

On inducing degenerate sums through 2-labellings

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GT GO, March 3, 2023

Labellings

Definition

A k -labelling of a graph G is a function $\ell : E(G) \rightarrow \{1, \dots, k\}$.

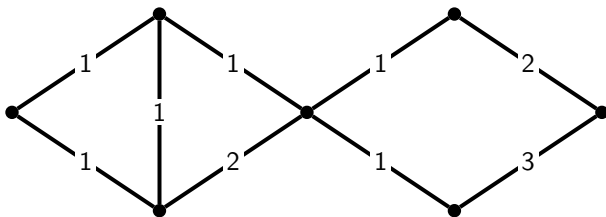
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We call *resulting sum* (relative to a labelling ℓ) of a vertex u the sum of labels on edges incident to u . We denote it $\sigma_\ell(u)$.

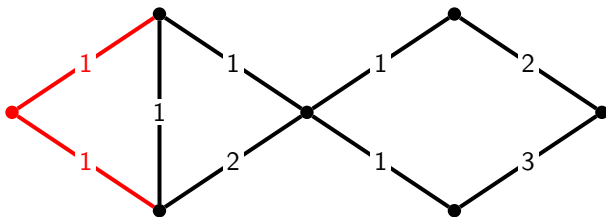
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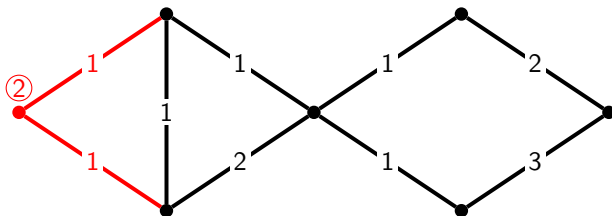
Small example



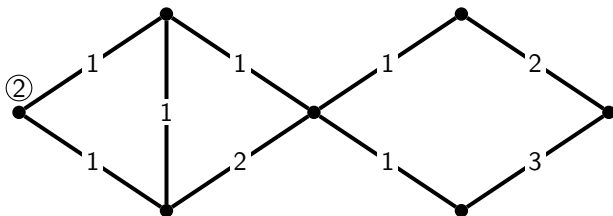
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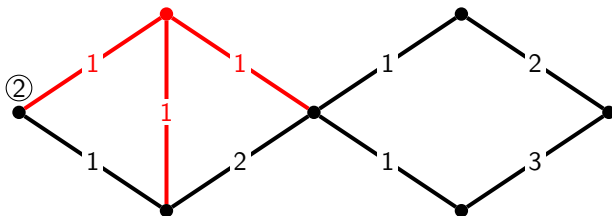
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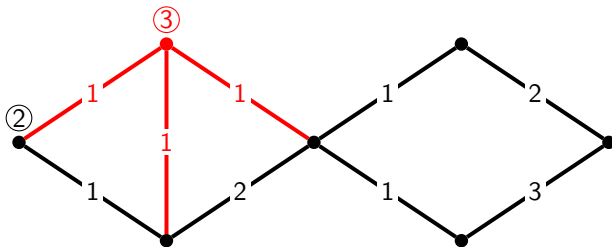
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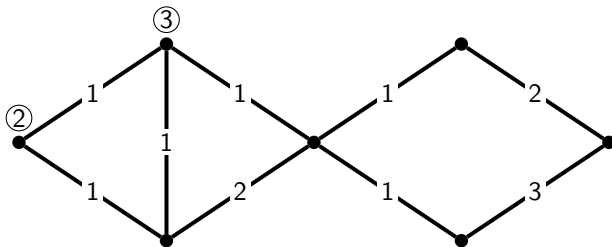
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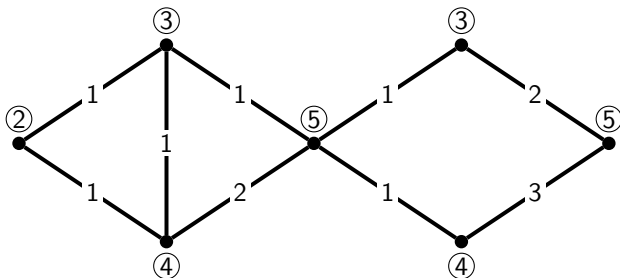
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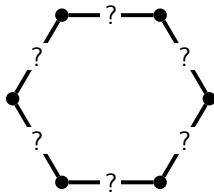
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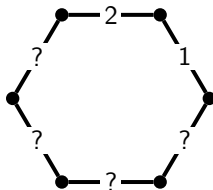
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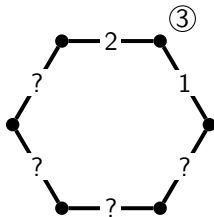
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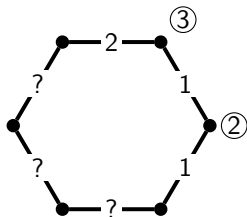
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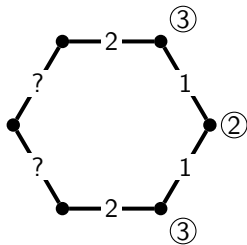
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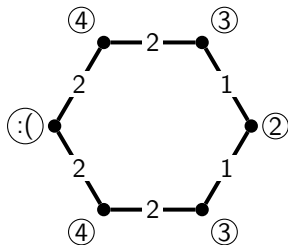
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Conjectures and knowledge

