Math 4610 Fundamentals of Computational Mathematics - Topic 17.

In this section we will talk about inevitable errors in number representations on computers. The finite limitations of computers and other devices mean that we cannot exactly represent most numbers on the real line.
In this section we will focus on the limitations of number representation on computers. As mentioned above, the capabilities of computers have advanced rapidly. Over the years computer archetecture has increased resolution by using more bits to account for smaller and larger numbers. Starting with 8-bit, 16-bit, and then onto 32-bit and 64-bit numbers, the resolution has gotten better, but will never be able to represent an infinite number of digits needed to store irrational numbers.
Even a number like 1/3 must be truncated after a finite number of digits due to the way computers store numbers using bits. Data is stored in a binary format on all computers. So, the number 1/3 becomes

$$\frac{1}{3} = 0.10101010101010\ldots$$

The representation repeats, but it is necessary to use an infinite number of digits to exactly represent the number.

Absolute and Relative Error: Introduction.

In most cases, the solution of a mathematical problem can only be approximated. So, it would be a good idea to have a way to measure the error between the exact solution and the approximation of that solution. We will define two types of error. These are absolute error and relative error. The **absolute error** is the absolute value of the difference between the approximation and the exact value for the solution. That is, if x^* is the exact value approximated by x, then

$$e_{abs} = |x - x^*|$$

defines the absolute error. The **relative error** is a scaled error defined by

$$e_{rel} = \frac{|x - x^*|}{|x^*|}$$

So, the relative error is a scaled or percent error based on the magnitude of the exact value.

Absolute and Relative Error: Examples.

If we are trying to find the roots of the polynomial

$$p(x) = x^5 + x^3 - 2\ x^2 + 5\ x$$

we can see that $x = 0$ is one solution. To find other roots we can use Newton's method to generate a sequence of approximations given a starting point. That is, we will generate a sequence

$$S = \{x_k\}_{k=0}^{\infty}$$

Newton's method will be covered a bit later in this course. However, what we will want is for the sequence to converge to a root, say x^*. This can be rephrased as

$$|x_k - x^*| \to 0$$

which implies the absolute error will tend to zero as k tends to ∞. Using the relative error we want

$$\frac{|x_k - x^*|}{|x^*|} \to 0$$

For the polynomial define in this example, there will be problems in using the relative error near the zero roots. So, if the sequence starts to converg to the zero root, we would need to use the absolute error as a measure.

Table 1: Absolute and Relative Error Values

x	x^*	**abs. err.**	**rel. err.**
0.01	0.1	0.09	0.9
1.01	1.0	0.01	0.01
2.0	3.0	1.0	0.5
10.0	9.0	1.0	0.1
100.0	99.0	1.0	0.01

Absolute and Relative Error: A Numerical Example.

As an illustration of how absolute and relative errors compare, we can consider some numerical examples. The following table gives a pair of numbers along with both the absolute and relative errors.
Note that when both the approximation and exact value are close to zero, the absolute error becomes a better measure of error than the relative error and for large values (at least greater than one) one should expect that the better measure of the error is the relative error. We will use both in the development of algorithms to numerically solve mathematical problems.
The following is a list of sources for error that need to be taken into account by compoutational scientists.

1. **Modeling Errors** These errors can occur when assumptions are made about the phenomena being studied. For example, one may consider a model of the solar system where the planets are assumed to be spheres, which is not the case.

2. **Measurement Errors:** These errors occur when instruments are used to measure physical quantities. For example, the temperature of molten lava might be measured to within one or two degrees based on the magnitude of the exact temperature. The fractional part of the measurement would characterize the error.

3. **Discretization Error:** In order to computer solutions to mathematical problems using computers, it necessary that the model be finite and discrete. For example, weather models based on systems of partial differential equations require a discretization of the continuous model to fit in the discrete framework of a computer simulation.

Note that Github will not allow you the luxury of creating empty folders. This is an advantage in using “git” on a local machine. When changes are “pushed” to Github, empty folders are ignored. So, let’s get started on the formatting the homework solutions portion of the repository for the class.