HINTS AND SOLUTIONS TO PROBLEMS IN

CALCULUS ON MANIFOLDS

A MODERN APPROACH TO CLASSICAL THEOREMS OF ADVANCED CALCULUS

by

Michael Spivak

Editor

DongPhD

For the day...

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Functions on Euclidean Space

NORM AND INNER PRODUCT

1-1. *Prove that* $|x| \leq \sum_{i=1}^{n} |x^i|$.

One has $|x|^2 = \sum_{i=1}^{n}(x^i)^2 \leq \sum_{i=1}^{n}(x^i)^2 + 2\sum_{i\neq j}|x^i||x^j| = (\sum_{i=1}^{n}|x^i|)^2$. Taking the square root of both sides gives the result.

1-2. *When does equality hold in Theorem 1-1 (3)?*

Equality holds precisely when one is a nonnegative multiple of the other. This is a consequence of the analogous assertion of the next problem.

1-3. *Prove that* $|x - y| \leq |x| + |y|$. *When does equality hold?*

The first assertion is the triangle inequality. I claim that equality holds precisely when one vector is a non-positive multiple of the other.

If $x = ay$ for some real a, then substituting shows that the inequality is equivalent to $|a - 1||y| \leq (|a| + 1)|y|$ and clearly equality holds if a is non-positive. Similarly, one has equality if $ax = y$ for some real a.

Conversely, if equality holds, then $\sum_{i=1}^{n}(x^i - y^i)^2 = |x - y|^2 = (|x| + |y|)^2 = \sum_{i=1}^{n}(x^i)^2 + (y^i)^2 + 2\sqrt{\sum(x^i)^2\sum(y^i)^2}$, and so $< x, -y >= |x||-y|$. By Theorem 1-1 (2), it follows that x and y are linearly dependent. If $x = ay$ for some real a, then substituting back into the equality shows that a must be non-positive or y must be 0. The case where $ax = y$ is treated similarly.

1-4. *Prove that* $||x| - |y|| \leq |x - y|$.

If $|x| \geq |y|$, then the inequality to be proved is just $|x| - |y| \leq |x - y|$ which is just the triangle inequality. On the other hand, if $|x| < |y|$, then the result follows from the first case by swapping the roles of x and y.

1-5. *The quantity $|y-x|$ is called the distance between x and y. Prove and interpret geometrically the "triangle inequality": $|z-x| \leq |z-y| + |y-x|$.*

The inequality follows from Theorem 1-1(3):

$$|z-x| = |(z-y) + (y-x)| \leq |z-y| + |y-x|$$

Geometrically, if x, y, and z are the vertices of a triangle, then the inequality says that the length of a side is no larger than the sum of the lengths of the other two sides.

1-6. *Let f and g be functions integrable on $[a,b]$.*

(a) *Prove that $|\int_a^b f \cdot g| \leq (\int_a^b f^2)^{1/2} (\int_a^b g^2)^{1/2}$.*

Theorem 1-1(2) implies the inequality of Riemann sums:

$$|\sum_i f(x_i) g(x_i) \Delta x_i| \leq (\sum_i f(x_i)^2 \Delta x_i)^{1/2} (\sum_i g(x_i)^2 \Delta x_i)^{1/2}$$

Taking the limit as the mesh approaches 0, one gets the desired inequality.

(b) *If equality holds, must $f = \lambda g$ for some $\lambda \in \mathbb{R}$? What if f and g are continuous?*

No, you could, for example, vary f at discrete points without changing the values of the integrals. If f and g are continuous, then the assertion is true. In fact, suppose that for each λ, there is an x with $(f(x) - \lambda g(x))^2 > 0$. Then the inequality holds true in an open neighborhood of x since f and g are continuous. So $\int_a^b (f - \lambda g)^2 > 0$ since the integrand is always non-negative and is positive on some subinterval of $[a,b]$. Expanding out gives $\int f^2 - 2\lambda \int f \cdot g + \lambda^2 \int g^2 > 0$ for all λ. Since the quadratic has no solutions, it must be that its discriminant is negative.

(c) *Show that Theorem 1-1 (2) is a special case of (a).*

Let $a = 0$, $b = n$, $f(x) = x^i$ and $g(x) = y^i$ for all x in $[i-1, i)$ for $i = 1, ..., n$. Then part (a) gives the inequality of Theorem 1-1 (2). Note, however, that the equality condition does not follow from (a).

1-7. *A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called norm preserving if $|T(x)| = |x|$, and inner product preserving if $< T(x), T(y) >=< x, y >$.*

(a) *Show that T is norm preserving if and only if T is inner product preserving.*

If T is inner product preserving, then one has by Theorem 1-1 (4):

$$|Tx| = \sqrt{< Tx, Tx >} = \sqrt{< x, x >} = |x|$$

Similarly, if T is norm preserving, then the polarization identity together with the linearity of T give:

$$\begin{aligned} < Tx, Ty > &= \frac{|Tx + Ty|^2 - |Tx - Ty|^2}{4} \\ &= \frac{|T(x+y)|^2 - |T(x-y)|^2}{4} \\ &= \frac{|x+y|^2 - |x-y|^2}{4} =< x, y > . \end{aligned}$$

(b) *Show that such a linear transformation T is 1-1, and that T^{-1} is of the same sort.*

Let T be norm preserving. Then $|Tx| = 0$ implies $x = 0$, i.e. the kernel of T is trivial. So T is 1-1. Since T is a 1-1 linear map of a finite dimensional vector space into itself, it follows that T is also onto. In particular, T has an inverse. Further, given x, there is a y with $x = Ty$, and so $|T^{-1}x| = |T^{-1}Ty| = |y| = |Ty| = |x|$, since T is norm preserving. Thus T^{-1} is norm preserving, and hence also inner product preserving.

