

Monitoring edge-geodetic sets on oriented graphs

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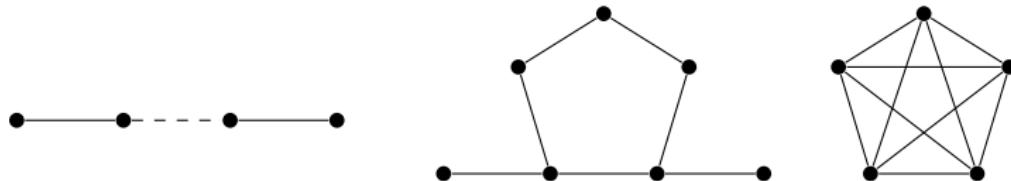
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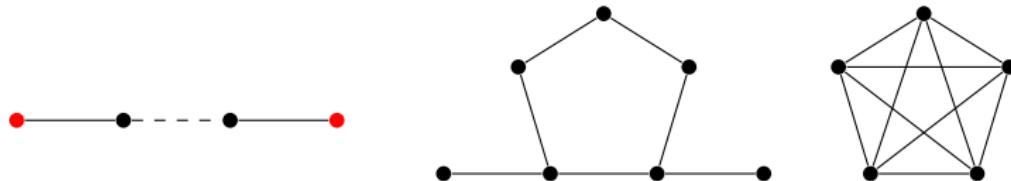
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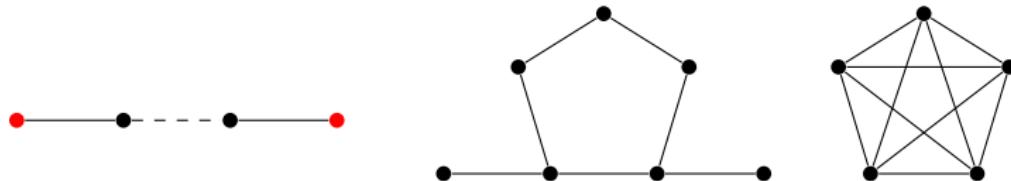
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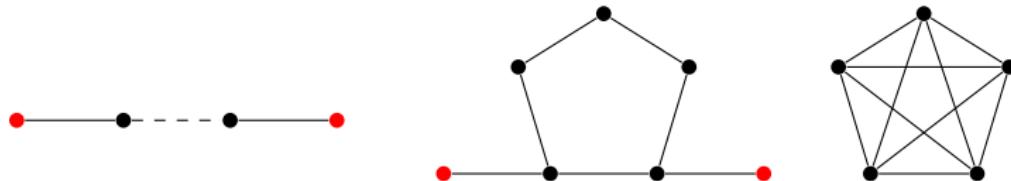
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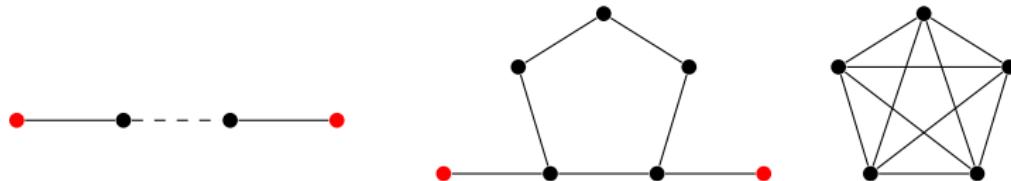
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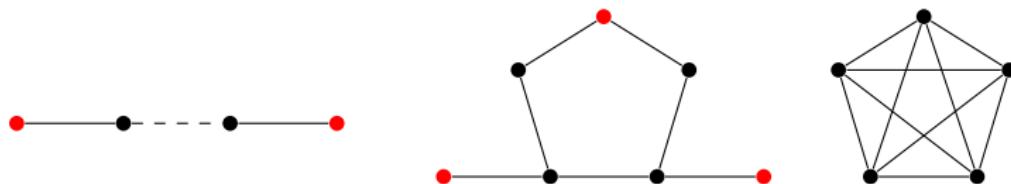
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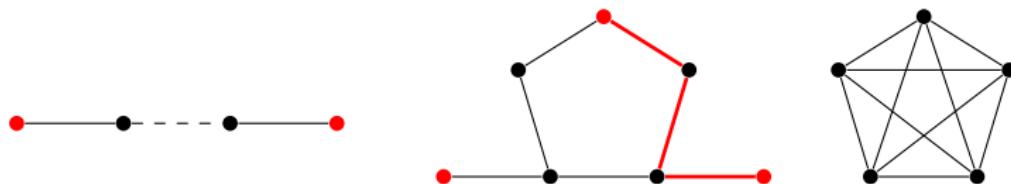
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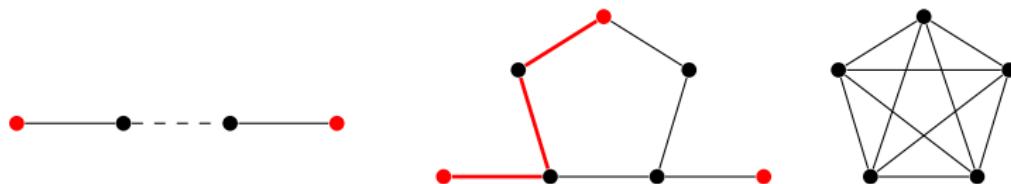
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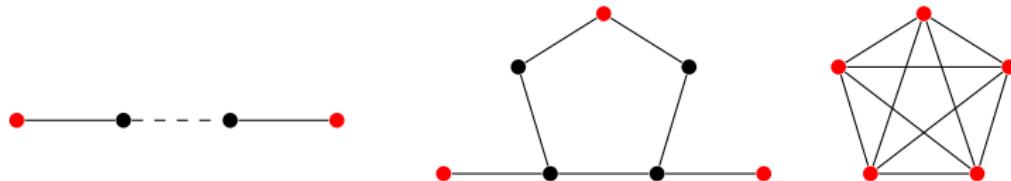
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A few results on the *meg* parameter

