Putnam Problems and Solutions

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1 Introduction

This is a compilation of the solutions to the Putnam problems from 1938-2023. They typically consist of a breakdown of the key theorems, facts, insights, tricks, and reasoning needed to reach the solution. They are, most of the time, not my own original solutions but rather a rewrite of the official solutions in my own words and in my own style. If you're anything like me, there's a good chance Putnam problems seem very elusive to you and often have very clever solutions that seem to come out of nowhere. My goal is to find some structure in the seemingly random and chaotic thought processes that are needed to find creative solutions to these very difficult problems. I seek to help the reader (and myself) reverse-engineer entire thought processes in order to train his/her intuition to think like a Putnam competitor. The first step to maximizing the gain you want to get out of this document is to believe that, with enough exposure and imitation, you too can solve hard problems. Contrary to popular advice found in most textbooks, I will not suggest that you attempt the problems yourselves before reading the solutions, especially if you are complete beginners to the world of the Putnam and math contests more generally. Unless the problem is easy enough (by Putnam standards), it is highly unlikely you will be able to figure out the solution just by thinking very hard. This should not be interpreted as an insult to your intelligence but rather an assertion of the fact that these problems are, by design, very difficult and sometimes require prerequisite knowledge (whether knowledge of a theorem or knowledge of a problem-solving technique) that would not occur to the inexperienced problem-solver. Instead, I suggest that the reader spend some time getting a feel for the question. It is important to really understand what makes the question meaningful and difficult. It is not important, and certainly not expected, that you have a feel for what to try or how to go about solving the problem. That intuition is what I seek to relay to you, and it will only come to you after a lot of exposure to problems and the various strategies used to crack them. Important questions you must ask yourself throughout your study of a solution is: How might I have thought of doing that? Why did the author decide to take this route instead of this other one? Why did this particular route not lead to a dead-end? What made the proof "click"? Do I have an intuitive understanding of why this solution makes sense? Can I translate this intuition into a formal argument? Am I fully convinced of a certain theorem/key fact?

Why is the solution I was thinking of wrong/uninsightful? What problem-solving strategy was used here and can be transferred to other problems? Happy Putnam training!

2 Problem A1 2022

A1 Determine all ordered pairs of real numbers (a,b) such that the line y = ax + b intersects the curve y = ln(1 + x^2) in exactly one point.

Solution We first list all relevant theorems and observations used to reach the answer and observe what we can conclude from each observation:

1. y = f(x) = ln(1 + x^2) is an even function thus its graph is symmetric about the y-axis

2. y = f(x) is always positive for all x > 0 and this is due to the fact that the natural logarithm of any *positive* whole number is positive

   From (1) we can reduce our search space by considering only the lines with positive slope (a > 0) as this takes advantage of the graph's symmetry. In other words, two lines that differ by the sign of their slopes cut the graph of y = f(x) the same number of times, thus it makes sense to consider only one sign for a (the positive sign for simplicity) and make the same arguments on the other sign.

   From (2) we note that f(0) = 0. Due to this fact and (1), as well as a quick sketch of f(x), the line y = 0 is tangent to the graph of f(x) at O (0, 0). Therefore, (a,b) = (0,0) is a solution.

3. **Mean Value Theorem:** Suppose f is a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then, there exists a number c in the interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

4. **Intermediate Value Theorem:** Suppose f is a continuous function on a closed interval $[a, b]$, and let y be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then, there exists a number c in the interval (a, b) such that $f(c) = y$.

5. $f'(x) = \frac{2x}{x^2+1}$ admits a maximum value of 1 and a minimum value of -1. (You can see this by calculating the roots of the second derivative and plugging those into the derivative function)

   We will demonstrate how to use (4) to prove that the line and the curve must intersect *at least* once for $a \geq 1$, and we will use (3) and (5) to prove that they cannot intersect *more* than once, also for $a \geq 1$. We can think of the MVT as setting an upper bound on the number of intersection points and the IVT as setting a lower bound for the number of intersection points.

3 Problem A1 2006

Find the volume of the region of points (x, y, z) such that

$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).$$

Solution This is a classic cylindrical coordinates problem. We note that

$$x^2 + y^2 = r^2$$

This reduces to

$$r^2 + z^2 + 8 \leq 6r,$$

or

$$(r - 3)^2 + z^2 \leq 1.$$

This defines a solid of revolution (a solid torus); the area being rotated is the disc $(x - 3)^2 + z^2 \leq 1$ in the xz-plane. By Pappus's theorem, the volume of this equals the area of this disc, which is π, times the distance through which the center of mass is being rotated, which is $(2\pi)3$. That is, the total volume is $6\pi^2$.

4 Problem A1 1986

Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.

Solution This is on the very easy side. Computation details will be left out. We must simply complete the square on the constraint equation, then rewrite the resulting expression as (a-b)(a+b), where its actual value is $(x^2 - 4)(x^2 - 9) \leq 0$. The 4 roots to this are -2, 2, -3, 3. Plugging these values into $f(x)$, the maximum is 18.

5 Problem A1 2020

How many positive integers N satisfy all of the following three conditions?

