

General pyknosis

MSRI

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Contents

Introduction	2
o Elements of general topology	3
o.1 Ultrafilters and compacta	3

Introduction

These are notes for a series of talks at MSRI about the *pyknotic formalism*.

The pyknotic formalism offers a way to coordinate ‘topological’ and ‘derived’ structures. This formalism is only partially developed, but it’s already clear that there is a lot to explore, and a lot of interesting possible applications. Many different points of view on pyknotic objects will be necessary to develop these applications.

o Elements of general topology

o.1 Ultrafilters and compacta

o.1.1 Notation. Write **Set** for the category of tiny finite sets. Write **Fin** \subset **Set** for the full subcategory of finite sets, and write i for the inclusion **Fin** \hookrightarrow **Set.**

o.1.2 Definition. For any tiny set S , write h^S for the functor **Fin** \rightarrow **Set given by $I \mapsto \text{Map}(S, I)$. An *ultrafilter* μ on S is a natural transformation**

$$\int_S (\cdot) d\mu: h^S \rightarrow i,$$

which for any finite set I gives a map

$$\begin{aligned} \text{Map}(S, I) &\longrightarrow I \\ f &\longmapsto \int_S f d\mu \end{aligned}$$

Write $\beta(S)$ for the set of ultrafilters on S . For any set S , the set $\beta(S)$ is the set

$$\beta(S) = \lim_{I \in \text{Fin}_S} I.$$

The functor

$$\beta: \mathbf{Set} \rightarrow \mathbf{Set}$$

is thus the right Kan extension of the inclusion **Fin** \hookrightarrow **Set** along itself.

o.1.3 Example. For any set S and any element $s \in S$, there is a *principal ultrafilter* δ_s , which is defined so that

$$\int_S f d\delta_s = f(s).$$

Every ultrafilter on a finite set is principal, but infinite sets have ultrafilters that are not principal. To prove the existence of these, let us look at a more traditional way of defining an ultrafilter on a set.

o.1.4 Definition. Let S be a set, $T \subseteq S$, and μ an ultrafilter on S . There is a unique *characteristic map* $\chi_T: S \rightarrow \{0, 1\}$ such that $\chi_T(s) = 1$ if and only if $s \in T$. Let us write

$$\mu(T) := \int_S \chi_T d\mu.$$

We say that T is μ -*thick* if and only if $\mu(T) = 1$. Otherwise (that is, if $\mu(T) = 0$), then we say that T is μ -*thin*.

For any $s \in S$, the principal ultrafilter δ_s is the unique ultrafilter relative to which $\{s\}$ is thick.

o.1.5. If S is a set and μ is an ultrafilter on S , then we can observe the following facts about the collection of thick and thin subsets (relative to μ):

- (1) The empty set is thin.
- (2) Complements of thick sets are thin.
- (3) Every subset is either thick or thin.
- (4) Subsets of thin sets are thin.
- (5) The intersection of two thick sets is thick.

o.1.6. Ultrafilters are functorial in maps of sets. Let $\phi: S \rightarrow T$ be a map, and let μ be an ultrafilter on S . The ultrafilter $\phi_*\mu$ is the ultrafilter on T given by

$$\int_T f d(\phi_*\mu) = \int_S (f \circ \phi) d\mu .$$

For any $U \subseteq T$, one has in particular

$$(\phi_*\mu)(U) = \mu(\phi^{-1}(U)) .$$

Thus U is $\phi_*\mu$ -thick if and only if $\phi^{-1}U$ is μ -thick.

o.1.7 Definition. A *system of thick subsets* of S is a collection $F \subseteq P(S)$ such that for any finite set I and any partition

$$S = \coprod_{i \in I} S_i ,$$

there is a unique $i \in I$ such that $S_i \in F$.

o.1.8 Construction. We have seen that an ultrafilter μ specifies the system F_μ of μ -thick subsets. In the other direction, attached to any system F of thick subsets is an ultrafilter μ_F : for any finite set I and any map $f: S \rightarrow I$, the element $i = \int_S f d\mu \in I$ is the unique one such that $S_i \in F$.

The assignments $\mu \mapsto F_\mu$ and $F \mapsto \mu_F$ together define a bijection between ultrafilters on S and systems of thick subsets.

o.1.9 Definition. If S is a set, and if $G \subseteq P(S)$, then an ultrafilter μ is said to be *supported on G* if and only if every element of G is μ -thick, that is, $G \subseteq F_\mu$.

o.1.10 Lemma. Let S be a set, and let $G \subseteq P(S)$. Assume that no finite intersection of elements of G is empty. Then there exists an ultrafilter μ on S supported on G .

