Lambda Calculi and You

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June 22, 2017

What you'll learn

What the λ calculus is, how to calculate with it, and lessons to draw from it

slides available at:
`https://github.com/clarissalittler/talks/`

History of λ calculus

$$\hat{} \rightarrow \Lambda \rightarrow \lambda$$

What is it?

- $\lambda x.M$
- MN
- x

Let's break it down

$\lambda x.M$

```
function (x) {
  M
}
```

```
(lambda x: M)
```

```
{ |x| M}
```

Sample programs

$$
\begin{aligned}
\mathrm{id} &= \lambda x.x \\
\mathrm{double} &= \lambda f.\lambda x.f\,(f\,x) \\
\mathrm{if} &= \lambda b.\lambda t.\lambda f.b\,t\,f
\end{aligned}
$$

Substitution: where computation happens

$$(\lambda x.M)N \to N[M/x]$$

You've seen this

```
def sillyFun(x):
    y = x + 2
    print(y)
    return (x*x)
```

You've seen this

```
sillyFun(3):
  y = 3 + 2
  print(y)
  return (3 * 3)
```

Evaluation order

Functions before arguments

$MN \to (\lambda x.I)N$

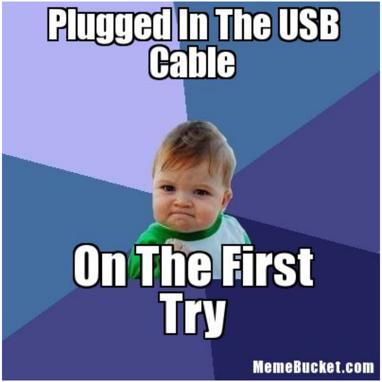
Capture avoidance

$(\lambda x.\lambda y.x\ y)y \to \lambda y.y\ y$
But *that* can't be right!

Rename variables

$$(\lambda x.\lambda z.x\, z)y \to \lambda z.y\, z$$

Implementing it



Is this a real language?

Believe it or not, everything we need is here

Church encodings

Church encodings are representations of *data* as *functions*

Natural numbers

Numbers are functions of the form $\lambda s.\lambda z.??$

$0 := \lambda s.\lambda z.z$
$S := \lambda n.\lambda s.\lambda z.s(n\, s\, z)$

How are these numbers?

$$1 := S(0) = \lambda s.\lambda z.s(0\, s\, z) = \lambda s.\lambda z.s\, z$$
$$2 := S(1) = \lambda s.\lambda z.s(1\, s\, z) = \lambda s.\lambda z.s(s\, z)$$

The meaning of a natural number

The number `N` represents doing *something* `N` times

double = 2

Definite iteration

Natural numbers encapsulate the act of definite iteration

Arithmetic

$$
\begin{aligned}
m + n &:= m(S)(n) \\
1 + 1 &= 1(S)(1) = S(1) = 2 \\
2 + 2 &= 2(S)(2) = S(S(2)) = 4 \\
3 + 5 &= 3(S)(5) = S(S(S(5))) = 8
\end{aligned}
$$

$$
\begin{aligned}
m * n &:= m(n(S))(0) \\
1 * 1 &= 1(1(S))(0) \\
&= 1(S)(0) = S(0) = 1 \\
2 * 2 &= 2(2(S))(0) \\
&= 2(S)(2(S)(0)) = 2(S)(2) = 4
\end{aligned}
$$

Booleans

We represent true and false as *functions*

$$\text{true} := \lambda t.\, \lambda f.\, t$$
$$\text{false} := \lambda t.\, \lambda f.\, f$$

The bool is the choice

if-expression:

$$\text{if} := \lambda b.\, \lambda t.\, \lambda f.\, b\, t\, f$$

examples:

$$\begin{aligned}\text{if}(\text{true})(x)(y) &= \text{true}(x)(y) = x\\ \text{if}(\text{false})(x)(y) &= \text{false}(x)(y) = y\end{aligned}$$

the choice is built into the booleans themselves

Lists

A list is empty, or an element followed by a list

$$\mathrm{nil} := \lambda c.\lambda n.n$$
$$\mathrm{cons}(x, xs) := \lambda c.\lambda n.c\,x(xs\,c\,n)$$

Understanding reduce/fold

List with three elements

$$
\begin{aligned}
ourList &:= cons(1, cons(2, cons(3, nil))) \\
ourList(+, 0) &= 1 + cons(2, cons(3, nil))(+, 0) \\
&= 1 + 2 + cons(3, nil)(+, 0) \\
&= 1 + 2 + 3 + nil(+, 0) \\
&= 1 + 2 + 3 + 0
\end{aligned}
$$

```
[1,2,3].reduce(function (x,y) {return x + y},0)
```

A lesson from Church encodings

Thinking inductively $\Rightarrow$ modular code

What about control flow?

We've *almost* shown Turing completeness

Recursion

$$Y := \lambda f.(\lambda x.f(x\,x))(\lambda x.f(x\,x))$$

A simple proof it works

$$\begin{aligned} Y(g) &= (\lambda x.\, g(xx))(\lambda x.\, g(xx)) \\ &= g((\lambda x.\, g(xx))(\lambda x.\, g(xx))) \\ &= g(Y(g)) \end{aligned}$$

Sequencing code

$$l_1; l_2 \Rightarrow (\lambda x. l_2) l_1$$

Variable binding*

$$\text{let}\, x \;=\; v \,\text{in}\, M \Rightarrow (\lambda x.M)v$$

Global variable binding*

Easiest with variable hoisting

$$\mathrm{var}\, x \ = \ v; M \Rightarrow (\lambda x. M) v$$

Compilation as language design

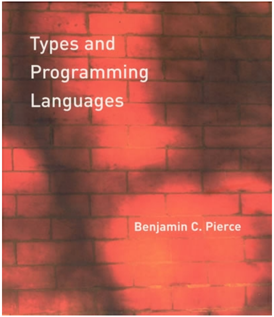
You can experiment with features via compilation between languages

λ : a common language

The λ calculus can be found inside many languages

λ : a PL toolkit

The common language of PL researchers



λ : a way to understand computation

Formal mathematical models let us get at the heart of computation

Questions

Any Questions?

Bonus slides

GUESS WE HAD MORE TIME!

What about *mutable variables*?

Mutable variables can be *simulated*

The secret origin of monads

Notions of Computation and Monads

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The λ-calculus is considered a useful mathematical tool in the study of programming languages, since programs can be *identified* with λ-terms. However, if one goes further and uses $\beta\eta$-conversion to prove equivalence of programs, then a gross simplification is introduced (programs are identified with total functions from *values* to *values*) that may jeopardise the applicability of theoretical results. In this paper we introduce calculi, based on a categorical semantics for *computations*, that provide a correct basis for proving equivalence of programs for a wide range of *notions of computation*. © 1991 Academic Press, Inc.

Closures + state

λ + state = everything