Modeling of a Private Monopolist Insurer: Theory and Numerical Analysis

Project report

April 15, 2025

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1 Introduction

Within the framework of the *Optimisation et calcul de Variation* course, we were tasked with a project modeling an insurer's strategy in a simplified scenario, initially with a single contract, later extended to multiple contracts. We first present the theoretical and numerical aspects of the single-contract case (mainly based on the work of Braouezec et al., 2023), then generalize to multiple contracts.

2 Model and Fundamental Equations

We consider a continuum of risk-averse agents indexed by their type $\theta \in [0, 1]$, where θ denotes the agent's probability of experiencing a financial loss of magnitude $L > 0$ within a given period. Each agent is endowed with the same initial wealth $W_0 > 0$ and possesses identical preferences represented by a twice continuously differentiable, strictly increasing and concave utility function $U(W)$. The insurer, acting as a monopolist, offers a single non-mandatory insurance contract $C = (P, R)$, where $P \in [0, L]$ is the premium paid upfront, and $R \in [0, L]$ is the indemnity received in the event of damage.

An agent of type θ evaluates the insurance offer by comparing it's expected utility with and without insurance.

- Without insurance:

$$V(\theta, 0) = \theta U(W_0 - L) + (1 - \theta)U(W_0).$$

- With contract $C = (P, R)$:

$$V(\theta, C) = \theta U(W_0 + R - L - P) + (1 - \theta)U(W_0 - P).$$

The agent accepts the contract if and only if $G(\theta, P, R) := V(\theta, C) - V(\theta, 0) \geq 0$.

3 One Unique Insurance Contract

3.1 Market Segmentation and Critical Type

Given a contract $C = (P, R)$, we define the *critical type* $\theta_c \in [0, 1)$ as the unique solution to:

$$G(\theta_c, P, R) = 0.$$

All agents with $\theta \geq \theta_c$ prefer the insurance contract, while those with $\theta < \theta_c$ opt out. The measure of the insured population is $Q(\theta_c) := 1 - F(\theta_c)$, where F is the CDF and $f = F'$.

Willingness-to-Pay and Premium Function

Assuming CARA utility $U(W) = 1 - e^{-\lambda W}$ with $\lambda > 0$, the willingness-to-pay $P(\theta, R)$ solves the indifference condition:

$$V(\theta, C) = V(\theta, 0).$$

Using the CARA form:

$$\begin{aligned} V(\theta, C) &= \theta\left(1 - e^{-\lambda(W_0+R-L-P)}\right) + (1-\theta)\left(1 - e^{-\lambda(W_0-P)}\right), \\ V(\theta, 0) &= \theta\left(1 - e^{-\lambda(W_0-L)}\right) + (1-\theta)\left(1 - e^{-\lambda W_0}\right). \end{aligned}$$

Subtracting both sides, eliminating the constants, and rearranging terms, we get:

$$\begin{aligned} \theta e^{-\lambda(W_0+R-L-P)} + (1-\theta)e^{-\lambda(W_0-P)} &= \theta e^{-\lambda(W_0-L)} + (1-\theta)e^{-\lambda W_0} \\ \Leftrightarrow \quad e^{-\lambda W_0}\left[\theta e^{\lambda(L-R)} + (1-\theta)\right] &= \theta e^{\lambda L} + (1-\theta), \\ \Leftrightarrow \quad e^{\lambda P}\left[\theta e^{\lambda(L-R)} + (1-\theta)\right] &= \theta e^{\lambda L} + (1-\theta). \end{aligned}$$

Finally, we arrive at the simplified expression:

$$P(\theta, R) = \frac{1}{\lambda} \ln\left(\frac{\theta e^{\lambda L} + (1-\theta)}{\theta e^{\lambda(L-R)} + (1-\theta)}\right), \tag{1}$$

which represents the maximum premium that an agent of type θ is willing to pay for a contract offering indemnity R against a potential loss L.

Average Probability of Damage and Insurer Profit

The average risk of the insured population:

$$A(\theta_c) := \frac{1}{1 - F(\theta_c)} \int_{\theta_c}^{1} x f(x)\, dx.$$

The insurer's total profit:

$$\Pi(\theta_c, R) = [P(\theta_c, R) - R \cdot A(\theta_c)] \cdot (1 - F(\theta_c)).$$

This function is continuous, and under mild regularity assumptions, there exists a unique optimal pair (θ^*, R^*) that maximizes profit. By the Extreme Value Theorem, the existence of such an optimum is guaranteed since the domain is compact and Π is continuous.

