Modeling of a Private Monopolist Insurer: Theory and Numerical Analysis

Project report

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1 Introduction

Within the framework of the *Optimisation et calcul de Variation* course, we were tasked with a project modeling an insurer's strategy in a simplified scenario, initially with a single contract, later extended to multiple contracts. We first present the theoretical and numerical aspects of the single-contract case (mainly based on the work of Braouezec et al., 2023), then generalize to multiple contracts.

2 Model and Fundamental Equations

We consider a continuum of risk-averse agents indexed by their type $\theta \in [0,1]$, where θ denotes the agent's probability of experiencing a financial loss of magnitude $L > 0$ within a given period. Each agent is endowed with the same initial wealth $W_0 > 0$ and possesses identical preferences represented by a twice continuously differentiable, strictly increasing and concave utility function $U(W)$. The insurer, acting as a monopolist, offers a single non-mandatory insurance contract $C = (P, R)$, where $P \in [0, L]$ is the premium paid upfront, and $R \in [0, L]$ is the indemnity received in the event of damage.

An agent of type θ evaluates the insurance offer by comparing her expected utility with and without insurance. Without insurance:

$$V(\theta, 0) = \theta U(W_0 - L) + (1 - \theta)U(W_0).$$

With contract $C = (P, R)$:

$$V(\theta, C) = \theta U(W_0 + R - L - P) + (1 - \theta)U(W_0 - P).$$

The agent accepts the contract if and only if $G(\theta, P, R) := V(\theta, C) - V(\theta, 0) \geq 0$.

3 One Unique Insurance Contract

3.1 Market Segmentation and Critical Type

Given a contract $C = (P, R)$, we define the *critical type* $\theta_c \in [0, 1)$ as the unique solution to:

$$G(\theta_c, P, R) = 0.$$

All agents with $\theta \geq \theta_c$ prefer the insurance contract, while those with $\theta < \theta_c$ opt out. The measure of the insured population is $Q(\theta_c) := 1 - F(\theta_c)$, where F is the CDF and $f = F'$.

Willingness-to-Pay and Premium Function

Assuming CARA utility $U(W) = 1 - e^{-\lambda W}$ with $\lambda > 0$, the willingness-to-pay $P(\theta, R)$ solves the indifference condition:

$$V(\theta, C) = V(\theta, 0).$$

Using the CARA form:

$$\begin{aligned} V(\theta, C) &= \theta \left(1 - e^{-\lambda(W_0 + R - L - P)}\right) + (1 - \theta)\left(1 - e^{-\lambda(W_0 - P)}\right), \\ V(\theta, 0) &= \theta \left(1 - e^{-\lambda(W_0 - L)}\right) + (1 - \theta)\left(1 - e^{-\lambda W_0}\right). \end{aligned}$$

Subtracting both sides, eliminating the constants, and rearranging terms, we get:

$$\begin{aligned} \theta e^{-\lambda(W_0 + R - L - P)} + (1 - \theta) e^{-\lambda(W_0 - P)} &= \theta e^{-\lambda(W_0 - L)} + (1 - \theta) e^{-\lambda W_0} \\ \Leftrightarrow \quad e^{-\lambda W_0}\left[\theta e^{\lambda(L - R)} + (1 - \theta)\right] &= \theta e^{\lambda L} + (1 - \theta), \\ \Leftrightarrow \quad e^{\lambda P}\left[\theta e^{\lambda(L - R)} + (1 - \theta)\right] &= \theta e^{\lambda L} + (1 - \theta). \end{aligned}$$

Finally, we arrive at the simplified expression:

$$P(\theta, R) = \frac{1}{\lambda} \ln\left(\frac{\theta e^{\lambda L} + (1 - \theta)}{\theta e^{\lambda(L - R)} + (1 - \theta)}\right), \tag{1}$$

which represents the maximum premium that an agent of type θ is willing to pay for a contract offering indemnity R against a potential loss L.

Average Probability of Damage and Insurer Profit

The average risk of the insured population:

$$A(\theta_c) := \frac{1}{1 - F(\theta_c)} \int_{\theta_c}^{1} x f(x)\, dx.$$

The insurer's total profit:

$$\Pi(\theta_c, R) = [P(\theta_c, R) - R \cdot A(\theta_c)] \cdot (1 - F(\theta_c)).$$

This function is continuous, and under mild regularity assumptions, there exists a unique optimal pair (θ^*, R^*) that maximizes profit. By the Extreme Value Theorem, the existence of such an optimum is guaranteed since the domain is compact and Π is continuous.

