∞-categories for the working mathematician

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Higher categories & disposable ladders

> My propositions elucidate in the sense that anyone who understands me will eventually recognize them as nonsense, once they have climbed through them – on them – beyond them. (They must, as it were, throw the ladder away after they have climbed up it.)
> They must surmount these propositions; then they will see the world aright.
>
> – Ludwig Wittgenstein, *Tractatus Logico-Philosophicus*, 6.54.

According to most set-theoretic foundations, the number 6 might refer to:

- a set with 6 elements (when viewed as a natural number),
- a set with countably many elements (when viewed as an integer or as a rational number), or
- a set with uncountably many elements (when viewed as a real number).

Working mathematicians do not fret much about this ambiguity, because they know there are preferred injections

$$\boldsymbol{N} \hookrightarrow \boldsymbol{Z} \hookrightarrow \boldsymbol{Q} \hookrightarrow \boldsymbol{R}\,.$$

These injections are in fact *unique* with certain simple properties.

Better still, each of these number systems satisfies a universal property:

- $\boldsymbol{N}$ is the free commutative monoid on one generator;
- $\boldsymbol{Z}$ is the group completion of $\boldsymbol{N}$;
- $\boldsymbol{Q}$ is the field of fractions of $\boldsymbol{Z}$;
- $\boldsymbol{R}$ is the Cauchy completion of $\boldsymbol{Q}$.

Each of these universal properties characterizes maps from these objects in certain categories.

To *construct* these objects set-theoretically and to prove that they have the desired properties is sometimes a chore. There are often many constructions of the same object. For example, are real numbers Dedekind cuts, or are they equivalence classes of Cauchy sequences of rational numbers? If we ask questions that involve the *elements* of a real number, then we can observe differences between these two options. But if the kinds of questions we study only involve the things for which we usually use real numbers – adding, subtracting, multiplying, dividing, and taking limits – then any differences among the various constructions of $\boldsymbol{R}$ become irrelevant.

As a result, working mathematicians don't generally regard a question such as 'is $3 \in \pi$?' as *meaningful*, despite the fact that in set-theory-founded mathematics, it technically will admit an answer. Precisely *because* the answer depends upon the set-theoretic model one chooses of $\boldsymbol{R}$, we know we aren't supposed to ask that sort of question. That is, we adopt a *structuralist* attitude to the real numbers. Every complete ordered field has a unique element that deserves the name π, and the *meaningful* questions about π make reference only to that structure. Since any pair of complete ordered fields are topologically isomorphic in exactly one way, the answers to meaningful questions will not depend on any particular construction of the complete ordered field.

Now we may still *answer* meaningful questions by appealing to a particular model. One is certainly allowed to deduce facts about π by thinking of it and working with it as an equivalence class of Cauchy sequences. But those facts are only taken to be meaningful if they can be stated only with reference to the structures present in a complete ordered field.

This pattern of mathematical development resembles Wittgenstein's parable of the disposable ladder. We climb the ladder by constructing a mathematical object X set-theoretically. Once we've done this, we identify the salient *structures* and *properties* of X. We write enough of these down to characterize X uniquely, up to an appropriate notion of *isomorphism*. The structures will often reveal to us a category $\boldsymbol{C}$ of which X should be viewed as an object, and the properties will identify X uniquely up to unique isomorphism in $\boldsymbol{C}$. We then restrict our attention to the questions about X that can be formulated in terms provided by the category $\boldsymbol{C}$, and we regard questions about X that do *not* admit such formulations as meaningless. Information about our initial construction of X – the ladder we used to climb to our understanding of it as an object of $\boldsymbol{C}$ – become not just irrelevant but *uninterrogable*. We throw away the ladder.

Here is another example. Let R be a (commutative, unital) ring, and let M and N be R-modules. The first time many students see the tensor product $M \otimes_R N$, it is defined via generators and relations: given a set S of generators of M and a set T of generators of N, define $M \otimes_R N$ as the R-module generated by pairs (x, y) – usually written as $x \otimes y$ and called *simple tensors* – with $x \in S$ and $y \in T$, subject to the relations $x \otimes (ry) = (rx) \otimes y = r(x \otimes y)$ and $x \otimes 0 = 0 \otimes y = 0$. This description is a disposable ladder: eventually students will understand it as a specific *presentation* of the true object $M \otimes_R N$, which is the object representing the functor that carries an R-module T to the set of R-bilinear mays $M \times N \to T$.

The theory of *higher categories* also follows this pattern, to a point. There are many ways to construct a theory of higher categories. These constructions vary significantly in detail. However, there are also today a number of different *unicity theorems*; these state that any two purported models of higher categories satisfying a few sensible axioms are equivalent in an appropriate sense. In other words, all of these are models of the same structure. That structure is the *higher category of higher categories*. We will have more to say about this below.

Before we do, though, we must address the *ouroboros* that makes our story unusual. After all, $\boldsymbol{R}$ is not itself a real number; one can define complete ordered fields without making reference to an existing theory of real numbers. And $M \otimes_R N$ is an R-module, not an element of $M \otimes_R N$. But higher categories *do* form a higher category. One feels a palpable sense of discomfort. Part of this is a response to the cautionary tale Russell told us about the dangers of self-membership. But there's something else: is it not somehow paradoxical to assert that there is a unique higher category of higher categories? Are we not trying to throw away our ladder while we are still standing on it? After all, how can one characterize the theory of higher categories as a higher category before one has fully worked out what a higher category is? And how can one claim to have fully worked out what a higher category is without having characterized the theory of higher categories?

It's not quite as vicious a circle as it may seem. The strategy is to begin with a definition (in set-theoretic terms) of *putative* higher categories. We show that putative higher categories form a putative higher category **Put**. (Technically, we will need avoid Russell's paradox with large/small distinctions, but let's ignore this for now.) Then we uniquely characterize an object **True** of **Put**, making reference only to the structure available in **Put**. We call **True** the putative higher category of *true* higher categories. Then we prove that **Put** satisfies the conditions of our characterization. Hence the putative higher category **Put** of putative higher categories is equivalent (in a unique fashion) to the putative higher category **True** of true higher categories.

But it is easy to tell a degenerate version of this story. I define a putative higher category to be a one-point set. The category **Put** of one-point sets is, up to equivalence, a one-point set itself. We characterize **True** as the unique object of **Put** (up to unique isomorphism). Necessarily, **Put** $\cong$ **True**. We were hoping for a theory of higher categories with more content than this!

So the theory of higher categories, as we can develop it today, differs from many other mathematical abstractions in one important respect. Whereas we eventually find new (and often clearer) structuralist definitions of many mathematical objects, we do not have a way of defining higher categories without introducing a model of the theory at some point. We can, for example, give a structuralist definition of 'real number' as an element of a complete ordered field $\boldsymbol{R}$; such a definition doesn't exhibit any preference for any particular construction of $\boldsymbol{R}$. We can also define $M \otimes_R N$ as the object representing the functor $T \mapsto \mathbf{Bilin}_R(M \times N, T)$; no need to choose generators.

By contrast, we do not currently know how to give a completely structuralist definition of 'higher category'. What we have is a structuralist *characterization* of the theory of higher categories, when this theory is regarded as an object in some extant model of higher categories. So we cannot yet avoid the difficult task of constructing and studying an explicit model of higher categories. Such a model remains entangled with the theory of higher categories, even when the goal is a model-independent understanding of the theory.

What this means for higher category theory: some tools beyond the usual arsenal of categorical techniques will have to be deployed in order to provide an account of higher category theory that is untethered from set-theoretical models. (Homotopy type theory is probably the most promising approach at the moment.)

What this means for us: if we are to undertake a well-motivated study of higher categories, then we will have to take a dialectical approach. We will spend some time experimenting with some provisional definitions in order to develop good intuitions about the kinds of structures and examples we actually want the theory to capture. We do not, however, intend to take a historical approach to the theory of higher categories. The development of higher category theory has been complex and discursive, and it would be easy for our narrative to degenerate into omphaloskepsis.

Instead, in this book, we will focus on one essential challenge in the theory – *the question of sameness* – which will in turn motivate *the homotopical turn* in higher category theory.

Why is higher category theory subtle? At first it's hard to see what all the fuss is about. We already know what 0-categories are: they're *sets*. We also know what 1-categories are: they're *categories*. So a 1-category has a collection of objects and, between every pair x, y of objects, one has a 0-category $\mathrm{Mor}(x, y)$ of maps. Composition is then a map

$$\mathrm{Mor}(x, y) \times \mathrm{Mor}(y, z) \to \mathrm{Mor}(x, z)$$

given by $(f, g) \mapsto g \circ f$.

The way to iterate this definition seems uncontroversial. Let $\boldsymbol{V}$ be a category with all finite products. A *category* $\boldsymbol{C}$ *enriched in* $\boldsymbol{V}$ consists of:

- a collection $\operatorname{Obj}\boldsymbol{C}$ of objects;
- between every pair $x, y \in \operatorname{Obj}\boldsymbol{C}$, an object
$$\operatorname{Mor}_{\boldsymbol{C}}(x, y) \in \operatorname{Obj}\boldsymbol{V}\ ;$$
- for every $x \in \operatorname{Obj}\boldsymbol{C}$, a morphism $\mathrm{id}_x : 1_{\boldsymbol{V}} \to \operatorname{Mor}_{\boldsymbol{C}}(x, x)$ (where $1_{\boldsymbol{V}}$ denotes the terminal object of $\boldsymbol{V}$); and
- for every $x, y, z \in \operatorname{Obj}\boldsymbol{C}$, a morphism
$$\operatorname{Mor}_{\boldsymbol{C}}(x, y) \times \operatorname{Mor}_{\boldsymbol{C}}(y, z) \to \operatorname{Mor}_{\boldsymbol{C}}(x, z)\ ,$$
written $(f, g) \mapsto g \circ f$.

These data are subject to the usual axioms: composition is associative ($(h \circ g) \circ f = h \circ (g \circ f)$), and unital ($f \circ \mathrm{id}_x = \mathrm{id}_y \circ f = f$). There's an attached notion of $\boldsymbol{V}$*-enriched functor*, whose definition is predictable.

Now we can give an easy iterative definition. Our base case is the notion of *strict* 0*-categories* and the category $\mathbf{Cat}_0^{str} \coloneqq \mathbf{Set}$. Then we define a *strict* n*-category* as a category enriched in $\mathbf{Cat}_{n-1}^{str}$, and we define $\mathbf{Cat}_n^{str}$ as the category of $\mathbf{Cat}_{n-1}^{str}$-enriched categories and $\mathbf{Cat}_{n-1}^{str}$-enriched functors.

Unwinding the recursion, we see that a strict n-category $\boldsymbol{C}$ has:

- a collection $\operatorname{Obj}\boldsymbol{C}$ of *objects* or 0*-morphisms*;
- for every pair $x, y \in \operatorname{Obj}(\boldsymbol{C})$ of objects, a collection $\operatorname{Mor}_{\boldsymbol{C}}(x, y)$ of *morphisms* or 1*-morphisms* with *source* x and *target* y;
- for every pair $f, g \in \operatorname{Mor}_{\boldsymbol{C}}(x, y)$ of 1-morphisms (which we say are *parallel* because they have the same source, and they have the same target), a collection $\operatorname{Mor}_{\operatorname{Mor}_{\boldsymbol{C}}(x,y)}(f, g)$ of 2*-morphisms* with source f and target g;
- for every parallel pair $\alpha, \beta \in \operatorname{Mor}_{\operatorname{Mor}_{\boldsymbol{C}}(x,y)}(f, g)$ of 2-morphisms, a collection
$$\operatorname{Mor}_{\operatorname{Mor}_{\operatorname{Mor}_{\boldsymbol{C}}(x,y)}(f,g)}(\alpha, \beta)$$
of 3*-morphisms* with source α and target β;

…

- for every parallel pair ϕ, ψ of $(n-1)$-morphisms, a collection
$$\operatorname{Mor}_{\operatorname{Mor}_{\ddots_{\operatorname{Mor}_{\boldsymbol{C}}(x,y)}\,\iddots}}(\phi, \psi)$$
$\operatorname{Mor}(\phi, \psi)$ of n*-morphisms* with source ϕ and target ψ.

Each of these kinds of morphisms can be composed, in all sorts of ways. As n increases, the combinatorics can get a little heady, but it's nothing we can't handle.

But as the word 'strict' suggests, this is demanding too much. To see how, let's look more carefully at the case $n = 2$. A strict 2-category consists of a set $\operatorname{Obj} \boldsymbol{C}$ of objects; between every $x, y \in \operatorname{Obj} \boldsymbol{C}$, a set $\operatorname{Mor}_{\boldsymbol{C}}(x, y)$ of morphisms $x \to y$; and between every $f, g \in \operatorname{Mor}_{\boldsymbol{C}}(x, y)$, a set $\operatorname{Mor}_{\operatorname{Mor}_{\boldsymbol{C}}(x,y)}(f, g)$ of 2-morphisms $f \to g$. Morphisms and 2-morphisms each have composition laws, which are associative and unital.

Associativity means that if $f\colon x \to y$, $g\colon y \to z$, and $h\colon z \to u$ are 1-morphisms of $\boldsymbol{C}$, then $h \circ (g \circ f) = (h \circ g) \circ f$. There's something strange about this. On either side of the equals sign here are *objects* of the category $\operatorname{Mor}_{\boldsymbol{C}}(x, u)$, and here we are asking them to be *equal*. This is just the kind of unreasonable request that our structuralist disposition is supposed to deny. Let's illustrate this issue with an interesting class of examples that 'ought' to be 2-categories, but are not strict 2-categories.

We take our inspiration from a construction we meet early in a study of category theory: if M is a monoid, then we define a category BM with exactly one object $*$ and $\operatorname{Mor}_{BM}(*, *) = M$; composition in BM is multiplication in M. An action of M on an object of a category $\mathbf{C}$ is then precisely a functor $BM \to \mathbf{C}$. The construction $M \mapsto BM$ identifies the category of monoids with the category of categories with exactly one object.

Let's try to tell this story again one 'category level' up. There are some interesting examples that look like monoid objects in categories. For example, consider the category $\mathbf{Mod}(R)$ of modules over our ring R. The tensor product $\otimes_R$ is a multiplication law on $\mathbf{Mod}(R)$. We cannot quite call this a monoid structure though, because it's not strictly associative. If A, B, and C are three R-modules, then it depends on some rather pedantic set-theoretic points of your precise definition of the tensor product as to whether we really have *equality*

$$(A \otimes_R B) \otimes_R C = A \otimes_R (B \otimes_R C)\,.$$

More dramatically, in our preferred, ladder-free understanding of the tensor product, we came to the conclusion that we should simply *define* it as the object representing the functor $T \mapsto \mathbf{Bilin}_R(A \times B, T)$. Of course, representing objects are only unique up to canonical isomorphism, not up to set-theoretic equality. So from this perspective, strict associativity for $\otimes_R$ is no longer *meaningful*.

