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Optimal Stopping and Hard Terminal Constraints Applied to a Missile Guidance Problem

Jason J. Ford* and **Peter M. Dower**[†]

*Weapons Systems Division
Defence Science and Technology Organisation,
and The School of Information Technology and Electrical Engineering,
Australian Defence Force Academy.
e-mail: Jason.Ford@dsto.defence.gov.au

[†] Department of Electrical and Electronic Engineering,
The University of Melbourne.
e-mail: p.dower@ee.unimelb.edu.au

Abstract

This paper describes two new types of deterministic optimal stopping control problems: optimal stopping control with hard terminal constraints only and optimal stopping control with both minimum control effort and hard terminal constraints. Both problems are initially formulated in continuous-time (a discrete-time formulation is given towards the end of the paper) and solutions given via dynamic programming. A numeric solution to the continuous-time dynamic programming equations is then briefly discussed.

The optimal stopping with terminal constraints problem in continuous-time is a natural description of a particular type of missile guidance problem. This missile guidance application is introduced and the presented solutions used in missile engagements against targets.

1 Introduction

Techniques for design of control system (whether optimal or robust) have typically involved integral-type (or soft constraint) performance criteria on control actions [14]. While these types of criteria are suitable in many situations, in some applications, it is important that hard constraints on the terminal performance of the system be met.

In this paper we consider a related problem posed in an optimal stopping setting, where the objective is to design both a control action sequence and a optimal stopping time that ensures that the optimal stopped terminal performance is less than some specified performance level. This problem is related to the (hard constraint) l^{∞} optimisation problem considered in [14]. Motivation for this problem is provided by the need to consider a robust version of the missile guidance problem.

Guidance is the term used to describe the process of determining the desired engagement trajectory for an intercepting missile against a target. These trajectories are typically designed to ensure some predetermined performance requirements are achieved. Historically, these performance requirements have placed less importance on mid-course performance compared to terminal properties of the engagement. Furthermore, in most applications, the time taken for interception, as long as interception occurs, is considered of much less importance.

In new emerging guidance applications, the achieved terminal properties have become so critical that they can now be characterised as hard constraints. In these applications, if these hard terminal performance requirements are not achieved the engagement is considered a failure. These types of guidance problems are naturally suited to an optimal stopping problem formulation with terminal hard constraints.

This paper is organised as follows: In Section 2 a continuous-time dynamic model is introduced and two types of hard terminal constraints problems are presented. In Section 3 dynamic programming solutions are provided for both types of hard terminal constraint problems. In Section 4, one of the hard terminal constraint controllers is then applied to a missile guidance problem. A numeric approximation of the optimal controller and guaranteed performance level sets are presented. In Section 5, the equivalent discrete-time problem is introduced and dynamic programming solutions are given. Finally, Section 6 provides some brief concluding remarks.

2 Dynamics and Control Objectives

Consider the following nonlinear continuous-time dynamical system defined for $t \in R^+$:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), w(t)) \\ z(t) &= g(x(t)) \end{aligned} \qquad (1)$$

where $x(t) \in R^n$, $u(t) \in R^m$, $w(t) \in R^p$ and $z(t) \in R^q$ are the state, control input, disturbance input and performance output quantity, respectively.

We assume that $u(t) \in U$ and $w(t) \in W$ are non-anticipating maps of the state, where U and W are compact bounded sets of the admissible controls and disturbances.

We consider the finite stopping time horizon $[0, \tau]$ for some (non-anticipating) $\tau \leq T$, with T fixed, and assume the following:

1. $x(\ell) \in \mathcal{X}$ for all $\ell \in [0, \tau]$ for some bounded set $\mathcal{X}$.
2. $u(\ell) \in U$ for all $\ell \in [0, \tau]$.
3. $w(\ell) \in W$ for all $\ell \in [0, \tau]$.
4. $\sup_{x \in \mathcal{B}} g(x) < \infty$ for any compact set $\mathcal{B}$.
5. $\sup_{x \in \mathcal{B}} -g(x) < \infty$ for any compact set $\mathcal{B}$.
6. there exists $x \in \mathcal{X}$ such that $g(x) < \lambda$, where λ is some given required performance level.

We employ the following notation: $\mathcal{U}_{0,\tau}$ is the set of admissible control sequences on $[0, \tau]$ in which $u(\ell) \in \mathcal{U}$ for all $\ell \in [0, \tau]$. Similarly define $\mathcal{W}_{0,\tau}$. We take the infimum over a empty set to be equal to ∞.

