1 - 1) Demostrar que $f(x,y) = x \oplus y$ NO es umbral

Función umbral: $f(x_1, x_2) \begin{cases} 1 & \text{si } p_1x_1 + p_2x_2 \geq F \\ 0 & \text{si } p_1x_1 + p_2x_2 < F \end{cases}$

x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

$f(0,1) = 1 \rightarrow 0 \cdot p_1 + 1 \cdot p_2 \geq F;$

$p_2 \geq F$

$f(1,0) = 1 \rightarrow p_1 \geq F$

$f(1,1) = 0 \rightarrow p_1 + p_2 < F$

Esto es imposible para todo p_1, p_2, F, incumple las reglas de las desigualdades

2) Estudiar si son función umbral:

a) $f(x,y) = x \downarrow y$

x (p_1)	y (p_2)	f
0	0	1
0	1	0
1	0	0
1	1	0

$\Rightarrow$

$0 \geq F$
$p_1 < F$
$p_2 < F$
$p_1 + p_2 < F$

$F = 1$
$p_1 = -1$
$p_2 = -2$

$0 \geq 1$ ✓
$-1 < 1$ ✓
$-2 < 1$ ✓
$-1-2 < 1$ ✓

Sí, es f umbral

b) $f(x\,y\,z) = y + xz$

x (p_1)	y (p_2)	z (p_3)	f
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$\Rightarrow$

$0 < F$
$p_3 < F$
$p_2 \geq F$
$p_2 + p_3 \geq F$
$p_1 < F$
$p_1 + p_2 \geq F$
$p_1 + p_2 \geq F$
$p_1 + p_2 + p_2 \geq F$

$F = 10$
$p_1 = 6$
$p_2 = 11$
$p_3 = 5$

Sí, es umbral

$0 < 10$
$5 < 10$
$11 \geq 10$
$11 + 5 \geq 10$
$6 < 10$
$6 + 5 \geq 10$
$6 + 11 \geq 10$
$6 + 11 + 5 \geq 10$

c) $f(xyz) = \bar{z} + xy$

x (p_1)	y (p_2)	z (p_3)	f
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

$\Rightarrow$

$0 \geq F$
$p_3 < F$
$p_2 \geq F$
$p_2 + p_3 < F$
$p_1 \geq F$
$p_1 + p_3 < F$
$p_1 + p_2 \geq F$
$p_1 + p_2 + p_3 \geq F$

$F = -3$
$p_1 = 1$
$p_2 = 1$
$p_3 = -5$

Sí, es umbral

$0 \geq -3$
$-5 < -3$
$1 \geq -3$
$-5 + 1 < -3$
$1 \geq -3$
$-5 + 1 < -3$
$1 + 1 \geq -3$
$-5 + 1 + 1 \geq -3$

d) $f(xyz) = xz + yt$

x (p_1)	y (p_2)	z (p_3)	t (p_4)	f	
0	0	0	0	0	$\rightarrow 0 < F$
0	0	0	1	0	$\rightarrow p_4 < F$
0	0	1	0	0	$\rightarrow p_3 < F$
0	0	1	1	0	$\rightarrow p_3 + p_4 < F$ •
0	1	0	0	0	$\rightarrow p_2 < F$
0	1	0	1	1	$\rightarrow p_2 + p_4 \geq F$ •
0	1	1	0	0	$\rightarrow p_2 + p_3 < F$ •
0	1	1	1	1	
1	0	0	0	0	$\rightarrow p_1 < F$
1	0	0	1	0	$\rightarrow p_1 + p_4 < F$ •
1	0	1	0	1	$\rightarrow p_1 + p_3 \geq F$ •
1	0	1	1	1	
1	1	0	0	0	$\rightarrow p_1 + p_2 < F$ •
1	1	0	1	1	
1	1	1	0	1	
1	1	1	1	1	

- $0 < F$
- $p_1, p_2, p_3, p_4 < F$

Podemos sustituir la F por aquellas expresiones que son $\geq F$:

- $p_3 + p_4 < F \Rightarrow p_3 + \cancel{p_4} < p_2 + \cancel{p_4}$
- $p_2 + p_3 < F \Rightarrow p_2 + \cancel{p_3} < p_1 + \cancel{p_3}$
- $p_1 + p_4 < F \Rightarrow \cancel{p_1} + p_4 < \cancel{p_1} + p_3$
- $p_1 + p_2 < F \Rightarrow p_1 + \cancel{p_2} < \cancel{p_2} + p_4$

