Topological Charge and Cooling Scales in Pure SU(2) Lattice Gauge Theory

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B. A. Berg and D. A. Clarke, Phys. Rev. D **97** 054506 (2018)

Background

- To reach the continuum limit, send the lattice spacing to zero (compared to a physical length)
- In the continuum limit, one can predict dimensionless mass ratios $c_k = m_k/m_0$ (m_0^{-1} can serve as a length scale)
- Choosing the reference m_0 is called scale setting
- Lüscher introduces the gradient flow[1], suggests gradient scale as a new reference scale; scale setting gains renewed interest
- Bonati and D'Elia[2] suggest standard cooling[3] can be used similarly for scale setting; advantage in computational efficiency
- We verify[4] this can be done in pure SU(2)
- This project: Investigate whether there is any dependence of the cooling scales on topological charge
- This project: Obtain a new estimate for topological susceptibility

[1] M. Lüscher, J. High Energy Phys. 2010 (2010).

[2] C. Bonati and M. D'Elia, Phys. Rev. D, 89 (2014).

[3] B. A. Berg, Phys. Lett. B, 104, 475 (1981).

[4] B. A. Berg and D. A. Clarke, Phys. Rev. D, 95 (2017).

Outline

1. Topological charge
2. Brief overview of lattice
3. Results
4. Conclusion

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Topological sectors[5]

Consider classical, euclidean SU(2) gauge theory

$$\mathcal{L} = -\frac{1}{2g^2}\,\mathrm{tr}\,F_{\mu\nu}F_{\mu\nu} = \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}.$$

The quantum theory has infinitely many vacuum states.

1. Classical vacuum state: $F^a_{\mu\nu} = 0 \Rightarrow A_\mu = U\partial_\mu U^\dagger$
2. If you can smoothly deform U into U', a topological charge Q is conserved
3. If you can't smoothly deform U into U', there is an energy barrier between A_μ and A'_μ (topological sectors)
4. There are infinitely many U all having different Q
5. This corresponds to different vacuum states in quantum theory

[5] M. Srednicki *Quantum Field Theory* Cambridge: Cambridge, 2007.

Topological charge

We can use topological winding number

$$Q \equiv -\frac{1}{24\pi^2}\int d^3x\,\epsilon_{ijk}\,\mathrm{tr}\,U\partial_i U^\dagger\,U\partial_j U^\dagger\,U\partial_k U^\dagger$$

- Invariant under coordinate changes
- Invariant under smooth deformations of U
- Counts number of times U covers spatial three-sphere
- We can rewrite Q as

$$Q = \frac{1}{32\pi^2}\int d^4x\,\epsilon_{\mu\nu\rho\sigma}\,\mathrm{tr}\,F_{\mu\nu}F_{\rho\sigma}\,,$$

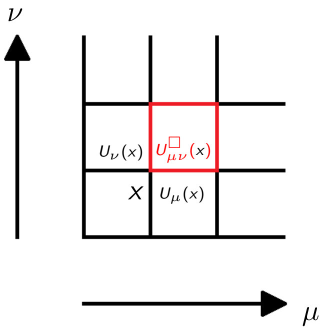
which will be useful on the lattice

Pocket dictionary for pure SU(2) LGT

- 4D space-time with Euclidean metric and periodic BCs
- Sites $x = (an_1, an_2, an_3, an_4)$ with n_4 in time direction
- Regularization through lattice spacing a
- Link variables $U_\mu(x) = e^{-aA_\mu(x)} \in \mathrm{SU}(2)$ on links
- Plaquette $U^\square_{\mu\nu}(x)$
- Hypercube of volume $(aN)^4$
- Discretized calculus

$$\int d^4x \leftrightarrow a^4 \sum_x$$

$$\partial_\mu f(x) \leftrightarrow \frac{f(x + a\hat{\mu}) - f(x)}{a}$$



Pocket dictionary for pure SU(2) LGT, continued

- Bare coupling controls lattice spacing

$$a\Lambda_L = \exp\left(-\frac{1}{2b_0g^2}\right)(b_0g^2)^{-b_1/2b_0^2}\left(1+\mathcal{O}(g^2)\right)$$

