On the influence of exogenous factors on the game theoretic process of resource exploitation*

Ekaterina Viktorovna Gromova[1] José Daniel López-Barrientos[2]
Ekaterina Miroshnichenko[3]

March 23, 2018

Abstract

2010 Mathematics Subject Classification: 91A10, 91A15, 91A25
Keywords and phrases: *Differential games in random-time horizon, discounted equilibrium, Weibull distribution, Chen distribution.*

Abstract

In this work, we analyze an n-person differential competitive game with random horizon to model the situation of the extraction of a non-renewable resource. We proceed by means of standard dynamic programming arguments to obtain Nash equilibria in the form of feedback controllers and closed formulas for the value functions. We assume that the hazard rate of the so-called *random horizon* is a *U-shaped* function and we compare our results for two cases of general interest in the Risk Theory: the Weibull and Chen distributions. Our goal is to show the impact of a priori knowledge on the random variables describing real-life systems of resource extraction.

1 Introduction

When it comes to consider the time horizon over which a dynamic game takes place, there are mainly two approaches. In the first approach, the game develops in time on a fixed interval $[t_0, T]$, where the end time of the game T is known in advance and the game has a prescribed duration. The other approach has a large number of economic applications; in it, the game develops over an infinite time interval. In practice, however, conflict-controlled processes have a random end-point, which is associated with a variety of reasons that might be exogenous to the game itself. For this reason, the consideration of games with random duration seems to

*This research was supported by Katya's grant and by Anáhuac's help.
[1]St. Petersburg State University, Faculty of Applied Mathematics and Control Processes Universitetskii prospekt 35, Petergof, Saint Petersburg, Russia 198504. ekaterina.shevkoplyas@gmail.com.
[2]Facultad de Ciencias Actuariales. Universidad Anáhuac México. Av. Universidad Anáhuac, No. 46 Col. Lomas Anáhuac, Huixquilucan, Edo. de México. Tel. +52 (55) 5627-0210 ext. 8506. daniel.lopez@anahuac.mx.
[3]Bwin Interactive Entertainment AG, Vienna, Austria. ekaterina.miroshnichenko@gmail.com.

be plausible and realistic.

We consider an n-player differential game $\Gamma(x(t_0))$ with random duration $T-t_0$ and initial state x_0. Namely, it is assumed that the game develops on the interval $[t_0, T]$, where T is a random variable whose distribution function $F(t)$, $t \in [t_0, \infty[$, is known and satisfies the normalization condition:

$$\int_{t_0}^{\infty} \mathrm{d}F(t) = 1. \tag{1.1}$$

Thus, the formulation of a differential game with a random duration is a generalization of the setting of the game with a prescribed duration.

The dynamics of the game is given by the ordinary differential equation:

$$\dot{x} = g(x, u_1, ..., u_n),\ x \in \mathbb{R}, u_i \in U,\ x(t_0) = x_0, \tag{1.2}$$

$i = 1, ..., n$; where U is a compact set. The game starts at time t_0 from state x_0, and is terminated at the random time T. We impose the following hypothesis on the distribution function of the random terminal time T.

Assumption 1.1. *The cumulative distribution function of T satisfies the following:*

(a) *it is absolutely continuous with respect to Lebesgue's measure;*

(b) $(1-F(t))\int_{t_0}^{t} h(x(t), u_1(t), ..., u_n(t))dt \to 0$ *as* $t \to \infty$, *for all* $h : \mathbb{R} \times U^n \to \mathbb{R}$.

This section is written as if all the paper were about an n-player game. Maybe we should change it... The instantaneous payoff of the i-th player at time τ, $\tau \in [t_0, \infty[$ depends on the time τ and the phase variable $x(t_0, x_0, u(\cdot))$, where $u(\cdot) := (u_1(\cdot), ..., u_n(\cdot))$ is the vector of strategies of the players. For brevity, we denote such instantaneous payoff function as $h_i(\tau, x(\tau), u(\tau))$.

We assume that h_i is a continuous function on $\mathbb{R}$ for $i = 1, ..., n$. Then the expected integral payoff of the i-th player has the form:

$$\begin{aligned} K_i(x_0, t_0, u_1, \ldots, u_n) &= \mathbb{E}\left[\int_{t_0}^{t} h_i(\tau, x(\tau), u(\tau))\mathrm{d}\tau\right] \qquad (1.3)\\ &= \int_{t_0}^{\infty}\int_{t_0}^{t} h_i(\tau, x(\tau), u(\tau))\mathrm{d}\tau f(t)\mathrm{d}t, \qquad (1.4) \end{aligned}$$

for $i = 1, ..., n$, where $f(t) := F'(t)$ is a density function of the random variable T, whose existence follows from Assumption 1.1(a). In (1.3), of course, the expectation is taken with respect to the random variable T.

If $h_i(\tau, x(\tau), u(\tau))$ is a nonnegative function, then, by the Fubini-Tonelli theorem, (1.4) can be reduced to a simple functional form[4]. Indeed, by Assumption 1.1(b), we can apply

[4]If we can not guarantee the non-negativity of the instantaneous gain function h_i, but the condition

$$\int_0^{\infty}\int_0^{\infty} |f(t)h_i(\tau, x(\tau), u(\tau))|\mathrm{d}t\mathrm{d}\tau < \infty$$

holds, then we can still change the order of integration in (1.4) by applying Fubini-Tonelli's theorem.

Proposition 2 in [11] (or some result in [7]), and an integration by parts, to obtain

$$K_i(x_0, 0, u_1, ..., u_n) = \int_0^\infty (1 - F(\tau))h_i(\tau, x(\tau), u(\tau))\mathrm{d}\tau. \tag{1.5}$$

As soon as the game starts to develop in time, at some intermediate moment $\theta \in]t_0, \infty[$, the players fall into the subgame $\Gamma(x(\theta))$ with initial state $x(\theta) = x$. Obviously, the game can end up precisely at the moment θ with probability $F(\theta)$. Thus, the probability of continuing the game after the moment θ is $1 - F(\theta)$. Then, the integral payoff for the i-th player in the subgame $\Gamma(x(\theta))$, is the mathematical conditional expectation of his earnings. Namely:

$$K_i(x, \theta, u_1, ..., u_n) = \frac{1}{1 - F(\theta)} \int_\theta^\infty (1 - F(\tau))h_i(\tau, x(\tau), u(\tau))\mathrm{d}\tau. \tag{1.6}$$

2 Dynamic models for the extraction of natural resources by one agent

Consider the model [4, Chapter 10.3] for extracting non-renewable natural resources (in particular, oil). It should be noted that, due to the specific nature of oil extraction, especially when it takes place offshore, there is a direct dependence of losses with the rate of accidents of the enterprise. Accidents at the wells lead to both:

- downtime during the repairs and/or replacement of the equipment,
- severe environmental consequences.

The costs of eliminating these problems often lead to enormous losses. In most of the previously known game-theoretic models describing the dynamic process of oil extraction by one or several players, it is assumed that the extraction tasks go on in an infinite time interval with constant discounting. In this work we compare the results of a random time terminal T for two particular cases: The Weibull and Chen laws.

2.1 The failure rate function

In the mathematical reliability (risk) theory, one of the most important random variables studied is the time-until-failure T of a system; and the failure (hazard) rate function

$$\lambda(t) := \frac{f(t)}{1 - F(t)}. \tag{2.1}$$

is one of its main characteristics (see [2, Chapter 3], [9, Chapters 4 and 5] and [15, Section 8.5]). We can use this notion to draw an analogy between the theories of risk and dynamic games. We consider games with a random terminal time T, to which the basic terminology of reliability theory can be applied directly. In fact, the failure rate function (2.1) can be thought of as a conditional distribution density provided that the system did not fail until the moment t. In our terminology, we would talk about the distribution density of the terminal time of the game, provided that the game was not terminated before the moment t. The failure rate function $\lambda(t)$ that describes the life cycle of the system, has the following form: The first phase is called the run-in phase. According to the theory of reliability, the failures in this phase arise due to undetectable latent defects. Specificity of this problem is understandable not only from the point of view of the application of elements to technical systems, in actuarial risk

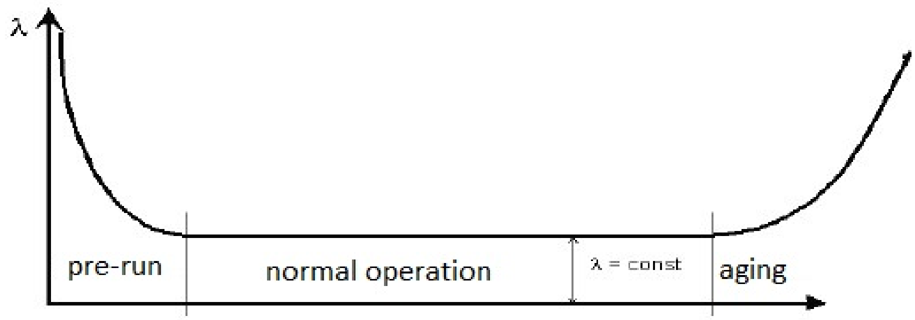


Figure 1: A U-shaped (bathtub) hazard rate function.

theory, for such a period the terms "newborn period", "infant mortality" and "early failures" can be used (see [2, Chapter 3]). From the point of view of game theory, early failures can be caused by inexperience, i.e. inconsistency of the players just who just entered the game. The failure rate function λ in this phase is a decreasing function of time.

