General topology

The Problems

Autumn 2020

The one-point compactification

Definition. Let (X, τ) be a topological space. Define X^* as the set $X \cup \{\infty\}$, where $\infty \notin X$. We endow X^* with a topology τ^* defined as follows. Let $S \subseteq X^*$ be a subset.

$$\tau^*(S) := \begin{cases} \tau(S \cap X) \cup \{\infty\} & \text{if for every closed and compact } Z \subseteq X\text{, there is a point } x \in S \cap (X \smallsetminus Z); \\ \tau(S \cap X) & \text{otherwise.} \end{cases}$$

In other words, a subset $Z \subseteq X^*$ is closed if and only if either $Z \subseteq X$ is closed and compact or $Z = Z_0 \cup \{\infty\}$, where $Z_0 \subseteq X$ is closed.

In still other words, a subset $U \subseteq X^*$ is open if and only if either $U \subseteq X$ is open or $U = U_0 \cup \{\infty\}$, where $U_0 \subseteq X$ is an open set whose complement $X \smallsetminus U_0$ is compact.

We call X^* the *one-point compactification* of X.

Problem 1

Prove that the one-point compactification X^* is compact.

Problem 2

Assume that X is Hausdorff and *locally compact* – that is, that every point $x \in X$ is contained in an open neighborhood whose closure is compact. Prove that the one-point compactification X^* is Hausdorff.

Definition. Let X and Y be topological spaces. A map $f\colon X \to Y$ is *closed* if and only if, for any closed subset $V \subseteq X$, the image $f(V) \subseteq Y$ is closed. A map $f\colon X \to Y$ is *proper* if and only if, for any compact subspace $K \subseteq Y$, the inverse image $f^{-1}K \subseteq X$ is compact.

Problem 3

Let X and Y be locally compact Hausdorff topological spaces, and let $f\colon X \to Y$ be a continuous map. Show that the following are equivalent.

- The map f is proper.
- For any topological space Z, the map $\mathrm{id} \times f\colon Z \times X \to Z \times Y$ is closed.
- The map f is closed, and for any point $y \in Y$, the inverse image $f^{-1}\{y\}$ is compact.

Problem 4

Which of the following maps are proper? You do not need to justify your answer.

- any continuous map $f\colon X \to Y$ from a compact topological space X to a Hausdorff topological space Y;
- any inclusion $j\colon U \hookrightarrow X$ of an open subspace $U \subseteq X$;
- any inclusion $i\colon Z \hookrightarrow X$ of a closed subspace $Z \subseteq X$;
- the map $\boldsymbol{R} \to \boldsymbol{R}$ given by $t \mapsto 2^t$;
- the product
$$\prod_{\alpha\in\Lambda} f_\alpha\colon \prod_{\alpha\in\Lambda} X_\alpha \to \prod_{\alpha\in\Lambda} Y_\alpha$$
of any family $\{f_\alpha\colon X_\alpha \to Y_\alpha\}_{\alpha\in\Lambda}$ of proper continuous maps;
- for any $n \geq 1$, the quotient map $\boldsymbol{C}^{n+1} \smallsetminus \{0\} \to \boldsymbol{P}^n_{\boldsymbol{C}}$;
- for any $n \geq 1$, the quotient map $S^{2n+1} \to \boldsymbol{P}^n_{\boldsymbol{C}}$.

Problem 5

Let X and Y be locally compact Hausdorff spaces, and let X^* and Y^* be the one-point compactifications. Let $f\colon X \to Y$ be a proper continuous map, and define $F\colon X^* \to Y^*$ by the formula

$$F(x) \coloneqq \begin{cases} f(x) & \text{if } x \in X; \\ \infty & \text{if } x = \infty\,. \end{cases}$$

Prove that F is continuous.

Problem 6

Give an example of two locally compact Hausdorff topological spaces X and Y and a continuous map $f\colon X \to Y$ such that if we define $F\colon X^* \to Y^*$ by the formula

$$F(x) \coloneqq \begin{cases} f(x) & \text{if } x \in X; \\ \infty & \text{if } x = \infty\,, \end{cases}$$

then F is not continuous.