What do you know about indices?

Exercise 1: The Basics – Worked Solutions and Explanations

Laws of Indices

1. $a^x a^y = a^{x+y}$ Multiplication
2. $a^x \div a^y = a^{x-y}$ Division
3. $(a^x)^y = a^{xy}$ Raising to a power

Warm up

1. Simplify each of these

Question		Working	Commentary
a)	$x^7 \times x^4$	$= x^{7+4}$ $= x^{11}$	For this example it is a case of applying the first law listed above: $a^x a^y = a^{x+y}$ The base is the same for both terms being multiplied so indices are simply added together.
b)	$(3a^4)^3$	$= 3^3 a^{4\times 3}$ $= 27a^{12}$	The index outside the bracket has to be applied to all of the terms being multiplied together inside the bracket. This means that the 3 as well as the a^4 being raised to the power of 3. 3^4 is 27 and can be written in directly. To raise a^4 to the power of 3 requires the application of the third law listed above: $(a^x)^y = a^{xy}$. The indices are multiplied together to give the answer.
c)	$p^7 \div p^3$	$= p^{7-3}$ $= p^4$	For this example it is a case of applying the second law listed above: $a^x \div a^y = a^{x-y}$ The base is the same for both terms being multiplied so the index for the divisor is subtracted from the index for the dividend.

d)	$4y^4 \times (3y)^2$	$= 4y^4 \times 9y^2$ $= 36y^{4+2}$ $= 36y^6$	The first step for this question is to remove the brackets from the second term. The index applies to all terms being multiplied in the bracket. 3 squared is 9, which can be written in directly, and y squared is simply y^2. To multiply the two terms together, the numbers are multiplied, $4 \times 9 = 36$, and the first law of indices is used for $y^4 \times y^2$.
e)	$x^3y^4z^2 \times x^2y^3z$	$= x^{3+2}y^{4+3}z^{2+1}$ $= x^5y^7z^3$	This requires the application of the first law to each variable. It is important to realise that z is the same as z^1 for the purpose of adding the indices.
f)	$\dfrac{(4mn^2)^3}{2m^2n}$	$= \dfrac{64m^3n^6}{2m^2n}$ $= 32mn^5$	The top of the fraction is simplified first by applying the index outside the bracket to each term in the multiplication inside. 4^3 is 64. The third index law is used for the n terms. The index for m is straightforward. The division is done term by term starting with the numbers $64 \div 2$ is 32. For the rest of the terms, the second law is used and the indices are subtracted.
g)	$\left(\dfrac{6x^2y \times 3y^2z^3}{9xyz}\right)^4$	$= \left(\dfrac{18x^2y^3z^3}{9xyz}\right)^4$ $= (2xy^2z^2)^4$ $= 16x^4y^8z^8$ alternative $= 16x^4(yz)^8$	The top of the fraction is simplified first. The numbers multiply to give 18. The first law is used with the y terms. The x^2 and z^3 will be present in the result as it is a multiplication. The division in the bracket is done next. $18 \div 9$ is 2. The second law of indices is used with the other terms and the indices are subtracted. The final step is raising everything in the bracket to the power 4. This uses the third index law. It is worth noting that there are other ways to write the answer and one alternative is given. As both y and z are raised to the power of 8, they could be put together in a bracket. It is arguable which of the two forms is the more simplified.

h)	$\left(\frac{8a^3b^4}{(2ab^3)^2}\right)^3 \times \left(\frac{3a^8b^3}{a^3b}\right)^4$	$= \left(\frac{8a^3b^4}{4a^2b^6}\right)^3 \times \left(\frac{3a^8b^3}{a^3b}\right)^4$ $= \left(\frac{2a}{b^2}\right)^3 \times (3a^5b^2)^4$ $= \frac{8a^3}{b^6} \times 81a^{20}b^8$ $= 648a^{23}b^2$	The first step is to simplify the denominator of the first fraction by applying the index outside the bracket to every term inside. This uses the third index law. Each fraction is simplified further by dividing numbers where possible and using the second index law. It looks like negative indices are needed for the b terms in the first fraction. This can be avoided by treating it as the reciprocal of b^2. The indices outside of each bracket are applied to the terms inside using the third index law. The final step requires the first index law for the a terms and the second for the b terms.

