Stratified homotopy theory

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0 Introduction

Our work here stems from a general principle: if X is a (nice) geometric object (a topological space, a variety, *etc.*), then a suitable class A of sheaves on X determines and is determined by a certain 'homotopy type' X_A attached to X. The larger the class A, the finer we can expect the homotopy type X_A to be.

In this introduction, our aim is to examine this principle in a four relatively familiar situations:

- when X is a topological space, and A is the family of all sheaves (Subsection 0.1);
- when X is a topological space, and A is the family of constant sheaves;
- when X is a topological space, and A is the family of locally constant sheaves;
- when X is a scheme, and A is the family of locally constant étale sheaves.

In these circumstances, the object X_A is a homotopy type (or piece thereof) in more or less the usual sense.

In the body of this text, we will study two further situations:

- when X is a stratified topological space, and A is the family of constructible sheaves;

- when X is a scheme, and A is the family of constructible sheaves.

In these situations, the object X_A will be a *stratified homotopy type*. We will see that this is a dramatically different object, but it comes with much finer information about the structure of X.

0.1 Notes on sobriety

The most elementary illustration of the principle above is the consideration of topological spaces X that can be recovered from their entire categories of sheaves. These are called the *sober* topological spaces.

Definition 0.1.1. Let X be a topological space. A closed subset $Z \subseteq X$ is *irreducible* if and only if it is nonempty and if, for any closed subsets $Z_1, Z_2 \subseteq X$ such that $Z = Z_1 \cup Z_2$, either $Z = Z_1$ or $Z = Z_2$. Dually, an open subset $U \subset X$ is *irreducibly open* if and only if it is proper and if, for any open subsets $U_1, U_2 \subseteq X$ such that $U_1 \cap U_2 \subseteq U$, either $U_1 \subseteq U$ or $U_2 \subseteq U$.

Example 0.1.2. If X is a topological space with a point $x \in X$, then the closure $\overline{x}$ is irreducible. In this case, one says that x is a *generic point* of X. The union of all the open subsets that do not contain x is irreducibly open.

Definition 0.1.3. Let X be a topological space, and let $I(X)$ be the set of irreducible closed subsets of X. Then we say that X is *sober* if the map $X \to I(X)$ given by $x \mapsto \overline{x}$ is a bijection; that is, X is sober if and only if every irreducible closed subset of X has a unique generic point.

0.1.4. Sober topological spaces are always Kolmogoroff (*i.e.*, any distinct points are topologically distinguishable). Indeed, Kolmogoroff topological spaces are exactly those with the property that the map $x \mapsto \overline{x}$ is an injection.

Example 0.1.5. Any Hausdorff space is sober.

Example 0.1.6. The Zariski topology of any scheme is sober. In particular, not all sober topological spaces are T1.

Example 0.1.7. The set $\boldsymbol{N}$ of natural numbers, when equipped with the cofinite topology, is T1 but not sober.

Construction 0.1.8. Let X be a topological space, and let $I(X)$ be the set of irreducible closed subsets thereof. Write γ for the map $x \mapsto \overline{x}$. We topologise $I(X)$ with the finest topology such that γ_X is continuous; in other words, a subset $U \subseteq I(X)$ is open if and only if $\gamma_X^{-1}U$ is open in X. Equivalently, U is open if and only if there exists an open subset $V \subseteq X$ such that $U = \{F \in I(X) : F \cap V \neq \varnothing\}$.

Thus γ induces a bijection between the opens of $I(X)$ and the opens of X, and $I(X)$ is sober.

The assignment $X \mapsto I(X)$ along with the natural transformation γ together define a left adjoint to the forgetful functor from sober topological spaces to all topological spaces. Thus one may *sober up* any topological space X.

Now let us demonstrate that a sober space is completely controlled by its lattice of open sets.