- Continuum limit $a \to 0$, *with care*
- Wilson action:

$$S = \beta \sum_{x,\mu<\nu}\left(1-\frac{1}{2}\operatorname{tr} U^{\square}_{\mu\nu}(x)\right) \approx -\frac{\beta}{8}\sum_x a^4 \operatorname{tr} F_{\mu\nu}(x)F_{\mu\nu}(x)$$

- Identify $\beta = 4/g^2$ so that in continuum limit

$$S \to -\frac{1}{2g^2}\int d^4x \operatorname{tr} F_{\mu\nu}F_{\mu\nu}$$

SU(2) LGT simulations

Calculate expected value of operator O

- In QFT expectation values given by

$$\langle O\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}\phi\, O(\phi)\, e^{-S(\phi)}, \qquad \mathcal{Z} \equiv \int \mathcal{D}\phi\, e^{-S(\phi)}$$

- Markov Chain Monte Carlo basic idea:
  - Each configuration generated depending on last one only
  - Accept new configuration with probability $\min\{1, e^{-\Delta S}\}$
  - Create a time series of measurements O_n of O
  - Multiple sweeps between measurements to reduce correlation
- The estimator for $\langle O\rangle$ on the lattice is

$$\bar{O} = \frac{1}{N_{\text{conf}}}\sum_{n=1}^{N_{\text{conf}}} O_n$$

Q on the lattice

Naive discretization using pocket dictionary:

$$Q = \frac{1}{32\pi^2} \int d^4x \, \epsilon_{\mu\nu\rho\sigma} \operatorname{tr} F_{\mu\nu} F_{\rho\sigma}$$

becomes

$$Q = \frac{1}{2^9\pi^2} \, a^4 \sum_x \sum_{\mu\nu\rho\sigma=\pm1}^{\pm4} \epsilon_{\mu\nu\rho\sigma} \operatorname{tr} U^{\square}_{\mu\nu}(x) U^{\square}_{\rho\sigma}(x)$$

- Sum over backward directions to symmetrize
- Suffers from short-ranged renormalization effects

Topological susceptibility

$$\chi = \frac{1}{V} \left\langle Q^2 \right\rangle = \frac{1}{N^4} \frac{1}{N_{\text{conf}}} \sum_{i=1}^{N_{\text{conf}}} Q_i^2$$

- Provides information about distribution of Q
- Phenomenological interest

Cooling

We use "smoothe" using standard cooling

$$V_\mu(x, n_c) = \frac{V_\mu^{\sqcup}(x, n_c - 1)}{\sqrt{\det V_\mu^{\sqcup}(x, n_c - 1)}}.$$

- Locally minimizes action
- Suppresses local fluctuations without changing global charge
- Extract Q and χ from cooled configurations, after these quantities become "metastable"
- Q will "freeze out" at large enough β

Given target value y, a cooling scale $s(\beta)$ is defined by

$$y(t) = t^2 \langle E_t \rangle \quad \text{with} \quad s(\beta) = \sqrt{t(\beta)}$$

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Lattices and statistics

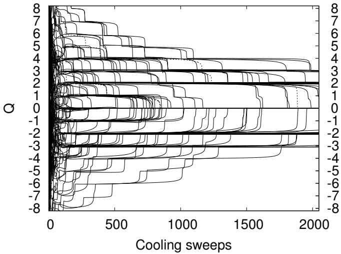
Table: Lattices used for both χ and $L(Q)$ analysis.

β	Measurements $\times$ sweeps	N
2.710	128×2^{13}	16, 28, **40**
2.751	128×2^{13}	16, 28, **40**
2.816	128×2^{13}	28, 40, **44**
2.875	128×2^{13}	28, 40, 44, **52**
2.928	128×2^{13} $(\times 1.5)$	28, 40, 44, 52, **60**

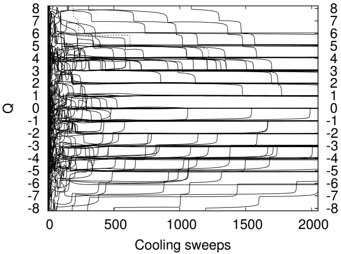
- Multiple N to extrapolate $N \to \infty$
- Multiple β to extrapolate $a \to 0$
- Generate 128 configurations by MCMC, then cool them
- Largest in study of χ in pure SU(2)