Proof. Consider all the families $A \subseteq P(S)$ with the following properties:

- (1) A contains G ;
- (2) no finite intersection of elements of A is empty.

By Zorn's lemma there is a maximal such family, F .

We claim that F is a system of thick subsets. For this, let $S = \coprod_{i \in I} S_i$ be a finite partition of S . Condition 2 ensures that at most one of the summands S_i can lie in F . Now suppose that none of the summands S_i lies in F . Consider, for each $i \in I$, the family

$F \cup \{S_i\} \subseteq P(S)$; the maximality of F implies that none of these families can satisfy Condition 2. Thus for each $i \in I$, there is an empty finite intersection $S_i \cap \bigcap_{j=1}^{n_i} T_{ij} = \emptyset$. But this implies that the intersection $\bigcap_{i \in I} \bigcap_{j=1}^{n_i} T_{ij}$ is empty, contradicting Condition 2 for F itself. Hence at least one – and thus exactly one – of the summands S_i lies in F . Thus F is a system of thick subsets of S . \square

0.1.11. It is not quite accurate to say that the Axiom of Choice is *necessary* to produce nonprincipal ultrafilters, but it is true that their existence is independent of Zermelo–Fraenkel set theory.

0.1.12. If ϕ is a functor $\mathbf{Set} \rightarrow \mathbf{Set}$, then a natural transformation $\phi \rightarrow \beta$ is the same thing as a natural transformation $\phi \circ i \rightarrow i$. Please observe that we have a canonical identification $\beta \circ i = i$.

It follows readily that the functor β is a monad: the unit $\text{id} \rightarrow \beta$ corresponds to the identification $\text{id} \circ i = i$, and the multiplication $\beta^2 \rightarrow \beta$ corresponds to the identification $\beta^2 \circ i = i$.

The unit for the monad β structure is the assignment $s \mapsto \delta_s$ that picks out the principal ultrafilter at a point.

To describe the multiplication $\tau \mapsto \mu(\tau)$, let us write T^\dagger for the set of ultrafilters supported on $\{T\}$. Now if τ is an ultrafilter on $\beta(S)$, then $\mu(\tau)$ is the ultrafilter on S such that

$$\mu(\tau)(T) = \tau(T^\dagger).$$

0.1.13 Construction. Let \mathbf{Top} denote the category of tiny topological spaces. If S is a set, we can introduce a topology on $\beta(S)$ simply by forming the inverse limit $\lim_{I \in \mathbf{Fin}_S} I$ in \mathbf{Top} . That is, we endow $\beta(S)$ with the coarsest topology such that all the projections $\beta(S) \rightarrow I$ are continuous. We call this the *Stone topology* on $\beta(S)$. By Tychonoff, this limit is a compact Hausdorff topological space. This lifts β to a functor $\mathbf{Set} \rightarrow \mathbf{Top}$.

0.1.14. Let's be more explicit about the topology on $\beta(S)$. The topology on $\beta(S)$ is generated by the sets T^\dagger (for $T \subseteq S$). In fact, since the sets T^\dagger are stable under finite intersections, they form a base for the Stone topology on $\beta(S)$. Additionally, since the sets T^\dagger are stable under the formation of complements, they even form a base of clopens of $\beta(S)$.

0.1.15 Definition. A *compactum* is an algebra for the monad β . Hence a compactum consists of a set K and a map $\lambda_K: \beta(K) \rightarrow K$, which is required to satisfy the usual identities:

$$\lambda_K(\lambda_{K,*} \tau) = \lambda_K(\mu(\tau)) \quad \text{and} \quad \lambda_K(\delta_s) = s,$$

for any ultrafilter τ on $\beta(S)$ and any point $s \in S$. The image $\lambda_K(\mu)$ will be called the *limit* of the ultrafilter μ . The category \mathbf{Comp} is the category of compacta.

0.1.16 Construction. If K is a compactum, then we use the limit map $\lambda_K: \beta(K) \rightarrow K$ to topologise K as follows. For any subset $T \subseteq K$, we define the closure of T as the image $\lambda_K(T^\dagger)$.

A subset $Z \subseteq K$ is thus closed if and only if the limit of any ultrafilter relative to which Z is thick lies in Z . Dually, a subset $U \subseteq K$ is open if and only if it is thick with respect to any ultrafilter whose limit lies in U .

