1. $J_1-J_2: 2-0$, $J_1-J_3: 2-1$, $J_2-J_4: 2-0$, $J_3-J_4: 2-1$ (segons l'enunciat) $\Rightarrow$

(a)
$$\boxed{A = \begin{pmatrix} 1 & 20 & 10 & 1 \\ 1 & 1 & 1 & 20 \\ 1 & 1 & 1 & 10 \\ 1 & 1 & 1 & 1 \end{pmatrix}}$$

(b) Vector inicial (victòries/partits) $\Rightarrow$ $\boxed{x^T = \left(\frac{2}{2}, \frac{1}{2}, \frac{1}{2}, 0\right) = (1, 0.5, 0.5, 0)}$

Normalització en $\|\ \|_1$: $\|x\|_1 = 2$ $\Rightarrow$ $\boxed{v^T = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0\right) = (0.5, 0.25, 0.25, 0)}$

(c) Primera iteració.

$$x = Av = \begin{pmatrix} 1 & 20 & 10 & 1 \\ 1 & 1 & 1 & 20 \\ 1 & 1 & 1 & 10 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/4 \\ 1/4 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 + 20/4 + 10/4 \\ 1/2 + 1/4 + 1/4 \\ 1/2 + 1/4 + 1/4 \\ 1/2 + 1/4 + 1/4 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \boxed{x^T = (8, 1, 1, 1)}$$

$\|x\|_1 = 11$ $\Rightarrow$ $\boxed{v^T = \left(\frac{8}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}\right) \approx (0.\widehat{72}, 0.\widehat{09}, 0.\widehat{09}, 0.\widehat{09})}$

(d) Segona iteració

$$x = Av = \begin{pmatrix} 1 & 20 & 10 & 1 \\ 1 & 1 & 1 & 20 \\ 1 & 1 & 1 & 10 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 8/11 \\ 1/11 \\ 1/11 \\ 1/11 \end{pmatrix} = \begin{pmatrix} (8+20+10+1)/11 \\ (8+1+1+20)/11 \\ (8+1+1+10)/11 \\ (8+1+1+1)/11 \end{pmatrix} = \begin{pmatrix} 39/11 \\ 30/11 \\ 20/11 \\ 11/11 \end{pmatrix} \Rightarrow \boxed{x^T = \left(\frac{39}{11}, \frac{30}{11}, \frac{20}{11}, 1\right) = (3.\widehat{54}, 2.\widehat{72}, 1.\widehat{81}, 1)}$$

$\|x\|_1 = \frac{1}{11}(39+30+20+11) = \frac{100}{11}$ $\Rightarrow$ $\boxed{v^T = \left(\frac{39}{100}, \frac{30}{100}, \frac{20}{100}, \frac{11}{100}\right) = (0.39, 0.3, 0.2, 0.11)}$

<u>Nota</u>. Es convergeix a $v^T \simeq (0.5481, 0.2193, 0.1483, 0.0843)$

③ $Ax=b$, 5×5, $A=\begin{pmatrix} 2 & -1 & & & \\ 0 & 2 & -1 & & \\ -1 & 0 & 2 & -1 & \\ & -1 & 0 & 2 & -1 \\ & & -1 & 0 & 2 \end{pmatrix}$ $b=\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}$

ⓐ Jacobi: $(L+D+U)x=b \Leftrightarrow Dx=-(L+U)x+b \Leftrightarrow x=+D^{-1}(-L-U)x+D^{-1}b$

$J=D^{-1}(-L-U)=\boxed{\begin{pmatrix} 0 & 1/2 & & & \\ 0 & 0 & 1/2 & & \\ 1/2 & 0 & 0 & 1/2 & \\ & 1/2 & 0 & 0 & 1/2 \\ & & 1/2 & 0 & 0 \end{pmatrix}}$; $\boxed{\|J\|_\infty=\text{màx}\{\frac{1}{2},1\}=1}$

(no podem assegurar que Jacobi convergeixi)