Numerical Implementation

We assume a Beta distribution:

$$f(\theta) \propto \theta^{a-1}(1-\theta)^{b-1}, \quad \theta \in [0,1],$$

with $a = 2, b = 5, L = 1$ and $\lambda = 3$.

We solve the following problem using the L-BFGS-B algorithm, which is well-suited for strictly convex functions with bound constraints:

$$\max_{\theta \in [0,1), R \in (0,L)} [P(\theta, R) - R \cdot A(\theta)] \cdot Q(\theta).$$

The results of the implementation are shown below :

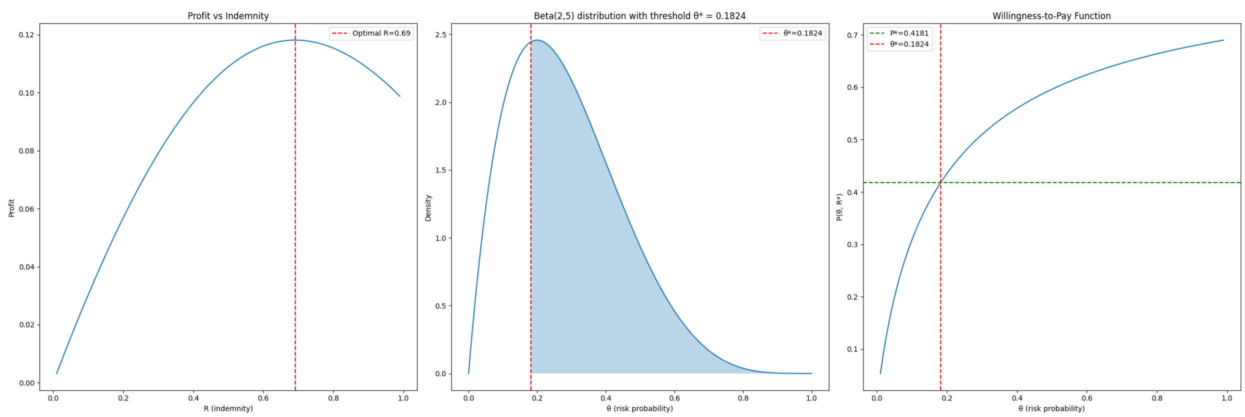

Figure 1: From left to right: $\Pi(R)$ vs R, Beta distribution with threshold θ^*, $P(\theta, R^*)$ with WTP curve.

The optimal indemnity is $R^* = 0.6913$ and threshold $\theta^* = 0.1824$, corresponding to an optimal profit $\Pi^*(\theta^*, R^*) = 0.1181$. The majority (69%) purchase the contract.

3.2 Risk aversion influence

The utility function $U(W) = 1 - e^{-\lambda W}$, models the agent behavior, where $\lambda > 0$ $\lambda > 0$ represents the agent's absolute risk aversion. As $\lambda \to 0$, $U(W)$ converges to a linear utility function, $U(W) \approx W$. In this limit, is "risk-neutral". But as λ increases, the agent has a greater aversion to uncertainty. Consequently, the agent is willing to pay a higher risk premium to mitigate potential losses.

We can see for instance how the variables of our problem evolve as λ changes.

