Staggered Fermions[1,2,3]

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Journal Club

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[1] C. Gattringer and C. B. Lang *QCD on the Lattice* Springer, 2010.
[2] H. J. Rothe *Lattice Gauge Theories* 3rd ed. World Scientific, 2005.
[3] E. Follana et al., Phys. Rev. D, 75.5, 054502 (2007).

Quick Outline

Staggered:

1. Motivation
2. Broad strokes
3. Detailed strokes
4. Some discussion

HISQ:

1. Improving staggered actions
2. How HISQ does it

Summary

Staggered: Motivation

- Last time saw fermion discretization has doubling problem.

$$D^{-1}(p) = \frac{\frac{i}{a}\sum_{\mu=1}^{4}\gamma_\mu \sin(p_\mu a)}{\frac{1}{a^2}\sum_{\nu=1}^{4}\sin^2(p_\nu a)}$$

  in massless case, where there are unphysical poles at the corners of the Brillouin Zone, where at least one $p_\mu \neq 0$.
- Wilson: Add Wilson term that kills doublers but vanishes for $a \to 0$

$$a\sum_{\mu=1}^{4}\frac{1}{2a^2}\left(2\delta_{x,y} - \delta_{x,y-a\hat{\mu}} - \delta_{x,y+a\hat{\mu}}\right)$$

  in the free case. This is discretized $\partial_\mu\partial_\mu$.
- **Drawback:** Breaks $\mathrm{SU}_A(N_f) \times \mathrm{U}_A(1)$ explicitly.
- **Desire:** To remove doublers in a way that preserves at least some remnant of chiral symmetry.

Staggered: Broad strokes

- **Idea:** Increase the effective lattice spacing, thereby reducing the Brillouin Zone.
- One way to do this: Distribute d.o.f. on each corner of unit hypercube. Each d.o.f. is then separated by $2a$.
- Define new quarks (tastes) by linear combinations of the d.o.f.
- Reduces 15 unphysical flavors to 3.
- There will remain $\mathrm{U}(1) \times \mathrm{U}(1)$.

Staggered: Detailed strokes

Starting point: Naive free fermion action

$$S_F = a^4 \sum_n \bar{\psi}(n) \left(\sum_{\mu=1}^{4} \gamma_\mu \frac{\psi(n+\hat{\mu}) - \psi(n-\hat{\mu})}{2a} + m\psi(n) \right).$$

and introduce staggered transformation

$$\psi(n) = \gamma_1^{n_1} \gamma_2^{n_2} \gamma_3^{n_3} \gamma_4^{n_4} \psi(n)'$$

$$\bar{\psi}(n) = \bar{\psi}(n)' \gamma_4^{n_4} \gamma_3^{n_3} \gamma_2^{n_2} \gamma_1^{n_1}$$

Now write S_F in terms of these transformed fields...

Staggered: Detailed strokes

$$S_F = a^4 \sum_n \bar{\psi}(n)' \mathbf{1} \left(\sum_{\mu=1}^{4} \eta_\mu(n) \frac{\psi(n+\hat{\mu})' - \psi(n-\hat{\mu})'}{2a} + m\psi(n)' \right)$$

$$\eta_1(n) = 1, \quad \eta_2(n) = (-1)^{n_1}, \quad \ldots, \quad \eta_4(n) = (-1)^{n_1+n_2+n_3}$$

$$\bar{\psi}(n)\gamma_4\psi(n+\hat{4}) = (-1)^{n_1+n_2+n_3}\bar{\psi}(n)'\mathbf{1}\psi(n+\hat{4})$$

$$\bar{\psi}(n) = \bar{\psi}(n)'\gamma_4^{n_4}\gamma_3^{n_3}\gamma_2^{n_2}\gamma_1^{n_1} \qquad \psi(n) = \gamma_1^{n_1}\gamma_2^{n_2}\gamma_3^{n_3}\gamma_4^{n_4}\psi(n)'$$

Staggered: Detailed strokes

$$S_F = a^4 \sum_n \bar{\psi}(n)' \mathbf{1} \left(\sum_{\mu=1}^{4} \eta_\mu(n) \frac{\psi(n+\hat{\mu})' - \psi(n-\hat{\mu})'}{2a} + m\psi(n)' \right)$$

- Diagonal in Dirac space, with identical components
- Take only one copy, call it χ

Carrying out the second bullet point...

