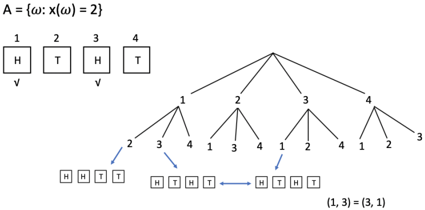
Example 4: Coin flipping



HHHH, THHH, HTHT, TTHT,
HHHT, HHTT, THHT, THTT,
HHTH, TTHH, HTTH, HTTT,
HTHH, THTH, TTTH, TTTT

$|A_2| = \binom{4}{2} = \frac{4\times 3}{2} = 6.$

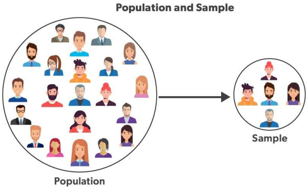
Do not confuse order of picking blanks with order of coin flippings.

In general, flip a fair coin n times independently,

$$P(A_k) = P(\{\omega : X(\omega) = k\}) = P(X = k) = \frac{\binom{n}{k}}{2^n}.$$

Survey sampling

Population of N people, M males.
Repeat random sampling n times independently



$\rightarrow N^n$ equally likely sequences.
For a sequence ω, $X(\omega) =$ number of males in ω.
$A_m = \{\omega : X(\omega) = m\}$: sequences with m males.
$|A_m| = \binom{n}{m} M^m (N-M)^{n-m}$. n blanks. Choose m blanks for males, the rest $n-m$ blanks for females. Each male blank has M choices. Each female blank has $N-M$ choices.

Survey sampling

Population of N people. M males.
Sample a person, $p = M/N$ = Prob(male).

$$P(A_m) = P(X = m) = \frac{|A_m|}{|\Omega_n|}$$
$$= \frac{\binom{n}{m} M^m (N - M)^{n-m}}{N^n}$$
$$= \binom{n}{m} p^m (1 - p)^{n-m}.$$

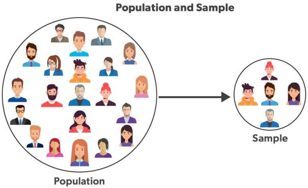
Most sequences are representative, $X/n \approx M/N = p$.

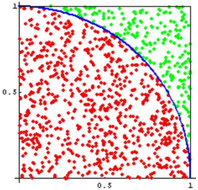
Binomial distribution, probability mass function

Flip a coin n times independently, p = probability of head.

$$p(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$
$$x = 0, 1, ..., n.$$

$p(x)$: probability mass function, probability distribution.

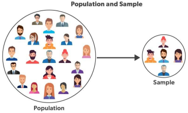




Survey sampling, poll before election, $p = M/N$.
Monte Carlo, $p = \pi/4$.

Law of large number

Survey sampling, poll before election, $p = M/N$.



Among all N^n sequences in the hyper-population of sequences Ω_n, let

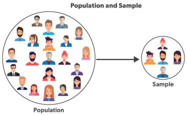
$$A = \left\{\omega : \left|\frac{X(\omega)}{n} - p\right| \leq .01\right\}.$$

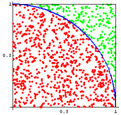
consist of representative sequences.

$$P(A) = \frac{|A|}{|\Omega_n|} = \sum_{x \in [n(p-.01), n(p+.01)]} p(x) \to 1,$$

($x \in [49, 51]$ ($n = 100$), [490, 510] ($n = 1000$), ...)
$X/n \to p$ in probability.

Definition of probability





Right: Population proportion, $P(A) = \frac{|A|}{|\Omega|}$, normalized measure, subjective belief or common sense of uncertainty.
Wrong: Long run frequency, $P(A) = \lim_{n \to} \frac{X}{n}$, under independent repetitions of the same experiments.
Limit does not always exist nor is the same for any sequence of repetitions. Independence not defined.
Right: Hyper-population of sequences of repetitions.
Uniform + Independence: all sequences are equally likely.
Proportion of representative sequences within hyper-population $\to 1$ as $n \to \infty$.

Special case: flip fair coin

$p = 1/2$, or $N = 2$.

