An Improved Bound for Equitable Proper Labellings

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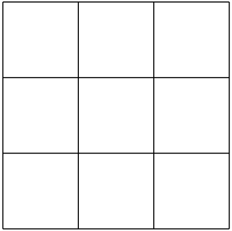
IWOCA, July 2, 2024

Magic labellings



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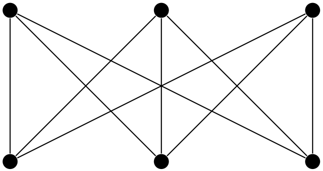

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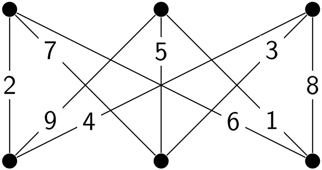

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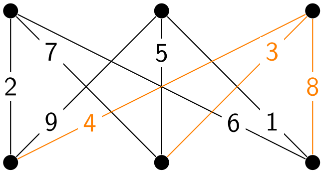

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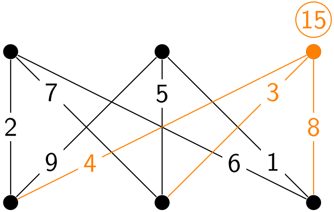

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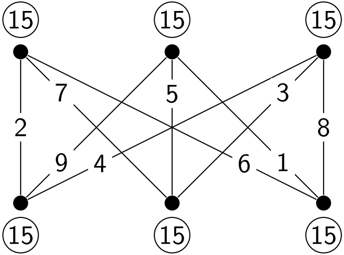

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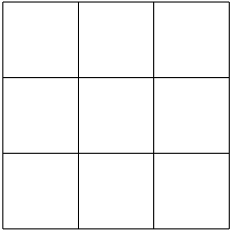

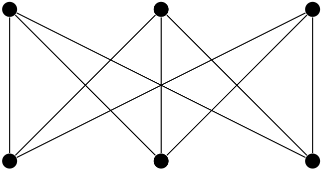

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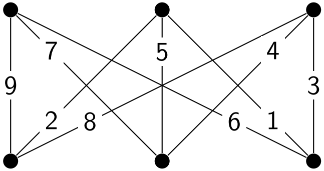

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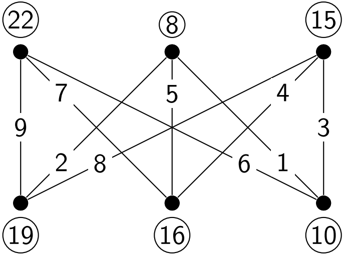

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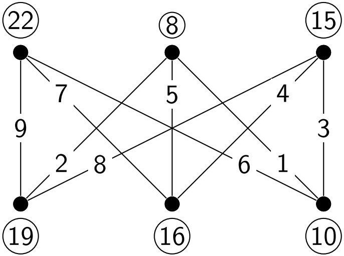

Definition

A *labelling* is said to be (locally) *distinguishing* if any pair of (adjacent) vertices have different *resulting sum*.

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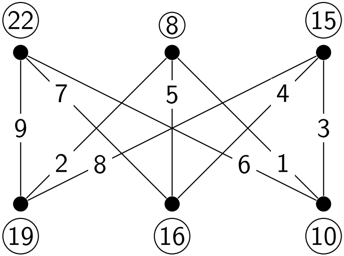

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We will get back to all this terminology later.

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A *k-labelling* of a graph G is a function $\ell : E(G) \to \{1, \dots, k\}$.

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Definition (related to Irregularity Strength)

We say a labelling ℓ is *distinguishing* if for every two vertices u and v of G, $\sigma_\ell(u) \neq \sigma_\ell(v)$.

