On the Modal Account of Forcing

Oberseminar Logik der Universität Bonn

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Independence Proofs

- A large area of set theory focuses on consistency and independence proofs:

  Is φ provable from ZFC?
  Is $\neg\varphi$ provable from ZFC?
  Are neither provable from ZFC, i.e. is φ *independent*?

- In other words, we want to prove statements of the form

$$\mathrm{Con}(\mathrm{ZFC}) \implies \mathrm{Con}(\mathrm{ZFC} + \varphi)$$

  i.e. $\mathrm{ZFC} + \varphi$ is *relatively consistent*.

- We need a large toolbox of ways to construct new models! One such tool is *forcing*.

What is Forcing?

- We start off with a ground model W of ZFC. By doing lots of "technical stuff", we can extend W to a new model $W[G]$ of ZFC in a very specific way. [1]
- The "technical stuff" allows us to:
  - *force* certain sentences to be true in $W[G]$, and
  - reason about $W[G]$ *from within W*, even though a lot of $W[G]$ lives outside of W.

$$\begin{aligned}\mathrm{Con}(\mathrm{ZFC}) &\Longrightarrow \text{there is a model } W \models \mathrm{ZFC} \\ &\Longrightarrow \text{there is a model } W[G] \models \mathrm{ZFC} + \varphi \\ &\Longrightarrow \mathrm{Con}(\mathrm{ZFC} + \varphi)\end{aligned}$$

[1] G denotes the *generic filter* of a forcing notion used in the construction.

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Modal Logic

- Modal Logic is the study of the modalities *necessarily* ($\Box$) and *possibly* ($\Diamond$). It gives a framework for describing to what extent a formula φ is true.

- There are many other interpretations of $\Box$ and $\Diamond$, for instance:
  - Epistemic: Alice *knows* φ ($\Box\varphi$); Alice *believes* φ ($\Diamond\varphi$)
  - Deontic: It is *obligatory* that φ; it is *permissible* that φ
  - Temporal: At *every* future moment φ; at *some* future moment φ

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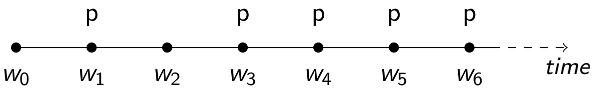
  → Note that $\Box$ and $\Diamond$ are dual, so $\Diamond\varphi \iff \neg\Box\neg\varphi$.

Kripke frames and Kripke models

Temporal example: $w_0 \models \Diamond p, \quad w_0 \not\models \Box p, \quad w_0 \models \Diamond\Box p$



In general, we study *frames* (W, R),

- where W is a set of *worlds*,
- R an *accessibility* relation,

and *models on frames* (W, R, ν),

- where $\nu : \mathrm{Prop} \times W \to \{0, 1\}$ is a valuation function.

Kripke frames and Kripke models

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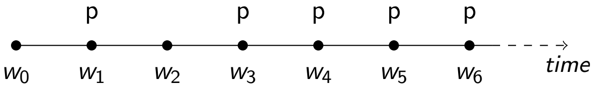

In this example $\mathcal{M} = (W, R, \nu)$ is given by:

- $W = \{w_n \mid n \in \omega\}$
- $w_n R w_m \iff n < m$
- $\nu(p, w_n) = 1 \iff (n \neq 0 \wedge n \neq 2)$

We say $\mathcal{M}, w \models \Box\varphi$ if and only if for all v with wRv we have $\mathcal{M}, v \models \varphi$.

For a frame $\mathcal{F}$, we may write $\mathcal{F} \models \varphi$ if $\mathcal{M}, w \models \varphi$ for every model $\mathcal{M}$ on $\mathcal{F}$ and every world w on the frame.

Interpretations for studying mathematical structures

Suppose

- $\mathcal{C}$ is the collection of $\mathcal{L}$-structures for some first-order language $\mathcal{L}$
- and $\preceq$ is some accessibility relation on $\mathcal{C}$.

Then $(\mathcal{C}, \preceq)$ is a Kripke frame which we can study.

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Some examples that have been studied include

- All *abelian groups* together with the relation $\preceq$ that holds between G and H whenever G is isomorphic to a subgroup of H.
- All transitive set models of ZFC together with $M \preceq N$ if and only if M is an *inner model* in N.
- In general, $\mathrm{Mod}(\Gamma)$ for some set of axioms Γ together with a specified type of embedding.
- **All set models of** ZFC **together with** $M \preceq N$ **if and only if** N **is a forcing extension of** M**.**
  - ➔ See for instance [8], [9], [10], [1], [2].