The next period of the life cycle of the system is the so-called period of normal system operation. The failure rate function $\lambda(\cdot)$ in this period is constant (or approximately constant), and the sudden failures themselves are caused by imperfection of the system itself, or are caused by some external factor. This is called the "adult" period of the process (see [8, what part?]). The game in the period under consideration can stop under the influence of some unforeseen circumstances of the external world.

In the last period, the system goes into the aging phase. The system failures in this period are associated with how the system ages, and that's why the failure rate function $\lambda(t)$ is an increasing function. THIS VERSION OF THE WORK DOES NOT HAVE ANYHING ON THIS: In addition, the agreements reached before the start of the game, in general, are not realizable in long-term projects. This fact (first noted in [14]) are sometimes dubbed as *dynamic instability of solutions*, or *insolvency in time*.

2.2 The Weibull distribution

In the mathematical theory of risk, various probability distributions, namely, exponential, normal, log-normal, Weibull, power-law distribution, Gamma distribution and others (see [2, 15]) are used for the time-until-failure random variable T.

Weibull distribution is widely used in reliability theory. Customarily, this distribution is used to model the operating time-until-failure of many non-renewable electronic devices (electronic lamps, semiconductor devices, some microwave devices). The rule of correspondence of the distribution function is:

$$F(t) = 1 - \mathrm{e}^{-\lambda t^{\delta}}, \tag{2.2}$$

where $\lambda > 0$ is a scale parameter[5], and $\delta > 0$ corresponds to one of the three phases in which the game can located at. Namely, the value $\delta < 1$ corresponds to the pre-run period, here the failure rate function $\lambda(t)$ is a decreasing function. At $\delta = 1$, the system is in the normal

[5]Recall that, if X is a random variable that follows the distribution function $F(x;\lambda)$, and λ is a scale parameter, then cX $(x \neq 0)$ will be driven by the distribution function $F(x; c\lambda)$ (see [9, Appendix A.3] and [15, Chapter 17])

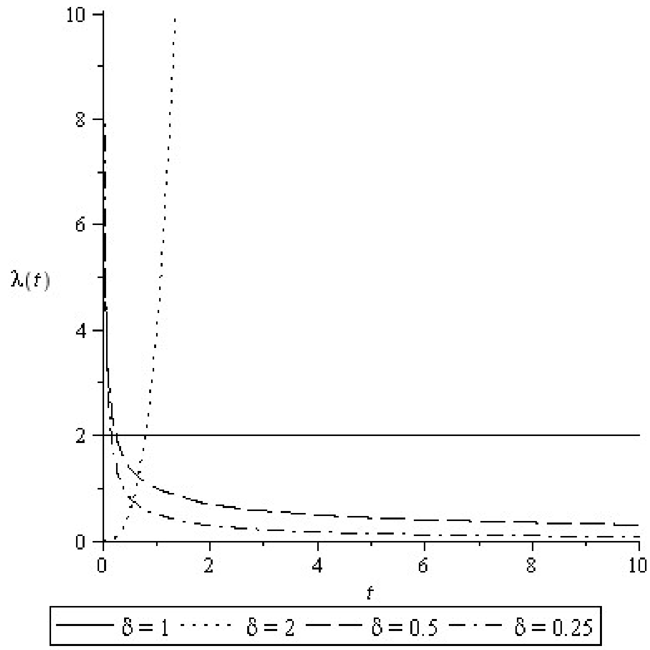


Figure 2: Weibull's hazard rate function for $\lambda = 2$.

operation mode, here $\lambda(t)$ equals a constant value of $\lambda > 0$. We note that for $\delta = 1$, the Weibull distribution corresponds to an exponential distribution. For $\delta > 1$, the system is in a aging state, $\lambda(t)$ is an increasing function. A special case of the Weibull distribution for this instance is the so-called *Rayleigh* distribution. We are in this case when $\delta = 2$.
Now, by (2.1), the failure rate function is

$$\lambda(t) = \lambda\delta t^{\delta-1}. \tag{2.3}$$

A graphic representation of the Weibull distribution's failure rate function for all three game scenarios for a fixed scale parameter $\lambda = 2$ is shown in Figure 2. It should be noted that the disadvantage of the Weibull distribution is the description of the failure probability of the system in the pre-run mode ($\delta < 1$). As can be seen from Figure 2, for small values of the parameter δ (that is, for $\delta < 1$), the function $\lambda(t)$ is decreasing over all the horizontal axis, whereas any technical system in a finite horizon passes from the state of pre-run to the following modes of the life cycle of the system and "early" failures are replaced by failures of other types.

2.3 The Chen distribution

Chen's distribution (see [3]) is a fairly new two-parameter distribution for random variables with increasing *U-shaped* failure rate functions (such as the one displayed in Figure 1). The rule of correspondence of this distribution is

$$F(t) = 1 - \exp\left(\lambda\left(1 - \mathrm{e}^{t^{\delta}}\right)\right), \tag{2.4}$$

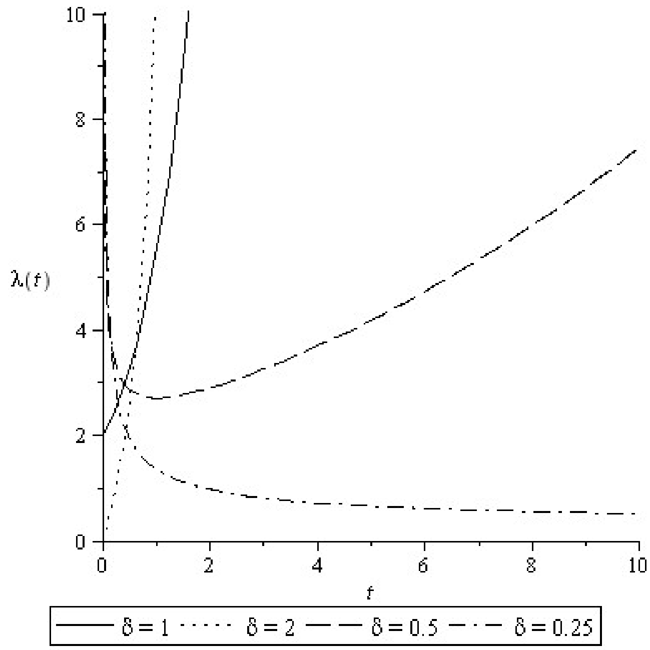


Figure 3: The failure rate function $\lambda(t)$ for the Chen distribution with the value $\lambda = 2$.

where $\lambda \geq 0$ is a scale parameter, and $\delta \geq 0$ is a shape parameter (see [9, 15]). It is straightforward from (2.1), that the failure rate function acquires the form

$$\lambda(t) = \lambda\delta t^{\delta-1}\mathrm{e}^{t^{\delta}}. \tag{2.5}$$

Note that

$$\lambda'(t) = \lambda\delta t^{\delta-1}\mathrm{e}^{t^{\delta}} \text{ for } t > 0. \tag{2.6}$$

If $\delta < 1$, we will be at the "newborn" phase. Here, the failure rate function $\lambda(\cdot)$ is bathtub-shaped. This corresponds to a realistic process of extracting natural resources. When $\delta = 1$, the system is in the normal operation mode and the hazard rate function $\lambda(\cdot)$ is increasing. At $\delta > 1$, the system is in aging state, and $\lambda(\cdot)$ is also an increasing function, but it's easy to see (from (2.6)) that the growth rate of $\lambda(\cdot)$ is noticeably larger than in the case of normal operation. This implies that there is a greater probability of failure at this stage of the extraction.

