Optimal Control, Guidance and Estimation

Lecture – 18

Linear Optimal Missile Guidance using LQR

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Topics

- Basic Fundamentals
- Philosophy of PN Guidance
- Linear-Quadratic Optimal Missile Guidance
  - Proportional Navigation Guidance
  - Augmented PN (APN) Guidance
  - ZEM Derivation of PN Guidance
  - Aspect Angle Constrained Guidance
- Concluding Remarks

Basic Fundamentals

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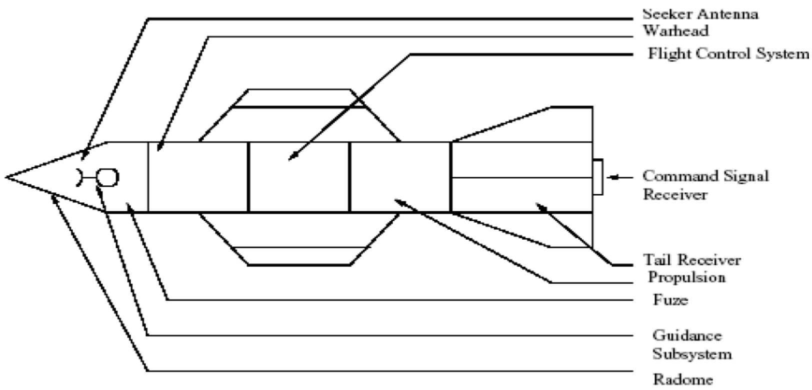
Classification of Missiles

- Surface-to-Surface
  - Strategic (large impact, but rarely used)
  - Tactical (limited & targeted impact, usually used)
- Surface-to-Air
- Air-to-Air
- Air-to-Surface

Classification of Missiles

- Tactical vs. Strategic
- Exo-Atmospheric vs. Endo-Atmospheric
- Ballistic vs. Cruise
- Long range vs. Short range
- Solid propulsion vs. Liquid propulsion

Components of a Missile



Reference: D. Ghose, Lecture Notes on Missile Guidance, Dept. of AE, IISc-Bangalore.

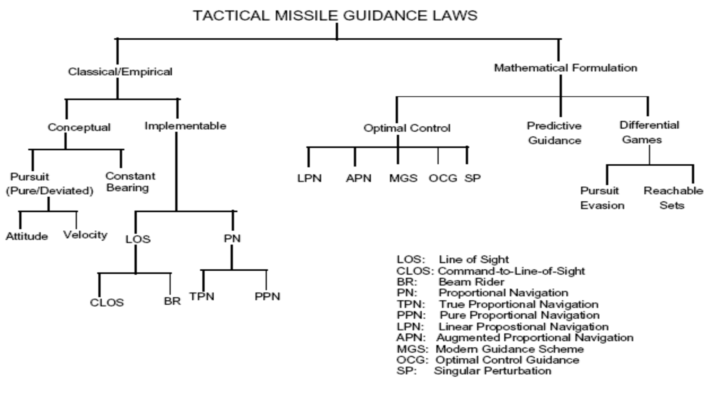
Mode of Operation

- Navigation
  - Determination of current position using sensors
- Guidance
  - Determination of a strategy to dictate a flight path (which will lead to successful interception)
- Control (Autopilot)
  - Determination of the required control forces and moments so as to enforce the missile to follow the guidance command.

What is a Guided Missile?

- A guided missile is a unmanned flight vehicle which is usually fired in a direction approximately towards the target and subsequently receives steering commands (either from onboard sensors or with external aid) to improve its accuracy

Tactical Missile Guidance Laws



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Flight Control System

- Objective: To deliver a flight vehicle from its current state to a desirable final state using the available information and available actuation, while satisfying all physical constraints on the vehicle
- Desired Terminal State
  - Rendezvous
  - Interception
  - Orbital parameters

Flight Control Systems

- Available information
  - Own state vector
  - Terminal position (for stationary targets)
  - Instantaneous relative position and velocity (for moving targets): Need to estimate where the target is likely to be at the final time
- Actuators
  - Aerodynamic: Fins, Canards, Spoilers
  - Thrust vector control (engine swiveling, side-injection, nozzle deflection)
  - Reaction jets (Cold/Hot RCS)
  - Divert thrusters (e.g. Vernier thrusters)