Theorem [Foucaud *et al.*, 2023]

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Theorem [Haslegrave, 2023]

The decision problem of determining for a graph G and a natural number k whether $\text{meg}(G) \leq k$ is NP-complete.

Oriented version

We consider orientations of simple graphs, without digones.

Definition

In an oriented graph \vec{G} , two vertices x and y are said to **monitor** an arc \vec{a} if \vec{a} belongs to all oriented shortest paths from x to y or from y to x .

Definition

A **monitoring arc-geodetic set**, or MAG-set, of an oriented graph \vec{G} is a vertex subset $M \subseteq V(\vec{G})$ such that given any arc \vec{a} of $A(\vec{G})$, \vec{a} is monitored by x, y , for some $x, y \in M$. For an oriented graph \vec{G} , we denote $mag(\vec{G})$ the size of a smallest MAG-set of \vec{G} .

First results

First note that for an oriented graph \vec{G} , the relation between $mag(\vec{G})$ and $meg(G)$ is not clear:



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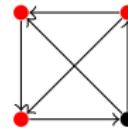
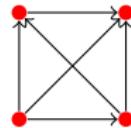
Let \vec{G} be an oriented tree. There is a unique minimal MAG-set to \vec{G} , and it is exactly the set of sources and sinks of \vec{G} .

Tournaments

Theorem [Das et al., 2023+]

Let \vec{G} be an orientation of K_n for some $n \in \mathbb{N}^*$. Then $mag(\vec{G}) \in \{n - 1, n\}$.

Since one can check in polynomial type if a set of vertices of \vec{G} is an MAG-set, we can now easily characterize all tournaments for this parameter.



Complexity of computing the MAG-set size

We consider the following decision problem:

MAG-SET problem

Instance: An oriented graph \vec{G} , an integer k .

Question: Does there exist an MAG-set of \vec{G} of size k ?

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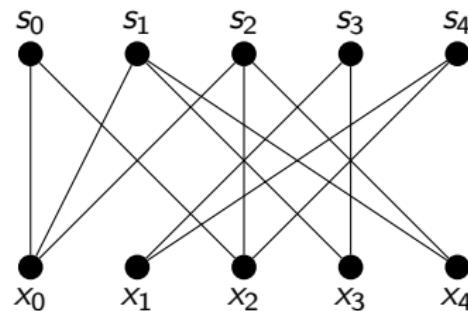
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Theorem [Das et al., 2023+]

The MAG-SET problem is NP-complete.

