Putnam Problems and Solutions

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1 Introduction

This is a compilation of the solutions to the Putnam problems. They typically consist of a breakdown of the key theorems or facts needed to reach the solution. They are not my own original solutions but rather a rewriting of the official solutions in my own words so as to also match my own flow of thought as a self-check for my own understanding and in order to gain deeper insight into which steps or tricks in the solution I had the most trouble with.

2 Problem A1 2022

A1 Determine all ordered pairs of real numbers (a,b) such that the line y = ax + b intersects the curve y = ln(1 + x^2) in exactly one point.

Solution We first list all relevant theorems and observations used to reach the answer and observe what we can conclude from each observation:

1. y = f(x) = ln(1 + x^2) is an even function thus its graph is symmetric about the y-axis

2. y = f(x) is always positive for all x > 0 and this is due to the fact that the natural logarithm of any *positive* whole number is positive

   From (1) we can reduce our search space by considering only the lines with positive slope (a > 0) as this takes advantage of the graph's symmetry. In other words, two lines that differ by the sign of their slopes cut the graph of y = f(x) the same number of times, thus it makes sense to consider only one sign for a (the positive sign for simplicity) and make the same arguments on the other sign.

   From (2) we note that f(0) = 0. Due to this fact and (1), as well as a quick sketch of f(x), the line y = 0 is tangent to the graph of f(x) at O (0, 0). Therefore, (a,b) = (0,0) is a solution.

3. **Mean Value Theorem:** Suppose f is a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then,

there exists a number c in the interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

4. **Intermediate Value Theorem:** Suppose f is a continuous function on a closed interval $[a, b]$, and let y be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then, there exists a number c in the interval (a, b) such that $f(c) = y$.

5. $f'(x) = \frac{2x}{x^2+1}$ admits a maximum value of 1 and a minimum value of -1. (You can see this by calculating the roots of the second derivative and plugging those into the derivative function)

   We will demonstrate how to use (4) to prove that the line and the curve must intersect *at least* once for $a \geq 1$, and we will use (3) and (5) to prove that they cannot intersect *more* than once, also for $a \geq 1$. We can think of the MVT as setting an upper bound on the number of intersection points and the IVT as setting a lower bound for the number of intersection points.

3 Problem A1 2006

Find the volume of the region of points (x, y, z) such that

$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).$$

Solution This is a classic cylindrical coordinates problem. We note that

$$x^2 + y^2 = r^2$$

This reduces to

$$r^2 + z^2 + 8 \leq 6r,$$

or

$$(r - 3)^2 + z^2 \leq 1.$$

This defines a solid of revolution (a solid torus); the area being rotated is the disc $(x - 3)^2 + z^2 \leq 1$ in the xz-plane. By Pappus's theorem, the volume of this equals the area of this disc, which is π, times the distance through which the center of mass is being rotated, which is $(2\pi)3$. That is, the total volume is $6\pi^2$.

4 Problem A1 1986

Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.

Solution This is on the very easy side. Computation details will be left out. We must simply complete the square on the constraint equation, then rewrite the resulting expression as (a-b)(a+b), where its actual value is $(x^2-4)(x^2-9) \leq 0$. The 4 roots to this are -2, 2, -3, 3. Plugging these values into $f(x)$, the maximum is 18.

5 Problem A1 2020

How many positive integers N satisfy all of the following three conditions?

1. N is divisible by 2020.
2. N has at most 2020 decimal digits.
3. The decimal digits of N are a string of consecutive ones followed by a string of consecutive zeros.

Solution We note that $2020 = 2^2 \times 5 \times 101$.

