$\boxed{\text{Exercici 1}}$

$$\left.\begin{array}{c|c|c|c} x_k & -1 & 0 & 1 \\ \hline y_k & 0,3 & 0,6 & 0,8 \end{array} \quad f(x) = \frac{1}{1+a\,e^{-bx}} \right\} E(a,b) = \sqrt{\sum_{k=0}^{2} \left[f(x_k) - y_k\right]^2}$$

ⓐ $a_0 = b_0 = 1$

$$E(1,1)^2 = \left(\frac{1}{1+e^{1}} - 0.3\right)^2 + \left(\frac{1}{1+e^{0}} - 0.6\right)^2 + \left(\frac{1}{1+e^{-1}} - 0.8\right)^2 = 1,571755\times10^{-2}$$

$$\boxed{E(1,1) = 1,253697\times10^{-1}}$$

ⓑ $h = 0.1$

($a_0 \pm h, b_0 \pm h$), canviant 1 sol dels 2 paràmetres

Cal avaluar $E(a,b)$ en 4 "veïns". Els càlculs es fan de la mateixa manera

$E(1.1,1)^2 = 2,552666\times10^{-2} \Rightarrow E(1.1,1) = 1,597706\times10^{-1}$ (no millora)

$E(0.9,1)^2 = 7,901600\times10^{-3} \Rightarrow E(0.9,1) = \boxed{8,889094\times10^{-2}} \leftarrow \boxed{\text{mínim}}$

$E(1,1.1)^2 = 1,500014\times10^{-2} \Rightarrow E(1,1.1) = 1,224750\times10^{-1}$ (sí millora)

$E(1,0.9)^2 = 1,804988\times10^{-2} \Rightarrow E(1,0.9) = 1,343499\times10^{-1}$ (no millora)

$\Rightarrow$ 1a. actualització $\boxed{a = 0,9,\ b = 1}$

Nota. Hi ha convergència cap a

$a = 0,714932$

$b = 1,137547$

$\text{err} = 2,3673\times10^{-2}$

(2) $Ax=y$ amb $A=\begin{pmatrix}1&a&0&0\\0&1&a&0\\b&0&1&a\\0&b&0&1\end{pmatrix}$ $a,b\in\mathbb{R}$ paràmetres

$A=L+D+U$, $D=\text{Ident.}$

(a) Jacobi

$$B_J=D^{-1}(-L-U)=-L-U=\boxed{\begin{pmatrix}0&-a&0&0\\0&0&-a&0\\-b&0&0&-a\\0&-b&0&0\end{pmatrix}}$$

Espectre:

$$0=-\lambda\underbrace{\begin{vmatrix}-\lambda&-a&0\\0&-\lambda&-a\\-b&0&-\lambda\end{vmatrix}}_{-\lambda^3-a^2b}+a(+b)\underbrace{\begin{vmatrix}-\lambda&0\\0&-a\end{vmatrix}}_{a\lambda}=\lambda^4+2a^2b\lambda\Rightarrow\begin{cases}\lambda=0\\\lambda^3+2a^2b=0\iff\lambda^3=-2a^2b\end{cases}$$

$\Rightarrow \text{Spec}(B_J)=\boxed{\{0,\ \text{3 arrels cúbiques de } -2a^2b\}}$

$\begin{cases}\text{si } a\neq0 \text{ i } b\neq0\text{, són les 3 diferents} \begin{cases}\text{1 real}\neq0\\\text{2 complexes conjugades (no reals)}\end{cases}\\ \text{si } a=0 \text{ o } b=0\text{, són totes 0}\end{cases}$

Per tant, $\rho(B_J)=\boxed{\sqrt[3]{|2a^2b|}}$ (les 3 arrels cúbiques tenen el mateix mòdul)

(b) Gauss-Seidel

$$B_G=(L+D)^{-1}(-U)=\underbrace{\begin{pmatrix}1&0&0&0\\0&1&0&0\\b&0&1&0\\0&b&0&1\end{pmatrix}^{-1}}_{\begin{pmatrix}1&&&\\0&1&&\\-b&0&1&\\0&-b&0&1\end{pmatrix}}\begin{pmatrix}0&-a&0&0\\&0&-a&0\\&&0&-a\\&&&0\end{pmatrix}=\boxed{\begin{pmatrix}0&-a&0&0\\0&0&-a&0\\0&ab&0&-a\\0&0&ab&0\end{pmatrix}}$$