$\Downarrow$

$p_3 < p_2$
$p_2 < p_1$ — $p_3 < p_2 < p_1$
$p_4 < p_3$ — $p_4 < p_3 < p_2 < p_1$
$\boxed{p_1 < p_4}$ ⊗

Hemos encontrado una contradicción, así que NO es función umbral

e) $f(xyzt) = xy + x\bar{z}t$

p_1 x	p_2 y	p_3 z	p_4 t	f
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	1	1	0
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	1
1	0	1	0	0
1	0	1	1	0
1	1	0	0	1
1	1	0	1	1
1	1	1	0	1
1	1	1	1	1

$\Rightarrow$

1 •
- $0 < F$
- $p_4 < F$
- $p_3 < F$
- $p_2 < F$
- $p_1 < F$

2 •
- $p_3 + p_4 < F$
- $p_2 + p_4 < F$
- $p_2 + p_3 < F$
- $p_1 + p_4 \geq F$
- $p_1 + p_3 < F$
- $p_1 + p_2 \geq F$

$p_3 + \cancel{p_4} < p_1 + \cancel{p_4}$; $p_3 < p_1$

$p_2 + \cancel{p_4} < p_1 + \cancel{p_4}$; $p_2 < p_1$

$\cancel{p_2} + p_4 < p_1 + \cancel{p_2}$; $p_4 < p_1$

$\cancel{p_2} + p_3 < p_1 + \cancel{p_2}$; $p_3 < p_1$

$\cancel{p_1} + p_3 < \cancel{p_1} + p_4$; $p_3 < p_4$

$\cancel{p_1} + p_3 < \cancel{p_1} + p_2$; $p_3 < p_2$

$\cancel{p_1} + p_4 \geq \cancel{p_1} + p_3$; $p_4 < p_3$

3 •
- $p_2 + p_3 + p_4 < F$
- $p_1 + p_3 + p_4 < F$
- $p_1 + p_2 + p_4 \geq F$
- $p_1 + p_2 + p_3 \geq F$

$\cancel{p_2} + p_3 + \cancel{p_4} < p_1 + \cancel{p_2} + \cancel{p_4}$; $p_3 < p_1$

$\cancel{p_2} + \cancel{p_3} + p_4 < p_1 + p_2 + \cancel{p_3}$; $p_4 < p_2$

$\cancel{p_1} + p_3 + \cancel{p_4} < \cancel{p_1} + p_2 + \cancel{p_4}$; $p_3 < p_2$

$\cancel{p_1} + \cancel{p_3} + p_4 < \cancel{p_1} + p_2 + \cancel{p_3}$; $p_4 < p_2$

$\Downarrow$

$p_4 < p_3 < p_2 < p_1$

$F \geq 0$

4 • $p_1 + p_2 + p_3 + p_4 \geq F$

$$\begin{cases} F = 10 \\ p_1 = 7 \\ p_2 = 6 \\ p_3 = -2 \\ p_4 = 3 \end{cases}$$

$\hookrightarrow$

1 •
- $0 < 10$
- $3 < 10$
- $-2 < 10$
- $6 < 10$
- $7 < 10$

2 •
- $-2 + 3 < 10$
- $6 + 3 < 10$
- $6 - 2 < 10$
- $7 + 3 \geq 10$
- $7 - 2 < 10$
- $7 + 6 \geq 10$

3 •
- $6 - 2 + 3 < 10$
- $7 - 2 + 3 < 10$
- $7 + 6 + 3 \geq 10$
- $7 + 6 - 2 \geq 10$

