A Comparison of Scales in Pure SU(2) LGT

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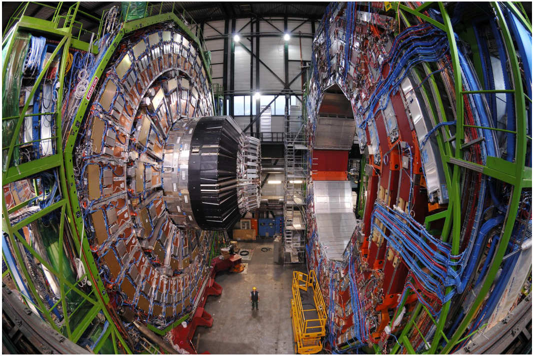
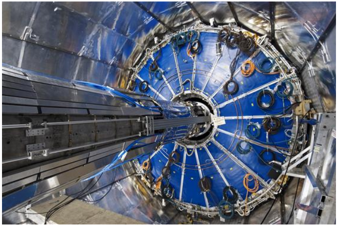
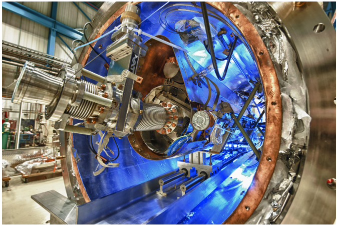
Outline

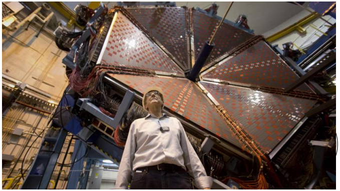
Lots of cool experiments

- Hadronic structure
- Hadron spectroscopy
- Studying confinement
- New physics searches
- ...and more









Lattice calculations are useful

- Lattice spacing provides a regularization of QFT
- Allows one to calculate non-perturbative quantities
- In principle arbitrary precision can be achieved provided enough computing power is available
- That can help guide experimental searches
- Can help confirm Standard Model with high precision, or otherwise hint at new physics

Scale setting

- To reach the continuum limit, send the lattice spacing to zero (compared to a physical length)
- In the continuum limit, one can predict dimensionless mass ratios $c_k = m_k/m_0$ (m_0^{-1} can serve as a length scale)
- Choosing the reference m_0 is called scale setting
- Many challenging calculations rely on scales of previous work; in some cases inaccuracy of reference scale greatly influenced result
  - 2008 and subsequent 2010 HPQCD calculation of f_{D_s} [1, 2]
- Ideally, reference scale can be calculated with high precision using modest computational resources

Gauge symmetries and SU(2): A quick review

- Noether's Theorem: each continuous symmetry of the Lagrangian implies a conserved current
- SU(N) group of $N \times N$ unitary matrices with determinant 1
  - $N^2 - 1$ generators
  - Corresponding to each generator T^a is a gauge field A_μ^a
  - Particles associated with these fields are gauge bosons, but in SU(N) LGT we just call them "gluons"
- SU(2) has $2^2 - 1 = 3$ generators, the Pauli matrices σ^a
- The kinetic term

$$\mathcal{L}_G = -\frac{1}{2g^2} \operatorname{Tr} F_{\mu\nu} F^{\mu\nu}$$

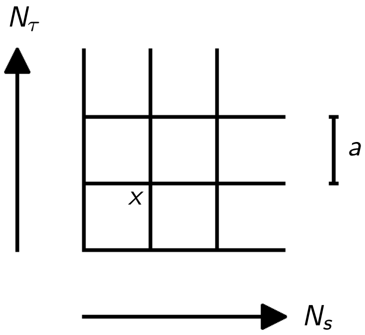
is invariant under local SU(2) transformations

Why pure SU(2)?