The above definitions and following control problems are motivatived by the trajectory design problem in which the dynamics must be controlled, in finite time, to a finite set. An example of a suitable performance output quantity for these problems is $g(x) = |x|$.

2.1 Objective 1: Hard Terminal Stopping Constraint

The objective of the hard terminal stopping constraint problem is to design a causal state feedback control $u \in \mathcal{K}_{st}$ such that, given a fixed stopping time constraint $T < \infty$ and any initial state $x_\circ \in \mathcal{X}$, there exists a stopping time $\tau(x_\circ, u) \in [0, T]$ such that

$$z(\tau(x_\circ, u)) \leq \lambda, \qquad (2)$$

where system (1) is initialised at $x_\circ \in \mathcal{X}$. This problem appears somewhat related to the l^∞ bounded problem introduced in [14].

If more that one control sequence (and stopping time) achieves this hard constraint, then minimise the achieved terminal performance. That is, select u^* and τ^* where

$$z(\tau^*(x_\circ, u^*)) \leq z(\tau(x_\circ, u)) \text{ for all } u \in \mathcal{K}_{st}, \tau \in [0, T]$$

Remark: It is also possible to consider a slightly modified problem, where if more than one control sequence (and stopping time) achieves this hard constraint, then minimise the stopping time. That is,

$$\tau^*(x_\circ) = \inf\{\tau(x_\circ) : z(\tau(x_\circ)) \leq \lambda\}$$

with an associated controller.

2.2 Objective 2: Hard Terminal Stopping Constraint with Minimum Control

The objective of the hard terminal stopping constraint and minimum control problem is to design a causal state feedback control $\bar{\mathcal{K}}_{st}$ such that, given a fixed stopping time constraint $T < \infty$ and any initial state $x_\circ \in \mathcal{X}$, there exists a $\tau_T^*(x_\circ) \in [0, T]$ such that

$$z(\tau_T^*(x_\circ)) \leq \lambda, \qquad (3)$$

where system (1) is initialised at $x_\circ \in \mathcal{X}$ and

$$\int_0^{\tau_T^*(x_\circ)} \gamma(|u(t)|) dt \quad \text{is minimised.} \qquad (4)$$

We assume that $\gamma \in \mathcal{K}_\infty$. That is, $\gamma : \mathcal{R} \to \mathcal{R}_{\geq 0}$ is continuous, strictly increasing, and satisfies $\gamma(0) = 0$.

3 Dynamic Programming Solutions

In this section we solve both introduced hard terminal constraint problems using a dynamic programming approach.

3.1 Terminal Stopping Constraint Only

Let us consider the following cost function:

$$J(x_0, u, w, \tau) = g(x_{x_0,u,w}(\tau))$$

where $x_{x_0,u,w}(\tau)$ will denote the solution at time τ of (1) initialised at x_0 with input sequences u and w. We will use the shorthand $x(\tau)$ in the following and assume the meaning is clear from context.

The optimal control problem considered here is to design a stopping time, τ, and a control sequence, u, to minimise the worst cost $J(x_0, u, w, \tau)$ against disturbance inputs w.

If $J(x_0, u^*, w, \tau^*) \leq \lambda$ for all w then this optimal choice of τ^* and u^* is considered to be a candidate solution to the hard terminal constraint problem introduced in the previous section.

Let us introduce the following value function:

$$v_T(t,x) = \inf_{\tau\in[t,T]} \inf_{u\in\mathcal{U}_{t,\tau}} \sup_{w\in\mathcal{W}_{t,\tau}} g(x(\tau)). \tag{5}$$

The motivation for considering this value function is that if $v_T(0,x_0) > \lambda$, then no control sequence u can be designed that meets the hard terminal constraint for all disturbances.

The dynamic programming equation for $v_T(t,x)$ (in a viscosity sense [12, 11]) is the following variational inequality:

$$\max\left(v_T(t,x) - g(x), -\frac{\partial v_T}{\partial t} - H(x, Dv_T(t,x))\right) = 0 \tag{6}$$

with $v_T(T,x) = g(x)$. Here $H(x,p)$ is the Hamiltonian given by

$$H(x,p) = \inf_{u\in U} \sup_{w\in W} p.f(x,u,w) \tag{7}$$

and

$$Dv_T(t,x) = \left(\frac{\partial v_T}{\partial x_1}, \ldots, \frac{\partial v_T}{\partial x_n}\right)'. \tag{8}$$

Remark:

1. The partial differential equation (PDE) (6) may not have solutions in a classical sense (smooth). In general, non-smooth solutions to (6) have to be understood in a generalised sense such as viscosity solutions [11].
2. In practical applications, numeric solutions to the PDE (6) can be obtained using numeric techniques such as the Markov chain approximation approach described in [10].
3. When $f(x,u,w)$ is separable in u and w and not lower bounded in u or upper bounded in w then bang-bang optimal controls result. We will see this in a later example.