1-8. *If x and y in $\mathbb{R}^n$ are both non-zero, then the angle between x and y, denoted $\angle(x, y)$, is defined to be* $\arccos(< x, y > /|x||y|)$ *which makes sense by Theorem 1-1 (2). The linear transformation T is angle preserving if T is 1-1 and for $x, y \neq 0$, one has $\angle(Tx, Ty) = \angle(x, y)$.*

(a) *Prove that if T is norm preserving, then T is angle preserving.*

Assume T is norm preserving. By Problem 1-7, T is inner product preserving. So $\angle(Tx, Ty) = \arccos(< Tx, Ty > /|Tx||Ty|) = \arccos(< x, y > /|x||y|) = \angle(x, y)$.

(b) *If there is a basis $x_1, ..., x_n$ of $\mathbb{R}^n$ and numbers $\lambda_1, ..., \lambda_n$ such that $Tx_i = \lambda_i x_i$, prove that T is angle preserving if and only if all $|\lambda_i|$ are equal.*

The assertion is false. For example, if $n = 2$, $x_1 = (1, 0)$, $x_2 = (1, 1)$, $\lambda_1 = 1$, and $\lambda_2 = -1$, then $T(0, 1) = T(x_2 - x_1) = T(x_2) - T(x_1) = -x_2 - x_1 = (-2, -1)$. Now, $\angle((0, 1), (1, 0)) = \pi/2$, but $\angle(T(0, 1), T(1, 0)) = \angle((-2, -1), (1, 0)) = \arccos(-2/\sqrt{5})$ showing that T is not angle preserving.

To correct the situation, add the condition that the x_i be pairwise orthogonal, i.e. $< x_i, x_j >= 0$ for all $i \neq j$. Using bilinearity, this means that: $< \sum a_i x_i, \sum b_i x_i >= \sum a_i b_i |x_i|^2$ because all the cross terms are zero.

Suppose all the λ_i are equal in absolute value. Then one has

$$\begin{aligned}\angle(T(\sum a_i x_i), T(\sum b_i x_i)) &= \arccos((\sum a_i b_i \lambda_i^2 |x_i|^2)/\sqrt{\sum a_i^2 \lambda_i^2 |x_i|^2 \sum b_i^2 \lambda_i^2 |x_i|^2}) \\ &= \angle(\sum a_i x_i, \sum b_i x_i)\end{aligned}$$

because all the λ_i^2 are equal and cancel out. So, this condition suffices to make T be angle preserving.

Now suppose that $|\lambda_i| \neq |\lambda_j|$ for some i and j and that $\lambda_i \neq 0$. Then

$$\angle(T(x_i + x_j), Tx_i) = \arccos(\frac{\langle \lambda_i x_i + \lambda_j x_j, \lambda_i x_i \rangle}{(|\lambda_i x_i + \lambda_j x_j||\lambda_i x_i|)})$$

$= \arccos(1/(|x_i|^2 + (\lambda_j/\lambda_i)^2 |x_j|^2)) \neq \arccos(1/(|x_i|^2 + |x_j|^2)) = \angle(x_i + x_j, x_i)$

since $|x_j| \neq 0$. So, this condition suffices to make T not be angle preserving.

(c) *What are all angle preserving $T : \mathbb{R}^n \to \mathbb{R}^n$?*

The angle preserving T are precisely those which can be expressed in the form $T = UV$ where U is angle preserving of the kind in part (b), V is norm preserving, and the operation is functional composition.

Clearly, any T of this form is angle preserving as the composition of two angle preserving linear transformations is angle preserving. For the converse, suppose that T is angle preserving. Let $x_1, x_2, ..., x_n$ be an orthogonal basis of $\mathbb{R}^n$. Define V to be the linear transformation such that $V(x_i) = T(x_i)|x_i|/|T(x_i)|$ for each i. Since the x_i are pairwise orthogonal and T is angle preserving, the $T(x_i)$ are also pairwise orthogonal. In particular, $< V(\sum a_i x_i), V(\sum a_i x_i) >=< \sum a_i T(x_i)|x_i|/|T(x_i)|, \sum a_i T(x_i)|x_i|/|T(x_i)| >= \sum a_i^2 |x_i|^2 =< \sum a_i x_i, \sum a_i x_i >$ because the cross terms all cancel out. This proves that V is norm preserving. Now define U to be the linear transformation $U = TV^{-1}$. Then clearly $T = UV$ and U is angle preserving because it is the composition of two angle preserving maps. Further, U maps each $T(x_i)$ to a scalar multiple of itself; so U is a map of the type in part (b). This completes the characterization.

1-9. *If $0 \leq \theta < \pi$, let $T : \mathbb{R}^2 \to \mathbb{R}^2$ have the matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Show that T is angle preserving and that if $x \neq 0$, then $\angle(x, Tx) = \theta$.*

The transformation T is 1-1 by Cramer's Rule because the determinant of its matrix is 1. Further, T is norm preserving since

$$< T(a,b), T(a,b) >= (\cos(\theta)a + \sin(\theta)b)^2 + (-\sin(\theta)a + \cos(\theta)b)^2 = a^2 + b^2$$

by the Pythagorean Theorem. By Problem 8(a), it follows that T is angle preserving.

If $x = (a, b)$, then one has $< x, Tx >= a(a\cos(\theta) + b\sin(\theta)) + b(-a\sin(\theta) + b\cos(\theta)) = (a^2+b^2)\cos(\theta)$. Further, since T is norm preserving, $|x||Tx| = a^2+b^2$. By the definition of angle, it follows that $\angle(x, Tx) = \theta$.

1-10. *If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, show that there is a number M such that $|T(h)| \leq M|h|$ for $h \in \mathbb{R}^m$.*

Let N be the maximum of the absolute values of the entries in the matrix of T and $M = nN$. One has

$$\begin{aligned} |T(h)| &= |(\sum a_{1i}h^i, ..., \sum a_{ni}h^i)| \\ &= (\sum_{j=1}^{n}(\sum_{i=1}^{m} a_{ji}h^i)^2)^{1/2} \leq \sum_{j=1}^{n} N|h| = nN|h| = M|h|. \end{aligned}$$

1-11. *For $x, y \in \mathbb{R}^n$ and $z, w \in \mathbb{R}^m$, show that $< (x, z), (y, w) >= (x, y) + (z, w)$ and $|(x, z)| = \sqrt{|x|^2 + |z|^2}$. Note that (x, z) and (y, w) denote points in $\mathbb{R}^{n+m}$.*

This is a perfectly straightforward computation in terms of the coordinates of x, y, z, w using only the definitions of inner product and norm.