These are the main conjectures and theorems in the field:

1-2-3 Conjecture (Karoński *et al.*, 2004)

All graphs admit a distinguishing 3-labelling.

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And some specific cases. For instance, the 1-2-3 Conjecture is known to hold for 3-colorable graphs.

Labellings (again)

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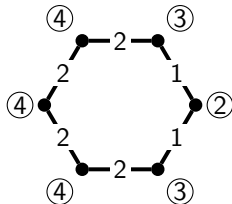
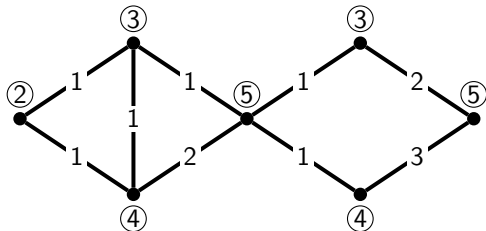
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We said a labelling ℓ is *degenerate* if the resulting sums by ℓ induce forests.

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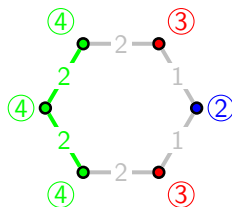
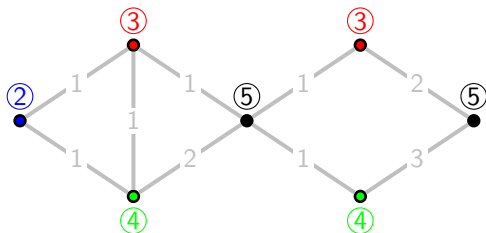
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All graphs admit a degenerate 2-labelling.

They proved the conjecture for:

- Graphs with $\text{mad} \leq 3$.
- Series-parallel graphs.
- Complete bipartite graphs.
- Cycles.
- Complete graphs.

Our contribution

From this exploratory work, our contribution consists in multiple things:

- Re-defining the problem in usual terms;
- Link it to several well-known graph notions;
- Improve the result on complete bipartite graphs to all bipartite graphs;
- Improve the results on series-parallel graphs to all 2-degenerate graphs;
- Improve the *mad* bound to $\frac{10}{3}$.

Theorem (Bensmail *et al.*, 2023+)

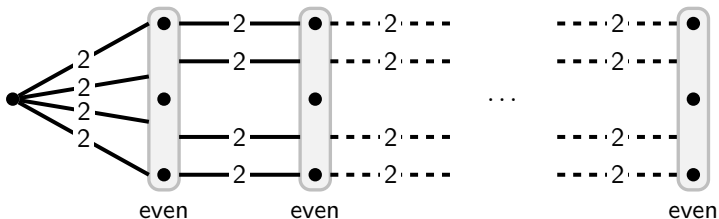
Let G be a graph. If G is bipartite, 2-degenerate or of $mad < \frac{10}{3}$, then G admits a degenerate 2-labelling.

Bipartite graphs

Theorem

Let G be a bipartite graph. Then G admits a degenerate 2-labelling.

We use a classical technique of path swapping from the field.

