Miscellany

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Balmer spectrum

Let C be a symmetric monoidal stable presentable category.

An *idempotent algebra* in C is an object A along with a map $\eta\colon 1 \to A$ such that $\eta \otimes A\colon A \xrightarrow{\sim} A \otimes A$. The category of idempotent algebras forms a poset, which is the poset of closeds in a locale $\operatorname{Spec} C$. So an idempotent algebra A corresponds to a closed $Z_A \subseteq \operatorname{Spec} C$, which corresponds to an open complement $U_A \subseteq \operatorname{Spec} C$.

The category C now localizes over $\operatorname{Spec} C$: for every open set $U_A \subseteq \operatorname{Spec} C$, let

$$O_{\operatorname{Spec} C}(U_A) = C/\mathbf{Mod}(A)\ .$$

Accordingly, $\Gamma(\operatorname{Spec} C, O) = C$.

The pair $(\operatorname{Spec} C, O)$ is

Geometric backgrounds

Let $\boldsymbol{X}$ be a topos, and let $\boldsymbol{X}_0 \subseteq \boldsymbol{X}$ be a subcategory that is stable under pullback. Let $\boldsymbol{C}$ be a symmetric monoidal stable presentable category.

A *six functor formalism* for $(\boldsymbol{X}, \boldsymbol{X}_0)$ valued in $\boldsymbol{C}$-modules is a lax symmetric monoidal functor

$$\boldsymbol{A}\colon\ \operatorname{Span}(\boldsymbol{X}; \boldsymbol{X}, \boldsymbol{X}_0) \to \mathbf{Mod}(\boldsymbol{C})$$

that restricts to a sheaf

$$\boldsymbol{X}^{op} = \operatorname{Span}(\boldsymbol{X}; \boldsymbol{X}, \boldsymbol{X}^{\simeq}) \to \mathbf{Mod}(\boldsymbol{C})$$

Such a triple $(\boldsymbol{X}, \boldsymbol{X}_0, \boldsymbol{A})$ is what we call a *geometric background*.