Conventional Flight Control Systems

- Outer loop guidance (one loop) with inner loop autopilot (two or more loops)
  - Developed in 1950s
  - Accounts for limited onboard computational capability
  - Limited analysis and design tools are adequate
  - Lower expectation on the vehicle performance
  - Adequate for stationary and non-maneuvering (or less maneuvering) targets: True for earlier targets
  - **Simpler guidance laws (PN, Pursuit etc.)**

Objectives and Mechanics of Missile Guidance

- Objective: To control the position and velocity vectors to cause interception, subject to constraints

- Mechanics of Guidance:
  - Manipulate the position through changes in the velocity vector
  - Control velocity vector magnitude either by manipulating the thrust (if possible) or by minimizing the drag
  - Control velocity vector magnitude by applying forces normal to the velocity vector

Difficulties of Missile Guidance

- Target related information is not readily available: Target state estimation is a must (unless it is a stationary target)
- Mechanisms for applying forces normal to the velocity vector
  - Direct lateral thrust (divert thrusters)
  - Lateral components of main thrust vectors
  - Aerodynamic forces: Angle of attack & Side-slip angle (skid-to-turn) or Angle of attack & Bank angle (bank-to-turn)

Difficulties of Missile Guidance

- Undesirable consequences of applied lateral forces
  - Lateral force components not at the CG of the vehicle: Undesirable moments
  - Non-minimum phase behaviour
  - Vehicles are designed aerodynamically unstable (for better L/D ratio and better maneuverability). Hence, motion arising from maneuver forces may destabilize the vehicle
  - Large lateral acceleration command is a sad reality, where as Rapid command response is a bad necessity.

Applications of Guidance Theory

- Missile guidance
- Torpedo guidance
- Robotics
- Smart cars (autonomous driving)
- Collision avoidance
- Formation flying
- Automatic landing
- Spacecraft docking
- Air-to-air refueling

Fundamental Problem of Tactical Missile Guidance

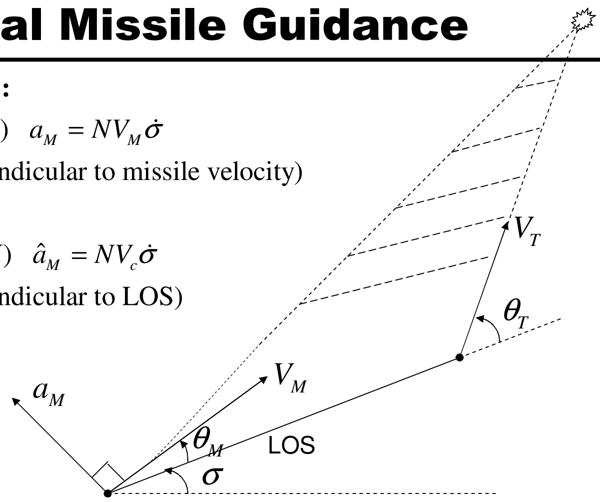
PN Guidance :

Pure PN (PPN) $a_M = NV_M\dot{\sigma}$

(applied perpendicular to missile velocity)

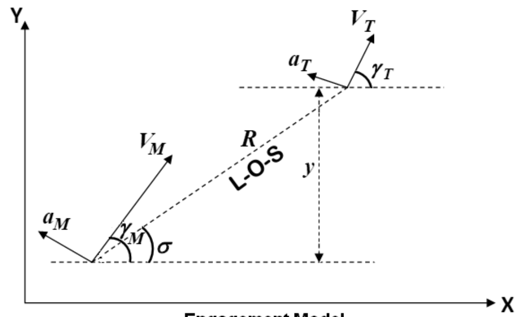
True PN (TPN) $\hat{a}_M = NV_c\dot{\sigma}$

(applied perpendicular to LOS)



