Putnam and Beyond

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1 Introduction

In this document, I hope to make an accompanying guide to *Putnam and Beyond* by Razvan Gelca and Titu Andreescu. My main goal is to rewrite the solutions and proofs presented so as to give motivation to every step of the proof to demystify seemingly randomly chosen, clever "tricks" into more deliberate choices consciously made by the mathematician to reach the desired result that needs proof. I will not copy the author's interpretation of the proof. Rather, I will expand upon it and break it down step-by-step, showcasing the strategy behind each idea. This is why I suggest reading my notes side-by-side with the authors' proofs as they may not make much intelligible sense as stand-alones.

2 Euclid's Theorem

Euclid's theorem. *There are infinitely many prime numbers.*

The newly contrived number N is constructed by adding 1 to the product of all the primes, whose number was assumed to be finite. By our assumption of the finiteness of primes, this newly contrived number N must be composite, as all the primes in existence have already been listed, and N is not among them but is 1 more than the product of all of them. By definition of compositeness, N is divisible by at least 2 primes. However, looking at its construction, it cannot be divisible by any of the primes from p1 to pn: in fact, it is 1 modulo pi where i ranges from from 1 to n. Therefore, since it is necessarily composite but not divisible by any of the listed primes, there must exist at least 2 more primes outside this finite set. Even if these 2 primes are "found", we can reconstruct this whole argument again and keep arguing for the existence of more primes than we have listed ad infinitum, thus proving the infinitude of the primes. Note that the way N was contrived is key, as adding 1 to the sum of all primes for example would not help yield the divisibility argument.

3 Example - Proof by Contradiction

Prove that there is no polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$$

with integer coefficients and of degree at least 1 with the property that $P(0), P(1), P(2)$, ... are all prime numbers.

It may be tempting, blindly following the template of proof by contradiction, to make the assumption that all the values $P(0), P(1), P(2)$... are prime and see what conclusions we can draw from that. However, this will not be very insightful as one is making too general a statement to gain any new insight. Instead, when presented with problems dealing with very large, in this case infinite, number of cases, it is best to proceed by dealing with only a few small ones, cleverly chosen, so as to discover a pattern or trend. Here, as we are dealing with a polynomial, we must always keep in mind the advantage of $P(0)$, as the 0 cancels all the terms having x and its output is the constant term or coefficient a_0. By our assumption, P(0) $= a_0 = a$ prime p. Now, noting that a polynomial is nothing but the sum of products of constant coefficients by variable x's, it is useful to view it modulo some number. Because a_0 is some prime p hence obviously divisible by p, then the entire polynomial P will be divisible by p for any input being a non-zero multiple of p. But for any such input, the output must also be some prime by our assumption. We have found our contradiction: the outputs must be prime but divisible by the prime p at the same time. Hence, they must all be none other than p itself. A polynomial with integer coefficients and degree at least 1 cannot result in the same output given multiple different inputs, hence the statement is impossible.

4 Example - Proof by Contradiction

Let $F = E_1, E_2, ..., E_s$ be a family of subsets with r elements of some set X. Show that if the intersection of any $r + 1$ (not necessarily distinct) sets in F is nonempty, then the intersection of all sets in F is nonempty

5 Problems - Proof by Contradiction

1. *Prove that $\sqrt{2} + \sqrt{3} + \sqrt{5}$ is an irrational number*

   Assume it is rational. By definition, it can be written in the form $\frac{p}{q}$ where p, q are nonzero integers. While this is at the heart of the argument that Pythagoras gave to prove that the square root of 2 is irrational, it will not be of good use just yet, primarily because the value we are dealing with is not as clean as the singular value of $\sqrt{2}$ but is a sum of square roots. Thus, using the definition directly is messy and not insightful. Instead, we will assume that the sum is a rational number r. Ultimately, starting the problem in this manner will reduce the problem from having a sum of three terms to a problem with just one singular square root which can be proven to be irrational via Pythagoras' argument.

We proceed in this manner: Square both sides of the equality

$$\sqrt{2} + \sqrt{3} = r - \sqrt{5}$$

to obtain

$$5 + 2\sqrt{6} = r^2 + 5 - 2r\sqrt{5}$$

We now utilize the fact that arithmetic operations on rational numbers must yield rational numbers, so every term in this squared equality is rational, including the square root terms $2\sqrt{6}$ and $2r\sqrt{5}$. We have not yet reached a place with one single square root term, so let us take a leap of faith and square again to obtain

$$24 + 20r^2 + 8r\sqrt{30}$$

which is rational by our fact hence $\sqrt{30}$ is rational. We may finally use Pythagoras' argument to contradict this. By our assumption, $\sqrt{30}$ can be written as $\frac{p}{q}$ in lowest terms (i.e. p and q are coprime and nonzero integers). Rewrite this as $p^2 = 30q^2$. The aim is to show that they cannot be coprime. Looking at our last equality, p^2 is a multiple of 30 and so is divisible by 2 (and 3 and 5). This is possible only if p itself is even. This immediately forces p^2 to be divisible by 4, implying that n^2 must divisible by 2 as well, meaning that n^2 and therefore n must be even. Hence, p and q are both even and not coprime, and the fraction was not in lowest terms, contradicting our initial claim, and the sum is irrational.
Note that applying Pythagoras' famous argument was the easy part of this problem. They key was knowing to square the equation twice with the aim of reducing the problem to one square root term. We should be convinced of this technique because we know how squaring the sum of two square roots ends up giving us just one square root. So to obtain one square root from three square roots, we square twice, each time reducing the number of square roots by one.