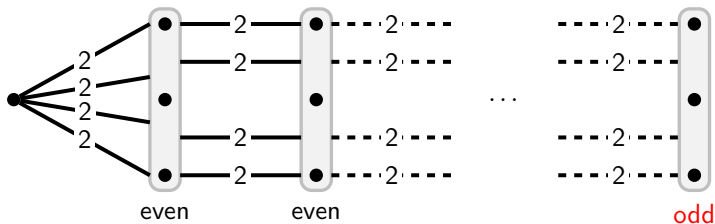


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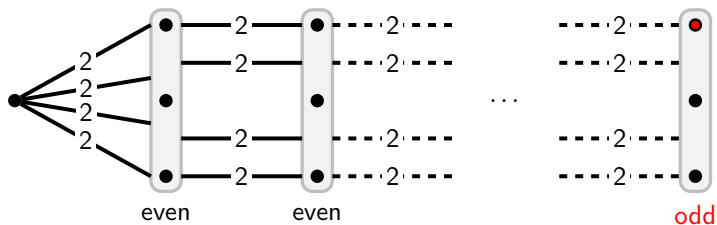


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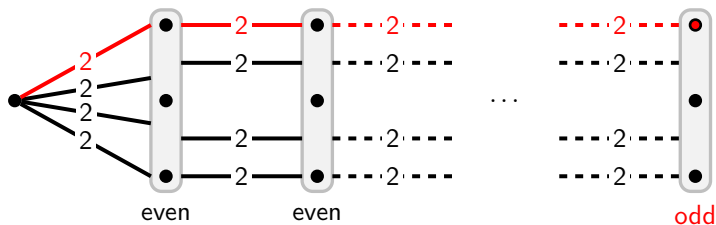


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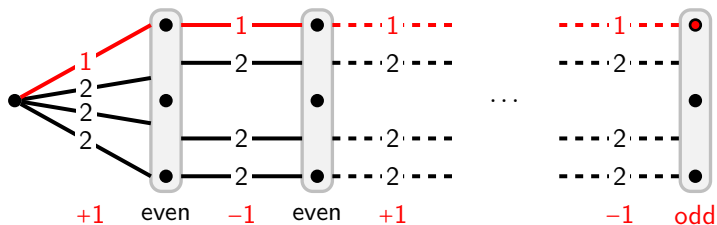


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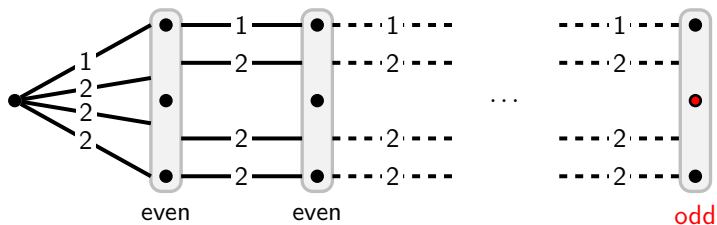


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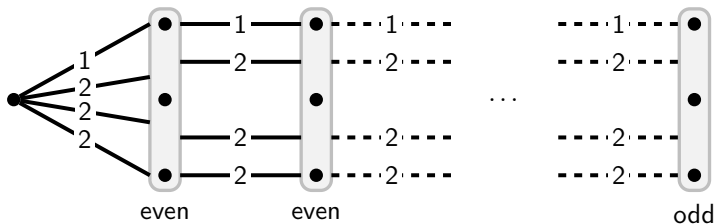


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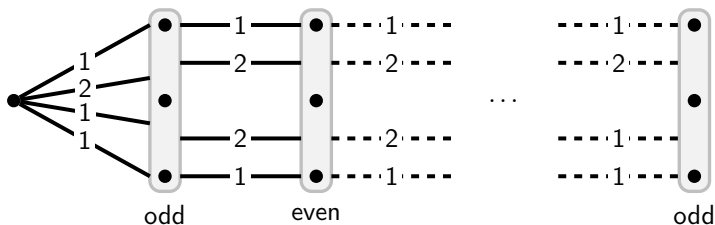


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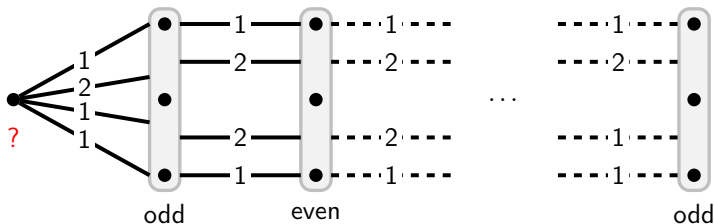


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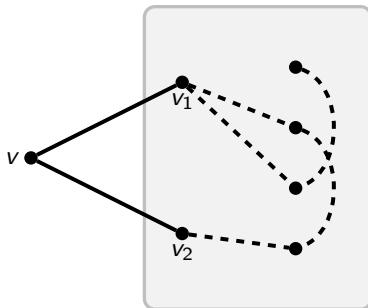


2-degenerate graphs

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Let G be a 2-degenerate graph. Then G admits a degenerate 2-labelling.