On the other hand, there clearly is *some* kind of associativity here, because we have an isomorphism

$$\alpha_{A,B,C}\colon (A \otimes_R B) \otimes_R C \cong A \otimes_R (B \otimes_R C)$$

that is natural in A, B, and C, since these objects each represent the functor $T \mapsto \mathbf{Trilin}_R(A \times B \times C, T)$. In other words, associativity is no longer a *property*, but a piece of *structure* – namely, the natural isomorphism α. The category $\mathbf{Mod}(R)$ is a *monoidal category*.

We might now try to construct a 2-category $B\mathbf{Mod}(R)$. There will be exactly one object, $*$. A 1-morphism $A: * \to *$ will be a module over a ring R. A 2-morphism $\phi: A \to B$ will be an R-linear map. In other words, $\mathrm{Mor}(*, *) = \mathbf{Mod}(R)$. The composition functor

$$\mathrm{Mor}(*, *) \times \mathrm{Mor}(*, *) \to \mathrm{Mor}(*, *)$$

is the formation of the tensor product $(A, B) \mapsto A \otimes_R B$. But of course this isn't a strict 2-category, because this composition is not strictly associative.

The difference between 2-categories and strict 2-categories reduces, in the one-object case, to the difference between monoidal categories and monoid objects in categories. In order to come to grips with 2-categories in general, it's a good start to understand monoidal 1-categories. In particular, how does one deal with the failure of associativity?

Why is associativity important anyhow? This is not a frivilous question. In a monoid M, if we have three elements $x, y, z \in M$, then what associativity buys us is the ability to talk about the element $xyz \in M$ without ambiguity. More generally, for any n, and for any collection $\{x_1, x_2, \dots, x_n\}$, associativity lets us make unique sense of the product

$$x_1 x_2 \cdots x_n \in M .$$

This even makes sense when $n = 0$, because the empty product in M is the unit for the multiplication.

We are used to thinking of the *structure* of a monoid as an element $e \in M$ and a binary multiplication $M \times M \to M$. These data are then required to satisfy two *properties*: $xe = x = ex$ and $(xy)z = x(yz)$. From one perspective, this is a strange thing to do: the structure we really want out of a monoid is the ability to make sense of any product $x_1 x_2 \cdots x_n \in M$. We get that as a result of the associativity and unitality, but if we were more direct in our intentions, we would define the structure of a monoid as a set M along with products

$$\prod_{i \in I} : M^I \to M ,$$

one for every totally ordered finite set I, all arriving in a single packet. The binary multiplication map is what we get for $I = \{1, 2\}$, and the unit is what we get for $I = \varnothing$.

Of course, we've just introduced a lot more structure here, and in general we have to pay for new pieces of structure with more properties. In our case, we require that if I is a singleton, then $\prod_{i\in I}$ is the identity. More importantly, we demand that these multiplication maps satisfy a higher associativity: for every monotonic map $\phi\colon J \to I$ between totally ordered finite sets, we have

$$\prod_{i\in I} \prod_{j\in\phi^{-1}\{i\}} x_j = \prod_{j\in J} x_j \, .$$

Applied to the two monotonic surjections $\{1,2,3\} \to \{1,2\}$, we recover associativity $(xy)z = xyz = x(yz)$; applied to the two injections $\{1\} \to \{1,2\}$, we recover the unit condition $xe = x = ex$.

This sort of definition of 'monoid' is inefficient, but it is equivalent to the usual definition. (That is, they define equivalent categories.) Tom Leinster calls this definition *unbiased*, because the usual definition exhibits a bias for small totally ordered finite sets – those of cardinality less than 3.

Let us *categorify* this story in order to make sense of monoidal categories. A monoidal category consists of a category $\boldsymbol{C}$ and a family of functors

$$\otimes_I = \bigotimes_{i\in I} \colon \boldsymbol{C}^I \to \boldsymbol{C} \, ,$$

one for each totally ordered finite set I. This is not, however, all the structure we need. We also need to define, for every monotonic map $\phi\colon J \to I$, a natural isomorphism

$$\alpha_\phi \colon \bigotimes_{i\in I} \bigotimes_{j\in\phi^{-1}\{i\}} x_j \xrightarrow{\sim} \bigotimes_{j\in J} x_j \, .$$

This is a great deal of structure, and it must be made to satisfy a great many conditions, called *coherences*. These identify α_{id} with id and express the relationship among $\alpha_{\psi\circ\phi}$,α_ϕ, and α_ψ. A *monoidal functor* between two monoidal categories C and D is then a functor $F\colon C \to D$ along with isomorphisms $\gamma_I \colon \otimes_I \circ F^I \xrightarrow{\sim} F \circ \otimes_I$ that are suitably compatible with the associators for C and D.

Once we already *have* the categorified notion – the notion of 'monoidal category' and 'monoidal functor', our unbiased definition of monoid becomes natural and trivial to state cleanly. We observe that *concatenation* is a monoidal structure on the category $\boldsymbol{O}$ of totally ordered finite sets. A monoid is then exactly a monoidal functor $\boldsymbol{O} \to \mathbf{Set}$, where the monoidal structure on **Set** is the cartesian product. This phenomenon is a little expression of what John Baez called the *macrocosm/microcosm principle*.

In order to establish a workable framework for *weak* higher categories, we can follow a similar strategy to the one taken here to contemplate monoidal categories.

1 We identify a large family of composition shapes we can form in a strict n-category.
2 We define a category of these composition shapes. The morphisms of this category should control the higher associativity laws of these compositions.
3 We then weaken these associative laws from *equations* to pieces of *structure* (like isomorphisms) that identify the two sides of the equations.
4 These isomorphisms will have to satisfy coherences that are controlled by compositions in our category of composition shapes. These coherences might themselves be properties that would be more naturally expressed as higher isomorphisms, which would in turn be subject to yet higher coherences (controlled by composable sequences of maps), *etc*.

The transition from equations to pieces of structure to exhibit *sameness* is the heart of our approach. This is what we call the *homotopical turn* in higher category theory. As a result of this transition, the theory of higher categories will unfold in a more robust and appealing way than the more direct approach – explicitly weakening the theory of n-categories – can achieve.

When are two mathematical objects the same? Let's start with natural numbers a and b. Perhaps you want to prove that a and b are the same. *How might you go about that?* Well, it depends upon how a and b arose in your thinking. Since they are natural numbers, it's likely you found them by counting something. In other words, a is the cardinality of a finite set A, and b is the cardinality of a finite set B.

Perhaps A is the set of nontrivial ways of (correctly and nonredundantly) parenthesizing an expression like *abcd* with two pairs of matching parentheses, such as $(a(bc))d$. Perhaps B is the set of full binary trees with 4 leaves. It would be perfectly valid to prove that $a = b$ by computing each of these numbers and comparing the answers, but this sort of proof is unsatisfying, even contentless: such a proof doesn't actually inform us of anything about *why* these two numbers are equal. The equation $a = b$ is a shadow of something more interesting (and more general): a *bijection* between the sets A and B.

Whereas '$a = b$' is a *property*, a bijection between A and B is *structure*. A bijection $\phi\colon A \to B$ is an isomorphism in the category of sets; its inverse is map $\psi\colon B \to A$ such that $\psi\phi = \mathrm{id}_A$ and $\phi\psi = \mathrm{id}_B$.

If we now want to operate in a 2-category, these equalities are no longer a natural thing to demand. For example, we know we should not expect two categories to be isomorphic. The correct notion of sameness for categories (or any pair of objects in a 2-category is *equivalence*. An equivalence $\phi\colon A \to B$ has an inverse $\psi\colon B \to A$ in the sense that $\psi\phi$ is isomorphic to id_A and $\phi\psi$ is isomorphic to id_B. That means that there are 2-morphisms

$$\alpha\colon \mathrm{id}_A \to \psi\phi\,, \qquad \beta\colon \psi\phi \to \mathrm{id}_A$$

such that $\beta\alpha = \mathrm{id}_{\mathrm{id}_A}$ and $\alpha\beta = \mathrm{id}_{\psi\phi}$, and there are 2-morphisms

$$\gamma\colon \mathrm{id}_B \to \phi\psi\,, \qquad \delta\colon \phi\psi \to \mathrm{id}_B$$

such that $\delta\gamma = \mathrm{id}_{\mathrm{id}_B}$ and $\gamma\delta = \mathrm{id}_{\phi\psi}$.

The mission of homotopy theory is to iteratively enhance every *property* of a mathematical object to *structure* on it. The homotopy theorist deconstructs the equals sign by this process: they no longer regard '$x = y$' as a property that x and y together possess, but rather as piece of structure that connects x and y. That structure is then a *path* between x and y. Semantically, we consider this path as a 'reason' for – or as a 'witness' to – the equality $x = y$.

If α and β are paths connecting x and y, then again we do not wish to speak of '$\alpha = \beta$' as a property, but as a further piece of structure – a *homotopy* between α and β. We iterate: two homotopies are no longer merely 'equal', but they may be connected by *higher homotopies*; two higher homotopies may be connected by further higher homotopies, *etc.*, *etc.*, *ad infinitum*.

The data of all these points and paths and homotopies and higher homotopies, taken together, constitute an *anima* (pl. *animae*). Animae are also called *spaces*, *homotopy types*, *Kan complexes*, or ∞-*groupoids*. These terms each reflect a certain attitude toward these objects. Terms like 'space' and 'homotopy type' acknowledge that these objects were first modelled and understood using topological spaces and topological notions of homotopy. A 'Kan complex' (named for Dan Kan) is then a combinatorial blueprint for these homotopies and their relations. The phrase '∞-groupoid' then reinterprets the (higher) homotopies as (higher) *symmetries*. The fact that these terms can all be used interchangeably is a nontrivial insight – Grothendieck's *homotopy hypothesis*. We will formulate and prove a version of this sentence in this chapter – surely not the version Grothendieck had in mind, but one that is better-adapted to the needs of contemporary mathematicians. In our formulation, it becomes a theorem of Kan.

Our use of the term *anima* reflects our desire not to favor any one of these attitudes. Animae play the same role in *homotopical mathematics* that sets play in 'ordinary' mathematics: virtually all objects in homotopical mathematics are described in terms of animae.

Symmetries of objects were certainly central to the mathematics of the 19th century, but it's a distinctly 20th century notion that symmetries might meaningfully have their own symmetries. (It is interesting to reflect on the origins of this idea, but it would be difficult to pinpoint the first person who seriously considered this possibility.) In any case, the 20th century provided three realizations about homotopy theory's iterative enhancement process.

First was the promise of interesting new phenomena to study. Homological algebra appears at first to be a relatively featureless outgrowth of linear algebra. The Bockstein homomorphism is an early illustration that ordinary modules interact in new ways in derived settings. But it's the nontriviality of the Hopf element η in the stable homotopy group π_1^s that decisively separates *homological algebra* – ordinary algebra that is then derived – from *homotopical algebra* – algebra done in a natively homotopical setting. The first signal from the mysterious world of homotopical algebra was a short message: '$\eta \neq 0$'.

In spite of our vague description of homotopy theory as an inductive enhancement of properties into structure, this can all be made precise. In fact, it can be made precise in different ways. On one hand, we can model homotopical structures entirely via ordinary mathematical objects. We'll do that here: we'll adopt the Kanian approach and encode homotopy types as *simplicial sets*. Alternatively – and in a spirit closer to that of this introduction – one might instead attempt to rewrite the logical foundations of mathematics in a way that bakes in our preference for structure over properties. This is the approach of *homotopy type theory*, a stirring vision of new foundations in mathematics. As of this writing, homotopy type theory is still in its infancy. In later editions of this book, perhaps the combinatorics of simplices will be replaced by fundamental facts about type theory.

Once it was understood how to model homotopical structures accurately, new questions arose: *how do we construct models that are maximally useful? what does it mean to say that two models represent the same homotopy theory? shouldn't this notion of sameness, whatever it is, be subject to the same inductive refinement process that got us here?* These questions lay at the heart of the many foundational developments in homotopy theory starting in the 1970s. By the end of the millennium it was clear that one would need to take a further step, and contemplate a *homotopy theory of homotopy theories*. Just as Grothendieck had seen that ordinary homotopy theory should be equivalent to the theory of ∞-groupoids, people like Joyal, Kan, Rezk, Simpson, and Toen recognized that the homotopy theory of homotopy theories should be equivalent to the theory of ∞-*categories*.

Frustratingly, we can't yet give a thorough and precise account of higher category theory in the same terms that experienced practitioners use it. Instead, we have to construct a model of higher categories within established set-theoretic foundations. We will then work within that model to develop a slate of fundamental definitions, constructions, and theorems. Once enough of this development is complete, the corresponding higher category of higher categories is then unique. At that point, we are free to throw our ladder away, to ignore set-level specifics of the chosen model, and to work contentedly in a natively higher-categorical way.

Accordingly, this text is divided into two parts. The first part climbs the ladder by developing the theory of quasicategories as a model of $(\infty, 1)$-categories. The second part throws the ladder away by treating these objects with minimal reference to the explicit definitions.

PART ONE

THE LADDER

1

Grothendieck's homotopy hypothesis

Our approach to constructing the homotopy theory of animae follows a general recipe, which will inform our work throughout this book. The recipe provides a strategy for designing a homotopy theory $\boldsymbol{E}$, using only partial or imperfect information about it:

1 Select a piece $\boldsymbol{E}_0$ of $\boldsymbol{E}$ that is simple enough that you can understand it completely, but complex enough so that any object $X \in \boldsymbol{E}$ is completely determined by the sets/groups/whatevers of maps $T \to X$ with $T \in \boldsymbol{E}_0$. In many cases, you'll want $\boldsymbol{E}_0$ to generate $\boldsymbol{E}$ under suitable colimits.

2 Now select a small category J, usually combinatorial in nature, along with an essentially surjective functor $R\colon J \to \boldsymbol{E}_0$ (which need not be fully faithful). In many cases, it will be helpful if the objects of J come equipped with some notion of *degree*. The category J and functor R should be chosen so that some key salient features of general objects $X \in \boldsymbol{E}$ can be read off from mapping objects $\mathrm{Map}(R(j), X)$.