6 Problem A1 2018

Find all ordered pairs (a, b) of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}.$$

Solution Another classic problem involving diophantine equations. The obvious thing to first do here is clear denominators to make the equation easier to work with. Therefore, we multiply both sides by the product of the denominators $a \times b \times 2018$ to obtain

$$3ab - 2018a - 2018b = 0$$

At this point, this is a classic factoring problem. We should be reminded of the product of binomials of the form:

$$(x-p)(y-q) = xy - qx - py + pq$$

The coefficients in front of x and y were omitted for simplicity. In our case, the product of the x and y coefficients is 3, and the pq term should be 2018^2 but seems to be missing from our expression. We thus add it to both sides of the equation to be able to exploit the "nice" factorization trick. But, before that, let us neatly multiply both sides by 3 so that we may break up the a and b with 3 as the coefficient in front of each like so:

$$9ab - 2018 \times 3a - 2018 \times 3b + 2018^2 = 2018^2$$

The complete factorization is thus:

$$(3a - 2018)(3b - 2018) = 2018^2$$

Note that the reason for multiplying by 3 was to make the coefficient of ab a perfect square which then helps preserve the symmetry of the factors which allows us to cleanly break them up into 3a and 3b, instead of 3a and b, for instance.

Now, we must examine the possible factorizations of 2018^2 knowing that we are restricted by the fact that a and b must be positive integers. Note that we could simply notice that $2018^2 = 2^2 \times 1009^2$ and has only 9 factors which makes it easy enough to check the solutions by hand by setting each of the factors we found equal to each of the factors and seeing which pair of factors does not yield to a contradiction to the integer requirement. However, this is a tedious and slow process and has no conceptual value. Instead, we will restrict our search space by making use of number theoretic observations. For example, if we are clever enough to notice that $(3a - 2018)$ and $(3b - 2018)$ are both of the form $3k + m$ where $3k$ is trivially a multiple of 3 (hence 0 mod 3) while m = - 2018 is congruent to 1 mod 3, then we know that each of the two factors is congruent to 1 mod 3 and hence cannot be set equal to 2018 for example, which is obviously a factor of 2018^2 and is congruent to 2 mod 3 yielding a contradiction. By similar arguments, we eliminate the factors: 2, 2018, and 2×1009^2. Our 2 factors can each therefore be: 1, 2018^2, 4, 1009, 1009^2, and 4×1009. Finding the exact possible values of (a,b) is trivial from here.

Note that we omitted checking the negative factors of 2018^2 simply because that would give negative values for a and b.

7 Problem A1 2005

Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, 23 = 9 + 8 + 6.)

Solution From the wording, this is a problem needing strong induction.

Base Case: The smallest positive integer $1 = 2^0 3^0$.
Inductive Step: Suppose the hypothesis is true for all positive integers upto n-1. This one is a bit tricky.

8 Problem A1 1995

Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any *three* (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

Solution There is one crucial observation to be made/inferred from the given that will prove extremely useful to not getting stuck while solving the problem:

- S is a group (by the group theory definition) and its binary operation multiplication is therefore associative

Now, one must begin with the correct strategy to approach this problem. The only approach that would work for proving the above is the method of proof by contradiction. The wording of the problem should have inspired us to think of this strategy because we are given a list of restrictions or properties that our sets must strictly obey, and we are asked to show that a certain, contrived property must necessarily hold under these conditions. The most intuitive approach should therefore be to assume that the property we are asked to prove does not hold and see if that yields any violation of the given rules that our mathematical objects must obey. If we do reach such a contradiction, it would mean that our initial assumption was false. In fact, we will even show that *exactly* one of the subsets T, U is closed under multiplication.

Let us assume that neither one of T, U is closed under multiplication. By definition,

$$t_1, t_2 \in T \implies t_1 \times t_2 \notin T.$$

Similarly for U,

$$u_1, u_2 \in U \implies u_1 \times u_2 \notin U.$$

However, because the set S is closed under $\times$, and because the union of T and U comprises all of S, then it is necessary that

$$t_1 \times t_2 \in U$$

$$u_1 \times u_2 \in T$$

In other words, the product of any two elements belonging to S which is not in T must be in U, and the product of any two elements belonging to S which is not in U must be in T.