$C=\text{diag}(1,1,c,c,1)$, $c\neq 0$

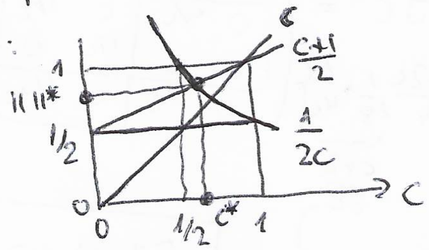
$\tilde{J}=CJC^{-1}=\begin{pmatrix} 0 & 1/2 & & & \\ 0 & 0 & 1/2c & & \\ c/2 & 0 & 0 & 1/2 & \\ & c/2 & 0 & 0 & c/2 \\ & & 1/2c & 0 & 0 \end{pmatrix}$. Si $c>0 \Rightarrow \|\tilde{J}\|_\infty=\text{màx}\{\frac{1}{2},\frac{1}{2c},\frac{c+1}{2},c\}$

Valor de $c>0 \mid \|\tilde{J}\|_\infty<1$?

Cal $\left\{\begin{array}{l} \frac{1}{2c}<1 \Leftrightarrow c>\frac{1}{2} \\ \frac{c+1}{2}<1 \Leftrightarrow c<1 \\ c<1 \; (=) \end{array}\right\} \Leftrightarrow \boxed{\begin{array}{l}\|\tilde{J}\|_\infty<1 \\ (c>0)\end{array} \Leftrightarrow c\in\left(\frac{1}{2},1\right)}$

Valor de $c>0 \mid \|\tilde{J}\|_\infty$ mínima?

Gràficament:



$\|\tilde{J}\|_\infty$ és mínima quan c és el valor d'intersecció

$\frac{c+1}{2}=\frac{1}{2c}$

$\Rightarrow c^2+c-1=0 \Rightarrow c=\frac{-1\pm\sqrt{5}}{2}$

Només interessa $c>0$ $\Rightarrow \boxed{c^*=\frac{\sqrt{5}-1}{2}\approx 0{,}618}$

El valor corresponent de $\|\tilde{J}\|_\infty$? $\frac{c+1}{2}=\frac{\frac{\sqrt{5}-1}{2}+1}{2}=\frac{\sqrt{5}+1}{4} \Rightarrow \boxed{\|\tilde{J}\|_\infty=\frac{\sqrt{5}+1}{4}\approx 0{,}809}$

Convergència Jacobi?

$\left.\begin{array}{l}\text{convergència} \Leftrightarrow \rho(J)<1 \\ \text{però } \rho(J)=\rho(\tilde{J})\leq\|\tilde{J}\|_\infty\approx 0{,}809<1\end{array}\right\} \Rightarrow$ $\boxed{\text{Jacobi sí convergeix}}$

(b) Gauss-Seidel: $(L+D+U)x=b \iff (L+D)x=-Ux+b \iff x=(L+D)^{-1}(-U)x+(L+D)^{-1}b$

$$G=(L+D)^{-1}(-U)=\underbrace{\begin{pmatrix}2&&&&\\0&2&&&\\-1&0&2&&\\&-1&0&2&\\&&-1&0&2\end{pmatrix}^{-1}}_{\begin{pmatrix}1/2&&&&\\0&1/2&&&\\1/4&0&1/2&&\\0&1/4&0&1/2&\\1/8&0&1/4&0&1/2\end{pmatrix}}\begin{pmatrix}0&1&&&\\&0&1&&\\&&0&1&\\&&&0&1\\&&&&0\end{pmatrix}=\boxed{\begin{pmatrix}0&1/2&0&0&0\\0&0&1/2&0&0\\0&1/4&0&1/2&0\\0&0&1/4&0&1/2\\0&1/8&0&1/4&0\end{pmatrix}}$$

$$\|G\|_\infty=\max\left\{\frac{1}{2},\underbrace{\frac{1}{4}+\frac{1}{2}}_{3/4},\underbrace{\frac{1}{8}+\frac{1}{4}}_{3/8}\right\}$$

$$\Rightarrow \boxed{\|G\|_\infty=\frac{3}{4}=0.75}\equiv\beta$$

Iteracions k?

$$x^0=0\in\mathbb{R}^5\Rightarrow x^1=(L+D)^{-1}b=\Big(\frac{1}{2},\frac{1}{2},\underbrace{\frac{1}{4}+\frac{1}{2}}_{3/4},\underbrace{\frac{1}{4}+\frac{1}{2}}_{3/4},\underbrace{\frac{1}{8}+\frac{1}{4}+\frac{1}{2}}_{7/8}\Big)^T\Rightarrow\|x^1\|_\infty=\frac{7}{8}$$

