Signal à Temps discret.

1) Définition, propriété

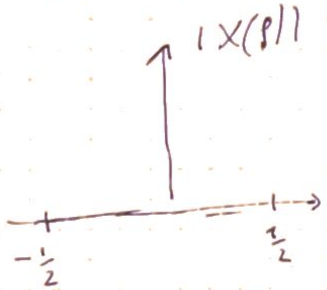
$x(t)$: Temps continue $\xrightarrow{TF}$ $X(f) = \int_{-\infty}^{+\infty} x(t) e^{-2\pi j f t} dt$

x_k, $k \in \mathbb{Z}$: Temps discret $\quad X(f) = \sum_{k=-\infty}^{+\infty} x_k e^{-2\pi j k f}$

$$X(f+1) = \sum_{k=-\infty}^{+\infty} x_k e^{-2\pi j k (f+1)}$$

$$= \sum_{k=-\infty}^{+\infty} x_k e^{-2\pi j k f} \underbrace{e^{-2\pi j k}}_{1} = X(f)$$

Les $X(f)$ sont périodiques de période 1



$$x_k = \int_{-1/2}^{1/2} X(f) e^{2\pi j k f} df$$

Produit de convolution

TC $\quad x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau$

TD $\quad x_k * h_k = \sum_{m=-\infty}^{+\infty} x_m h_{k-m}$

2) Transformé en $\mathcal{Z}$ (T z)

$$x_k , k \in \mathbb{Z} \xrightarrow{Tz} X(z) = \sum_{k=-\infty}^{+\infty} x_k z^{-k}$$

$$z \in \mathbb{C}$$

On doit associé un domaine de convergence.

$$X(z) = \sum_{k=-\infty}^{-1} x_k z^{-k} + \sum_{k=0}^{+\infty} x_k z^{-k}$$

$$\left| \sum_{k=0}^{\infty} x_k z^{-k} \right| \leq \sum_{k=0}^{\infty} |x_k| |z^{-1}|^k \qquad |z| > |R_1|$$

$$\left| \sum_{k=-\infty}^{-1} x_k z^{-k} \right| \leq \sum_{k=1}^{\infty} |x_{-k}| |z|^k \qquad |z| < |R_2|$$

$$\Rightarrow \quad |r_1| < |z| < |r_2|$$

Exemple

$$x_k = a^k u_k \qquad \text{où} \quad u_k = \begin{cases} 1 & \text{si } k \geq 0 \\ 0 & \text{sinon} \end{cases}$$

$$X(z) = \sum_{k=-\infty}^{+\infty} a^k u_k z^{-k}$$

$$= \sum_{k=0}^{+\infty} a^k z^{-k} = \sum_{k=0}^{+\infty} (a z^{-1})^k$$

converge ssi $\frac{a}{z} < 1$

$$\Rightarrow |z| > |a|$$

$$= \frac{1}{1 - az^{-1}}$$

$$y_k = -a^k u_{-k-1} \qquad u_{-k-1} = \begin{cases} 1 & \text{si } -k-1 \geq 0 \\ 0 & \text{sinon} \end{cases} \quad k \leq -1$$

$$= \begin{cases} 1 & \text{si } -k \geq 1 \\ 0 & \text{sinon} \end{cases}$$

$$Y(z) = -\sum_{k=-\infty}^{-1} a^k z^{-k}$$

$$= -\sum_{k=1}^{\infty} a^{-k} z^k$$

$$= -\left[\sum_{k=0}^{+\infty} (a^{-1}z)^k - 1 \right]$$

converge ssi $|a^{-1}z| < 1$

$|z| < |a|$

$$= -\left[\frac{1}{1-a^{-1}z} - 1 \right] = -\frac{1-(1-a^{-1}z)}{1-a^{-1}z}$$

$$= -\frac{z/a}{1-z/a} = -\frac{1/a}{z^{-1}-1/a} = -\frac{1}{az^{-1}-1}$$

$$= \frac{1}{1-az^{-1}}$$

$$a^k u_k \xrightarrow{TZ} \frac{1}{1-az^{-1}} \qquad |z| > |a|$$

$$-a^k u_{-k-1} \xrightarrow{TZ} \frac{1}{1-az^{-1}} \qquad |z| < |a|$$

$$\begin{cases} x(t) \text{ causal} \Rightarrow x(t) = 0 \text{ si } t \leq 0 \\ x(t) \text{ anticausal} \Rightarrow x(t) = 0 \text{ si } t \geq 0 \end{cases}$$

x_k est un signal causal si $x_k = 0$ si $k \leq -1$

x_k est un signal anti-causal si $x_k = 0$ si $k \geq 0$

$$X(z) = \sum_{k=-\infty}^{-1} x_k z^{-k} + \sum_{k=0}^{+\infty} x_k z^{-k}$$

signal causal $X(z) = \sum_{k=0}^{+\infty} x_k z^{-k}$

signal anticausal $X(z) = \sum_{k=-\infty}^{-1} x_k z^{-k}$

En résumé :

$$\frac{1}{1-az^{-1}} \xrightarrow{TZ^{-1}} a^k u_k \text{ si } |z| > |a|$$

$$\frac{1}{1-az^{-1}} \xrightarrow{TZ^{-1}} -a^k u_{-k-1} \text{ si } |z| < |a|$$