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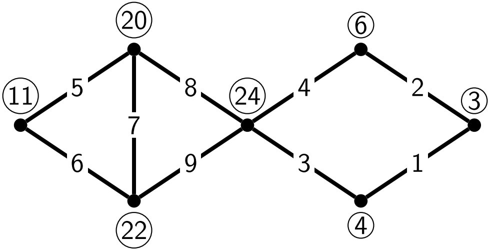
We denote $\overline{\chi_\Sigma}(G)$ the smallest integer k such that G admits an equitable k-labelling.

Theorem (consequence of Lyngsie and Zhong, 2018)

If G is a "nice" graph, then $\overline{\chi_\Sigma}(G) \leq |E(G)|$.

Examples

An equitable labelling which happens to be antimagic:



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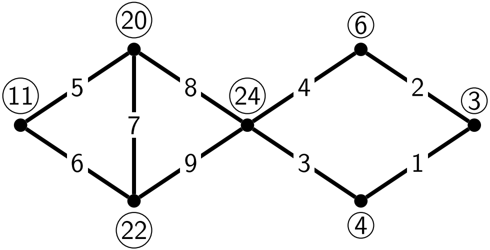
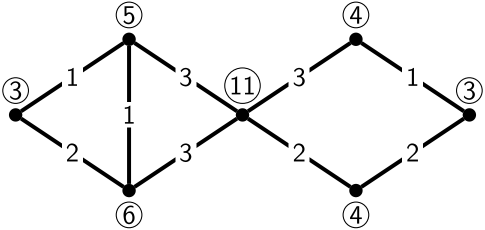


An equitable 3-labelling:



Conjecture and contribution

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Theorem (Bensmail, M., 2024+)

If G is a nice graph, then $\overline{\chi_\Sigma}(G) \leq \left\lfloor \frac{|E(G)|}{2} \right\rfloor + 2$.

Preliminary work

We consider the *sequence of labels* $L = (1, 1, 2, 2, \ldots, k+1, k+1, k+2, k+2)$ (where $k = \left\lfloor \frac{|E(G)|}{2} \right\rfloor$).

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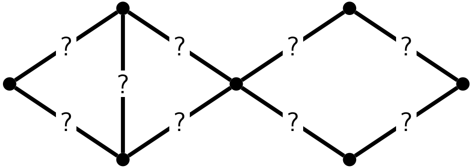

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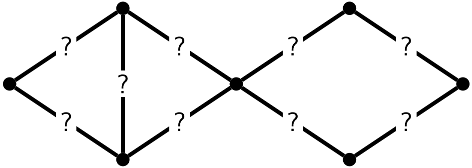

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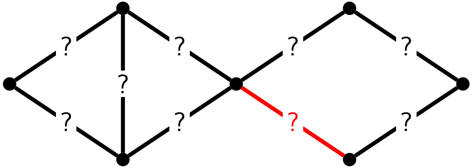

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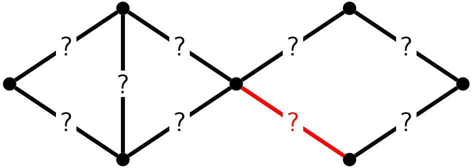

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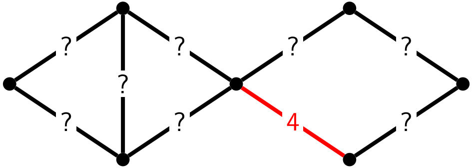

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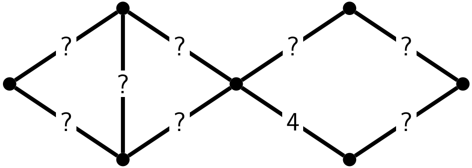

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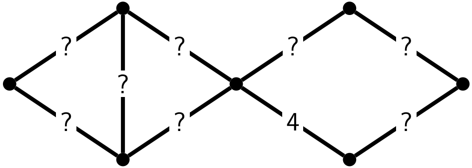

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Question

How do we carry the fact that some edges received a label?

