∞-categories for the working mathematician

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Contents

PART ONE

HIGHER GROUPOIDS & HIGHER CATEGORIES

1
Grothendieck's homotopy hypothesis

The mission of homotopy theory is to iteratively enhance every *property* of a mathematical object to *structure* on it. The homotopy theorist deconstructs the equals sign by this process: they no longer regard '$x = y$' as a property that x and y together possess, but rather as piece of structure that connects x and y. That structure is then a *path* between x and y. Semantically, we consider this path as a *reason* for – or as a *witness* to – the equality '$x = y$'.

If α and β are paths connecting x and y, then again we do not wish to speak of '$\alpha = \beta$' as a property, but as a further piece of structure – a *homotopy* between α and β. We iterate: two homotopies are no longer merely 'equal', but they may be connected by *higher homotopies*; two higher homotopies may be connected by further higher homotopies, *etc*., *etc*., *ad infinitum*.

The data of all these points and paths and homotopies and higher homotopies, taken together, constitute an *anima* (plural: *animae*). Animae are also called *spaces*, *homotopy types*, *Kan complexes*, or ∞-*groupoids*. These terms each reflect a certain attitude toward these objects. The terms 'space' and 'homotopy type' acknowledge that these objects were first modelled and understood using topological spaces and topological notions of homotopy. A 'Kan complex' is then a combinatorial blueprint for these homotopies and their relations. The phrase '∞-groupoid' then reinterprets the (higher) homotopies as (higher) *symmetries*. The fact that these terms can all be used interchangeably is a nontrivial insight – Grothendieck's *homotopy hypothesis*. We will formulate and prove a version of this sentence in this chapter – surely not the version Grothendieck had in mind, but one that is better-adapted to the needs of contemporary mathematicians. In our formulation, it becomes a theorem of Kan.

Our adoption of 'anima' reflects our desire not to favor any one of these attitudes. Animae are for us fundamental objects: they play the same role in *homotopical mathematics* that sets play in 'ordinary' mathematics. In ordinary set-theoretic mathematics, every object is ultimately defined in terms of sets;

similarly, every object in homotopical mathematics is defined in terms of animae.

Symmetries of objects were certainly central to the mathematics of the 19th century, but it's a distinctly 20th century concept that symmetries might meaningfully have their own symmetries. (It is interesting to consider the origins of this idea, but it would be difficult to pinpoint the first person who seriously considered it.) In any case, the 20th century provided three realizations about homotopy theory's iterative enhancement process.

1. Foremost was the promise of interesting new phenomena to study. Homological algebra appears at first to be a relatively featureless outgrowth of linear algebra. The Bockstein homomorphism is an early illustration that ordinary modules interact in new ways in derived settings. But it's the nontriviality of the Hopf element η in the stable homotopy group π_1^s that decisively separates *homological algebra* – ordinary algebra that is then derived – from *homotopical algebra* – algebra done in a natively homotopical setting. The first signal from the mysterious world of homotopical algebra was a short message: '$\eta \neq 0$'.
2. In spite of our informal description of homotopy theory as an inductive enhancement of properties into structure, by the middle of the 20th century, the idea had been made precise, and it had become part of the standard toolkit of algebraic topology.

   In fact, it can now be made precise in a number of different ways. On one hand, we can model homotopical structures entirely via ordinary mathematical objects. We could, for example, model them using topological structures. For reasons we discuss below, we will adopt the Kanian approach and encode homotopical structures using *simplicial diagrams*.

   Alternatively – and in a spirit closer to that of this introduction – one might instead attempt to rewrite the logical foundations of mathematics in a way that bakes in our preference for structure over properties. This is the approach of *homotopy type theory*, a stirring vision of new foundations in mathematics. As of this writing, homotopy type theory is still in its infancy. In later editions of this book, perhaps the combinatorics of simplices will be replaced by fundamental facts about type theory.
3. Once it was understood how to model homotopical structures accurately, new questions arose: *how do we construct models that are maximally useful? what does it mean to say that two models represent the same homotopy theory? shouldn't this notion of sameness, whatever it is, be subject to the same inductive refinement process that got us here?* These questions lay at the heart

of the many foundational developments in homotopy theory starting in the 1970s.