- When the Lagrangian has the SU(N) kinetic term and nothing else, we call the theory pure SU(N)
- SU(2) is a gauge group of the Standard Model
- Computationally simple compared to complex systems like QCD
  - Accessible to our resources
  - No fermions, which are challenging to put on the lattice
  - Comparatively short simulations provide large statistics
  - Allows one to study a wider range of lattice size and bare coupling combinations
- General insight discovered in pure SU(2) can perhaps be applied valuable for complex theories such as QCD

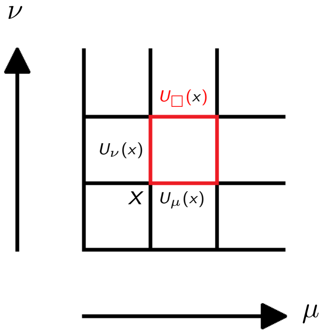
Pure SU(2) LGT

- Non-perturbative approach to QFT
- 4D space-time with Euclidean metric and periodic BCs
- Sites $x = (n_1, n_2, n_3, n_4)$ with n_4 in time direction
- Regularization through lattice spacing a
  - Finite volume $a^4 N_\tau N_s^3$
  - Ultraviolet cutoff $\sim a^{-1}$
  - Infrared cutoff $\sim (aN)^{-1}$
- Continuum limit $a \to 0$



Pure SU(2) LGT

- Link variables or "gluons" $U_\mu = e^{-aA_\mu(x)} \in \mathrm{SU}(2)$ on links
- For a closed path $\mathcal{C}$, the corresponding Wilson loop is $\mathrm{Tr}\,\mathrm{U}(\mathcal{C})$
  - Gauge invariant
  - Used to construct observables
  - Plaquette $\Box$ is the smallest Wilson loop



Pure SU(2) LGT

- Wilson action:

$$S_W = \beta \sum_{\square} \left(1 - \frac{1}{2} \mathrm{Re}\, \mathrm{Tr}\, U_{\square} \right) \approx -\frac{\beta}{8} \sum_{x} a^4\, \mathrm{Tr}\, F_{\mu\nu}(x) F_{\mu\nu}(x)$$

  - $\beta \equiv 4/g^2$
  - In limit $a \to 0$ agrees with

$$S_G = -\frac{1}{2g^2} \int d^4x\, \mathrm{Tr}\, F_{\mu\nu} F_{\mu\nu}$$

  - Equivalence with 4D statistical mechanics
    - $\beta \leftrightarrow$ "inverse temperature"
    - Physical temperature defined by $T \equiv 1/aN_\tau$
  - As we will see, there are other possible energy definitions

Taking the continuum limit

- How does a approach 0?
  - Renormalization group equation

$$B(g) = a\frac{dg}{da} = -b_0 g^3 - b_1 g^5 + \mathcal{O}(g^7)$$

  - Solve the differential equation:

$$\begin{aligned} a\Lambda_L &= \exp\left(\int^g \frac{dg'}{B(g')}\right) \\ &= \exp\left(-\frac{1}{2b_0 g^2}\right)(b_0 g^2)^{-b_1/2b_0^2}\Big(1+\mathcal{O}(g^2)\Big) \\ &\equiv f_{\text{as}}(g^2)\Big(1+\mathcal{O}(g^2)\Big) \end{aligned}$$

  - Λ_L integration constant, so this shows how $g \to 0$ drives $a \to 0$
  - Hence $\beta \to \infty$ brings the system to its continuum limit

Scaling in LGT

- Lattice Λ-parameter

$$\Lambda_L = \lim_{g\to 0} \frac{1}{a} f_{\text{as}}\left(g^2\right)$$

- $m_k = c_k \Lambda_L$ in continuum limit
- Ratios of masses scale as

$$\frac{m_j}{m_i} = \frac{c_j}{c_i}\left[1 + \mathcal{O}\left(a^2\Lambda_L^2\right)\right]; \quad \text{equivalently} \quad \frac{L_i}{L_j} \approx r_{ij} + k_{ij} a^2 \Lambda_L^2$$

- $\mathcal{O}\left(a^2\Lambda_L^2\right)$ deviations from constant ratios are scaling violations
- Want reference scale to depend at most weakly on a

SU(2) LGT simulations

- Calculate expected value of operator O
  - In QFT expectation values given by

$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\phi O[\phi] e^{-S[\phi]}, \qquad \mathcal{Z} \equiv \int \mathcal{D}\phi e^{-S[\phi]}$$