"Freezing out" of cooling trajectories

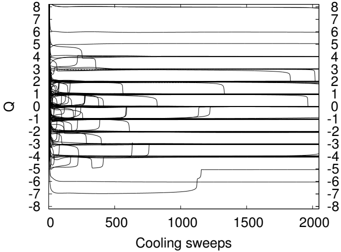
$\beta = 2.300,\ N^4 = 16^4$



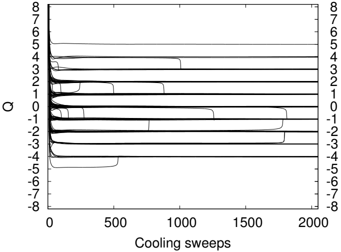
$\beta = 2.510,\ N^4 = 28^4$



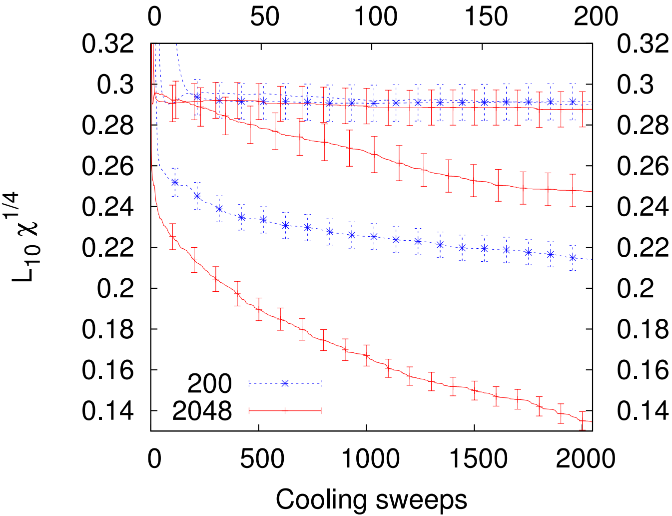
$\beta = 2.751,\ N^4 = 40^4$



$\beta = 2.928,\ N^4 = 60^4$



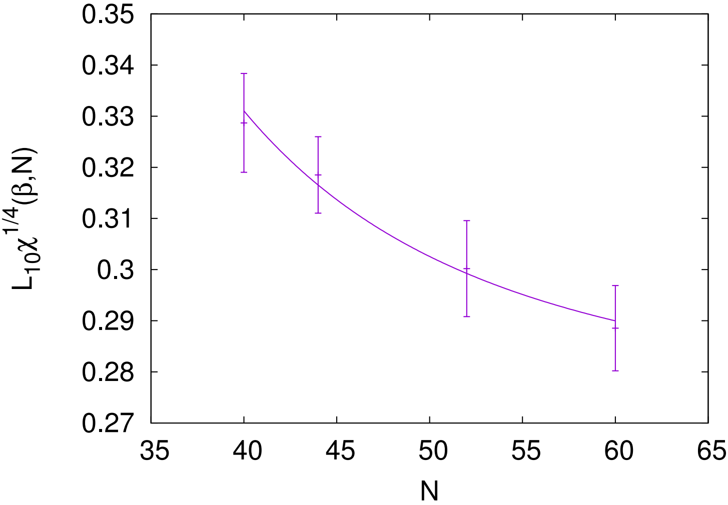
Stabilization of χ



χ and Q stable for $\beta \gtrsim 2.751$

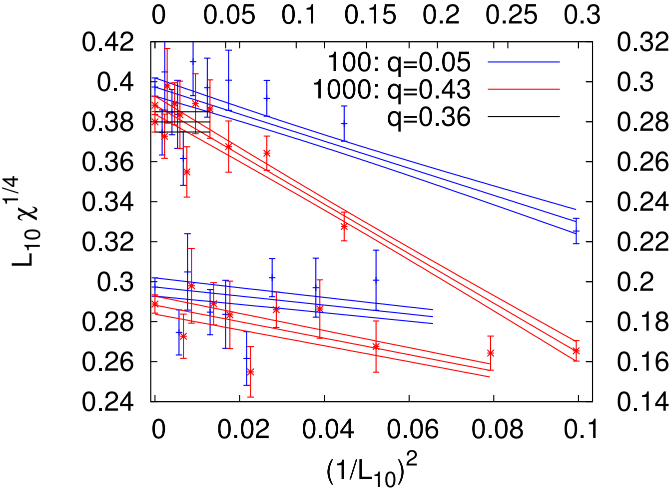
Systematics: Finite size extrapolation

$$L_{10}\chi^{1/4}(\beta, N) = L_{10}\chi^{1/4}(\beta) + \frac{c}{N^4}$$



Systematics: Continuum limit extrapolation

$$L_{10}\chi^{1/4}(\beta) \approx L_{10}\chi^{1/4} + k\, a^2\Lambda_L^2 = {\color{blue}L_{10}\chi^{1/4}} + {\color{blue}c}\left(\frac{1}{L_{10}(\beta)}\right)^2$$



Prediction for topological susceptibility

$$\chi^{1/4}/\sqrt{\sigma} = 0.4446(82) \text{ for } n_c = 1000$$
$$\chi^{1/4}/\sqrt{\sigma} = 0.4655(87) \text{ for } n_c = 100$$

Table: Comparisons with past predictions using Gaussian difference tests.

$\chi^{1/4}/\sqrt{\sigma}$	q_{1000}	q_{100}
0.501(45)[6]	0.31	0.44
0.528(21)[7]	0.00	0.01
0.480(23)[8]	0.32	0.56
0.4831(56)[9]	0.01	0.09
0.4745(63)[9]	0.07	0.40
0.4742(56)[9]	0.06	0.40

[6] P. De Forcrand, M. G. Perez, and I.-O. Stamatescu, Nucl. Phys. B, 499, 409 (1997).
[7] T. DeGrand, A. Hasenfratz, and T. G. Kovacs, Nuclear Physics B, 505, 417 (1997).