We swap paths according to a degenerate 2-coloring layout.

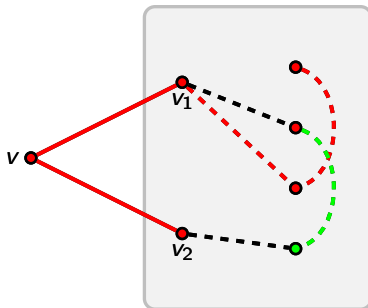


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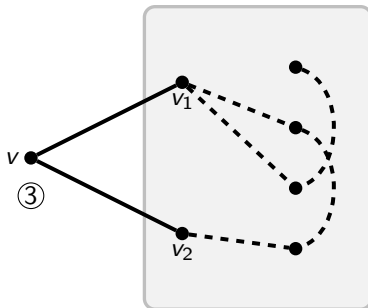


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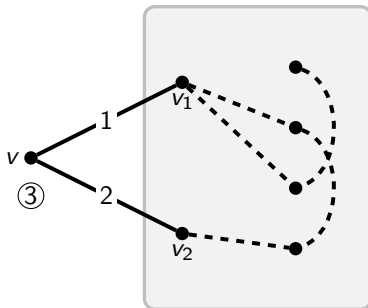


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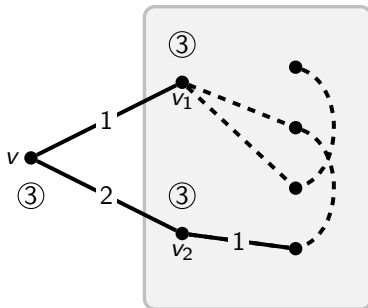


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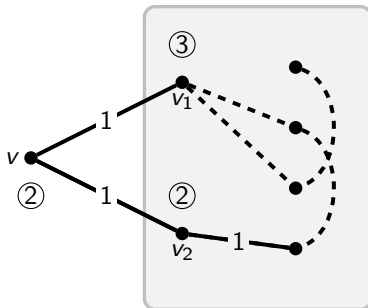


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Main result

$mad(G)$ is the maximum density of an induced subgraph of G .

Theorem (Gao *et al.*, 2015)

Let G be a graph of $mad \leq 3$. Then G admits a degenerate 2-labelling.

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Theorem (Gao *et al.*, 2015)

Let G be a graph of $\text{mad} \leq 3$. Then G admits a degenerate 2-labelling.

Corollary

Let G be a planar graph of girth 6. Then G admits a degenerate 2-labelling.

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Corollary

Let G be a planar graph of girth 5. Then G admits a degenerate 2-labelling.

Main ideas

Definition

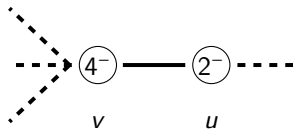
We say a graph G with $\text{mad}(G) < \frac{10}{3}$ is a *minimal counter example* (relative to a theorem) if there is no H with $\text{mad}(H) < \frac{10}{3}$ such that $|E(H)| + |V(H)| < |E(G)| + |V(G)|$.

- Suppose we have a minimal counterexample (minimal CE) to the theorem;
- Prove that it cannot contain some sparse structures;
- Put charge $d(v) - \frac{10}{3}$ on every vertex v ;
- Move charges between vertices;
- Prove that the mad is too big.

An example of a reducible configuration

Theorem

Let G be a minimal CE. Then G does not contain a 4^- -vertex adjacent to a 2^- -vertex.

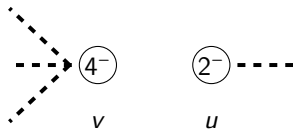


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Let G be a minimal CE. Then G does not contain a 4^- -vertex adjacent to a 2^- -vertex.

By minimality of G , we can compute ℓ'
a degenerate 2-labelling of $G - \{uv\}$.

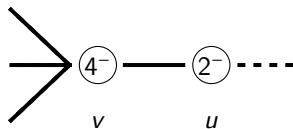


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Theorem

Let G be a minimal CE. Then G does not contain a 4^- -vertex adjacent to a 2^- -vertex.

Suppose $d(v) = 4$.

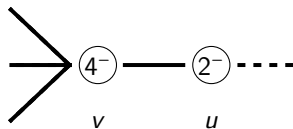


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Suppose $d(v) = 4$.
Then $\sigma_\ell(v) > \sigma_\ell(u)$.

