

# Monitoring edge-geodetic sets on oriented graphs

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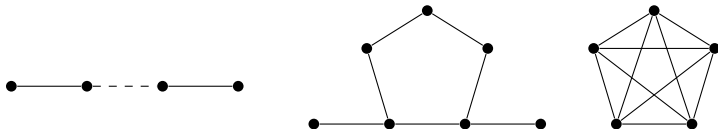
Indian Institute of Technology Dharwad, India

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## Definition

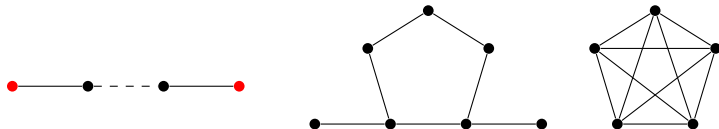
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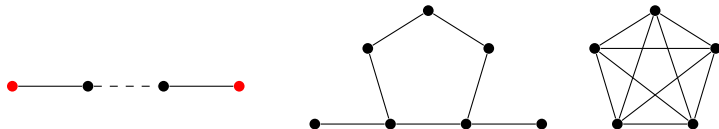
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This object simulates probes monitoring a network: if the value of the ping between two probes increases, then one can know where the failure happened.

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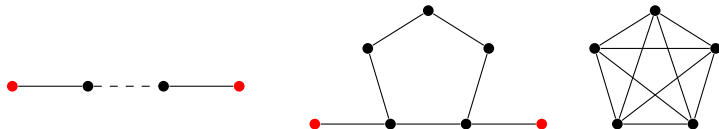
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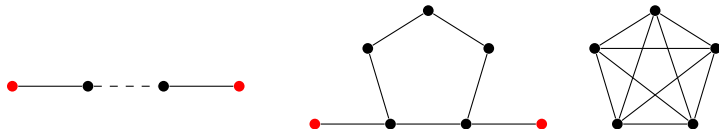
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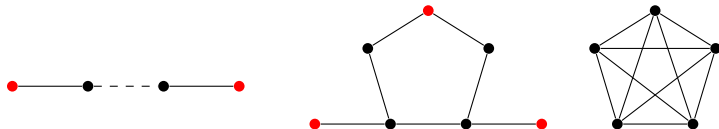
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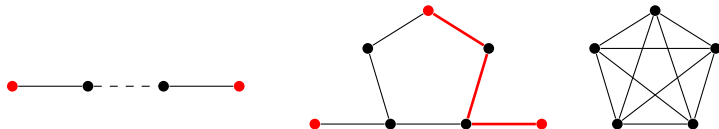
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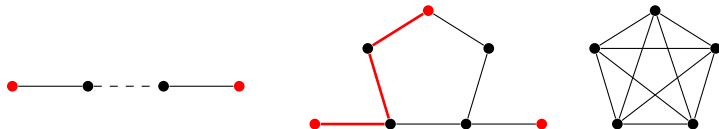
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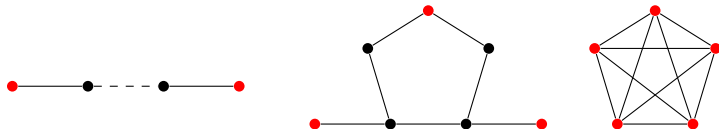
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# A few results on the *meg* parameter

Theorem [Foucaud *et al.*, 2023]

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## Theorem [Haslegrave, 2023]

The decision problem of determining for a graph  $G$  and a natural number  $k$  whether  $\text{meg}(G) \leq k$  is NP-complete.

# Oriented version

We consider orientations of simple graphs, without digones.

## Definition

In an oriented graph  $\vec{G}$ , two vertices  $x$  and  $y$  are said to **monitor** an arc  $\vec{a}$  if  $\vec{a}$  belongs to all oriented shortest paths from  $x$  to  $y$  or from  $y$  to  $x$ .

## Definition

A **monitoring arc-geodetic set**, or MAG-set, of an oriented graph  $\vec{G}$  is a vertex subset  $M \subseteq V(\vec{G})$  such that given any arc  $\vec{a}$  of  $A(\vec{G})$ ,  $\vec{a}$  is monitored by  $x, y$ , for some  $x, y \in M$ . For an oriented graph  $\vec{G}$ , we denote  $mag(\vec{G})$  the size of a smallest MAG-set of  $\vec{G}$ .



# First results

First note that for an oriented graph  $\vec{G}$ , the relation between  $mag(\vec{G})$  and  $meg(G)$  is not clear:



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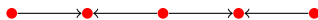


Remark [Das *et al.*, 2023+]

Let  $\vec{G}$  be an oriented graph, and  $x \in V(\vec{G})$ . If  $x$  is either a source or a sink, then  $x$  is in all MAG-set of  $\vec{G}$ .

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Theorem [Das *et al.*, 2023+]

Let  $\vec{G}$  be an oriented tree. There is a unique minimal MAG-set to  $\vec{G}$ , and it is exactly the set of sources and sinks of  $\vec{G}$ .