1-12. *Let $(\mathbb{R}^n)^*$ denote the dual space of the vector space $\mathbb{R}^n$. If $x \in \mathbb{R}^n$, define $\varphi_x(y) =< x, y >$. Define $T : \mathbb{R}^n \to (\mathbb{R}^n)^*$ by $T(x) = \varphi_x$. Show that T is a 1-1 linear transformation and conclude that every $\varphi \in (\mathbb{R}^n)^*$ is φ_x for a unique $x \in \mathbb{R}^n$.*

One needs to verify the trivial results that (a) φ_x is a linear tranformation and (b) $\varphi_{ax+by} = a\varphi_x + b\varphi_y$. These follow from bilinearity; the proofs are omitted. Together these imply that T is a linear transformation.

Since $\varphi_x(x) = |x|^2 \neq 0$ for $x \neq 0$, T has no non-zero vectors in its kernel and so is 1-1. Since the dual space has dimension n, it follows that T is also onto. This proves the last assertion.

1-13. *If $x, y \in \mathbb{R}^n$, then x and y are called perpendicular (or orthogonal) if $< x, y >= 0$. If x and y are perpendicular, prove that $|x + y|^2 = |x|^2 + |y|^2$.*

By bilinearity of the inner product, one has for perpendicular x and y:

$$|x+y|^2 =< x+y, x+y >=< x, x > +2 < x, y > + < y, y >= |x|^2 + |y|^2$$

SUBSETS OF EUCLIDEAN SPACE

1-14. *Prove that the union of any (even infinite) number of open sets is open. Prove that the intersection of two (and hence finitely many) open sets is open. Give a counterexample for infinitely many open sets.*

Let $\{U_i : i \in I\}$ be a collection of open sets, and U be their union. If $x \in U$, then there is an $i \in I$ with $x \in U_i$. Since U_i is open, there is an open rectangle $R \subseteq U_i \subseteq U$ containing x. So U is open.

Let U and V be open, and $W = U \cap V$. If $x \in U \cap V$, then there are open rectangles R (resp. S) containing x and contained in U (resp. V). Since the intersection of two open rectangles is an open rectangle (Why?), we have $x \in R \cap S \subseteq W$; so W is open. The assertion about finitely many sets follows by induction.

The intersection of the open intervals $(-1/n, 1/n)$ is the set containing only 0, and so the intersection of even countably many open sets is not necessarily open.

1-15. *Prove that $U = \{x \in \mathbb{R}^n : |x - a| < r\}$ is open.*

If $x \in U$, then let R be the open rectangle centered at x with sides of length $2(r - |x - a|)/\sqrt{n}$. If $y \in R$, then

$$\begin{aligned} |y - a| &\le |y - x| + |x - a| \\ &< \sqrt{\sum_{i=1}^{n} (r - |x - a|)^2/n} + |x - a| = r \end{aligned}$$

and so $R \subseteq U$. This proves that U is open.

1-16. *Find the interior, exterior, and boundary of the sets:*

$$U = \{x \in \mathbb{R}^n : |x| \le 1\}$$

$$V = \{x \in \mathbb{R}^n : |x| = 1\}$$

$$W = \{x \in \mathbb{R}^n : \text{ each } x^i \text{ is rational}\}$$

The interior of U is the set $\{x : |x| < 1\}$; the exterior is $\{x : |x| > 1\}$; and the boundary is the set V.

The interior of V is the empty set $\emptyset$; the exterior is $\{x : |x| \neq 1\}$; and the boundary is the set V.

The interior of W is the empty set $\emptyset$; the exterior is the empty set $\emptyset$; and the boundary is the set $\mathbb{R}^n$.

In each case, the proofs are straightforward and omitted.

1-17. *Construct a set $A \subset [0,1] \times [0,1]$ such that A contains at most one point on each horizontal and each vertical line but the boundary of A is $[0,1] \times [0,1]$.* Hint:*It suffices to ensure that A contains points in each quarter of the square $[0,1] \times [0,1]$ and also in each sixteenth, etc.*

To do the construction, first make a list L_1 of all the rational numbers in the interval [0, 1]. Then make a list L_2 of all the quarters, sixteenths, etc. of the unit sqare. For example, L_1 could be made by listing all pairs (a, b) of integers with b positive, a non-negative, $a \le b$, in increasing order of $\max(a,b)$, and amongst those with same value of $\max(a,b)$ in increasing lexicographical order; then simply eliminate those pairs for which there is an earlier pair with the same value of a/b. Similarly, one could make L_2 by listing first the quarters, then the sixteenths, etc. with an obvious lexicographical order amongst the quarters, sixteenths, etc. Now, traverse the list L_2: for each portion of the square, choose the point (p,q) such that (p,q) is in the portion, both p and q are in the list L_1, neither has yet been used, and such that the latter occurring (in L_1) of them is earliest possible, and amongst such the other one is the earliest possible.

To show that this works, it suffices to show that every point (p, q) in the square is in the boundary of A. To show this, choose any open rectangle containing (p, q). If it is $(a_1, b_1) \times (a_2, b_2)$, let $r = \min(|a_1 - p|, |a_2 - p|, |b_1 - q|, |b_2 - q|)$. Let n be chosen so that $2^n < r$. Then there is some $(4^n)^{\text{th}}$ portion of the square in L_2 which is entirely contained within the rectangle and containing (p, q). Since this part of the square contains an element of the set A and elements not in A (anything in the portion with the same x-coordinate p works), it follows that (p, q) is in the boundary of A.