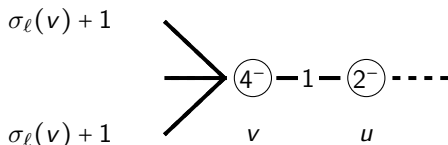


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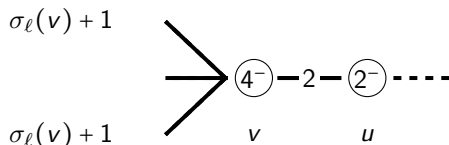


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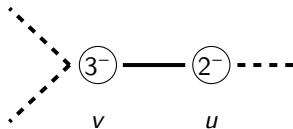
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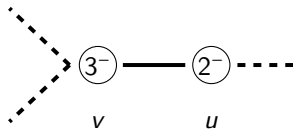


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Suppose $d(u) = 1$.



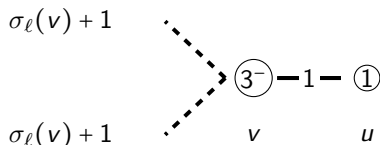
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By arguments very similar.



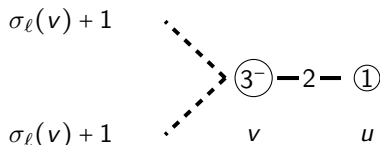
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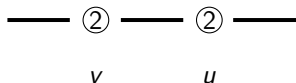


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Suppose $d(v) = 2$.



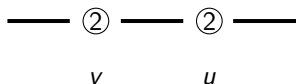
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Suppose $d(v) = 2$.

We can choose a label such that $\sigma_\ell(v) + \ell'(uv)$ is not equal to the resulting sum of the only neighbour of v other than u .

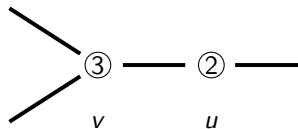


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We can now assume $d(v) = 3$.

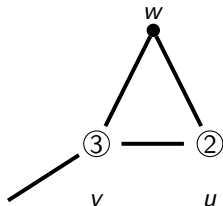


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We can now assume $d(v) = 3$.
We just pick a label by ℓ' for uv such
that $\sigma_\ell(v) + \ell'(uv) \neq \sigma_\ell(w)$.



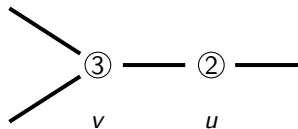
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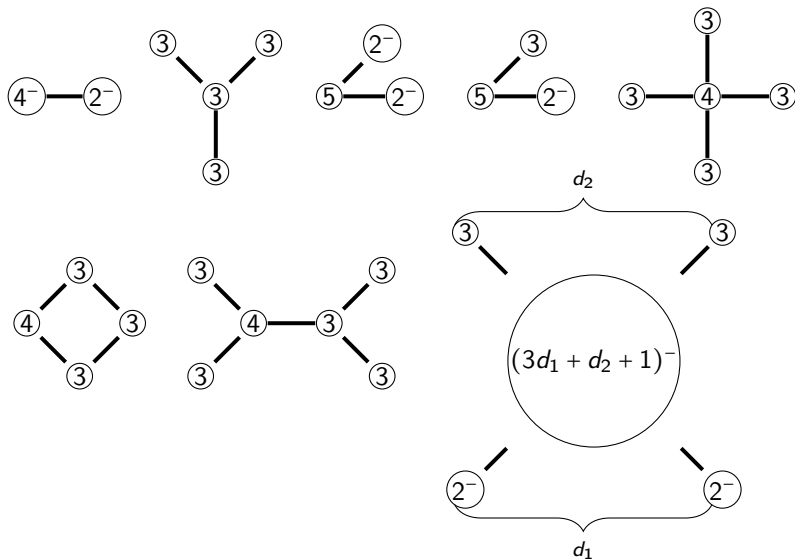
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We generalize the idea in a lemma.



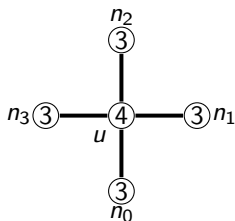
reducible configurations



Some more reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

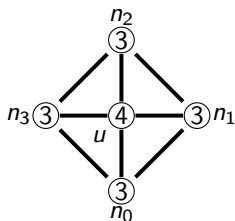


We need to consider possible inner edges, and complete a labelling of $G - \{u\}$ and without the inner edges.