3.1.1 Optimal Stopping and Optimal Control: The optimal stopping rule for this terminal constraint problem can be expressed as

$$\begin{cases} \text{if } v_T(t,x) \geq g(x) & \text{stop} \\ \text{if } v_T(t,x) < g(x) & \text{continue.} \end{cases} \tag{9}$$

An example interpretation is that if $v_T(t,x) \geq g(x)$ then the optimal terminal constraint has been obtained. Otherwise the dynamics should continue. If continuing, the optimal control action is the minimising u in (7). This control action will be in a (time-varying) state feedback form.

To complete the optimal control solution a verification theorem is required. This is not done here, but readers are refered to [11] where verification results for similar problems are presented.

3.1.2 Feasible Initial Conditions: Let the set of feasible initial states for a particular stopping terminal constraint level λ be denoted by

$$\mathcal{S}_{\lambda,T} = \{x_0 : v_T(0,x_0) \leq \lambda\}. \tag{10}$$

3.2 Terminal Constraints and Minimum Control Action

Let us introduce the following indicator function for the terminal constraint:

$$\delta_\lambda(x) = \begin{cases} 0 & \text{if } g(x) \leq \lambda \\ \infty & \text{otherwise} \end{cases}$$

Then introduce the following value function:

$$\begin{aligned}\bar{v}_T(t,x) &= \inf_{\tau\in[t,T]} \inf_{u\in\mathcal{U}_{\tau,T}} \sup_{w\in\mathcal{W}_{\tau,T}} \\ &\left\{\int_0^T \gamma(|u(t)|)dt + \delta_\lambda(x(\tau))\right\}\end{aligned} \tag{11}$$

The motivation for this choice of value function is that if $\bar{v}_T(0,x_0) < \infty$ (ie. $x_o \in S_{\lambda,T}$) then the hard terminal constraint can be satisfied for some choice of $\tau \in [0,T]$ and $u \in \mathcal{U}_{0,\tau}$.

In a manner parallel to above, $\bar{v}_T(t,x)$ is the solution to the following variational inequality:

$$\max\left(\bar{v}_T(t,x) - \delta_\lambda(x), -\frac{\partial \bar{v}_T}{\partial t} - \bar{H}(x, D\bar{v}_T(t,x))\right) = 0 \tag{12}$$

with $\bar{v}_T(T,x) = \delta_\lambda(x)$. Here $\bar{H}(x,p)$ is an Hamiltonian given by

$$\bar{H}(x,p) = \inf_{u\in U} \sup_{w\in W} \{p.f(x,u,w) + \gamma(|u|)\} \tag{13}$$

3.2.1 Optimal Stopping and Optimal Control: The optimal stopping rule for minimum cost and terminal constraint problem can be expressed, assuming $\bar{v}_T(t,x)$ is finite, as

$$\begin{cases} \text{if } g(x) \leq \lambda & \text{stop} \\ \text{if } g(x) > \lambda & \text{continue.} \end{cases} \tag{14}$$

If continuing, the optimal control action is the minimising u in (13). This control action will be in a (time-varying) state feedback form.

3.2.2 Feasible Initial Conditions: Let the set of feasible initial states for a particular stopping terminal constraint level λ be denoted by

$$\bar{\mathcal{S}}_{\lambda,T} = \{x_0 : \bar{v}_T(0,x_0) < \infty\}. \tag{15}$$

4 Application: Missile Guidance

In this section we consider a missile guidance problem as a optimal stopping hard terminal constraint problem. Problems with and without target disturbance inputs will be considered to highlight several features of the hard terminal constraint optimal stopping problem.

4.1 Engagement Dynamics

Consider the following non-linear continuous time state-space model defined for $t \in R^+$ (see Figure 1):

$$\begin{aligned} \dot{r}(t) &= V_m\left[\rho\cos(\theta_t(t)) - \cos(\theta_m(t))\right], \\ r(t)\dot{\sigma}(t) &= V_m\left[\rho\sin(\theta_t(t)) - \sin(\theta_m(t))\right], \\ \dot{\gamma}_m(t) &= u(t), \\ \dot{\gamma}_t(t) &= w(t), \end{aligned} \tag{16}$$

where (without loss of generality) $r(t) > \epsilon > 0$ for ϵ a known small constant and angles are measured in a counter-clockwise direction. Here V_m is the forward velocity of the interceptor missile at angle γ_m and $0 \leq \rho < 1$ is defined so that ρV_m is the forward velocity of the target at angle γ_t. The angle σ is called the line of sight angle and we define two angles relative to the line-of-sight as $\theta_m = \gamma_m - \sigma$ and $\theta_t = \gamma_t - \sigma$. In a general sense, the objective of the missile guidance problem is to drive $r(\tau) \to 0$ for some $\tau \in [0, T]$.