1-18. *If $A \subset [0, 1]$ is the union of open intervals (a_i, b_i) such that each rational number in $(0, 1)$ is contained in some (a_i, b_i), show that the boundary of A is $[0, 1] - A$.*

Clearly, the interior of A is A itself since it is a union of open sets; also the exterior of A clearly contains $\mathbb{R} - [0, 1]$ as $A \subset [0, 1]$. Since the boundary is the complement of the union of the interior and the exterior, it suffices to show that nothing in $[0, 1]$ is in the exterior of A. Suppose $x \in [0, 1]$ is in the exterior of A. Let $I = (a, b)$ be an open interval containing x and disjoint from A. Let r be a rational number in $[0, 1]$ contained in I. Then there is a $(a_i, b_i) \subset A$ which contains r, which is a contradiction.

1-19. *If A is a closed set that contains every rational number $r \in [0, 1]$, show that $[0, 1] \subset A$.*

Suppose $x \in [0, 1] - A$. Since $\mathbb{R} - A$ is open, there is an open interval I containing x and disjoint from A. Now $[0, 1] \cap I$ contains a non-empty open subinterval of $(0, 1)$ and this is necessarily disjoint from A. But every non-empty open subinterval of $(0, 1)$ contains rational numbers, and A contains all rational numbers in $[0, 1]$, which is a contradiction.

1-20. *Prove the converse of Corollary 1-7: A compact subset of $\mathbb{R}^n$ is closed and bounded.*

Suppose A is compact. Let $\mathcal{O}$ be the open cover consisting of rectangles $R_n = (-n, n) \times ... \times (-n, n)$ for all positive integers n. Since A is compact, there is a finite subcover $\{R_{n_1}, ..., R_{n_k}\}$. If $r = \max(n_1, ..., n_k)$, then $A \subset R_r$ and so A is bounded.

To show that A is closed, it suffices its complement is open. Suppose x is not in A. Then the collection $\mathcal{O} = \{S_n : n = 1, 2, ...\}$ where $S_n = \{y \in \mathbb{R}^n : |y - x| > 1/n\}$ is an open cover of A. Let $\{S_{n_1}, ..., S_{n_k}\}$ be a finite subcover. Let $r = \max(n_1, ..., n_k)$. Then $\{y \in \mathbb{R}^n : |y - x| < 1/r\}$ is an open neighborhood of x which is disjoint from A. So the complement of A is open, i.e. A is closed.

1-21. (a) *If A is closed and $x \notin A$, prove that there is a number $d > 0$ such that $|y - x| \geq d$ for all $y \in A$.*

Such an x is in the exterior of A, and so there is an open rectangle $(a_1, b_1) \times ... \times (a_n, b_n)$ containing x and disjoint from A. Let $d = \min(|a_1 - x^1|, |b_1 - x^1|, ..., |a_n - x^n|, |b_n - x^n|)$. This was chosen so that $S_d = \{y \in \mathbb{R}^n : |y - x| < d\}$ is entirely contained within the open rectangle. Clearly, this means that no $y \in S_d$ can be A, which shows the assertion.

(b) *If A is closed, B is compact, and $A \cap B = \emptyset$, prove that there is a $d > 0$ such that $|y - x| \geq d$ for all $y \in A$ and $x \in B$.*

For each $x \in B$, choose d_x to be as in part (a). Then $\mathcal{O} = \{B_x = \{y \in \mathbb{R}^n : |y - x| < d_x/2\} : x \in B\}$ is an open cover of B. Let $\{B_{x_1}, ..., B_{x_k}\}$ be a finite subcover, and let $d = \min(d_{x_1}, ..., d_{x_k})/2$. Then, by the triangle inequality, we know that d satisfies the assertion.

(c) *Give a counterexample in $\mathbb{R}^2$ if A and B are required both to be closed with neither compact.*

A counterexample: A is the x-axis and B is the graph of the exponential function.

1-22. *If U is open and $C \subset U$ is compact, show that there is a compact set D such that C is contained in the interior of D and $D \subset U$.*

Let d be as in Problem 1-21 (b) applied with $A = \mathbb{R}^n - U$ and $B = C$. Let $D = \{y \in \mathbb{R}^n : \exists x \in C \ni |y - x| \leq d/2\}$. It is straightforward to verify that D is bounded and closed; so D is compact. Finally, $C \subset D \subset U$ is also true.

FUNCTIONS AND CONTINUITY

1-23. *Prove that* $f : A \to \mathbb{R}^m$ *and* $a \in A$*, show that* $\lim_{x\to a} f(x) = b$ *if and only if* $\lim_{x\to a} f^i(a) = b^i$ *for each* $i = 1, ..., m$.

Suppose that $\lim_{x\to a} f^i(x) = b^i$ for each i. Let $\epsilon > 0$. Choose for each i, a positive δ_i such that for every $x \in A - \{a\}$ with $|x - a| < \delta_i$, one has $|f^i(x) - b^i| < \epsilon/\sqrt{n}$. Let $\delta = \min(\delta_1, ..., \delta_n) > 0$. Then, if $x \in A - \{a\}$ satisfies $|x - a| < \delta$, then $|f(x) - b| < \sqrt{\sum_{i=1}^n \epsilon^2/n} = \epsilon$. So, $\lim_{x\to a} f(x) = b$.

Conversely, suppose that $\lim_{x\to a} f(x) = b$, $\epsilon > 0$, and δ is chosen as in the definition of $\lim_{x\to a} f(x) = b$. Then, for each i, if x is in $A - \{a\}$ and satisfies $|x - a| < \delta$, then $|f^i(x) - b^i| \le |f(x) - b| < \epsilon$. So $\lim_{x\to a} f^i(x) = b^i$.

1-24. *Prove that* $f : A \to \mathbb{R}^m$ *is continuous at* a *if and only if each* f^i *is.*

This is an immediate consequence of Problem 1-23 and the definition of continuity.

1-25. *Prove that a linear transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$ *is continuous.*

By Problem 1-10, there is an $M > 0$ such that $|T(x)| < M|x|$ for all x. Let $a \in \mathbb{R}^n$ and $\epsilon > 0$. Let $\delta = \epsilon/M$. If x satisfies $|x - a| < \delta$, then $|T(x) - T(a)| = |T(x - a)| \le M|x - a| < M\delta = \epsilon$. So T is continuous at a.