Espectre:

A més de $\lambda=0$, cal $0=\begin{vmatrix}-\lambda&-a&0\\ab&-\lambda&-a\\0&ab&-\lambda\end{vmatrix}=-\lambda^3-2a^2b\lambda\Rightarrow\begin{cases}\lambda=0\\\lambda^2=-2a^2b\end{cases}$

$\Rightarrow \text{Spec}(B_G)=\boxed{\{0\ (\text{doble}),\ \text{2 arrels quadrades de } -2a^2b\}}$

$\begin{cases}\text{si } a\neq0 \text{ i} \begin{cases}b>0\Rightarrow\text{2 complexos conjugats de la forma } \pm i d\\b<0\Rightarrow\text{2 reals diferents del mateix mòdul}\end{cases}\\ \text{si } a=0 \text{ o } b=0\Rightarrow\text{són totes 0}\end{cases}$

$$\Rightarrow\boxed{\rho(B_G)=\sqrt[2]{|2a^2b|}}$$

(c) $a=1/2,\ b=3/4$ $\Rightarrow$ $\rho(B_J)=\left(2\cdot\frac{1}{4}\cdot\frac{3}{4}\right)^{1/3}=\left(\frac{3}{8}\right)^{1/3}\approx0{,}721125$
$\rho(B_G)=\left(2\cdot\frac{1}{4}\cdot\frac{3}{4}\right)^{1/2}=\left(\frac{3}{8}\right)^{1/2}\approx0{,}612372$ $\Bigg\}\Rightarrow$ són convergents

Observem que $\rho(B_J)^3=\rho(B_G)^2$. Per tant,

$\boxed{\text{3 iteracions de Jacobi "equivalen" a 2 iteracions de Gauss-Seidel}}$

③ $A=(a_{ij})$, $n\times n$, $a_{ii}=i$, $a_{ij}\in\{-0.2,+0.2\}$ $\forall i,j,\ i\neq j$

ⓐ $\boxed{n=2} \Rightarrow A=\begin{pmatrix}1 & \pm0.2\\ \pm0.2 & 2\end{pmatrix}$ té 2 discs de G. disjunts $\Rightarrow \exists 1$ vap a cadascun $\Rightarrow \exists$ 2 vaps reals i diferents $\lambda_1\in[0.8,1.2]$, $\lambda_2\in[1.8,2.2]$

$\boxed{n=3} \Rightarrow A=\begin{pmatrix}1 & \pm0.2 & \pm0.2\\ \pm0.2 & 2 & \pm0.2\\ \pm0.2 & \pm0.2 & 3\end{pmatrix}$ raonament idèntic, canviant 2 per 3 $\Rightarrow \begin{cases}\lambda_1\in[0.6,1.4]\\ \lambda_2\in[1.6,2.4]\\ \lambda_3\in[2.6,3.4]\end{cases}$

Cas $\boxed{n=4}$ No es pot fer el raonament anterior. $S_c=\mathrm{diag}(c,c,1,1)$ i $B_c=S_cAS_c^{-1}$, $c>0$

ⓑ
$$A=\begin{pmatrix}1 & \pm0.2 & \pm0.2 & \pm0.2\\ \pm0.2 & 2 & \pm0.2 & \pm0.2\\ \pm0.2 & \pm0.2 & 3 & \pm0.2\\ \pm0.2 & \pm0.2 & \pm0.2 & 4\end{pmatrix} \Rightarrow B_c=S_cAS_c^{-1}=\left(\begin{array}{cc|cc}1 & \pm0.2 & \pm0.2c & \pm0.2c\\ \pm0.2 & 2 & \pm0.2c & \pm0.2c\\ \hline \frac{\pm0.2}{c} & \frac{\pm0.2}{c} & 3 & \pm0.2\\ \frac{\pm0.2}{c} & \frac{\pm0.2}{c} & \pm0.2 & 4\end{array}\right)$$