1. N is divisible by 2020.
2. N has at most 2020 decimal digits.
3. The decimal digits of N are a string of consecutive ones followed by a string of consecutive zeros.

Solution We note that $2020 = 2^2 \times 5 \times 101$.

6 Problem A1 2018

Find all ordered pairs (a, b) of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}.$$

Solution Another classic problem involving diophantine equations. The obvious thing to first do here is clear denominators to make the equation easier to work with. Therefore, we multiply both sides by the product of the denominators $a \times b \times 2018$ to obtain

$$3ab - 2018a - 2018b = 0$$

At this point, this is a classic factoring problem. We should be reminded of the product of binomials of the form:

$$(x - p)(y - q) = xy - qx - py + pq$$

The coefficients in front of x and y were omitted for simplicity. In our case, the product of the x and y coefficients is 3, and the pq term should be 2018^2 but seems to be missing from our expression. We thus add it to both sides of the equation to be able to exploit the "nice" factorization trick. But, before that, let us neatly multiply both sides by 3 so that we may break up the a and b with 3 as the coefficient in front of each like so:

$$9ab - 2018 \times 3a - 2018 \times 3b + 2018^2 = 2018^2$$

The complete factorization is thus:

$$(3a - 2018)(3b - 2018) = 2018^2$$

Note that the reason for multiplying by 3 was to make the coefficient of ab a perfect square which then helps preserve the symmetry of the factors which allows us to cleanly break them up into 3a and 3b, instead of 3a and b, for instance.

Now, we must examine the possible factorizations of 2018^2 knowing that we are restricted by the fact that a and b must be positive integers. Note that we could simply notice that $2018^2 = 2^2 \times 1009^2$ and has only 9 factors which makes it easy enough to check the solutions by hand by setting each of the factors we found equal to each of the factors and seeing which pair of factors does not yield to a contradiction to the integer requirement. However, this is a tedious and slow process and has no conceptual value. Instead, we will restrict our search space by making use of number theoretic observations. For example, if we are clever enough to notice that $(3a - 2018)$ and $(3b - 2018)$ are both of the form $3k + m$ where $3k$ is trivially a multiple of 3 (hence 0 mod 3) while m = - 2018 is congruent to 1 mod 3, then we know that each of the two factors is congruent to

1 mod 3 and hence cannot be set equal to 2018 for example, which is obviously a factor of 2018^2 and is congruent to 2 mod 3 yielding a contradiction. By similar arguments, we eliminate the factors: 2, 2018, and 2×1009^2. Our 2 factors can each therefore be: 1, 2018^2, 4, 1009, 1009^2, and 4×1009. Finding the exact possible values of (a,b) is trivial from here.

Note that we omitted checking the negative factors of 2018^2 simply because that would give negative values for a and b.

7 Problem A1 2005

Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.)

Solution From the wording, this is a problem needing strong induction.

Base Case: The smallest positive integer $1 = 2^0 3^0$.
Inductive Step: Suppose the hypothesis is true for all positive integers upto n-1. This one is a bit tricky.

8 Problem A1 1995

Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any *three* (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

Solution There is one crucial observation to be made/inferred from the given that will prove extremely useful to not getting stuck while solving the problem:

- S is a group (by the group theory definition) and its binary operation multiplication is therefore associative

Now, one must begin with the correct strategy to approach this problem. The only approach that would work for proving the above is the method of proof by contradiction. The wording of the problem should have inspired us to think of this strategy because we are given a list of restrictions or properties that our sets must strictly obey, and we are asked to show that a certain, contrived property must necessarily hold under these conditions. The most intuitive approach should therefore be to assume that the property we are asked to prove does not hold and see if that yields any violation of the given rules that our mathematical objects must obey. If we do reach such a contradiction, it would mean that our initial assumption was false. In fact, we will even show that *exactly* one of the subsets T, U is closed under multiplication.

Let us assume that neither one of T, U is closed under multiplication. By definition,

$$t_1, t_2 \in T \implies t_1 \times t_2 \notin T.$$

Similarly for U,

$$u_1, u_2 \in U \implies u_1 \times u_2 \notin U.$$

However, because the set S is closed under $\times$, and because the union of T and U comprises all of S, then it is necessary that

$$t_1 \times t_2 \in U$$

$$u_1 \times u_2 \in T$$

In other words, the product of any two elements belonging to S which is not in T must be in U, and the product of any two elements belonging to S which is not in U must be in T.

Now, since the product of any 3 elements belonging to any one of the two subsets T, U belongs to the subset itself as well (closure under multiplication of three elements), we can say:

$$t_1 \times t_2 \times (u_1 \times u_2) \in T$$

$$(t_1 \times t_2) \times u_1 \times u_2 \in U$$

noting that $t_1, t_2, (u_1 \times u_2) \in T$ and $(t_1 \times t_2), u_1, u_2) \in U$

However, and this where our initial observation comes in handy, since multiplication in this context is associative, the above products of elements are the same. Therefore, we have found an element that belongs simultaneously to both subsets T and U, which is a contradiction as we are given that they are disjoint.

We may conclude that our initial hypothesis that neither T nor U is closed under multiplication is false.