By the end of the millennium it was clear that one would need to take a further step, and contemplate a *homotopy theory of homotopy theories*. Just as Grothendieck had seen that ordinary homotopy theory should be equivalent to the theory of ∞-groupoids, Joyal, Kan, Rezk, Simpson, and Toen all recognized that the homotopy theory of homotopy theories should be equivalent to the theory of ∞-*categories*.

Our approach to constructing the homotopy theory of animae follows a general recipe, which will inform our work throughout this book. The recipe provides a strategy for designing a homotopy theory $\boldsymbol{E}$, using only partial or imperfect information about it:

1 Select a piece $\boldsymbol{E}_0$ of $\boldsymbol{E}$ that is simple enough that you can understand it completely, but complex enough so that any object $X \in \boldsymbol{E}$ is completely determined by the sets/groups/whatevers of maps $T \to X$ with $T \in \boldsymbol{E}_0$. In many cases, you'll want $\boldsymbol{E}_0$ to generate $\boldsymbol{E}$ under suitable colimits.

2 Now select a small category $\boldsymbol{J}$, usually combinatorial in nature, along with an essentially surjective functor $R\colon \boldsymbol{J} \to \boldsymbol{E}_0$ (which need not be fully faithful). In many cases, it will be helpful if the objects of $\boldsymbol{J}$ come equipped with some notion of *degree*. The category $\boldsymbol{J}$ and functor R should be chosen so that some key salient features of general objects $X \in \boldsymbol{E}$ can be read off from mapping objects $\mathrm{Map}(R(j), X)$.

3 Encode these salient features as a set of properties $\boldsymbol{P}$ of the corresponding functors $\mathrm{Map}(R(-), X)$. For example, perhaps you'll want to require that for some particular $j \in \boldsymbol{J}$, the object $\mathrm{Map}(R(j), X)$ is trivial. Or perhaps such a property expresses $\mathrm{Map}(R(k), X)$ as a limit or colimit of various other mapping objects $\mathrm{Map}(R(j), X)$. The objects of your homotopy theory will be exactly the functors $\boldsymbol{J}^{op} \to \mathbf{Set}$ (or other enriching category) that enjoy these properties of $\boldsymbol{P}$.

In this chapter, we will construct the homotopy theory of animae in this way: $\boldsymbol{E}_0$ will be the homotopy theory of contractible animae (which is trivial), $\boldsymbol{J}$ is the category of *simplices* Δ (defined below), and $\boldsymbol{P}$ is Kan's *horn-filling condition*.

Other options for $\boldsymbol{J}$ (and therefore $\boldsymbol{P}$) are possible: there is an interesting class of categories – the *test categories* – that can stand in for Δ. These include the category of nonempty finite sets, Joyal's categories Θ_n, as well as various categories of cubes and trees.

One special feature of Δ is that it can also be used to define the larger homotopy theory of ∞-categories itself. For this homotopy theory, $\boldsymbol{E}_0$ is the homotopy theory of finite ordinals (which is not trivial, but is nevertheless very simple), $\boldsymbol{J}$ is again the category Δ, and $\boldsymbol{P}$ is the *inner horn-filling condition* first identified explicitly by Boardman–Vogt.

1.1 Simplicial objects

Our aim is to convert equality from a property into a structure. At a minimum, we want to be able to work with notions of equivalence that are not just set-theoretic equality. Traditionally, such an alternative is described as an *equivalence relation* on a set S.

A *relation* is encoded by a subset $R \subseteq S \times S$, which is the subset of related pairs of elements. An *equivalence* relation is one that is reflexive, symmetric, and transitive:

1 Reflexivity is the condition that R contains the image of the *diagonal map* $S \to S \times S$ given by $x \mapsto (x, x)$.
2 Symmetry is the condition that R is stable under the involution of $S \times S$ given by $(x, y) \mapsto (y, x)$.
3 Transitivity states that the projection $S \times S \times S \to S \times S$ given by $(x, y, z) \mapsto (x, z)$ carries the subset $R \times_S R \subseteq S \times S \times S$ to the subset R.

An equivalence relation on S can thus be converted into a diagram

$$R \times_S R \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; R \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; S\,, \tag{1.1.0.1}$$

The maps from left to right are various projections. The maps from right to left are various diagonal maps. The colimit of this diagram agrees with the coequalizer of the subdiagram

$$R \rightrightarrows S\,,$$

which is in turn the set of equivalence classes S/R.