A graphical representation of the failure rate function of the Chen distribution for all possible types of failures for a fixed scale parameter λ is shown in Figure 3 below. Figure 4 also shows the behavior of the failure rate function of the new distribution under consideration for the run-in period for different values of the shape parameter δ (we keep $\lambda = 1$). Indeed, one can clearly see that as $\delta \to 1$, the slope of the graph grows larger. This fact might be interpreted by arguing that Chen's distribution plausibly describes how the system goes from the pre-run state, into the normal operation mode.

2.4 Comparison

The greatest interest in the model we are considering is the pre-run stage, when the value of the shape parameter $\delta < 1$. In the models presented in [18, 19], the random terminal time

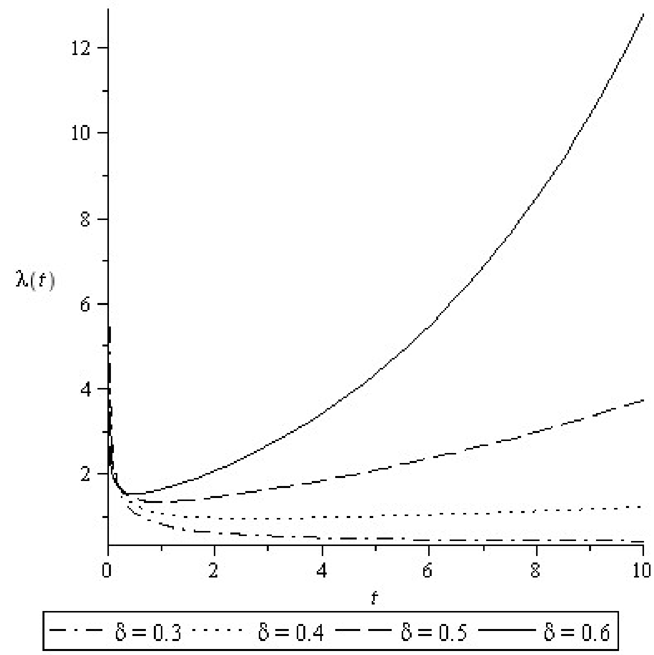


Figure 4: The failure rate function $\lambda(\cdot)$ for the "newborn" period ($\delta < 1$) in Chen's distribution.

of the game is distributed according to Weibull's law. There, the early failures are retained during the whole game. However, in the actuarial non-life models, this is certainly not the case: the system from the running-in state goes into normal operation mode, and then it ages.

Graphical representations of the failure rate functions for the Weibull and Chen distributions for fixed $\lambda = 2$, and the same values of the parameter δ are shown in Figures 5a-5c below. Note that we display these functions for each of the periods we have identified.

3 Optimal control problem

Consider the situation where only one agent performs extraction tasks, that is, the degenerate game where $n = 1$. Let $x(t)$ be the stock of the non-renewable resource of our interest. The control of the player is the amount of the resource that he plans to extract. We denote this amount as $u(t)$. We use the following ordinary differential equation to describe the dynamics of $x(t)$:

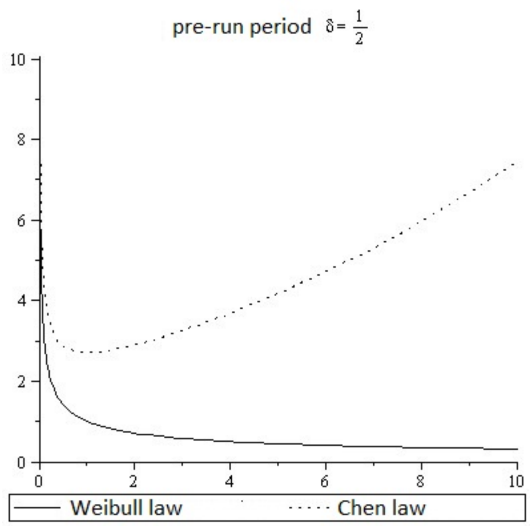
$$\dot{x}(t) = -u(t),\ x(t_0) = x_0 > 0, \tag{3.1}$$

where $u(t) \geq 0$. An application of formula (1.5) yields that the expected payoff of the agent, provided that the terminal random time follows the Weibull law, is

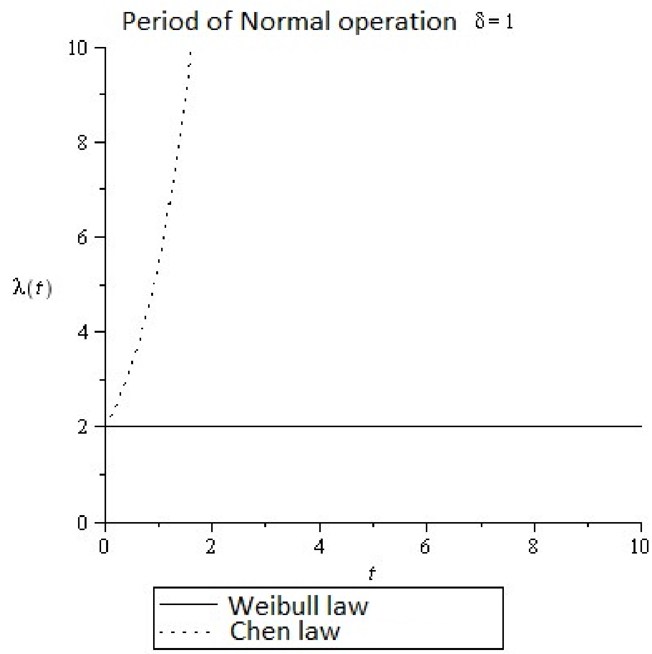
$$K(x_0, 0, u) = \int_0^\infty h(t, x(t), u(t)) \mathrm{e}^{-\lambda t^\delta} \mathrm{d}t. \tag{3.2}$$

If we use Chen's law for the random terminal time, the winnings of the extractor are given by

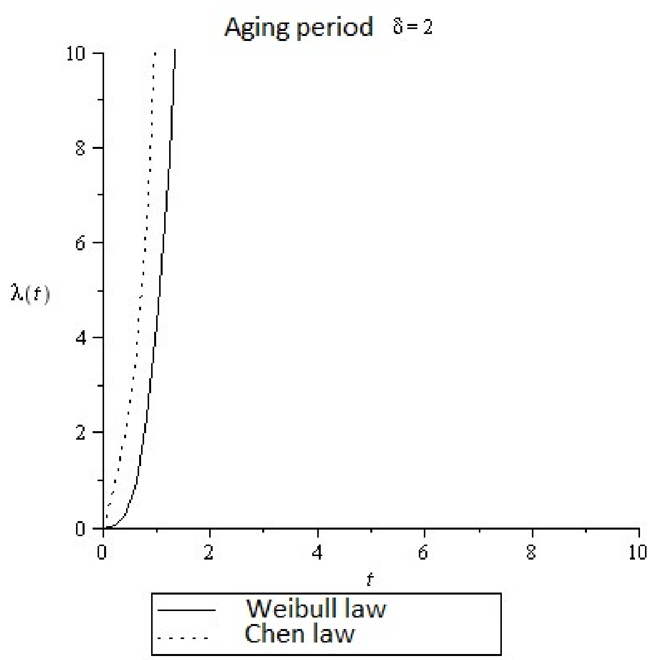
$$K(x_0, 0, u) = \int_0^\infty h(t, x(t), u(t)) \exp\left[-\lambda\left(1 - \mathrm{e}^{t^\delta}\right)\right] \mathrm{d}t. \tag{3.3}$$



(a) Comparison of the failure rate functions in the pre-run stage



(b) Comparison of the failure rate functions at the stage of normal operation



(c) Comparison of the failure rate functions during the system aging period

The problem of maximizing the expected gains in formulae (3.2) and (3.3) under the condition (3.1) can be solved using the following Bellman equation

$$\begin{aligned}\lambda(t)W(x,t) &= \frac{\partial}{\partial t}W(x,t) + \max_u \left(h(t,x,u) + g(x,u)\frac{\partial}{\partial x}W(x,t) \right) \\ &= \frac{\partial}{\partial t}W(x,t) + \max_u \left(h(t,x,u) - u\frac{\partial}{\partial x}W(x,t) \right). \end{aligned} \tag{3.4}$$