Mirem discs de Gerschgorin de B_c

Disc	Centre	Radi
1	1	$0.2+0.4c$
2	2	$0.2+0.4c$
3	3	$0.2+0.4/c$
4	4	$0.2+0.4/c$

$D_1\cap D_2=\emptyset \Leftrightarrow 1+0.2+0.4c < 2-0.2-0.4c \Leftrightarrow 0.8c<0.6 \Leftrightarrow \boxed{c<\frac{6}{8}=\frac{3}{4}=0.75}$

$D_1\cap D_3=\emptyset \Leftrightarrow 1+0.2+0.4c < 3-0.2-\frac{0.4}{c} \Leftrightarrow 0.4c+\frac{0.4}{c}<1.6 \underset{c>0}{\Leftrightarrow} 0.4c^2-1.6c+0.4<0 \Updownarrow \underbrace{c^2-4c+1}_{h(c)}<0$

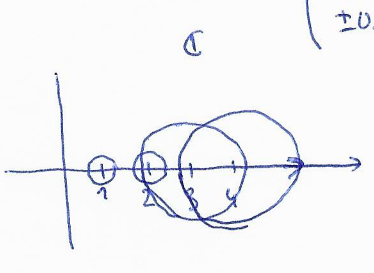
$D_1\cap D_4=\emptyset \Leftrightarrow 1+0.2+0.4c<4-0.2-\frac{0.4}{c}$ ← és redundant

$h(c)=0 \Leftrightarrow c=\frac{4\pm\sqrt{16-4}}{2}=2\pm\sqrt{3}\approx\begin{cases}0,27\\3,73\end{cases} \rightarrow$ 

Per tant, D_1 és disjunt dels altres 3 discs $\Leftrightarrow$ $\boxed{c\in(2-\sqrt{3},\ 3/4)\approx(0{,}267949,\ 0{,}75)}$

ⓒ $c=0.4 \Rightarrow B_c=\begin{pmatrix}1 & \pm0.2 & \pm0.08 & \pm0.08\\ \pm0.2 & 2 & \pm0.08 & \pm0.08\\ \pm0.5 & \pm0.5 & 3 & \pm0.2\\ \pm0.5 & \pm0.5 & \pm0.2 & 4\end{pmatrix} \Rightarrow$

Disc	Centre	Radi
1	1	0.36
2	2	0.36
3	3	1.2
4	4	1.2



$\exists\lambda_1\in D_1$ únic $\Rightarrow$ real $\Rightarrow \lambda_1\in[0.64,1.36]$

$\exists\lambda_2,\lambda_3,\lambda_4\in D_2\cup D_3\cup D_4 \Rightarrow \forall i=2,3,4.\ |\lambda_i|\begin{cases}\le 5.2\\ \ge 1.64\end{cases}$

Potència inversa $\equiv$ potència a B_c^{-1}

B_c^{-1} té vaps $\mu_i=1/\lambda_i$ $i=1\div4$ $\Rightarrow \begin{cases}\exists \text{ vap dominant}\overset{\text{real}}{}\ \frac{1}{\lambda_1}\in\left[\frac{1}{1.36},\frac{1}{0.64}\right]\approx[0.735,1.5625]\\ \exists\ 3 \text{ vaps més, amb mòduls } \frac{1}{|\lambda_i|}\begin{cases}\le 1/1.64 \approx 0.61\\ \ge 1/5.2 \approx 0.19\end{cases}\end{cases}$

La raó asimptòtica de conv. és $r=\frac{\max\{|\mu_2|,|\mu_3|,|\mu_4|\}}{|\mu_1|}$

$$\Rightarrow r\begin{cases}\le \frac{1/1.64}{1/1.36}=\frac{1.36}{1.64}=\frac{34}{41}\\ \ge \frac{1/5.2}{1/0.64}=\frac{0.64}{5.2}=\frac{8}{65}\end{cases}$$

$$\boxed{r\in\left[\frac{8}{65},\frac{34}{41}\right]\approx[0.123,0.829]}$$