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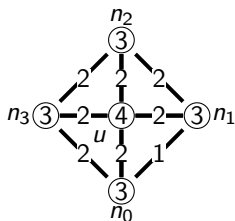


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 4$.

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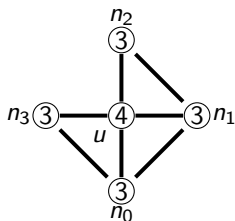


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 4$.
Then G is a wheel of order 5.

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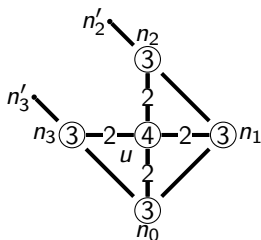


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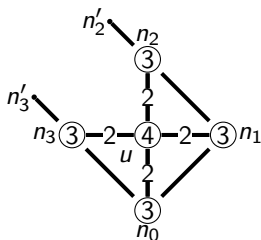


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 3$.
We ensure a large sum for u .

Some more reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

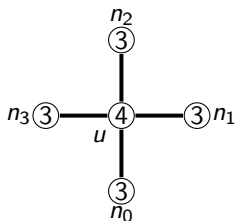


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 3$.
We ensure a large sum for u . We pick
remaining labels according to the
resulting sums of n'_2 and n'_3 .

Some more reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

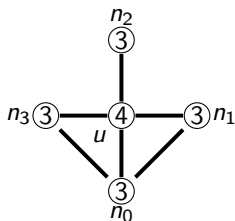


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 2$.
There are two subcases.

Some more reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

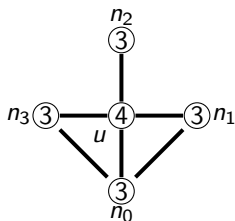


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 2$.
Assume inner edges are incident to a common vertex.

Some more reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

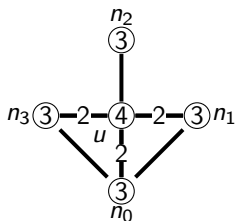


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 2$.
 Assume inner edges are incident to a common vertex. We have no control over the label of un_2 .

Some more reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

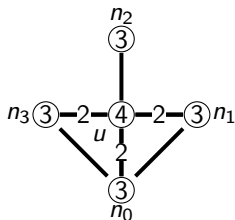


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 2$. Assume inner edges are incident to a common vertex. We can still ensure that the resulting sum of u is large.

Some more reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

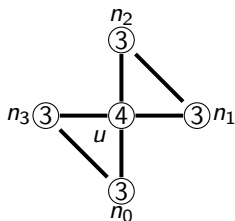


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 2$.
 Assume inner edges are incident to a common vertex. We can still ensure that the resulting sum of u is large.
 We conclude with Lemma.

Some more reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

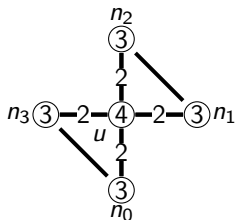


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 2$.
Now assume this configuration.

Some more reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

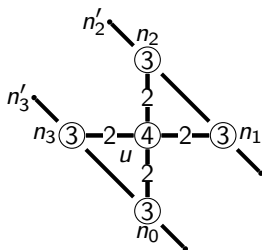


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 2$.
Now assume this configuration. The resulting sum of u is large.

Some more reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

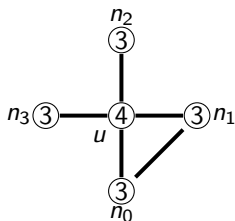


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 2$.
 Now assume this configuration. The resulting sum of u is large. We pick labels for $n_1 n_2$ and $n_0 n_3$ to differentiate them from their neighbours.

Some more reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

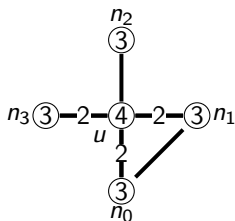


Assume
 $|E(G[\{n_0, n_1, n_2, n_3\}])| = 1.$

Some more reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.



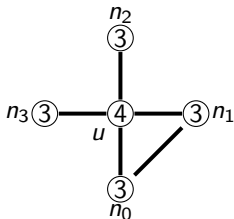
Assume

$|E(G[\{n_0, n_1, n_2, n_3\}])| = 1$. If we have the choice for at least one of un_2 or un_3 , we can ensure a large resulting sum for u , and pick a label for n_0n_1 like in last case.

Some more reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.