The SETCOVER problem

We proceed with a reduction from the SETCOVER problem.



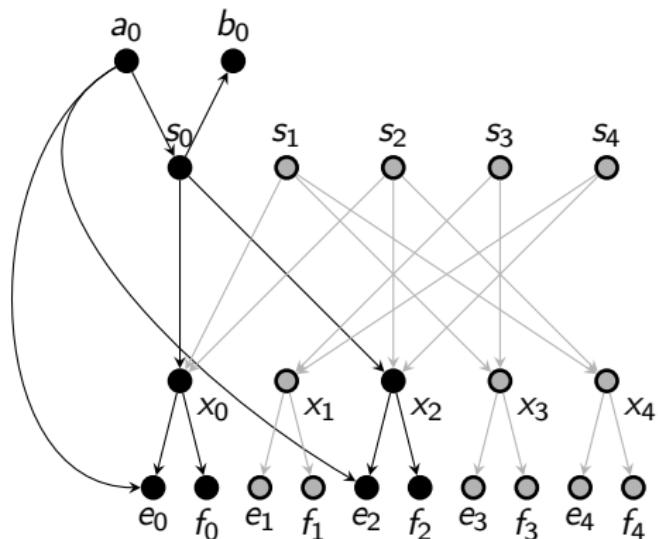
SETCOVER Problem:

Instance: A set $\{X_0, X_1, \dots, X_n\}$, sets $\{S_0, S_1, \dots, S_m\}$ such that $\cup_{i=0}^m S_i = \{X_0, X_1, \dots, X_n\}$ and an integer k .

Question: Does there exist a subcollection of at most k sets S_i 's such that their union is $\{X_0, X_1, \dots, X_n\}$.

Our gadget for the reduction

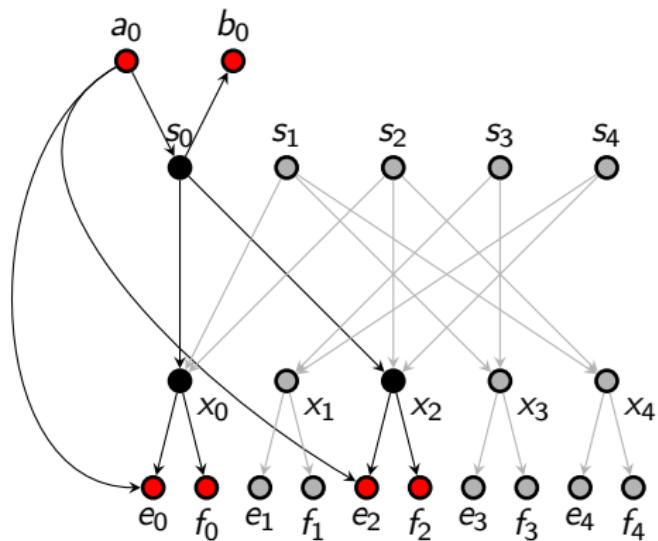
From any instance of SETCOVER I , we can compute $\overrightarrow{G}(I)$ an instance of MAG-SET. Assume we have M an MAG-set of $\overrightarrow{G}(I)$.



We now need to study the properties of any MAG-set of $\overrightarrow{G}(I)$.