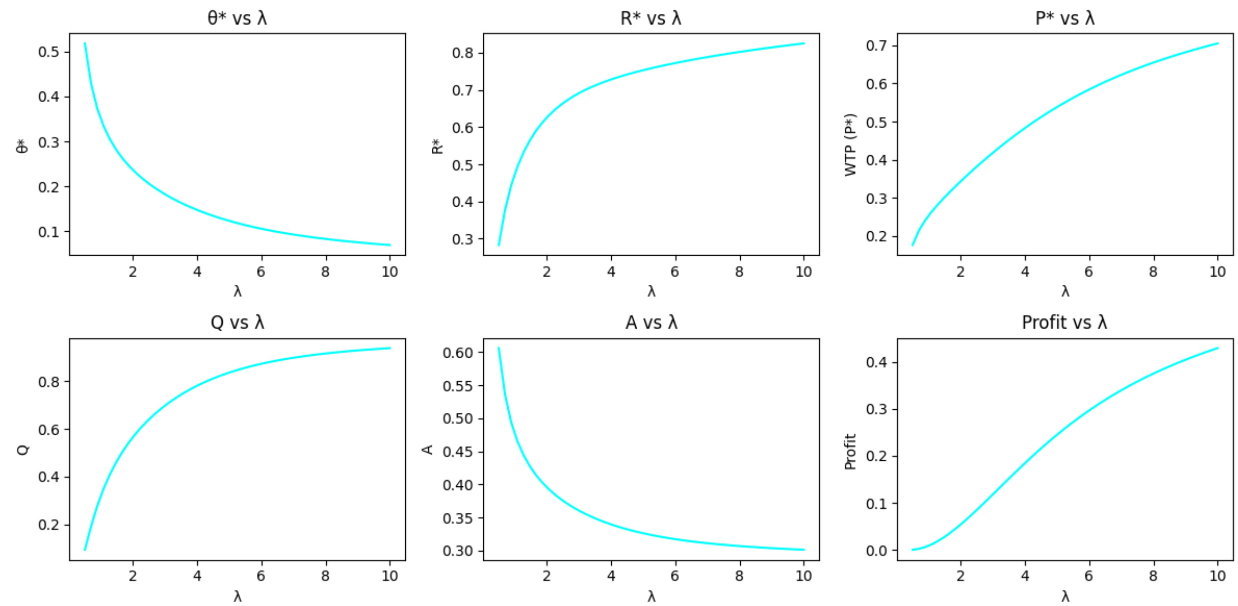

Figure 2: Evolution of variables with λ

All the functions are monotone. Naturally, as λ increases, the agent has greater aversion to risk and thus more agents buy the contract, increasing the insurer profit. However not all the variable have the same derivative: beyond a certain threshold ($\lambda \approx 4$), we can see that R^* and θ^* have weak slope.

4 Optimization of Multiple Contracts

4.1 Theoretical Analysis

We expand to N contracts, each with parameters (θ_i, R_i). The population is subdivided into segments $(\Theta_i)_{i \leq N}$ according to type of agents. We note $\Theta_i = [\theta_i, \theta_{i+1}[$. The agents of types in Θ_i are offered a contract (P_i, R_i) that suits them.

An agent of type $\theta \in \Theta_i$ who selects contract C_i obtains:

$$V(\theta, C_i) = \theta\, U\big(W_0 + R_i - L - P_i\big) + (1-\theta)\, U\big(W_0 - P_i\big). \tag{2}$$

and thus agent chooses to buy it if and only if :

$$V(\theta, C_i) \geq V(\theta, O)$$

The profit per segment $[\theta_i, \theta_{i+1}]$ (with the convention $\theta_{N+1} = 1$) is:

$$\Pi_i = (P(\theta_i, R_i) - R_i \cdot A_i) \cdot Q_i$$

with:

$$A_i = \frac{\int_{\theta_i}^{\theta_{i+1}} \theta' f(\theta')\, d\theta'}{F(\theta_{i+1}) - F(\theta_i)}$$

The total expected profit:

$$\max_{\{(\theta_i, R_i)\}} \sum_{i=1}^{N} \Pi_i$$

with the constraints :

- $\theta_1 < \cdots < \theta_N$,
- $\theta_i \in (0, 1]$,
- $R_i \in (0, L)$.

4.2 Numerical Results

Optimization is performed using the differential evolution method to minimize the negative total profit. This algorithm is particularly suited to handling functions with many local minima, as well as the increasing number of constraints as N grows.