  - An estimator for $\langle O \rangle$ on the lattice is

$$\bar{O} = \frac{1}{N} \sum_{n=1}^{N} O_n$$

- Markov Chain Monte Carlo basic idea:
  - Create a time series of measurements of O, each of which is calculated using the configuration
  - Configuration $n+1$ generated based on configuration n; usually sample from some distribution
  - Accept or reject new configuration with probability $e^{-\Delta S}$

SU(2) LGT simulations

- In practice, configurations n and $n+1$ are highly correlated
- Solution:
  - Update the lattice multiple times between measurements, e.g. measurements on configurations n and $n+5$
  - Divide measurements into N blocks, called repetitions, and compute average over data in the block
  - These averages are effectively statistically independent
- Updating every link on the lattice is called a sweep
- Length scale A is more efficient than length scale B if A requires fewer *total* sweeps to achieve the same fractional error bar

QUESTION: Which reference scale is most efficient?

Deconfining phase transition temperature

- Remember physical temperature is $T = 1/aN_\tau$
- Deconfining phase transition at T_c in $N_s \to \infty$ limit
  - Polyakov loop susceptibility diverges at β_c
  - "glueballs" for $T < T_c$
  - "gluon plasma" for $T > T_c$
- On finite lattices, Polyakov loop susceptibility has maximum at the pseudocritical coupling $\beta_c(N_\tau, N_s)$
- Determine critical coupling constant $\beta_c(N_\tau)$ from $\beta_c(N_s, N_\tau)$ using three parameter fit

$$\beta_c(N_s, N_\tau) = \beta_c(N_\tau) + A\, N_s^{-B}$$

- Inverting the results defines the deconfining length scale $N_\tau(\beta)$
- Clear physical meaning, but computationally expensive

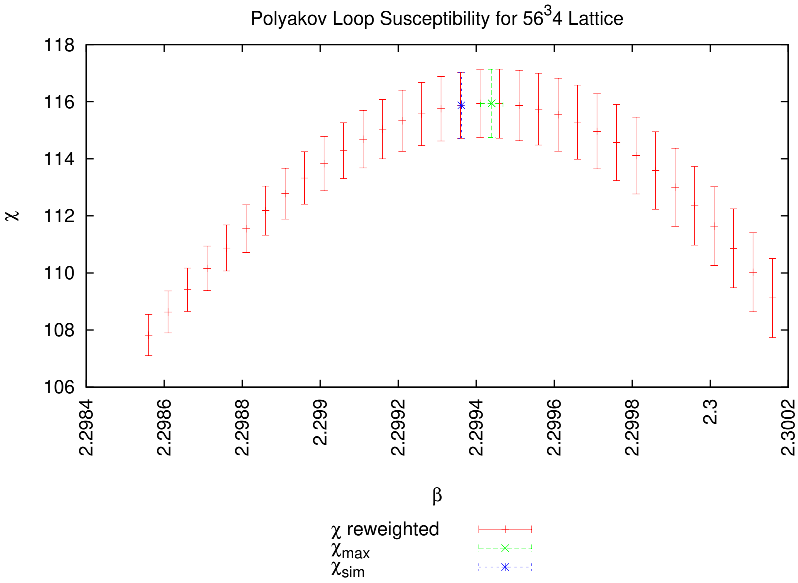

Polyakov Loop Susceptibility for 56^3 4 Lattice
118
116
114
112
110
108
106
χ
2.2984
2.2986
2.2988
2.299
2.2992
2.2994
2.2996
2.2998
2.3
2.3002
β
χ reweighted
χmax
χsim

Table: Pseudocritical β values N_s: β_c. Error bars of β_c are in parentheses.