3 Encode these salient features as a set of properties P of the corresponding functors $\mathrm{Map}(R(-), X)$. For example, perhaps you'll want to require that for some particular $j \in J$, the object $\mathrm{Map}(R(j), X)$ is trivial. Or perhaps such a property expresses $\mathrm{Map}(R(k), X)$ as a limit or colimit of various other mapping objects $\mathrm{Map}(R(j), X)$. The objects of your homotopy theory will be exactly the functors $J^{op} \to \mathbf{Set}$ (or other enriching category) that enjoy these properties of P.

In this chapter, we will construct the homotopy theory of animae in this way: $\boldsymbol{E}_0$ will be the homotopy theory of contractible animae (which is trivial), J is the category of *simplices* $\boldsymbol{\Delta}$ (defined below), and P is Kan's *horn-filling condition*.

Other options for J (and therefore P) are possible: there is an interesting class of categories – the *test categories* – that can stand in for $\boldsymbol{\Delta}$. These include the category of nonempty finite sets, Joyal's categories Θ_n, as well as various categories of cubes and trees.

One special feature of $\mathbf{\Delta}$ is that it can also be used to define the larger homotopy theory of ∞-categories itself. For this homotopy theory, $\boldsymbol{E}_0$ is the homotopy theory of finite ordinals (which is not trivial, but is nevertheless very simple), J is again the category $\mathbf{\Delta}$, and P is the *inner horn-filling condition* first identified explicitly by Boardman–Vogt.

1.1 Simplicial objects

Our aim is to convert equality from a property into a structure. At a minimum, we want to be able to work with notions of equivalence that are not just set-theoretic equality. Traditionally, such an alternative is described as an *equivalence relation* on a set S.

A *relation* is encoded by a subset $R \subseteq S \times S$, which is the subset of related pairs of elements. An *equivalence* relation is one that is reflexive, symmetric, and transitive:

1 Reflexivity is the condition that R contains the image of the *diagonal map* $S \to S \times S$ given by $x \mapsto (x, x)$.
2 Symmetry is the condition that R is stable under the involution of $S \times S$ given by $(x, y) \mapsto (y, x)$.
3 Transitivity states that the projection $S \times S \times S \to S \times S$ given by $(x, y, z) \mapsto (x, z)$ carries the subset $R \times_S R \subseteq S \times S \times S$ to the subset R.

An equivalence relation on S can thus be converted into a diagram

$$R \times_S R \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; R \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; S \,, \tag{1.1.0.1}$$

The maps from left to right are various projections. The maps from right to left are various diagonal maps. The colimit of this diagram agrees with the coequalizer of the subdiagram

$$R \rightrightarrows S \,,$$

which is in turn the set of equivalence classes S/R.

The case we want to make is that this diagram shape is the start of a more natural *simplicial diagram*, extending infinitely off to the left:

$$\cdots \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; R_2 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; R_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; R_0 \;.$$

This diagram can encode the higher forms of equivalence we sought in the introduction to this chapter.

Our diagram (1.1.0.1) does not capture the symmetry property of R. It would be a simple matter to incorporate actions of the symmetric groups Σ_2 and Σ_3 on R and $R \times_S R$ into our diagram. This would lead us to the theory of *symmetric simplicial sets*. We will see in our story that the symmetry is encoded not in the shape of a simplicial diagram of sets, but in the properties we demand of it when it models an anima. This asymmetry is as much a feature as it is a bug, however: it will be necessary when we want to connect the theory of simplicial sets to that of categories.

Nothing in the diagram (1.1.0.1) forces R to be a subset of $S \times S$; we may demand only that R map to $S \times S$. The effect of this is to permit the elements of S to be equivalent in many ways. For example, if $x, y \in S$, then the fiber of the map $R \to S \times S$ is a set

$$\ulcorner xRy \urcorner \, ,$$

which is the set of ways in which x and y are equivalent – or witnesses to their equivalence. The middle map from $R \times_S R \to R$ now carries two such witnesses, $\beta \in \ulcorner xRy \urcorner$ and $\gamma \in \ulcorner yRz \urcorner$, to a composite witness $\gamma\beta \in \ulcorner xRz \urcorner$. But now we encounter a kind of *higher transitivity*: if $\alpha \in \ulcorner wRx \urcorner$, then there ought to be only one composite witness $\gamma\beta\alpha \in \ulcorner wRz \urcorner$. In other words, this composition should be *associative*: $(\gamma\beta)\alpha = \gamma(\beta\alpha)$. If it is, then what we have is the data of a *groupoid* whose objects are the elements of S and whose (iso)morphisms are elements of R, so that $\mathrm{Isom}(x, y) = \ulcorner xRy \urcorner$.

The associativity also lets us extend our diagram (1.1.0.1) to a larger diagram

$$R \times_S R \times_S R \mathrel{\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}} R \times_S R \mathrel{\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}} R \leftrightarrows S \, . \tag{1.1.0.2}$$

These diagrams are now becoming sufficiently complicated that we need to be more pedantic about the indexing categories we're using.

1.1.1 The simplex category

Definition 1.1.1.1 The *simplex category* $\mathbf{\Delta}$ is the category whose objects are nonempty, totally ordered, finite sets, and whose morphisms the monotonic maps between these.

Every object of $\mathbf{\Delta}$ is uniquely isomorphic to a finite ordinal

$$[n] = \{0 < 1 < \cdots < n\}$$

for some integer $n \geq 0$. This entitles us to refer to objects of $\mathbf{\Delta}$ as if they are all of this form.

Between these, we have the following morphisms:

- for every $j \in [n]$, the *face map* is the injective map $\delta_j \colon [n-1] \to [n]$ whose image does not contain j;
- for every $i \in [n]$, the *degeneracy map* is the surjective map $\sigma_i \colon [n+1] \to [n]$ that carries $i+1$ to i.

Every other map in the simplex category can be expressed as a composite of face and degeneracy maps. It is elementary (but boring) to prove that $\mathbf{\Delta}$ is generated by these face and degeneracy maps, subject only to the following relations, called the *simplicial identities*:

- if $i \leq j$, then
$$\delta_i \delta_j = \delta_{j+1} \delta_i \ ;$$
- if $i \leq j$, then
$$\sigma_j \sigma_i = \sigma_i \sigma_{j+1} \ ;$$
- for every i, j,
$$\sigma_j \delta_i = \begin{cases} \delta_i \sigma_{j-1} & \text{if } i < j \ ; \\ \mathrm{id} & \text{if } i \in \{j, j+1\} \ ; \\ \delta_{i-1} \sigma_j & \text{if } j+1 < i \ . \end{cases}$$

In practice, the generators-and-relations description of $\mathbf{\Delta}$ is usually more trouble than it's worth, but it does provide a schematic picture of the category $\mathbf{\Delta}$:

$$[0] \ \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \ [1] \ \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \ [2] \ \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \ [3] \ \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \ \cdots \ ,$$

as well as its opposite $\mathbf{\Delta}^{op}$:

$$\cdots \ \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \ [3] \ \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \ [2] \ \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \ [1] \ \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \ [0]$$

Notation 1.1.1.2 For any integer $n \geq 0$, we define $\mathbf{\Delta}_{\leq n} \subset \mathbf{\Delta}$ as the full subcategory spanned by the objects $[k]$ with $k \leq n$.

We started by contemplating an equivalence relation R on S in a diagrammatic way. That gave us diagram (1.1.0.1), which we now can describe efficiently as a functor $X \colon \mathbf{\Delta}^{op}_{\leq 2} \to \mathbf{Set}$ that carries $[0]$ to S, $[1]$ to R, and $[2]$ to $R \times_S R$.

Our job is to replace properties with structure, so we then considered what happens if you allow the possibility of different ways of being equivalent. The equivalence relation becomes a groupoid; S becomes the set of objects; R becomes the set of morphisms. The transitivity condition becomes a composition structure. That new structure was encoded in a map $R \times_S R \to R$, and we asked it to satisfy a *coherence condition*, which asserted the associativity of composition. That associativity is then expressed in the larger diagram (1.1.0.2), which is a functor $\mathbf{\Delta}^{op}_{\leq 3} \to \mathbf{Set}$ that carries [3] to $R \times_S R \times_S R$.

But let's examine the meaning of associativity in our groupoid a little more carefully. The value of associative laws is that they permit us to make sense of composites not only of pairs of morphisms

$$x \to y \to z\,,$$

but also of triples of morphisms

$$w \to x \to y \to z\,,$$

and even of arbitrary finite sequences of morphisms

$$x_0 \to x_1 \to \cdots \to x_n\,.$$

That's a subtle shift of perspective here that we want to take a moment to appreciate. We were never *really* interested in the equation $(\gamma\beta)\alpha = \gamma(\beta\alpha)$; what we wanted was to say that there was an unambiguous meaning to the expression $\gamma\beta\alpha$ and even of $\alpha_n\alpha_{n-1}\cdots\alpha_1$.

In other words, the data in which we are ultimately interested is not that of a single multiplication law $R \times_S R \to R$, but in fact of a family of multiplication laws

$$R^{\times_S n} \coloneqq R \times_S R \times_S \cdots \times_S R \to R\,,$$

one for each n. Associativity is then the expression of the compatibility between these. This is all packaged up very neatly in the functor $\mathbf{\Delta}^{op} \to \mathbf{Set}$ that carries $[n]$ to $R^{\times_S n}$. This is our first example of a *simplicial object*.

1.1.2 Simplicial & cosimplicial

Definition 1.1.2.1 Let C be a category. A *simplicial object* of C is a functor $\mathbf{\Delta}^{op} \to C$. A *cosimplicial object* of C is a functor $\mathbf{\Delta} \to C$. We write

$$sC \coloneqq \mathrm{Fun}(\mathbf{\Delta}^{op}, C) \qquad \text{and} \qquad cC \coloneqq \mathrm{Fun}(\mathbf{\Delta}, C)\,.$$

If $X \in sC$, then we write X_n for the value $X([n])$, and if $Y \in cC$, then we write Y^n for the value $Y([n])$. At times it may be convenient to write $X_\bullet$ and $Y^\bullet$ instead of X and Y, just to emphasize the variance.

Thus a simplicial object $X_\bullet$ specifies object $X_0, X_1, \dots$, along with *face maps* $d_j \colon X_n \to X_{n-1}$ for each $j \in [n]$ and *degeneracy maps* $s_i \colon X_n \to X_{n+1}$ for each $i \in [n]$:

$$\cdots \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} X_3 \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} X_2 \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} X_1 \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} X_0 \quad .$$

We will be interested in simplicial objects of a number of different categories, but our big interest is the theory of simplicial *sets*.

Example 1.1.2.2 If $[m], [n] \in \mathbf{\Delta}$, then let us write

$$\Delta^n_m \coloneqq \mathrm{Mor}_{\mathbf{\Delta}}([m], [n]) \,.$$

Thus for each $[n] \in \mathbf{\Delta}$, we have the simplicial set it represents

$$\Delta^n \coloneqq \mathrm{Mor}_{\mathbf{\Delta}}(-, [n]) \colon \mathbf{\Delta}^{op} \to \mathbf{Set} \,.$$

We call Δ^n the *standard n-simplex*. The assignment $[n] \mapsto \Delta^n$ is a functor $\mathbf{\Delta} \to s\mathbf{Set}$. Equally, for each $[m] \in \mathbf{\Delta}$, we have the cosimplicial set it corepresents

$$\Delta_m \coloneqq \mathrm{Mor}_{\mathbf{\Delta}}([m], -) \colon \mathbf{\Delta} \to \mathbf{Set} \,,$$

and the assignment $[m] \mapsto \Delta_m$ is a functor $\mathbf{\Delta}^{op} \to c\mathbf{Set}$.

The standard simplices play a critical role in the theory of simplicial sets. The Yoneda lemma implies that for every simplicial set X, we have a natural isomorphism

$$X_n = \mathrm{Mor}_{s\mathbf{Set}}(\Delta^n, X) \,.$$

An element $\sigma \in X_n$ – or equivalently a map $\sigma \colon \Delta^n \to X$ – is called an *n-simplex of* X. If $n = 0$, we may call this a *vertex*; if $n = 1$, we may call this an *edge*.

Every simplicial set $X \in s\mathbf{Set}$ is the colimit of its simplices. More precisely, consider the Yoneda embedding $\mathbf{\Delta} \to s\mathbf{Set}$ given by $[n] \mapsto \Delta^n$, and let

$$\mathbf{\Delta}_{/X} \coloneqq \mathbf{\Delta} \times_{s\mathbf{Set}} s\mathbf{Set}_{/X} \,.$$

We call $\mathbf{\Delta}_{/X}$ the *category of simplices* of X. We now have a canonical isomorphism

$$\operatorname*{colim}_{[n] \in \mathbf{\Delta}_{/X}} \Delta^n = X \,.$$

Said differently, the left Kan extension of the Yoneda embedding $\mathbf{\Delta} \to s\mathbf{Set}$ is the identity functor on $s\mathbf{Set}$. We will discuss this much more carefully in 1.1.3.

Example 1.1.2.3 Let A be a small category. The *nerve* $N_\bullet A$ is the simplicial set that carries $[n]$ to the *set* of functors $[n] \to A$.

In other words, by regarding each nonempty totally ordered finite set $[n]$ as a category, we obtain a fully faithful inclusion $\mathbf{\Delta} \hookrightarrow \mathbf{Cat}$. The nerve $N_\bullet A$ is the composite of $\mathbf{\Delta}^{op} \hookrightarrow \mathbf{Cat}^{op}$ with the functor $\mathbf{Cat}^{op} \to \mathbf{Set}$ represented by A.

Thus $N_0 A$ is the set of objects of A, and $N_1 A$ is the set of morphims. If $f: x \to y$ is a morphism of A, then $d_0(f) = y$, and $d_1(f) = x$. For any object x of A, the degenerate 1-simplex $s_0(x)$ is the identity at x.

The set $N_2 A$ of 2-simplices is the set of commutative triangles

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd}$$

in A. If we call this 2-simplex α, then $d_0(\alpha) = g$, $d_1(\alpha) = h$, and $d_2(\alpha) = f$. For every morphism $f: x \to y$ of A, we also have two degenerate 2-simplices $s_0(f)$ and $s_1(f)$, which correspond to the commutative triangles

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "\mathrm{id}"] \arrow[rr, "f"'] & & y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & y \arrow[dr, "\mathrm{id}"] & \\ x \arrow[ur, "f"] \arrow[rr, "f"'] & & y \end{tikzcd} ,$$

respectively.

