LIMITS IN HIGHER CATEGORIES

Motivation: Role of limits in Alg Geom

- (X, τ) ⟵ some geom obj (eg: top space)
- $\mathcal{C}$ ⟵ collection of alg objects
- $\Gamma : \tau^{op} \to \mathcal{C}$ $\quad U \subseteq X \mapsto \Gamma(U) \in \mathcal{C}$

$\mathcal{C}$ could be...	then $\Gamma(U)$ could be...
a) {Abelian gps}	$\mathcal{F}(U; \mathbb{R})$ ⟵ scalar functions
b) {cochain cmplx}	$C^\bullet(U)$ ⟵ singular cochains
c) {categories}	Vect_U ⟵ vector bundles
d) {∞categories}	Sh_U ⟵ sheaves

Need language to express
gluing | descent | local to global condition

$\boxed{\Gamma(A \cup B) \text{ built from } \Gamma(A), \Gamma(B), \Gamma(A \cap B)}$

How? Request that, in $\mathcal{C}$, have

$$\Gamma(A \cup B) \xrightarrow{\sim} \Gamma(A) \times_{\Gamma(A \cap B)} \Gamma(B)$$

$$= \lim[\Gamma(A) \to \Gamma(A \cap B) \leftarrow \Gamma(B)]$$

$$= \lim[\boxed{\bullet \leftarrow \bullet \to \bullet} \to \mathcal{C}]$$

Making sense of this depends on $\mathcal{C}$!

$\mathcal{C}$ could be a ...	then Γ would be a ...
a) category	(pre)sheaf
b) ∞-category	∞ (pre)sheaf
c) 2-category	(pre)stack
d) $(\infty,2)$-category	∞-(pre)stack

Punchline: Need theory of limits for a–d.

a) classical MacLane ~1940

b) understood Joyal, Lurie, Riehl + Verity
~2000 (Joyal, Lurie), ~2010 (Riehl + Verity)

c) understood with recent surprises
Street, Kelly ~1980 Clingman-Moser ~2020

d) recent + work in progress
Gagna-Harpaz-Lanari ~2020 Moser-Rasekh-R ~2022

a) Limits in a category

Recall: A category $\mathcal{C}$ consists of

objects c, d + homsets $\mathcal{C}(c,d) \in \mathrm{Set}$
$$\begin{array}{c}\mathcal{C}(c,d)\\ \rotatebox{90}{$\in$}\\ c \xrightarrow{f} d\end{array}$$

+ composition associative & unital

Examples: $\mathcal{C} = \mathrm{Set}$, $\mathcal{C} = \mathrm{Ab}$

Def for $F : J \to C$ functor, $\lim F \in \mathrm{ob}\, C$ is s.t. ← limit of F

$$C(c, \lim F) \overset{\Phi}{\cong} C^J(\Delta c, F) \in \mathrm{Set}$$

$$c \xrightarrow{g} \lim F \quad \overset{\uparrow}{\text{naturally in } x} \quad \Delta c \overset{\gamma}{\Rightarrow} F \quad \therefore$$

$$\begin{tikzcd} & c \arrow[dl] \arrow[d] \arrow[dr] & \\ F_i \arrow[r] & F_k & F_j \arrow[l] \end{tikzcd}$$

$$[\lim F \xrightarrow{id} \lim F] \longleftrightarrow [\Delta \lim F \overset{\lambda}{\Rightarrow} F]$$

$$[c \xrightarrow{g} \lim F] \longleftrightarrow [\Delta c \overset{\Delta g}{\Rightarrow} \Delta \lim F \overset{\lambda}{\Rightarrow} F] \quad \therefore$$

$$\begin{tikzcd} & c \arrow[d, "g"] & \\ & \lim F \arrow[dl] \arrow[d] \arrow[dr] & \\ F_i & F_k & F_j \end{tikzcd} \quad \lambda \text{ (lower part)}, \ \gamma \text{ (whole)}$$

Example $\lim[X \to Z \leftarrow Y] \cong X \times_Z Y$

b) limits in an ∞-category

"Def" An *∞-category* $\mathcal{C}$ consists of

objects $c, d, \ldots$ + homspaces $\mathcal{C}(c,d) \in \mathrm{Space}$

+ composition associative + unital

all weakly w/ many models! eg quasicat's

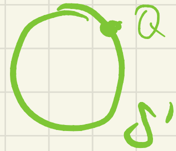
Example: $\mathcal{C} = \mathrm{Space}$ $\quad \mathcal{C} = \mathcal{D}(\mathrm{Ab})$ ← derived ∞cat