$N_\tau = 4$	$N_\tau = 6$	$N_\tau = 8$
08: 2.30859(53)	12: 2.43900(33)	16: 2.52960(90)
12: 2.30334(33)	18: 2.43096(43)	24: 2.51678(43)
16: 2.30161(30)	20: 2.42973(11)	32: 2.51296(20)
20: 2.30085(17)	24: 2.42873(35)	40: 2.51192(12)
24: 2.30060(16)	28: 2.427939(74)	44: 2.51150(11)
28: 2.30025(19)	30: 2.427690(87)	48: 2.51119(11)
32: 2.299754(99)	36: 2.427274(67)	52: 2.51130(11)
40: 2.299593(74)	44: 2.426827(67)	56: 2.511096(85)
48: 2.299452(83)	48: 2.426756(64)	64: 2.510635(83)
56: 2.299435(29)	56: 2.426605(62)	72: 2.510716(72)
	60: 2.426596(55)	80: 2.510517(79)
∞: 2.299188(61)	∞: 2.42636(52)	∞: 2.510363(71)
$q = 0.56$	$q = 0.73$	$q = 0.14$
$N_\tau = 4 \pm 0.00063$	$N_\tau = 6 \pm 0.0011$	$N_\tau = 8 \pm 0.0019$

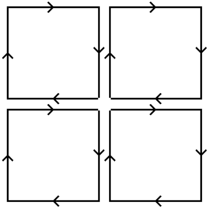
Lots of simulations needed for each point!

Energy operators

- Time dependent plaquettes with parameterization

$$\langle U_\square(t)\rangle = a_0(t)\mathbf{1} + i\sum_{i=1}^{3} a_i(t)\sigma_i$$

- Multiple possible energy discretizations, must agree when $a \to 0$
- Examined three discretizations for this project:
  - $E_0(t) \equiv 2\left[1 - a_0(t)\right]$
  - $E_1(t) \equiv \sum_{i=1}^{3} a_i(t)^2$
  - $E_4(t) \equiv \frac{1}{4}\sum_{i=1}^{3}\left(a_i^{(1)} + a_i^{(2)} + a_i^{(3)} + a_i^{(4)}\right)^2$
  - E_4 suggested by Lüscher [3]



QUESTION: Is there any scaling advantage to using E_4?

Gradient flow

- Lüscher's gradient flow equation [3]

$$\dot{V}_\mu(x,t) = -g^2 V_\mu(x,t)\partial_{x,\mu} S[V(t)]$$

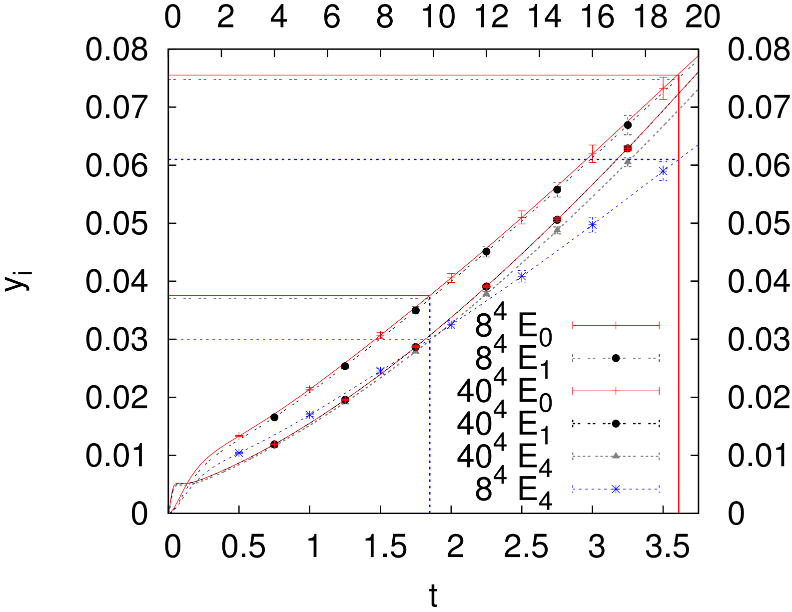
  - Flow time t and initial condition $V_\mu(x,t)\big|_{t=0} = U_\mu(x)$
  - Gradient flow equation drives action down
- Given target value y, length scale $s(\beta) = \sqrt{t_y(\beta)}$ defined implicitly via equation

$$t^2 \langle E_t \rangle \big|_{t=t_y} = y$$

- Requires no fits or extrapolations
- And at least 100 times more efficient than $N_\tau(\beta)$
- But no obvious physical meaning
- And some ambiguity in choosing y: $2 \times 3 = 6$ targets per scale

QUESTION: Does choosing one target over another result in seriously distinct scaling behavior?