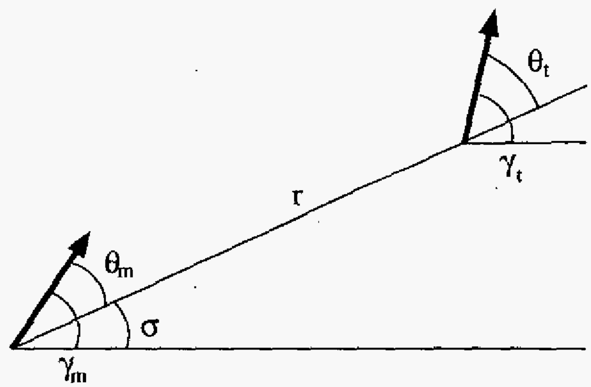

Figure 1: Geometry in two dimensions

We assume that $u(t) \in U$ (control of interceptor via a commanded turn rate) is a compact set of admissible controls. That is, the interceptor control action is assumed to be perpendicular to it's body axis (no thrust vector control). Likewise we assume that $w(t) \in W$ (disturbance behaviour of the target via a commanded turn rate) is a compact set of admissible disturbances.

Sometimes, a reduced order model with normalised range and θ_t, θ_m states is useful, so let $\bar{r}(t) = r(t)/V_m$. After some substitutions we note that the dynamics can be written as:

$$\begin{aligned} \dot{\bar{r}}(t) &= \left[\rho\cos(\theta_t(t)) - \cos(\theta_m(t))\right], \quad \bar{r}(t) > \bar{\epsilon} > 0 \\ \dot{\theta}_t(t) &= w - \frac{1}{\bar{r}(t)}\left[\rho\sin(\theta_t(t)) - \sin(\theta_m(t))\right] \\ \dot{\theta}_m(t) &= u - \frac{1}{\bar{r}(t)}\left[\rho\sin(\theta_t(t)) - \sin(\theta_m(t))\right] \end{aligned} \tag{17}$$

where $\bar{\epsilon}$ is a known small constant.

Let us denote the state of the reduced order model for the missile guidance problem as $x(t) = [\bar{r}(t)\ \theta_t(t)\ \theta_m(t)]'$. Also, let $f^G(x, u, w)$ denote the dynamics described in (17) so that $\dot{x} = f^G(x, u, w)$.

4.2 Dynamic Programming Equations for Missile Guidance

In this application we will consider a missile performance index with only a terminal constraint. Consider the following cost function:

$$J_T^G(x_0, u, w, \tau) = g(x(\tau)).$$

where $g(x) = |\bar{r}|$. This choice of terminal constraint is motivated by the requirement to achieve a hard constraint on the terminal range.

To design the optimal stopping time and control sequence we utilise the value function (5).

This value function can be solved using (6).

4.2.1 Optimal Control Action:

The control and disturbance are separated in the dynamics and the cost, so a Issac's condition can be established for this problem (see [4, 5, 6]). From (7) and (17) it is then clear that the optimal solution is bang-bang, see [15] for discussion of bang-bang control. Let u^* denote the optimal control action. Then

$$u^* = \begin{cases} \min(U) & \text{if } p_3 > 0, \\ \max(U) & \text{if } p_3 < 0, \\ \text{otherwise} & \text{any } u \in U. \end{cases} \tag{18}$$

where $p_3 = \frac{\partial v_T^G(t,x)}{d\theta_m}$.

Let w^* denote the optimal disturbance action (worst case). Then

$$w^* = \begin{cases} \min(W) & \text{if } p_2 < 0, \\ \max(W) & \text{if } p_2 > 0, \\ \text{otherwise} & \text{any } w \in W. \end{cases} \tag{19}$$

where $p_2 = \frac{\partial v_T^G(t,x)}{d\theta_t}$.

4.3 Numeric Approach:

For this guidance problem, the optimal guidance solution can be expressed as a close-form bang-bang control determined from the value function. Unfortunately, a general close form solution for the value function of the above terminal constraint optimal stopping problem is not known and numeric approaches that approximate the value function are required.

