

# CASS Completion Notes

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The goal here is to derive a closed-form solution to the  $Y$ -update of the completion sub-problem

$$\underset{Y}{\text{minimize}} \lambda/2 \|(YC) \odot W\|_F^2 + \mu/2 \sum_{i=1}^N \|L^i - Y \text{diag}(c_i) + U^i\|_F^2 \quad \text{s.t.} \quad P_\Omega(Y - X) = 0.$$

We will solve this problem separately for each row  $y_j$ ,  $j = 1, \dots, D$ . Let  $x_j$ ,  $w_j$ ,  $\omega_j$ ,  $\ell_j^i$ ,  $u_j^i$  similarly be rows for the corresponding matrices. We now want to solve

$$\underset{y_j}{\text{minimize}} \lambda/2 \|(\text{diag}(w_j)C^\top)y_j\|_2^2 + \mu/2 \sum_{i=1}^N \|\text{diag}(c_i)y_j - (\ell_j^i + u_j^i)\|_2^2 \quad \text{s.t.} \quad P_{\omega_j}(y_j - x_j) = 0.$$

Dropping subscripts and abbreviating  $A = \text{diag}(w_j)C^\top$ ,  $d_i = \ell_j^i + u_j^i$ , the problem becomes

$$\underset{y}{\text{minimize}} \lambda/2 \|Ay\|_2^2 + \mu/2 \sum_{i=1}^N \|\text{diag}(c_i)y - d_i\|_2^2 \quad \text{s.t.} \quad P_\omega(y - x) = 0.$$

Now, to encode the linear  $P_\omega$  constraint, we can pull out the constrained values from  $y$

$$\underset{y_{\omega^c}}{\text{minimize}} \lambda/2 \|A_{\omega^c}y_{\omega^c} + A_\omega x_\omega\|_2^2 + \mu/2 \sum_{i=1}^N \|\text{diag}(c_i)_{\omega^c}y_{\omega^c} + \text{diag}(c_i)_\omega x_\omega - d_i\|_2^2.$$

Next, form the first order optimality condition

$$\begin{aligned} \lambda A_{\omega^c}^\top (A_{\omega^c}y_{\omega^c} + A_\omega x_\omega) + \mu \sum_i \text{diag}(c_i)_{\omega^c}^\top (\text{diag}(c_i)_{\omega^c}y_{\omega^c} + \text{diag}(c_i)_\omega x_\omega - d_i) &= 0 \\ (\lambda A_{\omega^c}^\top A_{\omega^c} + \mu \sum_i \text{diag}(c_i)_{\omega^c}^\top \text{diag}(c_i)_{\omega^c})y_{\omega^c} &= -\lambda A_{\omega^c}^\top (A_\omega x_\omega) - \mu \sum_i \text{diag}(c_i)_{\omega^c}^\top (\text{diag}(c_i)_\omega x_\omega - d_i) \\ (\lambda A_{\omega^c}^\top A_{\omega^c} + \mu \text{diag}((C^{\circ 2} \mathbf{1})_{\omega^c}))y_{\omega^c} &= -\lambda A_{\omega^c}^\top (A_\omega x_\omega) + \mu \sum_i \text{diag}(c_i)_{\omega^c}^\top d_i \\ (\lambda A_{\omega^c}^\top A_{\omega^c} + \mu \text{diag}((C^{\circ 2} \mathbf{1})_{\omega^c}))y_{\omega^c} &= -\lambda A_{\omega^c}^\top (A_\omega x_\omega) + \mu ((C \odot D) \mathbf{1})_{\omega^c} \end{aligned}$$

where the second to last line holds in part because  $\text{diag}(c_i)_{\omega^c}^\top \text{diag}(c_i)_\omega = 0$ , and  $D = [d_1 \dots d_N]$  is the  $j$ th row-slice of the 3d tensor  $L + U$ , arranged in a matrix.