Numerical Methods for Ordinary Differential Equations

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Introduction

- What is an ODE?
  - Ordinary Differential Equations model dynamic systems
  - Set of equations all dependent on a single variable (Generally time)
- Why use numerical methods?
  - Save computational power over producing exact solutions
  - Simplify complex definite integrals

Problem Statement

- Numerical Methods for Integration and ODE
  - Part 1: Implement numerical methods to approximate definite integral
  - Part 2: Use Simpson's rule to approximate bags of mulch needed to fill an irregularly shaped plot
  - Part 3: Use Euler's method to Estimate population growth

Methods and Analysis: Part 1

- Solve Definite integral

  $\int_0^1 e^{-x^2} dx$

- Numerical Methods:
  - Midpoint Rule:

    $$\int_a^b f(x)dx \approx h\sum_{i=1}^{N} f(x_i), h = \frac{b-a}{N}, x_i = a + \frac{2i-1}{2}h$$

  - Simpson's Rule

    $$\int_a^b f(x)dx \approx \frac{h}{3}[f(x_0) + 4\sum_{i=1,3,5}^{N-1} f(x_i) + 2\sum_{i=2,4,6}^{N-2} f(x_i) + f(x_N), h = \frac{b-a}{N}, x_i = a + ih$$

  - Trapezoid Rule

    $\int_a^b f(x)dx \approx h[\frac{1}{2}f(x_0)$

  - 2-point Gauss-Quadrature Rule

$$\int_a^b f(x)dx \approx \frac{h}{2}\sum_{i=1}^{N}[f(x_i + f(x_r)]), x_l = \frac{1}{2}\left(x_i + x_{i-1} - \frac{h}{\sqrt{3}}\right), x_l = \frac{1}{2}\left(x_i + x_{i-1} + \frac{h}{\sqrt{3}}\right), h = \frac{b-a}{N}, x_i = a + \frac{2i-1}{2}h$$

Methods and Analysis: Part 2

Use Simpson’s rule to approximate bags of mulch needed to fill an irregularly shaped plot

- 3 inch covering of mulch over entire plot
- 3 cubic feet of Bark mulch per bag

Dimensions

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
y	8	10	11	15	16	16	16	16	15.5	15.5	15.5	15	15	15	14.5	14.5	14

x	17	18	19	20	21	22	23	24	25	26
y	14	14	13.5	13	13	13	12	11	19	6

Simpson's Rule

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[f(x_0) + 4\sum_{i=1,3,5}^{N-1} f(x_i) + 2\sum_{i=2,4,6}^{N-2} f(x_i) + f(x_N)\right],\ h = \frac{b-a}{N},\ x_i = a+ih.$$

Methods and Analysis: Part 3

- Uses Euler's Method to Estimate The World's Population from 1950 to 2020.

Given Data:

- p_0(in 1950) $= 2555$ million people
- $k_{rm} = 0.026/\text{yr}$
- $p_{max} = 12000$ million people

The differential equation is as follows:

$$\frac{dp}{dt} = k_{rm}\left(1 - \frac{p}{p_{\max}}\right)p$$

The Euler's Method is as follows:

$$p_{n+1} = p_n + h\,f(p_n)$$

Year	Population (millions)
1950	2555
1960	3040
1970	3708
1980	4454
1990	5276
2000	6079
2010	6922
2020	7753

Table 1: Given world population data from 1950–2020.

Iterations (N)	Midpoint	Simpson's	Trapezoid	Gauss
100	0.756827	0.753083	0.756818	0.746824
500	0.751825	0.748086	0.748824	0.746824
1000	0.747824	0.747456	0.747824	0.746824
2000	0.747324	0.747140	0.747324	0.746824
5000	0.747024	0.746951	0.747024	0.746824
10000	0.746924	0.746887	0.746924	0.746824

Figure 1: table showing values of x for each numerical method at different values of N