As for proving why exactly one of T and U is closed under multiplication, observe that, to prove this, we can also rely on proof by contradiction. This time, we make the assumption that *both* T and U are closed under multiplication. We have already proved that having neither of them be closed under multiplication is an impossibility. If we can also prove that both of them being closed under multiplication is impossible, we are left with the necessity that exactly one of them is.

Assuming both T and U are closed under multiplication, we have:

$$t_1, t_2 \in T \implies t_1 \times t_2 \in T.$$

$$u_1, u_2 \in U \implies u_1 \times u_2 \in U.$$

Let us now multiply $t_1 \times t_2$ by $u_1 \times u_2$, knowing that these two elements belong to set S and that S is closed under multiplication.

$$(t_1 \times t_2) \times (u_1 \times u_2) \in S$$

From the given, this element must land in either T or U. However, if the first factor belongs to T, and the second factor belongs to U, and assuming we place this product in T without loss of generality, we would effectively be saying

9 Problem A7 1954

Prove that the equation $m^2 + 3mn - 2n^2 = 122$ has no integral solutions.

Solution This problem is also another classic one in the sense that the tricks used to solve it appear frequently in other problems. The question is essentially asking us to prove that the given equation is not a diophantine equation. We know from prior experience that working with diophantine equations often involves neat manipulations that greatly simplify the equation and help reveal insights into its solutions. In this case, we will rely on completing the square as the equation looks very reminiscent of a quadratic in m and n. This will become clearer if we manipulate it in a clever way.

First, looking at the coefficient of the mn term, we want to make it a multiple of 2. However, only multiplying the equation by 2 will not give us a neat perfect square coefficient for m (instead we will get $\sqrt{2}$) which is undesirable to work with. Instead, to have a perfect square coefficient for m while making the mn term even, let us conveniently multiply the entire equation by 4 to obtain:

$$4m^2 + 12mn - 8n^2 = 488$$

Completing the square yields

$$(2m + 3n)^2 - 17n^2 = 488$$

At this point, we have manipulated the equation enough to make it insightful to work with. Now, we have a numerical value on the right-hand side and an expression in m and n on the left-hand side. Looking at the LHS more closely, we observe that we have a multiple of 17 added to the square of a sum of m and n terms. It should be clear by now that our next strategy is to look at this equation modulo 17 and hope to reach a contradiction that prevents m and n from admitting any integral solutions.

488 is congruent to 12 mod 17; therefore, we need the LHS to be congruent to 12 mod 17 as well. Since $-17n^2$ is obviously congruent to 0 mod 17, we need $(2m+3n)^2$ to be congruent to 12 modulo 17. Examining the possible remainders of $(2m+3n)^2$, which range from 0 to 16^2, the only unique possible remainders are 0, 1, 4, 9, 16, 8, 2, 15, 13. For anything beyond $8^2 = 64$ (which is 13 mod 17) up to 16^2, the cycle of remainders begins to repeat itself in reverse. As we can see, 12 is not a possible remainder mod 17 for $(2m+3n)^2$. Thus, we have found our awaited contradiction and proven that the given equation cannot be true if $m, n \in \mathbb{Z}$

10 Problem A1 1998

A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

Solution

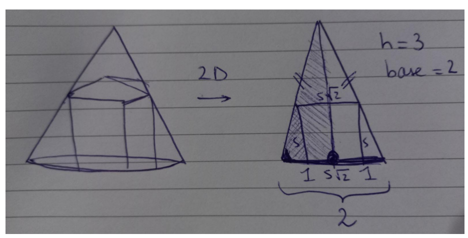

Figure 1: A quick sketch.

This is quite easy and straightforward if one draws the figure correctly. As can be seen from the sketch, to find the cube's side length s, we shall consider the two shaded similar right triangles sharing the common shaded angle.

$$\frac{s}{3} = \frac{\frac{1-s\sqrt{2}}{2}}{1}$$

That is because the height of the smaller triangle is the side length s, while that of the larger triangle is $h = 3$. The base of the smaller triangle is half of the base of the cone (which is the radius $r = 1$) minus half $s\sqrt{2}$. The latter comes from looking at the cube's square face at the bottom of the cone

11 Problem A1 1978

Let A be any set of 20 distinct integers chosen from the arithmetic progression 1,4, 7 , ... , 100. Prove that there must be two distinct integers in A whose sum is 104.

Solution The wording of the problem, in that it requires 20 integers to be certain that any two of them sum to 104, is highly suggestive of the Pigeonhole Principle. We look at all the ways in which two distinct integers from the set A having the form $1 + 3n$ sum to 104:

$$104 = 100 + 4$$

$$104 = 97 + 7$$

$$104 = 94 + 10$$

$$...$$

$$104 = 58 + 46$$

$$104 = 55 + 49$$

$$104 = 52 + 52$$

There is no point continuing since, by symmetry, we will obtain the same partitions but this time from the bottom up. Counting the partitions, there are 16 possible ways (16 without the last one) to sum two distinct integers from the set A into 104. One way one could start to choose 20 random distinct integers from the set A and not have any two of them sum to 104 would be to choose one number (without its partner) from each of the above listed sums. However, one quickly reaches a roadblock as he or she will exhaust the 17 possible sum choices and be forced to include 3 more addends from the above sums. Thus, three of the originally chosen numbers will be paired with their "sister" addend, forming a pair that sums to 104. To be precise, we will forcibly have three pairs in our chosen set of 20 that sum to 104.

