STATS 100A Homework 2

Instructor: Xiaowu Dai
Due date: Monday, May 5, 2025 at 11:59 pm on Gradescope

Please write down your name and UID clearly.

Notations: boldfaced symbols denote vectors and matrices.

Problem 1 (15 pts)

Suppose we flip a fair coin 100 times independently.

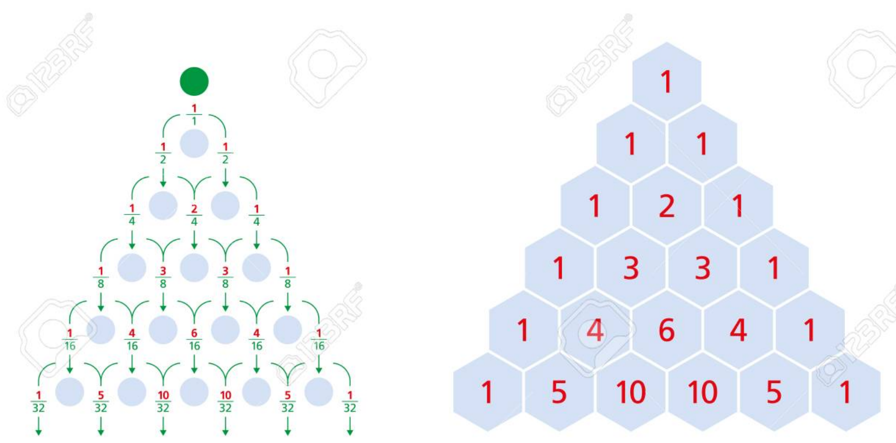
1. What is the probability we get 50 heads?
2. Let X be the number of heads. What is $P(40 \leq X \leq 60)$?
3. Let $Z_i = 1$ if the i-th flip is head, and $Z_i = 0$ if the i-th flip is tail. Express X in terms of Z_i.

Note: For (1) and (2), you only need to write down the math formulas. You do not need to calculate the concrete numbers.

Problem 2 (15 pts)

Draw a Galton board with 5 layers.

1. Write down the corresponding Pascal triangle. Explain the meaning of the numbers.



2. Suppose we label the bins by 0, 1, 2, 3, 4, 5. Calculate the probability that the ball drops into each bin.
3. Suppose we drop 1 million balls. What are the proportions of balls in these bins?

Problem 3 (20 pts)

Consider a random walk on integers. We start from $X_0 = 0$, and at each step, we flip a fair coin. If it is head, we move forward by 1, and if it is tail, we move backward by 1. In math notation, $X_{t+1} = X_t + \epsilon_t$, where $\epsilon_t = 1$ with probability 1/2, and $\epsilon_t = -1$ with probability 1/2.

1. At time $t = 5$, what are the possible values of X_t?

2. What is the probability of each possible value in (1)?

3. What is $P(X_{t+1} = j|X_t = i)$?

4. Interpret (2) in terms of 1 million people doing the random walk simultaneously and independently, all starting from 0.

Problem 4 (20 pts)

Consider a random walk over 3 webpages, 1, 2, 3. At any step, if the person is at webpage 1, then with probability 1/6, she will go to webpage 2, and with probability 1/6, she will go to webpage 3. If the person is at webpage 2, then with probability 1/2, she will go to webpage 1, and with probability 1/2, she will go to webpage 3. If the person is at webpage 3, then with probability 1/2, she will go to webpage 1, and with probability 1/2, she will go to webpage 2.

Let X_t be the webpage the person is browsing at time t, and let us assume she starts from webpage 1 at time 0, i.e., $X_0 = 1$.

1. Let $K_{ij} = P(X_{t+1} = j|X_t = i)$. Let $K = (K_{ij})$ be the 3×3 transition matrix. Write down K.

2. Let $p_i^{(t)} = P(X_t = i)$. Let $p^{(t)} = (p_i^{(t)}, i = 1, 2, 3)$ be the row vector. Calculate $p^{(t)}$ for $t = 1, 2, 3$ using vector matrix multiplication.

3. Let π_i be the stationary distribution at webpage i, so that $\pi_j = \sum_{i=1}^{3} \pi_i K_{ij}$. Let $\pi = (\pi_i, i = 1, 2, 3)$ be the row vector. Then $\pi = \pi K$. Given K, solve π from this equation. Is $p^{(3)}$ close to π?

4. Based on the above calculations, answer the following questions. Suppose there are 1 million people doing the above random walk independently, and suppose they all start from webpage 1 at time $t = 0$. Then on average, what is the distribution of these 1 million people for $t = 1, 2, 3$? What is the stationary distribution of these 1 million people? Which page is the most popular?

Problem 5 (20 pts)

Suppose at any moment, the probability of fire in a classroom is α. Suppose the conditional probability of alarm given fire is β, and the conditional probability of alarm given no fire is γ.

1. Calculate the conditional probability of fire given alarm.

2. Please explain your calculation by counting hypothetical repetitions. For instance, suppose $\alpha = 1/1000$, and $\beta = 99/100$, and $\gamma = 2/100$. Imagine we repeat the experiment 100,000 times. Then on average, 100 times there is fire. Out of these 100 times, 99 times there is alarm, and so on.

Problem 6 (10 pts)

Read about Google pagerank:

`https://en.wikipedia.org/wiki/PageRank.`

Write a brief summary, focusing on Markov chain and random walk conceptualization, and population migration interpretation.