HHHH, THHH, HTHT, TTHT,
HHHT, HHTT, THHT, THTT,
HHTH, TTHH, HTTH, HTTT,
HTHH, THTH, TTTH, TTTT

$$p(x) = \frac{\binom{n}{x}}{2^n}.\ x = 0, 1, ..., n.$$

Among all 2^n sequences, let

$$A = \left\{\omega : \left|\frac{X(\omega)}{n} - \frac{1}{2}\right| \le .01\right\}.$$

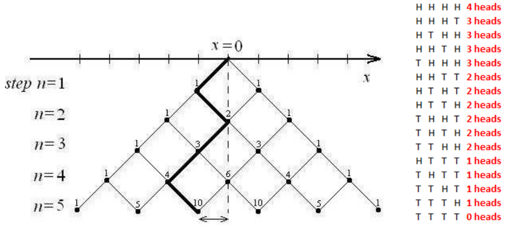
consist of representative sequences.

$$P(A) = \sum_{x \in [n(p-.01), n(p+.01)]} \frac{\binom{n}{x}}{2^n} \to 1,$$

($x \in [49, 51]$ ($n = 100$), [490, 510] ($n = 1000$), ...)
$X/n \to 1/2$ in probability.

Random walk based on coin flipping

Either go forward or backward by flipping a fair coin.
Walk n steps.



Number of heads $X = x$, then random walk ends up at
$Y = y = x - (n - x) = 2x - n$, $x = (y + n)/2$.

$$p_Y(y) = P(Y = y) = P(X = x) = p_X(x) = \frac{\binom{n}{x}}{2^n} = \frac{\binom{n}{(y+n)/2}}{2^n}.$$

Random walk

Either go forward or backward by flipping a fair coin.

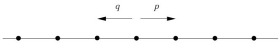


Figure 1: Simple random walk

The Symmetric Random Walk

$n \setminus x$	-5	-4	-3	-2	-1	0	1	2	3	4	5
0						1					
1					$\frac{1}{2}$	0	$\frac{1}{2}$				
2				$\frac{1}{4}$	0	$\frac{2}{4}$	0	$\frac{1}{4}$			
3			$\frac{1}{8}$	0	$\frac{3}{8}$	0	$\frac{3}{8}$	0	$\frac{1}{8}$		
4		$\frac{1}{16}$	0	$\frac{4}{16}$	0	$\frac{6}{16}$	0	$\frac{4}{16}$	0	$\frac{1}{16}$	
5	$\frac{1}{32}$	0	$\frac{5}{32}$	0	$\frac{10}{32}$	0	$\frac{10}{32}$	0	$\frac{5}{32}$	0	$\frac{1}{32}$

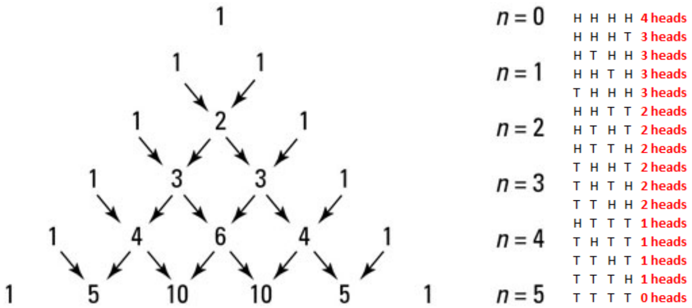
H H H H 4 heads
H H H T 3 heads
H T H H 3 heads
H H T H 3 heads
T H H H 3 heads
H H T T 2 heads
H T H T 2 heads
H T T H 2 heads
T H H T 2 heads
T H T H 2 heads
T T H H 2 heads
H T T T 1 heads
T H T T 1 heads
T T H T 1 heads
T T T H 1 heads
T T T T 0 heads

Number of heads $X = x$, then random walk ends up at $Y = y = x - (n - x) = 2x - n$, $x = (y + n)/2$.

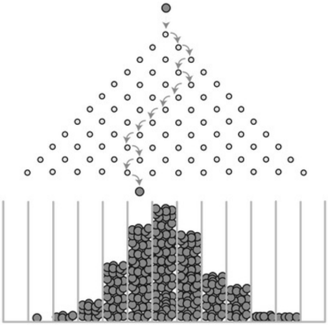
$$p_Y(y) = P(Y = y) = P(X = x) = p_X(x) = \frac{\binom{n}{x}}{2^n} = \frac{\binom{n}{(y+n)/2}}{2^n}.$$

Pascal triangle

Example 4: Coin flipping
Pascal triangle



Example 4: Coin flipping



All 2^n paths are equally likely (population of trajectories)
Number of paths that end up in x-th bin $= \binom{n}{x}$.
X: destination. $p(x) = P(X = x) = \binom{n}{x}/2^n$.
Drop 1 million balls, how often the balls fall into x-th bin.

