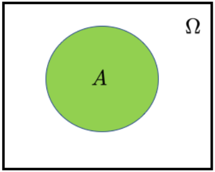
# Axiom 0

**Equally likely scenario**
A real population of people, under purely random sampling
**or imagined population of equally likely possibilities**

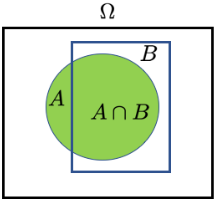



$$P(A) = \frac{|A|}{|\Omega|}.$$

Axiom 0.
Can always translate a problem into equally likely setting.

# Conditional probability

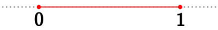
**Equally likely scenario**

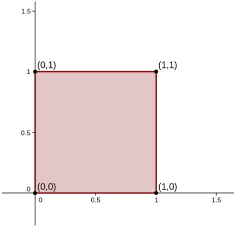


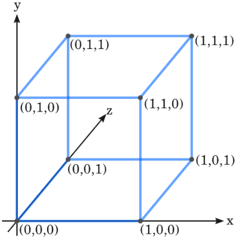

$$P(A|B) = \frac{|A \cap B|}{|B|} = \frac{|A \cap B|/|\Omega|}{|B|/|\Omega|} = \frac{P(A \cap B)}{P(B)}.$$

**Physical:** sample from $B$. $B$ defines condition.
**Mental:** know that $B$ happened, as if sample from $B$.
Axiom 4 or definition of conditional probability.

# Sample space is region

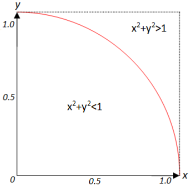



(1) $X$ is uniform random number in $[0, 1]$.
(2) $(X, Y)$ are two independent random numbers in $[0, 1]$.
(3) $(X, Y, Z)$ are three independent random numbers in $[0, 1]$.
$\Omega = [0, 1]$ or $[0, 1]^2$ or $[0, 1]^3$ = set of points.
**Region = population of points** (uncountably infinite).

**Random point in a region**
**Example 3: throwing point into region**

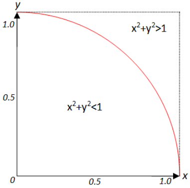



$X$ and $Y$ are independent uniform random numbers in [0, 1].
$(X, Y)$ is a random point in $\Omega = [0, 1]^2$.
$A = \{(x, y) : x^2 + y^2 \leq 1\}$.

$$P(A) = \frac{|A|}{|\Omega|} = \frac{\pi}{4}.$$

$|A|$ is the size of $A$, e.g., area (length, volume).

## Example 3: throwing point into region



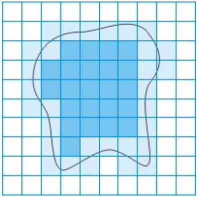

$X$ and $Y$ are independent uniform random numbers in [0, 1].
$(X, Y)$ is a random point in $\Omega = [0,1]^2$.
$A = \{(x, y) : x^2 + y^2 \leq 1\}$.

$$P(X^2 + Y^2 \leq 1) = \pi/4.$$

$$P(X^2 + Y^2 = 1) = 0.$$

Capital letters for random variables.

# Measuring by counting

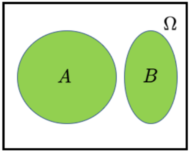


**Discretization $\rightarrow$ finite population of tiny squares.**
**Area** $=$ number of tiny squares $\times$ area of each tiny square.
**Inner measure**: fill inside by tiny squares $\rightarrow$ upper limit.
**Outer measure**: cover outside by tiny squares $\rightarrow$ lower limit.
**Measurable**: inner measure $=$ outer measure.
The collection of all measurable sets, $\sigma$-algebra.
**Integral**: area under curve.

# Axioms



**Probability as measure**, i.e., count, length, area, volume ...
Axiom 0: $P(A) = \frac{|A|}{|\Omega|}$ in equally likely scenario.
**Axiom 1**: $P(\Omega) = 1$.
**Axiom 2**: $P(A) \geq 0$.

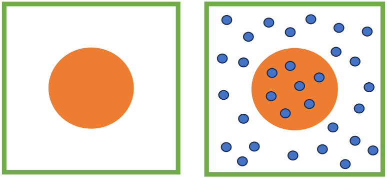



**Axiom 3**: Additivity: If $A \cap B = \phi$ (empty), then

$$P(A \cup B) = P(A) + P(B).$$

Axiom 4: $P(A|B) = \frac{P(A \cap B)}{P(B)}$, assuming $P(B) > 0$.

# Counting repetitions



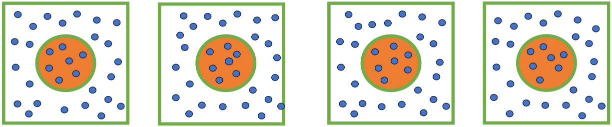
Throw $n$ points into $\Omega$. $m$ of them fall into $A$.

$$P(A) = \frac{|A|}{|\Omega|} \approx \frac{m}{n}.$$

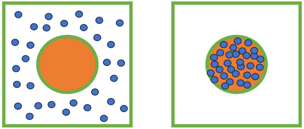
As $n \to \infty$, $\frac{m}{n} \to P(A)$ in probability.
$P(A)$ can be interpreted as **long run frequency**.

# Fluctuations

Repeat random sampling $n$ times independently.
Throw $n$ points into $\Omega$. $m$ of them fall into $A$.
Among all equally likely possibilities, 99.999$\%$ are like below, where $m/n$ is close to $P(A)$.



.00000001$\%$ are like below, where $m/n$ are far from $P(A)$.



Can prove $P(|\frac{m}{n} - P(A)| > \epsilon) \to 0$ for any fixed $\epsilon > 0$.