Our gadget for the reduction

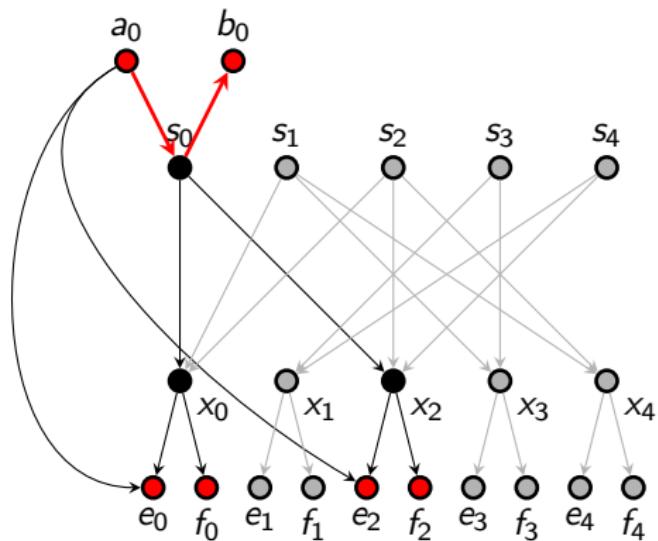
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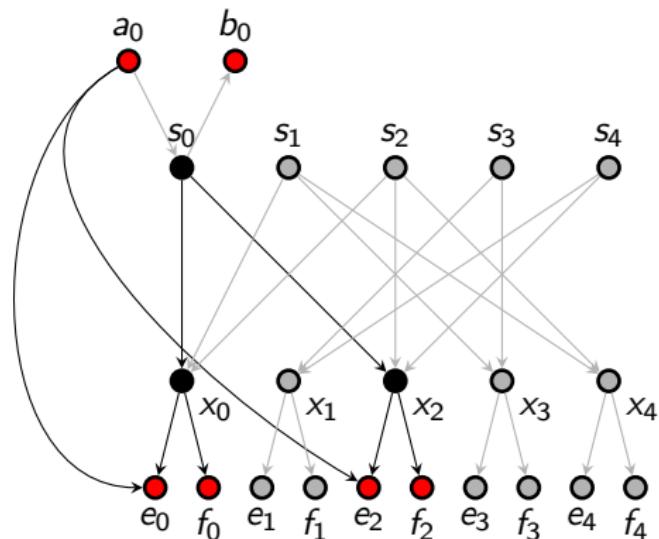
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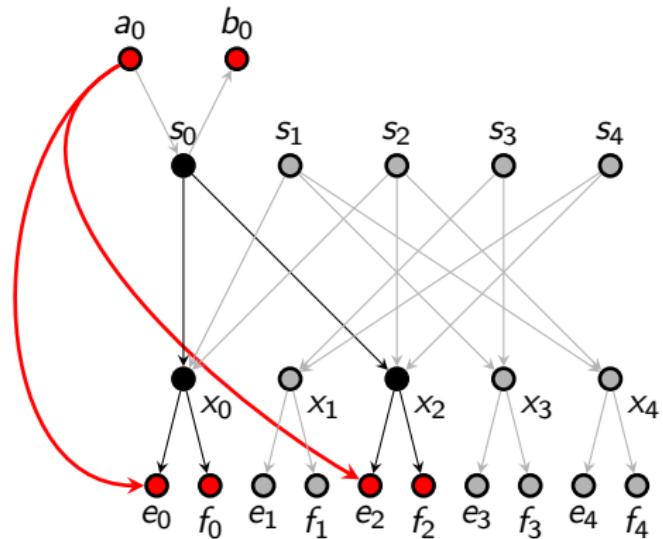
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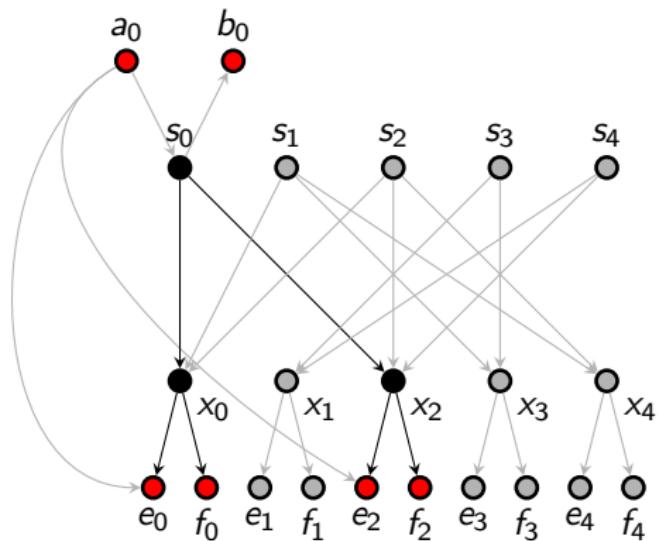
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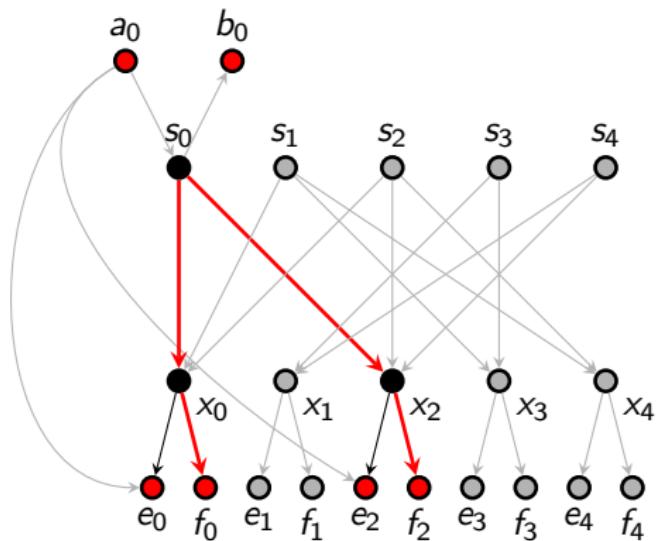
