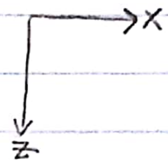
Simple 2D TPBVP

Y

Vehicle is well controlled by arbitrary CS inputs
No limit on CS or vehicle AoA
Constant Thrust, Don't consider aero forces

State Variables



$x = x_1 = x$ coord location (abs), meters
$z = x_2 = z$ coord location (abs), meters

$\dot{x} = x_3 = x$ vel (abs), m/s
$\dot{z} = x_4 = z$ vel (abs), m/s

Controls

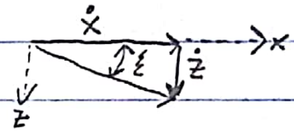
$u = \theta =$ missile pitch euler angle (not the same as AoA), deg

Cost Function

$$J = A\left(\tan^{-1}\left(\frac{x_4}{x_3}\right) - \xi\right)^2 + B\, t_f$$

A, B = user specified weights

ξ = greek tornado = desired impact angle relative to horizontal, deg



Dynamics

$\dot{x}_1 = x_3$

$\dot{x}_2 = x_4$

$\dot{x}_3 = \frac{m}{T} \cos\theta$

$\dot{x}_4 = -\frac{m}{T} \sin\theta + g$

m = missile mass, kg
T = missile thrust, N
g = gravity, m/s^2

Constraints

$x_1(t_f) = \mathbb{X}$

$x_2(t_f) = \mathbb{Z}$

$\mathbb{X}, \mathbb{Z}$ = target coordinates, constant, m

IC's

x_1, x_2, x_3, x_4 @ $t=0$ given

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Solution

$$\psi = \begin{bmatrix} x_1(t_f) - \mathbb{X} \\ x_2(t_f) - \mathbb{Z} \end{bmatrix}$$

$$G(x(t_f), t_f, \mu) = A\left(\tan^{-1}\left(\frac{x_4}{x_3}\right) - \xi\right)^2 + B t_f + \mu_1 (x_1(t_f) - \mathbb{X}) + \mu_2 (x_2(t_f) - \mathbb{Z})$$

$$\mathcal{L} = 0 \qquad = A\left[\left(\tan^{-1}\left(\frac{x_4}{x_3}\right)\right)^2 - 2\xi \tan^{-1}\left(\frac{x_4}{x_3}\right) + \xi^2\right] + \ldots\ldots$$

$$H(x(t), u(t), \lambda(t)) = \cancelto{0}{\mathcal{L}} + \lambda(t)^T f(x(t), u(t))$$

$$= \lambda_1(t)\, x_3(t) + \lambda_2(t)\, x_4(t) + \lambda_3(t) \frac{m}{T} \cos u + \lambda_4(t)\left(g - \frac{m}{T} \sin u\right)$$

$$\lambda(t_f) = \left[\frac{\partial G(x(t_f), t_f, \mu)}{\partial x(t_f)}\right]^T = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$1 = \frac{\partial G}{\partial x_1(t_f)} = \mu_1 \qquad 2 = \frac{\partial G}{\partial x_2(t_f)} = \mu_2$$

$$3 = \frac{\partial G}{\partial x_3(t_f)} = A\left[\frac{2\tan^{-1}\left(\frac{x_4(t_f)}{x_3(t_f)}\right)}{\left(\frac{x_4(t_f)}{x_3(t_f)}\right)^2 + 1}\left(-\frac{x_4(t_f)}{(x_3(t_f))^2}\right) + \frac{2\xi}{\left(\frac{x_4(t_f)}{x_3(t_f)}\right)^2 + 1}\left(+\frac{x_4(t_f)}{(x_3(t_f))^2}\right)\right]$$

$$= A\left[\left(2\xi - 2\tan^{-1}\left(\frac{x_4(t_f)}{x_3(t_f)}\right)\right)\left(\frac{1}{\left(\frac{x_4(t_f)}{x_3(t_f)}\right)^2 + 1} \frac{x_4(t_f)}{(x_3(t_f))^2}\right)\right]$$

$$= A \frac{2 x_4(t_f)}{x_4(t_f)^2 + x_3(t_f)^2}\left(\xi - \tan^{-1}\left(\frac{x_4(t_f)}{x_3(t_f)}\right)\right)$$

$$4 = \frac{\partial G}{\partial x_4(t_f)} = A\left[\frac{2\tan^{-1}\left(\frac{x_4(t_f)}{x_3(t_f)}\right)}{\left(\frac{x_4(t_f)}{x_3(t_f)}\right)^2 + 1} \frac{1}{x_3(t_f)} + \frac{2\xi}{\left(\frac{x_4(t_f)}{x_3(t_f)}\right)^2 + 1} \frac{1}{x_3(t_f)}\right]$$

$$= \frac{2A}{\frac{x_4(t_f)^2}{x_3(t_f)} + x_3(t_f)}\left(\xi + \tan^{-1}\left(\frac{x_4(t_f)}{x_3(t_f)}\right)\right)$$

Scanned by CamScanner

Simple 2D TPBVP Solution

7

$$\dot{\lambda}(t) = -\left[\frac{\partial H}{\partial x}\right]^T = \begin{bmatrix} 0 \\ 0 \\ -\lambda_1(t) \\ -\lambda_2(t) \end{bmatrix} \implies \lambda_1 + \lambda_2 \text{ are constant}$$

$$\therefore \lambda(t) = \lambda(t_f) + \int_{t_f}^{t} \dot{\lambda}(t)\,dt = \begin{bmatrix} \mu_1 \\ \mu_2 \\ ③ + \lambda_1(t_f - t) \\ ④ + \lambda_2(t_f - t) \end{bmatrix}$$

$$\frac{\partial H}{\partial u} = 0 = -\lambda_3(t)\cancel{\tfrac{m}{T}}\sin u - \lambda_4(t)\cancel{\tfrac{m}{T}}\cos u \implies \lambda_3(t)\sin u + \lambda_4(t)\cos u = 0$$

$$u = \tan^{-1}\left(\frac{-\lambda_4}{\lambda_3}\right)$$

$$\frac{\partial G(x(t_f), t_f, \mu)}{\partial t_f} + H\Big|_{t=t_f} = 0 = B + \lambda_1(t_f)x_3(t_f) + \lambda_2(t_f)x_4(t_f) + \lambda_3(t_f)\frac{m}{T}\cos u(t_f) + \lambda_4(t_f)\left(g - \frac{m}{T}\sin u(t_f)\right)$$

$\checkmark$

Convert $\tau = t/t_f$ for MATLAB solution

$$\dot{x}(t) = x(0) + \int_0^t \dot{x}(t)\,dt \qquad t = \tau t_f \qquad dt = t_f\,d\tau$$

$$\dot{x}(\tau) = x(0) + \int_0^{\tau} \dot{x}(\tau)\,d\tau$$

no $\dot{x}$ equations are functions of t so

$$\dot{x}(\tau) = t_f\,\dot{x}(t)$$

$$\dot{\lambda}(t) = \lambda(t_f) + \int_{t_f}^{t} \dot{\lambda}(t)\,dt$$

$$\dot{\lambda}(\tau) = \lambda(1) + \int_1^{\tau} \dot{\lambda}(\tau)\,d\tau \qquad \dot{\lambda}(\tau) = t_f\,\dot{\lambda}(t)$$