The discrete-time Markov chain approximation approach for control problems in continuous time is based on an approximation of the original continuous time problem by a Markov chain optimal control problem in discrete-time.

We used the approach presented in [10] to pose a locally consistent (in a particular sense) Markov chain problem in discrete-time to approximate the original continuous-time problem.

4.4 Missile Guidance Simulation Results

In this section we present simulation results for the presented terminal cost optimal stopping guidance law. To illustrate the proposed guidance law in a difficult engagement we consider guidance against a non-stationary target (with $\rho = 0.4$).

We use the reduced order model (17) to describe the engagement dynamics and solve the terminal stopping constraint control problem for $T \to \infty$. This is equivalent to considering the missile guidance problem in which there is *no maximum time limit to the engagement* (but there must be a finite stopping time). In the missile guidance problem considered here, as $T \to \infty$ the optimal control surface becomes time-invariant.

4.4.1 Deterministic Target with Disturbance Behaviour: We choose a bounded region of state-space ($\bar{r} \leq 20$ dimensionless, $-0.4 \leq \theta_m, \theta_t \leq 0.4$ radians) with $h_r = 1$ dimensionless, $h_\theta = h_\sigma = 0.04$ radians. Here h_r, h_θ and h_σ are the size of discretisation used for the $\bar{r}_t$, θ_t and θ_t variables respectively to create a Markov chain approximation of the continuous state space. The control action is assumed to be from the bounded set $U = \{u : -0.5 \leq u \leq 0.5\}$ radians/second. And the disturbance is assumed to be from the bounded set $W = \{w : -0.04 \leq w \leq 0.04\}$ radians/second.

In this guidance application, with $\rho > 0$, the optimal control is a $R^3 \to R$ mapping.

Figure 2 shows a numeric representation of the optimal control for one value of target attitude angle, $\theta_t = 0$ radians. That is, a $R^2 \to R$ mapping (there is a separate $R^2 \to R$ mapping for each value of θ_t). Hence, for each σ_m and $\bar{r}$ value, the Figure 2 shows the numerically calculated optimal control action when $\theta_t = 0$ radians. There is one of these figures for each value of θ_t (ie. 7 figures in the current approximation). Together these 7 figures describe all the optimal control actions for this problem.

Note that Figure 2 illustrates the bang-bang characteristic of u^* described by (18), except for the dead-zone centred at $\theta_m = 0$. The dead-zone at the centre of the figure is a numeric artifact that is not an essential feature of the optimal control problem (there is more discussion of this issue later). Although not shown here, the calculated w^* also exhibits the expected bang-bang characteristics.

4.4.2 Level Sets: Level sets for this engagement are shown in Figure 3.

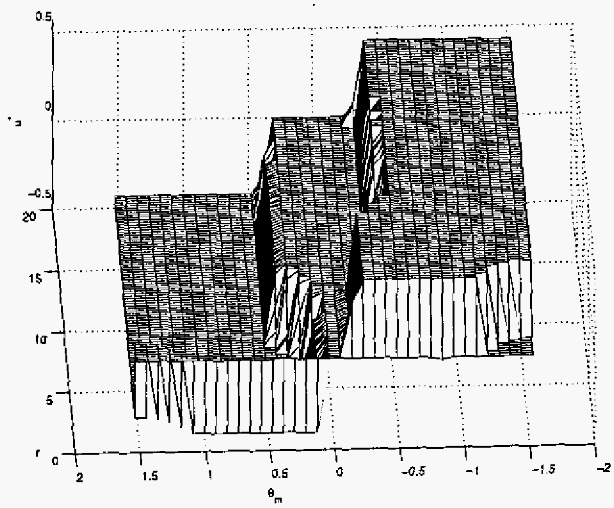

Figure 2: Optimal Guidance against a ($\rho = 0.4$) target with disturbance: $\theta_t = 0$ radians

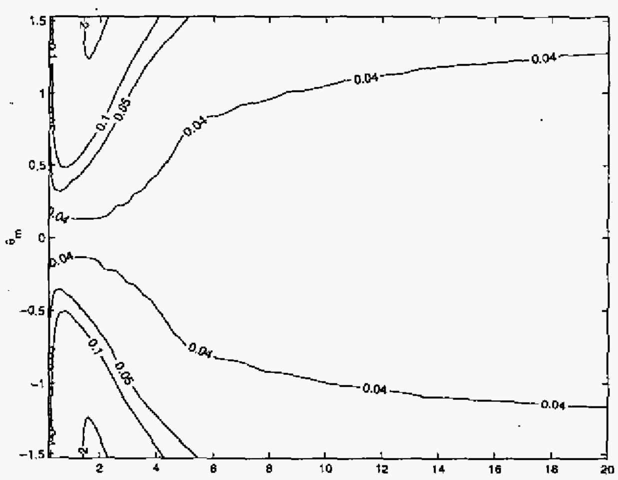

Figure 3: Level Sets against a ($\rho = 0.4$) target with disturbance: $\theta_t = 0$ radians

