CASS ALS L update

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Here I want to accelerate the very slow N-fold singular-value thresholding step that appears both in CASS self-expression and completion, and takes $\sim 80\%$ compute time. The idea will be to solve factorized variants of the problem jointly, for all $i = 1, \ldots, N$, using alternating-least-squares updates.

The optimization problem we want to solve is

$$\underset{L_i}{\text{minimize}} \, \|L_i\|_* + \lambda/2 \|L_i - Z_i\|_F^2$$

for each $i = 1, \ldots, N$. By introducing a rank-constraint $\operatorname{rank}(L_i) \leq d$, this problem becomes equivalent to

$$\underset{U_i \in \mathbb{R}^{D \times d}, V_i \in \mathbb{R}^{N \times d}}{\text{minimize}} \, 1/2(\|U_i\|_F^2 + \|V_i\|_F^2) + \lambda/2 \|U_i V_i^\top - Z_i\|_F^2.$$

We can solve this problem by alternating least squares, as in (Hastie et al., JMLR 2015). The alternating updates on the kth iteration are given by the linear equations

$$U_i^{k+1}(I + \lambda (V_i^k)^\top V_i^k) = \lambda Z_i V_i^k \qquad V_i^{k+1}(I + \lambda (U_i^{k+1})^\top U_i^{k+1}) = \lambda Z_i^\top U_i^{k+1}.$$

Now dropping k superscripts, and form concatenated matrices $\bar{U} = [U_1 \cdots U_N] \in \mathbb{R}^{D \times dN}$, $\bar{V} = [V_1 \cdots V_N] \in \mathbb{R}^{N \times dN}$, and $\bar{Z} = [Z_1 \cdots Z_N] \in \mathbb{R}^{D \times N^2}$. Similarly, define block diagonal matrices $\hat{U} \in \mathbb{R}^{DN \times dN}$, $\hat{V} \in \mathbb{R}^{N^2 \times dN}$. Let $\bar{Z}^* = [Z_1^\top \cdots Z_N^\top] \in \mathbb{R}^{N \times DN}$. Then the above equations can be combined in a single expression

$$\bar{U}(I + \lambda \hat{V}^\top \hat{V}) = \lambda \bar{Z} \hat{V} \qquad \bar{V}(I + \lambda \hat{U}^\top \hat{U}) = \lambda \bar{Z}^* \hat{U}.$$