Week 1

Rings and examples of rings

Consider a field F and a vector space V over F. Then $\mathcal{L}(V)$ itself is not a vector space: If $S, T \in \mathcal{L}(V)$ then it is not always the case that $T \circ S = S \circ T$, and it is not the case that for every $T \in \mathcal{L}(V)$ there is a T^{-1} such that $T \circ T^{-1} = I$. Then $\mathcal{L}(V)$ is a ring:

Definition 1 (Ring)**.** A ring $(R, +, \cdot)$ is a set R with two operations, denoted $+$ and $\cdot$ such that for all a, b, c in R:

(i) $(R, +)$ is an abelian group.

(ii) $(ab)c = a(bc)$.

(iii) $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$.

If $ab = ba$ for all $a, b \in R$ then R is a commutative ring. If $1_R a = a = a 1_R$ for all $a \in R$ then R is a ring with identity.

Example. $(\mathbb{Z}, +, \cdot)$ is a commutative ring with identity.

Example. $(\mathcal{L}(V), +, \circ)$ is a ring with identity

A nonzero element $a \in R$ is a left (right) zero divisor if there is a nonzero element $b \in R$ such that $ab = 0$ ($ba = 0$). A zero divisor is both a left and right zero divisor.

An element a in a ring R with identity is said to be left (right) invertible if there exists a $c \in R$ such that $ca = 1_R$ ($ac = 1_R$). The element c a left (right) inverse of a. An element $a \in R$ that is both left and right invertible is invertible or a unit.

The set of units in a ring R with identity forms a group under multiplication.

There are rings with identity and inverses that do not commute:

Example. $\mathbb{H}$, the quaternions, is a non-commutative ring with identity and inverses, also known as division ring or skew field.

Consider $\mathcal{F}(X, R)$, where $X \neq \varnothing$ and R is a ring. This is the set of functions from X to R. Then for f, g we can define addition as $(f+g)(x) = f(x) + g(x)$ and multiplication as $(f+g)(x) = f(x)g(x)$. Then $\mathcal{F}(X, R)$ is a ring.

Definition 2 (Integral domain)**.** A commutative ring R with identity $1_R \neq 0$ and no zero ddivisors is called an integral domain. A ring D with identity $1_D \neq 0$ in which every nonzero element is a unit is called a division ring. A field is a commutative division ring.

Ring homomorphisms

Let R, S be rings and $\phi : R \to S$ a function such that for all $a, b \in R$, $\phi(a+b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$.