[8] B. Allés, M. D'Elia, and A. Di Giacomo, Phys. Lett. B, 412, 119 (1997).
[9] B. Lucini and M. Teper, J. of High Energy Phys. 2001, 050 (2001).

Dependence of cooling scale on Q

- Compared scales using Student difference tests
- Found $L(Q)$ statistically compatible with $L(-Q)$
- Found scales with $|Q| > 1$ to be statistically compatible
- Therefore combine into three bins

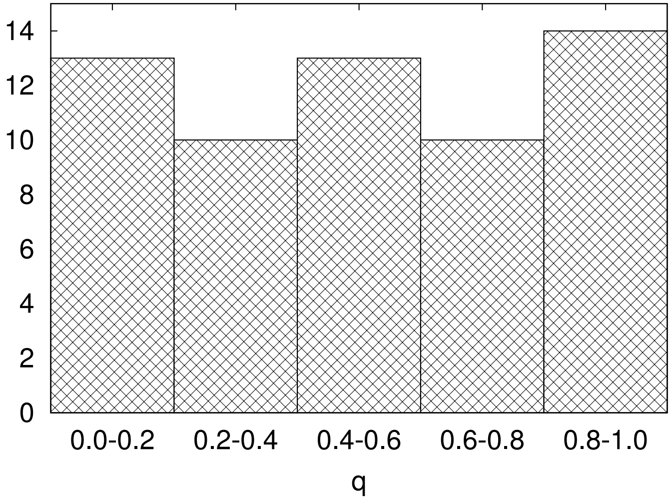
$$|Q| = 0, \quad |Q| = 1, \text{ and } |Q| \geq 2.$$

- L_{10} highly correlated with L_{11}; average them
- Statistical analysis on remaining $4 \times 15 = 60$ scales

Table: Example cooling scales on largest lattice, by sector, at $n_c = 1000$.

β	$\|Q\|$	n	L_{10}	L_{11}	L_{12}
2.928	0	26	12.61(23)	12.55(23)	11.66(21)
	1	49	12.74(18)	12.68(17)	11.66(18)
	≥ 2	53	12.39(14)	12.34(14)	11.64(14)

Dependence of cooling scale on Q



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Conclusions

- For large enough β and N, standard cooling can be used to obtain stable topological sectors
- Best estimate for topological susceptibility:

$$\chi^{1/4}/\sqrt{\sigma} = 0.4446(82),$$

which is surprisingly close to past predictions
- Within our statistics, we find no evidence of correlations between cooling scales due to topological charges

Thanks for listening