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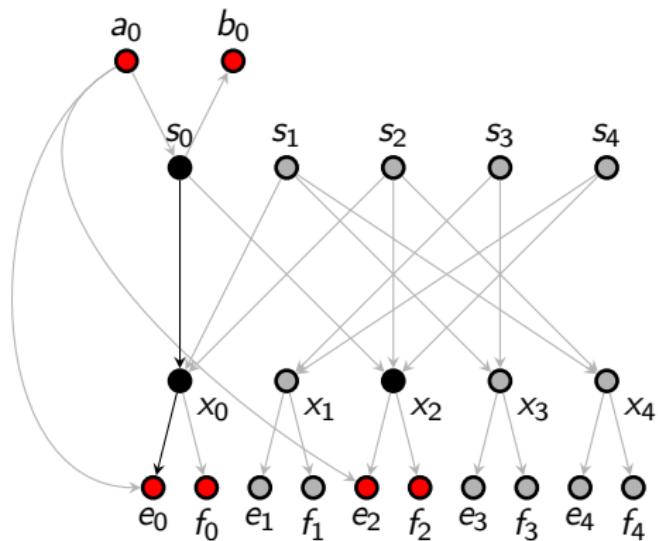
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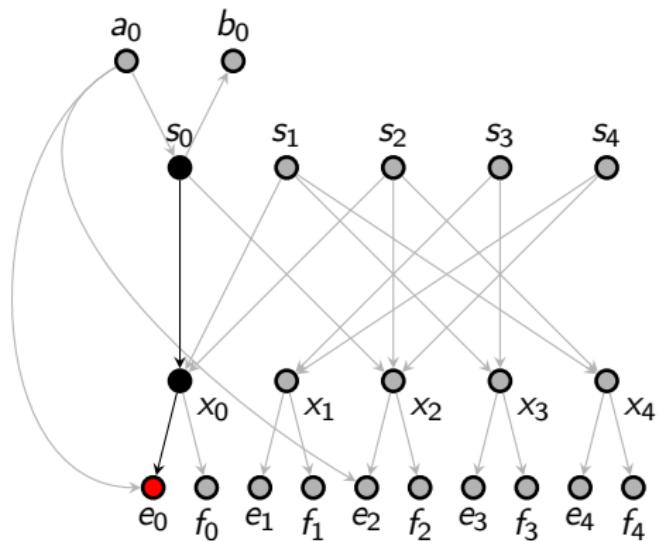
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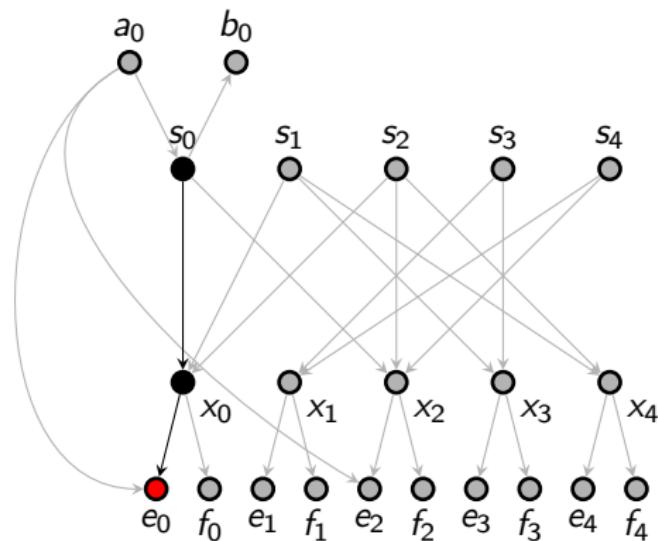
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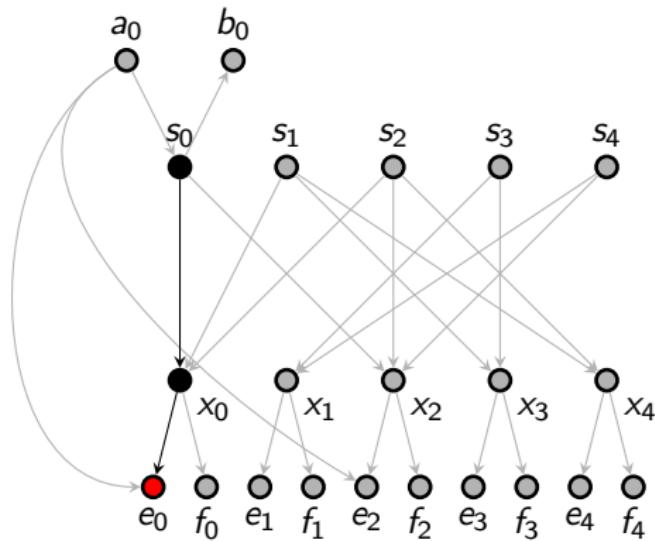
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For every X_i , either x_i or some s_j with $X_i \in S_j$ is in M .