Notation 0.1.9. Let X be a topological space. Then we write $\Omega(X)$ for the poset of open subsets of X; this is a complete lattice. If Y is also a topological space, then we write $\mathrm{Map}^{cl}(\Omega(Y), \Omega(X))$ for the set of *complete lattice homomorphisms* – monotonic maps $\Omega(Y) \to \Omega(X)$ that preserve finite intersections and arbitrary unions.

Proposition 0.1.10. *Let X and Y be sober topological spaces; Then the assignment $f \mapsto f^{-1}$ is a bijection*

$$\mathrm{Map}(X, Y) \xrightarrow{\sim} \mathrm{Map}^{cl}(\Omega(Y), \Omega(X)) .$$

Proof. Let $F \colon \Omega(Y) \to \Omega(X)$ be a monotonic map that preserves finite intersections and arbitrary unions. Now let us define a map $f : X \to Y$: let $x \in X$. For each of X and Y, the points are in bijection with the irreducible closed subsets, which are in turn in bijection with the irreducibly open subsets. Now the map F induces a map from the irreducibly open subsets of X to those of Y: this map carries an irreducible open $U \subseteq X$ to the largest open $V \subseteq Y$ such that $F(V) \subseteq U$. Following these bijections, we obtain a unique map f with the property that $f^{-1} = F$.

To be explicit, let $V \in \Omega(Y)$ be the largest open subset such that $x \notin F(V)$. This subset is irreducibly open, and so its complement is an irreducible closed. The value $f(x)$ is the unique generic point of $Y \smallsetminus V$.

This provides a fully faithful embedding of sober topological spaces into complete lattices. We now show that this provides a similar fully faithful embedding of sober topological spaces into topoi.

Notation 0.1.11. Let X be a topological space. We shall write $\widetilde{X}_{\leq 0}$ for the topos of sheaves of sets on X. (This notation may seem a little strange, but soon we will have to contend with sheaves of *spaces* on X, and we are reserving this notation for that notion.)

For any two topoi $\boldsymbol{X}$ and $\boldsymbol{Y}$, we write $\mathrm{Fun}_*(\boldsymbol{X}, \boldsymbol{Y})$ for the category whose objects are geometric morphisms $f_* : \boldsymbol{X} \to \boldsymbol{Y}$ and whose morphisms are natural transformations $f_* \to g_*$.

0.1.12. If X is a topological space, then we can recover $\Omega(X)$ from $\widetilde{X}_{\leq 0}$ as the poset of subobjects of the terminal object $1_{\widetilde{X}_{\leq 0}}$. Now we can make the following little observation: if X and Y are topological spaces, then the left adjoint of a geometric morphism $f_* : \widetilde{X}_{\leq 0} \to \widetilde{Y}_{\leq 0}$ restricts to a complete lattice homomorphism $\Omega(X) \to \Omega(Y)$. This is an inverse to the obvious functor $\mathrm{Map}^{cl}(\Omega(Y), \Omega(X)) \to \mathrm{Fun}_*(\widetilde{X}_{\leq 0}, \widetilde{Y}_{\leq 0})$.

In particular, please observe that the geometric morphisms $\widetilde{X}_{\leq 0} \to \widetilde{Y}_{\leq 0}$ form a discrete category.

Corollary 0.1.13. *Let X and Y be sober topological spaces. Then the assignment $f \mapsto f_*$ is an equivalence of categories*

$$\mathrm{Map}(X, Y) \xrightarrow{\sim} \mathrm{Fun}_*(\widetilde{X}_{\leq 0}, \widetilde{Y}_{\leq 0}) .$$

0.2 Connectedness and constant sheaves

Let us begin by understanding the nature of connectedness.

Notation 0.2.1. We well begin by contemplating sheaves of *sets* on topological spaces. For any topological space X, let $\widetilde{X}_{\leq 0}$ be the category of sheaves of sets on X. (Eventually, we shall have to consider sheaves of *spaces* on X, and we are saving the notation $\widetilde{X}$ for that.)