12 Problem A2 1991

Let A and B be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible?

Solution The first thing that comes to mind is the two (most common) ways to prove that a matrix is invertible: either we try showing that the determinant is nonzero, which is very hard in this case because the formula for the determinant if n greater than 3 is too complicated, or that the matrix multiplied by any nonzero vector in a basis is nonzero, which also won't work in this case as we need to find a basis of n vectors. At this point, it is worth looking into proving *non-invertibility*: that is, we shall try finding a counterexample where the matrix multiplied by a nonzero vector gives 0. We do not necessarily have to find *the* such vector, as it most likely is not one singular particular vector but rather a general form of vectors as we shall see.

Keeping in mind that a matrix multiplied by a vector always gives a vector and that matrix multiplications by a vector is associative will be central to the solution. These might be obvious but are mentioned to refresh a rusty Linear Algebra memory. Also, the given information is meant to be used in our manipulations.

Via some wishful thinking, and always keeping in mind that we want to contrive a construction of matrix multiplications (by other matrices or by vectors) that result in 0, we should be prompted to multiply $A^2 + B^2$ by $A - B$ to give:

$$(A^2 + B^2)(A - B) = A^3 - A^2B + B^2A - B^3 = 0$$

Since $A \neq B$, the matrix $A - B \neq 0$. Therefore, there exists a vector u such that $(A - B)u \neq 0$. Now just set $v = (A - B)u$, which is nonzero, and we have $(A^2 + B^2)v = ((A^2 + B^2)(A - B))u = 0u = 0$. Thus $A^2 + B^2$ is always non-invertible since its product with a nonzero vector yielded 0.

13 Problem B1 1990

Find all real-valued continuously differentiable functions f on the real line such that for all x,

$$(f(x))^2 = \int_0^x [(f(t))^2 + (f'(t))^2]\, dt + 1990.$$

Solution We would like to simplify the gruesome expression by differentiating both sides to get rid of the integral like so

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14 Problem A1 1994

(a_n) is a sequence of positive reals satisfying

$$a_n \leq a_{2n} + a_{2n+1}$$

for all n. Prove that the infinite series

$$\sum_{n=1}^{\infty} a_n$$

diverges.

Solution We are not given an explicit formula for (a_n). Rather, we only have an inequality relating some terms of the sequence with each other. So let us just fiddle around with it to make some substitutions and see if that takes us anywhere. In particular, notice that

$$a_1 \leq a_2 + a_3$$

Letting

$$S = a_1 + a_2 + a_3 + a_4 + a_5 + ... + a_n + ...$$

, we can use the inequality to note that

$$S \leq a_2 + a_3 + a_2 + a_3 + a + 4 + a_5 + ... = S - a_1$$

But since all terms of the sequence (a_n) are positive real numbers, the last inequality is impossible unless S diverges to infinity.
Note we could have also solved it like in the official solution:

$$S = a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + ...$$

and each bracket has sum at least $a_1 \geq 0$. thus S is greater or equal to a sum that diverges to infinity. Hence S itself must diverge to infinity. Fun fact: Johann Bernoulli used a proof very similar to the first method to show that the harmonic series diverges in 1689!

15 Problem A3 1951

Evaluate

$$1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \frac{1}{13} - ...$$

Solution My attacks at this problem consisted of trying to combine some fractions, make some cancellations, find a telescoping series... Needless to say no clever simplifications were found on my part, though I am wondering if there actually is such a trick that may work. Anyway, it didn't hurt to try. The key to this problem though was a knowledge of some famous integrals.
The motivation for using integrals, which, as we will see shortly, will replace each term of the sequence, is the observation that all the denominators are 1 away from being multiples of 3. Pair that with a keen memory from Calc 2 and you recall that:

$$\int_0^1 x^n \, dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}$$

Essentially, we will replace each term in the sequence with its corresponding integral. The reason this is is so useful is because now each exponent on each x term is a multiple of 3. Combining the now infinitely many integral terms into one integral involving sums and differences of powers of x, we notice we are integrating a geometric series of common ration $-x^3$.
More clearly, our series becomes:

$$\int_0^1 x^0 \, dx - \int_0^1 x^3 \, dx + \int_0^1 x^6 \, dx - \int_0^1 x^9 \, dx + \int_0^1 x^{12} \, dx ... = \int_0^1 x^0 - x^3 + x^6 - x^9 + x^{12} \, dx ...$$

Since x is bounded between 0 and 1, our well-known formula for geometric series can be used to rewrite the integral of the infinite sum as

$$\int_0^1 \frac{1}{1 - (-x^3)} \, dx$$

The rest would be a pain to write out and would not benefit you as much as reviewing your notes on integration from Calc 2. The gist of the remainder of this solution is factoring $1+x^3$, then using partial fraction decomposition, then making the numerator the derivative of the denominator to be able to use the $ln(denominator)$ technique of integration... all very exciting stuff that my notes would not do justice (Read: I am lazy).