The regions inside different contour lines of the value function shown in Figure 3 correspond to $\mathcal{S}_{\lambda,\infty}$ for different values of λ. For example, initialisation inside the inner contour, ie. in $\mathcal{S}_{0.04,\infty}$, guarantees $|\bar{r}_{T^*}| \leq 0.04$.

4.4.3 Dead-zone in Calculated Optimal Controller: Figure 2 shows that there is a dead-zone (centred on $\theta_m = 0$) exhibited in the numerically calculated optimal solution that is not an obvious consequence of the bang-bang solution described in (18). This dead-zone is a feature of the optimal solution to the time-discretised version of the problem.

Essentially, over the non-infinitesimal time discretisation, non-infinitesimal angular changes result from $\max(U)$ and $\min(U)$ control actions. Control actions that result in angular changes that cross the

$p_3 = 0$ boundary are non-optimal. Hence, in the time-discretised version of the problem, bang-bang controls are not optimal when close to the $p_3 = 0$ boundary.

The size of the dead-zone is dependent on the time discretisation. The numeric procedure used to solution the control problem adaptively chooses the time discretisation and this is why the size and shape of the dead-zone is different in Figures 2 and 4.

4.5 Non-manoeuvring Target

We then consider an engagement with no disturbance. That is, the disturbance is assumed to be from the set with the single element $W = \{0\}$ radians/second.

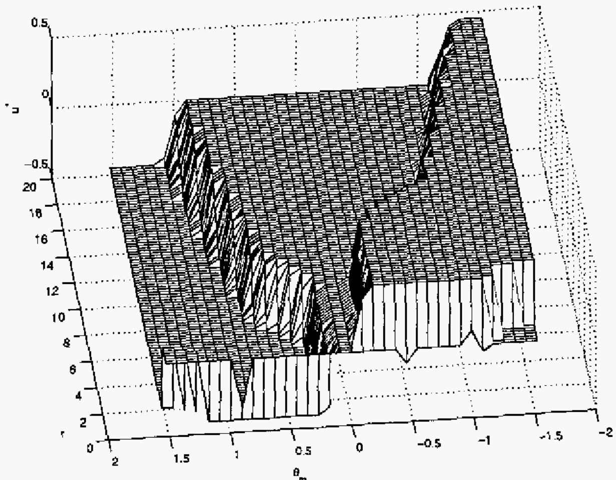

Figure 4: Optimal Guidance against a ($\rho = 0.4$) non-manoeuvring target: $\theta_t = 0$ radians

Some Critical Remarks on the Presented Missile Guidance Approach:

1. The most significant (and valid) criticism of the above missile guidance approach is that, in a practical setting, it is well known that large control actions are not sustainable over an extended period. All aerodynamic missile manoeuvres increase drag and hence reduce the missile's forward velocity. Too much aerodynamic control over an extended period can reduce the missile's forward velocity to a level where interception is no longer possible.

2. For these reasons, historical approaches have included, a somewhat arbitrary, soft running cost on the control energy. The resulting optimal control then matches the intuition that large control actions are not desirable. However, this soft running cost on control energy is not an essential feature of the missile guidance problem, and can result in overly conservative trajectories leading to the potential of guidance failures where more aggressive control actions, taken at appropriate times, would lead to success.

   However, because the energy of control actions must be supplied on-board the missile, there is some sense in which there is a total limit to how much control actuation energy can be expended during an engagement; hence, soft-constraints might have utility in describing this aspect of the design problem.

3. We suggest that this artificial soft running cost in control energy can be avoided by proposing a modified system description that includes the effect of manoeuvres on drag and missile's velocity. A new hard terminal constraint problem could then be solved that incorporates these practical concerns in a direct rather than heuristic manner. This is not done here due to computational issues. We hope to consider this approach in the near future.

4. One significant practical advantage of the presented guidance approaches is that the guidance demands depend primarily on simple angular quantities that are very likely to be available in most guidance problems (although knowledge of θ_t may be a little difficult to obtain).

5 Discrete-time Equivalent Problem

In this section we consider the equivalent hard terminal constraint problem in a discrete-time framework.