Imposem
$$\frac{\beta^k}{1-\beta}\|x^1-x^0\|<10^{-10}\iff\beta^k<\frac{(1-\beta)10^{-10}}{\|x^1-x^0\|}\iff k>\frac{-10+\log_{10}(1-\beta)-\log_{10}\|x^1-x^0\|}{\log_{10}\beta}$$

$\beta\in(0,1)\Rightarrow\log_{10}\beta<0$

Substituint
$$k>\frac{-10+\log_{10}(0.25)-\log_{10}(7/8)}{\log_{10}(0.75)}\approx 84.39\Rightarrow\boxed{k=85}$$

Similaritat

$$C=\operatorname{diag}(1,1,c,c,1),\ c\neq0\Rightarrow\tilde{G}=CGC^{-1}=\begin{pmatrix}0&1/2&0&0&0\\0&0&1/2c&0&0\\0&c/4&0&1/2&0\\0&0&1/4&0&c/2\\0&1/8&0&1/4c&0\end{pmatrix}$$

Quan $c>0$, $\|\tilde{G}\|_\infty=\max\left\{\frac{1}{2},\frac{1}{2c},\underbrace{\frac{c}{4}+\frac{1}{2}}_{\frac{c+2}{4}},\frac{1+2c}{4},\underbrace{\frac{1}{8}+\frac{1}{4c}}_{\frac{c+2}{8c}}\right\}$

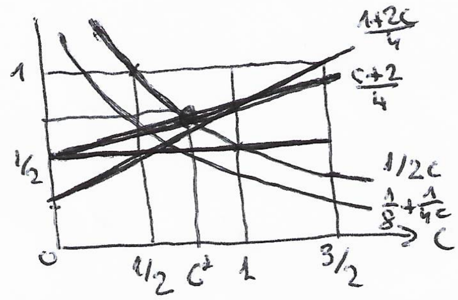
Condicions per a $\|\tilde{G}\|_\infty<1$:

$$\left.\begin{cases}\frac{1}{2c}<1\Rightarrow c>1/2\\ \frac{c+2}{4}<1\Rightarrow c<2\\ \frac{1+2c}{4}<1\Rightarrow c<\frac{3}{2}\\ \frac{1}{8}+\frac{1}{4c}<1\Rightarrow\frac{1}{4c}<\frac{7}{8}\Rightarrow 4c>\frac{8}{7}\Rightarrow c>\frac{2}{7}\end{cases}\right\}\Rightarrow\boxed{\begin{array}{l}\|\tilde{G}\|_\infty<1\iff c\in\left(\frac{1}{2},\frac{3}{2}\right)\\(c>0)\end{array}}$$

s'ha de complir tot

Valors òptims de c i de $\|\tilde{G}\|_\infty$?

Gràficament:



c^* òptim per a $\frac{1}{2c}=\frac{c+2}{4}$

$$\Rightarrow c^2+2c-2=0\Rightarrow c=\frac{-2\pm\sqrt{12}}{2}$$

Cal $c>0\Rightarrow\boxed{c^*=\sqrt{3}-1\approx0{,}732}$

Llavors

$$\boxed{\|\tilde{G}\|_\infty=\frac{c^*+2}{4}=\frac{\sqrt{3}+1}{4}\approx0{,}683}$$

4. Equació de Kepler: $f(E,M) \equiv E - e\sin(E) - M = 0 \quad \forall M, E \in \mathbb{R}$

($e \in [0,1)$ és constant)

(a) Es pot aplicar el TF Implícita en qualsevols E_0, M_0 tals que $f(E_0, M_0) = 0$? (per a aïllar E en funció de M)

$\frac{\partial f}{\partial E} = 1 - e\cos(E)$. Com que $e \in [0,1)$ i $|\cos(E)| \le 1 \Rightarrow \frac{\partial f}{\partial E} > 0$

En particular, $\frac{\partial f}{\partial E} \neq 0$ sempre $\Rightarrow$ $\boxed{\text{Sí que es pot aplicar}}$

$E = g(M) \quad \forall M \in I$, entorn de M_0

Sí que es pot prendre $I = \mathbb{R}$, ja que $M \equiv h(E) = E - e\sin(E)$ és injectiva ($h'(E) > 0 \ \forall E \in \mathbb{R}$) amb imatge = tot $\mathbb{R}$

i g és la inversa de h.

