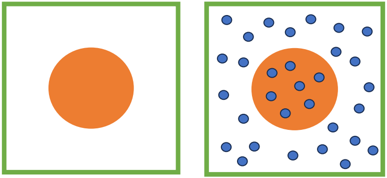
Counting repetitions



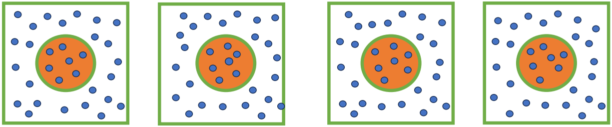
Throw n points into Ω. m of them fall into A.

$$P(A) = \frac{|A|}{|\Omega|} \approx \frac{m}{n}.$$

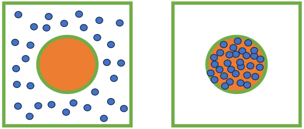
As $n \to \infty$, $\frac{m}{n} \to P(A)$ in probability.
$P(A)$ can be interpreted as **long run frequency**.

Fluctuations

Repeat random sampling n times independently.
Throw n points into Ω. m of them fall into A.
Among all equally likely possibilities, 99.999$\%$ are like below, where m/n is close to $P(A)$.

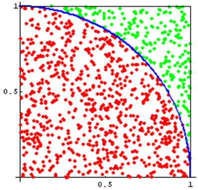


.00000001$\%$ are like below, where m/n are far from $P(A)$.



Can prove $P(|\frac{m}{n} - P(A)| > \epsilon) \to 0$ for any fixed $\epsilon > 0$.

Example 3: π



Throw n points into Ω. m of them fall into A.

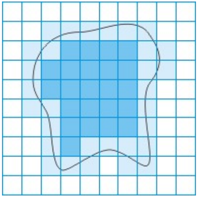
$$P(A) = \frac{|A|}{|\Omega|} = \frac{\pi}{4} \approx \frac{m}{n}.$$

Monte Carlo method:

$$\hat{\pi} = \frac{4m}{n}.$$

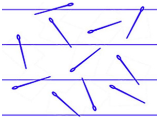
As $n \to \infty$, $\frac{m}{n} \to P(A)$ in probability.
$P(A)$ can be interpreted as **long run frequency**.

Deterministic method



Go over all the $n = 100 = 10^2$ tiny squares, count inner or outer measure, i.e., how many (m) fall into A.
3-dimensional? $n = 10^3$ tiny cubes.
4-dimensional? $n = 10^4$ tiny cells.
10000-dimensional? $n = 10^{10000}$ tiny cells.
Monte Carlo: sample $n = 1000$ points in the hyper-cube. Count how many (m) fall into A.

Example 3: π, buffon needle



Lazzarini threw $n = 3408$ times.

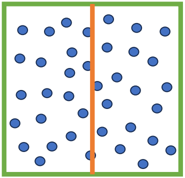
$$P(A) \approx \frac{m}{n}.$$

Monte Carlo method:

$$\hat{\pi} = \frac{355}{113}$$

Too accurate. m is random.
For fixed n, m is random. m/n fluctuates around $P(A)$.
As $n \to \infty$, $\frac{m}{n} \to P(A)$ in probability, law of large number.
$P(A)$ can be interpreted as long run frequency, how often A happens in the long run.

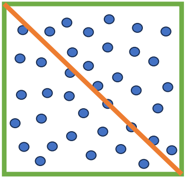
Example 3: throwing point into region



X and Y are independent uniform random numbers in [0, 1].
(X, Y) is a random point in $\Omega = [0, 1]^2$.
$A = \{(x, y) : x < 1/2\}$.

$$P(A) = P(X < 1/2) = \frac{|A|}{|\Omega|} = 1/2.$$

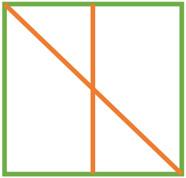
Example 3: throwing point into region



X and Y are independent uniform random numbers in [0, 1].
(X, Y) is a random point in $\Omega = [0,1]^2$.
$B = \{(x, y) : x + y < 1\}$.

$$P(B) = P(X + Y < 1) = \frac{|B|}{|\Omega|} = 1/2.$$

Example 3: throwing point into region

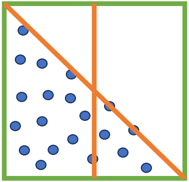


$$P(A|B) = \frac{|A \cap B|}{|B|} = \frac{1/2 - 1/8}{1/2} = 3/4.$$

$$P(X < 1/2 | X + Y < 1).$$

(1) randomly throw a point into B, as if B is the sample space. Then what is the probability the point falls into A?

