The Weak $(2, 2)$-Labelling Problem for graphs with forbidden induced structures

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Labellings and i-sums

Definition

An (α, β)-labelling of a graph G is a function $\ell : E(G) \to \{1, \dots, \alpha\} \times \{1, \dots, \beta\}$, where the label is a couple of a color and a value.

Definition

The i-sums of a vertex u, denoted $\sigma_i(u)$, is the sum of the values of its incident edges labelled with color i. Formally,

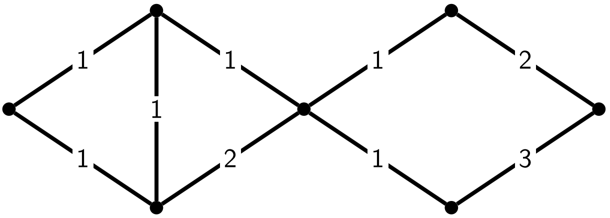
$$\sigma_i(u) = \sum_{\substack{e \in I(u) \\ \ell(e)[0]=i}} \ell(e)[1]$$

where $I(u)$ is the set of edges incident to u.

Distinguishing labellings

Definition

We said it is distinguishing if for every two adjacent vertices u and v of G, there is an $i \in \{1, \ldots, \alpha\}$ such that the i-sum of u and v differs.



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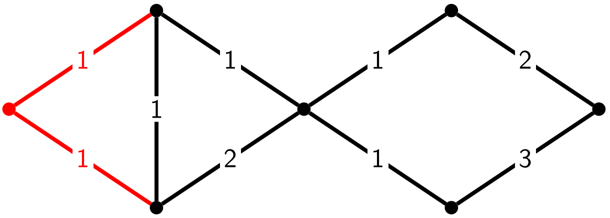

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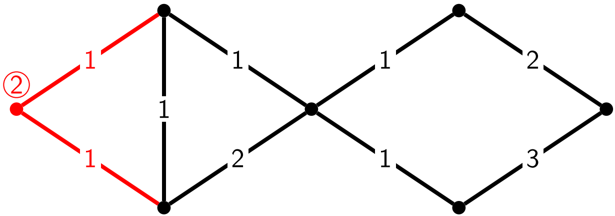

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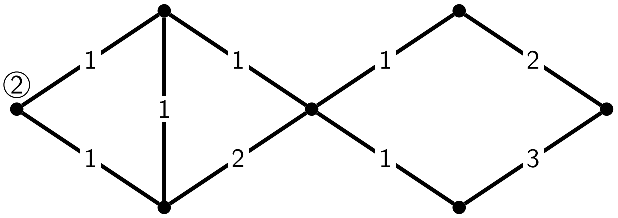

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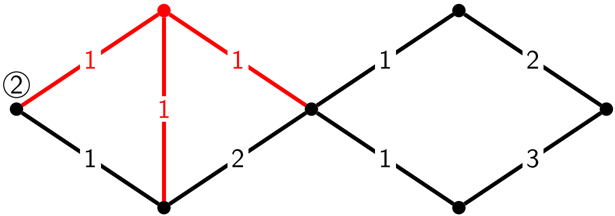

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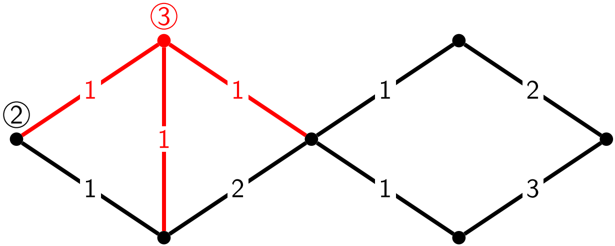

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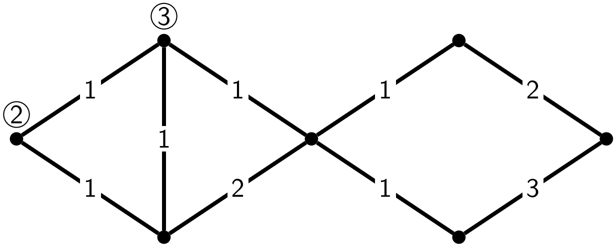