(b) Sigui $E = g(M) \ \forall M \in \mathbb{R} \mid g(M) - e\sin(g(M)) - M = 0$, $g(0) = 0$.

Anem derivant respecte M i substituint en $M=0$:

$g'(M) - e\cos(g(M))\, g'(M) - 1 = 0 \rightarrow g'(0) - e g'(0) - 1 = 0 \Rightarrow \boxed{g'(0) = \frac{1}{1-e}}$

$g''(M) + e\sin(g(M))(g'(M))^2 - e\cos(g(M)) \cdot g''(M) = 0 \rightarrow g''(0) - e g''(0) = 0 \Rightarrow \boxed{g''(0) = 0}$

$g'''(M) + e\cos(g(M))(g'(M))^3 + 3e\sin(g(M))\, g'(M)\, g''(M) - e\cos(g(M))\, g'''(M) = 0 \rightarrow$

$\rightarrow g'''(0) + \frac{e}{(1-e)^3} - e g'''(0) = 0 \Rightarrow \boxed{g'''(0) = \frac{-e}{(1-e)^4}}$

Llavors $E = g(M) \approx g(0) + g'(0)M + \frac{1}{2} g''(0) M^2 + \frac{1}{6} g'''(0) M^3 = \frac{1}{1-e} M - \frac{e}{6(1-e)^4} M^3$

$\Rightarrow \boxed{E = g(1) \approx \frac{1}{1-e} - \frac{e}{6(1-e)^4}}$. En el cas $e = 0{,}0167 \Rightarrow \boxed{E(M=1) \approx 1{,}0140063\ldots}$

(c) Iteració simple en el cas $M = 1$, $e = 0{,}0167$.

$E - 0.0167\sin(E) - 1 = 0 \iff E = 0{,}0167\sin(E) + 1 \equiv g(E)$

$\Rightarrow$ Iteració $\boxed{E_k = 0.0167\sin(E_{k-1}) + 1 \quad \forall k \ge 1}$

$g(E) - g(F) = 0{,}0167\underbrace{(\sin(E) - \sin(F))}_{=\cos(\xi)(E-F)}$

$K = |0.0167\cos(\xi)| \le 0.0167 \Rightarrow$ convergència molt ràpida

És evidentment una contracció a tot $\mathbb{R}$.
Començant per ex. a $E_0 = 0$ s'obté

$E_1 = 1$

$E_2 = 1{,}01405257$

$E_3 = 1{,}01417797$

$E_4 = 1{,}01417908$

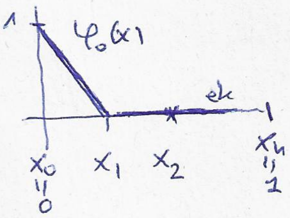
$\boxed{E_5 = 1{,}01417904}$ ja té 6 decimals com l'anterior

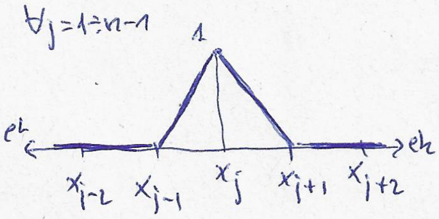
⑤ $E = \mathcal{C}^0([0,1])$, $\langle f,g\rangle = \int_0^1 f(x)g(x)dx$

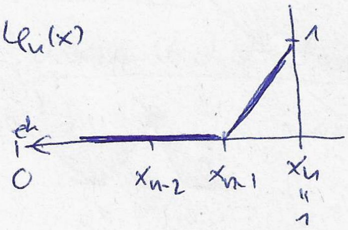
ⓐ $n>1$, $h=\frac{1}{n}$; $x_j = jh$ $\forall j=0\div n$

Cada φ_j { és contínua a $[0,1]$
val 1 en x_j, val 0 en els altres x_k's
en cada subinterval $[x_k,x_{k+1}]$ és un pol. de grau ≤ 1 $\equiv$ la gràfica és una recta

Per tant,







Independència lineal?