For the sake of illustration we propose use a logarithmic utility function, i.e.,

$$h(x,u,t) = \ln u(t). \tag{3.5}$$

The results we will obtain are interesting from the point of the actuarial sciences. To solve the Bellman equation (3.4) we can use the *ansatz*

$$W(x,t) = A(t)\ln x + B(t), \tag{3.6}$$

along with the transversality condition

$$\lim_{t\to\infty} W(x,t) = 0 \tag{3.7}$$

and therefore calculate

$$\frac{\partial}{\partial x}W(x,t) = \frac{A(t)}{x}, \tag{3.8}$$

$$\frac{\partial}{\partial t}W(x,t) = \dot{A}(t)\ln x + \dot{B}(t). \tag{3.9}$$

The substitution of (3.8)-(3.9) into (3.4) yields that the maximized control is of the form:

$$u^*(t,x) = \frac{x}{A(t)}, \tag{3.10}$$

and also the following system of differential equations:

$$\dot{A}(t) - \lambda(t)A(t) + 1 = 0; \tag{3.11}$$

$$\dot{B}(t) - \lambda(t)B(t) - \ln(A(t)) - 1 = 0. \tag{3.12}$$

The transversality condition (3.7) takes the form

$$\lim_{t\to\infty} A(t) = 0, \tag{3.13}$$

$$\lim_{t\to\infty} B(t) = 0. \tag{3.14}$$

Using the transversality condition (3.13) and (thus) the integrating factor

$$\exp\left(-\int_0^s \lambda(\tau)\mathrm{d}\tau \right) = 1 - F(s), \tag{3.15}$$

we can solve (3.11) and get

$$A(t) = \frac{1}{1-F(t)} \int_t^\infty \mathrm{e}^{-\int_0^s \lambda(\tau)\mathrm{d}\tau}\mathrm{d}s. \tag{3.16}$$

A substitution of (3.16) in (3.10) gives us

$$u^*(t,x) = \frac{1-F(t)}{\int_t^\infty \mathrm{e}^{-\int_0^s \lambda(\tau)\mathrm{d}\tau}\mathrm{d}s}x \tag{3.17}$$

$$= x \Bigg/ \frac{\int_t^\infty 1-F(s)\mathrm{d}s}{1-F(t)}. \tag{3.18}$$

Here, of course, $\lambda(\cdot)$ is the corresponding hazard rate function of the random terminal time. This controller is optimal for the degenerate game at hand by virtue of Theorem I.7.1(a) in [5].

Remark 3.1. *An interesting interpretation of this result is the fact that the function $A(t)$ agrees with the conditional expectation of the random terminal time T, given that the system continues to work at times t. In references, [9, Chapter 8] and [15, Chapter 17.10.3] this idea coincides with that of a* deductible *for an insurance an a loss whose distribution agrees with that of our random variable T. From this point of view, an interesting actuarial interpretation of* (3.17) *is that the optimal controller* (3.18) *is closely related to the concept of ... (see [2, Chapters 6 and 7]).*

The transversality condition (3.14) and again, the integrating factor (3.15) gives that

$$B(t) = \frac{\int_t^\infty \exp\left(-\int_0^s \lambda(\tau)\mathrm{d}\tau\right)(1+\ln(A(s)))\,\mathrm{d}s}{\exp\left(-\int_0^t \lambda(s)\mathrm{d}s\right)}$$

$$= \frac{\int_t^\infty (1-F(s))(1+\ln(A(s)))\,\mathrm{d}s}{1-F(t)}. \tag{3.19}$$

The susbtitution of (3.16) and (3.19) into (3.6) gives us solution of the Bellman equation (3.4), which, by the verification Theorem I.7.1(b) in [5], agrees with the maximum of (1.3), and, by the stability of the dynamic programming technique, can be used to calculate the maximum of (1.6).

3.1 Normal mode ($\delta = 1$)

As we already stated, for the case of Weibull distribution, when $\delta = 1$, the random terminal time is exponentially distributed with mean λ^{-1}. Then, the failure rate function $\lambda(t) = \lambda > 0$, and (3.17), the optimal strategy of the agent is $u^*(t,x) = \lambda x$ (see [19]). We solve (3.1) and get that the optimal trajectory is

$$x^*(\tau) = x_0\mathrm{e}^{-\lambda\tau}.$$

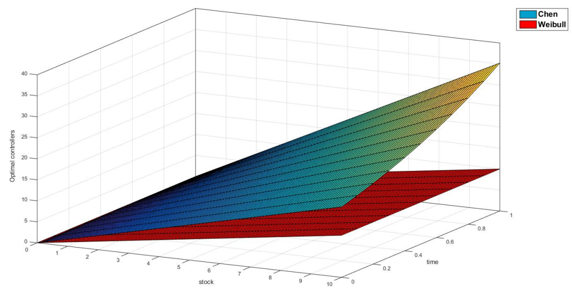
For the case of Chen's law, we let $\delta = 1$ and substitute (2.5) into (3.17) to get

$$u^*(t,x) = \frac{x\exp\left(-\lambda \mathrm{e}^t\right)}{\int_t^\infty \exp\left(-\lambda \mathrm{e}^s\right)\mathrm{d}s}.$$

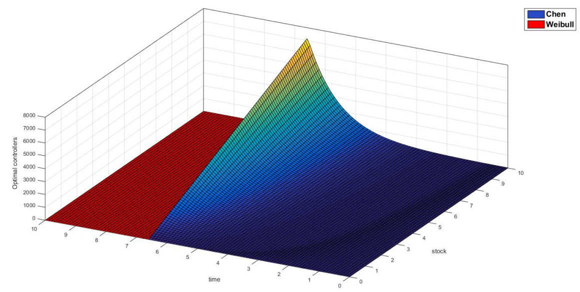
We substitute this controller into (3.1) and solve the differential equation to obtain the optimal trajectory when the random terminal time is distributed according to Chen's law and $\delta = 1$. That is:

$$x^*(\tau) = x_0 \exp\left(-\int_0^\tau \frac{\exp\left(\mathrm{e}^{-\lambda t}\right)}{\exp\left(-\int_\tau^\infty \mathrm{e}^{-\lambda s}\mathrm{d}s\right)}\mathrm{d}t\right).$$

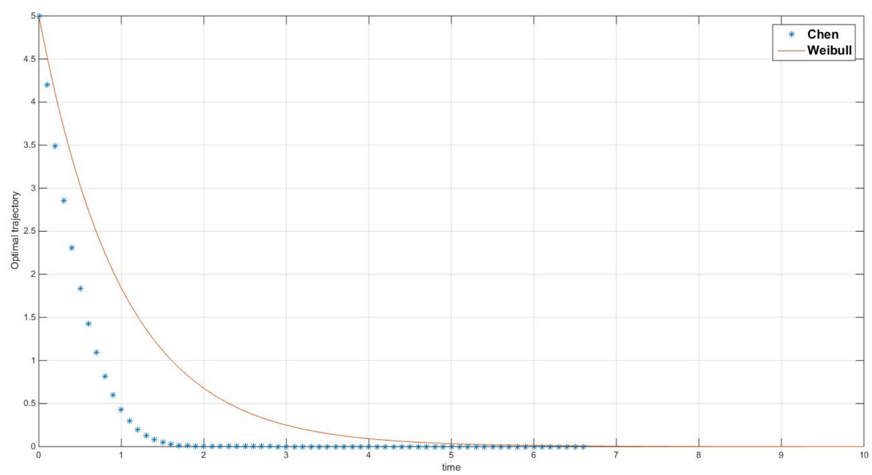
Figures 6a, 6b and 6c summarize these results when $\lambda = 1$.



(a) Optimal controllers for the laws of Weibull and Chen when $\delta = 1$ and $\lambda = 1$ when $(t, x) \in [0, 1] \times [0, 1]$.



(b) Optimal controllers for the laws of Weibull and Chen when $\delta = 1$ and $\lambda = 1$ when $(t, x) \in [0, 10] \times [0, 10]$.



(c) Optimal trajectories for the laws of Weibull and Chen when $\delta = 1$ and $\lambda = 1$.

3.2 Aging mode ($\delta = 2$)

We stated before that for $\delta = 2$, Weibull's law coincides with Rayleigh distribution. In this case, (2.3) gives that $\lambda(t) = 2\lambda t$; then from (3.17) we get

$$u^*(t,x) \quad = \quad \frac{x\mathrm{e}^{-\lambda t^2}}{\int_t^\infty \mathrm{e}^{-\lambda s^2}\mathrm{d}s}$$

For Chen's distribution, when we have an aging system, by (2.5), the failure rate function is $\lambda(t) = 2\lambda t\mathrm{e}^{t^2}$, and (3.17) gives that the optimal control is

$$u^*(t,x) = \frac{x\exp\left(-\lambda \mathrm{e}^{t^2}\right)}{\int_t^\infty \exp\left(-\lambda \mathrm{e}^{s^2}\right)\mathrm{d}s}.$$

We solve (3.1) and get that the optimal trajectories under each law are:

$$x^*(\tau) = \begin{cases} x_0 \exp\left(-\int_0^\tau \frac{\exp\left(-\lambda \mathrm{e}^{t^2}\right)}{\int_t^\infty \exp\left(-\lambda \mathrm{e}^{s^2}\right)\mathrm{d}s}\mathrm{d}t\right) & \text{for Chen's law with} \quad \delta = 2, \\ x_0 \exp\left(-\int_0^\tau \frac{\mathrm{e}^{-\lambda t^2}}{\int_t^\infty \mathrm{e}^{-\lambda s^2}\mathrm{d}s}\mathrm{d}t\right) & \text{for Weibull's law with} \quad \delta = 2. \end{cases}$$

Figures 7a, 7b and 7c summarize these results when $\lambda = 1$.