2. Write each of these as the product of powers of prime numbers

Question		Working	Commentary
a)	$12 \times 4 \times 18$	$= 3 \times 2^2 \times 2^2 \times 2 \times 3^2$ $= 2^5 \times 3^3$	The first step is to produce a prime factorisation for each of the numbers. $12 = 3 \times 4 = 3 \times 2^2$ $4 = 2^2$ $18 = 2 \times 9 = 2 \times 3^2$ The first index law is used to simplify terms with the same base by adding their indices together.
b)	$4^3 \times 9^3 \times 20^2$	$= (2^2)^3 \times (3^2)^3 \times (2^2 \times 5)^2$ $= 2^6 \times 3^6 \times 2^4 \times 5^2$ $= 2^{10} \times 3^6 \times 5^2$	The first step is to produce a prime factorisation for all of the numbers. The third index law is then used to remove the brackets. The first index law is used to find the final result.
c)	$\frac{35^4 \times 14^6 \times 15^4}{28^2 \times 21^3 \times 100}$	$= \frac{(5 \times 7)^4 \times (2 \times 7)^6 \times (3 \times 5)^4}{(2^2 \times 7)^2 \times (3 \times 7)^3 \times 2^2 \times 5^2}$ $= \frac{5^4 \times 7^4 \times 2^6 \times 7^6 \times 3^4 \times 5^4}{2^4 \times 7^2 \times 3^3 \times 7^3 \times 2^2 \times 5^2}$ $= \frac{2^6 \times 3^4 \times 5^8 \times 7^{10}}{2^6 \times 3^3 \times 5^2 \times 7^5}$ $= 3 \times 5^6 \times 7^5$	The first step is to produce a prime factorisation for all of the numbers. The third index law is then used to remove the brackets. The second index law is used to complete the calculation.

d)	$(3^2 \times 22^4 \times 24^3)^4 \times \frac{60^3 \times 45^2 \times 5^{13}}{(2^4 \times 75 \times 33)^9}$	$= (3^2 \times (2 \times 11)^4 \times (2^3 \times 3)^3)^4 \times \frac{(2^2 \times 3 \times 5)^3 \times (3^2 \times 5)^2 \times 5^{13}}{(2^4 \times 3 \times 5^2 \times 3 \times 11)^9}$ $= (3^2 \times 2^4 \times 11^4 \times 2^9 \times 3^3)^4 \times \frac{2^6 \times 3^3 \times 5^3 \times 3^4 \times 5^2 \times 5^{13}}{2^{36} \times 3^9 \times 5^{18} \times 3^9 \times 11^9}$ $= (2^{13} \times 3^5 \times 11^4)^4 \times \frac{2^6 \times 3^7 \times 5^{18}}{2^{36} \times 3^{18} \times 5^{18} \times 11^9}$ $= 2^{52} \times 3^{20} \times 11^{16} \times \frac{1}{2^{30} \times 3^{11} \times 11^9}$ $= 2^{22} \times 3^9 \times 11^7$	The first step is to produce a prime factorisation for all of the numbers. A gradual simplification is needed to avoid making mistakes. In each line of working a small part of the expression has been simplified using the appropriate index law. Reciprocals are used to avoid using negative indices.

Moving on

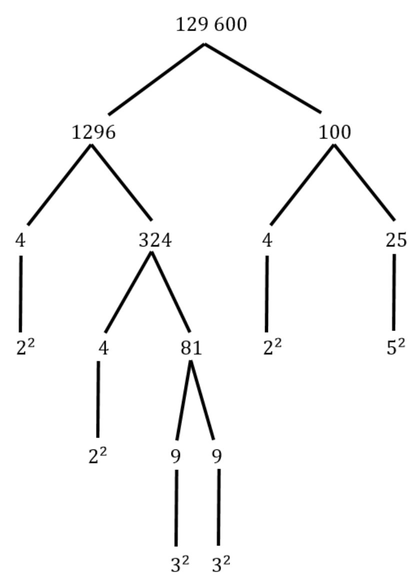
3. Multiply out these brackets

Question		Working	Commentary
a)	$a^2bc^3(ab^3c - bc^2)$	$a^2bc^3(ab^3c - bc^2)$ $= a^2b^4c^4 - a^2b^2c^5$	There is one term outside the brackets and two inside so the answer will have two terms. The first index law is used for each multiplication.
b)	$x^3y^4(xz + y^2z^3 - x^2z^2)$	$x^3y^4(xz + y^2z^3 - x^2z^2)$ $= x^4y^4z + x^3y^6z^3 - x^5y^4z^2$	There is one term outside the brackets and three inside so the answer will have three terms. The first index law is used for each multiplication.
c)	$(pq^2 + q^3r^2)(p^2r + p^2q^4r)$	$(pq^2 + q^3r^2)(p^2r + p^2q^4r)$ $= p^3q^2r + p^2q^7r^3 + p^2q^3r^3 + p^4q^6r$	There are two terms in the first bracket and two in the second so the answer will have four terms unless there are like terms that can be combined. The first index law is used for each multiplication.

