THE GEOMETRY OF ∞-CATEGORIES

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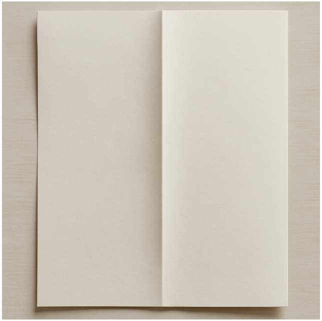
A CREASE

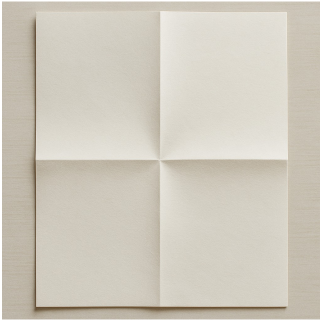
Take a piece of paper; fold it down the middle, creating a sharp crease.



The **topology** of the paper has not changed, but the **geometry** has:
at points on the crease, things like derivatives no longer work!
You have introduced a **singular locus**.

TWO CREASES

Fold your piece of paper again, perpindicular to the original fold.



Still the topology hasn’t changed.

You’ve changed the geometry by introducing a new singular locus, and in the process, you’ve also made that point in the center **more singular**.

STRATIFICATIONS: THE IDEA

A **stratification** of a topological space X partitions it into locally closed subsets $X_i \subseteq X$ – called *strata*.
These arose in work of Whitney, Thom, and Mather in order to cope with complicated singular loci in higher dimensions.

STRATIFICATIONS: PICTURES

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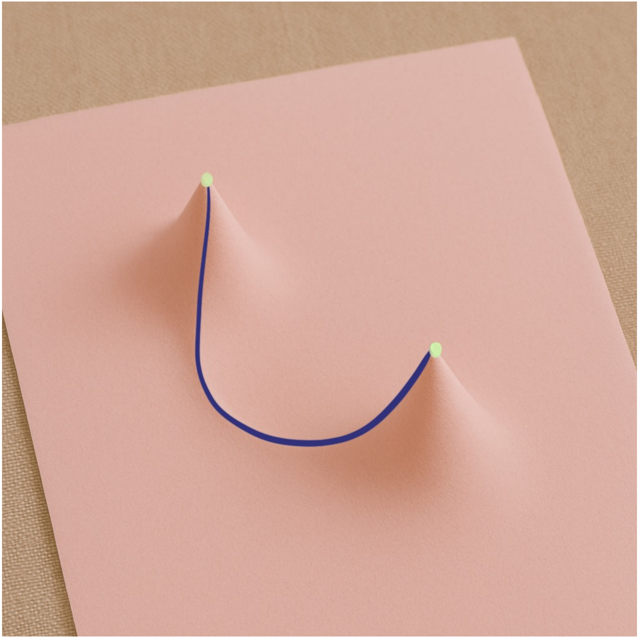


STRATIFICATIONS: THE GENERAL TOPOLOGY APPROACH

Let P be a **poset**. We declare $U \subseteq P$ **open** iff,

$$(a \in U \,\&\, a \leq b) \implies b \in U$$

Example

In $[1] := \{0 < 1\}$, the point 0 is closed, but 1 is not.

Now a **stratified topological space** X/P consists of:

- a topological space X,
- a poset P, and
- a continuous map $f\colon X \to P$.

The fiber $X_a = f^{-1}\{a\}$ is called the **a-th stratum**.

STRATIFICATIONS: TWO STRATA

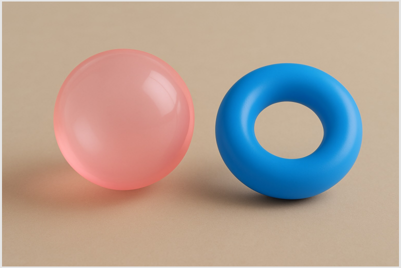
Example

A stratification of X over the poset $[1] = \{0 < 1\}$ is a decomposition $X = X_0 \cup X_1$, in which:

- the 0-th stratum X_0 is closed, and
- the 1-st stratum X_1 is the open complement.