Staggered: Detailed strokes

$$S_F = a^4 \sum_n \bar{\chi}(n) \left(\sum_{\mu=1}^{4} \eta_\mu(n) \frac{\chi(n+\hat{\mu}) - \chi(n-\hat{\mu})}{2a} + m\chi(n) \right)$$

We are now ready to increase the effective lattice spacing:

1. Divide lattice into disjoint unit hypercubes
2. The χ live on the corners of the hypercubes
3. New q will be linear combination of the χ
4. The new lattice sites will be indexed by hypercube number

Staggered: Detailed strokes (building q)

Introduce new vectors, h (hypercybe) and s (corner):

$$n_\mu = 2h_\mu + s_\mu \quad \text{with} \quad h_\mu \in \{0, 1, ..., N_\mu/2\} \quad s_\mu \in \{0, 1\}$$

In this indexing scheme,

$$\eta_\mu(n) = \eta_\mu(2h + s) = \eta_\mu(s).$$

Staggered: Detailed strokes (building q)

Introduce Γ, which will serve as the weights later

$$\Gamma^s = \gamma_1^{s_1}\gamma_2^{s_2}\gamma_3^{s_3}\gamma_4^{s_4}$$

Obeys, respectively, orthogonality and completeness relations:

$$\frac{1}{4}\,\mathrm{tr}\left[\Gamma^{s\dagger}\Gamma^{s'}\right] = \delta_{ss'} \qquad\qquad \frac{1}{4}\sum_s \Gamma^{s*}_{ba}\Gamma^{s}_{b'a'} = \delta_{aa'}\delta_{bb'}$$

$$\gamma_\mu = \gamma_\mu^\dagger = \gamma_\mu^{-1}, \quad \gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}$$

Staggered: Detailed strokes (building q)

The promised linear combination:

$$\boxed{q(h)_{ab} \equiv \frac{1}{8}\sum_s \Gamma^s_{ab}\chi(2h+s) \qquad \bar{q}(h)_{ab} \equiv \frac{1}{8}\sum_s \bar{\chi}(2h+s)\Gamma^{s*}_{ab}}$$

Can be inverted using orthogonality relation:

$$\boxed{\chi(2h+s) = 2\,\mathrm{tr}\left[\Gamma^{s\dagger}q(h)\right] \qquad \bar{\chi}(2h+s) = 2\,\mathrm{tr}\left[\bar{q}(h)\Gamma^s\right]}$$

$$\frac{1}{4}\,\mathrm{tr}\left[\Gamma^{s\dagger}\Gamma^{s'}\right] = \delta_{ss'}$$

Staggered: Detailed strokes (replace χ with q)

Next we want to replace χ with q in the staggered action

$$S_F = a^4 \sum_n \bar{\chi}(n) \left(\sum_{\mu=1}^{4} \eta_\mu(n) \frac{\chi(n+\hat{\mu}) - \chi(n-\hat{\mu})}{2a} + m\chi(n) \right)$$

1. Mass term easy: Plug in χ in terms of q.
2. Kinetic term mixes hypercubes...
3. Keep track of which hypercube χ belongs to!
4. Keeping track introduces some s dependence, which then keeps you from using completeness relations...
5. G&L suggest trick of shifting starting index for hypercubes by 1, which gives equivalent results, then averaging between this and original index convention.

Staggered: Detailed strokes (replace χ with q)

New action with spacing $b = 2a$, $q \equiv q(h)$, summation over μ:

$$\boxed{S_F = (2a)^4 \sum_h \Big(m\,\mathrm{tr}\,[\bar{q}q] + \mathrm{tr}\,[\bar{q}\gamma_\mu \partial_\mu q] - a\,\mathrm{tr}\,[\bar{q}\gamma_5 \Box_\mu q \gamma_\mu \gamma_5] \Big)}$$

$$\partial_\mu f(h) = \frac{f(h+\hat{\mu}) - f(h-\hat{\mu})}{2b}$$

$$\Box_\mu f(h) = \frac{f(h+\hat{\mu}) - 2f(h) + f(h-\hat{\mu})}{b^2}$$

Finally, we want to identify the tastes...

Staggered: Detailed strokes (getting a taste)

1. q_{ab} has 16 components, but Dirac spinors should have 4.
2. Tells us each q corresponds to 4 spinors.
3. Hence identify a taste index and Dirac index.
4. From e.g. $\mathrm{tr}[\bar{q}q] = \bar{q}_{ab}q_{ba}$ term, Dirac index is first.

Thus our taste spinors are

$$\psi^t(h)_\alpha \equiv q(h)_{\alpha t} \qquad \bar{\psi}^t(h)_\alpha \equiv \bar{q}(h)_{t\alpha}$$

Staggered: Detailed strokes

Staggered action with $\tau_\mu \equiv \gamma_\mu^T$, $\psi \equiv \psi(h)$ and sums over t, t', μ:

$$\boxed{S_F^{\mathrm{stagger}} = b^4 \sum_h \left(m\bar{\psi}^t \psi^t + \bar{\psi}^t \gamma_\mu \partial_\mu \psi^t - \frac{b}{2} \bar{\psi}^t \gamma_5 (\tau_5 \tau_\mu)_{tt'} \Box_\mu \psi^{t'} \right)}$$