PPN implementation

Engagement Model in 2-D



Engagement Model

$$R \cong V_c\left(t_f - t\right)$$
$$V_c \cong V_M - V_T$$

For small angles, $\sigma \approx \sin\sigma = \dfrac{y}{R}$

Optimal Missile Guidance

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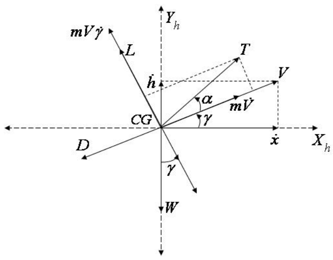
Motivation for Optimal Missile Guidance

- Many classical missile guidance laws are inspired from “observing nature”
  (e.g. Proportional Navigation (PN) guidance law is based on ensuring “collision triangle”)
- Control theoretic based guidance laws are usually based on “kinematics” and/or “linearized dynamics”. Hence, they are usually not very effective!
- Optimal control theory is a “natural tool” to obtain effective missile guidance laws

2-D Point-mass Model with Flat and Non-Rotating Earth



$$\dot{x} = V\cos\gamma$$

$$\dot{h} = V\sin\gamma$$

$$\dot{V} = \frac{1}{m}\left(T\cos\alpha - D - mg\sin\gamma\right)$$

$$\dot{\gamma} = \frac{1}{mV}\left(T\sin\alpha + L - mg\cos\gamma\right)$$

2-D Optimal Guidance Laws using LQR

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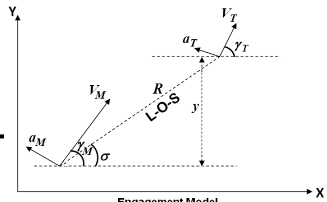
Engagement in 2-D



Engagement Model

$$\ddot{y} = a_T \cos\gamma_T - a_M \cos\gamma_M$$

For small flight path angle (i.e. $\gamma_T = \gamma_T \approx 0$)

$$\ddot{y} = a_T - a_M$$

where, $a_T -$ Target Acceleration

$a_M -$ Missile Acceleration

$$V_c \cong V_M - V_T,$$

where, $V_c -$ Closing Velocity

$V_M -$ Missile Velocity

$V_T -$ Target Velocity

$$R \cong V_c \left(t_f - t\right)$$

True PN Guidance Law using LQR

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Problem Formulation – 1
(Assumption: Target acceleration = 0)

System Dynamics:

$$\dot{y} = v$$
$$\dot{v} = a_M$$

Assumption : a_T is constant (i.e. circular target motion)

Performance Index: $J = \dfrac{1}{2}\int_0^{t_f} a_M^2 \, dt$

Boundary condition: Initial State: $y_0,\ v_0$ (Given)
Final State: $y_f = 0,\quad v_f$ is free

Optimality Conditions

Hamiltonian : $H = \frac{1}{2} a_M^2 + \lambda_1 v + \lambda_2 a_M$

Optimality Conditions :

State Equation : $\dot{y} = v$

$$\dot{v} = a_M$$

Optimal Control Equation :

$$\left(\frac{\partial H}{\partial a_M} \right) = a_M + \lambda_2 = 0$$

$$a_M = -\lambda_2$$

Optimality Conditions

Costate Equation : $\dot{\lambda}_1 = -\left(\frac{\partial H}{\partial y} \right) = 0$

$$\lambda_1(t) = c_1$$

$$\dot{\lambda}_2 = -\left(\frac{\partial H}{\partial v} \right) = -\lambda_1 = -c_1$$

$$\lambda_2(t) = -c_1 t + c_2$$

Transversality Condition :

$$\lambda_2(t_f) = -c_1 t_f + c_2 = 0$$

$$c_2 = c_1 t_f$$

Optimality Control

Optimal Control: $a_M = -\lambda_2 = -c_1\left(t_f - t\right)$

Q: How to find c_1? First, find v and apply boundary condition.

