STATS 100A Homework 1

Instructor: Xiaowu Dai
Due date: Monday, April 21, 2025 at 11:59 pm on Gradescope

Please write down your name and UID clearly.

Notations: boldfaced symbols denote vectors and matrices.

Problem 1 (20 pts)

Suppose a population has N people. Among them, N_1 are males, and N_0 are females. Among the males, T_1 are taller than 6 feet. Among the females, T_0 are taller than 6 feet. Suppose we randomly sample a person from the population. Let A be the event that the person is male. Let B be the event that the person is taller than 6 feet.

Using the numbers N, N_1, N_0, T_1, T_0, calculate or verify the following:

1. Calculate $P(A)$, $P(B)$, $P(A|B)$, $P(B|A)$, $P(A \cap B)$.

2. Verify that $P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$. This is called chain rule.

3. Verify that $P(B) = P(A)P(B|A) + P(A^c)P(B|A^c)$. This is called rule of total probability.

4. Verify that
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)}.$$
This is called the Bayes rule.

Problem 2 (10 pts)

A tennis player A has probability of $\frac{2}{3}$ of winning a set against player B. A match is won by the player who first wins three sets. Find the probability that A wins the match.

Problem 3 (10 pts)

Three identical fair coins are thrown simultaneously until all three show the same face. What is the probability that they are thrown more than three times.

Problem 4 (20 pts)

Suppose we generate X and Y from the uniform distribution over [0, 1] independently, i.e., (X, Y) is a random point within the unit square $[0, 1]^2$.

1. Calculate $P(X^2 + Y^2 \leq 1)$ by measuring the area.

2. Suppose we repeat the experiment n times. For large n, how often $X^2 + Y^2 \leq 1$? Suppose it happens m times. Can you approximate π based on m and n?

3. Calculate $P(X \geq 1/2)$ and $P(X \geq 1/2 | X + Y \geq 1)$.
4. Let A be the event that $X \in [.2, .6]$, and let B be the event that $Y \in [.3, .5]$. Show that $P(A \cap B) = P(A)P(B)$.

Problem 5 (20 pts)

Five identical bowls are labeled 1,2,3,4, and 5. Bowl i contains i white balls and $5 - i$ black balls, $i = 1, 2, 3, 4, 5$. A bowl is randomly selected and two balls are selected without replacement from the contents of the bowl.

1. What is the probability that both balls selected are white?
2. Given that both balls selected are white, what is the probability that bowl 3 was selected?

Problem 6 (10 pts)

Let A, B, and C be three arbitrary events. Show that the probability that exactly one of these three events will occur is

$$P(A) + P(B) + P(C) - 2P(A \cap B) - 2P(A \cap C) - 2P(B \cap C) + 3P(A \cap B \cap C).$$

Problem 7 (10 pts)

In a game of chance each player throws two unbiased dice, each one numbered $1, 2, 3, 4, 5, 6$, and scores the difference between the larger and smaller numbers which arise. Two players compete and one or the other wins if, and only if, he scores at least 4 more than his opponent. Find the probability that neither player wins.