# Tournaments

Theorem [Das *et al.*, 2023+]

Let  $\vec{G}$  be an orientation of  $K_n$  for some  $n \in \mathbb{N}^*$ . Then  $\text{mag}(\vec{G}) \in \{n-1, n\}$ .

Since one can check in polynomial type if a set of vertices of  $\vec{G}$  is an MAG-set, we can now easily characterize all tournaments for this parameter.





# Complexity of computing the MAG-set size

We consider the following decision problem:

MAG-SET problem

**Instance:** An oriented graph  $\vec{G}$ , an integer  $k$ .

**Question:** Does there exist an MAG-set of  $\vec{G}$  of size  $k$ ?

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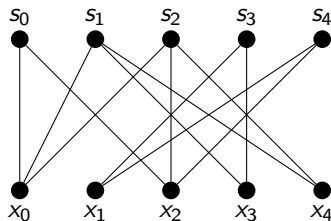
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Theorem [Das *et al.*, 2023+]

The MAG-SET problem is NP-complete.

# The SETCOVER problem

We proceed with a reduction from the SETCOVER problem.



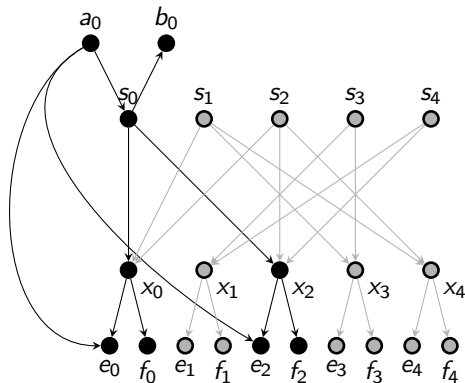
SETCOVER Problem:

**Instance:** A set  $\{X_0, X_2, \dots, X_n\}$ , sets  $\{S_0, S_1, \dots, S_m\}$  such that  $\cup_{i=0}^m S_i = \{X_0, X_2, \dots, X_n\}$  and an integer  $k$ .

**Question:** Does there exist a subcollection of at most  $k$  sets  $S_i$ 's such that their union is  $\{X_0, X_2, \dots, X_n\}$ .

# Our gadget for the reduction

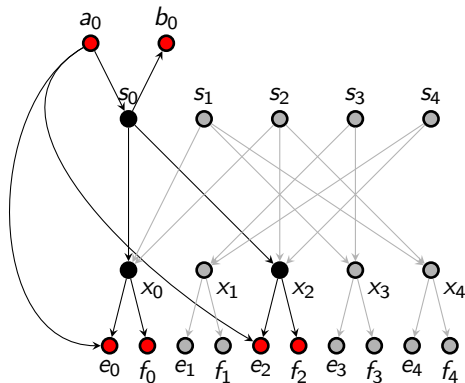
From any instance of SETCOVER  $I$ , we can compute  $\overrightarrow{G(I)}$  an instance of MAG-SET. Assume we have  $M$  an MAG-set of  $\overrightarrow{G(I)}$ .



We now need to study the properties of any MAG-set of  $\overrightarrow{G(I)}$ .

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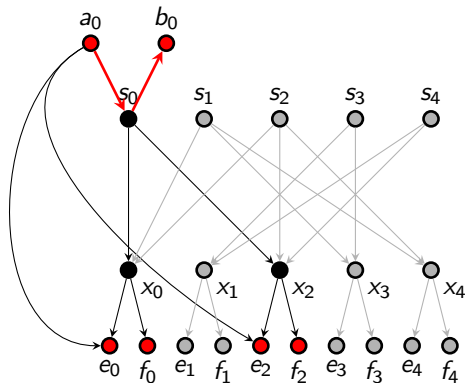
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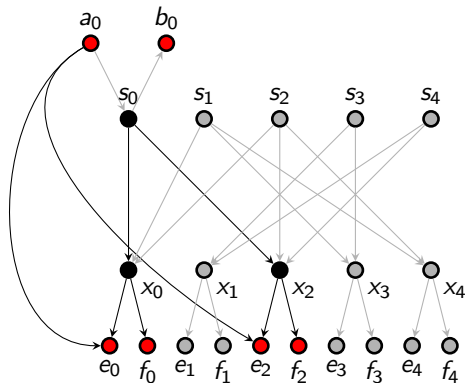
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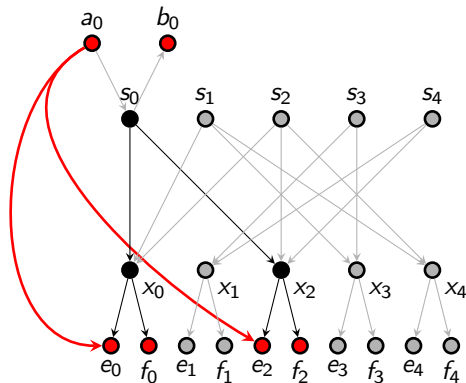
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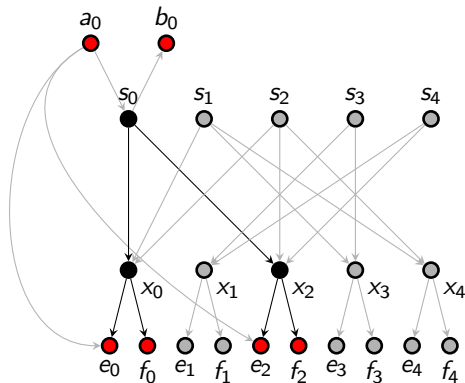