Exemple

- $X(z) = \dfrac{2}{(1-\frac{1}{2}z^{-1})(1-3z^{-1})}$

$$(1-\tfrac{1}{2}z^{-1})X(z) = \frac{a\,(1-\frac{1}{2}z^{-1})}{(1-\frac{1}{2}z^{-1})} + \frac{b\,(1-\frac{1}{2}z^{-1})}{(1-3z^{-1})}$$

$$\Rightarrow \frac{2}{1-3z^{-1}} = a + b\,\frac{(1-\frac{1}{2}z^{-1})}{1-3z^{-1}}$$

on prend $z^{-1} = 2$ pour plus avoir le b

$$\Rightarrow \left.\frac{2}{1-3z^{-1}}\right|_{z^{-1}=2} = a$$

$$a = \frac{2}{1-6} = -\frac{2}{5}$$

De même pour b

$$b = \left.\frac{2}{1-\frac{1}{2}z^{-1}}\right|_{z^{-1}=\frac{1}{3}}$$

$$= \frac{2}{1-\frac{1}{6}} = \frac{12}{5}$$

donc $X(z) = -\frac{2}{5}\frac{1}{1-\frac{1}{2}z^{-1}} + \frac{12}{5}\frac{1}{1-3z^{-1}}$

$$\frac{1}{1-\frac{1}{2}z^{-1}} \nearrow \left(\frac{1}{2}\right)^k u_k \quad \text{si } |z| > \left|\frac{1}{2}\right|$$ * signal causal

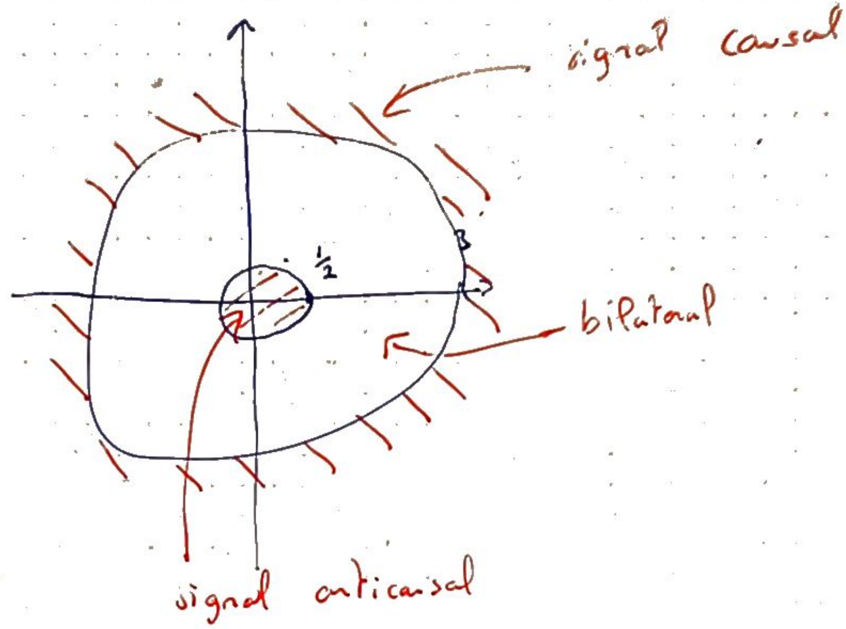
$$\frac{1}{1-\frac{1}{2}z^{-1}} \searrow -\left(\frac{1}{2}\right)^k u_{-k-1} \quad \text{si } |z| < \left(\frac{1}{2}\right)$$ Δ signal anticausal

$$\frac{1}{1-3z^{-1}} \nearrow 3^k u_k \quad \text{si } |z| > |3|$$ * signal causal

$$\frac{1}{1-3z^{-1}} \searrow -3^k u_{-k-1} \quad \text{si } |z| < |3|$$ Δ signal anticausal

* $\Rightarrow$ si $|z| > 3$ $\quad x_k = -\frac{2}{5}\left(\frac{1}{2}\right)^k u_k + \frac{12}{5}3^k u_k$ signal causal

