Central Difference Approximation of a Second Derivative

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

Taylor approximation of $f(x+h)$

$$f(x+h) = f(x) + f'(x)(x+h-x) + \frac{1}{2}f''(x)(x+h-x)^2 + \frac{1}{6}f'''(x)(x+h-x)^3 + \dots$$

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{6}f'''(x)h^3 + \frac{1}{24}f''''(x)h^4 + \dots$$

Taylor approximation of $f(x-h)$

$$f(x-h) = f(x) + f'(x)(x-h-x) + \frac{1}{2}f''(x)(x-h-x)^2 + \frac{1}{6}f'''(x)(x-h-x)^3 + \dots$$

$$f(x-h) = f(x) - f'(x)h + \frac{1}{2}f''(x)h^2 - \frac{1}{6}f'''(x)h^3 + \frac{1}{24}f''''(x)h^4 + \dots$$

$|\text{error}| = |\text{exact value - approximation}| \leq ch^r$

$$\begin{aligned} &= f''(x) - \frac{1}{h^2}(f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{6}f'''(x)h^3 + \frac{1}{24}f''''(x)h^4 + \dots - 2f(x) \\ &\quad + f(x) - f'(x)h + \frac{1}{2}f''(x)h^2 - \frac{1}{6}f'''(x)h^3 + \frac{1}{24}f''''(x)h^4 + \dots) \end{aligned}$$

$$= f''(x) - \frac{1}{h^2}(f''(x)h^2 + \frac{1}{12}f''''(x)h^4 \dots)$$
$$= -\frac{1}{12}f''''(x)h^2$$

The rate of convergence is h^2