Determination of gradient scale



Determination of target value

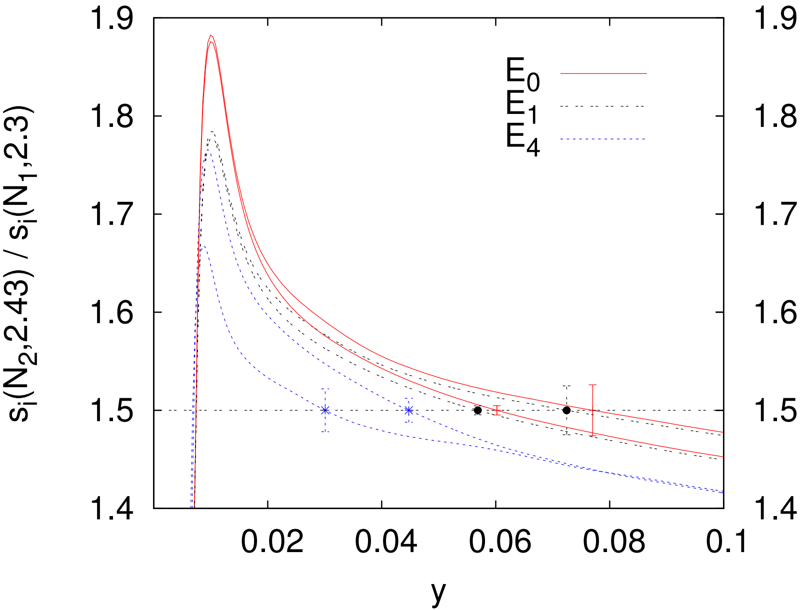


Table: Gradient length scales using energy operators E_0, E_1, E_4 and target values $y_0 = 0.0376$, $y_1 = 0.0370$, $y_4 = 0.0300$, respectively

β	Lattice	$L_1 = s_0$	$L_2 = s_1$	$L_3 = s_4$
2.3	16^4	1.3593(28)	1.3589(27)	1.2756(75)
2.43	28^4	2.1023(30)	2.0911(30)	1.9666(98)
2.51	28^4	2.7590(73)	2.7428(73)	2.570(14)
2.574	40^4	3.4103(72)	3.3896(71)	3.149(16)
2.62	40^4	3.954(10)	3.9293(99)	3.672(19)
2.67	40^4	4.622(17)	4.593(17)	4.297(24)
2.71	40^4	5.199(22)	5.167(22)	4.817(27)
2.751	40^4	5.909(34)	5.872(34)	5.457(41)
2.816	44^4	7.092(48)	7.049(47)	6.530(54)
2.875	52^4	8.510(64)	8.456(65)	7.883(68)

Each scale needs only one lattice, access large β!

Cooling flow

- Introduced by Berg [4] for $O(3)$ topological charge and used by Bonati and D'Elia [5] for topological observables in SU(3)
- Iterative process
  - Instead of a differential equation, just replace link variable with one that locally minimizes action

$$V_\mu(x, n_c) = \frac{V_\mu^{\sqcup}(x, n_c - 1)}{|V_\mu^{\sqcup}(x, n_c - 1)|}$$

  - n_c cooling sweeps corresponds to a gradient flow time [5]

$$t_c = n_c/3$$

- Length scale $x(\beta) = \sqrt{t_y(\beta)}$ defined implicitly like with the gradient flow

$$t^2 \langle E_t \rangle \big|_{t=t_y} = y$$

- Same advantages and disadvantages as the gradient flow
- And at least 34 times faster

QUESTION: Does the cooling flow experience significantly larger scaling violations than the gradient flow?