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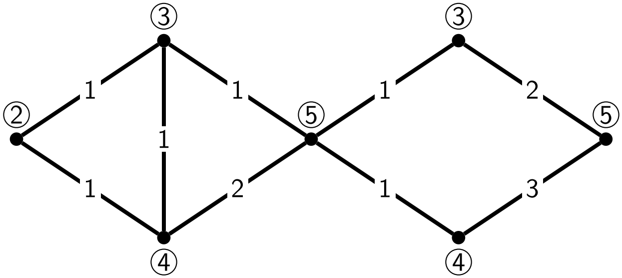

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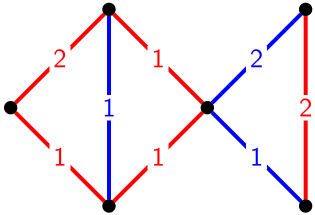

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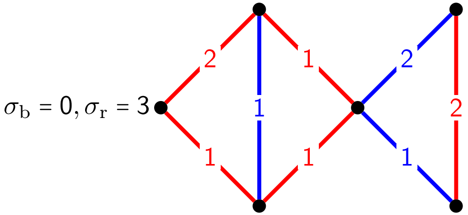

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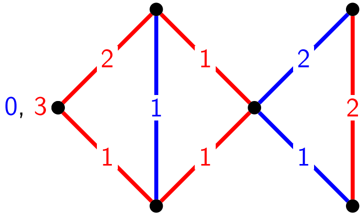

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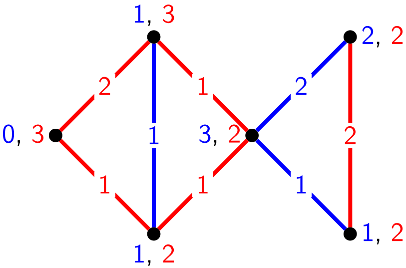

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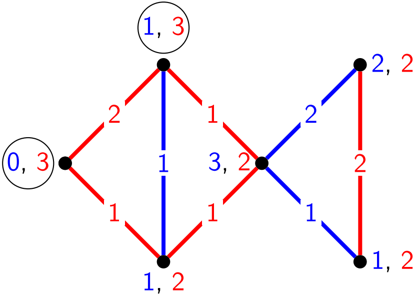

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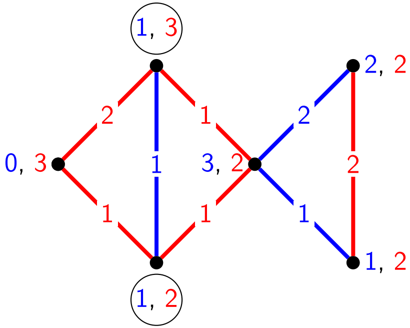


Conjectures

There are a few conjectures about "all" graphs admitting an (α, β)-labelling.

1-2-3 Conjecture (Karonski *et al., 2004)*

All graphs admit a $(1, 3)$-labelling.

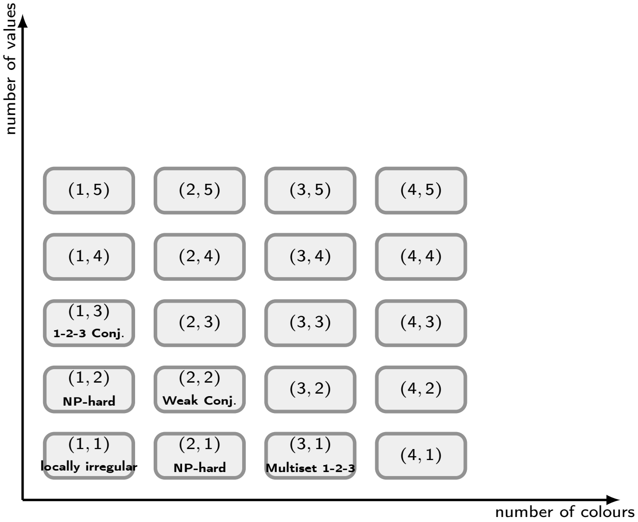
$(2, 2)$-Conjecture (Baudon *et al., 2019)*

All graphs admit a $(2, 2)$-labelling.

Observation

The 1-2-3 Conjecture implies the $(2, 2)$-Conjecture.