The case we want to make is that this diagram shape is the start of a more natural *simplicial diagram*, extending infinitely off to the left:

$$\cdots \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; R_2 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; R_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; R_0\,.$$

This diagram can encode the higher forms of equivalence we sought in the introduction to this chapter.

Our diagram (1.1.0.1) does not capture the symmetry property of R. It would be a simple matter to incorporate actions of the symmetric groups Σ_2 and Σ_3 on R and $R \times_S R$ into our diagram. This would lead us to the theory of *symmetric simplicial sets*. We will see in our story that the symmetry is encoded not in the shape of a simplicial diagram of sets, but in the properties we demand of it when it models an anima. This asymmetry is as much a feature as it is a bug, however: it will be necessary when we want to connect the theory of simplicial sets to that of categories.

Nothing in the diagram (1.1.0.1) forces R to be a subset of $S \times S$; we may demand only that R map to $S \times S$. The effect of this is to permit the elements of S to be equivalent in many ways. For example, if $x, y \in S$, then the fiber of the map $R \to S \times S$ is a set

$$\ulcorner xRy \urcorner\,,$$

which is the set of ways in which x and y are equivalent – or witnesses to their equivalence. The middle map from $R \times_S R \to R$ now carries two such witnesses, $\alpha \in \ulcorner xRy \urcorner$ and $\beta \in \ulcorner yRz \urcorner$, to a witness $\beta\alpha \in \ulcorner xRz \urcorner$. It now becomes natural to ask whether this operation is *associative*: is $\gamma(\beta\alpha) = (\gamma\beta)\alpha$? If so, then what we have is the data of a *groupoid* whose objects are the elements of S and whose (iso)morphisms are elements of R, so that $\mathrm{Isom}(x, y) = \ulcorner xRy \urcorner$.

The associativity also lets us extend our diagram (1.1.0.1) to a larger diagram

$$R \times_S R \times_S R \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; R \times_S R \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; R \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; S\,. \tag{1.1.0.2}$$

These diagrams are now becoming sufficiently complicated that we need to be more pedantic about the indexing categories we're using.

1.1.1 The simplex category

Definition 1.1.1.1 The *simplex category* Δ is the category whose objects are nonempty, totally ordered, finite sets, and whose morphisms the monotonic maps between these.

Every object of Δ is uniquely isomorphic to a finite ordinal

$$[n] = \{0 < 1 < \cdots < n\}$$

for some integer $n \geq 0$. This entitles us to refer to objects of Δ as if they are all of this form.

Between these, we have the following morphisms:

– for every $j \in [n]$, the *face map* is the injective map $\delta_j : [n-1] \to [n]$ whose image does not contain j;

– for every $i \in [n]$, the *degeneracy map* is the surjective map $\sigma_i : [n+1] \to [n]$ that carries $i + 1$ to i.

Every other map in the simplex category can be expressed as a composite of face and degeneracy maps. It is elementary (but boring) to prove that Δ is generated by these face and degeneracy maps, subject only to the following relations, called the *simplicial identities*:

– if $i \leq j$, then

$$\delta_i \delta_j = \delta_{j+1} \delta_i \ ;$$

– if $i \leq j$, then

$$\sigma_j \sigma_i = \sigma_i \sigma_{j+1} \ ;$$

– for every i, j,

$$\sigma_j \delta_i = \begin{cases} \delta_i \sigma_{j-1} & \text{if } i < j \ ; \\ \mathrm{id} & \text{if } i \in \{j, j+1\} \ ; \\ \delta_{i-1} \sigma_j & \text{if } j + 1 < i \ . \end{cases}$$

In practice, the generators-and-relations description of Δ is usually more trouble than it's worth, but it does provide a schematic picture of the category Δ:

$$[0] \ \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \ [1] \ \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \ [2] \ \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \ [3] \ \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \ \cdots \ ,$$

as well as its opposite Δ^{op}:

$$\cdots \ \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \ [3] \ \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \ [2] \ \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \ [1] \ \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \ [0]$$

Notation 1.1.1.2 For any integer $n \geq 0$, we define $\Delta_{\leq n} \subset \Delta$ as the full subcategory spanned by the objects $[k]$ with $k \leq n$.