3.3 Early period ($\delta = \frac{1}{2}$)

For the Weibull case, (2.3) yields that the hazard rate function is $\lambda(t) = \frac{\lambda}{2\sqrt{t}}$. A substitution in (3.17) gives us

$$u^*(t,x) \quad = \quad \frac{x\mathrm{e}^{-\lambda\sqrt{t}}}{\int_t^\infty \mathrm{e}^{-\lambda\sqrt{s}}\mathrm{d}s}$$

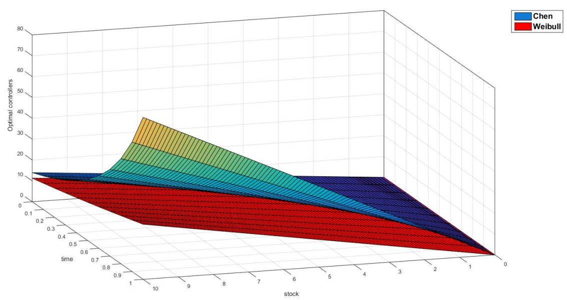
For Chen's law, the pre-run system has a hazard rate function of the form $\lambda(t) = \frac{\lambda}{2\sqrt{t}}\mathrm{e}^{\sqrt{t}}$. The corresponding optimal control has the form

$$u^*(t,x) \quad = \quad \frac{x\exp\left(-\lambda \mathrm{e}^{\sqrt{t}}\right)}{\int_t^\infty \exp\left(-\lambda \mathrm{e}^{\sqrt{s}}\right)\mathrm{d}s}.$$

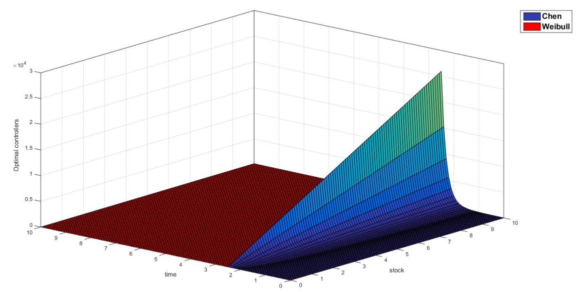
We solve (3.1) and get that the optimal trajectories under each law are:

$$x^*(\tau) = \begin{cases} x_0 \exp\left(-\int_0^\tau \frac{\lambda^2}{2(\lambda\sqrt{\tau}+1)}\mathrm{d}t\right) & \text{for Weibull's law with} \quad \delta = \frac{1}{2}, \\ x_0 \exp\left(-\int_0^\tau \frac{\exp\left(-\lambda \mathrm{e}^{\sqrt{t}}\right)}{\int_t^\infty \exp\left(-\lambda \mathrm{e}^{\sqrt{s}}\right)\mathrm{d}s}\mathrm{d}t\right) & \text{for Chen's law with} \quad \delta = \frac{1}{2}. \end{cases}$$

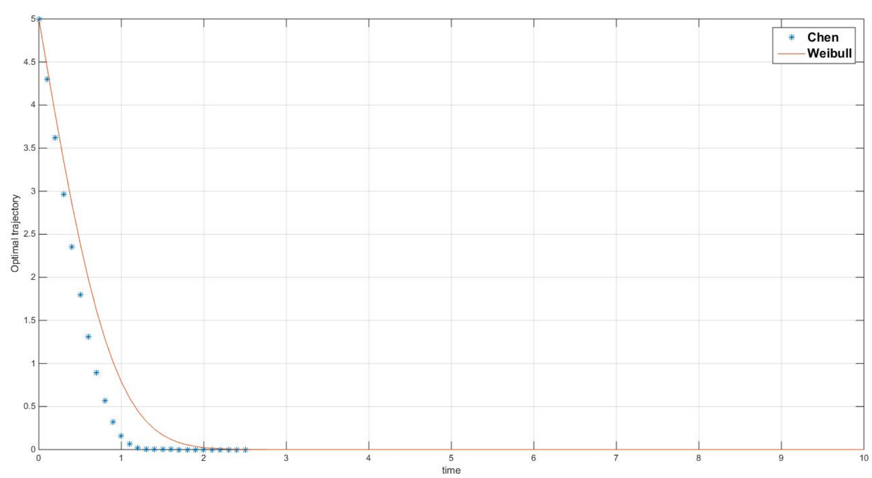
Figures 8a, 8b and 8c summarize these results when $\lambda = 1$.



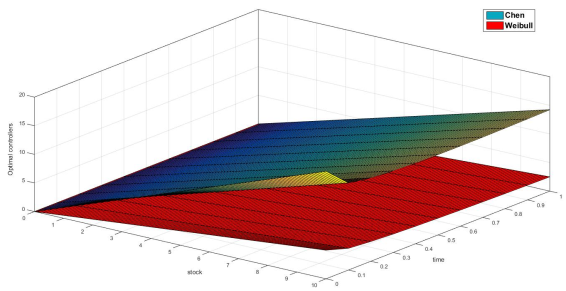
(a) Optimal controllers for Weibull's and Chen's laws when $\delta = 2$ and $\lambda = 1$ when $(t, x) \in [0, 1] \times [0, 1]$.



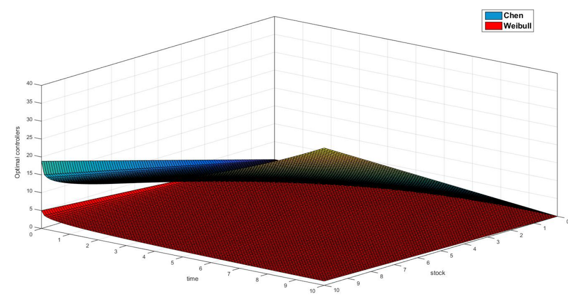
(b) Optimal controllers for Weibull's and Chen's laws when $\delta = 2$ and $\lambda = 1$ when $(t, x) \in [0, 10] \times [0, 10]$.



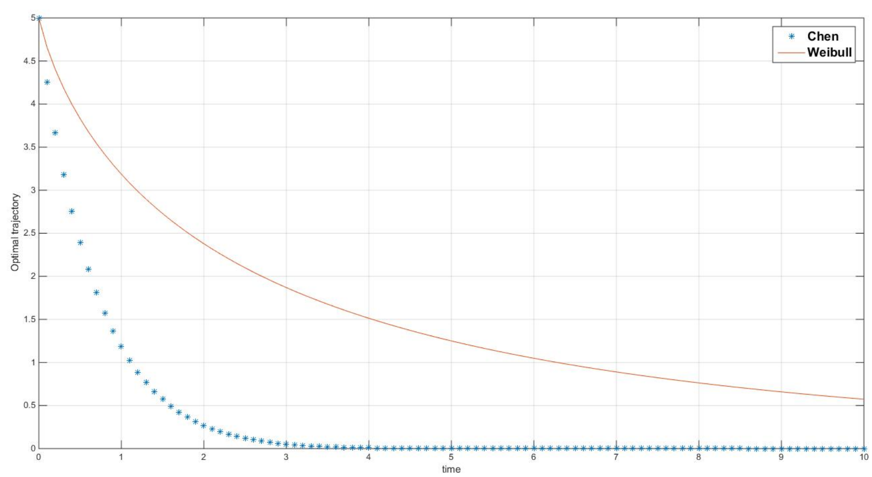
(c) Optimal trajectories for Weibull's and Chen's laws when $\delta = 2$ and $\lambda = 1$.



(a) Optimal controllers for Weibull's and Chen's laws when $\delta = \frac{1}{2}$ and $\lambda = 1$ when $(t, x) \in [0, 1] \times [0, 10]$.



(b) Optimal controllers for the laws of Weibull and Chen when $\delta = \frac{1}{2}$ and $\lambda = 1$ when $(t, x) \in [0, 10] \times [0, 10]$.