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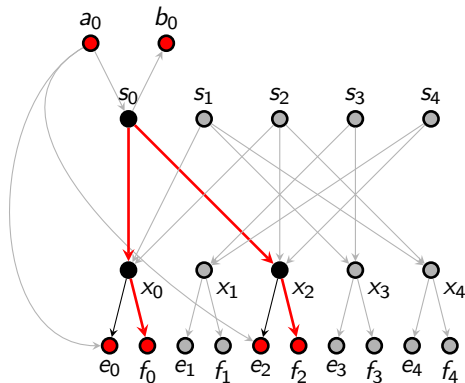
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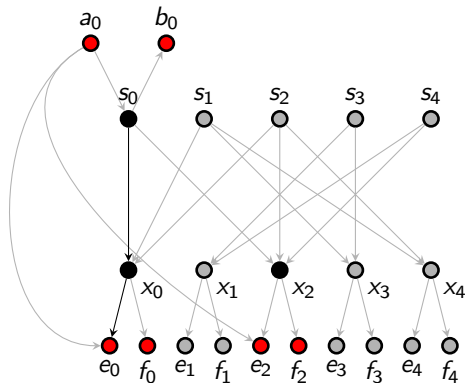
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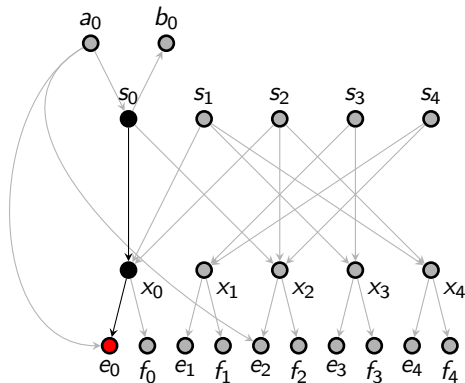
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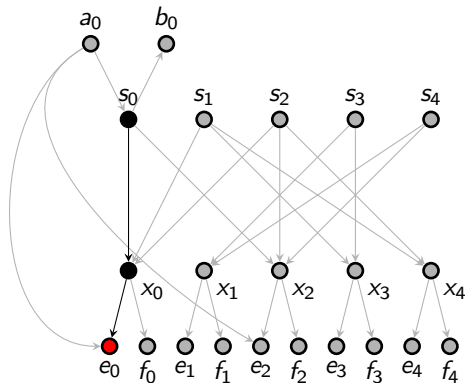
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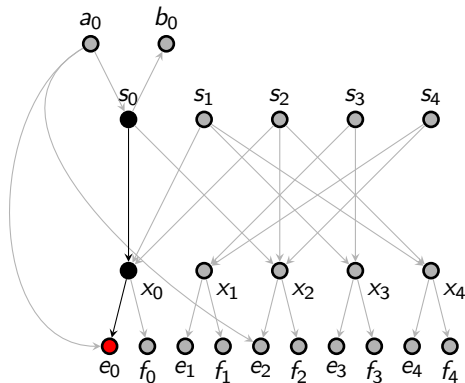
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For every  $X_i$ , either  $x_i$  or some  $s_j$  with  $X_i \in S_j$  is in  $M$ .

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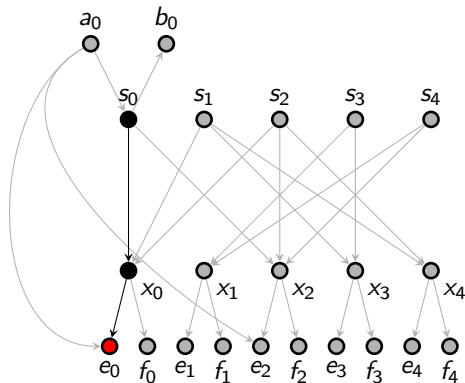
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If  $s_j \in M$  then we are done !

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If some  $x_i \in M$ , then we remove it and add an arbitrary  $s_j$  to  $M$ , with  $X_i \in S_j$ .

# Conclusion

We have proven the following results on oriented graphs:

	non-oriented	oriented
Trees	leaves	sources and sinks
Cycles	$3(4 \text{ for } C_4)$	$2 \leq \text{mag} \leq n$
$K_n$	$n$	either $n - 1$ or $n$
Decision problem	NP-hard	NP-hard



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A few perspectives:

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- Some other results have been proven for the non-oriented case and the bounds are not known in the oriented case.
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Thank you for your attention!