4 • $7 + 6 + 2 - 3 \geq 10$

Sí, es función umbral

5. Demostrar si son lógicamente equivalentes

1) $\alpha = (a \rightarrow \overbrace{\neg b \vee c}^{P}) \rightarrow d$; $\beta = \overbrace{(\neg d \rightarrow \overbrace{b \wedge a}^{Q})}^{R} \wedge \overbrace{(c \rightarrow d)}^{S}$

a	b	c	d	P	a→P	α	Q	R	S	β
0	0	0	0	1	1	0	0	0	1	0
0	0	0	1	1	1	1	0	1	1	1
0	0	1	0	1	1	0	0	0	0	0
0	0	1	1	1	1	1	0	1	1	1
0	1	0	0	0	1	0	0	0	1	0
0	1	0	1	0	1	1	0	1	1	1
0	1	1	0	1	1	0	0	0	0	0
0	1	1	1	1	1	1	0	1	1	1
1	0	0	0	1	1	0	0	0	1	0
1	0	0	1	1	1	1	0	1	1	1
1	0	1	0	1	1	0	0	0	0	0
1	0	1	1	1	1	1	0	1	1	1
1	1	0	0	0	0	1	1	1	1	1
1	1	0	1	0	0	1	1	1	1	1
1	1	1	0	1	1	0	1	1	0	0
1	1	1	1	1	1	1	1	1	1	1

Sí, son lógicamente equivalentes

2) $\alpha = \overbrace{a \vee \neg b}^{P} \rightarrow \overbrace{\neg(c \rightarrow a)}^{Q}$; $\beta = \overbrace{\neg(b \vee c)}^{R} \rightarrow a$

a	b	c	P	Q	α	R	β
0	0	0	0	0	1	1	0
0	0	1	0	1	1	0	1
0	1	0	0	0	1	0	1
0	1	1	0	1	1	0	1
1	0	0	1	0	0	1	1
1	0	1	1	0	0	0	1
1	1	0	0	0	1	0	1
1	1	1	0	0	1	0	1

No son lógicamente equivalentes

3) $\alpha = ((\neg b \to c) \vee \neg b \to a \wedge d) \to (\neg a \vee \neg b \vee \neg d \to c)$

$\beta = (\neg a \wedge d \to c) \to (\neg c \leftrightarrow a \vee d)$

- Si 2 fórmulas son equivalentes, la fórmula obtenida de $\alpha \leftrightarrow \beta$ es tautología
- La tautología puede escribirse como $\emptyset \models \alpha \leftrightarrow \beta$ (cons. lógica del vacío)
- Entonces, por los teoremas de deducción y negación:

$\emptyset \models \alpha \leftrightarrow \beta;$

$\emptyset \models (\alpha \to \beta) \wedge (\beta \to \alpha);$

$\{\alpha\} \models \beta \wedge \beta \to \alpha;$

$\{\alpha, \neg\beta\} \models \beta \to \alpha;$

$\{\alpha, \neg\beta, \beta\} \models \alpha;$

$\{\alpha, \neg\beta, \beta, \neg\alpha\} \models \emptyset$ Así que vamos a poner en forma clausulada $\alpha, \neg\alpha, \beta, \neg\beta$

▶ $\alpha = \neg(((\neg b \to c) \vee \neg b) \to (a \wedge d)) \vee ((\neg a \vee \neg b \vee \neg d) \to c)$

$= \neg(\neg((\neg b \to c) \vee \neg b) \vee (a \wedge d)) \vee (\neg(\neg a \vee \neg b \vee \neg d) \vee c))$

$= \neg(\neg(\cancel{\neg}(\neg b) \vee c) \vee \neg b) \vee (a \wedge d)) \vee ((a \wedge b \wedge d) \vee c)$

$= (\neg\neg(b \vee c) \wedge b) \wedge \neg(a \wedge d)) \vee ((a \wedge b \wedge d) \vee c)$

ABSORCIÓN:
$p \wedge (p \vee q) = p$
$p \vee (p \wedge q) = p$

$= ((\cancel{(b \vee c)} \wedge b) \wedge (\neg a \vee \neg d)) \vee ((a \vee c) \wedge (b \vee c) \wedge (d \vee c))$

$= (b \wedge (\neg a \vee \neg d)) \vee ((a \vee c) \wedge (b \vee c) \wedge (d \vee c))$

$= ((b \vee a \vee c) \wedge (\cancel{b} \vee b \vee c) \wedge (b \vee d \vee c)) \wedge$
$(\cancel{\neg a} \vee \neg d \vee \cancel{a} \vee c) \wedge (\neg a \vee \neg d \vee b \vee c) \wedge (\neg a \vee \cancel{\neg d} \vee \cancel{d} \vee c))$