I love and yet passionately despise this problem at the same time. This problem feels very impenetrable for various reasons. Most notably is the sheer innovativeness required to think of resorting to integrals in a problem about sums of reciprocals... But I guess the key is to have memorized well-known integrals until they become second-nature. What I mean to say is, take the product of conjugates $(a-b)(a+b)$ as an analogy. I don't think any reasonably seasoned problem solver would find it particularly clever to make the substitution with the difference of squares a^2-b^2. Any good mathematician has this identity stamped on her heart. I think that level of "memory" is crucial for this problem too, this time with dreaded integrals instead.

16 Problem B1 1977

Compute

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1}$$

Solution This one isn't too bad if you are familiar with the tricks used to manipulate these infinite products. This was also featured as a practice problem in my first Real Analysis course, and it should be very standard.

17 Problem A6 2021

Let $P(x)$ be a polynomial whose coefficients are all either 0 or 1. Suppose that $P(x)$ can be written as a product of two non-constant polynomials with integer coefficients. Does it follow that $P(2)$ is a composite integer?

Solution

18 Problem A1 1977

Show that if four distinct points of the curve $y = 2x^4+7x^3+3x-5$ are collinear, then their average x-coordinate is some constant k. Find k.

Solution *Trivial* By definition of collinearity, the four points lie on the same line, hence their coordinates satisfy the equation $y = ax+b$. At the same time, their coordinates satisfy the equation of the curve $y = 2x^4+7x^3+3x-5$. So if a point (x, y) lies on the curve and the line simultaneously, the expression for its y-coordinate in terms of the curve will be equal to that in terms of the line, i.e. we can say $y = y$ implies that

$$2x^4 + 7x^3 + 3x - 5 = ax + b$$

$$2x^4 + 7x^3 + 3x - 5 - (ax + b) = 0$$

$$2x^4 + 7x^3 + (3 - a)x - (5 + b) = 0$$

We have reduced the problem into a 4^{th} degree polynomial with at most 4 distinct roots x_1, x_2, x_3, x_4. We know the sum of these roots is $-a_{n-1}/a_n$ which in this case is $-7/2$. So the arithmetic mean of the four x-coordinates is $-7/8$.

19 Problem A1 2003

Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers,

$$n = a_1 + a_2 + \cdots + a_k$$

with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$, there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.

Solution Essentially, the question is asking for the number of ways that a positive number n can be partitioned as a sum of terms that are either all equal or having some (one or more) terms at most 1 greater than the rest of the (equal) terms. More intuitively, it is asking for the number of ways we can partition n into k terms that are as equally spaced as possible. We show that there are n such sums. To be more precise, there is (obviously) exactly one such sum with k summands for each of $k = 1, \ldots, n$. To see this, let $n = a_1 + ... + a_k$ with $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$. Then,

$$ka_1 = a_1 + a_1 + ... + a_1 \leq n \leq a_1 + (a_1 + 1) + ... + a_1 + 1 = ka_1 + k - 1$$

There is a unique integer a_1 satisfying this inequality. It is $a_1 = \left\lfloor \frac{n}{k} \right\rfloor$. This should make sense as we are essentially dividing n into k equal parts. One we fix a_1 in this manner, we can then notice that $n = a_1 + a_2 ... + a_k$ will in fact be a sum of k terms with some summands being equal to a_1, and the rest being at most $a_1 + 1$. Let i be the index of the last integer term a_i such that $a_i = a_1$. Beyond the index i, the terms $a_{i+1}, a_{i+2}, ..., a_{i+k-1}$ will be equal to $a_1 + 1$, so we can write that $\underbrace{n = a_1 + a_1 + ... + a_1}_{\text{k terms}} + (i - 1) = ka_1 + (i - 1)$. Here, i is the number of ones that were added to a_1 in the terms after a_i. i can take on any value between 1 and k, and exactly one of these possible sums equals n. So, indeed, there is exactly one such sum for each possible value of k, and k ranges from 1 to n, so there are n such sums in total.

20 Problem B1 2013

For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1) = 1$, $c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2013} c(n)c(n+2).$$

Solution We start with the belief that a Putnam question about finding the value of a complicated series should not be computational. Rather, we seek to simplify the series by hopefully cancelling terms via some telescoping. To see where the series is going, we can try computing by hand the first few terms. We notice that

$$\sum_{n=1}^{2013} c(n)c(n+2) = c(1)c(3)+c(2)c(4)+...+c(2012)c(2016)+c(2013)c(2015) = -1+1-1+1-1+...+1-1$$

where all the terms in the series get cancelled out except the first.

The telescoping property is made explicit as follows:

$$\begin{aligned} c(2k+1)c(2k+3) &= (-1)^k c(k)(-1)^{k+1}c(k+1) \\ &= -c(k)c(k+1) \\ &= -c(2k)c(2k+2). \end{aligned}$$

In other words, an odd term in the series is the negative of its preceding even term. The telescoping series can thus be written compactly as $\sum_{n=2}^{2013} c(n)c(n+2) = \sum_{k=1}^{1006}(c(2k)c(2k+2) + c(2k+1)c(2k+3)) = 0$, where we have paired off the even and odd terms that will cancel each other out, and so the desired sum is $c(1)c(3) = -1$

21 Problem B1 2007

Let f be a polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n)+1)$ if and only if $n = 1$. [Editor's note: one must assume f is nonconstant.]