Definition 0.2.2. The *constant sheaf* at a set S on a topological space X is the sheafification of the constant presheaf $U \mapsto S$.

0.2.3. The formation of the constant sheaf defines a left exact left adjoint

$$\Gamma_X^* : \mathbf{Set} \to \widetilde{X}_{\leq 0} .$$

Its right adjoint $\Gamma_{X,*}$ is the formation of global sections $F \mapsto F(X)$.

0.2.4. For any topological space X, any set S, and any point $x \in X$, the stalk of the constant sheaf $\Gamma_X^*(S)$ at X is canonically isomorphic to S.

Indeed, $x^* \Gamma_X^*$ is a left exact left adjoint from **Set** to itself; such a functor is isomorphic in a unique fashion to the identity.

Example 0.2.5. Constant sheaves are not constant as presheaves. Indeed, let X be the discrete space $\{0, 1\}$, and let S be any set. Then the constant sheaf at S on X certainly has the property that its values on $\{0\}$ and $\{1\}$ are each the set S, but now the sheaf condition requires that the global sections are given by

$$\Gamma_{X,*}\Gamma_X^*(S) = \Gamma_X^*(S)\{0, 1\} \cong \Gamma_X^*(S)\{0\} \times \Gamma_X^*(S)\{1\} \cong S \times S .$$

If S has at least two elements, then it follows that $\Gamma_X^*(S)$ is not constant as a presheaf.

The previous example does illustrate a general principle:

Lemma 0.2.6. *Let X be a discrete topological space, and let S be a set. Then* $\Gamma_{X,*}\Gamma_X^*(S) \cong \mathrm{Map}(X, S)$.

Thus in the discrete case, the endofunctor $\Gamma_{X,*}\Gamma_X^* : \mathbf{Set} \to \mathbf{Set}$ is actually representable. This lemma triggers a thought: If we want to understand various constant sheaves, we can do so by coming to grips with the various functors $\Gamma_{X,*}\Gamma_X^* : \mathbf{Set} \to \mathbf{Set}$ as X varies. The first thing we can notice is that this functor, though not always representable as in Lemma 0.2.6, is always *prorepresentable*; in other words, it is a left exact accessible functor.

Definition 0.2.7. A *proöbject* of a category C with all finite limits is a left exact accessible functor $C \to \mathbf{Set}$. The category $\mathrm{Pro}(C)$ of proöbjects is the *opposite* of the full subcategory of $\mathrm{Fun}(C, \mathbf{Set})$ spanned by the left exact accessible functors.

Example 0.2.8. If C is a category with all finite limits, then the Yoneda embedding provides a fully faithful functor $よ : C \hookrightarrow \mathrm{Pro}(C)$. Explicitly, if X is an object of a category C, then the proöbject it defines is $\mathrm{Map}(X, -)$.

Example 0.2.9. As a matter of terminology, we call the opposite A^{op} of a filtered category an *inverse* category; we call a diagram indexed by an inverse category an *inverse system*; and we call a limit of an inverse system an *inverse limit*.

If $X\colon A^{op} \to C$ is an inverse system in a category C with all finite limits, then the limit $\lim_{\alpha\in A^{op}} X_\alpha$ formed in $\mathrm{Fun}(C, Set)^{op}$ is a proöbject; this is the proöbject $Y \mapsto \operatorname{colim}_{\alpha\in A} \mathrm{Map}(X_\alpha, Y)$. Furthermore, *every* proöbject of C can be formed in such a manner.

Now if X and Y are two proöbjects that are exhibited as limits of inverse systems in this sense, then one has

$$\mathrm{Map}_{\mathrm{Pro(C)}}(X,Y) \cong \lim_{\beta\in B^{op}} \operatorname*{colim}_{\alpha\in A} \mathrm{Map}_C(X,Y)\,.$$

Example 0.2.10. Let X be a topological space. We obtain a proset $\pi_0^{top}(X) \coloneqq \Gamma_{X,*}\Gamma_X^*$. This defines a functor from topological spaces to prosets.