We started by contemplating an equivalence relation R on S in a diagrammatic way. That gave us diagram (1.1.0.1), which we now can describe efficiently as a functor $X : \Delta^{op}_{\leq 2} \to \mathbf{Set}$ that carries $[0]$ to S, $[1]$ to R, and $[2]$ to $R \times_S R$.

Our job is to replace properties with structure, so we then considered what happens if you allow the possibility of different ways of being equivalent. The equivalence relation becomes a groupoid; S becomes the set of objects; R becomes the set of morphisms. The transitivity condition becomes a composition structure. That new structure was encoded in a map $R \times_S R \to R$, and we asked it to satisfy a *coherence condition*, which asserted the associativity of composition. That associativity is then expressed in the larger diagram (1.1.0.2), which is a functor $\Delta^{op}_{\leq 3} \to \mathbf{Set}$ that carries [3] to $R \times_S R \times_S R$.

But let's examine the meaning of associativity in our groupoid a little more carefully. The value of associative laws is that they permit us to make sense of composites not only of pairs of morphisms

$$x \to y \to z \ ,$$

but also of triples of morphisms

$$w \to x \to y \to z \ ,$$

and even of arbitrary finite sequences of morphisms

$$x_0 \to x_1 \to \cdots \to x_n \ .$$

That's a subtle shift of perspective here that we want to take a moment to appreciate. We were never *really* interested in the equation $(\gamma\beta)\alpha = \gamma(\beta\alpha)$; what we wanted was to say that there was an unambiguous meaning to the expression $\gamma\beta\alpha$ and even of $\alpha_n\alpha_{n-1}\cdots\alpha_1$.

In other words, the data in which we are ultimately interested is not that of a single multiplication law $R \times_S R \to R$, but in fact of a family of multiplication laws

$$R \times_S R \times_S \cdots \times_S R \to R \ .$$

Associativity is then the expression of the compatibility between these. This is all packaged up very neatly in the functor $\Delta^{op} \to \mathbf{Set}$ that carries $[n]$ to the n-fold fiber product of R over S:

$$R \times_S R \times_S \cdots \times_S R \ .$$

This is our first example of a *simplicial object*.

1.1.2 Simplicial & cosimplicial

Definition 1.1.2.1 Let C be a category. A *simplicial object* of C is a functor $\Delta^{op} \to C$. A *cosimplicial object* of C is a functor $\Delta \to C$. We write

$$sC \coloneqq \mathrm{Fun}(\Delta^{op}, C) \qquad \text{and} \qquad cC \coloneqq \mathrm{Fun}(\Delta, C) \ .$$

If $X \in sC$, then we write X_n for the value $X([n])$, and if $Y \in cC$, then we write Y^n for the value $Y([n])$. At times it may be convenient to write $X_\bullet$ and $Y^\bullet$ instead of X and Y, just to emphasize the variance.

Thus a simplicial object $X_\bullet$ specifies object $X_0, X_1, \ldots$, along with *face maps* $d_j : X_n \to X_{n-1}$ for each $j \in [n]$ and *degeneracy maps* $s_i : X_n \to X_{n+1}$ for each $i \in [n]$:

$$\cdots \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} X_3 \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} X_2 \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} X_1 \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} X_0 \ .$$

We will be interested in simplicial objects of a number of different categories, but our big interest is the theory of simplicial *sets*.

Example 1.1.2.2 If $[m], [n] \in \Delta$, then let us write

$$\Delta^n_m \coloneqq \mathrm{Mor}_\Delta([m],[n]) \ .$$

Thus for each $[n] \in \Delta$, we have the simplicial set it represents

$$\Delta^n \coloneqq \mathrm{Mor}_\Delta(-,[n]) : \ \Delta^{op} \to \mathbf{Set} \ .$$

We call Δ^n the *standard n-simplex*. The assignment $[n] \mapsto \Delta^n$ is a functor $\Delta \to s\mathbf{Set}$. Equally, for each $[m] \in \Delta$, we have the cosimplicial set it corepresents

$$\Delta_m \coloneqq \mathrm{Mor}_\Delta([m],-) : \ \Delta \to \mathbf{Set} \ ,$$

and the assignment $[m] \mapsto \Delta_m$ is a functor $\Delta^{op} \to c\mathbf{Set}$.