Current knowledge



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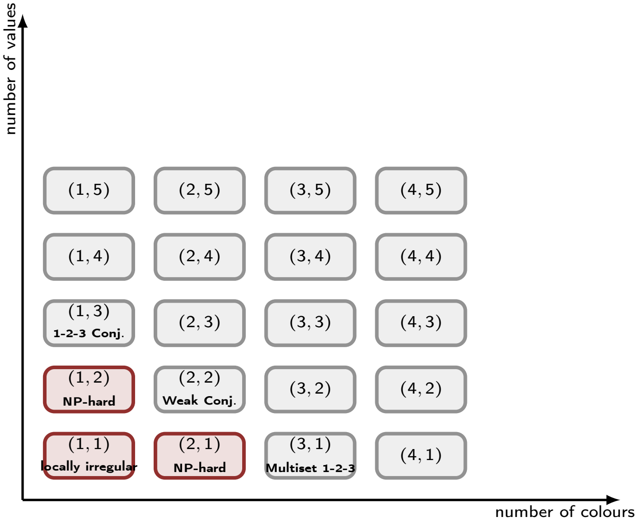

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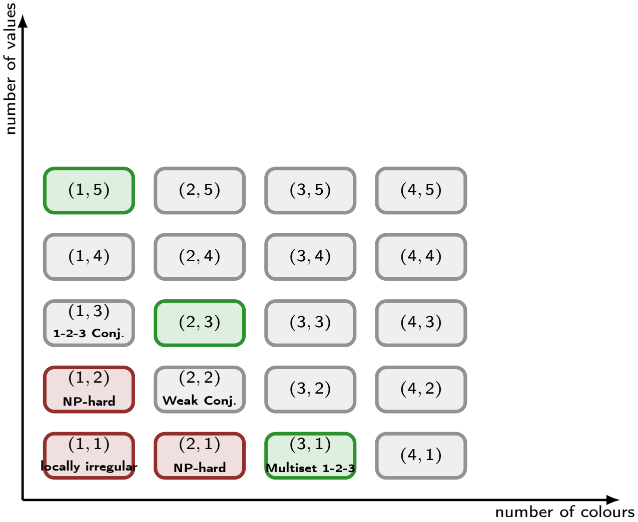

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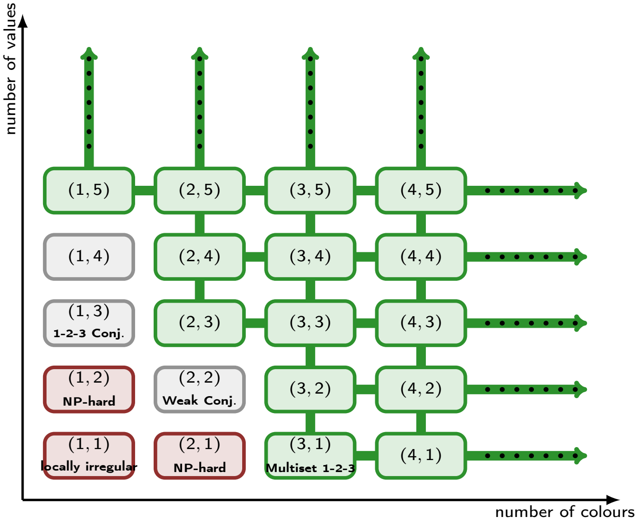

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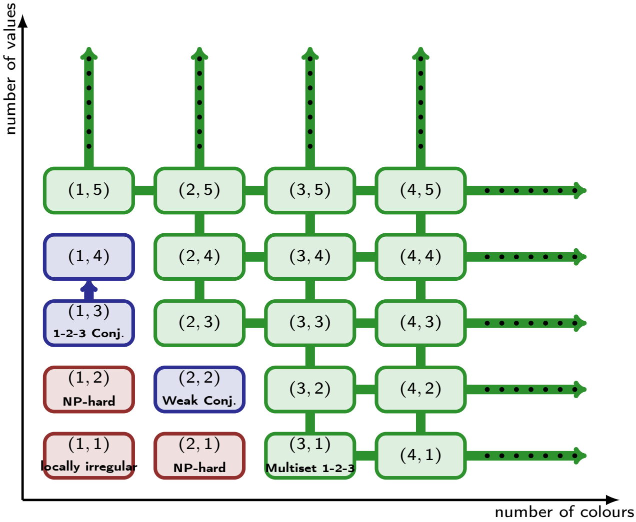