1-26. *Let* $A = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } 0 < y < x^2\}$.

(a) *Show that every straight line through* $(0, 0)$ *contains an interval around* $(0, 0)$ *which is in* $\mathbb{R}^2 - A$.

Let the line be $y = ax$. If $a \leq 0$, then the whole line is disjoint from A. On the other hand, if $a > 0$, then the line intersects the graph of $y = x^2$ at (a, a^2) and $(0, 0)$ and nowhere else. Let $f(x) = ax - x^2$. Then f is continuous and $f(a/2) = a/2 > 0$. Since the only roots of f are at 0 and a, it follows by the intermediate value theorem that $f(x) > 0$ for all x with $0 < x < a$. In particular, the line $y = ax$ cannot intersect A anywhere to the left of $x = a$.

(b) *Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x) = 0$ if $x \notin A$ and $f(x) = 1$ if $x \in A$. For $h \in \mathbb{R}^2$ define $g_h : \mathbb{R} \to \mathbb{R}$ by $g_h(t) = f(th)$. Show that each g_h is continuous at 0, but f is not continuous at $(0, 0)$.*

For each h, g_h is identically zero in a neighborhood of zero by part (a). So, every g_h is clearly continuous at 0. On the other hand, f cannot be continuous at $(0, 0)$ because every open rectangle containing $(0, 0)$ contains points of A and for all those points x, one has $|f(x) - f((0, 0))| = 1$.

1-27. *Prove that $B = \{x \in \mathbb{R}^n : |x - a| < r\}$ is open by considering the function $f : \mathbb{R}^n \to \mathbb{R}$ with $f(x) = |x - a|$.*

The function f is continuous. In fact, let $b \in \mathbb{R}^n$ and $\epsilon > 0$. Let $\delta = \epsilon$. If $0 < |x - b| < \delta$, then by Problem 1-4, one has: $|f(x) - f(b)| = ||x - a| - |b - a|| \leq |x - b| < \delta = \epsilon$. This proves that f is continuous.

Since $f^{-1}((-1, r)) = B$, it follows that B is open by Theorem 1-8.

1-28. *If $A \subset \mathbb{R}^n$ is not closed, show that there is a continuous function $f : A \to \mathbb{R}$ which is unbounded.*

As suggested, choose x to be a boundary point of A which is not in A, and let $f(y) = 1/|y - x|$. Clearly, this is unbounded. To show it is continuous at b, let $\epsilon > 0$ and choose $\delta = \min(|b - x|/2, \epsilon|b - x|^2/2)$. Then for any y with

$0 < |y-b| < \delta$, one has $|y-x| \geq |b-x|/2$. So,

$$\begin{aligned} |f(y)-f(b)| &= |\frac{1}{|y-x|} - \frac{1}{|b-x|}| \\ &= \frac{||b-x|-|y-x||}{|y-x||b-x|} \\ &\leq \frac{|b-y|}{|y-x||b-x|} \\ &< \frac{2\delta}{|b-x|^2} \leq \epsilon \end{aligned}$$

where we have used Problem 1-4 in the simplification. This shows that f is continuous at b.

1-29. *If A is compact, prove that every continuous function $f : A \to \mathbb{R}$ takes on a maximum and a minimum value.*

By Theorem 1-9, $f(A)$ is compact, and hence is closed and bounded. Let m (resp. M) be the greatest lower bound (respectively least upper bound) of $f(A)$. Then m and M are boundary points of $f(A)$, and hence are in $f(A)$ since it is closed. Clearly these are the minimum and maximum values of f, and they are taken on since they are in $f(A)$.

1-30. *Let $f : [a,b] \to \mathbb{R}$ be an increasing function. If $x_1, x_2, ..., x_n \in [a,b]$ are distinct, show that $\sum_{i=1}^{n} o(f, x_i) < f(b) - f(a)$.*

One has $0 \leq M(x_i, f, \delta) - m(x_i, f, \delta) \leq f(\min(x_i + \delta, b)) - f(\max(x_i + \delta, a))$. The function on the right is an increasing function of δ; in particular, $o(f, x_i)$ is bounded above by the quantity on the right for any $\delta > 0$. Now assume that the x_i have been re-ordered so that they are in increasing order; let $\delta < \min_{i=1,\ldots,n-1}(|x_{i+1} - x_i|/2)$. Now add up all the inequalities with this value of δ; it is an upper bound for the sum of the $o(f, x_i)$ and the right hand side “telescopes" and is bounded above by the difference of the two end terms which in turn is bounded above by $f(b) - f(a)$.

II

Integration

BASIC DEFINITIONS

2-1. *Prove that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, then it is continuous at a.*

If f is differentiable at a, then $\lim_{h \to 0} |f(a+h) - f(a) - Df(a)(h)| = 0$. So, we need only show that $\lim_{h \to 0} |Df(a)(h)| = 0$, but this follows immediately from Problem 1-10.

2-2. *A function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be independent of the second variable if for each $x \in \mathbb{R}$ we have $f(x, y_1) = f(x, y_2)$ for all $y_1, y_2 \in \mathbb{R}$. Show that f is independent of the second variable if and only if there is a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x)$. What is $f'(a, b)$ in terms of g'?*

The first assertion is trivial: If f is independent of the second variable, you can let g be defined by $g(x) = f(x, 0)$. Conversely, if $f(x, y) = g(x)$, then $f(x, y_1) = g(x) = f(x, y_2)$.

If f is independent of the second variable, then $f'(a, b)(c, d) = g'(a)(c)$ because:

$$\lim_{h \to 0} \frac{|f(a + h^1, b + h^2) - f(a, b) - g'(a)(c)|}{|h|}$$
$$= \lim_{h \to 0} \frac{|g(a + h^1) - g(a) - g'(a)(c)|}{|h^1|} \lim_{h \to 0} \frac{|h^1|}{|h|} = 0$$

Note: Actually, $f'(a, b)$ is the Jacobian, i.e. a 1 x 2 matrix. So, it would be more proper to say that $f'(a, b) = (g'(a), 0)$, but I will often confound Df with f', even though one is a linear transformation and the other is a matrix.