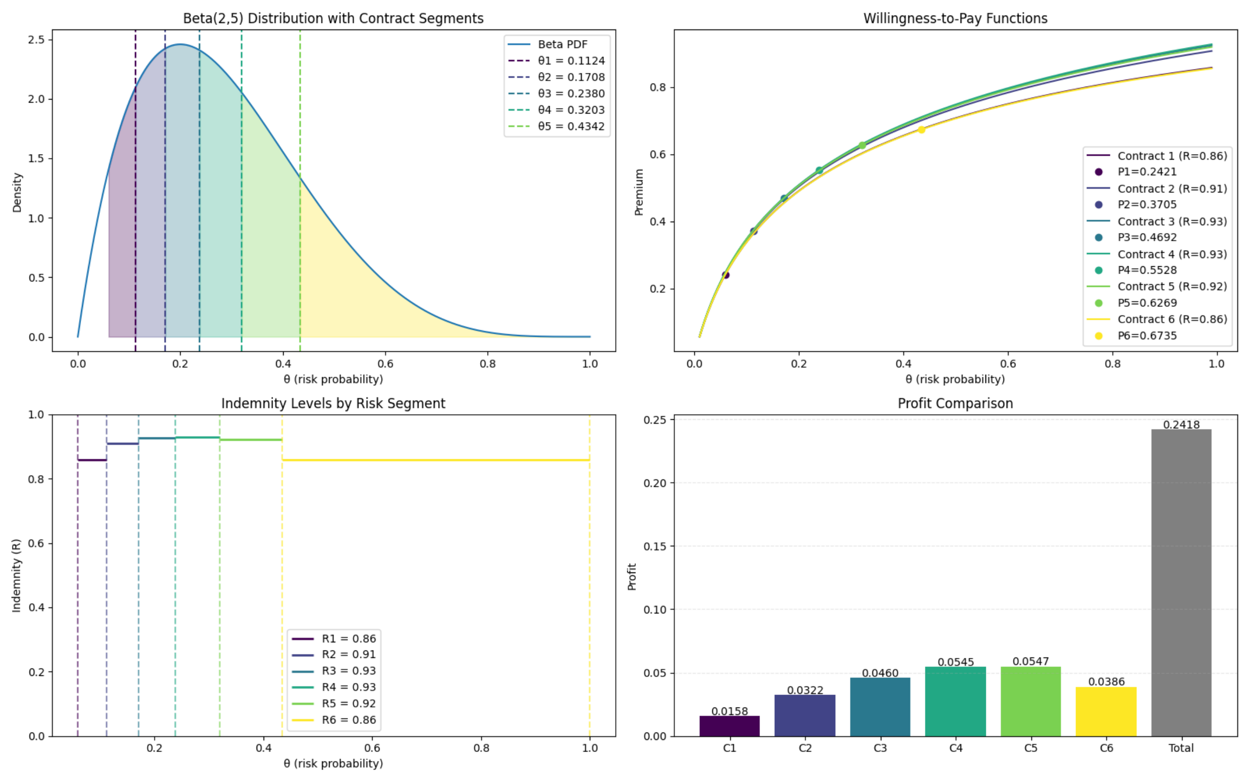

Figure 3: Results for $N = 6$: Beta distribution with the different thresholds, WTP per contract, indemnity levels by risk segment, and profit for each contracts.

We can see that suggesting multiple contracts instead of one increases the total profit of the insurer. Intuitively, more contract options have a higher chance to suit different needs of the population. This is confirmed by the numerical results.

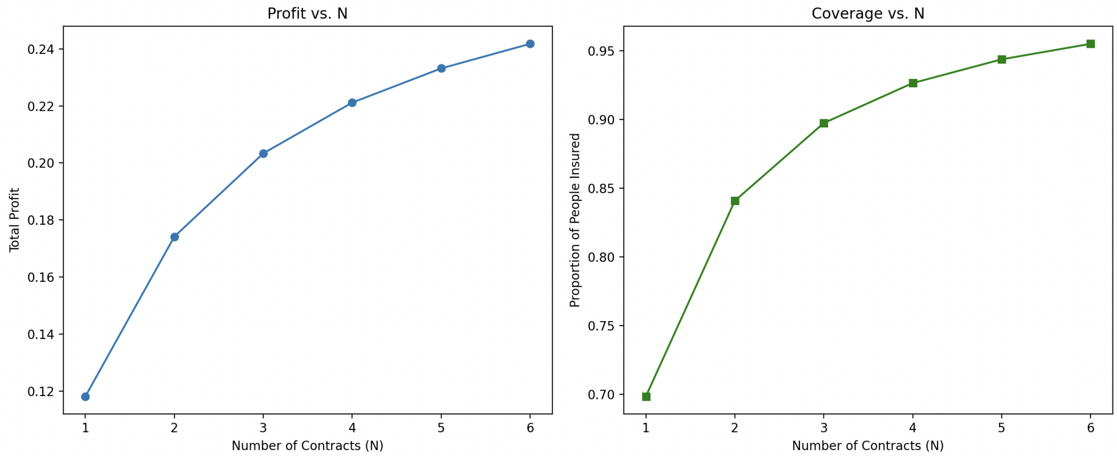

Figure 4: Evolution of the total profit and the proportion of the population covered with N.