Example

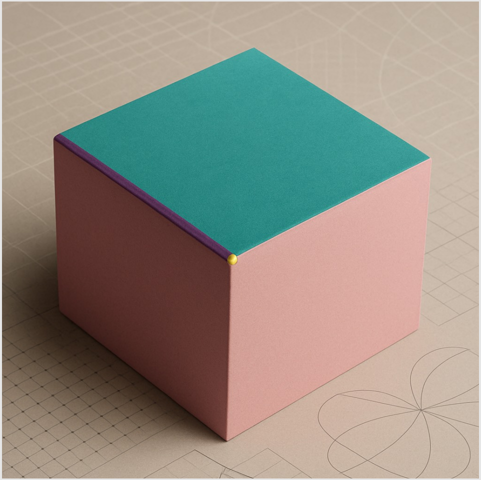
If P is a trivial poset (i.e., every pair of elements is incomparable), then a stratification of X over P is a decomposition of X as a disjoint union of clopens X_a for $a \in P$.



STRATIFICATIONS: LESS TRIVIAL EXAMPLE

Example

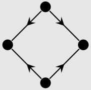
We can stratify the **corner** $[0,1)^n$ via the map $[0,1)^n \to [n] := \{0 < \cdots < n\}$ that carries t to the largest i such that $t_i \neq 0$.



Example

Stratify S^0 over the trivial poset $\{-1, +1\}$.

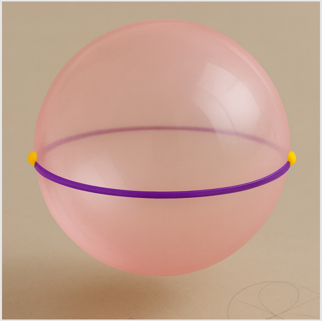
Now let's **suspend** this to stratify S^1:

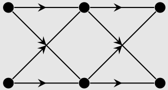


STRATIFICATIONS: SPHERE

Example

Let's suspend again. Now we have a stratification of S^2:

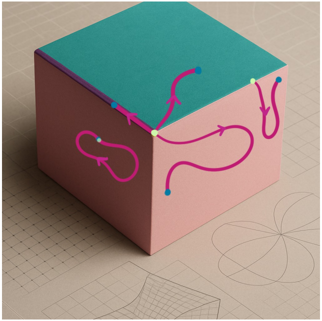




EXIT PATHS

There's an ∞-**category** attached to X/P:

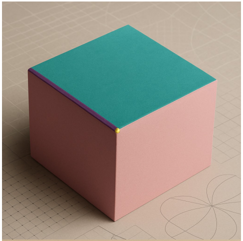
- an object is a point of X
    - a morphism from one point to another is an **exit path**:



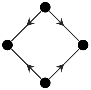
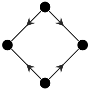
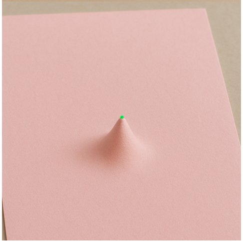
- a 2-morphism is a **homotopy** of exit paths
- etc.

This is the **exit-path** ∞-**category** $\mathrm{Exit}(X/P)$.

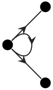
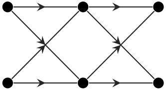
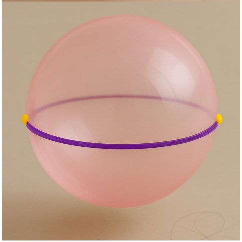
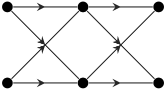
EXIT PATHS: EXAMPLES

Poset	Stratification	Exit-path ∞-category
$[n]$		$[n]$
$[1]$		$\bullet \rightrightarrows \bullet$

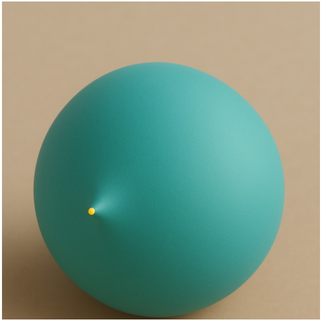
EXIT PATHS: EXAMPLES

Poset	Stratification	Exit-path ∞-category
		
$[1]$		

EXIT PATHS: EXAMPLES

Poset	Stratification	Exit-path ∞-category
$[1]$		
		

THESE AREN'T ALL 1-CATEGORIES



There are two strata, each of which is contractible. That means there are two objects, neither of which has automorphisms. But the space of ways to exit from the yellow point into the green region is an S^1.