Interpretations for studying mathematical structures

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Denote by $\mathcal{L}_\Box$ the language which contains infinitely many propositional variables and logical symbols $\wedge, \neg$ and $\Box$.

Question

For which $\mathcal{L}_\Box$ sentences $\varphi(p_0, ..., p_n)$ do we have

$$M \models \varphi(\psi_0/p_0, ..., \psi_n/p_n)$$

for all $M \in \mathcal{C}$ and all substitutions $p_i \mapsto \psi_i$ with $\mathcal{L}$ sentences ψ_i?

The Forcing Interpretation of $\Box$

A forcing translation is a function $\tau : \varphi \mapsto \varphi^\tau$ mapping formulas of $\mathcal{L}_\Box$ to $\mathcal{L}_\in$ such that Boolean connectives are preserved and $(\Box\varphi)^\tau$ is the $\mathcal{L}_\in$ formula expressing

"in all forcing extensions φ^τ holds"[2]

This is just a fancy way of saying that τ is a substitution of propositional variables in $\mathcal{L}_\Box$ for set-theoretic formulas.

Definition

- $\text{Force}^{\text{ZFC}} = \{\varphi \in \mathcal{L}_\Box \mid \text{ZFC} \vdash \varphi^\tau \text{ for all forcing translations } \tau\}$
- $\text{Force}^{W} = \{\varphi \in \mathcal{L}_\Box \mid W \models \varphi^\tau \text{ for all forcing translations } \tau\}$, where W is a model of set theory

[2] Note that this is indeed expressible in $\mathcal{L}_\in$

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What we already know

Theorem (Hamkins, Löwe [1])

If ZFC is consistent, then $\mathrm{Force}^{\mathrm{ZFC}} = \mathbf{S4.2}$.
If $W \models \mathrm{ZFC}$, then $\mathbf{S4.2} \subseteq \mathrm{Force}^{W} \subseteq \mathbf{S5}$.

$\mathbf{S4.2} = \mathbf{T} + \mathbf{4} + \mathbf{.2}$

- $\mathbf{T}$: $\Box p \to p$ (reflexivity)
- $\mathbf{4}$: $\Diamond\Diamond p \to \Diamond p$ (transitivity)
- $\mathbf{.2}$: $\Diamond\Box p \to \Box\Diamond p$ (directedness)

$\mathbf{S5} = \mathbf{S4.2} + \mathbf{5}$

- $\mathbf{5}$: $\Diamond\Box p \to \Box p$ (symmetry)

Control Statements

Proving $\mathrm{Force}^{\mathrm{ZFC}} \supseteq$ **S4.2** is easy: Just verify the axioms!
Proving $\mathrm{Force}^{\mathrm{ZFC}} \subseteq$ **S4.2** is significantly harder.

➔ This uses *control statements*.

Definition

Let w be a world in a Kripke model $\mathcal{M}$. In $(\mathcal{M}, w)$:

- φ is a button iff $\mathcal{M}, w \models \Box\Diamond\Box\varphi$
- φ is a switch iff $\mathcal{M}, w \models \Box\Diamond\varphi \wedge \Box\Diamond\neg\varphi$

Proposition

If **S4.2** holds, then every $\mathcal{L}_\Box$ formula is either a button, a negated button, or a switch.

If we view the forcing multiverse as a Kripke model, then the following propositions are control statements.

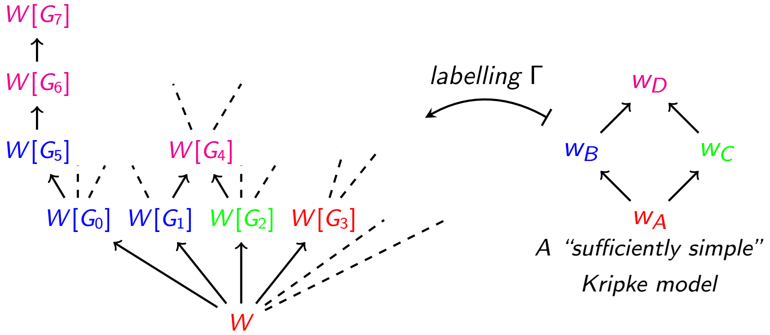
- $p =$ "$S \subseteq \omega_1$ is not stationary" is a button
- $p =$ "$\aleph_n^L$ is (not) collapsed" is a (negated) button
- $p =$ "Continuum hypothesis is true" is a switch