Weight function

Solution

We will keep the information by updating a *weight function*.

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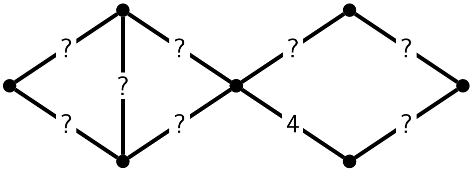

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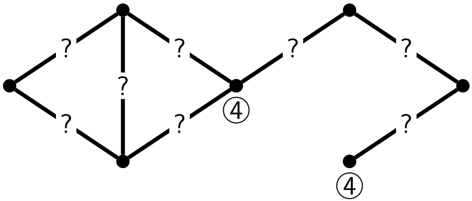

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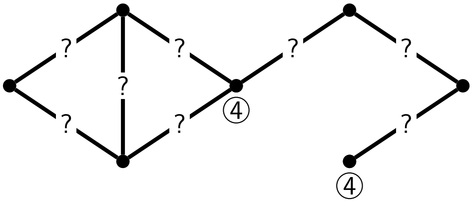

We now consider the weighted graph (G, c) where c is the weight function.

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Problem

Once all the edges incident to u have been labelled, there is no way to change $\sigma_\ell(u)$.

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- Ensure that the vertex at hand will have a resulting sum smaller than all vertices treated later in the induction.

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Problem

Maybe $G - u$ has a component isomorphic to K_2.

Ideas of the proof

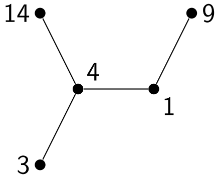
- Proceed by induction on the number of vertices.
- Build a partial labelling of the graph, and extend it.

- Ensure that a vertex will have a resulting sum smaller than the vertex treated later in the induction.

- Handle exceptions on the way.

Vertex of lowest potential

We want to find the vertex with the lowest potential resulting sum:



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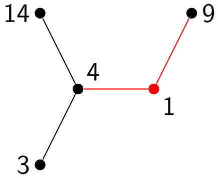

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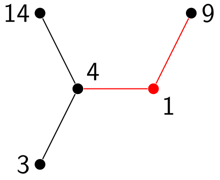

Here, the minimum possible resulting sum for this vertex is $1 + 6 + 7 = 14$.
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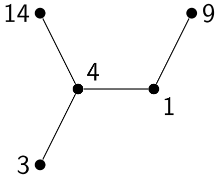

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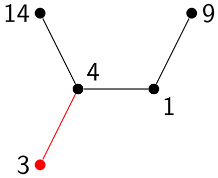

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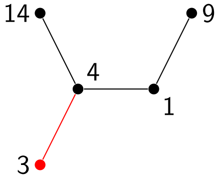

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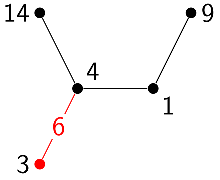

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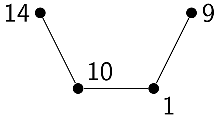
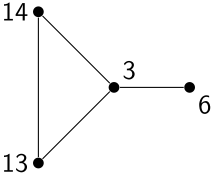


For instance, consider $L = (7, 7, 8, 8, 9, 9)$. Each vertex is annoted with its current weight.

Handling exceptions

Consider this weighted graph with $L = (1, 1, 2, 2, 3, 3, 4, 4)$:



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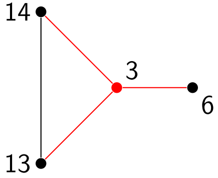

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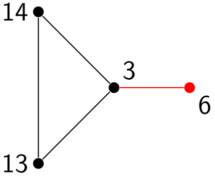

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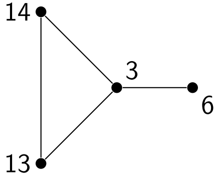

We choose a vertex of highest degree amongst vertices of lowest potential.

