CS251 Category Lens Problem Sets

Optional Enrichment (1–2 problems per week)

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1 Week 1: Arrows, Points, and Products

(a) Let $A = \{a, b\}$. List all morphisms $1 \to A$ in **Set** and match each to an element of A.

(b) Let $X = \{1, 2\}$, $A = \{a, b\}$, $B = \{c, d\}$. Define $f : X \to A$ by $f(1) = a, f(2) = b$ and $g : X \to B$ by $g(1) = d, g(2) = c$. Construct the unique map $\langle f, g \rangle : X \to A \times B$ and verify that $\pi_1 \circ \langle f, g \rangle = f$ and $\pi_2 \circ \langle f, g \rangle = g$.

2 Week 2: Isomorphisms, Sections, and Idempotents

(a) Suppose $f : A \to B$ has a right inverse $s : B \to A$ with $f \circ s = \mathrm{id}_B$. Prove that f is surjective.

(b) Let $A = \{1, 2, 3\}$ and define $p : A \to A$ by $p(1) = 1$, $p(2) = 1$, $p(3) = 3$. Show that p is idempotent. Find a subset $B \subseteq A$ and maps $r : A \to B$, $i : B \to A$ such that $p = i \circ r$ and $r \circ i = \mathrm{id}_B$.

3 Week 3: Preorders as Categories

(a) In the divisibility preorder on $\{1, 2, 3, 6\}$, compute the meet and join of 2 and 3.

(b) Let $F : \mathcal{P}(\{1, 2, 3\}) \to \mathbb{N}$ be $F(S) = |S|$. Prove that F is monotone with respect to $\subseteq$ and $\leq$.

4 Week 4: Products, Sums, and Exponentials

(a) Let $|A| = 2$ and $|B| = 3$. Compute $|B^A|$ and interpret it as the number of functions $A \to B$.

(b) Let $X = \{x, y\}$, $A = \{0, 1\}$, $B = \{a, b\}$. Define $g : X \times A \to B$ by $g(x, 0) = a$, $g(x, 1) = b$, $g(y, 0) = b$, $g(y, 1) = a$. Write the curried map $\tilde{g} : X \to B^A$ explicitly.

5 Week 5: Subsets as Maps to 2

(a) For $A = \{1, 2, 3, 4\}$, describe the bijection between subsets $S \subseteq A$ and characteristic maps $\chi_S : A \to 2$. How many maps have exactly two elements mapped to 1?

(b) Let $c : 2 \to 2$ swap 0 and 1. Show that the complement of S corresponds to the composite $c \circ \chi_S$.

6 Week 6: Parts and Graph Homomorphisms

(a) Prove that for any function $f : A \to B$ and subsets $S, T \subseteq B$, we have $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.

(b) Let G have vertices $\{1, 2, 3\}$ and edges $\{1, 2\}$, $\{2, 3\}$. Let H be the triangle graph on $\{a, b, c\}$. Show that the map $h(1) = a$, $h(2) = b$, $h(3) = c$ is a graph homomorphism. Give a different vertex map that is not a homomorphism.

7 Week 7: Free Categories and Components

(a) For the directed graph $1 \to 2 \to 3$, list all morphisms from 1 to 3 in $\mathbf{Path}(G)$. How many length-2 paths are there? Verify using the adjacency matrix.

(b) Let G have vertices $\{1, 2, 3, 4\}$ and edges $\{1, 2\}$, $\{2, 3\}$. Compute $\pi_0(G)$ (the connected components).

8 Week 8: Initial Algebras and Folds

(a) Define an algebra for binary trees that computes the number of leaves. Write the fold equations.

(b) Let t be a tree with a root whose left subtree is a single leaf and whose right subtree has two leaves. Use your fold to compute the number of leaves of t.

9 Week 9: Coalgebras and Automata

(a) Consider the DFA over $\Sigma = \{0, 1\}$ that accepts strings with an even number of 1's. Write the coalgebra map $Q \to 2 \times Q^{\Sigma}$ explicitly (name the states and indicate output and transitions).

(b) For the same DFA, compute $\delta^*(q_0, 1011)$ where q_0 is the start state.

10 Week 10: Monoids and Adjunctions

(a) Let $\Sigma = \{a, b\}$. Define a monoid homomorphism $h : \Sigma^* \to (\mathbb{N}, +, 0)$ by $h(a) = 1$ and $h(b) = 2$. Compute $h(\texttt{abba})$.

(b) Describe the one-object category corresponding to the monoid $(\mathbb{N}, +, 0)$: what are its morphisms and how is composition defined?