The standard simplices play a critical role in the theory of simplicial sets. The Yoneda lemma implies that for every simplicial set X, we have a natural isomorphism

$$X_n = \mathrm{Mor}_{s\mathbf{Set}}(\Delta^n, X) \ .$$

An element $\sigma \in X_n$ – or equivalently a map $\sigma : \ \Delta^n \to X$ – is called an *n-simplex of* X. If $n = 0$, we may call this a *vertex*; if $n = 1$, we may call this an *edge*.

Every simplicial set $X \in s\mathbf{Set}$ is the colimit of its simplices. More precisely, consider the Yoneda embedding $\Delta \to s\mathbf{Set}$ given by $[n] \mapsto \Delta^n$, and let

$$\Delta_{/X} \coloneqq \Delta \times_{s\mathbf{Set}} s\mathbf{Set}_{/X} \ .$$

We call $\Delta_{/X}$ the *category of simplices* of X. We now have a canonical isomorphism

$$\operatorname*{colim}_{[n]\in\Delta_{/X}} \Delta^n = X \ .$$

Said differently, the left Kan extension of the Yoneda embedding $\Delta \to s\mathbf{Set}$ is the identity functor on $s\mathbf{Set}$. (Exercise 1.1.1.)

Example 1.1.2.3 Let A be a small category. The *nerve* $N_\bullet A$ is the simplicial set that carries $[n]$ to the *set* of functors $[n] \to A$.

In other words, by regarding each nonempty totally ordered finite set $[n]$ as a category, we obtain a fully faithful inclusion $\Delta \hookrightarrow \mathbf{Cat}$. The nerve $N_\bullet A$ is the composite of $\Delta^{op} \hookrightarrow \mathbf{Cat}^{op}$ with the functor $\mathbf{Cat}^{op} \to \mathbf{Set}$ represented by A.

Thus $N_0 A$ is the set of objects of A, and $N_1 A$ is the set of morphims. If $f : x \to y$ is a morphism of A, then $d_0(f) = y$, and $d_1(f) = x$. For any object x of A, the degenerate 1-simplex $s_0(x)$ is the identity at x.

The set $N_2 A$ of 2-simplices is the set of commutative triangles

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd}$$

in A. If we call this 2-simplex α, then $d_0(\alpha) = g$, $d_1(\alpha) = h$, and $d_2(\alpha) = f$. For every morphism $f : x \to y$ of A, we also have two degenerate 2-simplices $s_0(f)$ and $s_1(f)$, which correspond to the commutative triangles

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "\mathrm{id}"] \arrow[rr, "f"'] & & y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & y \arrow[dr, "\mathrm{id}"] & \\ x \arrow[ur, "f"] \arrow[rr, "f"'] & & y \end{tikzcd} ,$$

respectively.

If S is the set of objects of A and R is its set of morphisms, then we find that $N_n A$ is the n-fold fiber product

$$R \times_S R \times_S \cdots \times_S R .$$

Thus we have recovered the simplicial objects we extracted from relations and groupoids.

In particular, the standard n-simplex Δ^n is precisely the nerve $N_\bullet[n]$.

Proposition 1.1.2.4 *The nerve functor* $N_\bullet : \mathbf{Cat} \to s\mathbf{Set}$ *is fully faithful.*

Proof Let C and D be categories, and let $f : N_\bullet C \to N_\bullet D$ be a morphism of simplicial sets. We aim to show that there exists a unique functor $F : C \to D$ such that $f = N_\bullet F$.

If $F : C \to D$ is a functor such that $N_\bullet F = f$, then on objects, F is the map $f_0 : N_0 C \to N_0 D$, and on morphisms, F is the map $f_1 : N_1 C \to N_1 D$. So our F is certainly unique if it exists.

So define F accordingly: on objects, take the map f_0, and on morphisms, take the map f_1. The compatibility of f with the face map $d_1 : N_2 \to N_1$ shows that F respects composition. The compatibility of f with the degeneracy map $s_0 \colon N_0 \to N_1$ shows that F respects identities. Hence F is indeed a functor. $\square$

This proposition guarantees that we lose no information when we pass from categories to simplicial sets.

1.1.3 Skeletal & coskeletal

The simplicial objects we have been discussing so far have all been extended from the finite subcategories $\Delta_{\leq n}$. Let's understand the mechanism for these extensions.