Numerical Implementation

Assuming a Beta distribution:

$$f(\theta) \propto \theta^{a-1}(1-\theta)^{b-1}, \quad \theta \in [0,1],$$

with a= 2, b=5, L=1 and $\lambda = 3$. We solve the following problem using the L-BFGS-B algorithm, which is well-suited for strictly convex functions with bound constraints:

$$\max_{\theta\in[0,1), R\in(0,L)} [P(\theta, R) - R \cdot A(\theta)] \cdot Q(\theta).$$

This is

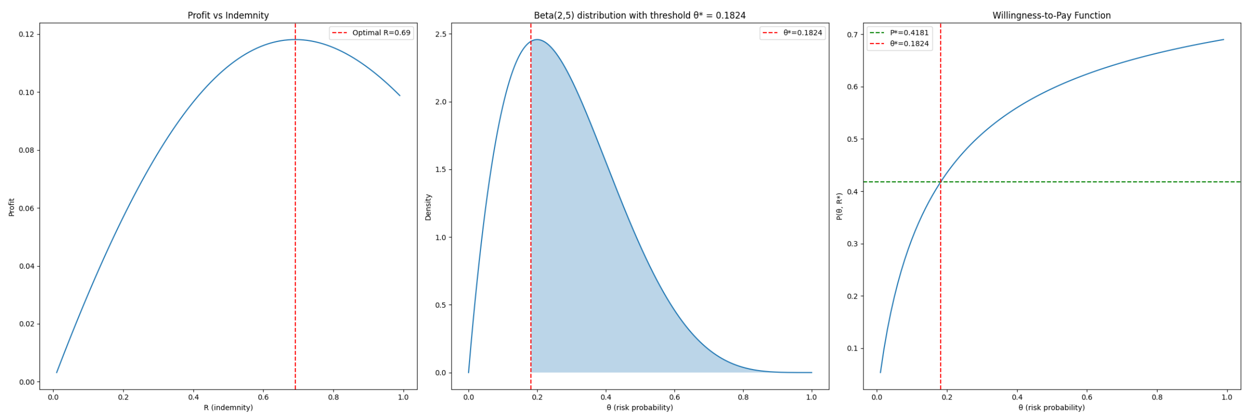

Figure 1: From left to right: $\Pi(R)$ vs R, Beta distribution with threshold θ^*, $P(\theta, R^*)$ with WTP curve.

The optimal indemnity is $R^* = 0.6913$ and threshold $\theta^* = 0.1824$, corresponding to an optimal profit $\Pi^*(\theta^*, R^*) = 0.1181$. The majority (69%) purchase the contract.

4 Optimization of Multiple Contracts

4.1 Theoretical Analysis

We expand to N contracts, each with parameters (θ_i, R_i). The profit per segment $[\theta_i, \theta_{i+1}]$ (with the convention $\theta_{N+1} = 1$) is:

$$\Pi_i = (P(\theta_i, R_i) - R_i \cdot A_i) \cdot Q_i$$

with:

$$A_i = \frac{\int_{\theta_i}^{\theta_{i+1}} \theta' f(\theta')\, d\theta'}{F(\theta_{i+1}) - F(\theta_i)}$$

The total expected profit:

$$\max_{\{(\theta_i, R_i)\}} \sum_{i=1}^{N} \Pi_i$$

with the constraints :

- $\theta_1 < \cdots < \theta_N$,
- $\theta_i \in (0, 1]$,
- $R_i \in (0, L)$.

4.2 Numerical Results

Optimization is performed using the differential evolution method to minimize the negative total profit. This algorithm is particularly suited to handling functions with many local minima, as well as the increasing number of constraints as N grows.