Assume

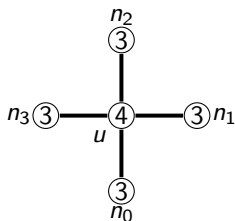
$$|E(G[\{n_0, n_1, n_2, n_3\}])| = 1.$$

Otherwise, we pick fitting labels for un_2 , un_3 , and we can make the resulting sum of one of n_0 or n_1 strictly bigger than the other.

Some more reductions

Configuration 5

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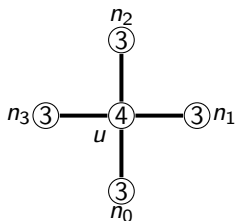


Assume
 $|E(G[\{n_0, n_1, n_2, n_3\}])| = 0$.

Some more reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

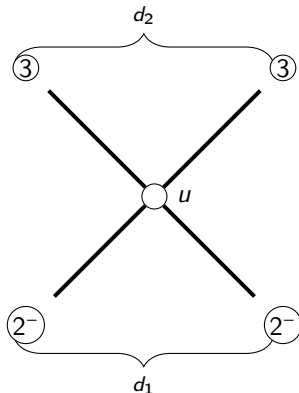


Assume
 $|E(G[\{n_0, n_1, n_2, n_3\}])| = 0$. We have
 a lemma.

Another reduction

Configuration 8

Let G be a minimal CE, d_1, d_2 two integers such that $3d_1 + d_2 + 1 \geq 6$. Then G does not contain a $(3d_1 + d_2 + 1)^-$ -vertex adjacent to d_2 3-vertices and d_1 2^- -vertices.

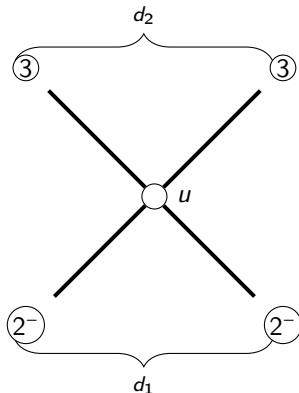


We consider a labelling of $G - \{uv_1, \dots, uv_{d_1}, uw_1, \dots, uw_{d_2}\}$ and we extend it.

Another reduction

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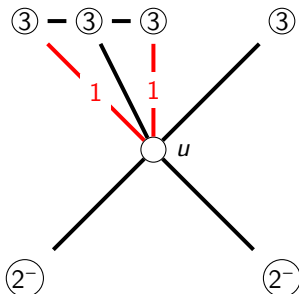


We consider a labelling of $G - \{uv_1, \dots, uv_{d_1}, uw_1, \dots, uw_{d_2}\}$ and we extend it. $G[\{w_1, \dots, w_{d_2}\}]$ has maximum degree 2. Its connected components are either cycles, paths or isolated vertices.

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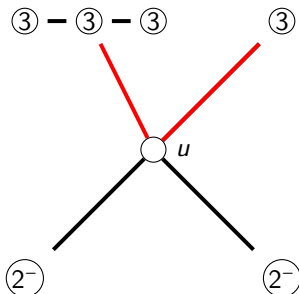


We consider a labelling of $G - \{uv_1, \dots, uv_{d_1}, uw_1, \dots, uw_{d_2}\}$ and we extend it. $G[\{w_1, \dots, w_{d_2}\}]$ has maximum degree 2. Its connected components are either cycles, paths or isolated vertices. For each component with edges, we pick one arbitrary w_i and assign label 1 to every other uw_j of the component.

Another reduction

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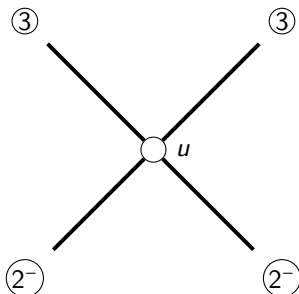


We consider a labelling of $G - \{uv_1, \dots, uv_{d_1}, uw_1, \dots, uw_{d_2}\}$ and we extend it. We assign a fitting label to every remaining uw_i .

Another reduction

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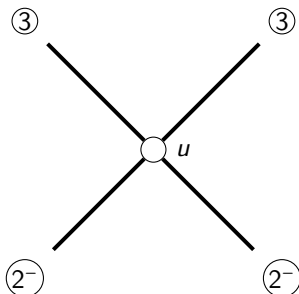


We consider a labelling of $G - \{uv_1, \dots, uv_{d_1}, uw_1, \dots, uw_{d_2}\}$ and we extend it. There can be no cycle with one of the w_i . We only need to be sure that u does not have the same resulting sum as two of its neighbours.