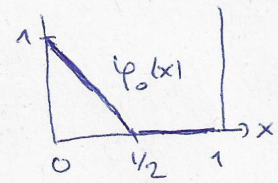
Suposem $\sum_{j=0}^{n} c_j\varphi_j = 0$. O sigui, $\forall x\in[0,1]$, $\sum_{j=0}^{n} c_j\varphi_j(x)=0$

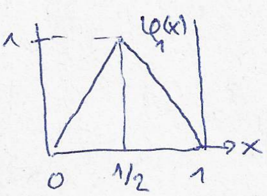
Aplicant-ho en $x=x_k$, com que $\varphi_j(x_k)=\delta_{jk}$ $\Rightarrow$ $c_k=0$.

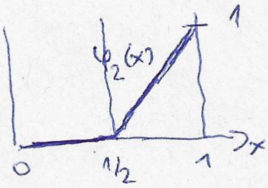
Això val $\forall k$ $\Rightarrow$ ja està.

E_n^* té dimensió $n+1$

ⓑ $n=2$







$\varphi_0(x) = \begin{cases}1-2x, & x\in[0,1/2]\\ 0, & x\in[1/2,1]\end{cases}$ $\quad\varphi_1(x) = \begin{cases}2x, & x\in[0,1/2]\\ 2-2x, & x\in[\frac12,1]\end{cases}$ $\quad\varphi_2(x) = \begin{cases}0 & x\in[0,1/2]\\ -1+2x & x\in[1/2,1]\end{cases}$

ⓒ Base ortogonal de E_2^*?

Observem que $\varphi_0(x)\cdot\varphi_2(x)=0$ $\forall x\in[0,1]$. Per tant, $\boxed{\Phi_0(x) = \varphi_0(x)}$ i $\boxed{\Phi_2(x)=\varphi_2(x)}$ $\boxed{\text{són ortogonals entre si}}$.

Busquem $\Phi_1(x) = \varphi_1(x) - a\varphi_0(x) - b\varphi_2(x)$ que sigui ortogonal a $\varphi_0(x)$ i $\varphi_2(x)$. Cal:

$$0=\langle\Phi_1,\varphi_0\rangle = \langle\varphi_1,\varphi_0\rangle - a\langle\varphi_0,\varphi_0\rangle - b\underbrace{\langle\varphi_2,\varphi_0\rangle}_{=0} \Rightarrow a = \frac{\langle\varphi_1,\varphi_0\rangle}{\langle\varphi_0,\varphi_0\rangle}$$

$$0 = \langle\Phi_1,\varphi_2\rangle = \langle\varphi_1,\varphi_2\rangle - a\underbrace{\langle\varphi_0,\varphi_2\rangle}_{=0} - b\langle\varphi_2,\varphi_2\rangle \Rightarrow b = \frac{\langle\varphi_1,\varphi_2\rangle}{\langle\varphi_2,\varphi_2\rangle}$$

Fem els càlculs:

$\langle\varphi_0,\varphi_0\rangle = \int_0^{1/2}(1-2x)^2dx = \int_0^{1/2}(1-4x+4x^2) = x - 4\frac{x^2}{2}+4\frac{x^3}{3}\Big]_0^{1/2} = \frac12 - 4\frac18 + 4\frac1{24} = \frac16$ } $\Rightarrow a = \frac{1/12}{1/6}=\frac12$

$\langle\varphi_1,\varphi_0\rangle = \int_0^{1/2}(1-2x)2x\,dx = \int_0^{1/2}(2x-4x^2) = x^2 - 4\frac{x^3}{3}\Big]_0^{1/2} = \frac14 - 4\frac1{24} = \frac1{12}$

$\langle\varphi_2,\varphi_2\rangle = \int_{1/2}^1(-1+2x)^2dx$ = --- (simetria respecte de φ_0) --- $= \frac16$ } $\Rightarrow b=\frac12$

$\langle\varphi_1,\varphi_2\rangle = \int_{1/2}^1(2-2x)(-1+2x)dx =$ ídem $= \frac{1}{12}$

$$\Rightarrow \boxed{\Phi_1(x) = \varphi_1(x) - \frac12\varphi_0(x) - \frac12\varphi_2(x)} = \boxed{\begin{cases}2x - \frac12(1-2x) = -\frac12+3x, & x\in[0,1/2]\\ 2-2x-\frac12(-1+2x) = \frac52-3x, & x\in[1/2,1]\end{cases}}$$

Millor aproximació f^* de $f(x)=x$ dins de E_2^*?