More generally, this is a functor from topoi to prosets: this carries a topos $\boldsymbol{X}$ to the composite

$$\pi_0(\boldsymbol{X}) \coloneqq \Gamma_{\boldsymbol{X},*}\Gamma_{\boldsymbol{X}}^*\colon \mathbf{Set} \to \boldsymbol{X} \to \mathbf{Set}\,.$$

Our claim is that the proset $\pi_0^{top}(X)$ is closely related to – and even identifiable with – the set $\pi_0(X)$. For a relatively nice class of topological spaces, $\pi_0(X)$ has a simple universal property.

0.2.11. Let us consider the category $\mathbf{Top}^{ng}$ of *numerically generated* topological spaces – these are topological spaces X with the property that a subset $U \subseteq X$ is open if and only if, for any continuous map $\phi\colon \boldsymbol{R}^n \to X$, the set $\phi^{-1}U$ is open.

Of course any discrete space is numerically generated, so the assignment $S \mapsto S^{disc}$ is a functor $\mathbf{Set} \to \mathbf{Top}^{ng}$. This functor has a left adjoint, $\pi_0\colon \mathbf{Top}^{ng} \to \mathbf{Set}$; in other words, for any numerically generated topological space X, the continuous maps $X \to S^{disc}$ are in bijective correspondence with the maps $\pi_0(X) \to S$.

Now we aim to prove the analogous universal property for the topos-theoretic π_0. For this, we need to understand a little about slice topoi. The following definition is not entirely necessary for our purposes right now, but the ideas will reappear later:

Definition 0.2.12. A morphism of topoi $p_*\colon \boldsymbol{X} \to \boldsymbol{Y}$ is *étale* if and only if the left adjoint p^* admits a further left adjoint $p_!$ that identifies $\boldsymbol{X}$ with the overcategory $\boldsymbol{Y}_{/p_!(1_X)}$.

Notation 0.2.13. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be topoi, and let $U \in \boldsymbol{Y}$ be an object. We write $p_*\colon \boldsymbol{Y}_{/U} \to \boldsymbol{Y}$ for the étale geometric morphism. In the presence of a geometric morphism $f_*\colon \boldsymbol{X} \to \boldsymbol{Y}$, we define the category $\mathrm{Fun}_{*,\boldsymbol{Y}}(\boldsymbol{X}, \boldsymbol{Y}_{/U})$ of pairs (F_*, β) consisting of a geometric morphism $F_*\colon \boldsymbol{X} \to \boldsymbol{Y}_{/U}$ along with an isomorphism $\beta\colon f_* \xrightarrow{\sim} p_* \circ F_*$.

Construction 0.2.14. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be topoi, and let $f_*\colon \boldsymbol{X} \to \boldsymbol{Y}$ be a geometric morphism. Let $U \in \boldsymbol{Y}$ be an object, and let $\alpha\colon 1_{\boldsymbol{X}} \to f^*U$ be a morphism of $\boldsymbol{X}$. Define a functor $F^*(\alpha)\colon \boldsymbol{Y}_{/U} \to \boldsymbol{X}$ that carries an object $[V \to U]$ of $\boldsymbol{Y}_{/U}$ to the object $1_{\boldsymbol{X}} \times_{f^*U} f^*V$; this functor admits a right adjoint $F_*(\alpha)$, which is a morphism of topoi. If we write $p_* : \boldsymbol{Y}_{/U} \to \boldsymbol{Y}$ for the canonical étale morphism, then we have an isomorphism $f^* \cong F^*(\alpha) \circ p^*$, and this defines a functor

$$\mathrm{Map}_{\boldsymbol{X}}(1_{\boldsymbol{X}}, f^*U) \to \mathrm{Fun}_{*,/\boldsymbol{X}}(\boldsymbol{X}, \boldsymbol{Y}_{/U}) .$$