As $\dot{v} = a_M \quad \Rightarrow \quad \dot{v} = -c_1\left(t_f - t\right)$

After integrating,

$$v = k_v - c_1\left(t_f t - \frac{t^2}{2}\right)$$

At $t = 0,\ v = v_0 = k_v$

Hence, $v = v_0 - c_1\left(t_f t - \frac{t^2}{2}\right)$

Optimal Guidance Law

Next, find y and apply boundary condition.

$$\dot{y} = v = v_0 - c_1\left(t_f t - \frac{t^2}{2}\right)$$

After integrating

$$y = k_y + v_0 t - c_1\left(\frac{t_f t^2}{2} - \frac{t^3}{6}\right)$$

At $t = 0,\ y = y_0 = k_y$

At $t = t_f,\ y_f = 0$ (zero miss distance)

$$\left[y_0 + v_0 t_f - \frac{c_1 t_f^3}{3}\right] = 0 \quad \Rightarrow \quad c_1 = \frac{3}{t_f^3}\left[y_0 + v_0 t_f\right]$$

Optimal Guidance Law

Optimal Guidance Law $\qquad a_M = -c_1\left(t_f - t\right)$

Time-to-go: $T = t_{go} = \left(t_f - t\right)$

Assumption: Initial time = Current time $t = 0$

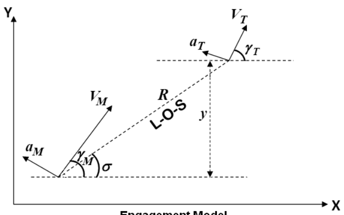
$$T \triangleq t_{go}\Big|_{t=0} = t_f$$

$$a_M\Big|_{t=0} = -c_1 T = \frac{-3}{T^3}\left[y_0 + v_0 T\right]T$$

$$= \frac{-3}{T^2}\left[y_0 + v_0 T\right]$$

Optimal Guidance

Proportional Navigation (PN)



Engagement Model

$$a_M = \frac{-3}{T^2}\left[y_0 + v_0 T\right]$$

$$= -3\left[\frac{y}{T^2} + \frac{v}{T}\right]$$

$$= -3V_c\dot{\sigma}$$

$$= a_M^{PN}$$

$$\sigma \cong \frac{y}{R} = \frac{y}{V_c\left(t_f - t\right)}$$

$$\dot{\sigma} = \frac{V_c\left(t_f - t\right)\dot{y} - y\left(-V_c\right)}{\left[V_c\left(t_f - t\right)\right]^2} = \frac{Tv + y}{V_c T^2}$$

$$V_c\dot{\sigma} = \frac{y}{T^2} + \frac{v}{T}$$

Note :

It does not requires estimation of Target acceleration.

Augmented PN Guidance Law using LQR

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Problem Formulation – 2
(Assumption: Target acceleration can be estimated)

System Dynamics:

$$\dot{y} = v$$
$$\dot{v} = a_M - a_T$$

Assumption : a_T is constant (i.e. circular target motion)

Performance Index: $J = \dfrac{1}{2}\int_0^{t_f} a_M^2 \, dt$

Boundary condition:

Initial State: $y_0,\ v_0$ (Given)

Final State: $y_f = 0,\quad v_f$ is free

Optimality Conditions

Hamiltonian : $H = \frac{1}{2}a_M^2 + \lambda_1 v + \lambda_2 (a_M - a_T)$

Optimality Conditions :

State Equation :

$$\dot{y} = v$$

$$\dot{v} = a_M - a_T$$

Optimal Control Equation :

$$\left(\frac{\partial H}{\partial a_M}\right) = a_M + \lambda_2 = 0$$

$$a_M = -\lambda_2$$

Optimality Conditions

Costate Equation :

$$\dot{\lambda}_1 = -\left(\frac{\partial H}{\partial y}\right) = 0$$

$$\lambda_1(t) = c_1$$

$$\dot{\lambda}_2 = -\left(\frac{\partial H}{\partial v}\right) = -\lambda_1 = -c_1$$

$$\lambda_2(t) = -c_1 t + c_2$$

Transversality Condition :

$$\lambda_2(t_f) = -c_1 t_f + c_2 = 0$$

$$c_2 = c_1 t_f$$

Optimality Control

Optimal Control: $a_M = -\lambda_2 = -c_1(t_f - t)$

Q: How to find c_1? First, find v and apply boundary condition.