Table: Cooling length scales using energy operators E_0, E_1, E_4 and target values $y_0 = 0.044$, $y_1 = 0.043$, $y_4 = 0.035$, respectively

β	Lattice	$L_7 = x_0$	$L_8 = x_1$	$L_9 = x_4$
2.3	16^4	1.3433(24)	1.3385(23)	1.2575(74)
2.43	28^4	2.0892(28)	2.0707(28)	1.9446(95)
2.51	28^4	2.7522(68)	2.7267(66)	2.548(15)
2.574	40^4	3.4048(69)	3.3730(67)	3.137(17)
2.62	40^4	3.9509(95)	3.9137(93)	3.645(22)
2.67	40^4	4.618(17)	4.574(16)	4.298(26)
2.71	40^4	5.203(21)	5.154(21)	4.794(28)
2.751	40^4	5.913(32)	5.857(32)	5.434(40)
2.816	44^4	7.105(45)	7.039(45)	6.511(55)
2.875	52^4	8.514(60)	8.433(59)	7.825(68)

Must check its scaling behavior

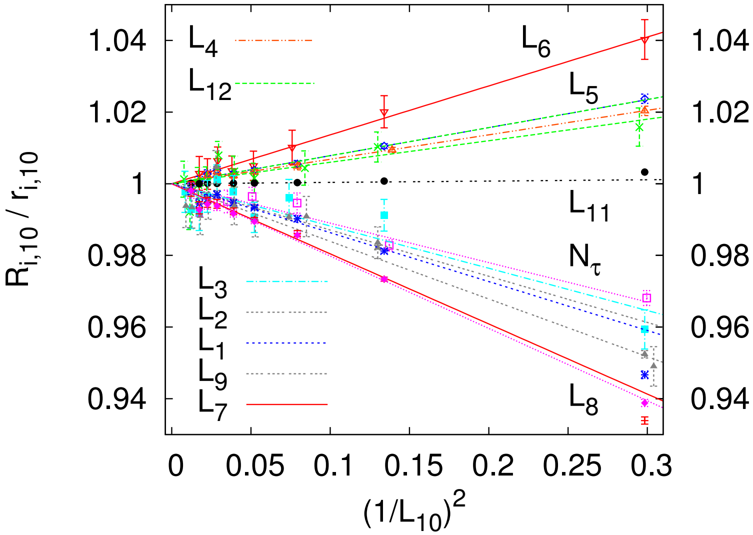
Scaling analysis

- Recall scaling behavior given by

$$R_{ij} \equiv \frac{L_i}{L_j} \approx r_{ij} + k_{ij} a^2 \Lambda_L^2 = r_{ij} + c_{ij} \left(\frac{1}{L_j}\right)^2$$

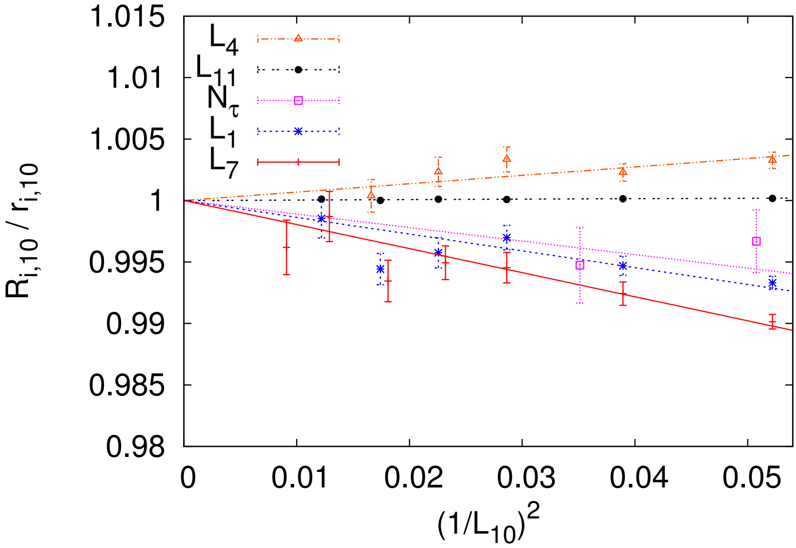
- Collect all length scales from the project
  - $L_1 - L_3$: gradient with first set of target values
  - $L_4 - L_6$: gradient with second set of target values
  - $L_7 - L_9$: cooling with first set of target values
  - $L_{10} - L_{12}$: cooling with second set of target values
- In the following we fix $L_j = L_{10}$ and plot