(c) Optimal trajectories for Weibull's and Chen's laws when $\delta = \frac{1}{2}$ and $\lambda = 1$.

4 Game-theoretic model for the extraction of natural resources

Now we consider a game model. For that purpose, we will make an extensive use of the results presented in [12]. Note that the utility function of the agent will depend only on its own control, and there are no payoff transferences among the players, i.e., on the extraction rate applied by the agent, and on $x(t)$, i.e., the stock of the resource at time t.

In what follows, there is no concordance among the notation we use: sometimes we write $h(t, x, ...)$; and some others we write $h(x, ...)$. We should decide which one we should use. Personally, I believe it's better to use the second one (because in our particular examples, we don't depend explicitly on time), however, I understand that one of the main ideas behind the approach of the conditional expectation and the random terminal time is the time-elapsed. What do you think, K?

4.1 Statement of the problem

Let $\Gamma(x_0)$ be a two-person differential game (in the context of this task, the players can be companies or countries), and the system satisfies the following conditions.

Assumption 4.1. (A) *Both players act simultaneously and start the game at some initial time $t_0 = 0$ from the state x_0.*

(B) *The control variables of the players are their respective extraction rates at every moment in time, namely $u_1, u_2 : [0, \infty[\to \in U$, where U is a compact subset of $\mathbb{R}$.*

(C) *The dynamics of the system is given by (1.2), i.e.,*

$$\dot{x} = g(x, u_1, u_2),\ \ x \in \mathbb{R}, u_1, u_2 \in U,\ \ x(t_0) = x_0, \tag{4.1}$$

where $g \in \mathcal{C}^2([0, x_0] \times U^2)$ is a function that satisfies

(i) $\frac{\partial}{\partial u_i} g < 0$ *for $i = 1, 2$; thus reflecting the effect of the firms' extraction efforts, and*

(ii) $\frac{\partial}{\partial x} g \leq 0$, *which mirrors the effect of the reduction of the resource's stock.*

Let T_i be the random terminal time for the i-player. We impose the following conditions on T_i. (Here, $i = 1, 2$.)

Assumption 4.2. (A) *Given the different characteristics of the firms, the terminal times of extraction of the same resource are mutually independent.*

(B) *The random terminal times of the firms are distributed according to some (known) functions $F_i : [0, \infty[\to [0, 1]$, $i = 1, 2$, that satisfy the normalization condition (1.1) and that are absolutely continuous with respect to Lebesgue's measure.*

(C) *As soon as one of the firms reaches its terminal time, it quits the game and the remaining one keeps extracting the resource until its terminal time is realized (which might happen when the resource becomes extinct).*

Assumptions 1.1 and 4.2 ensure that that the failure rate function associated with its the i-th firm is defined according to formula (2.1), i.e.,

$$\lambda_i(t) = \frac{f_i(t)}{1 - F_i(t)}, \tag{4.2}$$

where $f_i := F_i'$ is a density function for the random terminal time of the i-th player. We can profit from the well-known relation:

$$1 - F_i(t) = \exp\left(-\int_0^t \lambda_i(s)\mathrm{d}s\right),$$

(see [2, Chapter 3]) and mimic the argument that led us to (1.5), to get

$$K_i(x_0, u_1, ..., u_n) = \int_0^\infty h_i(t, x, u_1, ..., u_n)\exp\left(-\int_0^t \lambda_i(s)\mathrm{d}s\right)\mathrm{d}t.$$

The reward functions need to satisfy the following hypotheses.

Assumption 4.3. *Both firms have a concave utility function $h_i(x(t), u_1, u_2) \in \mathcal{C}^2([0, x_0] \times U^2)$ and tries to maximize the following enhanced version of the expected reward criterion we presented in* (1.3).

$$\begin{aligned} K_i(x_0, u_1, u_2) &= \mathbb{E}\left[\int_0^{T_i} h_i(x(\tau), u_1(\tau), u_2(\tau))\mathrm{d}\tau\chi_{\{T_i\le T_j\}}\right] &(4.3)\\ &+\mathbb{E}\left[\int_0^{T_j} h_i(x(\tau), u_1(\tau), u_2(\tau))\mathrm{d}\tau\chi_{\{T_i> T_j\}}\right] &(4.4)\\ &+\mathbb{E}\left[\Phi_i(x(T))\chi_{\{T_i> T_j\}}\right], &(4.5)\end{aligned}$$

for $i = 1, 2$, where the expectations are taken with respect to the joint distribution of (T_i, T_j); $\chi_{\{\cdot\}}$ is an indicator function; $T := \min\{T_i, T_j\}$; and $\Phi_i(\cdot)$ is the terminal payoff function.

Remark 4.4. *Note that the payoff of the game has two components: the integral payoff* (4.3)-(4.4)*; achieved while playing, and* (4.5)*, a final reward, which is assigned to the player that stayed longer in the system.*

In this game, each firm intends to maximize its profit. Then, we design the optimal trajectory as x^*; and the optimal strategies of the players as u_1^* and u_2^*. Now we can define $h_i^*(t) := h_i(x^*, u_1^*, u_2^*)$ to rewrite the optimal expected payoff resulting from the maximization of (4.3)-(4.5) as

$$\begin{aligned} K_i(x_0, u_1^*, u_2^*) &= \mathbb{E}\left[\int_0^{T_i} h_i^*(\tau)\mathrm{d}\tau\chi_{\{T_i\le T_j\}}\right] &(4.6)\\ &+\mathbb{E}\left[\int_0^{T_j} h_i^*(\tau)\mathrm{d}\tau\chi_{\{T_i> T_j\}}\right] &(4.7)\\ &+\mathbb{E}\left[\Phi_i(x^*(T))\chi_{\{T_i> T_j\}}\right]. &(4.8)\end{aligned}$$

Now we state the following result.

Proposition 4.5. *The optimal expected payoff for the problem starting at $t = 0$ is given by*

$$K_i(x_0, u_1^*, u_2^*) = \int_0^\infty h_i(\tau)(1 - F(\tau)) + \Phi_i(x^*(\tau))f_j(\tau)(1 - F_i(\tau))\mathrm{d}\tau, \tag{4.9}$$

where

$$F(\cdot) := (1 - F_1(\cdot))(1 - F_2(\cdot)) \tag{4.10}$$

|stands for the distribution function of the random variable T.

Proof. See [12, Corollary 3.1]. $\square$

If the game develops in time, at some intermediate moment in time $\theta > 0$, the players fall into the subgame $\Gamma(x, \theta))$ with initial state $x(\theta) = x$. Obviously, the game can generally end up to the moment θ with probability $F(\theta)$, if $0 < T < \theta$, and the probability of continuing the game after the moment t is $1 - F(\theta)$.
Then the expected integral payoff of the i-th player is calculated by

$$\mathbb{E}\left(\int_\theta^T h_i(\tau, x(\tau), u_1, ..., u_n)\mathrm{d}\tau\right) = \int_\theta^\infty \int_\theta^t h_i(\tau, x(\tau), u_1, ..., u_n)\mathrm{d}\tau \mathrm{d}F_\theta(\tau), \tag{4.11}$$

where $F_\theta(t)$, $t \geq \theta$ is the distribution function of the terminal time of the game in the subgame $\Gamma(x(\theta))$. Note that $F_\theta(t)$ is the conditional distribution function for the random terminal time, provided that the game does not end before the moment $\theta \in [0, \infty[$. It is necessary that $F_\theta(t)$ satisfies the standard normalization condition (1.1). The conditional distribution function is calculated as follows:

$$F_\theta(t) = \frac{F(t) - F(\theta)}{1 - F(\theta)}, \; t \in [\theta, \infty[.$$

Further, a density of the distribution of the angular momentum of the end of the game is given by $f(t) = F'(t)$. Obviously, in the subgame $\Gamma(x(\theta))$ the conditional distribution density $f_\theta(t)$ is defined as follows:

$$f_\theta(t) = \frac{f(t)}{1 - F(\theta)}, \; t \in [\theta, \infty[. \tag{4.12}$$

Thus, under Assumption 1.1(a), and taking into account (4.11) and (4.12), we obtain the integral payoff of the i-th player in the subgame $\Gamma(x(\theta))$ (recall (1.6)):

$$\begin{aligned} \mathbb{E}\left(\int_\theta^T h_i(\tau, x(\tau), u_1, u_2)\mathrm{d}\tau\right) &= \frac{1}{1 - F(\theta)} \int_\theta^\infty \int_\theta^t h_i(\tau, x(\tau), u(\tau))\mathrm{d}\tau f(t)\mathrm{d}t \quad (4.13) \\ &= \frac{1}{1 - F(\theta)} \int_\theta^\infty (1 - F(\tau))h_i(\tau, x(\tau), u(\tau))\mathrm{d}\tau. \quad (4.14) \end{aligned}$$

The goal of the i-player is to maximize the functional presented in (4.14) by using values that lie in the compact set U. The set of functions

$$U_i^* := \{u_i^*(t)\} := \arg\max_{u_i \in U} \mathbb{E}\left(\int_\theta^T h_i(\tau, x(\tau), u_1, u_2)\mathrm{d}\tau\right)$$

will be called *set of optimal strategies for the i-th player*, $i = 1, 2$; and the trajectory $x^*(t)$, corresponding to the optimal strategies $(u_1^*, u_2^*) \in U_1^* \times U_2^*$ is conditionally optimal.