In the other direction, let $F_* : \boldsymbol{X} \to \boldsymbol{Y}_{/U}$ be a geometric morphism over $\boldsymbol{Y}$. The diagonal morphism $U \to U \times U \cong p^*U$ in $\boldsymbol{Y}_{/U}$ can then be pulled back along this geometric morphism to obtain a morphism

$$\alpha_{F_*} : 1_{\boldsymbol{X}} \cong F^*U \to F^*(U \times U) \cong f^*U$$

of $\boldsymbol{X}$.

These constructions are inverse to one another: the natural transformation α above can be identified with the natural transformation $\alpha_{F_*(\alpha)}$, and to identify F_* with $F_*(\alpha_{F_*})$, the main point is that one must observe that for any object V in $\boldsymbol{Y}_{/U}$, the square

$$\begin{array}{ccc} V & \longrightarrow & V \times U \\ \downarrow & & \downarrow \\ U & \longrightarrow & U \times U \end{array}$$

is a pullback square in $\boldsymbol{Y}_{/U}$, and so one obtains a fomula for F^*:

$$F^*V \simeq 1_{\boldsymbol{X}} \times_{f^*U} f^*V .$$

The rather surprising upshot is that $\mathrm{Fun}_{*,/XX}(\boldsymbol{X}, \boldsymbol{Y}_{/U})$ is in fact discrete (*i.e.*, is equivalent to a set), and one has an equivalence of categories

$$\mathrm{Map}_{\boldsymbol{X}}(1_{\boldsymbol{X}}, f^*U) \equiv \mathrm{Fun}_{*,/\boldsymbol{X}}(\boldsymbol{X}, \boldsymbol{Y}_{/U}) .$$

0.2.15. In our case of interest, $\boldsymbol{Y} = \mathbf{Set}$, the object U is a set, and the topos $\boldsymbol{Y}_{/U}$ can be identified with the functor category $\widetilde{U} := \mathrm{Fun}(U, \mathbf{Set})$. We now unpack the identification above as

$$\mathrm{Fun}_*(\boldsymbol{X}, \widetilde{U}) \simeq \mathrm{Map}(1_{\boldsymbol{X}}, \Gamma^*U) \cong \Gamma_*\Gamma^*U \cong \mathrm{Map}(\pi_0\boldsymbol{X}, U) .$$

Theorem 0.2.16. *If X is a numerically generated sober topological space, then the proset $\pi_0^{top}(X)$ is nothing more than the set $\pi_0(X)$.*

Proof. Let U be a set. Numerical generation gives us

$$\mathrm{Map}(X, U^{disc}) \cong \mathrm{Map}(\pi_0(X), U) ,$$

sobriety gives us

$$\mathrm{Map}(X, U^{disc}) \simeq \mathrm{Fun}_*(\widetilde{X}, \widetilde{U}) ,$$

and the computation above gives us

$$\mathrm{Fun}_*(\widetilde{X}, \widetilde{U}) \simeq \mathrm{Map}(\pi_0^{top}(X), U) .$$

All of these are functorial equivalences, so we conclude.

0.3 Monodromy representations

Definition 0.3.1. Let X be a topological space.

A *locally constant sheaf* of sets L is a sheaf for which there exists an open cover $\{U_\alpha\}_{\alpha\in A}$ such that for any $\alpha \in A$, the sheaf $L|U_\alpha$ is constant on U_α. We will call such an open cover a *trivialising* open cover of X for L.

A *local system* is a locally constant sheaf for which there is a finite trivialising open cover.

0.3.2. Let X be a topological space, and let L be a locally constant sheaf on X. Any refinement of a trivialising open cover of X for L is again trivialising. Note also that if X is quasicompact, then any locally constant sheaf is a local system.