As $\dot{v} = a_M - a_T \quad \Rightarrow \quad \dot{v} = -c_1(t_f - t) - a_T$

After integrating,

$$v = k_v - c_1\left(t_f t - \frac{t^2}{2}\right) - a_T t$$

At $t = 0,\ v = v_0 = k_v$

Hence, $v = v_0 - c_1\left(t_f t - \frac{t^2}{2}\right) - a_T t$

Optimal Guidance Law

Next, find y and apply boundary condition.

$$\dot{y} = v = v_0 - c_1\left(t_f t - \frac{t^2}{2}\right) - a_T t$$

After integrating

$$y = k_y + v_0 t - c_1\left(\frac{t_f t^2}{2} - \frac{t^3}{6}\right) - a_T \frac{t^2}{2}$$

At $t = 0,\ y = y_0 = k_y$

At $t = t_f,\ y_f = 0$ (zero miss distance)

$$\left[y_0 + v_0 t_f - \frac{c_1 t_f^3}{3} - a_T \frac{t_f^2}{2}\right] = 0 \quad \Rightarrow \quad c_1 = \frac{3}{t_f^3}\left[y_0 + v_0 t_f - a_T \frac{t_f^2}{2}\right]$$

Optimal Guidance Law

Optimal Guidance Law $a_M = -c_1\left(t_f - t\right)$

Time-to-go: $T = t_{go} = \left(t_f - t\right)$

Assumption: Initial time = Current time $t = 0$

$$T \triangleq t_{go}\Big|_{t=0} = t_f$$

$$a_M\Big|_{t=0} = -c_1 T = \frac{-3}{T^3}\left[y_0 + v_0 T - a_T \frac{T^2}{2} \right] T$$

$$= \frac{-3}{T^2}\left[y_0 + v_0 T - a_T \frac{T^2}{2} \right]$$

Optimal Guidance: Augmented PN (APN)

$$a_M^{APN} = \frac{-3}{T^2}\left[y_0 + v_0 T - a_T \frac{T^2}{2} \right]$$

$$= \frac{-3}{T^2}\left[y_0 + v_0 T \right] + \frac{3}{2} a_T$$

$$= a_M^{PN} + \frac{3}{2} a_T$$

Note : It requires good estimation of Target acceleration. Else, performance can be inferior to PN.

Optimal Guidance Law: ZEM Derivation

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ZEM Derivative

Zero - Effort - Miss (ZEM) :

ZEM is the distance that the missile will miss the target, if there is no corrective measure.

Definition of ZEM (Z):

$$Z = y_f - y_0 - \dot{y}_0 t_{go}$$

where $t_{go} = \left(t_f - t\right)$

ZEM Derivative

ZEM Rate:

$$\dot{Z} = \cancel{\dot{y}_f} - \dot{y}_0 - \left(\ddot{y}_0 t_{go} + \dot{y}_0 \dot{t}_{go}\right)$$

$$= \cancel{-\dot{y}_0} - \ddot{y}_0 t_{go} \cancel{+\dot{y}_0}$$

where, $\dot{y}_f = 0, \quad \dot{t}_{go} = -1$

$$= -\ddot{y}_0 t_{go}$$

$$\boxed{\dot{Z} = -a_M t_{go}} \qquad (\text{since } a_M = \ddot{y})$$

Problem Formulation

Minimize : $J = \dfrac{1}{2}\int_0^{t_f} a_M^2 \, dt$

Subject to $\dot{Z} = -a_M t_{go}$

Hamiltonian : $H = \dfrac{1}{2} a_M^2 + \lambda\left(-a_M t_{go}\right)$

Optimality Condition :

Costate Equation : $\dot{\lambda} = 0$

So $\lambda = c$

Optimality Condition

Optimal Control Equation : $$\left(\frac{\partial H}{\partial a_M}\right) = a_M - t_{go}\lambda = 0$$

$$a_M = t_{go}\lambda$$

Zero-Effort-Miss

$$\dot{Z} = -a_M t_{go} = -t_{go}^2\lambda$$

$$Z = k_Z - \frac{t_{go}^3}{3}\lambda$$

Optimal Guidance Law

Initial Boundary Condition:

When $t_{go} = t_{go}(0),\ Z = Z_0 \quad \Rightarrow \quad k_Z = Z_0$

Hence $Z = Z_0 - \dfrac{t_{go}^3}{3}\lambda$

Initial Boundary Condition:

When $t_{go} = 0,\ Z = 0$

Hence $0 = Z_0 - \dfrac{t_{go}^3}{3}\lambda \quad \Rightarrow \quad \lambda = \dfrac{3}{t_{go}^3} Z_0$