If S is the set of objects of A and R is its set of morphisms, then we find that $N_n A$ is the n-fold fiber product $R^{\times_S n}$. The simplicial objects we extracted from relations and groupoids are therefore examples of nerves.

In particular, the standard n-simplex Δ^n is precisely the nerve $N_\bullet[n]$.

Proposition 1.1.2.4 *The nerve functor* $N_\bullet: \mathbf{Cat} \to s\mathbf{Set}$ *is fully faithful.*

Proof Let C and D be categories, and let $f: N_\bullet C \to N_\bullet D$ be a morphism of simplicial sets. We aim to show that there exists a unique functor $F: C \to D$ such that $f = N_\bullet F$.

If $F: C \to D$ is a functor such that $N_\bullet F = f$, then on objects, F is the map $f_0: N_0 C \to N_0 D$, and on morphisms, F is the map $f_1: N_1 C \to N_1 D$. So our F is certainly unique if it exists.

So define F accordingly: on objects, take the map f_0, and on morphisms, take the map f_1. The compatibility of f with the face map $d_1: N_2 \to N_1$ shows that F respects composition. The compatibility of f with the degeneracy map $s_0: N_0 \to N_1$ shows that F respects identities. Hence F is indeed a functor. □

This proposition guarantees that we lose no information when we pass from categories to simplicial sets.

1.1.3 Interlude: generators & relations of categories (incomplete)

We mentioned above that every simplicial set is the colimit of its simplices. In other words, every simplicial set is well approximated by simplices.

More generally, let $\boldsymbol{C}$ be a category with all small colimits. In most cases, $\boldsymbol{C}$ will be a large category.[1] We are interested in situations in which, even though $\boldsymbol{C}$ is large, all of its objects are well approximated – or even *determined* – by a small subcategory $A \subseteq \boldsymbol{C}$ via colimits. This will be an idea that plays a big role in our work, so let's discuss it carefully.

Definition 1.1.3.1 Let $\boldsymbol{C}$ be a category with all colimits, and let $A \subseteq \boldsymbol{C}$ be a small full subcategory. We say that A *generates* $\boldsymbol{C}$ *under colimits* if the smallest full subcategory $\boldsymbol{D} \subseteq \boldsymbol{C}$ that is closed under colimits and contains A is $\boldsymbol{C}$ itself.

We say that A *strongly generates* $\boldsymbol{C}$ *under colimits* – or that A is *dense* in $\boldsymbol{C}$ – if the left Kan extension of the inclusion $A \subseteq \boldsymbol{C}$ along itself is the identity functor on $\boldsymbol{C}$.

For every object $X \in \boldsymbol{C}$, we have the category

$$A_{/X} \coloneqq A \times_{\boldsymbol{C}} \boldsymbol{C}_{/X} \,,$$

whose objects are pairs (a, f) consisting of an object $a \in A$ and a morphism $f : a \to X$ in $\boldsymbol{C}$. The left Kan extension of the inclusion $A \hookrightarrow \boldsymbol{C}$ along itself is the functor that carries an object $X \in \boldsymbol{C}$ to the colimit $\operatorname{colim}_{a \in A_{/X}} a$. Thus A is dense in $\boldsymbol{C}$ if and only if, for every object $X \in \boldsymbol{C}$, the natural morphism

$$\operatorname*{colim}_{a \in A_{/X}} a \to X$$

is an isomorphism. So if A strongly generates $\boldsymbol{C}$, then every object of $\boldsymbol{C}$ is a colimit of objects in A in a canonical way. So certainly if A is dense in $\boldsymbol{C}$, then it generates $\boldsymbol{C}$ under colimits, but the converse is far from true. In fact, if A generates $\boldsymbol{C}$ under colimits, then it need not even be the case that every object of $\boldsymbol{C}$ can be written as a colimit of objects from A.

Example 1.1.3.2 Consider the full subcategory $\{*\} \subset \mathbf{Set}$ generated by the one-point set. The category $\{*\}$ is dense in $\mathbf{Set}$. To see this, let S be a set, and consider the category $\{*\}_{/S}$. An object of $\{*\}_{/S}$ is nothing more than an element of S, and there are no non-identity morphisms in $\{*\}_{/S}$ (because there are none

[1] This is an old argument of Peter Freyd. In fact, a small category $\boldsymbol{C}$ with all small coproducts is automatically a *preorder*. That is, if $x, y \in \boldsymbol{C}$ are two objects, then we claim that there is at most one map $x \to y$. To prove this, suppose that κ is the cardinality of the set of morphisms of $\boldsymbol{C}$, and consider the coproduct $\kappa \cdot x = \coprod_{\alpha<\kappa} x$. Now $\boldsymbol{C}(\kappa \cdot x, y) = \boldsymbol{C}(x, y)^{\kappa}$, so by Cantor, if $\#\boldsymbol{C}(x, y) \geq 2$, then $\#\boldsymbol{C}(\kappa \cdot x, y) \geq 2^{\kappa} > \kappa$, which contradicts our assumption.

in $\{*\}!$). Hence $\{*\}_{/S}$ is the set S, viewed as a category. Thus

$$\operatorname*{colim}_{s \in \{*\}_{/S}} s = \coprod_{s \in S} \{s\} = S\,.$$

Example 1.1.3.3 Let's generalize this example. Let A be a small category, and let $\boldsymbol{P}(A)$ be the category of *presheaves* on A – *i.e.*, functors $A^{op} \to \mathbf{Set}$. Thus

$$\boldsymbol{P}(A) = \mathrm{Fun}(A^{op}, \mathbf{Set})\,,$$

and in particular $s\mathbf{Set} = \boldsymbol{P}(\Delta)$. Now via the Yoneda embedding $よ\colon A \hookrightarrow \boldsymbol{P}(A)$, we consider A as a full subcategory of $\boldsymbol{P}(A)$.

The category A is dense in $\boldsymbol{P}(A)$. To prove this, consider a presheaf $F\colon A^{op} \to \mathbf{Set}$. An object of the category $A_{/F}$ is a pair (Y, f) consisting of an object $Y \in A$ and a natural transformation $f\colon よ_Y \to F$. By Yoneda, such a natural transformation is the same thing as an element $f \in F(Y)$. Thus $A_{/F}$ is sometimes called the *category of elements of F*. To prove that the canonical natural transformation

$$\eta\colon \operatorname*{colim}_{Y \in A_{/F}} よ_Y \to F$$

is an isomorphism, let $Z \in A$ be an object, and consider the map

$$\eta_Z\colon \operatorname*{colim}_{Y \in A_{/F}} \mathrm{Mor}_A(Z, Y) \to F(Z)\,.$$

The elements of the colimit on the left are equivalence classes of triples (Y, f, ϕ) consisting of an object $Y \in A$, an element $f \in F(Y)$, and a morphism $\phi\colon Z \to Y$ in A. The equivalence relation is generated by the demand that if (Y, f, ϕ) and (Y', f', ϕ') are such triples, and if there exists a map $\gamma\colon Y \to Y'$ such that $\phi' = \gamma\phi$ and $F(\gamma)(f') = f$, then (Y, f, ϕ) and (Y', f', ϕ') are equivalent. The map η_Z carries such a triple to $F(\phi)(f) \in F(Z)$. The aim is to show that this map is a bijection.

So let $g \in F(Z)$. Then $(Z, g) \in A_{/F}$ is an object, and $\mathrm{id}_Z \in A(Z, Z)$ provides an element (Z, g, id_Z) of the colimit on the left, and $\eta_Z(Z, g, \mathrm{id}_Z) = g$. Furthermore, if (Y, f, ϕ) is an element of the colimit such that $F(\phi)(f) = \eta_Z(Y, f, \phi) = g$, then by definition, $\phi\colon Z \to Y$ is a morphism such that $\phi = \phi\,\mathrm{id}_Z$ and $F(\phi)(f) = g$; thus (Y, f, ϕ) is equivalent to (Z, g, id_Z).

Example 1.1.3.4 Let **Ab** denote the category of abelian groups, and let $F \subset \mathbf{Ab}$ be the full subcategory spanned by the single object $\mathbf{Z}$. (In other words, F is the category of free abelian groups of rank 1.) What kinds of colimits can we form from F?

Certainly any free abelian group is a colimit of objects of F, since we can form the coproducts

$$\boldsymbol{Z}\{S\} := \bigoplus_{s\in S} \boldsymbol{Z}$$

for every set S. Now if E is any abelian group, then we claim that we can write E as a colimit of free abelian groups.

There's really only one way to prove that an object can be written as a colimit of some 'nice' objects: we have to somehow cook up a good supply of maps from the nice objects to our object. In our case, we can get away with a single map, as long as it is an epimorphism. We select a set $S \subseteq E$ of generators for E. (If we're feeling lazy or wasteful, we can choose $S = E$.) This defines an epimorphism

$$\phi\colon \boldsymbol{Z}\{S\} \to E\,.$$

Now the kernel of ϕ is a subgroup $K \subseteq \boldsymbol{Z}\{S\}$ such that $E = \boldsymbol{Z}\{S\}/K$. It turns out that K itself is automatically free, but that isn't obvious, and in any case we don't really need that fact. All we need to do is select some generators $T \subseteq K$. Now we have a homomorphism $\boldsymbol{Z}\{T\} \to \boldsymbol{Z}\{S\}$ whose image is K, so that E is a pushout:

$$\begin{tikzcd} \boldsymbol{Z}\{T\} \arrow[r] \arrow[d] & \boldsymbol{Z}\{S\} \arrow[d] \\ 0 \arrow[r] & E \end{tikzcd}$$

We have thus shown that every abelian group can be written as a colimit of free groups, and that every free group can be written as a colimit of copies of $\boldsymbol{Z}$. This implies that our subcategory F generates **Ab** under colimits.

But F does not *strongly* generate $\boldsymbol{Z}$. To see why not, let's consider the category $F_{/E}$ for an abelian group E. An object of $F_{/E}$ is a homomorphism $a\colon \boldsymbol{Z} \to E$; such a homomorphism is determined uniquely by the value $a(1)$, so we may regard the objects of $F_{/E}$ as *elements* of E. In this way, if $a, b \in E$ are two such objects, then a morphism $m\colon a \to b$ is an integer m such that $mb = a$ in E. Now if $E = \boldsymbol{Z} \oplus \boldsymbol{Z}$, then the elements of the form $(a, 1)$ or $(1, b)$ each give rise to infinite order elements in the colimit. Hence the colimit is much larger than $\boldsymbol{Z} \oplus \boldsymbol{Z}$.

Proposition 1.1.3.5 *Let $\mathbf{C}$ be a category with all colimits, and let $A \subseteq \mathbf{C}$ be a small full subcategory. Write $i\colon A \hookrightarrow \mathbf{C}$ for the inclusion functor. Then A is dense in $\mathbf{C}$ if and only if the restricted Yoneda*

$$i^* よ\colon \mathbf{C} \to \boldsymbol{P}(A)\,,$$

which carries $X \in \mathbf{C}$ to the functor $Y \mapsto \mathrm{Mor}_{\mathbf{C}}(Y, X)$, is fully faithful.

Proof The left Kan extension of $i\colon A \to \boldsymbol{C}$ along the Yoneda embedding is a left adjoint $よ_! i\colon \boldsymbol{P}(A) \to \boldsymbol{C}$ to the functor $i^* よ$. This functor can be evaluated as

$$(よ_! i)(F) = \operatorname*{colim}_{Y \in A_{/(F \circ i)}} Y = \int^{Y \in A} F(Y) \times Y\ .$$

The functor $i^* よ$ is fully faithful if and only if the counit $\eta\colon (よ_! i)(i^* よ) \to \mathrm{id}_{\boldsymbol{C}}$ is an isomorphism. For every object $X \in \boldsymbol{C}$, we observe that the category

$$A_{/i^* よ_X}$$

is equivalent to $A_{/X}$; hence

$$(よ_! i)(i^* よ_X) \cong \operatorname*{colim}_{Y \in A_{/X}} Y\ ,$$

and the morphism η_X is the natural maps. Since $A \subseteq \boldsymbol{C}$ is dense, it follows that this map is an isomorphism. ☐

Suppose that $\boldsymbol{C}$ is a category with colimits, and let us suppose that we want to write down a functor $s\mathbf{Set} \to \boldsymbol{C}$ that preserves colimits. We can restrict this functor along Yoneda to a functor $\boldsymbol{\Delta} \to \boldsymbol{C}$ along the Yoneda embedding. It turns out that this restriction loses no information.

Proposition 1.1.3.6 *Let $\boldsymbol{D}$ be a category with all colimits, and let A be a small category. Let*

$$\mathrm{Fun}^L(\boldsymbol{P}(A), \boldsymbol{D}) \subseteq \mathrm{Fun}(\boldsymbol{P}(A), \boldsymbol{D})$$

be the full subcategory spanned by those functors that preserve colimits. Then the restriction along the Yoneda embedding

$$\mathrm{Fun}^L(\boldsymbol{P}(A), \boldsymbol{D}) \to \mathrm{Fun}(A, \boldsymbol{D})$$

is an equivalence of categories.

Proof Exercise 1.1.1. ☐

1.1.4 Skeletal & coskeletal

The simplicial objects we have been discussing so far have all been extended from the finite subcategories $\boldsymbol{\Delta}_{\leq n}$. Let's understand the mechanism for these extensions.

We write

$$s_{\leq n}C \coloneqq \mathrm{Fun}(\Delta^{op}_{\leq n}, C) \qquad \text{and} \qquad c^{\leq n}C \coloneqq \mathrm{Fun}(\Delta_{\leq n}, C)\ .$$

Now restriction along the inclusion $\mathbf{\Delta}_{\leq n} \subset \mathbf{\Delta}$ defines functors

$$sC \to s_{\leq n}C \qquad \text{and} \qquad cC \to c^{\leq n}C\,,$$

which we will denote by $X \mapsto X_{\leq n}$ and $Y \mapsto Y^{\leq n}$.

Definition 1.1.4.1 If C has finite colimits, then these functors each admit a fully faithful left adjoint given by left Kan extension:

$$\mathrm{sk}_n : s_{\leq n}C \hookrightarrow sC \qquad \text{and} \qquad \mathrm{sk}_n : c^{\leq n}C \hookrightarrow cC\,.$$

These are called the *n-skeleton* functors.