We denote the resulting topological space K^{top} . The assignment $K \mapsto K^{top}$ defines a lift $\mathbf{Alg}(\beta) \rightarrow \mathbf{Top}$ of the forgetful functor $\mathbf{Alg}(\beta) \rightarrow \mathbf{Set}$.

0.1.17 Proposition. *The functor $K \mapsto K^{top}$ identifies the category of compacta with the category of compact Hausdorff topological spaces.*

We will spend the remainder of this section proving this claim. Please observe first that $K \mapsto K^{top}$ is faithful. What we will do now is prove:

- (1) that for any compactum K , the topological space K^{top} is compact Hausdorff;
- (2) that for any compact Hausdorff topological space X , there is a β -algebra structure K on the underlying set of X such that $X \cong K^{top}$; and
- (3) that for any compacta K and L , any continuous map $K^{top} \rightarrow L^{top}$ lifts to a β -algebra homomorphism $K \rightarrow L$.

To do this, it is convenient to describe a related idea: that of *convergence* of ultrafilters on topological spaces.

0.1.18 Definition. Let X be a topological space, and let $x \in X$. We say that x is a *limit point* of an ultrafilter μ on (the underlying set of) X if and only if every open neighbourhood of x is μ -thick. In other words, x is a limit point of μ if and only if, for every open neighbourhood U of x , one has $\mu \in U^\dagger$.

0.1.19 Lemma. *Let X be a topological space, and let $U \subseteq X$ be a subset. Then U is open if and only if it is thick with respect to any ultrafilter with limit point in U .*

Proof. If U is open, then U is by definition thick with respect to any ultrafilter with limit point in U .

Conversely, assume that U is thick with respect to any ultrafilter with limit point in U . Let $u \in U$. Consider the set $G := N(u) \cup \{X \setminus U\}$, where $N(u)$ is the collection of open neighbourhoods of u . If U does not contain any open neighbourhood of u , then no finite intersection of elements of G is empty. By [Lemma 0.1.10](#) there is an ultrafilter μ supported on the $N(u) \cup \{X \setminus U\}$, whence u is a limit point of μ , but U is not μ -thick. This contradicts our assumption, and so we deduce that U contains an open neighbourhood of u . \square

0.1.20 Lemma. *Let X and Y be topological spaces, and let $\phi: X \rightarrow Y$ be a map. Then ϕ is continuous if and only if, for any ultrafilter μ on X with limit point $x \in X$, the point $\phi(x)$ is a limit point of $\phi_*\mu$.*

Proof. Assume that ϕ is continuous, and let μ be an ultrafilter on X , and assume that $x \in X$ is a limit point of μ . Now assume that V is an open neighbourhood of $\phi(x)$. Since $\phi^{-1}V$ is an open neighbourhood of x , so it is μ -thick, whence V is $\phi_*\mu$ -thick. Thus $\phi(x)$ is a limit point of $\phi_*\mu$.

Assume now that if $x \in X$ is a limit point of an ultrafilter μ , then $\phi(x)$ is a limit point of $\phi_*\mu$. Let $V \subseteq Y$ be an open set. Let $x \in \phi^{-1}(V)$, and let μ be an ultrafilter on X with limit point x . Then $\phi(x)$ is a limit point of $\phi_*\mu$, so V is $\phi_*\mu$ -thick, whence $\phi^{-1}(V)$ is μ -thick. It follows from [Lemma 0.1.19](#) that $\phi^{-1}(V)$ is open. \square

0.1.21 Lemma. *Let X be a topological space. Then X is quasicompact if and only if every ultrafilter on X has at least one limit point.*

Proof. Assume first that X is quasicompact. Let μ be an ultrafilter on X , and assume that μ has no limit point. Select, for every point $x \in X$, an open neighbourhood U_x thereof that is not μ -thick. Quasicompactness implies that there is a finite collection $x_1, \dots, x_n \in X$ such that $\{U_{x_1}, \dots, U_{x_n}\}$ covers X . But at least one of U_{x_1}, \dots, U_{x_n} must be μ -thick. This is a contradiction.