Optimal Guidance Law: $a_M = t_{go}\lambda = 3\left[\dfrac{Z_0}{t_{go}^2}\right]$

Optimal Guidance Law with Terminal Aspect Angle Constraint

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Problem Formulation – 1
(Assumption: Target acceleration = 0)

System Dynamics:

$$\dot{y} = v$$
$$\dot{v} = a_M$$

Assumption : a_T is constant (i.e. circular target motion)

Performance Index: $J = \dfrac{1}{2}\int_0^{t_f} a_M^2 \, dt$

Boundary condition:

Initial State: $y_0,\ v_0$ (Given)

Final State: $y_f = 0,\ \ v(t_f) = V \tan\mu = v_f$

Optimality Conditions

Hamiltonian : $H = \frac{1}{2} a_M^2 + \lambda_1 v + \lambda_2 a_M$

Optimality Conditions :

State Equation : $\dot{y} = v$

$$\dot{v} = a_M$$

Optimal Control Equation :

$$\left(\frac{\partial H}{\partial a_M} \right) = a_M + \lambda_2 = 0$$

$$a_M = -\lambda_2$$

Optimality Conditions

Costate Equation : $\dot{\lambda}_1 = -\left(\frac{\partial H}{\partial y} \right) = 0$

$$\lambda_1(t) = c_1$$

$$\dot{\lambda}_2 = -\left(\frac{\partial H}{\partial v} \right) = -\lambda_1 = -c_1$$

$$\lambda_2(t) = -c_1 t + c_2$$

Optimal Control Equation :

$$a_M = -\lambda_2(t) = c_1 t - c_2$$

Optimal Guidance Law

$$\dot{v} = a_M = c_1 t - c_2$$

$$v = k_V + \frac{c_1 t^2}{2} - c_2 t$$

At $t = 0, \quad v = v_0 \quad \Rightarrow \quad k_V = v_0$

$$v = v_0 + \frac{c_1 t^2}{2} - c_2 t$$

Optimal Guidance Law

$$\dot{y} = v = v_0 + \frac{c_1 t^2}{2} - c_2 t$$

After integrating,

$$y = k_y + v_0 t + \frac{c_1 t^3}{6} - \frac{c_2 t^2}{2}$$

At $t = 0, \; y = y_0 \quad \Rightarrow \quad k_y = y_0$

$$y = y_0 + v_0 t + \frac{c_1 t^3}{6} - \frac{c_2 t^2}{2}$$

Optimal Guidance Law

Terminal Boundary Condition:

At $t = t_f$,

$y(t_f) = 0, \;\; v(t_f) = v_f$

This gives:

$$0 = y_0 + v_0 t_f + \frac{c_1 t_f^3}{6} - \frac{c_2 t_f^2}{2}$$

$$v_f = v_0 + \frac{c_1 t_f^2}{2} - c_2 t_f$$

Solving for c_1 and c_2,

$$c_1 = \frac{12}{t_f^3}\left(y_0 + v_0 t_f \right) - \frac{6}{t_f^2}\left(v_0 - v_f \right)$$

$$c_2 = \frac{6}{t_f^2}\left(y_0 + v_0 t_f \right) - \frac{2}{t_f}\left(v_0 - v_f \right)$$

Optimal Guidance Law

Time-to-go:

$T = t_{go} = \left(t_f - t \right)$

With $t = 0$ (i.e. current time is the initial time)

$t_{go} = t_f$

Hence,

$$a = -c_2 = \underbrace{-\frac{6}{t_{go}^2}\left(y_0 + v_0 t_{go} \right)}_{\text{Terminal Miss Component}} + \underbrace{\frac{2}{t_{go}}\left(v_0 - v_f \right)}_{\text{Aspect angle component}}$$

Concluding Remarks

- Missile guidance is a fascinating field
- The guidance concepts can be used for several applications, including collision avoidance
- Linear-Quadratic optimal control theory offers a good platform for optimal guidance, leading to closed form expressions
- Various time-to-go estimation ideas exists in the literature, which can be incorporated.
- Extension to nonlinear formulations are topics of current research: Leads to numerical solutions

Thanks for the Attention...!