2-3. *Define when a function $f : \mathbb{R}^n \to \mathbb{R}$ is independent of the first variable and find $f'(a,b)$ for such f. Which functions are independent of the first variable and also of the second variable?*

The function f is independent of the first variable if and only if $f(x_1, y) = f(x_2, y)$ for all $x_1, x_2, y \in \mathbb{R}$. Just as before, this is equivalent to their being a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(y)$ for all $x, y \in \mathbb{R}$. An argument similar to that of the previous problem shows that $f'(a,b) = g'(b)$.

2-4. *Let g be a continuous real-valued function on the unit circle $\{x \in \mathbb{R}^2 : |x| = 1\}$ such that $f(0,1) = g(1,0) = 0$ and $g(-x) = -g(x)$. define $f : \mathbb{R}^2 \to \mathbb{R}$ by*

$$f(x) = \begin{cases} |x|.g(\frac{x}{|x|}) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

(a) If $x \in \mathbb{R}^2$ and $h : \mathbb{R} \to \mathbb{R}$ is defined by $h(t) = f(tx)$, show that h is differentiable.

One has $h(t) = t|x|g(x/|x|)$ when $x \neq 0$ and $h(t) = 0$ otherwise. In both cases, h is linear and hence differentiable.

(b) Show that f is not differentiable at (0, 0) unless $g = 0$.

Suppose f is differentiable at $(0,0)$ with, say, $f'(0,0) = (a,b)$. Then one must have: $\lim_{h\to 0} \frac{|f(h,0)-ah|}{|h|} = 0$. But $f(h,0) = |h|g(1,0) = 0$ and so $a = 0$. Similarly, one gets $b = 0$. More generally, using the definition of derivative, we get for fixed θ: $\lim_{(h\cos(\theta),h\sin(\theta))\to(0,0)} \frac{|f(h\cos(\theta),h\sin(\theta))|}{|h|} = 0$. But $f(h\cos(\theta), h\sin(\theta)) = |h|g(\cos(\theta), \sin(\theta))$, and so we see that this just says that $g(\cos(\theta), \sin(\theta)) = 0$ for all θ. Thus g is identically zero.

2-5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{x|y|}{\sqrt{x^2+y^2}} & (x,y) \neq 0, \\ 0 & (x,y) = 0. \end{cases}$$

Show that f is a function of the kind considered in Problem 2-4, so that f is not differentiable at $(0,0)$.

Define g by $g(\cos(\theta), \sin(\theta)) = \cos(\theta)|\sin(\theta)|$ for all θ. Then it is trivial to show that g satisfies all the properties of Problem 2-4 and that the function f obtained from this g is as in the statement of this problem.

2-6. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \sqrt{|xy|}$. Show that f is not differentiable at 0.*

Just as in the proof of Problem 2-4, one can show that, if f were differentiable at 0, then $Df(0,0)$ would be the zero map. On the other hand, by approaching zero along the 45 degree line in the first quadrant, one would then have: $\lim_{h\to 0+} \frac{\sqrt{h^2}}{|h|} = 0$ in spite of the fact that the limit is clearly 1.

2-7. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that $|f(x)| \leq |x|^2$. Show that f is differentiable at 0.*

In fact, $Df(0,...,0) = 0$ by the squeeze principle using $0 \leq \frac{|f(h)|}{|h|} \leq |h|$.

2-8. *Let $f : \mathbb{R} \to \mathbb{R}^2$. Prove that f is differentiable at $a \in \mathbb{R}$ if and only if f^1 and f^2 are, and in this case*

$$f'(a) = \begin{pmatrix} (f^1)'(a) \\ (f^2)'(a)) \end{pmatrix}.$$

Suppose that $Df(a) = \begin{pmatrix} c^1 \\ c^2 \end{pmatrix}$. Then one has the inequality: $0 \leq \frac{|f^i(a+h) - f^i(a) - c^i h|}{|h|} \leq$

$\frac{|f(a+h)-f(a)-Df(a)(h)|}{|h|}$. So, by the squeeze principle, f^i must be differentiable at a with $(f^i)'(a) = c^i$.

On the other hand, if the f^i are differentiable at a, then use the inequality derived from Problem 1-1:

$$0 \le \frac{|f(a+h) - f(a) - ((f^1)'(a)h^1, (f^2)'(a)h^2)|}{|h|} \le \sum_{i=1}^{2} \frac{|f^i(a+h) - f^i(a) - (f^i)'(a)h^i|}{|h|}$$

and the squeeze principle to conclude that f is differentiable at a with the desired derivative.

2-9. *Two functions $f, g : \mathbb{R} \to \mathbb{R}$ are equal up to n^{th} order at a if*

$$\lim_{h \to 0} \frac{f(a+h) - g(a+h)}{h^n} = 0$$

(a) Show that f is differentiable at a if and only if there is a function g of the form $g(x) = a_0 + a_1(x-a)$ such that f and g are equal up to first order at a.

If f is differentiable at a, then the function $g(x) = f(a) + f'(a)(x-a)$ works by the definition of derivative.

The converse is not true. Indeed, you can change the value of f at a without changing whether or not f and g are equal up to first order. But clearly changing the value of f at a changes whether or not f is differentiable at a.

To make the converse true, add the assumption that f be continuous at a: If there is a g of the specified form with f and g equal up to first order, then

$\lim_{h\to 0}\frac{f(a+h)-a_0-a_1(h)}{h} = 0$. Multiplying this by h, we see that $\lim_{h\to 0} f(a+h) - a_0 = 0$. Since f is continuous, this means that $a_0 = f(a)$. But then the condition is equivalent to the assertion that f is differentiable at a with $f'(a) = a_1$.

(b) If $f'(a), ..., f^{(n)}(a)$ exist, show that f and the function g defined by

$$g(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

are equal up to n^{th} order at a.