We write

$$s_{\leq n}C \coloneqq \mathrm{Fun}(\Delta^{op}_{\leq n}, C) \qquad \text{and} \qquad c^{\leq n}C \coloneqq \mathrm{Fun}(\Delta_{\leq n}, C)\ .$$

Now restriction along the inclusion $\Delta_{\leq n} \subset \Delta$ defines functors

$$sC \to s_{\leq n}C \qquad \text{and} \qquad cC \to c^{\leq n}C\ ,$$

which we will denote by $X \mapsto X_{\leq n}$ and $Y \mapsto Y^{\leq n}$.

Definition 1.1.3.1 If C has finite colimits, then these functors each admit a fully faithful left adjoint given by left Kan extension:

$$\mathrm{sk}_n \colon s_{\leq n}C \hookrightarrow sC \qquad \text{and} \qquad \mathrm{sk}_n \colon c^{\leq n}C \hookrightarrow cC\ .$$

These are called the *n-skeleton* functors.

Dually, if C has finite limits, then these functors each admit a fully faithful right adjoint given by right Kan extension:

$$\mathrm{ck}_n \colon s_{\leq n}C \hookrightarrow sC \qquad \text{and} \qquad \mathrm{ck}_n \colon c^{\leq n}C \hookrightarrow cC\ .$$

These are called the *n-coskeleton* functors.

We have the usual formulas for these Kan extensions: if $X \in s_{\leq n}C$, then

$$\mathrm{sk}_n(X)_m = \operatorname*{colim}_{[k]\in((\Delta_{\leq n})_{[m]/})^{op}} X_k \qquad \text{and} \qquad \mathrm{ck}_n(X)_m = \lim_{[k]\in((\Delta_{\leq n})_{/[m]})^{op}} X_k\ ,$$

and if $Y \in c^{\leq n}C$, then

$$\mathrm{sk}_n(Y)^m = \operatorname*{colim}_{[k]\in(\Delta_{\leq n})_{/[m]}} Y^k \qquad \text{and} \qquad \mathrm{ck}_n(Y)^m = \lim_{[k]\in(\Delta_{\leq n})_{[m]/}} Y^k\ .$$

In the language of coends and ends:

$$\mathrm{sk}_n(X)_m = \int^{[k]\in\Delta_{\leq n}} \Delta^k_m \times X_k \qquad \text{and} \qquad \mathrm{ck}_n(X)_m = \int_{[k]\in\Delta_{\leq n}} X_k^{\Delta^m_k}\ ,$$

$$\mathrm{sk}_n(Y)^m = \int^{[k]\in\Delta_{\leq n}} \Delta^m_k \times Y^k \qquad \text{and} \qquad \mathrm{ck}_n(Y)^m = \int_{[k]\in\Delta_{\leq n}} (Y^k)^{\Delta^k_m} .$$

We will sometimes abuse notation slightly by writing $\mathrm{sk}_n(X) = \mathrm{sk}_n(X_{\leq n})$ and $\mathrm{ck}_n(X) = \mathrm{ck}_n(X_{\leq n})$ for a simplicial object X. That is, we will often regard the functors sk_n and ck_n as implicitly precomposed with the restriction $sC \to s_{\leq n}C$. The formulas above remain valid.

If C has all finite limits and colimits, and if $X, Y \in sC$, then we have natural bijections

$$\mathrm{Mor}_{sC}(\mathrm{sk}_n(X), Y) = \mathrm{Mor}_{s_{\leq n}C}(X_{\leq n}, Y_{\leq n}) = \mathrm{Mor}_{sC}(X, \mathrm{ck}_n(Y)) .$$

Definition 1.1.3.2 Let C be a category with all finite limits and colimits, and let $n \geq 0$ be an integer. A simplicial object $X \in sC$ is *n-skeletal* if and only if the natural map $\mathrm{sk}_n(X) \to X$ is an isomorphism. Accordingly, X is *n-coskeletal* if and only if the natural map $X \to \mathrm{ck}_n(X)$ is an isomorphism.

Similar definitions apply for cosimplicial objects.