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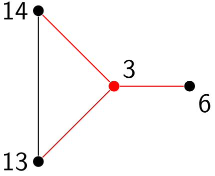

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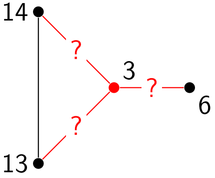

How to assign each label?

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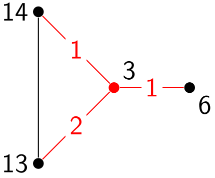

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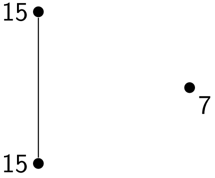
Consider this weighted graph with $L = ({\color{red}1}, {\color{red}1}, {\color{red}2}, 2, 3, 3, 4, 4)$:



Problem

A component isomorphic to K_2 with constant weight cannot be labelled.

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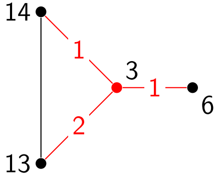

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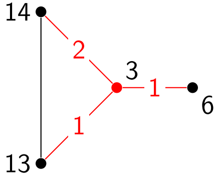

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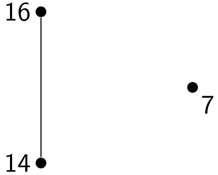

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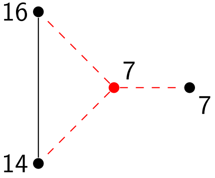

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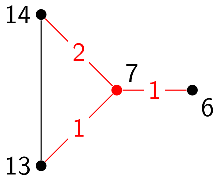

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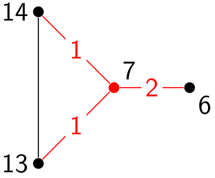

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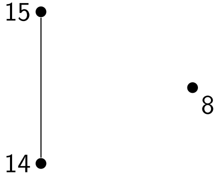

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Thank you for your attention!

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They are some configurations where you can not use the smallest labels. Assume for instance the sequence of labels is $(1, 1, 2, 2, 3, 3)$

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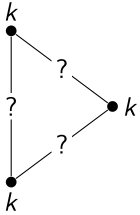

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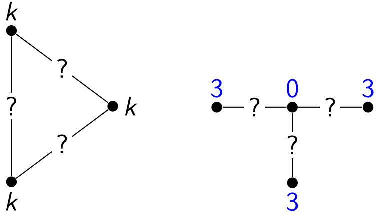

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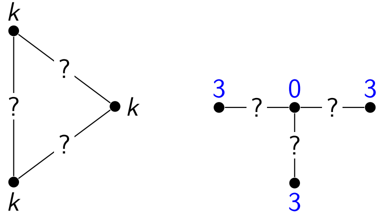

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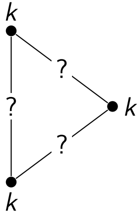

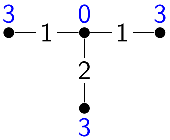

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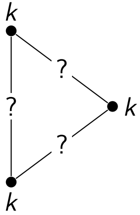

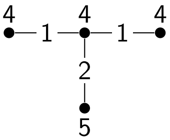

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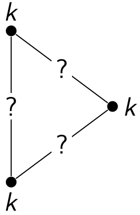

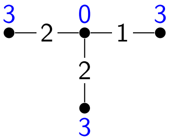

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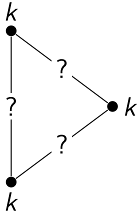

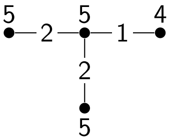

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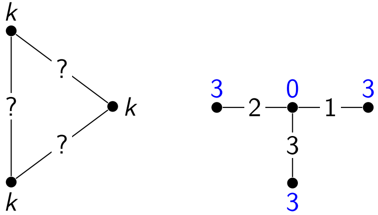

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