Dually, if C has finite limits, then these functors each admit a fully faithful right adjoint given by right Kan extension:

$$\mathrm{ck}_n : s_{\leq n}C \hookrightarrow sC \qquad \text{and} \qquad \mathrm{ck}_n : c^{\leq n}C \hookrightarrow cC\,.$$

These are called the *n-coskeleton* functors.

We have the usual formulas for these Kan extensions: if $X \in s_{\leq n}C$, then

$$\mathrm{sk}_n(X)_m = \operatorname*{colim}_{[k]\in((\mathbf{\Delta}_{\leq n})_{[m]/})^{op}} X_k \qquad \text{and} \qquad \mathrm{ck}_n(X)_m = \lim_{[k]\in((\mathbf{\Delta}_{\leq n})_{/[m]})^{op}} X_k\,,$$

and if $Y \in c^{\leq n}C$, then

$$\mathrm{sk}_n(Y)^m = \operatorname*{colim}_{[k]\in(\mathbf{\Delta}_{\leq n})_{/[m]}} Y^k \qquad \text{and} \qquad \mathrm{ck}_n(Y)^m = \lim_{[k]\in(\mathbf{\Delta}_{\leq n})_{[m]/}} Y^k\,.$$

In the language of coends and ends:

$$\mathrm{sk}_n(X)_m = \int^{[k]\in\mathbf{\Delta}_{\leq n}} \Delta^k_m \times X_k \qquad \text{and} \qquad \mathrm{ck}_n(X)_m = \int_{[k]\in\mathbf{\Delta}_{\leq n}} X_k^{\Delta^m_k}\,,$$

$$\mathrm{sk}_n(Y)^m = \int^{[k]\in\mathbf{\Delta}_{\leq n}} \Delta^m_k \times Y^k \qquad \text{and} \qquad \mathrm{ck}_n(Y)^m = \int_{[k]\in\mathbf{\Delta}_{\leq n}} (Y^k)^{\Delta^k_m}\,.$$

We will sometimes abuse notation slightly by writing $\mathrm{sk}_n(X) = \mathrm{sk}_n(X_{\leq n})$ and $\mathrm{ck}_n(X) = \mathrm{ck}_n(X_{\leq n})$ for a simplicial object X. That is, we will often regard the functors sk_n and ck_n as implicitly precomposed with the restriction $sC \to s_{\leq n}C$. The formulas above remain valid.

If C has all finite limits and colimits, and if $X, Y \in sC$, then we have natural bijections

$$\mathrm{Mor}_{sC}(\mathrm{sk}_n(X), Y) = \mathrm{Mor}_{s_{\leq n}C}(X_{\leq n}, Y_{\leq n}) = \mathrm{Mor}_{sC}(X, \mathrm{ck}_n(Y))\,.$$

Definition 1.1.4.2 Let C be a category with all finite limits and colimits, and let $n \geq 0$ be an integer. A simplicial object $X \in sC$ is *n-skeletal* if and only if the natural map $\mathrm{sk}_n(X) \to X$ is an isomorphism. Accordingly, X is *n-coskeletal* if and only if the natural map $X \to \mathrm{ck}_n(X)$ is an isomorphism.

Similar definitions apply for cosimplicial objects.

Example 1.1.4.3 Let $X \in sC$ be a simplicial object. Our X is 0-skeletal if and only if it is *constant*. It is 0-coskeletal if and only if it is determined by X_0 via the formula

$$X_m = X_0^{\times(m+1)} .$$

Example 1.1.4.4 The standard n-simplex Δ^n is n-skeletal. For a simplicial set X, the n-skeleton $\mathrm{sk}_n(X)$ is the colimit of the m-simplices of X with $m \leq n$:

$$\mathrm{sk}_n(X) = \operatorname*{colim}_{[m]\in(\mathbf{\Delta}_{\leq n})_{/X}} \Delta^m .$$

(Exercise 1.1.2.)

Example 1.1.4.5 If X is a simplicial set, then we have a formula for the m-simplices of the n-coskeleton:

$$\mathrm{ck}_n(X)_m = \mathrm{Mor}_{s\mathbf{Set}}(\mathrm{sk}_n(\Delta^m), X) .$$

Example 1.1.4.6 The nerve of any small category is 2-coskeletal (Exercise 1.1.3).

1.1.5 Geometric realization (incomplete)

We have seen that every simplicial set is the colimit of its simplices.

1.1.6 Higher groupoids & higher categories (incomplete)

Let's start with (2, 1)-*categories*. These are categories with the property that between two objects x and y, we have a *groupoid* of maps $x \to y$. Accordingly, between two maps $f, g\colon x \to y$ we have a set of isomorphisms $\mathrm{Isom}(f, g)$.

1.1.7 Fibrations (incomplete)

Exercises

1.1.1 Prove Proposition 1.1.3.6.

1.1.2 We have seen that every simplicial set X is the colimit of its simplices. Now show that X is n-skeletal if and only if the canonical map

$$\operatorname*{colim}_{[m]\in\mathbf{\Delta}_{\leq n,/X}} \Delta^m \to X$$

is an isomorphism.

1.1.3 Show that the composition of the nerve functor $N\colon \mathbf{Cat} \to s\mathbf{Set}$ with the restriction $s\mathbf{Set} \to s_{\leq 2}\mathbf{Set}$ is fully faithful. Conclude that the nerve of every small category is 2-coskeletal.

1.2 Basic constructions (incomplete)

1.2.1 Functor categories (incomplete)

1.2.2 Slice categories (incomplete)

1.2.3 Limits & colimits (incomplete)

1.2.4 Twisted arrow categories (incomplete)

1.2.5 Cartesian & cocartesian fibrations (incomplete)

1.2.6 Adjunctions (incomplete)

PART TWO

THROWING AWAY THE LADDER

PART THREE

APPENDICES

Appendix A
Set theory & cardinals

Mathematicians' 'stock' set theory, ZFC (Zermelo–Fraenkel set theory ZF plus the Axiom of Choice AC) doesn't quite have the expressive power one needs for work with categories and higher categories. The issue ultimately comes down to Cantor's diagonal argument: there is no surjection of a set onto its powerset. This is ultimately why no one can contemplate a set of all sets, and it's also the key to Freyd's observation that if C is a category and κ is the cardinality of its set of arrows, then C has all κ-indexed products only if C is a poset. This, in turn, is what's behind the 'solution set condition' in representability theorems or the Adjoint Functor Theorem. Hence one really must distinguish between 'large' and 'small' objects.

One improves matters by passing to von Neumann–Bernays–Gödel set theory (NBG), which is a conservative extension of ZFC. In NBG, the formal language consists of the symbols $\in$ and $=$; a constant V; suitable variables; the usual connectives of first-order logic ($\neg, \wedge, \vee, \implies$, and $\iff$); and the quantifiers $\forall$ and $\exists$. The objects of the theory are called *classes*. A class x is called a *set* if and only if $x \in V$; a *proper class* is a class X such that $X \notin V$. We summarize the axioms of NBG in informal language:

Extensionality Classes X and Y are equal if and only if, for any z, one has $z \in X$ if and only if $z \in Y$.

Regularity For every class X, there exists an element $z \in X$ such that $z \cap X = \varnothing$.

Infinity There is an infinite set.

Union If x is a set, then $\bigcup x = \bigcup_{z \in x} z$ is a set as well.

Pairing If x and y are sets, then $\{x, y\}$ is a set as well.

Powerset If x is a set, then the powerset $P(x)$ is a set as well.

Limitation of size A class X is a proper class if and only if there is a bijection between X and V.

Class comprehension For every first-order formula $\phi(x)$ with a free variable

x in which the quantifiers are over sets, there exists a class $\{x \in V : \phi(x)\}$ whose elements are exactly those sets x such that $\phi(x)$.

In matters of set theory, we will generally follow the notations and terminological conventions of the comprehensive monograph of Jech (2003). We will also refer to the texts of Drake (1974) and Kanamori (2009). In particular, **Ord** denotes the proper class of ordinal numbers. For any ordinal α, the set V_α is defined recursively as follows:

1 If $\alpha = 0$, then $V_\alpha \coloneqq \varnothing$;
2 if $\alpha = \beta + 1$ for an ordinal number β, then $V_\alpha \coloneqq P(V_\beta)$;
3 if α is a limit ordinal, then $V_\alpha \coloneqq \bigcup_{\beta<\alpha} V_\beta$.

If x is a set, then the *rank* of x is the smallest ordinal number α such that $x \in V_\alpha$. The proper class V is then the union $\bigcup_{\alpha\in\mathbf{Ord}} V_\alpha$.

In matters of (higher) category theory, we will generally follow the notations and terminological conventions of Lurie (2009). However, our set-theoretic conventions are slightly different:

Definition A.0.0.1 A *large-category*[1] C consists of a sequence $(C_n)_{n\in\mathbf{N}_0}$ of classes, along with a family of class maps $\phi^* : C_m \to C_n$ for each map $\phi : n \to m$, subject to all the formulas that express the statement that C_n is a simplicial class that satisfies the inner Kan condition.

If the large-category C contains a full subcategory $C' \subseteq C$ such that each C'_n is a set and such that every object of C is equivalent to an object of C', then we will call C a *category* or, for emphasis, a *small category*.[2]

A large-category C is said to be *locally small* if and only if, for every subset $C'_0 \subseteq C_0$, the full subcategory $C' \subseteq C$ spanned by the elements of C'_0 is small.

Limits and colimits are only considered for functors $J \to C$ in which the J is a small category. Hence when we refer to *all limits* or *all colimits*, we mean all limits or colimits of diagrams indexed on small categories.

If C is a category and D is a large-category, then the large-category $\mathrm{Fun}(C, D)$ is defined in the usual way, so that $\mathrm{Fun}(C, D)_n$ is the class of simplicial maps $C \times \Delta^n \to D$.

Notation A.0.0.2 We shall write $\mathbf{Set}^V$ for the large-category of all sets. The objects of $\mathbf{Set}^V$ are thus precisely the elements of V.

[1] We use the term 'category' for what other authors might call '∞-category', '$(\infty, 1)$-category', or 'quasicategory'. We will use the term '1-category' when the specification is needed.
[2] These are sometimes called *essentially small*.

We shall write $\mathbf{An}^V$ for the large-category of all animae.[3] The objects of $\mathbf{An}^V$ are thus precisely the Kan complexes $\Delta^{op} \to \mathbf{Set}^V$.

If C is a category, then we shall write $\boldsymbol{P}^V(C) \coloneqq \mathrm{Fun}(C^{op}, \mathbf{An}^V)$ and $\tau_0\boldsymbol{P}^V(C) \coloneqq \mathrm{Fun}(C^{op}, \mathbf{Set}^V)$. We will write

$$よ : C \hookrightarrow \boldsymbol{P}^V(C)$$

for the Yoneda embedding $X \mapsto よ_X$. Thus if X is an object of C, then $よ_X : C^{op} \to \mathbf{An}^V$ is the functor represented by X, so that $よ_X(U) = \mathrm{Map}(U, X)$.

We shall write $\mathbf{Cat}^V$ for the large-category of all categories. The objects of $\mathbf{Cat}^V$ are thus precisely the weak Kan complexes $\Delta^{op} \to \mathbf{Set}^V$.

The Class Comprehension Axiom Schema implies the *Axiom of Global Choice*, which ensures the existence of a choice function $\tau\colon V \to V$ such that $\tau(x) \in x$. One needs this to make sense of a construction like 'the' functor $- \times u\colon C \to C$ for a large-category C with all finite products and a fixed object $u \in C$.

Here, we will work with NBG as our base theory, so that we may speak of proper classes and large-categories whenever the occasion arises.

However, the whole project of higher category theory turns on the principle that we want to be able to deal with the collections of all objects of a given kind as a mathematical object in its own right, and that the passage up and down these category levels is a fruitful way to understand even completely 'decategorified' objects.[4] If we have a large-category C, NBG provides us with no mechanism to view C itself as an object of a still larger category of all categories. This limits the kinds of operations we are permitted to do to C.

To give ourselves the room to pass up and down category levels, we need to have a hierarchy of 'scales' at which we can work. These scales will be identified by inaccessible cardinals or, equivalently, Grothendieck universes. The existence of these cardinals is independent of NBG.

But now a different sort of concern arises. We will also want to have assurance that the results we obtain at one scale remain valid at other scales. We might prefer to prove sentences about *all* sets, animae, groups, etc. – not just those within a universe. This sort of 'scale-invariance of truth' is expressed by a *reflection principle* (**??**). The reflection principle we will use here states roughly that there is an inaccessible cardinal κ such that statements of set theory hold in the universe V_κ just in case they hold in V itself. This is the *Lévy scheme* LÉVY, which we will first formulate as a large cardinal axiom. In effect, this permits us to focus our attention on *categories* (as opposed to large-categories). This scheme has a

[3] We follow Clausen and Scholze, and we use the term 'anima' for what other authors might call 'space', '∞-groupoid', or '$(\infty, 0)$-category'.

[4] The *microcosm principle* of Baez–Dolan is a precise illustration of this principle.

strictly higher conistency strength than the existence of a proper class of inaccessible cardinals; however, its consistency strength is strictly lower than that of the existence of a single Mahlo cardinal.

A.1 Regularity & smallness

Definition A.1.0.1 If κ is a cardinal, then a set S is *κ-small* if and only if $|S| < \kappa$. We shall write $\mathbf{Set}^{\kappa} \subset \mathbf{Set}^{V}$ for the category of κ-small sets, Thus $\mathbf{Set}^{V}$ is the filtered union of the categories $\mathbf{Set}^{\kappa}$ over the proper class of regular cardinals.

A cardinal κ is *regular* if and only if, for every map $f : S \to T$ in which T and every fiber $f^{-1}\{t\}$ are all κ-small, the set S is κ-small as well. Equivalently, κ is regular if and only if $\mathbf{Set}^{\kappa} \subset \mathbf{Set}^{V}$ is stable under colimits indexed by κ-small posets.

Example A.1.0.2 Under this definition, 0 is a regular cardinal.[5] There are no 0-small sets.

Example A.1.0.3 The $\aleph$ family of cardinals is defined by a function from the class of ordinal numbers to the class of cardinal numbers, by transfinite induction:

1 The cardinal $\aleph_0$ is the ordinal number ω consisting of all finite ordinals.
2 For any ordinal α, one defines $\aleph_{\alpha+1}$ to be the smallest cardinal number strictly greater than $\aleph_\alpha$.
3 For any limit ordinal α, one defines $\aleph_\alpha \coloneqq \sup\left\{\aleph_\beta : \beta < \alpha\right\}$.