Solution The problem fails if f is allowed to be constant, e.g., take $f(n) = 1$. We thus assume that f is nonconstant. Write $f(n) = \sum_{i=0}^{d} a_i n^i$ with $a_i > 0$. Then just plug in $f(n) + 1$ as an argument to f:

$$\begin{aligned} f(f(n)+1) &= \sum_{i=0}^{d} a_i (f(n)+1)^i \\ &\equiv f(1) \pmod{f(n)}. \end{aligned}$$

If you don't see how we ended up with $f(1)$, recall that the remainder modulo $f(n)$ of a sum will be the sum of the remainders of the summands modulo $f(n)$.

And each summand has the form $a_i(f(n)+1)^i$. By modulo properties, the summand modulo $f(n)$ will be 1^i multiplied by a_i. So the remainder of the whole polynomial $f(f(n)+1)$ modulo $f(n)$ will be the sum of the coefficients $\sum_{i=0}^{d} a_i$, which is simply $f(1)$. Now if $n=1$, then this implies that $f(f(n)+1)$ is divisible by $f(n)$ because the remainder $f(1)$ and divisor $f(n=1)$ would be equal. To prove the reverse direction, proceed by contrapositive. Assume $n \neq 1$. Then we obtain the strict inequality $0 < f(1) < f(n)$ since f is nonconstant and has positive coefficients, so $f(f(n)+1)$ cannot be divisible by $f(n)$ since it would have a (valid) remainder upon division by $f(n)$.

22 Problem B1 2015

Let f be a three times differentiable function (defined on $\mathbb{R}$ and real-valued) such that f has at least five distinct real zeros. Prove that $f+6f'+12f''+8f'''$ has at least two distinct real zeros.

Solution Since we are asked about roots of derivatives, Rolle's theorem, which states that any real-valued differentiable function that attains equal values at two distinct points must have at least one stationary point somewhere between them—that is, a point where the first derivative is zero, must come to mind. By Rolle's theorem, if f is given to have at least 5 distinct real roots, say $a<b<c<d<e$, then we can pair them off like so:

$$f(a)=f(b)=0, f(b)=f(c)=0, f(c)=f(d)=0, f(d)=f(e)=0$$

and from each equality at two points we obtain one root for the first derivative of f, so f' has at least 4 distinct real roots. By a similar construction, we can see that f'', f''' will have at least 3, 2 distinct real roots respectively. Now it's important to think about where the given expression $f+6f'+12f''+8f'''$ might have come from. The coefficients in front of the higher-order derivatives seem to hint at repeated differentiation of a certain expression we would like to reverse-engineer. In addition, we just showed that the third-order derivative of f will have at least 2 distinct real roots. So we might somehow be inclined to show some sort of relationship between the third-order derivative of some function g and the given expression $f+6f'+12f''+8f'''$. We seek a three times real-valued, differentiable function g whose third derivative involves the expression $f+6f'+12f''+8f'''$. We can construct this function g any way we like so long as it fits the bill. The easiest way to proceed would be to scale f by an exponential function $e^a x$ (for some a we must find out. This a must help yield the given expression when we take the third derivative of g). Why? Because exponential functions are never zero, they never have any roots. This is crucial since we do not want to tamper with the number of roots provided by f already. Also, derivatives of exponential functions are very easy to work with. The exact details of how $a=\frac{1}{2}$ was found is left as an exercise. The worked out solution is provided here by an excellent Youtube video `https://www.youtube.com/watch?v=RkyGoWx3erk&t=422s&ab_channel=MichaelPenn`.

23 Problem A1 2005

Originally due to Paul Erdos Show that every positive integer is a sum of one or more numbers of the form 2^r3^s, where r and s are nonnegative integers and no summand divides another. (Note: This representation need not be unique. For example, 11 = 2+9 = 3+8) (For example, 23 = 9 + 8 + 6.)

Solution Not much creativity is needed here. We are asked to prove a claim about all positive integers. (Strong) induction is only a natural way to proceed. Base Case: $1 = 2^0 3^0$. Inductive Hypothesis: Suppose all positive integers less than $n-1$ can be represented in the desired manner. We then show how to construct such a representation for n by cleverly applying the inductive hypothesis to the numbers preceding n. If n is even, we can show that n can be written as a sum of terms having the form 2^r3^s by multiplying each summand in the representation of $\frac{n}{2}$ (which we know has the desired form by inductive assumption) by 2. If n is odd, How did we know to argue based on the parity of n?

24 Problem A1 2016

Find the smallest positive integer j such that, for every polynomial $p(x)$ with integer coefficients and for every integer k, the integer

$$p^{(j)}(k) = \frac{d^j}{dx^j} p(x)\Big|_{x=k}$$

(the j^{th} derivative of $p(x)$ at k) is divisible by 2016.