Now, since the product of any 3 elements belonging to any one of the two subsets T, U belongs to the subset itself as well (closure under multiplication of three elements), we can say:

$$t_1 \times t_2 \times (u_1 \times u_2) \in T$$

$$(t_1 \times t_2) \times u_1 \times u_2 \in U$$

noting that $t_1, t_2, (u_1 \times u_2) \in T$ and $(t_1 \times t_2), u_1, u_2) \in U$

However, and this where our initial observation comes in handy, since multiplication in this context is associative, the above products of elements are the same. Therefore, we have found an element that belongs simultaneously to both subsets T and U, which is a contradiction as we are given that they are disjoint.

We may conclude that our initial hypothesis that neither T nor U is closed under multiplication is false.

As for proving why exactly one of T and U is closed under multiplication, observe that, to prove this, we can also rely on proof by contradiction. This time, we make the assumption that *both* T and U are closed under multiplication. We have already proved that having neither of them be closed under multiplication is an impossibility. If we can also prove that both of them being closed under multiplication is impossible, we are left with the necessity that exactly one of them is.

Assuming both T and U are closed under multiplication, we have:

$$t_1, t_2 \in T \implies t_1 \times t_2 \in T.$$

$$u_1, u_2 \in U \implies u_1 \times u_2 \in U.$$

Let us now multiply $t_1 \times t_2$ by $u_1 \times u_2$, knowing that these two elements belong to set S and that S is closed under multiplication.

$$(t_1 \times t_2) \times (u_1 \times u_2) \in S$$

From the given, this element must land in either T or U. However, if the first factor belongs to T, and the second factor belongs to U, and assuming we place this product in T without loss of generality, we would effectively be saying

9 Problem A7 1954

Prove that the equation $m^2 + 3mn - 2n^2 = 122$ has no integral solutions.

Solution This problem is also another classic one in the sense that the tricks used to solve it appear frequently in other problems. The question is essentially asking us to prove that the given equation is not a diophantine equation. We know from prior experience that working with diophantine equations often involves neat manipulations that greatly simplify the equation and help reveal insights into its solutions. In this case, we will rely on completing the square as the equation looks very reminiscent of a quadratic in m and n. This will become clearer if we manipulate it in a clever way.

First, looking at the coefficient of the mn term, we want to make it a multiple of 2. However, only multiplying the equation by 2 will not give us a neat perfect square coefficient for m (instead we will get $\sqrt{2}$) which is undesirable to work with. Instead, to have a perfect square coefficient for m while making the mn term even, let us conveniently multiply the entire equation by 4 to obtain:

$$4m^2 + 12mn - 8n^2 = 488$$

Completing the square yields

$$(2m + 3n)^2 - 17n^2 = 488$$

At this point, we have manipulated the equation enough to make it insightful to work with. Now, we have a numerical value on the right-hand side and an expression in m and n on the left-hand side. Looking at the LHS more closely, we observe that we have a multiple of 17 added to the square of a sum of m and n terms. It should be clear by now that our next strategy is to look at this equation modulo 17 and hope to reach a contradiction that prevents m and n from admitting any integral solutions.

488 is congruent to 12 mod 17; therefore, we need the LHS to be congruent to 12 mod 17 as well. Since $-17n^2$ is obviously congruent to 0 mod 17, we need $(2m+3n)^2$ to be congruent to 12 modulo 17. Examining the possible remainders of $(2m + 3n)^2$, which range from 0 to 16^2, the only unique possible remainders are 0, 1, 4, 9, 16, 8, 2, 15, 13. For anything beyond $8^2 = 64$ (which is 13 mod 17) up to 16^2, the cycle of remainders begins to repeat itself in reverse. As we can see, 12 is not a possible remainder mod 17 for $(2m + 3n)^2$. Thus, we have found our awaited contradiction and proven that the given equation cannot be true if $m, n \in \mathbb{Z}$

10 Problem A1 1998

A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

Solution

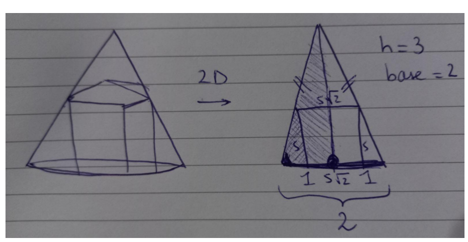

Figure 1: A quick sketch.

