THE GEOMETRY OF ∞-CATEGORIES

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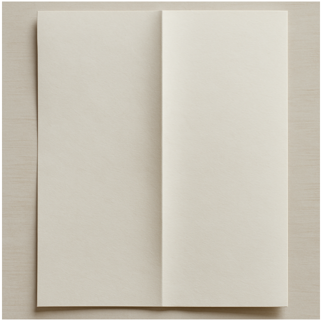
A CREASE

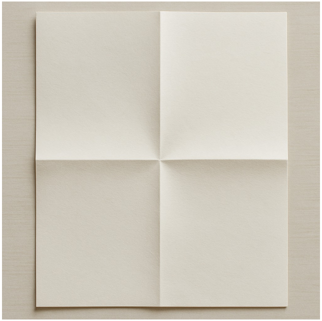
Take a piece of paper; fold it down the middle, creating a sharp crease.



The **topology** of the paper has not changed, but the **geometry** has:
at points on the crease, things like derivatives no longer work!
You have introduced a **singular locus**.

TWO CREASES

Fold your piece of paper again, perpindicular to the original fold.



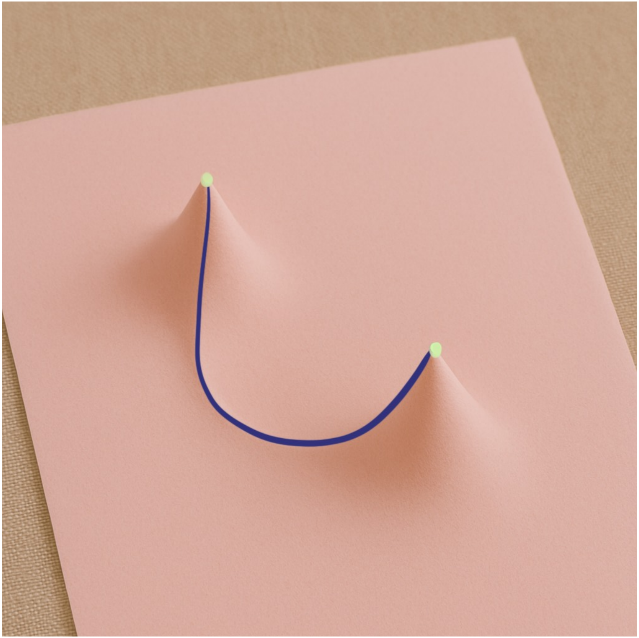
Still the topology hasn’t changed.

You’ve changed the geometry by introducing a new singular locus, and in the process, you’ve also made that point in the center **more singular**.

STRATIFICATIONS: THE IDEA

A **stratification** of a topological space X partitions it into locally closed subsets $X_i \subseteq X$ – called *strata*.
These arose in work of Whitney, Thom, and Mather in order to cope with complicated singular loci in higher dimensions.

STRATIFICATIONS: PICTURES



STRATIFICATIONS: THE GENERAL TOPOLOGY APPROACH

Let P be a **poset**. We declare $U \subseteq P$ **open** iff,

$$(a \in U \,\&\, a \leq b) \implies b \in U$$

Example

In $[1] := \{0 < 1\}$, the point 0 is closed, but 1 is not.

Now a **stratified topological space** X/P consists of:

- a topological space X,
- a poset P, and
- a continuous map $f\colon X \to P$.

The fiber $X_a = f^{-1}\{a\}$ is called the a**-th stratum**.

Example

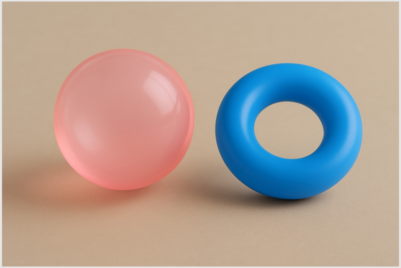
A stratification of X over the poset $[1] = \{0 < 1\}$ is a decomposition $X = X_0 \cup X_1$, in which:

- the 0-th stratum X_0 is closed, and
- the 1-st stratum X_1 is the open complement.

STRATIFICATIONS: INCOMPARABLE STRATA

Example

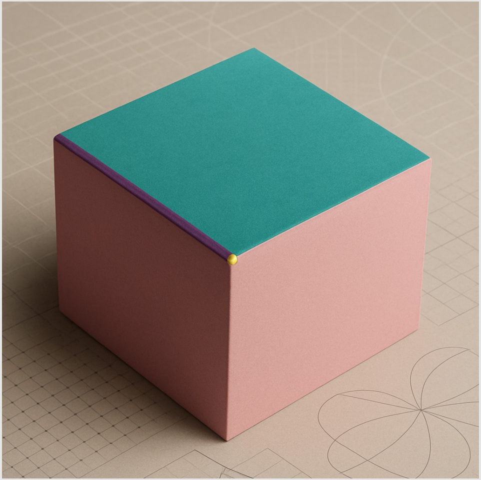
If P is a trivial poset (i.e., every pair of elements is incomparable), then a stratification of X over P is a decomposition of X as a disjoint union of clopens X_a for $a \in P$.