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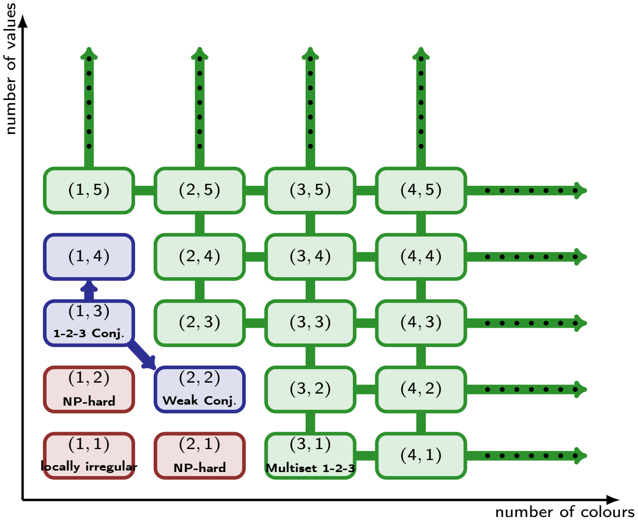
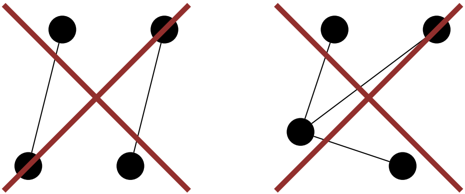


Some known cases and our contribution

- The 1-2-3 Conjecture is known to hold for 3-colorable graphs.
- Hence, the $(2,2)$-Conjecture holds for 3-colorable graphs.
- The $(2,2)$-Conjecture in fact holds for 4-colorable graphs.

Theorem (Bensmail, Hocquard, M., 2022+)

The $(2,2)$-Conjecture holds for $2K_2$-free graphs and $K_{1,3}$-free graphs.



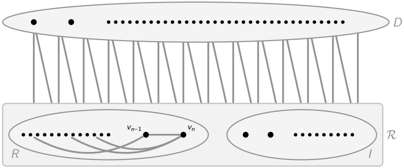
$2K_2$-free graphs

We want to show the $(2,2)$-Conjecture for those graphs. Remember it holds for 4-colorable graphs.

Theorem

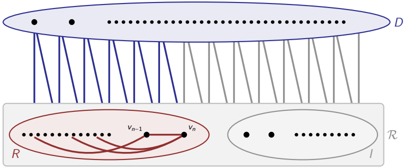
Every $2K_2$-free graph with chromatic number 5 admits a distinguishing $(2,2)$-labelling.

Outline of the proof



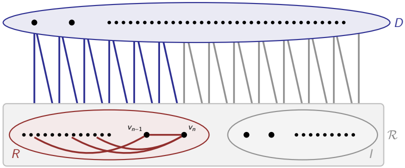
Isolate a maximum independent set to deal with a 4-colorable subgraph.

Outline of the proof



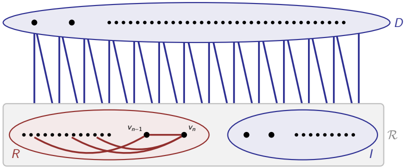
Use the 4-coloring to compute a $(2,2)$-labelling.

Outline of the proof



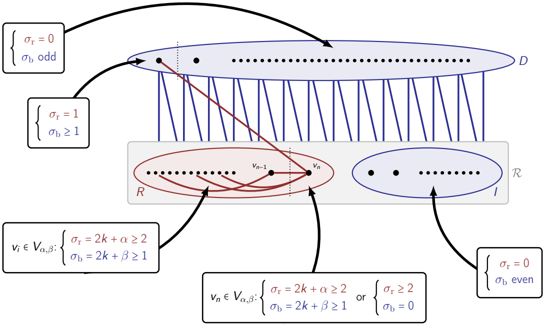
Ensure those vertices have a large red sum.

Outline of the proof



Label the remaining edges with mostly blue labels.

Outline of the proof



We labelled each edge so that every subset of the partition has these distinguishing properties.

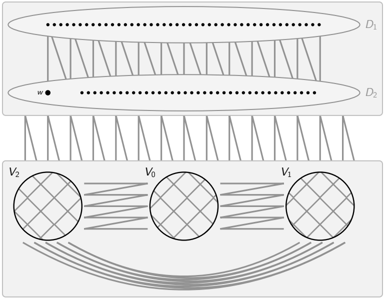
$2K_2$-free graphs of high chromatic number

Theorem

Every $2K_2$-free graph with chromatic number at least 6 admits a distinguishing $(2,2)$-labelling.