Example 1.1.3.3 Let $X \in sC$ be a simplicial object. Our X is 0-skeletal if and only if it is *constant*. It is 0-coskeletal if and only if it is determined by X_0 via the formula

$$X_m = X_0^{\times(m+1)} .$$

Example 1.1.3.4 The standard n-simplex Δ^n is n-skeletal. For a simplicial set X, the n-skeleton $\mathrm{sk}_n(X)$ is the colimit of the m-simplices of X with $m \leq n$:

$$\mathrm{sk}_n(X) = \operatorname*{colim}_{[m]\in(\Delta_{\leq n})_{/X}} \Delta^m .$$

(Exercise 1.1.2.)

Example 1.1.3.5 If X is a simplicial set, then we have a formula for the m-simplices of the n-coskeleton:

$$\mathrm{ck}_n(X)_m = \mathrm{Mor}_{s\mathbf{Set}}(\mathrm{sk}_n(\Delta^m), X) .$$

Example 1.1.3.6 The nerve of any small category is 2-coskeletal (Exercise 1.1.3).

1.1.4 Boundaries & horns

Definition 1.1.4.1 Let X be a simplicial set. A *simplicial subset* $Y \subseteq X$ is a choice of a subset $Y_n \subseteq X_n$ for each $n \geq 0$ such that for any map $\varphi : [m] \to [n]$ in Δ, the induced map $X_n \to X_m$ carries Y_n to Y_m.

We are particularly interested here in simplicial subsets of the simplicial set Δ^n.

Notation 1.1.4.2 Let $0 \leq i_0 < i_1 < \cdots < i_k \leq n$ be integers. Then we write

$$\Delta^{\{i_0,\ldots,i_k\}} \subseteq \Delta^n$$

for the corresponding simplicial subset of Δ^n. This simplicial subset is itself a k-simplex.

Families of simplicial subsets of a simplicial set X can be intersected or unioned, just as with subsets of a set.

Definition 1.1.4.3 Let $n \geq 0$ be an integer. For every integer $0 \leq i \leq n$, the *i-th face* is the simplicial subset

$$\Delta^{\hat{\imath}} \coloneqq \Delta^{\{0,\ldots,i-1,i+1,\ldots,n\}} \subset \Delta^n .$$

This is the unique $(n-1)$-simplex of Δ^n that does not contain the vertex $\Delta^{\{i\}}$.

The *boundary* of the n-simplex is the union of all the faces of Δ^n:

$$\partial\Delta^n \coloneqq \bigcup_{0\leq i\leq n} \Delta^{\hat{\imath}} \subset \Delta^n .$$

For every integer $0 \leq k \leq n$, the *k-th horn* is the union of all but the k-th face of Δ^n:

$$\Lambda^n_k \coloneqq \bigcup_{0\leq i\leq n, i\neq k} \Delta^{\hat{\imath}} \subset \Delta^n .$$

This is the union of all the faces of Δ^n that contain the vertex $\Delta^{\{k\}}$.

Equivalently, $\partial\Delta^n$ can be described as the $(n-1)$-skeleton of the n-simplex:

$$\partial\Delta^n = \mathrm{sk}_{n-1}(\Delta^n) .$$

Thus

$$\mathrm{Mor}_{s\mathbf{Set}}(\partial\Delta^n, X) = \mathrm{ck}_{n-1}(X)_n .$$

1.1.5 Higher groupoids & higher categories

1.1.6 Fibrations

Exercises

1.1.1 Use the Yoneda lemma to show that for any small category A, the left Kan extension of the Yoneda embedding $A \hookrightarrow \mathrm{Fun}(A^{op}, \mathbf{Set})$ along itself is the identity functor.

1.1.2 We have seen that every simplicial set X is the colimit of its simplices. Now show that X is n-skeletal if and only if the canonical map

$$\operatorname*{colim}_{[m]\in\Delta_{\leq n,/X}} \Delta^m \to X$$

is an isomorphism.

1.1.3 Show that the composition of the nerve functor $N : \mathbf{Cat} \to s\mathbf{Set}$ with the restriction $s\mathbf{Set} \to s_{\leq 2}\mathbf{Set}$ is fully faithful. Conclude that the nerve of every small category is 2-coskeletal.

1.2 Basic constructions

1.2.1 Functor categories

1.2.2 Slice categories

1.2.3 Limits & colimits

1.2.4 Twisted arrow categories

1.2.5 Cartesian & cocartesian fibrations

1.2.6 Adjunctions

PART TWO

BASIC HIGHER CATEGORY THEORY

References