Solution Suppose that j satisfies the given condition. Represent $p(x)$ as $p(x) = \sum_{m=0}^{n} a_m x^m$. Then the j^{th} derivative is

$$p^{(j)}(x) = \sum_{m=j}^{n} m(m-1)(m-2)...(m-j+1)a_m x^{m-j}$$

where the starting index in the series is $m = j$ since the first few terms of the polynomial having exponents between 1 and $j-1$ vanish when taking the j^{th} derivative. Notice now that the coefficient of each term in the j^{th} derivative, being $m(m-1)(m-2)...(m-j+1)$, is equal to $\binom{m}{j} \times j!$, so we can simplify the expression of $p^{(j)}(x)$ as follows:

$$p^{(j)}(x) = j! \sum_{m=j}^{n} \binom{m}{j} a_m x^{m-j}$$

Now, we want for 2016 to divide $j! \sum_{m=j}^{n} \binom{m}{j} a_m x^{m-j}$ for all $x \in \mathbb{Z}$. In particular, we want this to be true for the simplest value of x we can plug in, being $x = 0$.

All terms of the series except for the first one where $m = j$ will vanish if $k = 0$. This leaves us with the condition that 2016 must divide $j! \times \binom{j}{j} = j!$. Now, $2016 = 2^5 3^2 7$ which means that $2016 \nmid 7!$ but 2016 does divide 8! (verify this). So we can argue that $j = 8$ is the smallest j such that the j^{th} derivative of a polynomial $p(x)$ is divisible by 2016 for all values of $x \in \mathbb{Z}$.

25 Problem A1 2017

Let $S \subseteq \mathbb{N}$ be the smallest set such that:

1) $2 \in S$
2) If $n^2 \in S$, then $n \in S$
3) If $n \in S$, then $(n + 5)^2 \in S$

Which positive integers are not in S?

Solution We first make some trivial observations. Clearly, $1 \notin S$. By (3), all numbers of the form $(5k + 2)^2$ where k is a natural number are in S, and so by (2), all numbers of the form $5k + 2$ are in S. Careful that this does not mean these are the *only* numbers that are in S. We have not ruled out the possibility that numbers of the form $5k, 5k + 1, 5k + 3, 5k + 4$ may be in S. To do that, we can start by a simple by-hand computation to check for the presence of the simplest natural numbers having the one of the forms $5k, 5k + 1, 5k + 3, 5k + 4$. Namely, we must simply check if 3, 4, 5, 6 are in S or not, as these numbers would generate all other numbers of the forms $5k, 5k+1, 5k+3, 5k+4$ inside of S by properties (3) and (2). The details of the computation are included for completion, but the solution should be clear by this point. Starting at the smallest element 2 and following the given properties, we obtain $2 \to 49 \to 54^2 \to 56^2 \to 56 \to 121 \to 11 \to 16 \to 4 \to 9 \to 3$, and $16 \to 36 \to 6$. So in fact, 2, 3, 4, and 6 are in S, so the only numbers not in S are 1 and all multipled of 5 of the form $5k$.

26 Problem A1 2013

Recall that a regular icosahedron isa convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are 2 faces that share a vertex and have the same integer written on them.

Solution It is necessary to observe that an icosahedron consists of 4 collections of 5 faces which meet at a vertex. We wish to prove that there is a pair of faces that share a common vertex and which have the same integer written on them. We proceed by contradiction. Suppose there does not exist a pair of faces sharing a vertex and having the same number. What do we contradict here? Notice that we are constrained by the total sum, being 39, of the integers written on the 20 faces of the icosahedron. Surely, if we were not constrained by such a sum value it would be possible for ALL faces that meet at a vertex to NOT have the same integer written on them. So we should think to exploit the sum of 39 in our contradiction argument. If we are assuming that no two faces meeting at a vertex have the same number, then, considering a collection of 5 faces meeting at a vertex, these 5 faces must consist of 5 unique integers. If we take even the smallest possible sum these 5 faces can add up to, we obtain $0+1+2+3+4=10$. Now, each collection of 5 faces meeting at a vertex contains a pair of faces meeting at a vertex (clearly). So each integer can appear on the icosahedron at most 4 times, and so the sum of all the integers is at least $4(0+1+2+3+4)=40>39$, a contradiction.

My approach I wasn't really convinced by the above solution as I couldn't really visualize it. When I read the problem for the first time, my first instinct was to involve the Pigeonhole Principle somehow. Here is my solution.
For 20 integers to sum to 39, I claim that at least 5 of them must be equal.

27 Problem A2 2013

Let S be the set of all positive integers that are *not* perfect squares. For n in S, consider choices of integers $a_1, a_2, ...a_r$ such that $n < a_1 < a_2 < ... < a_r$ and $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minimum of a_r over all such choices. For example, $2 \in S$ and $2 \cdot 3 \cdot 6$ is a perfect square, while $2\cdot 3, 2\cdot 4, 2\cdot 5, 2\cdot 3\cdot 4, 2\cdot 3\cdot 5, 2\cdot 4\cdot 5, 2\cdot 3\cdot 4\cdot 5$ are not. Show that the function f from the set S to the natural numbers is injective.