STRATIFICATIONS: LESS TRIVIAL EXAMPLE

Example

We can stratify the **corner** $[0,1)^n$ via the map $[0,1)^n \to [n] := \{0 < \cdots < n\}$ that carries t to the largest i such that $t_i \neq 0$.

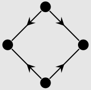


STRATIFICATIONS: CIRCLE

Example

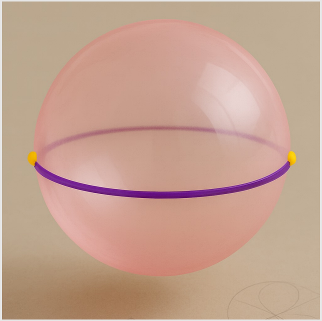
Stratify S^0 over the trivial poset $\{-1, +1\}$.

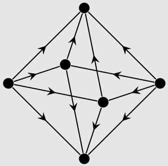
Now let's **suspend** this to stratify S^1:



Example

Let's suspend again. Now we have a stratification of S^2:



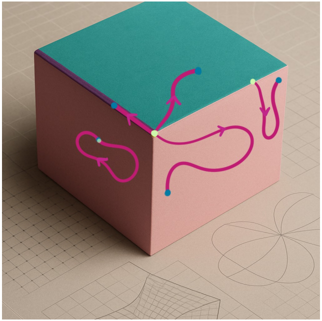


EXIT PATHS

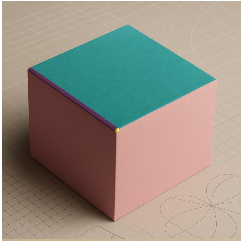
There's an ∞-**category** attached to X/P:

- an object is a point of X
- a morphism from one point to another is an **exit path**:
- a 2-morphism is a **homotopy** of exit paths
- etc.

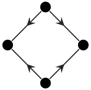
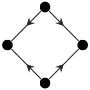
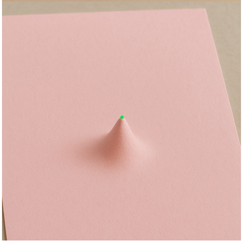
This is the **exit-path** ∞-**category** $\mathrm{Exit}(X/P)$.



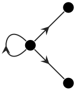
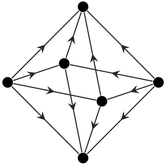
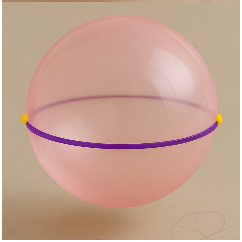
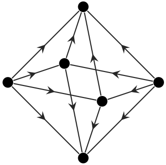
EXIT PATHS: EXAMPLES

Poset	Stratification	Exit-path ∞-category
$[n]$		$[n]$
$[1]$		$\bullet \rightrightarrows \bullet$

EXIT PATHS: EXAMPLES

Poset	Stratification	Exit-path ∞-category
		
$[1]$		

EXIT PATHS: EXAMPLES

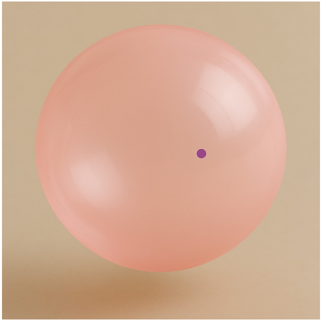
Poset	Stratification	Exit-path ∞-category
$[1]$		
		

EXIT PATHS: NOT ALL 1-CATEGORIES

Let's stratify $S^2 \to [1]$ as a point and its complement. The space of ways to exit from the point into the complement is an S^1.

$\mathrm{Exit}(S^2/[1])$ is a 2-category with

- two objects x_0 and x_1
- only one interesting morphism, which goes $\gamma\colon x_0 \to x_1$
- countably many 2-isomorphisms from γ to itself.



THIS WORKS FOR A SPHERE IN ANY DIMENSION

This story works for any sphere actually:
stratify S^n over $[1]$ so that

- the 0-th stratum is a single point
- the 1-st stratum is everything else.

Now $\mathrm{Exit}(S^n/[1])$ has two objects x_0 and x_1, with $\mathrm{Map}(x_0, x_1) = S^{n-1}$ and no other nontrivial maps.
In particular, for $n \geq 3$, **these are not *n*-categories for any finite *n*!**

Theorem (Ayala–Francis–Rozenblyum (2019), Haine (2018))
Let E be an ∞-category in which every endomorphism is an automorphism. Then there is a stratified topological space X/P, unique up to stratified homotopy equivalence, such that

$$E = Exit(X/P)$$

Furthermore, this construction can be performed functorially.

In other words, **these ∞-categories are stratified homotopy types**.

THE EXODROMY THEOREM

Theorem (Lurie (2012), Haine–Porta–Teyssier (2024))
Let X/P be a ***reasonable*** *stratified topological space, and let C be a* ***reasonable*** *category of coefficients. Then there is an equivalence of categories*

$$Constr(X/P, C) \simeq Fun(Exit(X/P), C)$$

We can think of this as an exceptionally general form of the **fundamental theorem of covering space theory**.

BEYOND

Stratified homotopy types also make sense in **algebraic geometry**, particularly for the **étale topology**.

Spec **Z** turns out to look something like this as a stratified space: