Francesco Lin :



Monopole Floer homology & theta characteristics

topology

geometry

Floer homology : topological invariants of 3-manifolds obtained by counting solutions to certain special non-linear PDEs (Yang-Mills, Seiberg-Witten, pseudo holomorphic curves)

⇝ very powerful!

Kronheimer ~~Kraah~~ - Mrowka 08 Khovanov homology detects the unknot
combinatorial

Manolescu '13 The triangulation conjecture is false in dim $\geq 5$

($\exists$ top. manifolds $\underset{\text{homeo}}{\not\sim}$ simplicial complex)

Drawback: very hard to compute/understand!

General theme of Thurston

one can study 3-manifolds using tools from Riemannian geometry

Q: Is there any relation between the geometry of $Y$ and its Floer homology ?

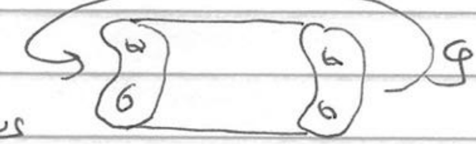
Rk: "most" 3-manifolds are hyperbolic $Y = \mathbb{H}^3/\Gamma$

(hyperbolic ints are top invariants)

let $\varphi \in MCG(\underset{=\Sigma}{M_g})$, $g \geq 2$

↳ mapping class group $= \mathrm{Diff}^{+} / \text{isotropy}$

$\rightsquigarrow M_\varphi$
mapping torus



$\varphi$

$[0,1] \times \Sigma$

<u>Thm</u> (Thurston) $\varphi$ is either:

today → 1) finite order $\rightsquigarrow$ $M_\varphi$ is modeled on $\mathbb{H}^2 \times \mathbb{R}$

2) reducible $\rightsquigarrow$ toroidal

generic case → 3) Pseudo-Anosov $\rightsquigarrow$ $M_\varphi$ is hyperbolic

<u>Rk</u>: if $\varphi$ has finite order,
then $\exists$ Riemann surface order on $\underset{=\Sigma_g}{\Sigma}$ s.t. $\varphi \circlearrowright \Sigma$ (acts) as automorphism

<u>Q</u>: is there a relation between Floer homology and the complex geometry of $(\Sigma, \varphi)$?

<u>Thm (L)</u>: yes when $\Sigma/\varphi = \mathbb{P}^1$
↳ $b_1(M_\varphi) = 1$

($\Sigma$ is Riemann surface of $w^d = p(z)$
$(z,w) \longmapsto (z, \zeta w)$
$\zeta$ $d$th root of unity)

<u>Objects of interest in $\mathbb{C}$-geometry side:</u>
theta characteristics

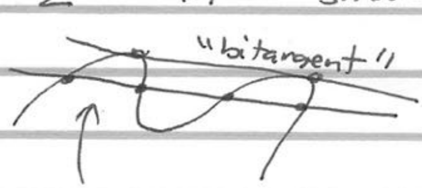
$K \to \Sigma$ canonical line bundle

theta characteristic $L$ is holomorphic line bdle s.t. $L^{\otimes 2} \simeq K$

$\Rightarrow$ there exactly $2^{2g}$ of them

$\hookrightarrow$ $H^0(L)$ space of holomorphic sections of $L$ (finite dim'l)
Interesting for classical problems in the geometry of algebraic curves.

Thm (Jacobi ~1850s)
$\Sigma \subset \mathbb{C}P^2$ smooth of degree 4, then $\exists$ exactly 28 bitangents




line $\cap \Sigma$ = 4 points by Bezout

$g(\Sigma) = 3 = \frac{(4-1)(4-2)}{2} \rightsquigarrow 2^6 = 64$ theta char

28 of these have dim $H^0(L) = 1$ $\rightsquigarrow$ the 28 bitangents

36 have dim $H^0(L) = 0$

In gen'l, dim $H^0(L)$ depends on the Riemann surface structure! (in a subtle way)

(Mumford) dim $H^0(L)$ mod 2 does not! (ie, when deform along family)
$\hookrightarrow$ odd/even

$\varphi \circlearrowright \Sigma$ automorphism $\Rightarrow \varphi \circlearrowright \{\text{theta char}\}$
$\Rightarrow$ (Serre) $\exists$ fixed points, say $\varphi$ fixes $L$
$\overset{\exists \text{ lift}}{\rightsquigarrow} \tilde{\varphi} : L \circlearrowleft \rightsquigarrow \tilde{\varphi}^* : H^0(L) \circlearrowleft$ finite order

$\rightsquigarrow \mathrm{Spec}(\tilde{\varphi}^*) \subset S^1$ (with multiplicities)
spectrum (eigenvalues)

On Floer homology:

$L$ $\varphi$ invariant $\rightsquigarrow$ spin$^c$ structure $S_L$ on $M_L$

$\rightsquigarrow$ $HM(M_\varphi, S_L) \to$ f.g. $\mathbb{R}[u]$-module

↑ monopole Floer homology

⇝ topological invariant!

Thm: $\varphi \circlearrowright \Sigma$, $\Sigma/\varphi = \mathbb{P}^1$, $L$ invariant theta char.

Then $\underbrace{\mathrm{Spec}(\tilde{\varphi}_*)}_{\text{geometric}}$ completely determines $\underbrace{HM(M_\varphi, S_L)}_{\text{topological}}$

For example

1) $\dim_{\mathbb{R}} HM(M_\varphi, S_L) = \dim_{\mathbb{C}} H^0(L)$

2) if $HM(M_\varphi, S_L)$ is cyclic $\Rightarrow$ $\pm\,\mathrm{Spec}(\tilde{\varphi}^*) \subseteq$ upper half plane
(can use, eg, surgery to compute)

Atiyah's proof (70s)
of Mumford's Thm

$$\begin{array}{ccc} \{\text{theta char}\} & \overset{1:1}{\longleftrightarrow} & \{\text{spin structures } s \text{ on } \Sigma\} \\ \downarrow & & \downarrow \\ \dim H^0(L) & \xrightarrow{\text{mod } 2} & \mathrm{Arf}(\Sigma, s) \in \mathbb{Z}_2 \\ \in \mathbb{N} & & \end{array}$$

⇝ using $\bar{\partial}$ = Dirac op in dim 2

↳ key protagonist
Seiberg–Witten eqns.