$$\frac{R_{i,10}}{r_{i,10}} = 1 + c'_{i,10} \left(\frac{1}{L_{10}}\right)^2$$

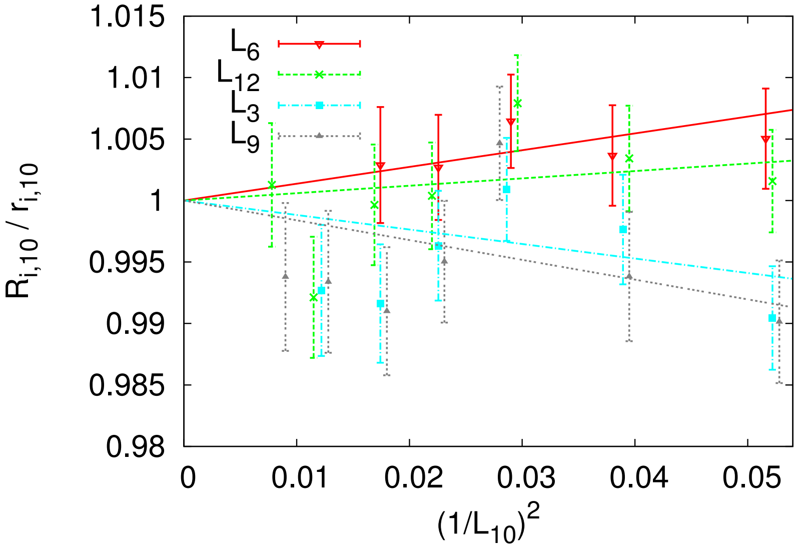


$R_{i,10} / r_{i,10}$
$(1/L_{10})^2$
L_4
L_{12}
L_6
L_5
L_{11}
N_τ
L_3
L_2
L_1
L_9
L_7
L_8
1.04
1.02
1
0.98
0.96
0.94
0
0.05
0.1
0.15
0.2
0.25
0.3

Closer look at lengths using E_0 and E_1



Closer look at lengths using E_4



Conclusions

- Does choosing different target values for the gradient and cooling flows lead to seriously distinct scaling behavior?
  - Not if using physical input to guide initial scaling values
  - Can use the deconfinement scale
- Is there any scaling advantage to using E_4?
  - None were observed
  - Estimates from the same statistics have smaller error bars using E_0 or E_1 than with E_4
- Does the cooling length experience significant scaling violations compared to the deconfining and gradient scales?
  - No noticeable loss of accuracy using cooling flow
- Which scale is most efficient?
  - Gradient scale at least 100 times more efficient than deconfining
  - Cooling at least 34 times more efficient than gradient

Thank you!

References

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- C. T. H. Davies *et al.*, "Update: Precision D s decay constant from full lattice QCD using very fine lattices", Physical Review D, **82** (2010).
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- B. Berg, "Dislocations and topological background in the lattice O(3) sigma model", Physics Letters, **104B** 475 (1981).
- C. Bonati and M. D'Elia, "Comparison of the gradient flow with cooling in SU(3) pure gauge theory", Physical Review D, **89** 105005 (2014).

Backup slides

- Deconfining scale error bars

$$\Delta N_\tau = \frac{N_\tau}{L_{10}^{1,3}(\beta_c)} \left[L_{10}^{1,3}(\beta_c) + L_{10}^{1,3}(\beta_c - \Delta\beta_c) \right]$$

- Polyakov loop definition $P(\vec{x}) = \prod U_\tau(\vec{x})$
- Polyakov loop susceptibility

$$\chi(\beta) = \left\langle P^2 \right\rangle - \langle |P| \rangle^2 , \quad \text{where} \quad P = \frac{1}{N_s^3} \sum_{\vec{x}} P(\vec{x})$$