Now assume that X is not quasicompact. Then there exists a collection $G \subseteq \mathcal{P}(X)$ of closed subsets of X such that the intersection of all the elements of G is empty, but no finite intersection of elements of G is empty. In light of [Lemma 0.1.10](#), there is an ultrafilter μ with the property that every element of G is thick. For any $x \in X$, there is an element $Z \in G$ such that $x \in X \setminus Z$. Since Z is μ -thick, $X \setminus Z$ is not. Thus μ has no limit points. \square

0.1.22 Lemma. *Let X be a topological space. Then X is Hausdorff if and only if every ultrafilter on X has at most one limit point.*

Proof. Assume that μ is an ultrafilter with two distinct limit points x_1 and x_2 . Choose open neighbourhoods U_1 of x_1 and U_2 of x_2 . Since they are both μ -thick, they cannot be disjoint; hence X is not Hausdorff.

Conversely, assume that X is not Hausdorff. Select two points x_1 and x_2 such that every open neighbourhoods U_1 of x_1 and U_2 of x_2 intersect. Now the set G consisting of open neighbourhoods of either x_1 or x_2 has the property that no finite intersection of elements of G is empty. In light of [Lemma 0.1.10](#), there is an ultrafilter μ with the property that every element of G is thick. Thus x_1 and x_2 are limit points of μ . \square

Let us now return to our functor $K \mapsto K^{top}$.

0.1.23 Lemma. *Let K be a compactum, and let μ be an ultrafilter on K . Then a point of K^{top} is a limit point of μ in the sense of [Definition 0.1.18](#) if and only if it is the limit of μ in the sense of [Definition 0.1.15](#).*

Proof. Let $x := \lambda_K(\mu)$. The open neighbourhoods U of x are by definition thick (relative to μ), so certainly x is a limit point of μ .

Now assume that $y \in K^{top}$ is a limit point of μ . To prove that the limit of μ is y , we shall build an ultrafilter τ on $\beta(K)$ with the following properties:

- (1) under the multiplication $\beta^2 \rightarrow \beta$, the ultrafilter τ is sent to μ ; and
- (2) under the map $\lambda_* : \beta^2 \rightarrow \beta$, the ultrafilter τ is sent to δ_y .

Once we have succeeded, it will follow that

$$\lambda_K(\mu) = \lambda_K(\mu_\tau) = \lambda_K(\lambda_{K,*}\tau) = \lambda_K(\delta_y) = y,$$

and the proof will be complete.

Consider the family G' of subsets of $\beta(K)$ of the form T^\dagger for a μ -thick subset $T \subseteq S$; since these are all nonempty and they are stable under finite intersections, it follows that no finite intersection of elements of G' is empty.

Now consider the set $G := G' \cup \{\lambda_K^{-1}\{y\}\}$. If T is μ -thick, then we claim that there is an ultrafilter $\nu \in \lambda_K^{-1}\{y\} \cap T^\dagger$. Indeed, consider the set $N(y) \cup \{T\}$, where $N(y)$ is the collection of open neighbourhoods of y . Since every open neighbourhood of y is μ -thick, no intersection of an open neighbourhood of y with T is empty. By [Lemma 0.1.10](#) there is an ultrafilter supported on $N(y) \cup \{T\}$, which implies that no finite intersection of elements of G is empty.

Applying [Lemma 0.1.10](#) again, we see that G supports an ultrafilter τ on $\beta(K)$. For any $T \subseteq K$,

$$\mu_\tau(T) = \tau(T^\dagger),$$

so since τ is supported on G' , it follows that $\mu_\tau = \mu$. At the same time, since τ is supported on $\{\lambda_K^{-1}\{y\}\}$, it follows that $\{y\}$ is thick relative to $\lambda_{K,*}\tau$, whence $\lambda_{K,*}\tau = \delta_y$. \square

Proof of [Proposition 0.1.17](#). Let K be a compactum. Combine [Lemmas 0.1.21](#) to [0.1.23](#) to conclude that K^{top} is a compact Hausdorff topological space.

Let X be a compact Hausdorff topological space with underlying set K . Define a map $\lambda_K : \beta(K) \rightarrow K$ by carrying an ultrafilter μ to its unique limit point in X . This is a β -algebra structure on X , and it follows from [Lemma 0.1.23](#) and the definition of the topology together imply that $X \cong K^{top}$.

Finally, let K and L be compacta, and let $\phi : K^{top} \rightarrow L^{top}$ be a continuous map. To prove that ϕ is a β -algebra homomorphism, it suffices to confirm that if μ is an ultrafilter on K , then

$$\lambda_L(\phi_*\mu) = \phi(\lambda_K(\mu)),$$

but this follows exactly from [Lemma 0.1.20](#). \square