CASS Completion Notes

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The goal here is to derive a closed-form solution to the Y-update of the completion sub-problem

$$\underset{Y}{\text{minimize}}\, \lambda/2\|(YC)\odot W\|_F^2 + \mu/2\sum_{i=1}^{N}\|L^i - Y\operatorname{diag}(c_i) + U^i\|_F^2 \quad \text{s.t.} \quad P_\Omega(Y-X)=0.$$

We will solve this problem separately for each row y_j, $j=1,\ldots,D$. Let x_j, w_j, ω_j, ℓ_j^i, u_j^i similarly be rows for the corresponding matrices. We now want to solve

$$\underset{y_j}{\text{minimize}}\, \lambda/2\|(\operatorname{diag}(w_j)C^\top)y_j\|_2^2 + \mu/2\sum_{i=1}^{N}\|\operatorname{diag}(c_i)y_j - (\ell_j^i + u_j^i)\|_2^2 \quad \text{s.t.} \quad P_{\omega_j}(y_j - x_j)=0.$$

Dropping subscripts and abbreviating $A=\operatorname{diag}(w_j)C^\top$, $d_i=\ell_j^i+u_j^i$, the problem becomes

$$\underset{y}{\text{minimize}}\, \lambda/2\|Ay\|_2^2 + \mu/2\sum_{i=1}^{N}\|\operatorname{diag}(c_i)y - d_i\|_2^2 \quad \text{s.t.} \quad P_\omega(y-x)=0.$$

Now, to encode the linear P_ω constraint, we can pull out the constrained values from y

$$\underset{y_{\omega^c}}{\text{minimize}}\, \lambda/2\|A_{\omega^c}y_{\omega^c} + A_\omega x_\omega\|_2^2 + \mu/2\sum_{i=1}^{N}\|\operatorname{diag}(c_i)_{\omega^c}y_{\omega^c} + \operatorname{diag}(c_i)_\omega x_\omega - d_i\|_2^2.$$

Next, form the first order optimality condition

$$\lambda A_{\omega^c}^\top(A_{\omega^c}y_{\omega^c} + A_\omega x_\omega) + \mu\sum_i \operatorname{diag}(c_i)_{\omega^c}^\top(\operatorname{diag}(c_i)_{\omega^c}y_{\omega^c} + \operatorname{diag}(c_i)_\omega x_\omega - d_i) = 0$$

$$(\lambda A_{\omega^c}^\top A_{\omega^c} + \mu\sum_i \operatorname{diag}(c_i)_{\omega^c}^\top\operatorname{diag}(c_i)_{\omega^c})y_{\omega^c} = -\lambda A_{\omega^c}^\top(A_\omega x_\omega) - \mu\sum_i \operatorname{diag}(c_i)_{\omega^c}^\top(\operatorname{diag}(c_i)_\omega x_\omega - d_i)$$

$$(\lambda A_{\omega^c}^\top A_{\omega^c} + \mu\operatorname{diag}((C^{\circ 2}\mathbf{1})_{\omega^c}))y_{\omega^c} = -\lambda A_{\omega^c}^\top(A_\omega x_\omega) + \mu\sum_i \operatorname{diag}(c_i)_{\omega^c}^\top d_i$$

$$(\lambda A_{\omega^c}^\top A_{\omega^c} + \mu\operatorname{diag}((C^{\circ 2}\mathbf{1})_{\omega^c}))y_{\omega^c} = -\lambda A_{\omega^c}^\top(A_\omega x_\omega) + \mu((C\odot D)\mathbf{1})_{\omega^c}$$

where the second to last line holds in part because $\operatorname{diag}(c_i)_{\omega^c}^\top\operatorname{diag}(c_i)_\omega = 0$, and $D=[d_1\ldots d_N]$ is the jth row-slice of the 3d tensor $L+U$, arranged in a matrix.