

The proof is essentially done by mimicking the proof of Shelah's theorem, using the following ZFC variant of $\diamondsuit(\kappa)$.

Proposition

Suppose κ is a regular uncountable cardinal. Then there is a continuous map $\varphi : 2^\kappa \to \prod_{\alpha<\kappa} 2^\alpha$ such that, for every continuous $\psi : 2^\kappa \to 2^\kappa$, there is $x \in 2^\kappa$ such that the set

$$\{\alpha < \kappa \mid \varphi(x)(\alpha) = \psi(x) \restriction \alpha\}$$

contains a club.

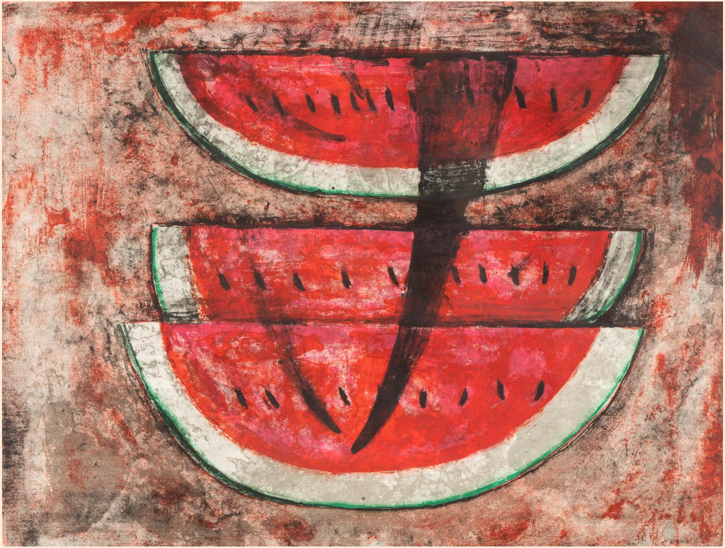
Proof.

Define φ as follows. For all $x \in 2^\kappa$ and $\alpha < \kappa$, let

$$\varphi(x)(\alpha) = x \restriction [\alpha, \alpha \cdot 2).$$

This works. $\square$

IV. Forcing



ED spaces and forcing

Recall one last characterization: ED is the class of Stone spaces of complete Boolean algebras.

Fact (folklore)

Suppose that $\mathbb{B}$ is a complete Boolean algebra and Y is a compact Hausdorff space. Then there is a correspondence between $\mathbb{B}$-names for elements of Y and continuous functions from the Stone space of $\mathbb{B}$ to Y.

What does it mean to interpret Y in a forcing extension? A point in Y corresponds to a maximal filter of nonempty closed subsets of Y. In the forcing extension, the points in the interpretation of Y are precisely these maximal filters of nonempty closed subsets of Y.

Proof of fact.

Given a $\mathbb{B}$-name $\dot{y}$ for an element of Y, define $f : \mathrm{St}(\mathbb{B}) \to Y$ as follows. Given $U \in \mathrm{St}(\mathbb{B})$, consider the set $\mathcal{D}$ of closed subsets D of Y such that $[\![D \in \dot{y}]\!] \in U$. Clearly, $\mathcal{D}$ is a filter. We claim that $\bigcap \mathcal{D}$ contains a single point. Suppose that y_0, y_1 are distinct points in $\bigcap \mathcal{D}$. Find disjoint open sets W_0, W_1 separating y_0 and y_1. Then $\{[\![Y \setminus W_0 \in \dot{y}]\!], [\![Y \setminus W_1 \in \dot{y}]\!]\}$ is a (finite) maximal antichain in $\mathbb{B}$, thus one of the sets must be in U. This yields a contradiction. Then set $f(U) = \bigcap \mathcal{D}$.

In the other direction, given a continuous $f : \mathrm{St}(\mathbb{B}) \to Y$, define a name $\dot{y}$ as follows: for all closed $D \subseteq Y$, let

$$[\![D \in \dot{y}]\!] = \bigvee \{b \in B\ [\forall U \in \mathrm{St}(\mathbb{B}).\ b \in U \Rightarrow f(U) \in D]\}.$$

$\square$

Condensed sets and forcing names

Recall that a topological space Y induces a condensed set $\underline{Y}$ such that $\underline{Y}(S) = \mathrm{Cont}(S, Y)$ for all $S \in \mathsf{ED}$. Therefore, if $Y \in \mathsf{CHaus}$, then $\underline{Y}$ is precisely an organized presentation of all names, in any possible set forcing extension, for elements of Y. A variation of the proof of the previous section, combined with these observations, yields:

Theorem (Bergfalk–LH–Šaroch)

Suppose that A is a nonfree abelian group, and let κ be the least cardinality of a nonfree subgroup of A. Then:

1. *A is not Whitehead in $V[\mathrm{Add}(\omega, \kappa)]$.*
2. *$\underline{\mathrm{Ext}}^1(\underline{A}, \underline{\mathbb{Z}})(S_\kappa) \neq 0$, where S_κ is the Stone space of the Boolean completion of $\mathrm{Add}(\omega, \kappa)$.*

I think there is a lot of potential in this direction, exploring the connections between condensed objects and forcing.

All artwork by Rufino Tamayo (1899–1991), born in Oaxaca.

Thank you!