This is quite easy and straightforward if one draws the figure correctly. As can be seen from the sketch, to find the cube's side length s, we shall consider the two shaded similar right triangles sharing the common shaded angle.

$$\frac{s}{3} = \frac{\frac{1-s\sqrt{2}}{2}}{1}$$

That is because the height of the smaller triangle is the side length s, while that of the larger triangle is $h = 3$. The base of the smaller triangle is half of the base of the cone (which is the radius $r = 1$) minus half $s\sqrt{2}$. The latter comes from looking at the cube's square face at the bottom of the cone

11 Problem A1 1978

Let A be any set of 20 distinct integers chosen from the arithmetic progression 1,4, 7 , ... , 100. Prove that there must be two distinct integers in A whose sum is 104.

Solution The wording of the problem, in that it requires 20 integers to be certain that any two of them sum to 104, is highly suggestive of the Pigeonhole Principle. We look at all the ways in which two distinct integers from the set A having the form $1 + 3n$ sum to 104:

$$104 = 100 + 4$$

$$104 = 97 + 7$$

$$104 = 94 + 10$$

$$...$$

$$104 = 58 + 46$$

$$104 = 55 + 49$$

$$104 = 52 + 52$$

There is no point continuing since, by symmetry, we will obtain the same partitions but this time from the bottom up. Counting the partitions, there are 16 possible ways (16 without the last one) to sum two distinct integers from the set A into 104. One way one could start to choose 20 random distinct integers from the set A and not have any two of them sum to 104 would be to choose one number (without its partner) from each of the above listed sums. However, one quickly reaches a roadblock as he or she will exhaust the 17 possible sum choices and be forced to include 3 more addends from the above sums. Thus, three of the originally chosen numbers will be paired with their "sister" addend, forming a pair that sums to 104. To be precise, we will forcibly have three pairs in our chosen set of 20 that sum to 104.

12 Problem A2 1991

Let A and B be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible?

Solution The first thing that comes to mind is the two (most common) ways to prove that a matrix is invertible: either we try showing that the determinant is nonzero, which is very hard in this case because the formula for the determinant if n greater than 3 is too complicated, or that the matrix multiplied by any nonzero vector in a basis is nonzero, which also won't work in this case as we need to find a basis of n vectors. At this point, it is worth looking into proving *non-invertibility*: that is, we shall try finding a counterexample where the matrix multiplied by a nonzero vector gives 0. We do not necessarily have to find *the* such vector, as it most likely is not one singular particular vector but rather a general form of vectors as we shall see.

Keeping in mind that a matrix multiplied by a vector always gives a vector and that matrix multiplications by a vector is associative will be central to the solution. These might be obvious but are mentioned to refresh a rusty Linear Algebra memory. Also, the given information is meant to be used in our manipulations.

Via some wishful thinking, and always keeping in mind that we want to contrive a construction of matrix multiplications (by other matrices or by vectors) that result in 0, we should be prompted to multiply $A^2 + B^2$ by $A - B$ to give:

$$(A^2 + B^2)(A - B) = A^3 - A^2B + B^2A - B^3 = 0$$

Since $A \neq B$, the matrix $A - B \neq 0$. Therefore, there exists a vector u such that $(A - B)u \neq 0$. Now just set $v = (A - B)u$, which is nonzero, and we have $(A^2 + B^2)v = ((A^2 + B^2)(A - B))u = 0u = 0$. Thus $A^2 + B^2$ is always non-invertible since its product with a nonzero vector yielded 0.

13 Problem B1 1990

Find all real-valued continuously differentiable functions f on the real line such that for all x,

$$(f(x))^2 = \int_0^x [(f(t))^2 + (f'(t))^2]\, dt + 1990.$$

Solution We would like to simplify the gruesome expression by differentiating both sides to get rid of the integral like so

+1990)