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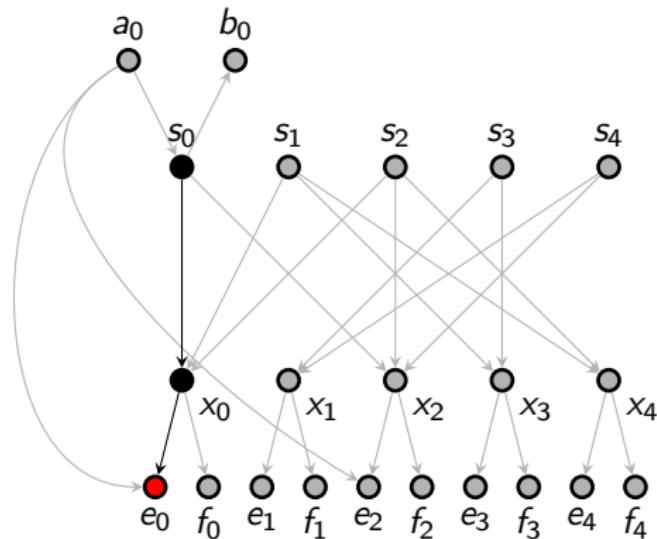
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If $s_j \in M$ then we are done !

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If some $x_i \in M$, then we remove it and add an arbitrary s_j to M , with $X_i \in S_j$.

Conclusion

We have proven the following results on oriented graphs:

	non-oriented	oriented
Trees	leaves	sources and sinks
Cycles	$3(4 \text{ for } C_4)$	$2 \leq \text{mag} \leq n$
K_n	n	either $n - 1$ or n
Decision problem	NP-hard	NP-hard

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A few perspectives:

- To follow up on the idea of networks, we can study the interaction of monitoring with local constraints on all subgraphs.
- Some other results have been proven for the non-oriented case and the bounds are not known in the oriented case.
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