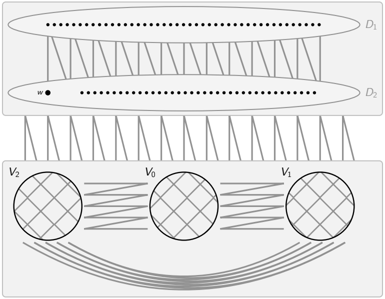
The proof is very similar...

Outline of the proof



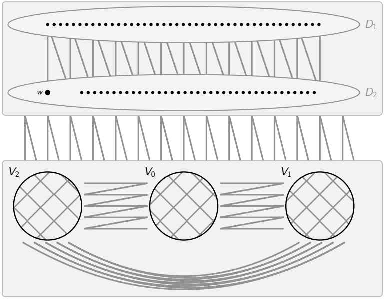
Separate a maximum independent set of the graph.

Outline of the proof



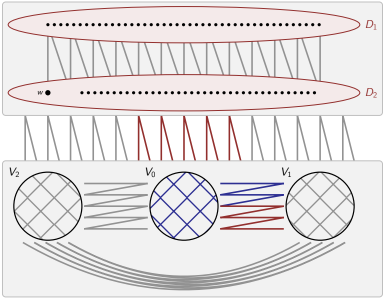
Separate another maximum independent set of the remaining subgraph.

Outline of the proof



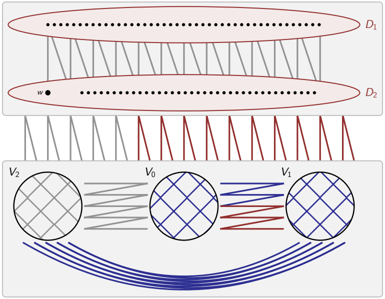
A tri-partition is a partition of a 4-colorable graph into 3 subsets V_0, V_1, V_2 such that every vertex has at least one neighbor in the next set, and more neighbors in the next set than in its own set.

Outline of the proof



Label each one of those subgraphs and the edges between them.

Outline of the proof



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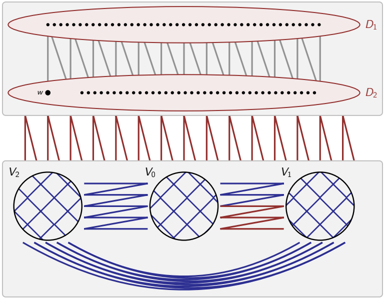

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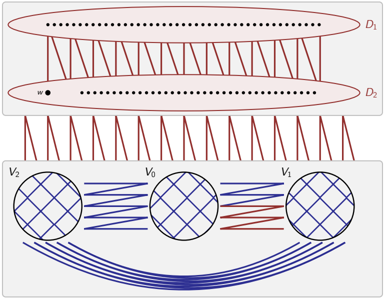

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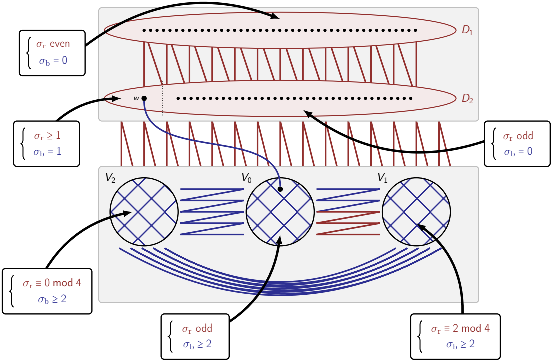

Verify some properties that ensure the distinguishing result.

Claw-free graphs

We also prove a similar result for $K_{1,3}$-free graphs.

Theorem

The $(2,2)$-Conjecture holds for $K_{1,3}$-free graphs.

The proof is very similar, but more technical:

- Each set can now have multiple connected components with edges.
- In the case of large chromatic number, the two independent sets can have multiple connected components.

Perspectives and conclusion

A few perspectives :

- Proving the $(2,2)$-Conjecture for other graph classes with forbidden induced structures, such as triangle-free graphs.
- Proving the 1-2-3 Conjecture for claw-free graphs and $2K_2$-free graphs.

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Thank you for your attention !