Δ $\Rightarrow$ si $|z| < \frac{1}{2}$ $\quad x_k = \frac{2}{5}\left(\frac{1}{2}\right)^k u_{-k-1} - \frac{12}{5}3^k u_{-k-1}$ signal anticausal


signal causal
$\frac{1}{2}$
3
bilateral
signal anticausal

Notation

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

$$X(z) = Q(z) + \sum_{k=1}^{M-N} \frac{\alpha_k}{1 - p_k z^{-1}}$$

où p_k : pôle

un pôle annule $D(z)$

zéro : annule $N(z)$

Si $M \geq N$ on effectue une division polynomiale

$$X(z) = Q(z) + \frac{N'(z)}{D(z)} \qquad \deg N'(z) < \deg D(z)$$

Exemple

$$X(z) = \frac{2 - \frac{1}{4} z^{-1} - \frac{1}{2} z^{-2} - \frac{3}{16} z^{-3} + \frac{1}{16} z^{-4}}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 + \frac{1}{2} z^{-1}\right)\left(1 - \frac{1}{4} z^{-1}\right)}$$

$\deg N(z) = 4 \qquad \deg D(z) = 3$

$p_1 = \frac{1}{2}$

$p_2 = -\frac{1}{2}$

$p_3 = \frac{1}{4}$

$$
\begin{array}{r|l}
\frac{1}{16}z^{-4} - \frac{3}{16}z^{-3} - \frac{1}{2}z^{-2} - \frac{1}{4}z^{-1} + 2 & \frac{1}{16}z^{-3} - \frac{1}{4}z^{-2} - \frac{1}{4}z^{-1} + 1 \\
\underline{-\frac{1}{16}z^{-4} + \frac{1}{4}z^{-3} + \frac{1}{4}z^{-2} - z^{-1}} & \overline{z^{-1} + 1} \\
\frac{1}{16}z^{-3} - \frac{1}{4}z^{-2} - \frac{3}{4}z^{-1} + 2 & \quad q \\
\underline{-\frac{1}{16}z^{-3} + \frac{1}{4}z^{-2} + \frac{1}{4}z^{-1} - 1} & \\
-z^{-1} + 1 &
\end{array}
$$

$$X(z) = 1 + z^{-1} + \frac{1 - z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}$$

décomposition en élément simple

et donc

$$\frac{1 - z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)} = \frac{\alpha}{1 - \frac{1}{2}z^{-1}} + \frac{\beta}{1 + \frac{1}{2}z^{-1}} + \frac{\gamma}{1 - \frac{1}{4}z^{-1}}$$

$$\alpha = \left.\frac{1 - z^{-1}}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}\right|_{z^{-1} = 2}$$

$$\beta = \left.\frac{1 - z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}\right|_{z = -2}$$

$$\gamma = \left.\frac{1 - z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)}\right|_{z = 4}$$

$$\frac{\alpha}{1-\frac{1}{2}z^{-1}} \begin{cases} \to \alpha\left(\frac{1}{2}\right)^k u_k & \text{si } |z| > \frac{1}{2} \\ \to -\alpha\left(\frac{1}{2}\right)^k u_{-k-1} & \text{si } |z| < \frac{1}{2} \end{cases}$$

$$\frac{\beta}{1+\frac{1}{2}z^{-1}} \begin{cases} \to \beta\left(-\frac{1}{2}\right)^k u_k & \text{si } |z| > \frac{1}{2} \\ \to -\beta\left(-\frac{1}{2}\right)^k u_{-k-1} & \text{si } |z| < \frac{1}{2} \end{cases}$$

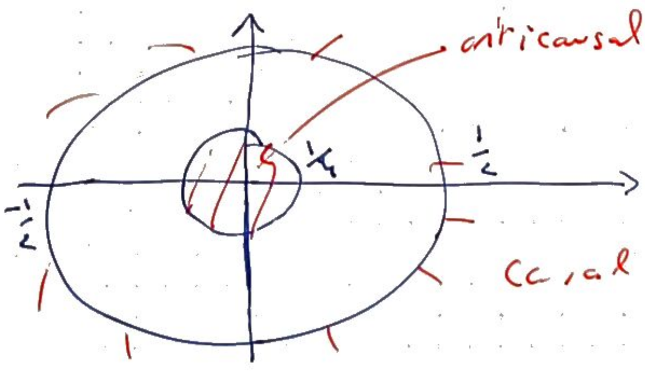
$$\frac{\gamma}{1-\frac{1}{4}z^{-1}} \begin{cases} \to \gamma\left(\frac{1}{4}\right)^k u_k & \text{si } |z| > \frac{1}{4} \\ \to -\gamma\left(\frac{1}{4}\right)^k u_{-k-1} & \text{si } |z| < \frac{1}{4} \end{cases}$$

si $|z| > \frac{1}{2}$

$$\left[\alpha\left(\frac{1}{2}\right)^k + \beta\left(\frac{-1}{2}\right)^k + \gamma\left(\frac{1}{4}\right)^k\right]u_k \quad \text{causal}$$