Example 0.3.3. On a discrete topological space S^{disc}, there are plenty of nonconstant local systems: in fact any sheaf on S^{disc} is locally constant.

Example 0.3.4. Consider the interval $I \coloneqq [0,1]$. Let L be a local system on I. A finite trivialising open cover can be refined to a finite trivialising open cover $\{U_1, \dots, U_n\}$ in which each U_i is an interval.

Now let U_i and U_j be two of these intervals; their intersection is either empty or else an inverval. Assume that their intersection is in fact nonempty; then their union is again an interval. Let S be a set such that $L|U_i$ is constant at S. Since U_i and U_j intersect, it follows that $L|U_j$ is constant at S as well. The sheaf condition gives an equaliser

$$L(U_i \cup U_j) \to S \times S \rightrightarrows S \times S \times S \times S\,,$$

where the top arrow is $(s,t) \mapsto (s,s,t,t)$ and the bottom arrom is $(s,t) \mapsto (s,t,s,t)$ which now implies that $L(U_i \cup U_j) \cong S$ in a way that is compatible with the restrictions to U_i and U_j. Thus we obtain a morphism η from the constant sheaf at S on $U_i \cup U_j$ to the restriction $L|(U_i \cup U_j)$ that restricts to an isomorphism on U_i and U_j. Thus it follows that η is an isomorphism.

Since any U_i must intersect *some* other U_j, any finite trivialising open cover consisting of $n > 1$ intervals can be replaced by a finite trivialising open cover consisting of $n-1$ intervals. By induction, it follows that I itself is a trivialising open cover, whence L is a constant sheaf.

We thus conclude that any local system on the interval I is in fact constant!

Counterexample 0.3.5. Let's take a moment to see that the argument above does not work for the circle S^1. The easiest example of a nontrivial local system on S^1 is the sheaf of sections of the Möbius band (and its canonical map to S^1). This will have a trivialising open cover consisting of two intervals. However, their intersection is a disjoint union of two intervals, and there are nonconstant local systems on these, and so the argument above does not work.

Here we would like to say that we are exhibiting S^1 as the homotopy pushout $* \cup^{*\cup *} *$, and we are seeing that pushout reflected somehow in the categories of local systems. That turns out to be true, and it will relate the category of local systems to the fundamental group.

In the course of this discussion, we encountered some facts that will be useful to us more generally.

Lemma 0.3.6. *Let X be a topological space, and let F be a sheaf of sets on X. Then if U and V are open sets such that $F|U$ and $F|V$ are constant, then*

1 Stratified topology

Posets and stratifications

Construction 1.0.1. Let P be a poset.

A *sieve* is a subset $Z \subseteq P$

Dually, a *cosieve*

The *Alexandroff topology* on P

Example 1.0.2. Let us consider the poset $[1] \coloneqq \{0 < 1\}$.

Example 1.0.3. More generally, let us consider the linearly ordered poset $[n]$.

Example 1.0.4. Let P and Q be posets, and let us consider the product poset $P \times Q$.

Example 1.0.5. For any set S, we can contemplate the trivial poset structure.

Construction 1.0.6. For any Kolmogoroff topological space X, the *specialisation poset*

Proposition 1.0.7. *The formation of the Alexandroff topology of a finite poset and the formation of the specialisation poset of a finite Kolmogoroff topological space are mutually inverse equivalences of categories.*

Notation 1.0.8. Let P be a finite poset P. We shall always regard P as endowed with its Alexandroff topology. We write $\widetilde{P}$ for the ∞-category of sheaves (of spaces) on P.

Construction 1.0.9. Let P be a finite poset. The stalk of a sheaf F on P at a point $p \in P$

Proposition 1.0.10. *The construction above defines an equivalence of ∞-categories*

$$\widetilde{P} \xrightarrow{\sim} \mathrm{Fun}(P, \mathbf{S}) \, .$$