Linear and quadratic inequalities

Exercise 1: Solving Inequalities - Solutions

1. Each of the following number lines represents an inequality. Express each inequality algebraically.

a) 

$x \geq 3$

b) 

$x < -2$

c) 

$4 \leq x \leq 9$

d) 

$-8 < x < 3$

e) 

$-4.5 < x \leq 3.2$

f) 

$x \leq 4 \; OR \; x \geq 7$

g) 

$x < -6 \; OR \; x > 6$

h) 

$x < -2 \; OR \; x \geq 0$

i) 

$x < -5 \; OR \; 0 < x < 5$

j) 

$-3 \leq x \leq 3 \; OR \; x = 7$

2. In the case of 1 (f) to (j), express the selected region of the number line in set language, using ∪ or ∩.

f) $x \leq 4 \; OR \; x \geq 7$

$\{x: x \leq 4\} \cup \{x: x \geq 7\}$

g) $x < -6 \; OR \; x > 6$

$\{x: x < -6\} \cup \{x: x > 6\}$

h) $x < -2 \; OR \; x \geq 0$

$\{x: x < 2\} \cup \{x: x \geq 0\}$

i) $x < -5 \; OR \; 0 < x < 5$

$\{x: x < -5\} \cup \{x: 0 < x < 5\}$

j) $-3 \leq x \leq 3 \; OR \; x = 7$

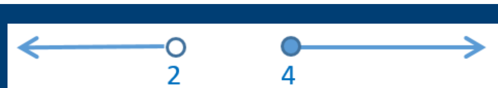
$\{x: -3 \leq x \leq 3\} \cup \{7\}$

3. Represent each of the following on a number line.

(a) $-2 < x < 3$

(b) $x \leq -3 \;\; OR \;\; x \geq -2$

(c) $x < 2 \;\; OR \;\; x \geq 4$



(d) $x < 4 \;\; AND \;\; x > 1$

4. Find the solution set for each of the following inequalities, using ∪ or ∩ where appropriate.

(a) $5(x-3) < 2x$

$\{x: x < 5\}$

(b) $3(x-2) \geq 4(2x+1)$

$\{x: x \leq -2\}$

(c) $x^2 + 12 \geq 7x$

$\{x: x \leq 3\} \cup \{x: x \geq 4\}$

(d) $1 < 5 - 2x < 11$

$\{x: -3 < x < 2\}$

(e) $3(x-7) > 7(x-3)$

$\{x: x < 0\}$

(f) $3(t-5) - 5(2t-7) > 6$

$\{t: t < 2\}$

(g) $m(m+1) \geq 5(m+1)$

$\{m: m \leq -1\} \cup \{m: m \geq 5\}$

(h) $2r^2 + 7r - 15 \leq 0$

$\{r: -5 \leq r \leq 1.5\}$

(i) $8 + 10s - 3s^2 < 0$

$\left\{x: x < -\frac{2}{3}\right\} \cup \{x: x > 4\}$

(j) $9 > p(6-p)$

$\{x: x \neq -3\}$

Alternatively $\{x: x < -3\} \cup \{x: x > -3\}$

5. In the case of 3(a), (a) find a quadratic inequality for which this is the solution.

$x^2 - x - 6 < 0$

(b) write the solution set in set language.

$\{x: -2 < x < 3\}$

6. Repeat Question 5 in the case of 3(b).

(a) find a quadratic inequality for which this is the solution.

$$x^2 + 5x + 6 \geq 0$$

(b) write the solution set in set language.

$$\{x: x \leq -3\} \cup \{x: x \geq -2\}$$

7. Repeat Question 5 in the case of 3(d).

(a) find a quadratic inequality for which this is the solution.

$$x^2 - 5x + 4 < 0$$

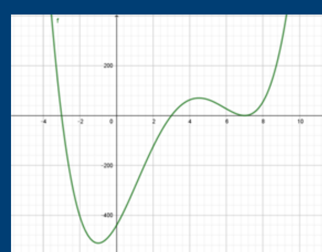
(b) write the solution set in set language.

$$\{x: 1 < x < 4\}$$

8. Explain why you cannot do the same for 3(c).

The interval $\{x: x < 2\}$ is <u>open</u> (does not contain 2) and the interval $\{x: x \geq 4\}$ is <u>closed</u> (does contain 4), so the quadratic would have to be ≥ 0 at one branch and > 0 at the other branch. These cannot be combined into a single inequality.

9. Can you imagine the sort of inequality or graph that would lead to 1(j) as its solution set?



We want a graph that crosses the x axis at -3 and 3.

The equation of the graph would need factors $(x + 3)$ and $(x - 3)$.

It also needs to <u>touch</u> the x axis at 7, so there is a repeated root at $x = 7$, which means a repeated factor.

The simplest graph with these properties is the graph

$$y = (x + 3)(x - 3)(x - 7)^2$$

We want the x values for which the graph is on or below the x axis, so the inequality is

$$(x + 3)(x - 3)(x - 7)^2 \leq 0.$$

10. In order to solve $\frac{x-2}{x+2} < 5$, Siobhan multiples by $(x + 2)$ to get $x - 2 < 5x + 10$, and deduces that $x > -3$. This means that her solution set includes -2. Why is that impossible? Can you improve Siobhan's method?

She has multiplied by $(x + 2)$, which will be negative if $x < -2$, making her method invalid.

She could EITHER consider the cases in which $x < -2$ and $x > -2$ separately OR multiply instead by $(x + 2)^2$.

FIRST METHOD

If $x < -2$,
$x - 2 > 5x + 10$
$\Rightarrow \quad -12 > 4x$
$\Rightarrow \quad x < -3.$
$x < -2$ is no extra constraint.
So $x < -3$.

If $x > -2$,
$x - 2 < 5x + 10$
$\Rightarrow \quad -12 < 4x$
$\Rightarrow \quad x > -3.$
But we need $x > -2$ as well.
So $x > -2$.

Combining these cases, $x < -3 \; OR \; x > -2$

SECOND METHOD

$(x - 2)(x + 2) < 5(x + 2)^2$
$\Rightarrow \; 0 < 5(x + 2)^2 - (x - 2)(x + 2)$
$\Rightarrow \; 0 < (x + 2)(5x + 10 - (x - 2))$
$\Rightarrow \; 0 < (x + 2)(4x + 12)$

Graph crosses x axis at -3 and -2.

We want the part above the x axis.

So $\quad x < -3 \; OR \; x > -2$