si $|z| < \frac{1}{4}$

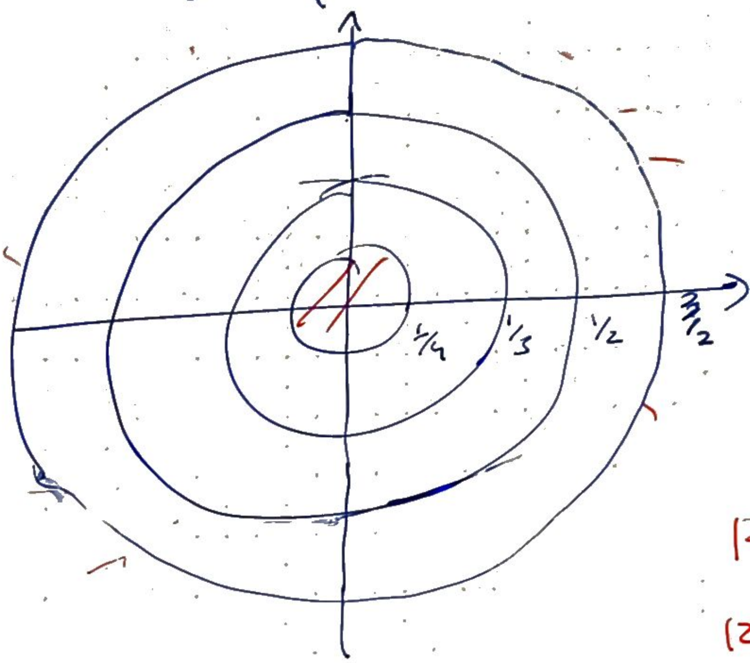
$$-\left[\alpha\left(\frac{1}{2}\right)^k + \beta\left(-\frac{1}{2}\right)^k + \gamma\left(\frac{1}{4}\right)^k\right]u_{-k-1} \quad \text{anticausal}$$



n pôles différents $\Rightarrow$ $n+1$ rayon de convergence

Dans l'exemple précédent, on avait 2 poles identiques donc 2 régions identique.

Mais on a toujours qu'un causal et qu'un anticausal



$p_1 = \frac{1}{4}$

$p_2 = \frac{1}{3}$

$p_3 = \frac{1}{2}$

$p_4 = \frac{3}{2}$

$|z| < \frac{1}{4}$ anticausal

$|z| > \frac{3}{2}$ causal

que des bilatéraux aux milieu

$$\delta_k \xrightarrow{T_z} \sum_k \delta_k z^{-k} = 1 \quad \text{car} \quad \delta_k = \begin{cases} 1 & \text{si } k=0 \\ 0 & \text{sinon} \end{cases}$$

$$x_k \xrightarrow{TZ} X(z)$$

$$y_k = x_{k-m} \qquad m > 0$$

$$Y(z) = \sum_{k=-\infty}^{+\infty} x_{k-m} z^{-k}$$ on pose $l = k - m$

$$Y(z) = \sum_{l=-\infty}^{+\infty} x_l \, z^{-(l+m)}$$

$$= z^{-m} \sum_{l=-\infty}^{+\infty} x_l \, z^{-l}$$

$$= z^{-m} X(z)$$

si $m > 0$ $\quad x_{k-m} \xrightarrow{TZ} z^{-m} X(z)$

$$1 + z^{-1} \xrightarrow{(TZ)^{-1}} \delta_k + \delta_{k-1}$$

$$w_k = x_{k+m} \qquad m > 0$$

$$W(z) = \sum_{k=-\infty}^{+\infty} x_{k+m} z^{-k} \qquad l = k + m$$

$$= \sum_{l=-\infty}^{+\infty} x_l \, z^{-(l-m)}$$

$$= z^{m} X(z)$$

Cas des signaux causaux

- $x_k \xrightarrow{TZ} \sum_{k=0}^{+\infty} x_k z^{-k}$

- $x_{k-m} \xrightarrow{TZ} \sum_{k=0}^{+\infty} x_{k-m} z^{-k}$ $\quad l = k-m$

$$= z^{-m} \sum_{l=-m}^{+\infty} x_l z^{-l}$$

$$= z^{-m} \sum_{l=0}^{+\infty} x_l z^{-l} + z^{-m} \cancel{\sum_{l=-m}^{-1} x_l z^{-l}}$$

- $x_{k+m} \xrightarrow{TZ} \sum_{k=0}^{+\infty} x_{k+m} z^{-l}$ $\quad l = k+m$

$$= \sum_{l=m}^{+\infty} x_l z^{-l+m}$$

$$= z^{m} \sum_{l=m}^{+\infty} x_l z^{-l}$$

$$= X(z) - \sum_{l=0}^{m-1} x_l z^{-l}$$