Apply L'Hôpital's Rule n - 1 times to the limit

$$\lim_{h\to 0} \frac{f(a+h) - \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!}(h)^i}{h^n}$$

to see that the value of the limit is $\lim_{h\to 0} \frac{f^{(n-1)}(a+h)-f^{(n-1)}(a)}{n!h} = \frac{f^{(n)}(a)}{n!}$. On the other hand, one has:

$$\lim_{h\to 0} \frac{\frac{f^{(n)}(a)}{n!}h^n}{h^n} = \frac{f^{(n)}(a)}{n!}$$

Subtracting these two results gives shows that f and g are equal up to n^{th} order at a.

BASIC THEOREMS

2-11. *Use the theorems of this section to find f' for the following:*

(a) $f(x, y, z) = x^y$

We have $f(x, y, z) = e^{y\ln(x)} = e^{\pi^2 \cdot \ln(\pi^1)}$ and so by the chain rule, one has: $Df(a, b, c) = e^{b\ln(a)} \cdot (\ln(a) \cdot \pi^2 + (b/a)\pi^1)$, i.e. $Df(a, b, c)(h, j, k) = a^b(\ln(a)j + (b/a)h)$.

(b) $f(x, y, z) = (x^y, z)$

Using Theorem 2-3 (3) and part (a), one has: $Df(a, b, c) = (e^{b\ln(a)} \cdot (\ln(a) \cdot \pi^2 + (b/a)\pi^1), \pi^3)$.

(c) $f(x, y) = \sin(x\sin(y))$.

One has $f(x, y) = \sin(\pi^1 \cdot \sin(\pi^2))$, and so by the chain rule: $Df(a, b) = \cos(a\sin(b))(\sin(b)\pi^1 + a\cos(b)\pi^2)$

(d) $f(x, y, z) = \sin(x\sin(y\sin(z))$

If $g(x, y) = \sin(x\sin(y))$ is the function of part (c), then

$$f(x, y, z) = \sin(\pi^1 g(\pi^2, \pi^3)).$$

Using the chain rule, we get:

$$\begin{aligned} Df(a, b, c) &= \cos(ag(b, c))D(\pi^1 g(\pi^2, \pi^3))(a, b, c) \\ &= \cos(a\sin(b\sin(c)))(aDg(\pi^2, \pi^3)(a, b, c) + g(b, c)D(\pi^1)) \\ &= \cos(a\sin(b\sin(c)))(a\cos(b\sin(c))(b\cos(c)\pi^3 + \sin(c)\pi^2) \\ &+ \sin(b\sin(c))\pi^1) \end{aligned}$$

(e) $f(x,y,z) = x^{y^z}$

If $g(x,y) = x^y$, then $f(x,y,z) = g(\pi^1, g(\pi^2,\pi^3))$ and we know the derivative of g from part (a). The chain rule gives: $Df(a,b,c) = a^{b^c}(\ln(a)(b^c(\ln(b)\pi^3 + (c/b)\pi^2)) + (b^c/a)\pi^1)$

(f) $f(x,y,z) = x^{y+z}$

If $g(x,y) = x^y$, then $f(x,y,z) = g(\pi^1, \pi^2+\pi^3)$. So one gets: $Df(a,b,c) = a^{b+c}(\ln(a)(\pi^2+\pi^3) + ((b+c)/a)\pi^1)$.

(g) $f(x,y,z) = (x+y)^z$

If $g(x,y) = x^y$, then $f(x,y,z) = g(\pi^1+\pi^2, \pi^3)$. So one gets: $Df(a,b,c) = (a+b)^c(\ln(a+b)\pi^3 + (c/(a+b))(\pi^1+\pi^2))$.

(h) $f(x,y) = \sin(xy)$

The chain rule gives: $Df(a,b) = \cos(ab)(b\pi^1 + a\pi^2)$.

(i) $f(x,y) = \sin(xy)^{\cos(3)}$

Using the last part: $Df(a,b) = \cos(3)\sin(ab)^{\cos(3)-1}\cos(ab)(b\pi^1 + a\pi^2)$.

(j) $f(x,y) = (sin(xy), sin(x\sin(y)), x^y)$

Using parts (h), (c), and (a), one gets

$$\begin{aligned} Df(a,b) = & \qquad (\cos(ab)(b\pi^1 + a\pi^2), \\ & = \cos(a\sin(b))(\sin(b)\pi^1 + a\cos(b)\pi^2), \\ & = a^b(\ln(a)\pi^2 + (b/a)\pi^1)) \end{aligned}$$

2-12. *Find f' for the following (where $g : \mathbb{R} \to \mathbb{R}$ is continuous):*

(a) $f(x,y) = \int_a^{x+y} g$.

If $h(x) = \int_a^x g$, then $f = h(\pi^1+\pi^2)$, and so: $Df(a,b) = g(a+b)(\pi^1+\pi^2)$.

(b) $f(x,y) = \int_a^{xy} g$.

If h is as in part (a), then $f = h(\pi^1\cdot\pi^2)$, and so: $Df(a,b) = g(ab)(b\pi^1 + a\pi^2)$.

(c) $f(x,y,z) = \int_{x^y}^{\sin(x\sin(y\sin(z)))} g$.

One has $Df(a,b,c) = g(\sin(a\sin(b\sin(c))))Dh(a,b,c) - g(a^b)Dk(a,b,c)$ where $h(a,b,c) = \sin(x\sin(y\sin(z)))$ and $k(x,y,z) = x^y$ have derivatives as given in parts (d) and (a) above.