Consider the nonlinear discrete-time dynamical system:

$$\begin{aligned} x_{k+1} &= f(x_k, u_k, w_k) \\ z_k &= g(x_k) \end{aligned} \tag{20}$$

where $x_k \in R^n$, $u_k \in R^m$, $w_k \in R^p$, and $z_k \in R^q$ are the state, control input, disturbance input and performance output quantity, respectively. The initial state value, x_0, is assumed given.

We consider the possibly finite stopping time horizon $[0, 1, \ldots, k]$ for some $k \leq T$, T fixed, and assume the following:

1. $x_\ell \in \mathcal{X}$ for all $\ell \in [0, 1, \ldots, k]$ where $\mathcal{X}$ is a bounded set.

2. $u_\ell \in U$ for all $\ell \in [0, 1, \ldots, k]$ where U is a bounded set.

3. $w_\ell \in W$ for all $\ell \in [0, 1, \ldots, k]$ where W is a bounded set.

4. $-\infty < g(x) < \infty$ for all $x \in \mathcal{X}$

5. there exists $x \in \mathcal{X}$ such that $g(x) < \lambda$, where λ is some given required perfromance level.

We employ the following notation: $\mathcal{U}[0,k]$ is the set of admissible control sequences on $[0,1,\ldots,k]$ in which $u_\ell \in U$ for all $\ell \in [0,1,\ldots,k]$. In a similar manner we defined $\mathcal{W}[0,k]$. We take the infimum over a empty set to be equal to ∞.

The above definitions and following control problems are motivatived by the trajectory design problem in which the dynamics must be controlled, in finite time, to a finite set. Again, an example of a suitable performance output quantity for these problems is $g(x) = |x|$.

5.1 Hard Terminal Stopping Constraint

The objective of the terminal stopping constraint problem is to design a causal state feedback control $u \in \mathcal{K}_{st}$ such that, given a fixed stopping time constraint $T < \infty$ and any initial state $x_0 \in \mathcal{X}$, there exists a stopping time $k^*(x_0, u) \in [0,1,\ldots,T]$ such that

$$z_{k^*(x_0,u)} \leq \lambda, \tag{21}$$

where system (20) is initialised at $x_0 \in \mathcal{X}$. This problem appears a dual problem to the l^∞ bounded problem introduced in [14].

If there is more than one control sequence and stopping time pair that satisfy this constraint, then use the pair that achieves the smallest $z_{k(x_0,u)}$. That is, u^* and k^* such that

$$z_{k^*(x_0,u^*)} < z_{k(x_0,u)} \text{ for all } u \in \mathcal{K}_{st}, k \in [0,1,\ldots,T] \tag{22}$$

5.2 Dynamic Programming for Hard Terminal Stopping Constraint

Let us introduce the following value function:

$$V_T(n,x) = \inf_{k\in[n,T]} \inf_{u\in\mathcal{U}[n,k-1]} \sup_{w\in\mathcal{W}[n,k-1]} g(x_{k,x,u,w}) \tag{23}$$

where $x_{k,x,u,w}$ is the solution at time k to the (20) from x at time n with input sequences u and w. In the following we will use the shorthand x_k as the meaning will be clear from context.

Lemma 5.1 *The dynamic programming principle for (23) is*

$$V_T(n,x) = \min\left(g(x), \min_{u\in U}\max_{w\in W} V_T(n+1, f(x,u))\right) \tag{24}$$

with $V_T(T,x) = g(x)$.

Proof: Rewrite $V_T(n,x)$ as

$$V_T(n,x_n) = \inf_{k\in[n,T]} \inf_{u\in\mathcal{U}[n,k-1]} \sup_{w\in\mathcal{W}[n,k-1]} g(x_k).$$

Then consider any $\eta \in [n, n+1, \ldots, T]$. The infimum over k then occurs in $[n, n+1, \ldots, \eta]$ or $[\eta+1, \eta+2, \ldots, T]$. Hence