$= \underbrace{(b \vee a \vee c)}_{C1} \wedge \underbrace{(b \vee c)}_{C2} \wedge \underbrace{(b \vee d \vee c)}_{C3} \wedge \underbrace{(\neg d \vee c)}_{C4} \wedge \underbrace{(\neg a \vee \neg d \vee b \vee c)}_{C5} \wedge \underbrace{(\neg a \vee c)}_{C6}$

$$\neg\alpha = \neg\left[(b\vee a\vee c)\wedge(b\vee c)\wedge(b\vee d\vee c)\wedge(\neg d\vee c)\wedge(\neg a\vee\neg d\vee b\vee c)\wedge(\neg a\vee c)\right]$$

$$= \neg(b\vee a\vee c)\vee\neg(b\vee c)\vee\neg(b\vee d\vee c)\vee\neg(\neg d\vee c)\vee\neg(\neg a\vee\neg d\vee b\vee c)\vee\neg(\neg a\vee c)$$

$$= \cancel{(\neg b\wedge\neg a\wedge\neg c)}\vee\underbrace{(\neg b\wedge\neg c)}_{ABS}\vee\cancel{(\neg b\wedge\neg d\wedge\neg c)}\vee(d\wedge\neg c)\vee$$
$$\cancel{(a\wedge d\wedge\neg b\wedge\neg c)}\vee\underbrace{(a\wedge\neg c)}_{ABS}$$

$$= (\neg b\wedge\neg c)\vee(d\wedge\neg c)\vee(a\wedge\neg c)$$

$$= \left((\neg b\vee d)\wedge\cancel{(\neg b\vee\neg c)}\wedge\cancel{(\neg c\vee d)}\wedge(\neg c\vee\cancel{\neg c})\right)\vee(a\wedge\neg c)$$

$$= ((\neg b\vee d)\wedge(\neg c))\vee(a\wedge\neg c)$$

$$= (\neg b\vee d\vee a)\wedge\cancel{(\neg b\vee d\vee\neg c)}\wedge\cancel{(\neg c\vee a)}\wedge(\neg c\vee\cancel{\neg c})$$

$$= \underbrace{(\neg b\vee d\vee a)}_{C7}\wedge\underbrace{(\neg c)}_{C8}$$

$$B = (\neg a\wedge d\rightarrow c)\rightarrow(\neg c\leftrightarrow a\vee d)$$

$$= \neg(\neg(\neg a\wedge d)\vee c)\vee((\neg c\rightarrow(a\vee d))\wedge((a\vee d)\rightarrow\neg c))$$

$$= (\neg\neg(\neg a\wedge d)\wedge\neg c)\vee((c\vee a\vee d)\wedge((\neg a\wedge\neg d)\vee\neg c)$$

$$= (\neg a\wedge d\wedge\neg c)\vee\left[(c\vee a\vee d)\wedge(\neg a\vee\neg c)\wedge(\neg d\vee\neg c)\right]$$

$$= \left[(\neg a\wedge d\wedge\neg c)\vee(c\vee a\vee d)\right]\wedge$$
$$\left[(\neg a\wedge d\wedge\neg c)\vee(\neg a\vee\neg c)\right]\wedge$$
$$\left[(\neg a\wedge d\wedge\neg c)\vee(\neg d\vee\neg c)\right]$$

$$= \left[(\cancel{\neg a}\vee c\vee a\vee d)\wedge(\cancel{d}\vee c\vee a\vee d)\wedge(\cancel{\neg c}\vee\cancel{c}\vee a\vee d)\right]\wedge$$
$$\left[(\cancel{\neg a}\vee\neg a\vee\neg c)\wedge(d\vee\neg a\vee\neg c)\wedge(\cancel{\neg c}\vee\neg a\vee\neg c)\right]\wedge$$
$$\left[(\neg a\vee\neg d\vee\neg c)\wedge(\cancel{d}\vee\cancel{\neg d}\vee\neg c)\wedge(\cancel{\neg c}\vee\neg d\vee\neg c)\right]$$

$$= \underbrace{(c\vee d)}_{C9}\wedge\underbrace{(c\vee a\vee d)}_{C10}\wedge\underbrace{(a\vee d)}_{C11}\wedge$$
$$\underbrace{(\neg a\vee\neg c)}_{C12}\wedge\underbrace{(d\vee\neg a\vee\neg c)}_{C13}\wedge\underbrace{(\neg a\vee\neg c)}_{C14}\wedge$$
$$\underbrace{(\neg a\vee\neg d\vee\neg c)}_{C15}\wedge\underbrace{(\neg c)}_{C16}\wedge\underbrace{(\neg d\vee\neg c)}_{C17}$$