The countable cardinal $\aleph_0$ is regular. Every infinite successor cardinal is regular; consequently, $\aleph_n$ for $n \in \boldsymbol{N}$ is regular as well. The cardinal $\aleph_\omega$ is the smallest infinite cardinal that is not regular.

Example A.1.0.4 A set is $\aleph_0$-small if and only if it is finite.

A.1.0.5 If κ is a regular cardinal, then $\mathbf{Set}^{\kappa} \subset \mathbf{Set}^{V}$ is the full subcategory generated by the singleton $\{0\}$ under colimits over κ-small posets.

Definition A.1.0.6 If κ is a regular cardinal, then we shall write $\mathbf{An}^{\kappa} \subset \mathbf{An}^{V}$ for the full subcategory generated by $\{0\}$ under colimits over κ-small posets. The objects of $\mathbf{An}^{\kappa}$ will be called *κ-small animae*.

Similarly, we shall write $\mathbf{Cat}^{\kappa} \subset \mathbf{Cat}^{V}$ for the full subcategory generated by $\{0\}$ and $\{0 < 1\}$ under colimits over κ-small posets. The objects of $\mathbf{Cat}^{\kappa}$ will be called *κ-small categories*.

[5] Many texts require that a regular cardinal be infinite.

Finally, a large-category C is said to be *locally λ-small* if and only if, for every λ-small subset $C_0' \subseteq C_0$ of objects of C, the full subcategory $C' \subseteq C$ that it spans is λ-small.

Example A.1.0.7 This turn of phrase above is slightly ambiguous when $\kappa = 0$. In that case, we take the phrase 'subcategory generated by … under colimits over the empty collection of posets' to mean the empty category. With this convention, there are no 0-small animae or categories:

$$\mathbf{Set}^0 = \mathbf{An}^0 = \mathbf{Cat}^0 = \varnothing\,.$$

Example A.1.0.8 An anima is $\aleph_0$-small if and only if it is weak homotopy equivalent to a simplicial set with only finitely many nondegenerate simplices.

A category C is $\aleph_0$-small if and only if it is Joyal equivalent to a simplicial set with only finitely many nondegenerate simplices.

Example A.1.0.9 Why does regularity arise so often in category theory? What role does this hypothesis play? Here is the sort of scenario that is often lurking in the background when we appeal to the regularity of a cardinal.

Let C be a large-category. Suppose that we have a *diagram of diagrams* in C, in the following sense. We have a category A; a functor $B\colon A \to \mathbf{Cat}^V$; and for each $\alpha \in A$, a functor $X_\alpha\colon B_\alpha \to C$. Furthermore, the colimits of each of these functors organize themselves into a functor $A \to C$:

$$\alpha \mapsto \operatorname*{colim}_{\beta \in B_\alpha} X_\alpha(\beta)\,.$$

We will often be in situations in which we need to analyze the *colimit of colimits*:

$$\operatorname*{colim}_{\alpha \in A} \operatorname*{colim}_{\beta \in B_\alpha} X_\alpha(\beta)\,.$$

In this case, we may reorganize these data. We first construct the cocartesian fibration corresponding to the functor B, which we will abusively write $B \to A$, since the fibers are the categories B_α. We now have a single functor $X\colon B \to C$ whose restriction to any fiber B_α is the functor X_α. Now the colimit of colimits above is a single colimit:

$$\operatorname*{colim}_{\alpha \in A} \operatorname*{colim}_{\beta \in B_\alpha} X_\alpha(\beta) \simeq \operatorname*{colim}_{\gamma \in B} X(\gamma)\,.$$

Now let κ be a cardinal. If A is κ-small, and if each category B_α is κ-small, then what can we conclude about B? In general, nothing. However, if κ is a regular cardinal, then B is also κ-small.

The motto here, then, is that *if κ is regular, then κ-small colimits of κ-small colimits are κ-small colimits.*

A.2 Accessibility & presentablity

Definition A.2.0.1 Let κ be a regular cardinal. A category Λ is *κ-filtered* if and only if it satisfies the following equivalent conditions:

1. For every κ-small category J, every functor $f\colon J \to \Lambda$ can be extended to a functor $F\colon J^{\triangleright} \to \Lambda$.
2. For every κ-small category J and every functor $H\colon \Lambda \times J \to \mathbf{An}^V$, the natural morphism
$$\operatorname*{colim}_{\lambda\in\Lambda} \lim_{j\in J} H(\lambda, j) \to \lim_{j\in J} \operatorname*{colim}_{\lambda\in\Lambda} H(\lambda, j)$$
is an equivalence.
3. For every κ-small category J, the diagonal functor $\Lambda \to \mathrm{Fun}(J, \Lambda)$ is cofinal.

Example A.2.0.2 For any regular cardinal κ, the ordinal κ, regarded as a category, is κ-filtered.

More generally, a poset is κ-filtered if and only if every κ-small subset thereof is dominated by some element.

Example A.2.0.3 A κ-small category is κ-filtered if and only if it contains a terminal object.

Example A.2.0.4 Since no category is 0-small, every category is 0-filtered.

Definition A.2.0.5 Let κ be a regular cardinal. A functor $f\colon C \to D$ between large-categories will be said to be *κ-continuous* if and only if it preserves κ-filtered colimits.

An object X of a locally small large-category C is said to be *κ-compact* if and only if the functor $よ^X\colon C \to \mathbf{An}^V$ corepresented by X (i.e., the functor $Y \mapsto \mathrm{Map}_C(X, Y)$) is κ-continuous. We write $C^{(\kappa)} \subseteq C$ for the full subcategory of κ-compact objects.

A large-category C is *κ-accessible* if and only if it satisfies the following conditions:

1. The category C is locally small.
2. The category C has all κ-filtered colimits.
3. The subcategory $C^{(\kappa)} \subseteq C$ is small.
4. The subcategory $C^{(\kappa)} \subseteq C$ generates C under κ-filtered colimits.

A κ-accessible large-category C is *κ-presentable*[6] if and only if $C^{(\kappa)}$ has all κ-small colimits.

[6] Some authors use the phrase *κ-compactly generated* instead.

Example A.2.0.6 A 0-continuous functor is one that preserves all colimits. Hence a 0-compact object X is one in which the natural map

$$\mathrm{Map}(X, \operatorname*{colim}_{\alpha \in A} Y_\alpha) \xrightarrow{\sim} \operatorname*{colim}_{\alpha \in A} \mathrm{Map}(X, Y_\alpha)$$

is an equivalence, irrespective of the category A or the diagram $Y \colon A \to C$.

The following are equivalent for a large-category C.

1. There exists a small full subcategory $D \subseteq C$ whose inclusion extends along the Yoneda embedding to an equivalence of categories (A.0.0.2)
$$\boldsymbol{P}^V(D) \xrightarrow{\sim} C\,.$$
2. The full subcategory $C^{(0)} \subseteq C$ of 0-compact objects is small, and its inclusion extends along the Yoneda embedding to an equivalence of categories
$$\boldsymbol{P}^V(C^{(0)}) \xrightarrow{\sim} C\,.$$
3. The large-category C is 0-accessible.
4. The large-category C is 0-presentable.

Example A.2.0.7 Let κ be an uncountable regular cardinal. Then the following are equivalent for a category C.

1. The category C is κ-small.
2. The set of equivalence classes of objects of C is κ-small, and for every morphism $f \colon X \to Y$ of C and every $n \in \boldsymbol{N}_0$, the set $\pi_n(\mathrm{Map}_C(X, Y), f)$ is κ-small.
3. The category C is κ-compact as an object of $\mathbf{Cat}^V$; that is, $\mathbf{Cat}^\kappa = \mathbf{Cat}^{V,(\kappa)}$.

In particular, an anima X is κ-small if and only if all its homotopy sets are κ-small, if and only if it is κ-compact as an object of $\mathbf{An}^V$.

Example A.2.0.8 The equivalence above is doubly false if $\kappa = \aleph_0$.

First, we certainly have a containment

$$\mathbf{Cat}^{\aleph_0} \subset \mathbf{Cat}^{V,(\aleph_0)}\,,$$

but this containment is proper. An $\aleph_0$-compact anima is a *retract* of an $\aleph_0$-small anima, but it may not be $\aleph_0$-small itself. If X is $\aleph_0$-compact and *simply connected*, then X is $\aleph_0$-small, but for non-simply-connected animae, we have the *de Lyra–Wall finiteness obstruction*, which lies in the reduced K_0 of the group ring $\boldsymbol{Z}[\pi_1(X)]$.

Second, the homotopy sets of an $\aleph_0$-small X anima are not generally finite. By a theorem of Serre, if each connected component $Y \subseteq X$ has finite fundamental group, then its homotopy groups are finitely generated. But if $\pi_1(X)$ isn't finite, this too fails; for example, $\pi_3(S^1 \vee S^2)$ is not finitely generated.

It is still true that the category $\mathbf{Cat}^V$ is $\aleph_0$-presentable.

A.2.0.9 Let $\kappa \leq \lambda$ be regular cardinals. A κ-small category is λ-small. A λ-filtered category is κ-filtered. A κ-continuous functor is λ-continuous. In general, however, there are κ-accessible categories that are not λ-accessible.

Definition A.2.0.10 Let κ and λ be regular cardinals. We write $\kappa \ll \lambda$ if and only if, for every pair of cardinals $\kappa_0 < \kappa$ and $\lambda_0 < \lambda$, one has $\lambda_0^{\kappa_0} < \lambda$. Equivalently, $\kappa \ll \lambda$ if and only if, for every κ-small set A and every λ-small set B, the set $\mathrm{Map}(A, B)$ is λ-small.

Example A.2.0.11 For every regular cardinal κ, one has $0 \ll \kappa$.

Example A.2.0.12 For every infinite regular cardinal κ, one has $\aleph_0 \ll \kappa$.

A.2.0.13 Let κ and λ be regular cardinals. How is the condition $\kappa \ll \lambda$ used in practice? The answer comes down to the following pair of manoeuvres, which we can do whenever $\kappa \ll \lambda$.

If J is a λ-small poset, then we can write

$$J = \bigcup_{\ell \in \Lambda} J_\ell \,,$$

where Λ is a λ-small and κ-filtered poset, and each $J_\ell \subseteq J$ is a κ-small poset. In this way, we may express any λ-small colimit as a λ-small and κ-filtered colimit of κ-small colimits:

$$\operatorname*{colim}_{j \in J} X(j) \simeq \operatorname*{colim}_{\ell \in \Lambda} \operatorname*{colim}_{j \in J_\ell} X(j)$$

(Lurie, 2009, Corollary 4.2.3.11).

On the other hand, if M is a κ-filtered poset, then we can write

$$M = \bigcup_{k \in K} M_k \,,$$

where K is a λ-filtered poset, and each $M_k \subseteq M$ is λ-small and κ-filtered. In this way, we may express any κ-filtered colimit as a λ-filtered colimit of λ-small and κ-filtered colimits:

$$\operatorname*{colim}_{m \in M} Y(m) \simeq \operatorname*{colim}_{k \in K} \operatorname*{colim}_{m \in M_k} Y(m)$$

(Lurie, 2009, Lemma 5.4.2.10).

Proposition A.2.0.14 (Lurie, 2009, Proposition 5.4.2.11) *If $\kappa \ll \lambda$ are regular cardinals, then every κ-accessible category is λ-accessible. Similarly, every κ-presentable category is λ-presentable.*

Definition A.2.0.15 A large-category C is *accessible* if and only if there exists a regular cardinal κ such that C is κ-accessible.

We shall say that C is *presentable* if and only if there exists a regular cardinal κ such that C is κ-presentable.

Example A.2.0.16 A small category is accessible if and only if it is idempotent-complete (Lurie, 2009, Corollary 5.4.3.6).

A.2.0.17 A large-category is presentable if and only if it is accessible and has all colimits. A presentable large-category automatically has all limits as well.

Definition A.2.0.18 A large-category C is *locally presentable*[7] if and only if every object $X \in C$ is contained in a presentable full large-subcategory $C' \subseteq C$ such that the inclusion $C' \hookrightarrow C$ preserves colimits.

A.2.0.19 In other words, a large-category C is locally presentable just in case it can be expressed as a class-indexed union of presentable large-categories, each of which is embedded in C via a colimit-preserving, fully faithful functor.

A.3 Presheaf categories

Let C be a large-category. What happens if we seek to make sense in NBG of the category $\tau_0 \boldsymbol{P}^V(C)$ of presheaves of sets $C^{op} \to \mathbf{Set}^V$?

Right away we encounter a problem: if the objects of C form a proper class C_0, then there is no class of class maps $\mathrm{Map}(C_0, V)$. Indeed, on one hand, in NBG, every element of a class is itself a set, and on the other hand, a class map $f : C_0 \to V$ cannot be a set.[8]

A.3.0.1 If C is a small category, then the large-category $\tau_0 \boldsymbol{P}^V(C)$ is locally small, and it enjoys many of the same good properties enjoyed by $\mathbf{Set}^V$ itself. For every regular cardinal κ, it is κ-presentable, and it is *cartesian closed*: for every pair of presheaves $X, Y : C^{op} \to \mathbf{Set}^V$, the morphisms $X \to Y$ form a presheaf $\mathrm{Mor}(X, Y) : D^{op} \to \mathbf{Set}^V$. The category $\tau_0 \boldsymbol{P}^V(C)$ is a *1-topos*.

Similarly, the category $\boldsymbol{P}^V(C)$ of presheaves $C^{op} \to \mathbf{An}^V$ is a κ-presentable topos for every regular cardinal κ.

Example A.3.0.2 Let C be a locally small category. If $Y \in C$ is an object, then $よ_Y : C^{op} \to \mathbf{Set}^V$ is the presheaf $X \mapsto \mathrm{Map}_C(X, Y)$ represented by Y.

[7] In the 1-category literature, the phrase *locally presentable category* is used for what we call *presentable category*.

[8] Worse still, the 'very large' category of classes is not cartesian closed, so there's no hope of defining $\mathrm{Map}(C_0, V)$ by means of some other artifice.

Dually, if $X \in C$ is an objects, then $よ^X : C \to \mathbf{Set}^V$ is the functor $Y \mapsto \mathrm{Map}_C(X, Y)$ corepresented by X.

Definition A.3.0.3 Let C be a locally small large-category. A *small presheaf* of sets on C is a functor $C^{op} \to \mathbf{Set}^V$ that is left Kan extended from its restriction to some small full subcategory $D \subseteq C$. We write $\tau_0 \boldsymbol{P}^V(C)$ for the locally small large-category of small presheaves of sets.