- Last (taste breaking) term kind of like Wilson term
- Taste-breaking term allows for taste mixing
- Otherwise tastes would be mass degenerate from first term
- Remaining symmetry in taste-breaking term is $\mathrm{U}(1) \times \mathrm{U}(1)$:

$$\psi' = e^{i\omega}\psi, \qquad \bar{\psi}' = \bar{\psi} e^{-i\omega}$$

$$\psi' = e^{i\omega\gamma_5 \otimes \tau_5}\psi, \qquad \bar{\psi}' = \bar{\psi} e^{i\omega\gamma_5 \otimes \tau_5}$$

Staggered: Some discussion

- **Harder to construct operators** that have the correct quantum numbers in the continuum limit.
- Taste breaking effects can be reduced by improved gauge actions or smearing[4,5].
- Smearing drives Dirac eigenvalues toward 4-fold degeneracy.
- Still would like just one fermion, though.
- This can be improved by **rooting**.
- Since the number of d.o.f. is reduced, staggered fermions are comparatively **numerically cheap**.
- Staggered results **agree with experimental results**.

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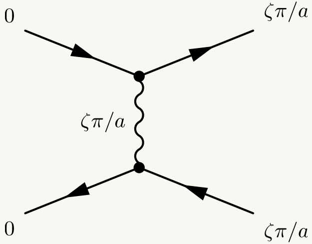
[4] S. Dürr, C. Hoelbling, and U. Wenger, Phys. Rev. D, 70, 094502 (2004).

[5] E. Follana, A. Hart, and C. T. H. Davies, Phys. Rev. Lett. 93, 241601 (2004).

HISQ: Improving staggered actions

Taste breaking can be thought of through taste exchange. (Figure from Follana *et al.* and ζ is a vector labelling the unphysical corners of the BZ.)



A low-energy quark absorbing a gluon with momentum $\zeta\pi/a$ can become a quark of another taste.

HISQ: Improving staggered actions

One way to reduce taste violation:

- Different tastes see different links.
- So the more links fluctuate, the greater the taste breaking.
- Taste exchange can be suppressed through smearing, where links are replace with local "averages", i.e. link takes more typical value given its neighbors.

Not directly related to fermions: but one can also improve derivative discretization to reduce lattice artifacts.

HISQ: Improving staggered actions

$$S_F^{\mathrm{stagger}} = b^4 \sum_h \left(m\bar{\psi}^t \psi^t + \bar{\psi}^t \gamma_\mu \partial_\mu \psi^t - \frac{b}{2} \bar{\psi}^t \gamma_5 (\tau_5 \tau_\mu)_{tt'} \Box_\mu \psi^{t'} \right)$$

- Without taste exchange, D would be block diagonal
- Then $\det D$ is the product of $\det D_{\mathrm{block}}$
- For $N_f = 2 + 1$ one can sample with

$$e^{-S_G} \det D(m_l)^{1/2} \det D(m_s)^{1/4}$$

- Better with smaller taste violations

HISQ: How HISQ does it

General strategy:

1. Improve finite difference derivatives (Naik term)

$$\partial_\mu \to \partial_\mu - \frac{a^2}{6}(1+\epsilon)\partial_\mu^3$$

   where ϵ depends on charm physics.

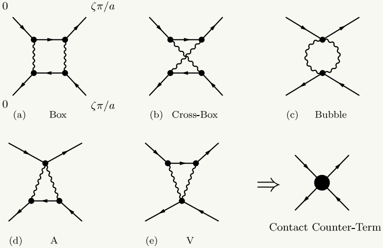
2. Find a smear $U_\mu \to \mathcal{F}_\mu U_\mu$ that vanishes for links carrying momentum π/a. We start with one called Fat7.
3. Multiple smearing reduces mass splittings more, so do that.
4. Remove new $\mathcal{O}(a^2)$ errors introduced by Fat7.
5. Smearing enhances some 1-loop taste exchange processes, which can be suppressed by reunitarizing.

$$\mathcal{F}^{\mathrm{HISQ}} \equiv \mathcal{F}^{\mathrm{Fat7}}_{\mathrm{corr}} \mathcal{U} \mathcal{F}^{\mathrm{Fat7}}$$

HISQ: How HISQ does it

A rough idea of how well HISQ does. The d are coefficients for 1-loop taste-exchange diagrams. The average is reported.

	Unimproved Gluons			Improved Gluons
	ASQTAD	HISQ	HYP	ASQTAD
Avg $d, \tilde{d}$	0.23	0.02	0.02	0.13



Summary

- Staggered quark d.o.f. are distributed on corners of unit hypercubes.
- Staggered quarks remove some doublers and preserve some chiral symmetry.
- They are numerically cheap and have agreed with experiment.
- Effects of multiple tastes can be reduced by rooting.
- There is some taste-symmetry breaking.
- HISQ suppresses this by smearing, improved "to one loop".