In other words, $\mathrm{Exit}(S^2/[1])$ is a 2-category with

- two objects x_0 and x_1
- only one interesting morphism, which goes $\gamma\colon x_0 \to x_1$
- countably many 2-isomorphisms from γ to itself.

THIS WORKS FOR A SPHERE IN ANY DIMENSION

This story works for any sphere actually:
stratify S^n over $[1]$ so that

- the 0-th stratum is a single point
- the 1-st stratum is everything else.

Now $\mathrm{Exit}(S^n/[1])$ has two objects x_0 and x_1, with $\mathrm{Map}(x_0, x_1) = S^{n-1}$ and no other nontrivial maps.
In particular, for $n \geq 3$, **these are not n-categories for any finite n!**

Theorem (Ayala–Francis–Rozenblyum (2019), Haine (2018))
Let E be an ∞-category in which every endomorphism is an automorphism. Then there is a stratified topological space X/P, unique up to stratified homotopy equivalence, such that

$$E = Exit(X/P)$$

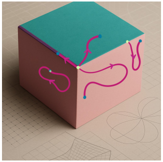
Furthermore, this construction can be performed functorially.

In other words, **these ∞-categories are stratified homotopy types**.

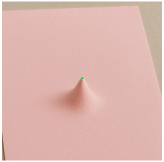
EXODROMY

Theorem (Lurie (2012), Haine–Porta–Teyssier (2024))
*Let X/P be a **reasonable** stratified topological space, and let C be a **reasonable** category of coefficients. Then there is an equivalence of categories*

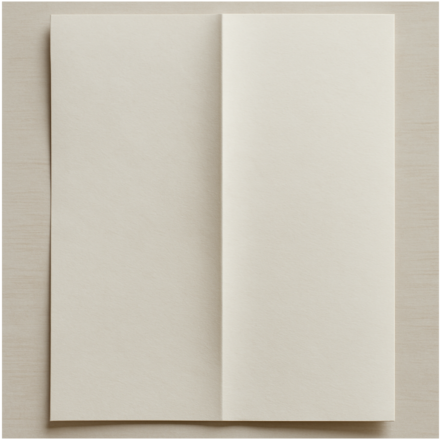
$$Constr(X/P, C) \simeq Fun(Exit(X/P), C)$$



Stratified homotopy types also make sense in **algebraic geometry**, particularly for the **étale topology**.

Spec $\mathbf{Z}$ looks something like this as a stratified space: 

COLLECTION OF FIGURES (ONE FIGURE PER CLICK)



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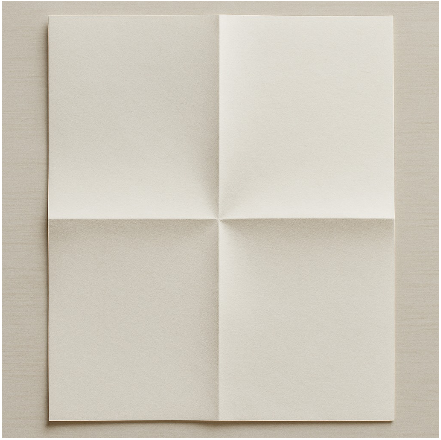

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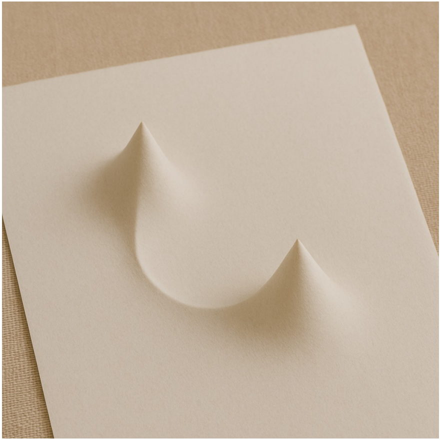

COLLECTION OF FIGURES (ONE FIGURE PER CLICK)