Another reduction

Configuration 8

Let G be a minimal CE, d_1, d_2 two integers such that $3d_1 + d_2 + 1 \geq 6$. Then G does not contain a $(3d_1 + d_2 + 1)^-$ -vertex adjacent to d_2 3-vertices and d_1 2^- -vertices.



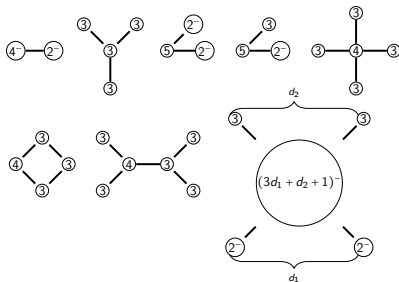
We consider a labelling of $G - \{uv_1, \dots, uv_{d_1}, uw_1, \dots, uw_{d_2}\}$ and we extend it.
We have a Lemma.

Our discharging process

Degree	1	2	3	4	5	6
Initial charge	$\frac{-7}{3}$	$\frac{-4}{3}$	$\frac{-1}{3}$	$\frac{2}{3}$	$\frac{5}{3}$	$\frac{9}{3}$
Final charge	$\frac{-4}{3}$	$\frac{2}{3}$	0	0	0	0

Weak 3-vertices

A weak 3-vertex is a 3-vertex adjacent to exactly one 4^+ -vertex, and it is a 4-vertex.



(R1) Every 5^+ -vertex sends 1 to each of its 2^- -neighbours.

(R2) Every 5^+ -vertex sends $\frac{1}{3}$ to each of its 3-neighbours.

(R3) Every 4-vertex sends $\frac{1}{3}$ to each of its weak 3-neighbours.

(R4) Every 4-vertex sends $\frac{1}{6}$ to each of its non-weak 3-neighbours.

The final weights

Consider for instance v a 3-vertex. Note that $\omega(v) = -\frac{1}{3}$, and that we need $\omega^*(v) \geq 0$:

- If v is weak, then v has only one 4^+ -neighbour, being a 4-vertex, which sent $\frac{1}{3}$ to v by Rule R3.
- If v is not weak, then either v neighbours at least one 5^+ -vertex, or v neighbours at least two 4-vertices. In the former case, at least one 5^+ -neighbour of v sent $\frac{1}{3}$ to v by Rule R2, while, in the latter case, at least two 4-neighbours of v both sent $\frac{1}{6}$ to v by Rule 4.

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Rule R3

Every 4-vertex sends $\frac{1}{3}$ to each of its weak 3-neighbours.

We prove the same result for every possible degree.

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Rule R2

Every 5^+ -vertex sends $\frac{1}{3}$ to each of its 3-neighbours.

We prove the same result for every possible degree.

The final weights

Consider for instance v a 3-vertex. Note that $\omega(v) = -\frac{1}{3}$, and that we need $\omega^*(v) \geq 0$:

- If v is weak, then v has only one 4^+ -neighbour, being a 4-vertex, which sent $\frac{1}{3}$ to v by Rule R3.
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Rule R4

Every 4-vertex sends $\frac{1}{6}$ to each of its non-weak 3-neighbours.

We prove the same result for every possible degree.

Results

In this presentation, we proved the degenerate 1-2 Conjecture:

- for 2-degenerate graphs;
- for bipartite graphs;
- for graphs of $mad < \frac{10}{3}$.

In fact, it also holds for graph of edge weight 7. We also have a corollary to the *mad* result:

Corollary

If G is a planar graph with $g(G) \geq 5$, then it admits a degenerate 2-labelling.

Perspectives

- Other classes of graph with vertex arboricity at most 2:
 - Graphs of maximum degree 4;
 - Graphs of degeneracy 3.
- Denser graphs (with bigger *mad*).

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Thank you for your attention !

Discharging

We will apply the following result to our minimal CE.

Theorem (Bonamy *et al.*, 2013)

Let G be a graph, m a value and (V_1, V_2) a partition of $V(G)$. Let ω be the charge function where $\omega(v) = d(v) - m$ for every $v \in V(G)$. If there is a discharging process resulting in a charge function ω^* such that:

- $\omega^*(v) \geq 0$ for every $v \in V_1$, and
- $\omega^*(V) \geq \omega(v) + d_{V_1}(v)$ for every $v \in V_2$,

Then $mad(G) \geq m$.

We consider V_2 the set of all the 2^- -vertices of G . Note that by Configuration 1, $d_{V_1}(v) = d(v)$ for every $v \in V_2$.