Example 3: throwing point into region



$$P(A|B) = \frac{|A \cap B|}{|B|} = \frac{1/2 - 1/8}{1/2} = 3/4.$$

$$P(X < 1/2 | X + Y < 1).$$

(2) Consider throwing a lot of points into Ω.
How often A happens? How often B happens?
When B happens, how often A happens?
Among all the points in B, what is the fraction belongs to A?

Coin flipping

Experiment → outcome → number

Example 4: Coin flipping

(4.1) Flip a coin → head or tail → 1 or 0

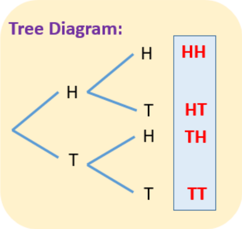
(4.2) Flip a coin twice → (head, head), or (head, tail), or (tail, head) or (tail, tail) → 11 or 10 or 01 or 00

List:

HH HT TH TT

Table:

	H	T
H	HH	HT
T	TH	TT

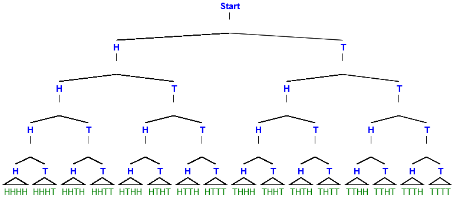


The sample space is {HH, HT, TH, TT}

Experiment $\rightarrow$ outcome $\rightarrow$ number

Example 4: Coin flipping

(4.3) Flip a coin n times $\rightarrow$ 2^n binary sequences.



Sample space Ω: all 2^n sequences.

Each $\omega \in \Omega$ is a sequence.

Randomly pick a sequence from 2^n sequences.

$Z_i(\omega) = 1$ if i-th flip is head; $Z_i(\omega) = 0$ if i-th flip is tail.

Example 4: Coin flipping

$Z_i(\omega) = 1$ if i-th flip is head; $Z_i(\omega) = 0$ if i-th flip is tail.

$$\begin{array}{llll} \text{HHHH}, & \text{THHH}, & \text{HTHT}, & \text{TTHT}, \\ \text{HHHT}, & \text{HHTT}, & \text{THHT}, & \text{THTT}, \\ \text{HHTH}, & \text{TTHH}, & \text{HTTH}, & \text{HTTT}, \\ \text{HTHH}, & \text{THTH}, & \text{TTTH}, & \text{TTTT} \end{array}$$

Flip a fair coin 4 times independently, let A be the event that there are 2 heads.

Randomly pick a sequence from 16 sequences.

$$P(A) = \frac{|A|}{|\Omega|} = \frac{6}{2^4} = \frac{3}{8}.$$

$$A = \{\omega : Z_1(\omega) + Z_2(\omega) + Z_3(\omega) + Z_4(\omega) = 2\}.$$

Number of heads

Example 4: Coin flipping

$Z_i(\omega) = 1$ if i-th flip is head; $Z_i(\omega) = 0$ if i-th flip is tail.