d)	$\left(x^5y^4+\dfrac{x^5y^3}{xy^2}\right)\left(x^6y^5-\dfrac{x^2y^3}{x^6y^4}\right)$	$=(x^5y^4+x^4y)\left(x^6y^5-\dfrac{1}{x^4y}\right)$ $=(x^5y^4+x^4y)\left(x^6y^5-\dfrac{1}{x^4y}\right)$ $=x^{11}y^9-1+x^{10}y^6-xy^3$	Before any multiplication is done, the fractions in the brackets are simplified. There are two terms in the first bracket and two in the second so the answer will have four terms unless there are like terms that can be combined. The first index law is used for each multiplication. Care is needed to ensure the signs (+/-) are correct. A reciprocal is used to avoid using negative indices. Two of the terms multiply together to give -1.

Challenge

4. Without using a calculator, find the positive square root of 129600

Working	Commentary
 $129\,600 = 2^6 \times 3^4 \times 5^2$ $= (2^3 \times 3^2 \times 5) \times (2^3 \times 3^2 \times 5)$ $\sqrt{129\,600} = 2^3 \times 3^2 \times 5$ $= 8 \times 9 \times 5$ $= 360$	A factor tree is used to find the prime factorization. As soon as powers of prime numbers are spotted they are written using indices. The prime factorization can be split into two identical groups. Calculating the value of one of the groups gives the square root.

5. Find the smallest number that each of these needs to be multiplied by in order to make a square number

Question		Working	Commentary
a)	2160	$2160 = 2^4 \times 3^3 \times 5$ Need $3 \times 5 = 15$	The first stage is to produce the prime factorization of 2160. This can be done using a factor tree. The factorization of a square number splits into two groups that multiply together to give the same result (the square root). For this to be possible there must be an even quantity of each prime number in the prime factorization. For 2160, there is not an even number of 3s or 5s. To make a square number multiply by another 3 and another 5 to make even quanitites.
b)	2430	$2430 = 2 \times 3^5 \times 5$ Need $2 \times 3 \times 5 = 30$	The prime factorization has an odd number of 2s, 3s and 5s. To make them all even we need one more of each.
c)	1500	$1500 = 2^2 \times 3 \times 5^3$ Need $3 \times 5 = 15$	The prime factorization has an odd number of 3s and 5s. To make them all even we need one more of each.

6. How many factors does each of these integers have

Question		Working	Commentary
a)	1740	$1740 = 2^2 \times 3 \times 5 \times 29$ There are $3 \times 2 \times 2 \times 2 = 24$ factors	The prime factorization shows that factors of 1740 can include any power of 2 from 2^0 to 2^2, any power of 3 from 3^0 to 3^1, any power of 5 from 5^0 to 5^1 and any power of 29 from 29^0 to 29^1. So for the 2s there are 3 possibilities 2^0, 2^1 and 2^2. For the 3s there are 2 possibilities 3^0 and 3^1. For the 5s there are 2 possibilities 5^0 and 5^1. For the 29s there are 2 possibilities 29^0 and 29^1. This means that there are $3 \times 2 \times 2 \times 2 = 24$ possibilities.
b)	$81\,675$	$81\,675 = 3^3 \times 5^2 \times 11^2$ There are $4 \times 3 \times 3 = 36$ factors	Using the prime factorization of 81675: For the 3s there are 4 possibilities 3^0, 3^1, 3^2 and 3^3. For the 5s there are 3 possibilities 5^0, 5^1 and 5^2. For the 11s there are 3 possibilities 11^0, 11^1 and 11^2. This means that there are $4 \times 3 \times 3 = 36$ possibilities.

c)	876 096	$876096 = 2^6 \times 3^4 \times 13^2$ $7 \times 5 \times 3 = 105$ factors	Using the prime factorization of 876 096: For the 2s there are 7 possibilities from 2^0 to 2^6. For the 3s there are 5 possibilities from 3^0 to 3^4. For the 13s there are 3 possibilities 13^0, 13^1 and 13^2. This means that there are $7 \times 5 \times 3 = 105$ possibilities.