Results: Part 1

Considering exact answers:

x = 0.74682413

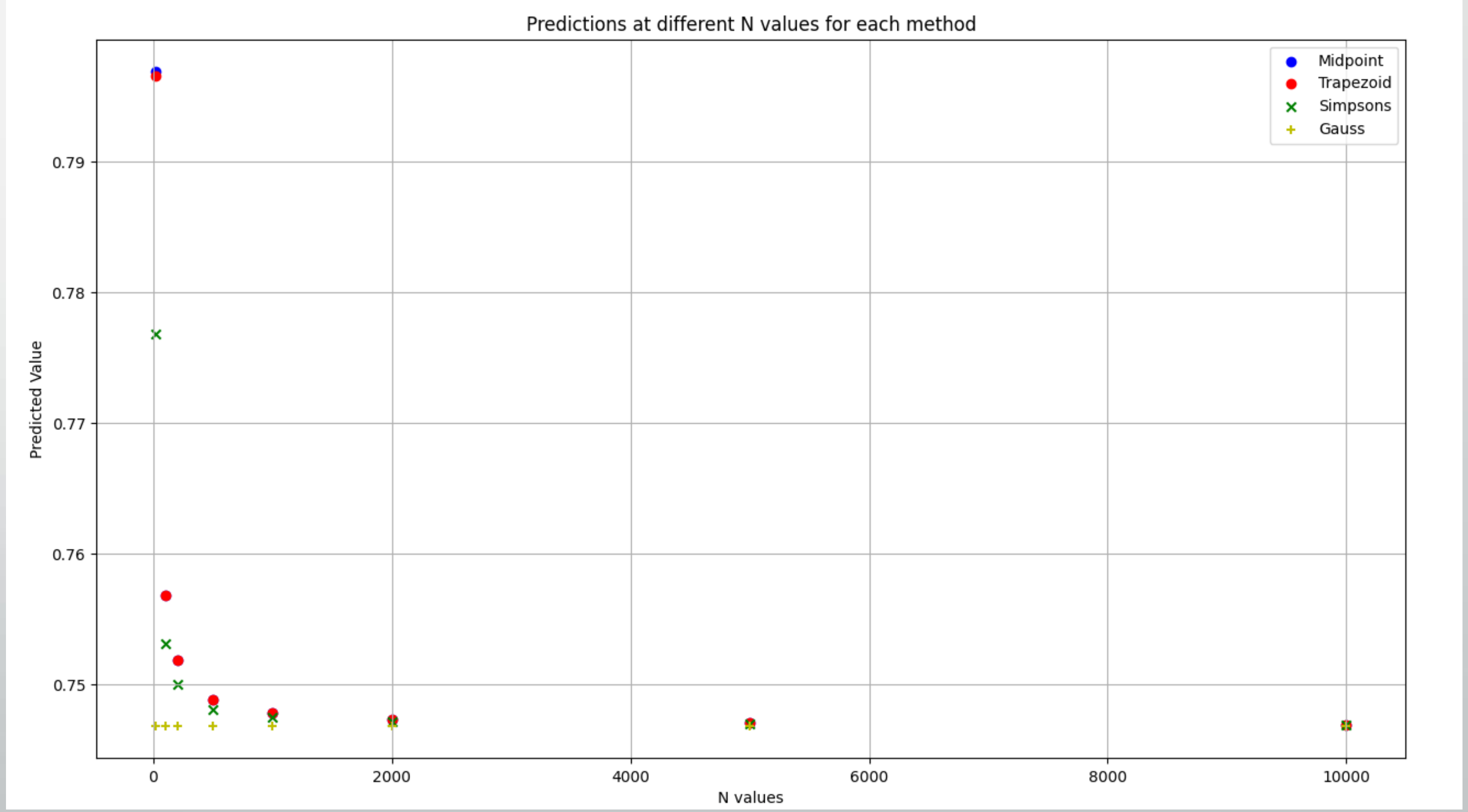

Figure 2: plot of predicted values of x with increasing N values

Results

Considering Exact Answers:

- Part 2

Area of land: 359.82716049382714 ft2

Volume of Mulch: 89.95679012345678 ft^3

Bags needed: 30

Results

Considering exact answers:

- Part 3:

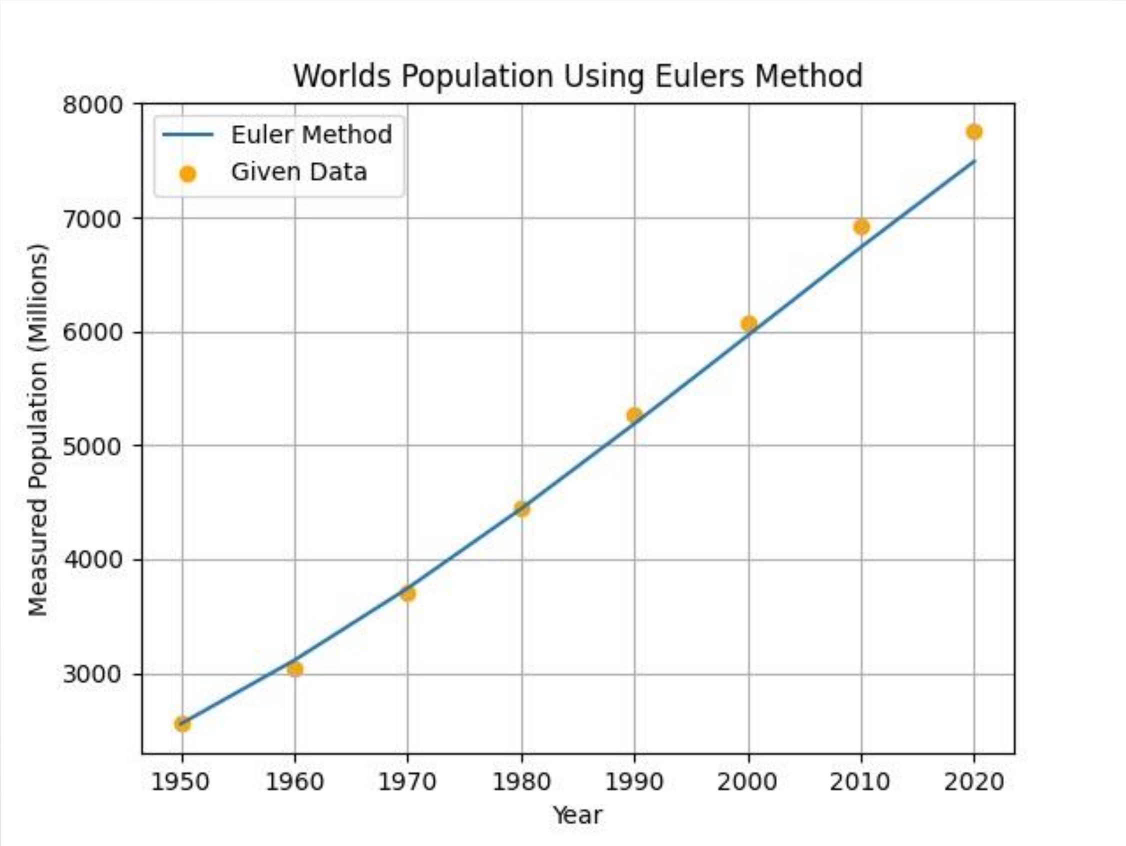

Figure 2: Plot of exact population data vs. Euler method approximation

Discussion: Part 1

- Considering exact answer x = 0.74682413
  - 2-Point Gauss Quadrature Rule is the most accurate
  - Midpoint and Trapezoid rules are nearly identical – also least accurate

N	Midpoint	Simpson's	Trapezoid	Gauss
100	1.339	0.8381	1.33815	$4.561e^{-10}$
500	0.2678	0.1690	0.2678	$7.284e^{-13}$
1000	0.1139	0.08456	0.1339	$2.973e^{-14}$
2000	0.06695	0.04230	0.06695	0
5000	0.02678	0.01693	0.02678	$1.4866e^{-14}$
10000	0.01339	0.008463	0.01339	$1.4866e^{-14}$

Figure 4: table showing percent difference of each numerical method at different values of N

Discussion: Part 3

For Part 3, Euler's Method shows that the population trend provides growth from 1950 to 2020. However, the first-order nature of Euler's Method gives inaccuracy the longer the time period. As far as the scope of this class, Euler's Method is adequate.

Conclusion

- 2-Point Gauss Quadrature Rule is the most accurate numerical method for evaluating ODEs
- Simpson's Rule is useful for approximating area and volumetric sums while maintaining simple algorithms and low computing power
- Euler's method, while more complicated, is adequate at reproducing results of an exponential growth function

References