H	H	H	H	4 heads
H	H	H	T	3 heads
H	T	H	H	3 heads
H	H	T	H	3 heads
T	H	H	H	3 heads
H	H	T	T	2 heads
H	T	H	T	2 heads
H	T	T	H	2 heads
T	H	H	T	2 heads
T	H	T	H	2 heads
T	T	H	H	2 heads
H	T	T	T	1 heads
T	H	T	T	1 heads
T	T	H	T	1 heads
T	T	T	H	1 heads
T	T	T	T	0 heads

Let $X(\omega)$ be the number of heads in the sequence ω.

$$X(\omega) = Z_1(\omega) + Z_2(\omega) + Z_3(\omega) + Z_4(\omega).$$

$$P(A_k) = P(\{\omega : X(\omega) = k\}) = P(X = k) = p_k.$$

$$(p_k, k = 0, 1, 2, 3, 4) = (1, 4, 6, 4, 1)/16.$$

Example 4: Coin flipping

$$\begin{array}{llll} \text{HHHH}, & \text{THHH}, & \text{HTHT}, & \text{TTHT}, \\ \text{HHHT}, & \text{HHTT}, & \text{THHT}, & \text{THTT}, \\ \text{HHTH}, & \text{TTHH}, & \text{HTTH}, & \text{HTTT}, \\ \text{HTHH}, & \text{THTH}, & \text{TTTH}, & \text{TTTT} \end{array}$$

$|A_2| = 6.$
$|A_2| = \binom{4}{2} = \frac{4\times 3}{2}.$
4 positions, choose 2 of them to be heads, and the rest are tails.

Multiplication: table

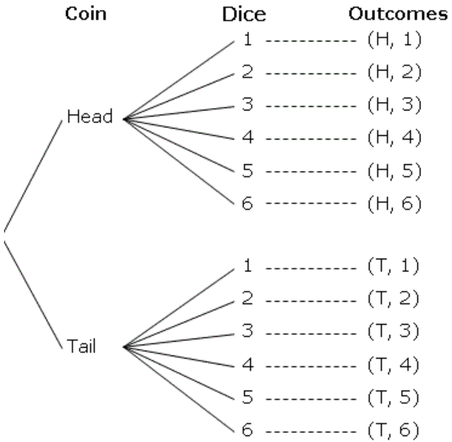
Ordered pair: roll a die twice

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Experiment 1 has n_1 outcomes. For each outcome of experiment 1, experiment 2 has n_2 outcomes. The number of all possible pairs is $n_1 \times n_2$.

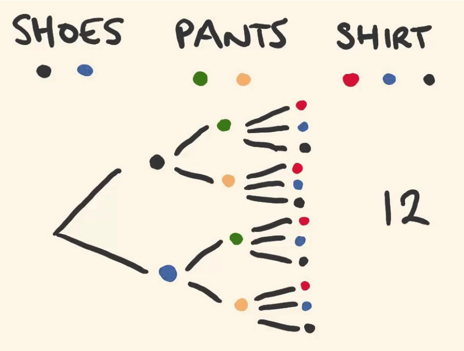
Multiplication

Ordered pair: flip a coin and roll a die



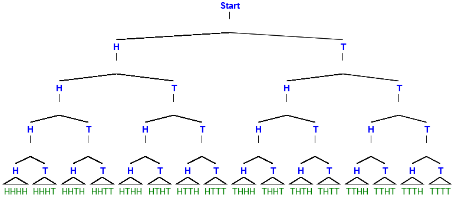
Multiplication

Ordered triplet



Sample space of sequences: coin

Flip a fair coin n times independently.
Sample space Ω_n: all possible sequences of heads and tails.



$|\Omega_n| = 2^n$.
$\Omega_n = \Omega_1 \times \Omega_1 \times ... \times \Omega_1 = \Omega_1^n$.
Ω_1: base sample space of flipping the fair coin once.
Ω_n: hyper sample space of flipping n times independently.
Population of sequences.

Sample space of sequences: die

Roll a fair die n times independently.
Sample space Ω_n: all possible sequences of numbers.

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

$|\Omega_n| = 6^n$.
$\Omega_n = \Omega_1 \times \Omega_1 \times \ldots \times \Omega_1 = \Omega_1^n$.
Ω_1: base sample space of rolling the fair die once.
Ω_n: hyper sample space of rolling n times independently.
Population of sequences.

Combination

Each combination corresponds to $k!$ permutations.

$$\binom{n}{k} = \frac{P_{n,k}}{k!} = \frac{n(n-1)...(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}.$$

$$\binom{4}{2} = \frac{4 \times 3}{2} = 6.$$