An invocation to Theorem 3.1 in [12] allows us to state the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations associated with the optimization problem for the i-th player.

$$\begin{aligned} &-\frac{\partial}{\partial t} W_i(x, t) + (\lambda_1(t) + \lambda_2(t))W_i(x, t) \\ &= \max_{u_i \geq 0} \left(h_i(x, u_1, u_2) + \Phi_i(x)\lambda_j(t) + \frac{\partial}{\partial x} W_i(x, t)g(x, u_1, u_2)\right), \end{aligned} \tag{4.15}$$

where $\lambda_i(\cdot)$ is as in (4.2), and $i = 1, 2$.

We will find explicit solutions for (4.15) when g and h_i are analogous to (3.1) and (3.5). That is, when $g(x, u_1, u_2) = -u_1 - u_2$ and $h_i(x, u_1, u_2) = \ln(u_i)$. We also suppose that $\Phi_i(x) = c_i \ln(x)$, for some positive constant value c_i. In this case, the HJBI equation (4.15) turns out to be

$$
\begin{aligned}
&-\frac{\partial}{\partial t} W_i(x,t) + (\lambda_1(t) + \lambda_2(t)) W_i(x,t) \\
&= \max_{u_i \geq 0} \left(\ln(u_i) + c_i \ln(x) \lambda_j(t) - \frac{\partial}{\partial x} W_i(x,t)(u_1 + u_2) \right),
\end{aligned}
$$

for $i = 1, 2$. We can find the optimal strategies and the value function for this problem by proceeding in the same way that led us to (3.6) and (3.17). That is, by taking an informed guess on the form of the value function:

$$
W_i(x,t) = A_i(t) \ln x + B_i(t), \tag{4.16}
$$

for $i = 1, 2$; substituting it into (4.15); and solving the following Cauchy problem (which is analogous to (3.11)-(3.14)):

$$
\begin{aligned}
-\dot{A}_i(t) + A(t)(\lambda_i(t) + \lambda_j(t)) &= 1 + c_i \lambda_j(t), && (4.17) \\
\dot{B}_i(t) - B_i(t)(\lambda_i(t) + \lambda_j(t)) &= \ln A_i(t) + 1 + \frac{A_i(t)}{A_j(t)} && (4.18) \\
\lim_{t \to \infty} A(t) &= 0, && (4.19) \\
\lim_{t \to \infty} B(t) &= 0. && (4.20)
\end{aligned}
$$

We apply the technique of the integrating factor in (4.17); use the transversality condition (4.19), and get

$$
\begin{aligned}
A_i(t) &= \frac{\int_t^\infty (1 + c_i \lambda_j(\tau)) \exp\left(-\int_0^\tau \lambda_i(s) + \lambda_j(s) \mathrm{d}s\right) \mathrm{d}\tau}{\exp\left(-\int_0^t \lambda_i(\tau) + \lambda_j(\tau) \mathrm{d}\tau\right)} \\
&= \frac{\int_t^\infty (1 + c_i \lambda_j(\tau))(1 - F(\tau)) \mathrm{d}\tau}{1 - F(t)}.
\end{aligned} \tag{4.21}
$$

The last inequality holds by virtue of the well-known relation

$$
1 - F(\tau) = \exp\left(-\int_0^\tau \lambda_i(s) + \lambda_j(s) \mathrm{d}s\right), \tag{4.22}
$$

where $F(\cdot)$ is as in (4.10) (see, for instance [2, Chapter 9.3]).

Again, we apply the technique of the integrating factor to (4.18), and use the transversality condition (4.20) to get:

$$
\begin{aligned}
B_i(t) &= \frac{\int_t^\infty \left(\ln A_i(s) + 1 + \frac{A_i(s)}{A_j(s)}\right) \exp\left(-\int_0^s \lambda_i(\tau) + \lambda_j(\tau) \mathrm{d}\tau\right) \mathrm{d}s}{\exp\left(-\int_0^t \lambda_i(s) + \lambda_j(s) \mathrm{d}s\right)}, \\
&= \frac{\int_t^\infty \left(\ln A_i(s) + 1 + \frac{A_i(s)}{A_j(s)}\right) (1 - F(s)) \mathrm{d}s}{1 - F(t)}
\end{aligned} \tag{4.23}
$$

where $A_i(\cdot)$ and $A_j(\cdot)$ are as in (4.21), and we have used (4.22) again.

The substitution of (4.21) and (4.23) in (4.16) gives us the value function for the i-th player.

Then, we proceed as we did to get (3.10) and obtain the strategies of the players:

$$u_i^*(t,x) = \frac{x\exp\left(-\int_0^t(\lambda_1(s)+\lambda_2(s))\mathrm{d}s\right)}{\int_t^\infty(1+c_i\lambda_{-i}(s))\exp\left(-\int_0^s(\lambda_1(r)+\lambda_2(r))\mathrm{d}r\right)\mathrm{d}s}, \tag{4.24}$$

where

$$\lambda_{-i}(\cdot) := \begin{cases}\lambda_2(\cdot) & \text{if } i=1,\\ \lambda_1(\cdot) & \text{if } i=2.\end{cases}$$

(See the proof of Proposition 4.1 in [12] to see the full procedure.) By Theorem I.7.1(a) in [5], we know that (4.24) is optimal for the game with no payoff transference among the players.

The optimal trajectory can be found by plugging (4.24) into (4.1) and solving. That is,

$$x^*(\tau) = x_0\exp\left(-\sum_{i=1}^{2}\int_0^\tau \frac{\exp\left(-\int_0^t(\lambda_1(s)+\lambda_2(s))\mathrm{d}s\right)}{\int_t^\infty(1+c_i\lambda_{-i}(s))\exp\left(-\int_0^s(\lambda_1(r)+\lambda_2(r))\mathrm{d}r\right)\mathrm{d}s}\mathrm{d}t\right). \tag{4.25}$$

4.2 An illustration

Now, we devote our efforts to analyzing the particular cases of our interest. We start by assuming that the terminal times of both players are distributed according to Weibull's law. Then we will revise Chen's case, and finally we will see what happens when one of them is Weibull-distributed, while the other follows Chen's distribution.

Assume that the terminal times of both players are distributed according to Weibull's law. Plugging (2.3) into (4.24) yields:

$$u_i^*(t,x) = \frac{x\exp\left(-\int_0^t(\lambda_1\delta_1 s^{\delta_1-1}+\lambda_2\delta_2 s^{\delta_2-1})\mathrm{d}s\right)}{\int_t^\infty(1+c_i\lambda_{-i}\delta_{-i}s^{\delta_{-i}-1})\exp\left(-\int_0^s(\lambda_1\delta_1 r^{\delta_1-1}+\lambda_2\delta_2 r^{\delta_2-1})\mathrm{d}r\right)\mathrm{d}s},$$

for $i=1,2$. The optimal trajectories are obtained by inserting (2.3) into (4.25):

$$x^*(\tau) = x_0\exp\left(-\sum_{i=1}^{2}\int_0^\tau \frac{\exp\left(-\int_0^t(\lambda_1\delta_1 s^{\delta_1-1}+\lambda_2(s))\mathrm{d}s\right)}{\int_t^\infty(1+c_i\lambda_{-i}(s))\exp\left(-\int_0^s(\lambda_1\delta_1 r^{\delta_1-1}+\lambda_2(r))\mathrm{d}r\right)\mathrm{d}s}\mathrm{d}t\right).$$