Similarly, a *small presheaf* (of animae) is a functor $C^{op} \to \mathbf{An}^V$ that is left Kan extended from its restriction to some small full subcategory $D \subseteq C$. We write $\boldsymbol{P}^V(C)$ for the locally small large-category of small presheaves.

Example A.3.0.4 Of course if C is a small category, then every presheaf on C is small. Thus the notation above does not conflict with the one established in Notation A.0.0.2.

A.3.0.5 Let C be a locally small large-category. For any small full subcategory $D \subseteq C$, we may contemplate the large-category $\boldsymbol{P}^V(D)$ of presheaves $D^{op} \to \mathbf{An}^V$. If we have an inclusion of full subcategories $D' \subseteq D \subset C$, then left Kan extension identifies $\boldsymbol{P}^V(D')$ with a full subcategory of $\boldsymbol{P}^V(D)$.

The (class-indexed) filtered union $\bigcup_D \boldsymbol{P}^V(D)$ over the class of small full subcategories of C is precisely the large-category $\boldsymbol{P}^V(C)$.

The categories $\tau_0 \boldsymbol{P}^V(C)$ and $\boldsymbol{P}^V(C)$ are thus locally presentable large-categories.

Example A.3.0.6 Let C be a locally small large-category. For any object $Y \in C$, the representable presheaf $よ_Y$ is left Kan extended from any full subcategory that contains Y. In particular, $よ_Y$ is small.

Thus the assignment $Y \mapsto よ_Y$ is the fully faithful *Yoneda embedding*

$$よ : C \hookrightarrow \boldsymbol{P}^V(C) .$$

Example A.3.0.7 Let C be a locally small large-category. If C^{op} is accessible, then $\tau_0 \boldsymbol{P}^V(C)$ and $\boldsymbol{P}^V(C)$ are the categories of accessible functors $C^{op} \to \mathbf{Set}^V$ and $C^{op} \to \mathbf{An}^V$, respectively.

A.3.0.8 Let C be a locally small large-category. The categories $\tau_0 \boldsymbol{P}^V(C)$ and $\boldsymbol{P}^V(C)$ may not enjoy all the same good features that $\mathbf{Set}^V$ and $\mathbf{An}^V$ have. The categories $\tau_0 \boldsymbol{P}^V(C)$ and $\boldsymbol{P}^V(C)$ possess all colimits, but they do not generally have all limits. For example, if C has no nonidentity arrows, then there is no terminal object in $\tau_0 \boldsymbol{P}^V(C)$.

If C^{op} is accessible or small, then $\tau_0 \boldsymbol{P}^V(C)$ and $\boldsymbol{P}^V(C)$ do have all limits.

Definition A.3.0.9 Let A be a class of categories. Let C be a locally small large-category, and let $C' \subseteq C$ be a full subcategory. Then we say that C' *generates* C

freely under A-shaped colimits if and only if, for every large-category D that has all A-shaped colimits, the following assertions obtain.

1 Every functor $C' \to D$ extends to a functor $C \to D$ that preserves A -shaped colimits.
2 For every pair of functors $F, G\colon C \to D$ that preserve A-shaped colimits, the map $\mathrm{Map}(F, G) \to \mathrm{Map}(F|C', G|C')$ is an equivalence.

If $f\colon C'' \hookrightarrow C$ is a fully faithful functor, then we will say that f *generates* C *freely under A-shaped colimits* if and only if its image $f(C'') \subseteq C$ does so.

Remark A.3.0.10 If C is not small, then in NBG we can make sense neither of $\mathrm{Fun}(C, D)$, nor of the full subcategory $\mathrm{Fun}^A(C, D) \subseteq \mathrm{Fun}(C, D)$ consisting of those functors that preserve A-shaped colimits. If however we are in a situation in which these objects *can* be made sensible, then C' generates C freely under A-shaped colimits if and only if the restriction induces an equivalence

$$\mathrm{Fun}^A(C, D) \simeq \mathrm{Fun}(C', D) .$$

Proposition A.3.0.11 *Let C be a locally small large-category. Then the Yoneda embedding $よ\colon C \hookrightarrow \boldsymbol{P}^V(C)$ generates $\boldsymbol{P}^V(C)$ freely under all colimits.*

The theory of small presheaves can be relativized to a regular cardinal κ:

Definition A.3.0.12 Let κ be a regular cardinal. Let C be a locally κ-small large-category. A *κ-small presheaf* of sets on C is a functor $C^{op} \to \mathbf{Set}^\kappa$ that is left Kan extended from its restriction to some κ-small full subcategory $D \subseteq C$. The large-category of κ-small presheaves of sets will be denoted $\tau_0\boldsymbol{P}^\kappa(C)$.

Similarly, a *κ-small presheaf* (of animae) is a functor $C^{op} \to \mathbf{An}^\kappa$ that is left Kan extended from its restriction to some κ-small full subcategory $D \subseteq C$. The large-category of κ-small presheaves will be denoted $\boldsymbol{P}^\kappa(C)$.

A.3.0.13 If C is small, then so is $\boldsymbol{P}^\kappa(C)$.

A.3.0.14 Since we have assumed that C is locally κ-small, it follows that the Yoneda embedding lands in $\boldsymbol{P}^\kappa(C)$. We can therefore characterize $\boldsymbol{P}^\kappa(C)$ as the smallest full large-subcategory of $\boldsymbol{P}^V(C)$ containing $よ(C)$ and closed under κ-small colimits.

Proposition A.3.0.15 *Let κ be a regular cardinal. Let C be a locally κ-small large-category. Then the Yoneda embedding $よ\colon C \hookrightarrow \boldsymbol{P}^\kappa(C)$ generates $\boldsymbol{P}^\kappa(C)$ freely under κ-small colimits.*

A.3.0.16 The category $\boldsymbol{P}^\kappa(C)$ has all κ-small colimits, but in general, it does not have κ-small limits, and it is not cartesian closed. To ensure these properties as well, we must turn to a discussion of inaccessible cardinals.

A.4 Strong limit & inaccessible cardinals

Definition A.4.0.1 One says that κ is a *weak limit cardinal* if and only if, for every cardinal ξ, if $\xi < \kappa$, then $\xi^+ < \kappa$.

A cardinal κ is said to be a *strong limit cardinal* if and only if, for every cardinal ξ, if $\xi < \kappa$, then $2^\xi < \kappa$ as well. Equivalently, κ is a strong limit cardinal if and only if, for every pair of κ-small sets X and Y, the set $\mathrm{Map}(X, Y)$ of maps $X \to Y$ is κ-small as well.

One says that κ is *weakly inaccessible* if and only if it is a regular, uncountable, weak limit cardinal.

One says that κ is *inaccessible*[9] if and only if it is a regular, uncountable,[10] strong limit cardinal. Equivalently, an uncountable cardinal κ is inaccessible if and only if $\mathbf{Set}^\kappa$ has all κ-small colimits and is cartesian closed. Equivalently again, an uncountable cardinal κ is inaccessible if and only if $\mathbf{Set}^\kappa$ has all κ-small colimits and all κ-small colimits.

The Generalized Continuum Hypothesis (GCH) is equivalent to the statement that the classes of strong and weak limit cardinals coincide, and similarly the classes of inaccessible and weakly inaccessible cardinal coincide.

Example A.4.0.2 A cardinal κ is a weak limit cardinal if and only if, for some limit ordinal α, one has $\kappa = \aleph_\alpha$.

Example A.4.0.3 The $\beth$ family of cardinals is defined by a function from the class of ordinal numbers to the class of cardinal numbers. It's defined by transfinite induction:

1. By definition, $\beth_0 = \aleph_0$.
2. For any ordinal α, one defines $\beth_{\alpha+1} \coloneqq 2^{\beth_\alpha}$.
3. For any limit ordinal α, one defines $\beth_\alpha \coloneqq \sup \left\{\beth_\beta : \beta < \alpha\right\}$.

The cardinal $\beth_\alpha$ is the cardinality of $V_{\omega+\alpha}$.

The Generalized Continuum Hypothesis (GCH) is equivalent to the statement that $\aleph_\alpha = \beth_\alpha$ for each ordinal α,

A cardinal κ is a strong limit cardinal if and only if, for some limit ordinal α, one has $\kappa = \beth_\alpha$.

The cardinal $\beth_\omega$ is the smallest uncountable strong limit cardinal. It is not inaccessible, however, because it is not regular.

An inaccessible cardinal κ is a $\beth$-*fixed point*: that is, $\beth_\kappa = \kappa$.

9 Some authors say *strongly inaccessible* instead of *inaccessible*.

10 We include the condition of uncountability only for convenience. It is not unreasonable to regard 0 and $\aleph_0$ as inaccessible as well.

A.4.0.4 A regular uncountable cardinal κ is inaccessible if and only if one has $\kappa \ll \kappa$.

Definition A.4.0.5 (SGA 4 I, Exposé I, §0 and Appendix) An uncountable set U is a *Grothendieck universe* if it satisfies the following conditions.

1 The set U is *transitive*: if $X \in Y \in U$, then $X \in U$ as well.
2 If $X, Y \in U$, then $\{X, Y\} \in U$ as well.
3 If $X \in U$, then the powerset $P(X) \in U$ as well.
4 If $A \in U$ and $X : A \to U$ is a map, then

$$\bigcup_{\alpha \in A} X(\alpha) \in U$$

as well.

Grothendieck universes are essentially the same thing as inaccessible cardinals. This was effectively proved by Tarski (1938). See also Bourbaki, SGA 4 I, Exposé I, Appendix.

Proposition A.4.0.6 *If κ is an inaccessible cardinal, then the set V_κ of all sets of rank less than κ is a Grothendieck universe of rank and cardinality κ.*

If U is a Grothendieck universe, then there exists an inaccessible cardinal κ such that $U = V_\kappa$.

Theorem A.4.0.7 *If κ is an inaccessible cardinal, then $V_\kappa \models$ ZFC, and $V_{\kappa+1} \models$ NBG.*

Assuming that ZFC (respectively, NBG) is consistent, then the existence of inaccessible cardinals is not provable by methods formalizable in ZFC (resp., NBG).

Axiom A.4.0.8 The *Axiom of Universes* (AU) is the assertion that every cardinal is dominated by an inaccessible cardinal, or, equivalently, every set is an element of some Grothendieck universe. *Tarski–Grothendieck set theory* is the schema TG = NBG + AU.

Under AU, the proper class of inaccessible cardinals can be well ordered. It will be helpful for us to have a notation for this.

Definition A.4.0.9 Assume AU. Let us define the $\daleth$ family of cardinals as a function from the class of ordinal numbers to the class of cardinal numbers:

1 By definition, $\daleth_0 = \aleph_0$.
2 For any ordinal α, one defines $\daleth_{\alpha+1}$ as the smallest inaccessible cardinal greater than $\daleth_\alpha$.
3 For any limit ordinal α, one defines $\daleth_\alpha \coloneqq \sup\left\{\daleth_\beta : \beta < \alpha\right\}$.

Thus $\daleth_0 = \aleph_0$, and for $\alpha \geq 1$ an ordinal, $\daleth_\alpha$ is the 'α-th inaccessible cardinal'.

A.5 Echelons of accessibility

The notions of smallness, accessibility, and presentability of categories can all be relativized to a Grothendieck universe.

Definition A.5.0.1 The *echelon* of a category C is the smallest ordinal α such that C is both locally $\beth_\alpha$-small and $\beth_{\alpha+1}$-small.

Example A.5.0.2 The category of finite sets is of echelon 0. More generally, for any ordinal number α, the category of $\beth_\alpha$-small sets is of echelon α.

Notation A.5.0.3 Let α be an ordinal number. We will denote by $\mathbf{Cat}_\alpha$ the category of categories that are $\beth_\alpha$-small.

Accordingly, we will denote by $\mathbf{Set}_\alpha$ and $\mathbf{An}_\alpha$ the categories of $\beth_\alpha$-small sets and animae, respectively.

The categories $\mathbf{Cat}_\alpha$, $\mathbf{Set}_\alpha$, and $\mathbf{An}_\alpha$ are all of echelon α.

Definition A.5.0.4 Let $\alpha \geq 1$ be an ordinal number, and let $\kappa < \beth_\alpha$ be a regular cardinal. Let C and D be categories of echelon $\leq \alpha$.

A functor $f : C \to D$ is *κ-continuous of echelon* $\leq \alpha$ if and only if it preserves all $\beth_\alpha$-small, κ-filtered colimits.

An object X of C is said to be *κ-compact of echelon* $\leq \alpha$ if and only if the functor $\text{よ}^X : C \to \mathbf{An}_\alpha$ corepresented by X is κ-continuous of echelon $\leq \alpha$. We write $C_\alpha^{(\kappa)} \subseteq C$ for the full subcategory of κ-compact objects of echelon $\leq \alpha$.

A category C is *κ-accessible of echelon* $\leq \alpha$ if and only if it satisfies the following quartet of conditions:

1 The category C is of echelon $\leq \alpha$.
2 The category C has all $\beth_\alpha$-small, κ-filtered colimits.
3 The subcategory $C_\alpha^{(\kappa)} \subseteq C$ is $\beth_\alpha$-small.
4 The subcategory $C_\alpha^{(\kappa)}$ generates C under $\beth_\alpha$-small and κ-filtered colimits.

A category C is *κ-presentable of echelon* $\leq \alpha$ if and only if it is κ accessible of echelon $\leq \alpha$, and $C_\alpha^{(\kappa)}$ has all κ-small colimits.

A category C is *accessible of echelon* $\leq \alpha$ if and only if there exists a regular cardinal $\kappa < \beth_\alpha$ such that C is κ-accessible of echelon $\leq \alpha$. It is *presentable of echelon* $\leq \alpha$ if and only if there exists a regular cardinal $\kappa < \beth_\alpha$ such that C is κ-presentable of echelon $\leq \alpha$.

Example A.5.0.5 Let $\alpha \geq 1$ be an ordinal number. Let C be a $\beth_\alpha$-small category. The category $\boldsymbol{P}^{\beth_\alpha}(C)$ is then 0-presentable of echelon $\leq \alpha$.

Conversely, if D is a 0-presentable category of echelon $\leq \alpha$, then there exists a $\beth_\alpha$-small category C and an equivalence $D \simeq \boldsymbol{P}^{\beth_\alpha}(C)$.

Notation A.5.0.6 Let $\alpha \geq 1$ be an ordinal number, and let $\kappa < \beth_\alpha$ be a regular cardinal. We shall write $\mathbf{Acc}_\kappa^\alpha \subset \mathbf{Cat}_{\alpha+1}$ for the following subcategory.

