General topology

The Problems

Autumn 2020

Abstract topological spaces

Problem 1

Let $X \subseteq \boldsymbol{R}^n$ be a nonempty subspace, and let $S \subseteq X$ be a subset. Show that the following conditions are equivalent.

- There is a point $x \in X$ such that for every $N \in \boldsymbol{R}$, there exists $s \in S$ such that $d(x, s) > N$.
- For every point $x \in X$ and every $N \in \boldsymbol{R}$, there exists $s \in S$ such that $d(x, s) > N$.

We'll say that S is *unbounded* if either (and therefore both) of these conditions is satisfied. Otherwise, we'll say that S is *bounded*.

Notation. The next two problems refer to the following notation. Let $X \subseteq \boldsymbol{R}^n$ be a nonempty subspace; denote by τ the subspace topology on X. Let X^+ be the set $X \cup \{\infty\}$, where $\infty \notin X$; define $\tau^+ \colon \boldsymbol{P}(X^+) \to \boldsymbol{P}(X^+)$ as follows: for any set $S \subseteq X^+$,

$$\tau^+(S) \coloneqq \begin{cases} \tau(S) & \text{if } S \subseteq X \text{ and } S \text{ is bounded;} \\ \tau(S) \cup \{\infty\} & \text{if } S \subseteq X \text{ and } S \text{ is unbounded;} \\ \tau(S \smallsetminus \{\infty\}) \cup \{\infty\} & \text{if } \infty \in S. \end{cases}$$

Problem 2

Prove that τ^+ is a topology on X^+.

Problem 3

Let $\phi \colon S^n \to (\boldsymbol{R}^n)^+$ be the map given by the rule

$$\phi(x_0, x_1, \dots, x_n) \coloneqq \begin{cases} \left(\frac{x_1}{1-x_0}, \frac{x_2}{1-x_0}, \dots, \frac{x_n}{1-x_0}\right) & \text{if } x_0 \neq 1; \\ \infty & \text{otherwise.} \end{cases}$$

Prove that ϕ is a homeomorphism, where $(\boldsymbol{R}^n)^+$ is given the topology τ^+ described above.

(With this in mind, let's reflect on the 3-sphere S^3, which is homeomorphic to $(\boldsymbol{R}^3)^+$. Now $\boldsymbol{R}^3$ is pretty easy to visualize, so all you have to imagine is that you've added a single point at ∞ to $\boldsymbol{R}^3$. Try to picture it!)

Problem 4

Define the subspace

$$D^2 \coloneqq \{x \in \boldsymbol{R}^2 : \|x\| \leq 2\} \subset \boldsymbol{R}^2 .$$

Construct a homeomorphism from the *solid torus* $ST^2 = D^2 \times S^1 \subset \boldsymbol{R}^4$ and the subspace

$$S \coloneqq \{(x, y, z) \in \boldsymbol{R}^3 : (2 - \sqrt{x^2 + y^2})^2 + z^2 \leq 1\} \subset \boldsymbol{R}^3 .$$

(I suggest drawing a picture of this!)

Problem 5

Keep the notations from the previous problem. Consider the interior ιS of S as a subset of $\boldsymbol{R}^3$, and therefore as a subset of $(\boldsymbol{R}^3)^+$, which is (as you've proved) homeomorphic to S^3. Prove that $(\boldsymbol{R}^3)^+ \smallsetminus \iota S$ is homeomorphic to ST^2.

(Reflect on the meaning of the following claim: S^3 is the union of two solid tori along a torus T^2.)

Problem 6

Let X be a topological space. Construct a topological space P_X and a continuous surjection $f : X \to P_X$ such that for every $p \in P_X$, the fiber $f^{-1}\{p\}$ is connected.

Problem 7

Construct a basis for the Cantor space C that consists of clopen subsets.

Extra problems (not to be handed in)

Problem 8

Let (X, d) be a metric space. Let $D > 0$. Define a new metric $d' : X \times X \to \boldsymbol{R}$ by the formula

$$d'(x, y) \coloneqq \min(d(x, y), D) .$$

Prove that the topology τ_d on X corresponding to d *coincides* with the topology $\tau_{d'}$ on X corresponding to d'.

Problem 9

The *Sierpiński topological space* S is the Alexandroff topological space attached to the poset $\{0, 1\}$, where $0 < 1$. For any topological space X, construct a bijection between the set $\mathscr{C} \subseteq \boldsymbol{P}(X)$ of closed sets of X and the set $\mathrm{Map}(X, S)$ of continuous maps $X \to S$.