On the other hand, if the terminal times of both players are distributed according to Chen's law, we plug (2.5) into (4.24) to get:

$$u_i^*(t,x) = \frac{x\exp\left(-\int_0^t\left(\lambda_1\delta_1 s^{\delta_1-1}\mathrm{e}^{s^{\delta_1}}+\lambda_2\delta_2 s^{\delta_2-1}\mathrm{e}^{s^{\delta_2}}\right)\mathrm{d}s\right)}{\int_t^\infty\left(1+c_i\lambda_{-i}\delta_{-i}s^{\delta_{-i}-1}\mathrm{e}^{s^{\delta_{-i}}}\right)\exp\left(-\int_0^s\left(\lambda_1\delta_1 r^{\delta_1-1}\mathrm{e}^{r^{\delta_1}}+\lambda_2\delta_2 r^{\delta_2-1}\mathrm{e}^{r^{\delta_2}}\right)\mathrm{d}r\right)\mathrm{d}s},$$

for $i = 1, 2$. We substitute (2.5) in (4.25) to obtain the optimal trajectory of the system when both terminal times are Chen-distributed:

$$x^*(\tau) =$$
$$x_0 \exp\left(-\sum_{i=1}^{2}\int_0^{\tau} \frac{\mathrm{e}^{-\int_0^t\left(\lambda_1\delta_1 s^{\delta_1-1}\mathrm{e}^{s^{\delta_1}}+\lambda_2\delta_2 s^{\delta_2-1}\mathrm{e}^{s^{\delta_2}}\right)\mathrm{d}s}}{\int_t^{\infty}\left(1+c_i\lambda_{-i}\delta_{-i}s^{\delta_{-i}-1}\mathrm{e}^{s^{\delta_{-i}}}\right)\mathrm{e}^{-\int_0^s\left(\lambda_1\delta_1 r^{\delta_1-1}\mathrm{e}^{r^{\delta_1}}+\lambda_2\delta_2 r^{\delta_2-1}\mathrm{e}^{r^{\delta_2}}\right)\mathrm{d}r}\mathrm{d}s}\mathrm{d}t\right)$$

Finally, if the random terminal time of player 1 has a Weibull distribution, and player 2's follows Chen's law, the optimal controllers are:

$$u_1^*(t,x) = \frac{x\exp\left(-\int_0^t(\lambda_1\delta_1 s^{\delta_1-1}+\lambda_2\delta_2 s^{\delta_2-1}\mathrm{e}^{s^{\delta_2}})\mathrm{d}s\right)}{\int_t^{\infty}(1+c_1\lambda_2\delta_2 s^{\delta_2-1}\mathrm{e}^{s^{\delta_2}})\exp\left(-\int_0^s(\lambda_1\delta_1 r^{\delta_1-1}+\lambda_2\delta_2 r^{\delta_2-1}\mathrm{e}^{r^{\delta_2}})\mathrm{d}r\right)\mathrm{d}s},$$

and

$$u_2^*(t,x) = \frac{x\exp\left(-\int_0^t(\lambda_1\delta_1 s^{\delta_1-1}+\lambda_2\delta_2 s^{\delta_2-1}\mathrm{e}^{s^{\delta_2}})\mathrm{d}s\right)}{\int_t^{\infty}(1+c_2\lambda_1\delta_1 s^{\delta_1-1})\exp\left(-\int_0^s(\lambda_1\delta_1 r^{\delta_1-1}+\lambda_2\delta_2 r^{\delta_2-1}\mathrm{e}^{r^{\delta_2}})\mathrm{d}r\right)\mathrm{d}s}.$$

The optimal trajectory is:

$$\begin{aligned} x^*(\tau) \;=\; & x_0\exp\Bigg(-\int_0^{\tau}\frac{\exp\left(-\int_0^t(\lambda_1\delta_1 s^{\delta_1-1}+\lambda_2\delta_2 s^{\delta_2-1}\mathrm{e}^{s^{\delta_2}})\mathrm{d}s\right)}{\int_t^{\infty}(1+c_1\lambda_2\delta_2 s^{\delta_2-1}\mathrm{e}^{s^{\delta_2}})\exp\left(-\int_0^s(\lambda_1\delta_1 r^{\delta_1-1}+\lambda_2\delta_2 r^{\delta_2-1}\mathrm{e}^{r^{\delta_2}})\mathrm{d}r\right)\mathrm{d}s} \\ & +\frac{\exp\left(-\int_0^t(\lambda_1\delta_1 s^{\delta_1-1}+\lambda_2\delta_2 s^{\delta_2-1}\mathrm{e}^{s^{\delta_2}})\mathrm{d}s\right)}{\int_t^{\infty}(1+c_2\lambda_1\delta_1 s^{\delta_1-1})\exp\left(-\int_0^s(\lambda_1\delta_1 r^{\delta_1-1}+\lambda_2\delta_2 r^{\delta_2-1}\mathrm{e}^{r^{\delta_2}})\mathrm{d}r\right)\mathrm{d}s}\mathrm{d}t\Bigg). \end{aligned}$$

5 Conclusion

The optimal behavior differs for different game scenarios and moments of its completion. For the run-in phase (i.e. when the equipment is not yet established and the overall picture of the process is not fully clear), the development speed should be the smallest, which corresponds to the agent's caution. The speed of development differs for two different moments of the end of the game, as expected at the beginning of the reasoning, i.e. Chen's distribution allows the most realistic description of the life cycle of the system. In normal operation, it is necessary to "dig" at a constant rate, but in the case where the end of the process is subject to of Chen, the pace of resource development still needs to be gradually increased. In the equipment aging mode, when the failure rate function increases, it does not matter what distribution law determines the completion of the development process, it is necessary to increase the rate of development of the fields.

References

[1] Boukas, E.K.; Haurie, A.; Michel, P. (1990) *An Optimal Control Problem with a Random Stopping Time.* Journal of Optimization Theory and Applications, vol. 64, no. 3, 471-480.

[2] Bowers, N.; Gerber, H.; Hickman, J.; Jones, D.; Nesbitt, C. (1997) *Actuarial mathematics.* SoA.

[3] Chen, Z. (2000) *A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function.* Statistics & Probability Letters 49, pp. 155-161.

[4] Dockner, E.J.; Jorgensen, S.; van Long, , N.; Sorger, G. (2000) *Differential Games in Economics and Management Science.* Cambridge University Press.

[5] Fleming, W.; Soner, H. M. (2005) *Controlled Markov Processes and Viscosity Solutions.* Springer. 2nd edition.

[6] Gromova, E.V. (2017) *Doctor of State dissertation.* SPbSU

[7] Gromova, E.V.; Tur, A. (2017) *On the form of integral payoff in differential games with random duration.* DOI: 10.1109/ICAT.2017.8171597.

[8] Henley, E.J.; Kumamoto, H. (1981) *Reliability engineering and risk assessment.* Prentice-Hall, Inc.

[9] Klugman, S.A., Panjer, H.H., Willmot, G.E. (2012) *Loss Models.* Wiley.

[10] Marin-Solano, J.; Shevkoplyas, E.V. (2011) *Non-constant discounting and differential games with random time horizon.* Automatica, vol. 47, no. 12, 2626-2638.

[11] Kostyunin, S.; Shevkoplyas, E. (2011) *On simplification of integral payoff in differential games with random duration*, Vestnik S. Petersburg Univ. Ser. 10. Prikl. Mat. Inform. Prots. Upr., 4, 47-56.

[12] Kostyounin, S.; Palestini, A.; Shevkoplyas, E.V. (2014) *On a Nonrenewable Resource Extraction Game Played by Asymmetric Firms* J. Optim. Theory Appl. 163, 660-673.

[13] Petrosyan, L.A.; Murzov, N.V. (1966) *Game-theoretic problems of mechanics.* Litovsk. Mat. Sb. 6, 423-433. (In Russian.)

[14] Petrosyan, L.A. (1977) *Differential persecution games.* Leningrad: Leningrad University Press. (In Russian.)

[15] Promislow, D.S. (2011) *Fundamentals of Actuarial Mathematics.* Wiley.

[16] Weibull, W. (1951) *A statistical distribution function of wide applicability*, J. Appl. Mech. Trans. ASME O. 18 (3): 293-297.

[17] Yaari, M.E. (1965) *Uncertain Lifetime, Life Insurance, and the Theory of the Consumer.* The Review of Economic Studies, 32(2): 137-150.

[18] Shevkoplyas, E.V. (2009) *The Hamilton-Jacobi-Bellman equation for a class of differential games with random duration.* Mathematical Game Theory and its Applications, 1(2): 98-118. (In Russian.)

[19] Shevkoplyas, E.V. (2010) *Stable cooperation in differential games with random duration.* UBS, 31(1): 162-190. (In Russian.)