1 The objects of $\mathbf{Acc}_\kappa^\alpha$ are the κ-accessible categories of echelon $\leq \alpha$.
2 The morphisms $f : C \to D$ of $\mathbf{Acc}_\kappa^\alpha$ are the κ-continuous functors of echelon $\leq \alpha$ such that $f(C_\alpha^{(\kappa)}) \subseteq D_\alpha^{(\kappa)}$.

Similarly, we shall write $\mathbf{Pr}_\kappa^{\alpha,L} \subset \mathbf{Acc}_\kappa^\alpha$ for the following subcategory.

1 The objects of $\mathbf{Pr}_\kappa^{\alpha,L}$ are the κ-presentable categories of echelon $\leq \alpha$.
2 The morphisms of $\mathbf{Pr}_\kappa^{\alpha,L}$ are those functors in $\mathbf{Acc}_\kappa^\alpha$ that preserve all $\beth_\alpha$-small colimits.

Now we may write

$$\mathbf{Acc}^\alpha = \bigcup_{\kappa < \beth_\alpha} \mathbf{Acc}_\kappa^\alpha \qquad \text{and} \qquad \mathbf{Pr}^{\alpha,L} = \bigcup_{\kappa < \beth_\alpha} \mathbf{Pr}_\kappa^{\alpha,L}$$

for the category of accessible categories of echelon α and the category of presentable categories of echelon α, respectively.

We specify some corresponding subcategories of $\mathbf{Cat}_\alpha$. Let $\mathbf{Cat}_\alpha^{idem} \subset \mathbf{Cat}_\alpha$ denote the full subcategory consisting of the idempotent-complete $\beth_\alpha$-small categories. Let $\mathbf{Cat}_\alpha^\kappa \subseteq \mathbf{Cat}_\alpha$ denote the subcategory whose objects are $\beth_\alpha$-small categories that possess all κ-small colimits and whose morphisms are functors that preserve κ-small colimits. Finally, let

$$\mathbf{Cat}_\alpha^{\kappa,idem} \coloneqq \mathbf{Cat}_\alpha^\kappa \cap \mathbf{Cat}_\alpha \ .$$

Please observe that if κ in uncountable, then in fact $\mathbf{Cat}_\alpha^{\kappa,idem} = \mathbf{Cat}_\alpha^\kappa$, because one can split an idempotent using a colimit that is $\aleph_1$-small and $\aleph_0$-filtered (Lurie, 2009, Corollary 4.4.5.15 & Example 5.3.1.9)

A.5.0.7 Let $\alpha \geq 1$ be an ordinal number, and let $\kappa < \beth_\alpha$ be a regular cardinal. The assignment $C \mapsto C_\alpha^{(\kappa)}$ defines a functor

$$\mathbf{Acc}_\kappa^\alpha \to \mathbf{Cat}_\alpha^{idem} \ .$$

This functor is an equivalence; furthermore, it restricts to an equivalence

$$\mathbf{Pr}_\kappa^{\alpha,L} \xrightarrow{\sim} \mathbf{Cat}_\alpha^{\kappa,idem} \ .$$

We will describe the inverses of these equivalences in the next section.

A.6 Indization

Definition A.6.0.1 Let $\alpha \geq 1$ be an ordinal number, and let $\kappa < \beth_\alpha$ be a regular cardinal. Let C be a category of echelon $\leq \alpha$.

Then $\mathrm{Ind}^\alpha_\kappa(C)$ is the smallest full subcategory $D \subseteq \boldsymbol{P}^{\beth_\alpha}(C)$ such that D contains the image of the Yoneda embedding よ$: C \hookrightarrow \boldsymbol{P}^{\beth_\alpha}(C)$, and D is stable under $\beth_\alpha$-small, κ-filtered colimits.

Accordingly, if C is a locally small large-category, then $\mathrm{Ind}^V_\kappa(C)$ is the smallest full subcategory $D \subseteq \boldsymbol{P}^V(C)$ such that D contains the image of the Yoneda embedding よ$: C \hookrightarrow \boldsymbol{P}^V(C)$, and D is stable under κ-filtered colimits.

Example A.6.0.2 The category $\mathrm{Ind}^\alpha_0(C)$ is equivalent to the presheaf category $\boldsymbol{P}^{\beth_\alpha}(C)$.

Proposition A.6.0.3 *Let $\alpha \geq 1$ be an ordinal number, and let $\kappa < \beth_\alpha$ be a regular cardinal. Let C be a category of echelon $\leq \alpha$. Then the Yoneda embedding* よ$: C \hookrightarrow \mathrm{Ind}^\alpha_\kappa(C)$, *generates* $\mathrm{Ind}^\alpha_\kappa(C)$ *freely under $\beth_\alpha$-small, κ-filtered colimits.*

Similarly, if C is a locally small large-category, then the Yoneda embedding よ$: C \hookrightarrow \mathrm{Ind}^V_\kappa(C)$, *generates* $\mathrm{Ind}^V_\kappa(C)$ *freely under κ-filtered colimits.*

A.6.0.4 Let $\alpha \geq 1$ be an ordinal number, and let $\kappa < \beth_\alpha$ be a regular cardinal. Let C and D be categories that contain all $\beth_\alpha$-small, κ-filtered colimits. Denote by $\mathrm{Fun}^\alpha_\kappa(C, D)$ the full subcategory of $\mathrm{Fun}(C, D)$ consisting of the functors $C \to D$ that preserve all $\beth_\alpha$-small, κ-filtered colimits.

If C' is a category of echelon $\leq \alpha$, then restriction along the Yoneda embedding induces an equivalence of categories

$$\mathrm{Fun}^\alpha_\kappa(\mathrm{Ind}^\alpha_\kappa(C'), D) \xrightarrow{\sim} \mathrm{Fun}(C', D)\,.$$

Example A.6.0.5 Let $\alpha \geq 1$ be an ordinal number, and let $\kappa < \beth_\alpha$ be a regular cardinal. Let C be a $\beth_\alpha$-small category. Then the category $\mathrm{Ind}^\alpha_\kappa(C)$ is κ-accessible of echelon $\leq \alpha$.

Hence we obtain a functor

$$\mathrm{Ind}^\alpha_\kappa\colon \mathbf{Cat}_\alpha \to \mathbf{Acc}^\alpha_\kappa\,.$$

This functor exhibits $\mathbf{Acc}^\alpha_\kappa$ as a localization of $\mathbf{Cat}_\alpha$, and it restricts to the inverse

$$\mathbf{Cat}^{idem}_\alpha \xrightarrow{\sim} \mathbf{Acc}^\alpha_\kappa$$

of the equivalence $C \mapsto C^{(\kappa)}_\alpha$ constructed in (A.5.0.7). It also restricts further to the equivalence

$$\mathbf{Cat}^{\kappa,idem}_\alpha \xrightarrow{\sim} \mathbf{Pr}^{\alpha,L}_\kappa$$

inverse to the restriction of $C \mapsto C^{(\kappa)}_\alpha$.

Construction A.6.0.6 Let $\beta > \alpha \geq 1$ be two ordinal numbers. Then we can use indization to define a change-of-universe functor

$$\boldsymbol{I}_\alpha^\beta \coloneqq \mathrm{Ind}_{\beth_\alpha}^\beta \ .$$

If $\kappa < \beth_\alpha$ is a regular cardinal, then this is a fully faithful functor

$$\boldsymbol{I}_\alpha^\beta \colon \mathbf{Acc}_\kappa^\alpha \hookrightarrow \mathbf{Acc}_\kappa^\beta \ ,$$

which is equivalent to the inclusion $\mathbf{Cat}_\alpha^{idem} \hookrightarrow \mathbf{Cat}_\beta^{idem}$. The functor $\boldsymbol{I}_\alpha^\beta$ restricts to a fully faithful functor

$$\boldsymbol{I}_\alpha^\beta \colon \mathbf{Pr}_\kappa^{\alpha,L} \hookrightarrow \mathbf{Pr}_\kappa^{\beta,L} \ ,$$

which is equivalent to the inclusion $\mathbf{Cat}_\alpha^{\kappa,idem} \hookrightarrow \mathbf{Cat}_\beta^{\kappa,idem}$.

Example A.6.0.7 For any category C of echelon α, one has

$$\boldsymbol{I}_\alpha^\beta(\boldsymbol{P}^{\beth_\alpha}(C)) \simeq \boldsymbol{P}^{\beth_\beta}(C) \ .$$

Notation A.6.0.8 Let $\alpha \geq 1$ be an ordinal number. Then we will abberviate

$$\mathrm{Ind}^\alpha \coloneqq \mathrm{Ind}_{\aleph_0}^\alpha \ .$$

A.7 Higher inaccessibility

A.7.0.1 We shall endow an ordinal with its order topology. This may be described recursively as follows:

1 The ordinal 0 is the empty topological space.
2 For any ordinal α with its order topology, the order topology on the ordinal $\alpha + 1$ is the one-point compactification of α.
3 For any limit ordinal α, the order topology is the colimit topology $\mathrm{colim}_{\beta<\alpha} \beta$.

We will use terminology that treats **Ord** itself as a topological space, even though it is not small.

Definition A.7.0.2 If $W \subseteq \mathbf{Ord}$ is a subclass, then a *limit point* of A is an ordinal α such that $\alpha = \sup(W \cap \alpha)$. The class W will be said to be *closed* if and only if it contains all its limit points.

An *ordinal function* is a class map $f \colon \mathbf{Ord} \to \mathbf{Ord}$. We say that f is *continuous* if and only if its restriction to any subset is continuous. Equivalently, f is continuous if and only if, for every subclass $W \subseteq \mathbf{Ord}$ and every limit point α of W, the ordinal $f(\alpha)$ is a limit point of $f(W)$.

We say that f is *normal* if and only if it is continuous and strictly increasing.

A.7.0.3 If f is a normal ordinal function, then its image is a closed and unbounded class[11] of ordinals. Conversely, if $W \subseteq \mathbf{Ord}$ is a closed and unbounded class, then we can define a normal ordinal function f by

$$f(\alpha) = \min \{\gamma \in W : (\forall \beta < \alpha)(f(\beta) < \gamma)\} .$$

Definition A.7.0.4 Let f be an ordinal function. A regular cardinal κ is said to be f*-inaccessible* if and only if, for every ordinal α, if $\alpha < \kappa$, then $f(\alpha) < \kappa$ as well.

Example A.7.0.5 If f is the ordinal function that carries an ordinal α to the cardinal $2^{|\alpha|}$, then an f-inaccessible cardinal is precisely an inaccessible cardinal.

Construction A.7.0.6 Let f be an increasing ordinal function such that for every ordinal β, one has $\beta < f(\beta)$. For every ordinal ξ, the normal ordinal function $\alpha \mapsto f^{\alpha}(\xi)$ in uniquely specified by the requirements that $f^0(\xi) = \xi$ and $f^{\alpha+1}(\xi) = f(f^{\alpha}(\xi))$.

Jorgensen (1970) proves that an f-inaccessible cardinal greater than an ordinal ξ is precisely a regular cardinal that is a *fixed point* for the ordinal function $\alpha \mapsto f^{\alpha}(\xi)$.

Example A.7.0.7 If f is the ordinal function $\beta \mapsto 2^{|\beta|}$, then $f^{\alpha}(\omega) = \beth_{\alpha}$. An inaccessible cardinal is thus precisely a regular $\beth$-fixed point.

If f is the ordinal function $\beta \mapsto |\beta|^{+}$, then $f^{\alpha}(\omega) = \aleph_{\alpha}$. A weakly inaccessible cardinal is precisely a regular $\aleph$-fixed point.

Example A.7.0.8 Assume AU. Consider the ordinal function f that carries an ordinal β to the smallest inaccessible cardinal greater than β. For any ordinal α, we have $\daleth_{\alpha} = f^{\alpha}(\omega)$.

An f-inaccessible cardinal is precisely a $\daleth$-fixed point. These are called *1-inaccessible cardinals*. If κ is 1-inaccessible, then $V_{\kappa} \vDash (\text{ZFC} + \text{AU})$. If ZFC + AU is consistent, then the existence of 1-inaccessible cardinals is not provable by methods formalizable in ZFC + AU.

Iterating this strategy, one can now proceed to define α-inaccessibility for every ordinal α. Iterating the iteration, one can define notions of hyperinaccessibility, hyper$^{\alpha}$inaccessibility, *etc*. We cut to the chase:

Axiom A.7.0.9 The *Lévy scheme* (LÉVY) is the assertion that for every ordinal function f and every ordinal ξ, there exists an f-inaccessible cardinal κ such that $\xi < \kappa$.

[11] This is often abbreviated *club class* in set theory literature.

Theorem A.7.0.10 (Lévy (1960); Montague (1962); Jorgensen (1970)) *The following are equivalent.*

1. *The Lévy scheme.*
2. *Every normal ordinal function has a regular cardinal in its image.*
3. *Every closed unbounded subclass $W \subseteq$ **Ord** contains a regular cardinal.*
4. *Every normal ordinal function has an inaccessible cardinal in its image.*
5. *Every closed unbounded subclass $W \subseteq$ **Ord** contains an inaccessible cardinal.*

A.7.0.11 The Lévy scheme implies the Axiom of Universes, and the consistency strength of NBG + LÉVY is strictly greater than that of NBG + AU.

The consistency strength of the Lévy scheme is also strictly greater than the existence of α-inaccessible, hyperinaccessible, hyper$^\alpha$inaccessible, *etc.*, cardinals.

Definition A.7.0.12 Let κ be a regular cardinal. One says that κ is *Mahlo* if and only if every closed unbounded subset $W \subseteq \kappa$ contains a regular cardinal.

A.7.0.13 Assume that κ is a Mahlo cardinal. Then κ is f-inaccessible for every ordinal function f. Accordingly, κ is a fixed point of every normal ordinal function.

Additionally, if κ is a Mahlo cardinal, then $V_\kappa \models$ (ZFC + LÉVY), and similarly $V_{\kappa+1} \models$ (NBG + LÉVY). The consistency strength of the axiom 'a Mahlo cardinal exists' is strictly greater than the Lévy scheme.

A.7.0.14 The Lévy scheme and its equivalents and slight variants have appeared under various names: 'Mahlo's principle' (Gloede, 1973), 'Axiom F' (Drake, 1974), '**Ord** is Mahlo' (Hamkins, 2003).

For our purposes, one of the main appeals of the Lévy scheme is the following.

Theorem A.7.0.15 *Assume* LÉVY.

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