Problem 10

A *filtration* on a topological space X is a sequence of subsets

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X$$

such that for each $i \in \boldsymbol{N}$, the subset $X_i \subseteq X$ is closed, and the union

$$\bigcup_{i \in \boldsymbol{N}} X_i$$

is again X.

Construct a topological space Z such that for any topological space X, the set $\mathrm{Map}(X, Z)$ of continuous maps $X \to Z$ is in bijection with the set $\mathscr{F}$ of filtrations on X.

Notation. let (X, τ) be a topological space. The formation of the *closure* is an operation

$$\tau\colon \boldsymbol{P}(X) \to \boldsymbol{P}(X)$$

on the power set $\boldsymbol{P}(X)$ (i.e., a map from $\boldsymbol{P}(X)$ to itself). The formation of the *complement* is an operation

$$\kappa\colon \boldsymbol{P}(X) \to \boldsymbol{P}(X)\ .$$

Thus $\kappa(S) = X \smallsetminus S$.

Please note that τ is *inclusion-preserving*,[1] and κ is *inclusion-reversing*;[2] also of course $S \subseteq \tau(S)$. Finally, please observe that τ is *idempotent*,[3] and that κ is *involutive*.[4]

We are interested in the operations $\boldsymbol{P}(X) \to \boldsymbol{P}(X)$ that we can obtain by composing τ and κ repeatedly. For example, the *interior* operator is

$$\iota \coloneqq \kappa\tau\kappa\colon \boldsymbol{P}(X) \to \boldsymbol{P}(X)\ .$$

Note that ι is inclusion-preserving[5] and ι is idempotent.

Many of the most important kinds of subsets of topological spaces are identified using τ and κ. For example, a subset $S \subseteq X$ is *closed* if and only if it is its own closure: $S = \tau(S) = S$; it is *open* if and only if it is its own interior: $S = \iota(S) = \kappa\tau\kappa(S)$.

[1] That is, if $S \subseteq T$, then $\tau(S) \subseteq \tau(T)$.

[2] That is, if $S \subseteq T$, then $\kappa(S) \supseteq \kappa(T)$.

[3] That is, $\tau^2 = \tau$

[4] That is, $\kappa^2 = \mathrm{id}$.

[5] Indeed, if you write down a sequence of τ's and κ's, then that operator will be inclusion-preserving if and only if there are an even number of κ's and inclusion-reversing if and only if there are an odd number of κ's.

Problem 11

Write down all the subsets of $\boldsymbol{R}$ (always with the standard topology) you can obtain by repeatedly applying the closure τ and the interior ι to the set

$$S \coloneqq \{-30\} \cup\,]-20, 0[\, \cup\,]0, 20[\, \cup (\boldsymbol{Q} \cap [25, 30[)\ .$$

Problem 12

A subset $S \subseteq X$ is said to be *dense* if $\tau(S) = X$. Find a countable dense subset of $\boldsymbol{R}$.

Problem 13

A subset $S \subseteq X$ is said to be *co-dense* if it has empty interior, so that $\iota(S) = \varnothing$. Give an example of an uncountable co-dense subset $S \subseteq \boldsymbol{R}$.

Problem 14

A subset $S \subseteq X$ is said to be *nowhere dense* if the interior of its closure is empty; that is, S is nowhere dense if $\iota\tau(S) = \varnothing$, or equivalently, $\kappa\tau\kappa\tau(S) = \varnothing$. Any nowhere dense subset of a topological space is co-dense, but give an example of a co-dense subset of $\boldsymbol{R}$ that is not nowhere dense.

Problem 15

Show that if $T \subseteq X$ is a closed co-dense subset, then any subset $S \subseteq T$ is nowhere dense.

Problem 16

Let $Z \subseteq X$. Prove that Z is the closure of some open subset of X if and only if Z is the closure of its interior, so that $Z = \tau\iota(Z)$, or equivalently, $Z = \tau\kappa\tau\kappa(Z)$.

Problem 17

Show that

$$\tau\kappa\tau = \tau\kappa\tau\kappa\tau\kappa\tau\,.$$

Deduce that

$$\iota\tau = \iota\tau\iota\tau \qquad \text{and} \qquad \tau\iota = \tau\iota\tau\iota$$

Problem 18

Let $S \subseteq X$. What is the maximum number of sets one can form by repeatedly applying the closure and complement operators to S?