$$\begin{aligned}
V_T(n,x_n) &= \min\Bigg(\inf_{k\in[n,\eta]} \inf_{u\in\mathcal{U}[n,k-1]} \sup_{w\in\mathcal{W}[n,k-1]} g(x_k), \\
&\qquad \inf_{k\in[\eta+1,T]} \inf_{u\in\mathcal{U}[n,k-1]} \sup_{w\in\mathcal{W}[n,k-1]} g(x_k)\Bigg) \\
&= \min\Bigg(\inf_{k\in[n,\eta]} \inf_{u\in\mathcal{U}[n,k-1]} \sup_{w\in\mathcal{W}[n,k-1]} g(x_k), \\
&\qquad \inf_{u\in\mathcal{U}[n,\eta]} \inf_{k\in[\eta+1,T]} \inf_{u\in\mathcal{U}[\eta+1,k-1]} \sup_{w\in\mathcal{W}[\eta+1,k-1]} g(x_k)\Bigg) \\
&= \min\Bigg(\inf_{k\in[n,\eta]} \inf_{u\in\mathcal{U}[n,k-1]} \sup_{w\in\mathcal{W}[n,k-1]} g(x_k), \\
&\qquad \inf_{u\in\mathcal{U}[n,\eta]} \sup_{w\in\mathcal{W}[n,\eta]} V_T(n+(\eta+1), x_{\eta+1})\Bigg)
\end{aligned}$$

where $V_T(T,x) = g(x)$. By considering $\eta = n$ we obtain the Lemma result

$$V_T(n,x) = \min\left(g(x), \inf_{u\in U}\sup_{w\in W} V_T(n+1, f(x,u))\right).$$

■

5.3 Feasible Hard Terminal Stopping Control

Let the set of feasible initial states for a particular stopping terminal constraint level λ be denoted by

$$\mathcal{S}^D_{\lambda,T} = \{x_0 : V_T(0,x_0) \leq \lambda\}. \tag{25}$$

For $x_0 \in \mathcal{S}^D_{\lambda,T}$, the stopping controller $u^*(x_0) \in \mathcal{K}_{st}$ and stopping time $k^*(x_0) \in [0,1,\ldots,T]$ is defined by the minimising u in (24) and $k^* = \inf\{k : g(x_k) = V_T(T-k, x_k)\}$

6 Conclusions

This paper presented two new types of deterministic continuous-time optimal stopping control problems involving hard terminal constraints. Dynamic programming solutions, involing variational inequalities, were presented for both problems.

A missile guidance problem was then used to illustrate one of the hard terminal constraint problems. An numeric approximation to the optimal control was calculated for the missile guidance problem and guaranteed performance levels presented.

Dynamic programming solutions for the equivalent discrete-time problem were also presented.

References

[1] D.B. Ridgely and M.B. McFarland, *Tailoring Theory to Practice in Tactical Missile Control*, IEEE Control Systems, pp. 49-55, Dec. 1999.

[2] D.P. Bertekas, *Dynamic Programming and Optimal Control*, Vol. 1 & 2, 2nd Ed., Athena, Massachusetts, 2000.

[3] J.R. Cloutier, J.H. Evers and J.J. Feeley, *Assessment of Air-to-Air Missile Guidance and Control Technology*, IEEE Control Systems Magazine, pp. 27-34, Oct. 1989.

[4] R. Isaacs, *Differential Games: A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization*, (Original Publisher: John Wiley & Sons, 1965), Dover Publications, New York, 1999.

[5] T. Basar and P. Bernhard, *H^∞-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach*, 2nd Ed., Birkhauser, Boston, 1995.

[6] T. Basar and G.J. Olsder, *Dynamic Noncooperative Game Theory*, 2nd Ed., (Original Publisher: Academic Press, 1982), SIAM, Classic in Applied Mathematics, Vol. 23, Philadelphia, 1999.

[7] P. Zarchan, *Tactical and Strategic Missile Guidance*, 4th Ed., Vol. 199, Progress in Astronautics and Aeronautics, AIAA, Washington, DC, 2002.

[8] J.Z. Ben-Asher and I. Yaesh, *Advances in Missile Guidance Theory*, Vol. 180, Progress in Astronautics and Aeronautics, AIAA, Virginia, 1998.

[9] N.A. Shneydor, *Missile Guidance and Pursuit: Kinematics, Dynamics and Control*, Horword Publishing, Chichester, 1998.

[10] H.J. Kushner and P. Dupuis, *Numerical Methods for Stochastic Control Problems in Continuous Time*, 2nd Ed., Springer, New York, 2001.

[11] M. Bardi and I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Birkhauser, Boston, 1997.

[12] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, (Original Publisher: Academic Press, 1980), SIAM, Classics in Applied Mathematics, Vol. 31, Philadelphia, 2000.

[13] J.J. Ford, *Precision Guidance with Impact Angle Requirements*, DSTO Series Publication, DSTO-TR-1219, Oct. 2001.

[14] Huang S., James M.R., *l^∞ Bounded Robustness for Nonlinear Systems: Analysis and Synthesis*, Submitted to IEEE Transactions on Automatic Control, 2002.

[15] Kirk, D.E. *Optimal Control Theory: an Introduction*, Prentice Hall, 1970.