"Def": For $F : J \to \mathcal{C}$ ∞-functor, $\lim^{\infty} F$ ← ∞-limit is s.t.

$$\mathcal{C}(c, \lim^{\infty} F) \simeq \mathcal{C}^J(\Delta c, F) \in \mathrm{Space}$$

$\boxed{\textbf{Note}: \text{limits \& ∞-limits are different!!}}$

Example: $f = [\{Q\} \hookrightarrow S^1 \hookleftarrow \{Q\}]$



$\rightsquigarrow f : \boxed{\bullet \leftarrow \bullet \rightarrow \bullet} \to \mathrm{Space}$ ⟵ category or ∞-category

Prop: $\lim f \not\simeq \lim^{\infty} f$!

- $\lim f \cong \{Q\} \times_{S^1} \{Q\} \cong \{Q\}$
- $\lim^{\infty} f \simeq \{Q\} \times^{h}_{S^1} \{Q\} \simeq P_* S^1 \times_{S^1} \{Q\} = \Omega S^1$

Example: A, B open sets in (X, τ)

$F = [C^\bullet(A) \to C^\bullet(A \cap B) \leftarrow C^\bullet(B)]$

Consider $f : \boxed{\bullet \leftarrow \bullet \rightarrow \bullet} \to Ch(Ab)$ ⟵ cat

Generally $\lim f \not\cong C^\bullet(A \cup B)$

Consider $f : \boxed{\bullet \leftarrow \bullet \rightarrow \bullet} \to \mathcal{D}(Ab)$ ⟵ ∞-cat

[Mayer-Vietoris] $\Rightarrow$ $\lim^{\infty} f \simeq C^\bullet(A \cup B)$

c) limits in a ~~2~~ (∞,2)-category

Recall: A ~~2~~ (∞,2)-category $\mathcal{C}$ consists of

objects + hom categories

$c, d, \ldots$ $\qquad$ $\mathcal{C}(c,d) \in$ ~~Cat~~ ∞Cat

$$\boxed{\begin{tikzcd} c \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & d \end{tikzcd}}$$

+ composition associative & unital

all weak ⇒ many models!

Example: Cat

Def: For $F : J \to C$ a ~~2~~ (∞,2)-functor, $\lim^{(\infty,2)} F$ is s.t.

(∞,2)-limit

$$\mathcal{C}(c, \lim^{(\infty,2)} F) \simeq \mathcal{C}^J(\Delta c, F) \in \text{~~Cat~~ } \infty\text{Cat}$$

$\left(c \xrightarrow{j} \lim^2 F\right) \longleftrightarrow \Delta c \overset{\alpha}{\Rightarrow} F$ $\qquad$ on obj

$$\left(\begin{tikzcd} c \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow"] & \lim^2 F \end{tikzcd}\right) \longleftrightarrow \begin{tikzcd} \Delta c \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\delta"'] \arrow[r, phantom, "\Downarrow"] & F \end{tikzcd} \qquad \text{on mor}$$

Example: $\lim^2[\mathrm{Vect}_A \to \mathrm{Vect}_{A\cap B} \leftarrow \mathrm{Vect}_B] \simeq \mathrm{Vect}_{A\cup B}$

Problem: In ∞- & $(\infty,2)$-case,

$\mathcal{C}(c,d)$ is really hard to access

Want alternative viewpoints

~~a)~~ $\boxed{b}$ ~~Prop~~ Thm for $F: J \to C$ ∞-functor

(Riehl-Verity) $\ell \underset{?}{\simeq} \lim^{\infty} F$

$\Updownarrow$

(ℓ, λ) ∞-terminal in $\mathrm{Cone}^{\infty}_{/F}$ $\leftarrow$ ∞-category of cones over F

Joyal's

c) $F : J \to C$ 2-functor.

<u>Thm</u> [Clingman Moser] $\ell \cong \lim^2 F$

$$\Downarrow \not\Uparrow$$

(ℓ, λ) 2-terminal in $\mathrm{Cone}^2_{/F}$ ⟵ 2-cat of cones

<u>Thm</u> [CM] $\ell \cong \lim^2 F$

$$\Updownarrow$$

(ℓ, λ) double-terminal in $\mathrm{Cone}^{dbl}_{/F}$ ⟵ double cat of cones (Grandis-Paré)

d) $F : J \to C$ $(\infty,2)$-functor

work in progress [Moser-Rasekh-R] $\ell \simeq \lim F$

$$\Updownarrow$$

(ℓ, λ) double ∞-terminal in $\mathrm{Cone}^{dbl\infty}_{/F}$ ⟵ suitable double ∞-cat of cones