2. *Show that no set of nine consecutive integers can be partitioned into two sets with the product of the elements of the first set equal to the product of the elements of the second set*

   My first instinct was to, of course, write out the 9 consecutive integers as x, $x+1$, $x+2$, $x+3$, $x+4$, $x+5$, $x+6$, $x+7$, $x+8$. This makes it a bit clearer to realize that we will have four or five multiples of 2, three multiples of 3, two or three multiples of 4, one or two multiples of 5, one or two multiples of 7, one or to multiples of 8, and one multiple of 9. These deductions are useful because the problem requires us to prove that there cannot exist two products equal to each other. For that, we should naturally think of examining prime factorizations and try to find factors in one product that cannot exist in the other. Note that it is futile to consider multiples of primes larger than 7, because in a set of nine consecutive integers, we cannot find more than one multiple of any such prime. This means we cannot divide the set into two partitions with equal products of integers because one product will have a prime that the other does not. On that note, notice how the primes, namely 5 and 7, for which we can have at least one or at most two multiples of, require us to have exactly two such multiples because we need one of each in each of the two partitions for the products to have the same prime factorization and thus be equal. For that to be fulfilled, our first multiple of 7 must be either the first or second integer in the sequence, because the second multiple of 7 must be 7 more than the first, so it will be the eighth or ninth. By a similar line of reasoning, our first multiple of 5 must be among the first four integers in the sequence so that the second multiple of 5 may also land within the bounds of the nine consecutive integers. Next, why restrict ourselves to considering multiples of numbers from 2 to 9? We should also think of repeated primes, namely 4, 25, and 49. We already found that at most 4 numbers will be divisible by 4. Of 25 and 49 we can only have at most one multiple. We thus can have at most six numbers out of the nine being multiples of repeated primes. The rest, at least three, must be among the divisors of the product $2 \times 3 \times 5 \times 7$, namely among

$$2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105, 210.$$

Recall that this is because, as discussed in the beginning, if we go any higher than multiples of the prime 7, we will not be able to partition the sequence into two and satisfy the problem. By looking at the possibilities listed, we must choose three numbers but with the restriction that no two of them are more than 9 apart otherwise they would not belong to the same consecutive sequence of nine integers. We can restrict our search even further by remembering we cannot include 11. The only possibility is the numbers from 1 to 9, which do not satisfy the problem (and we verify this by brute force). So the key to this problem was to simplify our search space as much as possible, using prime factorizations to guide the restrictions we place on the sequence we wish to find.

3. *Find the least positive integer n such that any set of n pairwise relatively prime integers greater than 1 and less than 2005 contains at least one prime number.*

   At first, it did not seem obvious to me at all how exactly to utilize the method of proof by contradiction. I was also even more dumbfounded and frustrated upon reading the solution. However, after careful and thoughtful considera- (OK, OK I searched it up on math stack exchange), I had my epiphany. The subtelty lies in the wording of the problem.

Notice the word **any** in the problem. The idea is that, if we can find an **arbitrary** set of however many (number n not important just yet) pairwise coprime integers that **do not** have any prime number among them, then we have effectively found a number n that does not satisfy the requirements imposed by the problem on the set. Since the goal is to find the least n or cardinality for which the set satisfies the requirements of the problem, the n we find that does not work can serve as a lower bound for n. To be less vague, let us think of how to cleverly choose the numbers in our pairwise coprime set.

Upon rereading the definition of a set being pairwise relatively prime, it should be clear that we want to choose our numbers to either be all primes or have some numbers be multiples of primes which do not appear again in the set in order to guarantee that no two elements of the set share common factors. Now, a set of all primes will not be very interesting as it will contain "at least one prime", whereas we are searching to contrive a pairwise coprime set with no primes at all. Cleverly, we can square all the primes in our set, keeping the pairwise coprimality condition but wiping out any prime element. This is where the restriction on our elements being less than 2005 comes in: only the first 14 primes (from 2 to 43) can be squared and still remain less than 2005 ($47^2 = 2209$). This shows that n must be at least 15. The problem is surely not done! We still have to prove that $n = 15$ is exactly the smallest cardinality possible.

We have a lower bound on n: 14 to be specific. If we can show that 16 is an upper bound, we prove that $n = 15$.

6 Example - Pigeonhole Principle

This problem is from the 1972 International Mathematical Olympiad, proposed by Russia.

Prove that every set of 10 two-digit integer numbers has two disjoint subsets with the same sum of elements.

Let S be the set of 10 integers. We first determine the possible number of subsets S can have. Because S has 10 elements, and each element has a binary choice of either being in the subset or not, that gives us $2^{10} - 2 = 1022$ possible subsets, where we subtract 2 to exclude S and the empty set. Let $A \subsetneq S$ be a proper subset of S. The maximum sum of the elements of A, knowing that it can have at most 9 elements, is $91 + 92 + ... + 99 = 855$. These sums, ranging from 1 to 855, are our holes. Because the number of pigeons or subsets (1022) exceeds the number of holes or sums (855), there must be two pigeons in the same hole. In other words, there will surely be two subsets with the same sum because there are many more subsets than there are sums to place them in. Even more intuitively, we cannot "distribute" the sums onto the subsets without experiencing a "collision" because we have 1022 nonempty subsets waiting to be filled by only 855 possible sums. Evidently, we will run out of sums to assign to the subsets, forcing us to start placing the same sum in different subsets. Therefore, we can guarantee having two disjoint subsets with the same sum of elements. Note how we deliberately chose to consider the case of the largest possible subset with the largest possible sum, as this helps us show that even the maximum number of pigeons and holes fall short of assigning unique subsets to unique sums.