Proving Force$^{\mathrm{ZFC}} \subseteq$ **S4.2**

- We want to show that if $\varphi \notin$ **S4.2**, then there is a ZFC model W and a forcing translation τ, such that $W \not\models \varphi^\tau$
- Idea: If we have completeness of **S4.2** with respect to a class of "sufficiently simple" Kripke models, then we can translate the failure of φ in a "sufficiently simple" Kripke model into the failure of φ^τ in the set-theoretic forcing multiverse.
- If we have a collection of *independent*[3] buttons and switches, then the possible patters (pushed/unpushed, on/off) form a pre-Boolean algebra.
  - ➔ This allows us to create a so-called *labelling of worlds*, which in turn gives us τ.

[3] A set of control statements is independent if manipulating the state (pushed/unpushed, on/off) of one of them does not change any others

A labelling of worlds



A "sufficiently simple" Kripke model

Set-theoretic forcing multiverse of W

If we have such a labelling, we can define τ such that W *mimics* w_A, $W[G_0]$ *mimics* w_B, $W[G_2]$ *mimics* w_C etc.

If φ fails in w_A, then φ^τ fails in W.

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What about the predicate modal logic of forcing?

- In the previous slides we only considered formulas φ^τ where $\Box$ does not occur in the scope of a quantifier, since quantifiers are only added to φ^τ through the substitution of propositional variables.
- Let's expand our modal language: $\mathcal{L}^\Box$ now consists of countably many variables and countably many *predicate* symbols P_i of each arity, and is closed under $\wedge, \neg$, $\Box$ and $\forall$.
- In this context, every world in a Kripke model now has a domain.

Question

What are the *predicate* modal principles of forcing?

One example is the converse Barcan formula:

$$\Box\forall x\varphi(x) \to \forall x\Box\varphi(x)$$

→ What follows is based on joint work with Joel David Hamkins

Predicate Modal Principles of Forcing

Definition

A predicate forcing translation τ maps n-ary predicate symbols $P_i(\bar{x})$ to set theoretic formulas $\psi_i(\bar{x})$ with n free variables.

$$\varphi(P_0(\bar{x}_0), \ldots, P_n(\bar{x}_n)) \quad \xmapsto{\tau} \quad \varphi(\psi_0(\bar{x}_0)/P_0, \ldots, \psi_n(\bar{x}_n)/P_n)$$

Definition

$\mathrm{Force}_\forall^{\mathrm{ZFC}} = \{\varphi \in \mathcal{L}_\Box | \mathrm{ZFC} \vdash \varphi^\tau \text{for all predicate forcing translations } \tau\}$

$\mathrm{Force}_\forall^{W} = \{\varphi \in \mathcal{L}_\Box \mid W \models \varphi^\tau \text{ for all predicate forcing translations } \tau\}$

In other words, we now consider all predicate substitution instances instead of propositional substitution instances.

Conjecture

Conjecture

$\mathrm{Force}_{\forall}^{\mathrm{ZFC}} = \mathbf{QS4.2}$ [4]

Proof Idea: Again, $\mathrm{Force}_{\forall}^{\mathrm{ZFC}} \supseteq \mathbf{QS4.2}$ is easy but $\mathrm{Force}^{\mathrm{ZFC}} \subseteq \mathbf{QS4.2}$ is hard.

- Expand the definition of labelling and prove that it still works.
- Prove a completeness result with respect to "nice" Kripke models.
  - ➔ What does "sufficiently simple" mean in this context? Not so easy since finite models will no longer do the job!
- Given a "sufficiently simple" Kripke model, provide a labelling with respect to some model W of ZFC.

[4] **QS4.2** is the quantified analogue of **S4.2**.

Conjecture

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Proof Idea: Again, $\mathrm{Force}_{\forall}^{\mathrm{ZFC}} \supseteq \mathbf{QS4.2}$ is easy but $\mathrm{Force}^{\mathrm{ZFC}} \subseteq \mathbf{QS4.2}$ is hard.

- ✓ Expand the definition of labelling and prove that it still works.
- ✓ Prove a completeness result with respect to "sufficiently simple" Kripke models.
  - ➔ What does "sufficiently simple" mean in this context? Not so easy since finite models will no longer do the job!
- Given a "sufficiently simple" Kripke model, provide a labelling with respect to some model W of ZFC.
  - ➔ 1. and 2. are done! Still figuring out some details for 3...

[4] **QS4.2** is the quantified analogue of **S4.2**.

Thank you for listening! Any questions?

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