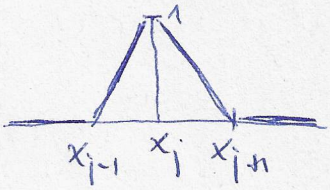
Com que $f\in E_2^*$ (ja que és un pol. de grau ≤ 1 en $[0,\frac12]$, i un pol. de grau ≤ 1 en $[\frac12,1]$)

la seva millor aproximació és $\boxed{\text{ella mateixa}}$ $\boxed{f^*=f}$

De fet, per la condició $\varphi_j(x_k)=\delta_{jk}$, serà $f^* = \underbrace{f(0)}_{0}\varphi_0(x) + \underbrace{f(1/2)}_{1/2}\varphi_1(x) + \underbrace{f(1)}_{1}\varphi_2(x)$ (es pot comprovar)

(d) $n>1$, $h=\frac{1}{n}$, $x_j=jh$ $\forall j=0\div n$

$\varphi_j(x)$ dibuixada a l'apartat (a)



Fórmula:

$$\varphi_j(x)=\begin{cases} n(x-x_{j-1}) & x\in[x_{j-1},x_j] \quad (1)\\ -n(x-x_{j+1}) & x\in[x_j,x_{j+1}] \quad (2)\\ 0 & \text{altrament}\end{cases}$$

en els casos $\begin{cases} j=0, \text{ no hi ha } (1) \\ j=n, \text{ no hi ha } (2)\end{cases}$

Per tant, si $|j-k|\geq 2$ llavors $\varphi_j(x)\cdot\varphi_k(x)=0$ $\forall x\in[0,1]$

$$\Rightarrow \langle\varphi_j,\varphi_k\rangle=0$$

Això vol dir que $\boxed{\text{la matriu de Gram és tridiagonal}}$

<u>Element de la diagonal ppal</u>

$$\boxed{\langle\varphi_j,\varphi_j\rangle}=\underbrace{\int_{x_{j-1}}^{x_j} n^2(x-x_{j-1})^2dx}_{=\underbrace{n^2\left.\frac{(x-x_{j-1})^3}{3}\right]_{x_{j-1}}^{x_j}}_{=n^2\frac{h^3}{3}\underset{nh=1}{=}\frac{h}{3}}}+\underbrace{\int_{x_j}^{x_{j+1}} n^2(x-x_{j+1})^2dx}_{=\text{igual}=h/3}=\boxed{\frac{2h}{3}}$$

$\boxed{\text{Això val } \forall j=1\div n-1.}$
En els casos $j=0$ i $j=n$, només hi ha 1 sumand, i no 2, i dona $h/3$

$$\boxed{\langle\varphi_0,\varphi_0\rangle=\langle\varphi_n,\varphi_n\rangle=\frac{h}{3}}$$

<u>Element de les diagonals per sobre i per sota de la ppal</u>

$$\langle\varphi_j,\varphi_{j+1}\rangle=\int_{x_j}^{x_{j+1}}[-n(x-x_{j+1})][n(x-x_j)]=-n^2\int_{x_j}^{x_{j+1}}[x^2-(x_j+x_{j+1})x+x_jx_{j+1}]\,dx=$$

$$=\text{ok (llarg)}=h/6$$

Per tant la matriu de Gram és

$$\begin{pmatrix} h/3 & h/6 & & & & \\ h/6 & 2h/3 & h/6 & & 0 & \\ & h/6 & 2h/3 & h/6 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & h/6 & 2h/3 & h/6 \\ & & & & h/6 & h/3 \end{pmatrix}$$

<u>Nota</u>: Una matriu tridiagonal és molt barat de resoldre (sistema lineal amb)

$\Rightarrow$ Per a aquestes funcions, <u>no</u> val la pena ortogonalitzar