Transition probability

Either go forward or backward

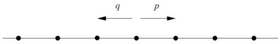


Figure 1: Simple random walk

The Symmetric Random Walk

$n \setminus x$	-5	-4	-3	-2	-1	0	1	2	3	4	5
0						1					
1					$\frac{1}{2}$	0	$\frac{1}{2}$				
2				$\frac{1}{4}$	0	$\frac{2}{4}$	0	$\frac{1}{4}$			
3			$\frac{1}{8}$	0	$\frac{3}{8}$	0	$\frac{3}{8}$	0	$\frac{1}{8}$		
4		$\frac{1}{16}$	0	$\frac{4}{16}$	0	$\frac{6}{16}$	0	$\frac{4}{16}$	0	$\frac{1}{16}$	
5	$\frac{1}{32}$	0	$\frac{5}{32}$	0	$\frac{10}{32}$	0	$\frac{10}{32}$	0	$\frac{5}{32}$	0	$\frac{1}{32}$

H H H H 4 heads
H H H T 3 heads
H T H H 3 heads
H H T H 3 heads
T H H H 3 heads
H H T T 2 heads
H T H T 2 heads
H T T H 2 heads
T H H T 2 heads
T H T H 2 heads
T T H H 2 heads
H T T T 1 heads
T H T T 1 heads
T T H T 1 heads
T T T H 1 heads
T T T T 0 heads

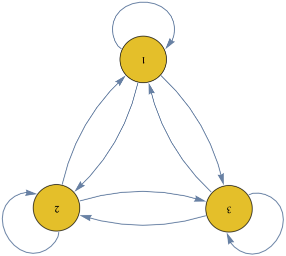
$$X_t = Z_1 + Z_2 + \ldots + Z_t.$$

$Z_k = 1$ or -1 with probability $1/2$ each.

$$X_{t+1} = X_t + Z_{t+1}.$$

$$P(X_{t+1} = x + 1 | X_t = x) = P(X_{t+1} = x - 1 | X_t = x) = 1/2.$$

Example 5: Random walk over three states

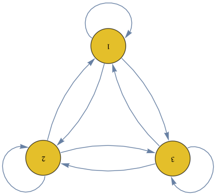


With probability 1/2, stay. With probability 1/4, go to either states.

$$K_{ij} = P(X_{t+1} = j | X_t = i).$$

Markov property: past history before X_t does not matter.

Example 5: Random walk over three states



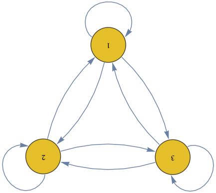
With probability $1/2$, stay. With probability $1/4$, go to either of the other two states.

$$K_{ij} = P(X_{t+1} = j | X_t = i).$$

Forward conditional probability, from cause to effect.
Imagine 1 million people migrating. At each step, for each state, half of the people stay, $1/4$ go to each of the other two states. 1 million trajectories.

Transition matrix

Example 5: Random walk over three states



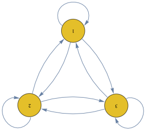
With probability 1/2, stay. With probability 1/4, go to either of the other two states.

$$K_{ij} = P(X_{t+1} = j | X_t = i).$$

$$\mathbf{K} = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

Marginal probability

Example 5: Random walk over three states



With probability 1/2, stay. With probability 1/4, go to either of the other two states.

$$K_{ij} = P(X_{t+1} = j | X_t = i).$$

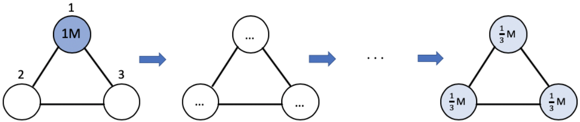
$$p_i^{(t)} = P(X_t = i).$$

Imagine 1 million people migrating. $p_i^{(t)}$ is the number of people (in million) in state i at time t.

$$\mathbf{p}^{(t)} = (p_1^{(t)}, p_2^{(t)}, p_3^{(t)}).$$

Population migration

Example 5: Random walk over three states

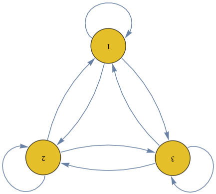


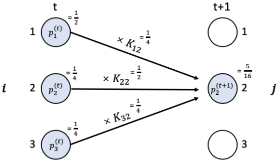
$$p_i^{(t)} = P(X_t = i).$$

Imagine 1 million people migrating. $p_i^{(t)}$ is the number of people (in million) in state i at time t.

$$\mathbf{p}^{(t)} = (p_1^{(t)}, p_2^{(t)}, p_3^{(t)}).$$

Example 5: Random walk over three states





$$K_{ij} = P(X_{t+1} = j | X_t = i).$$

$$p_i^{(t)} = P(X_t = i).$$

$$p_j^{(t+1)} = \sum_i p_i^{(t)} K_{ij}.$$

Number of people in state j at time $t+1$ = sum number of people in state i at time t $\times$ fraction of those in i who go to j.