$\neg B = \neg\{[(c \vee c) \wedge (c \vee c \vee d) \wedge (a \vee d)] \wedge$

$[(\neg a \vee \neg c) \wedge (d \vee \neg c \vee \neg c) \wedge (\neg c \vee \neg a)] \wedge$

$[(\neg a \vee \neg d \vee \neg c) \wedge (\neg c) \wedge (\neg d \vee \neg c)]$

$= [(\neg c \wedge \neg d) \vee (\neg c \wedge \neg a \wedge \neg d) \vee (\neg a \wedge \neg d)] \vee$

$[(a \wedge c) \vee (\neg d \wedge a \wedge c) \vee (c \wedge a)] \vee$

$[(a \wedge d \wedge c) \vee (c) \vee (d \wedge c)]$

$= [((\neg c \vee \neg c) \wedge \cancel{(\neg c \vee \neg a)} \wedge \cancel{(\neg c \vee \neg d)} \wedge \cancel{(\neg d \vee \neg c)} \wedge \cancel{(\neg d \vee \neg a)} \wedge (\neg d)) \vee (\neg a \wedge \neg d)] \vee$

$[(\cancel{(a \vee \neg d)} \wedge (a) \wedge (a \vee c) \wedge \cancel{(c \vee \neg d)} \wedge \cancel{(c \vee a)} \wedge (c)) \vee (c \wedge a)] \vee$

$[(\cancel{(a \vee c)} \wedge \cancel{(d \vee c)} \wedge (c)) \vee (d \wedge c)]$

$= [(\neg c \wedge \neg d) \vee (\neg a \wedge \neg d)] \vee$

$[(a \wedge c) \vee (c \wedge a)] \vee$

$[c \vee \cancel{(d \wedge c)}]$

$= [(\neg c \vee \neg a) \wedge \cancel{(\neg c \vee d)} \wedge \cancel{(\neg d \vee \neg a)} \wedge (\neg d)] \vee [a \wedge c] \vee [c]$

$= [(\neg c \vee \neg a) \wedge (\neg d)] \vee [\cancel{(a \vee c)} \wedge (c \vee c)]$

$= (\neg c \vee \neg a \vee c) \wedge (\neg d \vee c)$

$= \underbrace{(\neg a)}_{C18} \wedge \underbrace{(\neg d \vee c)}_{C19}$

La lista de clausulas obtenida de $\alpha, \neg\alpha, \beta, \neg\beta$ es:

$\{(b \vee a \vee c), (b \vee c), (b \vee d \vee c), (\neg d \vee c), (\neg a \vee \neg d \vee b \vee c), (\neg a \vee c),$

Repetidas → $(\neg b \vee d \vee a), (\neg c), (c \vee d), (c \vee a \vee d), (a \vee d), (\neg a \vee \neg c),$

$(d \vee \neg c \vee \neg c), \cancel{(\neg a \vee \neg c)}, (\neg a \vee \neg d \vee \neg c), \cancel{(\neg c)}, (\neg d \vee \neg c),$

$(\neg a), (\neg d \vee c)\}$

$\lambda = \neg c$

$\{(b \vee a), (b), (b \vee d), (\neg d), (\neg a \vee \neg d \vee b), (\neg a),$

$(\neg b \vee d \vee a), (d), (a \vee d), (\neg a)\}$

$\lambda = \neg d$

$\{(b \vee a), (b), (\neg a), (\neg b), \Box, (a)\}$

$\lambda = b$

$\{(\neg a), \Box, \Box, (a)\}$

$\lambda = a$

$\{\Box, \Box, \Box\}$

$\lambda = \Box$

$\emptyset$ → el cjto es satisfacible, así que NO es taut. y, por tanto, NO es lógicamente equivalente