The proportion of the population also seems to increase with N, which again makes sens because of the availability of multiple contract options.

Unfortunately, the optimization algorithm fails above $N = 6$ due to constraint violations on θ order. No solution has been found to resolve this issue.

5 Continuum of contracts

This case naturally extends the previous one, pushing $N \to \infty$. For that we define a contract function:

$$R : [0, 1] \to \mathbb{R}$$

where each individual with risk type $\theta \in [0, 1]$ is offered a personalized contract $R(\theta)$. That is, the principal offers a continuously varying reimbursement depending on each individual's risk level.

The total expected profit, when contracts are personalized over a continuum of types, is given by the functional

$$\Pi[R(\cdot)] = \int_0^1 [P(\theta, R(\theta)) - R(\theta)A(\theta)]\, f(\theta)\, d\theta \tag{3}$$

However, in this case, $A(\theta) = \theta$. Therefore, we want to maximize :

$$\max_{R(\cdot)} \Pi[R(\cdot)] = \int_0^1 [P(\theta, R(\theta)) - R(\theta)\theta]\, f(\theta)\, d\theta,$$

For each θ the integrand, noted $\mathcal{L}(\theta, R) = P(\theta, R(\theta)) - R(\theta)\theta$ should be maximized with respect to $R(\theta)$ within the interval $(0, L)$. Therefore, we consider the following necessary condition :

$$\partial_R P(\theta, R(\theta)) - \theta = 0$$

Using equation 1, we get

$$\frac{\theta e^{\lambda(L-R)}}{\theta e^{\lambda(L-R)} + 1 - \theta} - \theta = 0$$

Solving the first-order condition yields $R^*(\theta) = L$ implying the optimal strategy is to provide the maximum reimbursement to all individuals — i.e., full insurance coverage. This holds for all $\theta \in (0, 1)$. The second derivative of $\mathcal{L}$ is with respect to R is negative, confirming the concavity of the objective and validating the sufficiency of the first-order condition. Figure 5 shows the value of $\mathcal{L}$ with respect to θ. Finally, using eq 3, we find an optimal profit $\Pi^* = 0.29$.

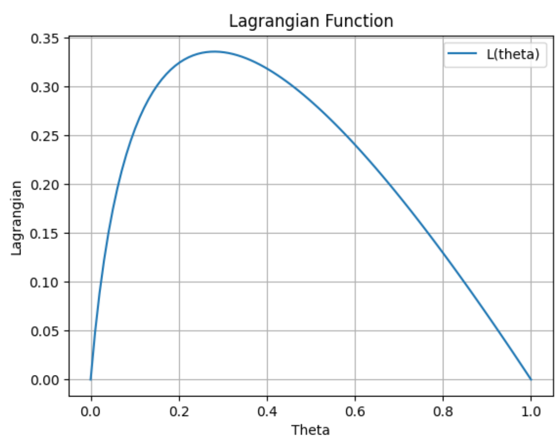

Figure 5:

Once again, the total profit is even greater for the insurer in this theoretical case of personalized contract $R(\theta)$.

6 Conclusion

This project studies a relatively simple yet realistic model of the behavior of a population under a monopolist insurer offering non-mandatory contracts. The single-contract case admits an explicit solution for the profit, which makes calculations easier. The general case is more complex, and the corresponding optimization problem becomes significantly harder as N increases. The numerical results obtained were coherent and logical, illustrating the behavior of agents who rationally choose to get covered depending on their expected utility. Intuitively, the more the insurer adapts to the individual needs of each agent, the more profit he will make, and the larger the portion of the population he will cover. This naturally leads to the final case, where we theoretically consider a continuum of contracts, resulting in maximal profit for the insurer and full insurance coverage for the agents.

This model can lay the groundwork for future studies with more complex and realistic modeling of the market, both rational and irrational agent behavior, the presence of multiple insurance companies, etc.