Solution Getting the setup right is crucial. To prove $f : S \to \mathbb{N}$ must be injective, we will suppose that $f(n) = f(m)$, and we will argue by way of contradiction to force that $n = m$. We assume that $f(n) = f(m)$ with $n < m$ WLOG. Let $n \cdot a_1 \cdots a_r, m \cdot b_1 \cdots b_s$ be perfect squares where $n < a_1 < a_2 < ... < a_r, m < b_1 < b_2 < ... < b_s, a_r, b_s$ minimal over all choices of sequences $(a_r), (b_s)$. The minimality of a_r, b_s will be what we exploit in this proof by contradiction. So we want to try to find a sequence of positive integers (a_r) (or (b_s)) by which we can multiply n (or m) such that the largest positive integer of this sequence is even less than the supposed largest being (a_r) (or (b_s)). (For clarity I will call these "f values from now on, and will assume we will contradict the minimality of a_r WLOG) It is not feasible to find such a sequence constructively. Instead, if we are observant enough, we notice that the contradiction is given to us for free. We already have a smaller f value than a_r which we can use in our contradiction argument, namely any of the smaller a_{r-k}, $k \in \mathbb{N}$ of the sequence are at our disposal. Now in order to use this to contradict the minimality of a_r, the product $m \cdot a_1 \cdots a_{r-k}$ must be a perfect square. This also implies that some of the terms of the sequence (a_r) have vanished. How? Well, if we multiply $(n \cdot a_1 \cdots a_r) \times (m \cdot b_1 \cdots b_s)$, we notice that we (clearly) get back a perfect square. We can then delete all duplicate factors in this new product i.e. if $a_i = b_j$ for some i, j then delete both a_i, b_j from this product. The product that remains after this deletion is also still a perfect square (this should make sense since the quotient of a perfect square by the square of some number is still a perfect square). Also notice that by this procedure we are cancelling $a_r = b_s$ from the product. But now we are left with a perfect square made of distinct factors, the smallest of which is n and the largest of which is an integer that is strictly smaller than $a_r = b_s$, contradicting the minimality of $a_r (= b_s)$.

Note: I have tried as much as I can to give motivation for the original solution written by Kiran Kedlaya and Lenny Ng, as the thought process associated with the steps of the solution seemed too contrived to me. I have written it in a way that attempts to reverse-engineer the steps, but I realize this makes it harder to read and understand. I suggest you read the above solution only after you have read and understood the official one.

28 Problem A3 2013

Suppose that the real numbers $a_0, a_1, ..., a_n$ and x, with $0 < x < 1$, satisfy

$$\frac{a_0}{1-x} + \frac{a_1}{a-x^2} + ... + \frac{a_n}{a-x^{n+1}} = 0$$

Prove that there exists a real number y with $0 < y < 1$ such that

$$a_0 + a_1 y + \cdots + a_n y^n = 0$$

Solution There isn't much to go off of for now, and the wording of the question seems to suggest a proof by contradiction, so we proceed by contradiction. Suppose that for all $0 < y < 1$, we have that $a_0 + a_1 y + ... + a_n y^n$ is nonzero. We are dealing with a polynomial function and the question of whether or not it attains a value of zero on some interval. This should remind us of the intermediate value theorem, which states that a continuous function over an interval (a, b) takes on all intermediate values between $f(a)$ and $f(b)$. The implication of the IVT to this problem is that, for the given polynomial to never take on the value of zero on the interval $y \in (0, 1)$, it cannot change sign over the interval $(0, 1)$, since, if it did, then by IVT we require the polynomial to take on the intermediate value of 0 in the function's transition from being positive to negative, but we are assuming the polynomial is always nonzero. Thus we assume WLOG that $a_0 + a_1 y + ... + a_n y^n > 0$ for $0 < y < 1$. For the given value of x i.e. $0 < x < 1$, we can rightly say that

$$a_0 + a_1 x + ... + a_n x^n >= 0$$

And since higher powers of x (say x^m for $m = 0, 1, 2, ...,$) get smaller if $0 < x < 1$ and remain within the interval $(0, 1)$, we can claim that $a_0 x^m + a_1 x^{2m} + ... + a_n x^{(n+1)m} >= 0$ with strict inequality if $m > 0$. Now we will take the sum of the expression $a_0 x^m + a_1 x^{2m} + ... + a_n x^{(n+1)m}$ over all possible (countably infinite) values of m. This results in an infinite series of terms. But since $0 < x < 1$, it follows that this series is absolutely convergent and geometric, so the whole summation over all values of m is given by

$$\frac{a_0}{1-x} + \frac{a_1}{a-x^2} + ... + \frac{a_n}{a-x^{n+1}} \quad (*)$$

by repeatedly using the formula for geometric series when $0 < x < 1$. (If it's not clear how the sum was obtained, convince yourself that each term in $(*)$ is the geometric series formula of the infinite series given by $a_i(1 + x + x^2 + ...)$ which comes about by simply rearranging (we are allowed to do this since the series is absolutely convergent!!!) and grouping the x-terms of same degree together). But $(*)$ is the absolutely convergent sum of all-positive sub-sums, so $(*)$ must evidently also be positive, a contradiction to the given that

$$\frac{a_0}{1-x} + \frac{a_1}{a-x^2} + ... + \frac{a_n}{a-x^{n+1}} = 0$$

.