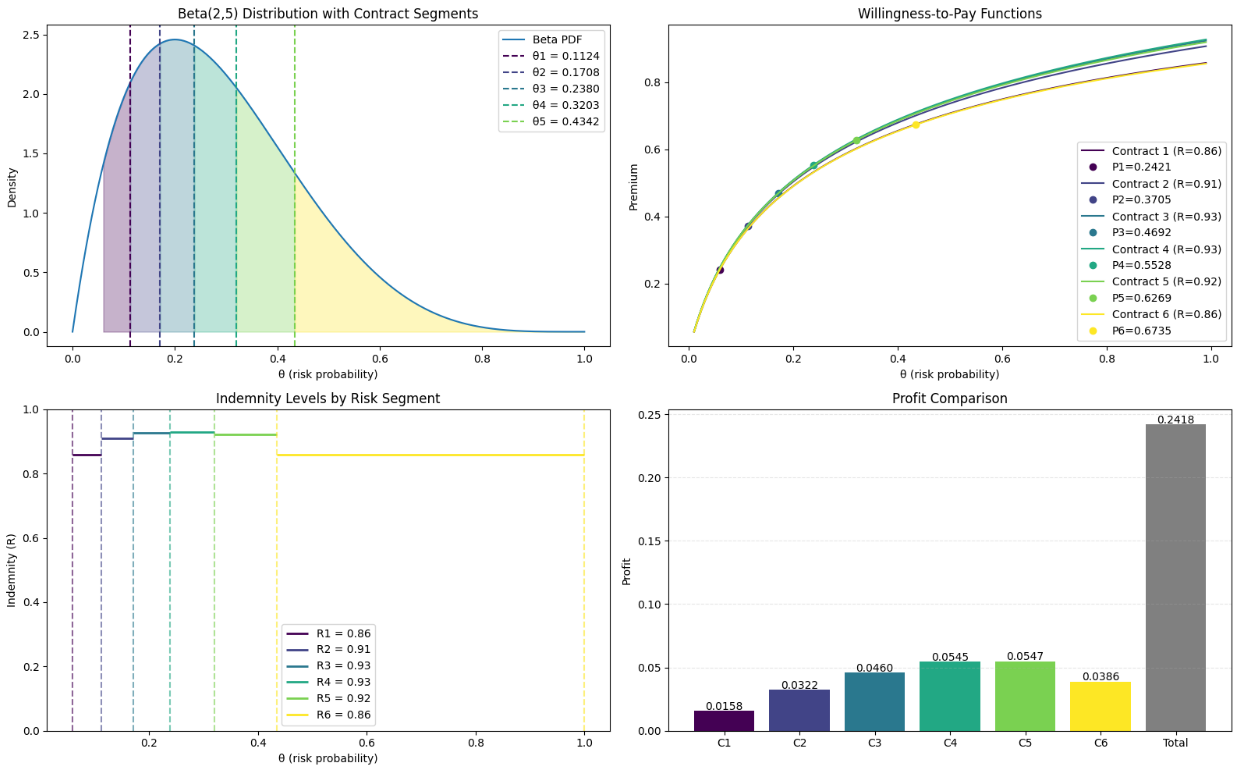

Figure 2: Results for N=6: Beta distribution with the different thresholds, WTP per contract, indemnity levels by risk segment, and profit for each contracts.

Unfortunately, the optimization algorithm fails above $N = 6$ due to constraint violations on θ order. No solution has been found to resolve this issue.

5 Continuum of contracts

This case naturally extends the previous one, pushing $N \to \infty$. For that we define a contract function:

$$R : [0,1] \to \mathbb{R}$$

where each individual with risk type $\theta \in [0,1]$ is offered a personalized contract $R(\theta)$. That is, the principal offers a continuously varying reimbursement depending on each individual's risk level.

The total expected profit, when contracts are personalized over a continuum of types, is given by the functional

$$\Pi[R(\cdot)] = \int_0^1 [P(\theta, R(\theta)) - R(\theta)A(\theta)]\, f(\theta)\, d\theta \tag{2}$$

However, in this case, $A(\theta) = \theta$. Therefore, we want to maximize :

$$\max_{R(\cdot)} \Pi[R(\cdot)] = \int_0^1 [P(\theta, R(\theta)) - R(\theta)\theta]\, f(\theta)\, d\theta,$$

For each θ the integrand, noted $\mathcal{L}(\theta, R) = P(\theta, R(\theta)) - R(\theta)\theta$ should be maximized with respect to $R(\theta)$ within the interval $(0, L)$. Therefore, we consider the following necessary condition :

$$\partial_R P(\theta, R(\theta)) - \theta = 0$$

Using equation 1, we get

$$\frac{\theta e^{\lambda(L-R)}}{\theta e^{\lambda(L-R)} + 1 - \theta} - \theta = 0$$

Solving the first-order condition yields $R^*(\theta) = L$ implying the optimal strategy is to provide the maximum reimbursement to all individuals — i.e., full insurance coverage. This holds for all $\theta \in (0,1)$. The second derivative of $\mathcal{L}$ is with respect to R is negative, confirming the concavity of the objective and validating the sufficiency of the first-order condition. Figure 3 shows the value of $\mathcal{L}$ with respect to θ. Finally, using eq 2, we find an optimal profit $\Pi^* = 0.29$.

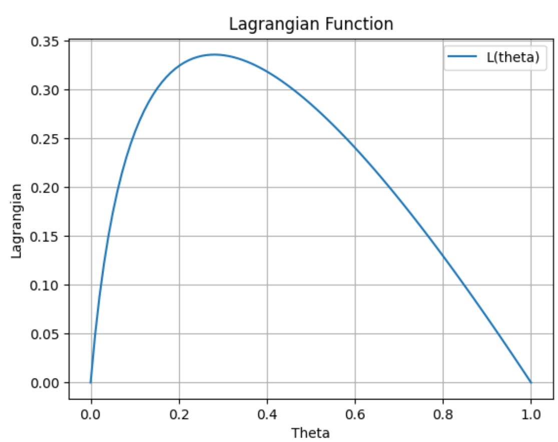

Figure 3: Lagrangian integrand $\mathcal{L}(\theta, R)$ evaluated at $R^*(\theta) = L$