2-13. *A function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is bilinear if for $x, x_1, x_2 \in \mathbb{R}^n$, $y, y_1, y_2 \in \mathbb{R}^m$, and $a \in \mathbb{R}$, we have*

$$f(ax,y) = af(x,y) = f(x,ay)$$
$$f(x_1+x_2,y) = f(x_1,y) + f(x_2,y)$$
$$f(x,y_1+y_2) = f(x,y_1) + f(x,y_2)$$

(a) *Prove that if f is bilinear, then*

$$\lim_{(h,k)\to 0} \frac{|f(h,k)|}{|(h,k)|} = 0.$$

Let $e_i = (0,...,0,1,0,...,0)$ have a 1 in the i^{th} place only. Then we have $f(h,k) = \sum_{i=1}^n \sum_{j=1}^m h^i k^j f(e_i,e_j)$ by an obvious induction using bilinearity. It follows that there is an $M > 0$ depending only on f such that: $|f(h,k)| \le M\max(|h^i|)\max(|k^j|) \le M|h||k|$. Since $|(h,k)| = \sqrt{|h|^2+|k|^2}$, we see that it suffices to show the result in the case where $m = n = 1$ and the bilinear function is the product function. But, in this case, it was verified in the proof of Theorem 2-3 (5).

(b) *Prove that $Df(a,b)(x,y) = f(a,y) + f(x,b)$.*

One has $\lim_{(h,k)\to 0} \frac{|f(a+h,b+k)-f(a,b)-(f(a,k)+f(h,b))|}{|(h,k)|} = \lim_{(h,k)\to 0} \frac{|f(h,k)|}{|(h,k)|} = 0$ by bilinearity and part (a).

(c) *Show that the formula for $Dp(a,b)$ in Theorem 2-3 is a special case of (b).*

This follows by applying (b) to the bilinear function p.

2-14. *Define $IP : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $IP(x,y) = \langle x,y\rangle$.*

(a) *Find $D(IP)(a,b)$ and $(IP)'(a,b)$.*

By Problem 2-12 and the fact that IP is bilinear, one has $D(IP)(a,b)(h,k) = \langle b,h\rangle + \langle a,k\rangle$. So $(IP)'(a,b) = (b,a)$.

(b) *If $f,g : \mathbb{R} \to \mathbb{R}^n$ are differentiable, and $h : \mathbb{R} \to \mathbb{R}$ is defined by $h(t) = \langle f(t), g(t)\rangle$, show that*

$$h'(a) = \left\langle f'(a)^T, g(a)\right\rangle + \left\langle f(a), g'(a)^T\right\rangle.$$

(Note that $f'(a)$ is an $n \times 1$ matrix; its transpose $f'(a)^T$ is an $1 \times n$ matrix, which we consider as a member of $\mathbb{R}^n$.)

Since $h = (IP) \cdot (f,g)$, one can apply the chain rule to get the assertion.

(c) *If $f : \mathbb{R} \to \mathbb{R}^n$ is differentiable and $|f(t)| = 1$ for all t, show that $\langle f'(t)^T, f(t)\rangle = 0$.*

Use part (b) applied to $\langle f(t), f(t)\rangle = 1$ to get $\langle f'(t)^T, f(t)\rangle + \langle f(t)^T, f'(t)^T\rangle = 0 = 0$. This shows the result.

(d) Exhibit a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that the function |f| defined by |f|(t) = |f(t)| is not differentiable.

Trivially, one could let $f(t) = t$. Then $|f|$ is not differentiable at 0.

2-15. *Let $E_i, i = 1, ..., k$ be Euclidean spaces of various dimensions. A function $f : E_1 \times ... \times E_k \to \mathbb{R}^p$ is called multilinear if for each choice of $x_j \in E_j, j \neq i$ the function $f : E_i \to \mathbb{R}^p$ defined by $g(x) = f(x_1, ..., x_{i-1}, x, x_{i+1}, ..., x_k)$ is a linear transformation.*

(a) *If f is multilinear and $i \neq j$, show that for $h = (h_1, ..., h_k)$, with $h_i \in E_i$, we have*

$$\lim_{h\to 0} \frac{|f(a_1, ..., h_i, ..., h_j, ..., a_k)|}{|h|} = 0.$$

This is an immediate consequence of Problem 2-12 (b).

(b) *Prove that* $Df(a_1,...,a_k)(x_1,...,x_k) = \sum_{i=1}^{k} f(a_1,...,a_{i-1},x_i,a_{i+1},...,a_k)$.

This can be argued similarly to Problem 2-12. Just apply the definition expanding the numerator out using multilinearity; the remainder looks like a sum of terms as in part (a) except that there may be more than two h_i type arguments. These can be expanded out as in the proof of the bilinear case to get a sum of terms that look like constant multiples of

$$\lim_{h\to 0} \frac{|h_{i_1}|...|h_{i_k}|}{|h|}$$

where m is at least two and the $i_1,...,i_m$ are distinct. Just as in the bilinear case, this limit is zero. This shows the result.

2-16. *Regard an* $n\times n$ *matrix as a point in the* n*-fold product* $\mathbb{R}^n\times...\times\mathbb{R}^n$ *by considering each row as a member of* $\mathbb{R}^n$.

(a) *Prove that* $\det : \mathbb{R}^n \times ... \times \mathbb{R}^n \to \mathbb{R}$ *is differentiable and*

$$D(det)(a_1,...,a_n)(x_1,...,x_n) = \sum_{i=1}^{n} \det \begin{pmatrix} a_1 \\ \vdots \\ x_i \\ \vdots \\ a_n \end{pmatrix}$$

This is an immediate consequence of Problem 2-14 (b) and the multilinearity of the determinant function.

(b) *If* $a_{ij} : \mathbb{R} \to \mathbb{R}$ *are differentiable and* $f(t) = det(a_{ij}(t))$, *show that*

$$f'(t) = \sum_{i=1}^{n} \det \begin{pmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \dots & \dots & \dots \\ a'_{j1}(t) & \dots & a'_{jn}(t) \\ a_{n1}(t), & \dots & a_{nn}(t) \end{pmatrix}$$

This follows by the chain rule and part (a).

(c) *If* $\det(a_{ij}(t)) \neq 0$ *for all* t *and* $b_1, ..., b_n : \mathbb{R} \to \mathbb{R}$ *are differentiable, let* $s_1, ..., s_n : \mathbb{R} \to \mathbb{R}$ *be the functions such that* $s_1(t), ..., s_n(t)$ *are the solutions of the equations:* $\sum_{j=1}^{n} a_{ji}(t)s_j(t) = b_i(t)$ *for* $i = 1, ..., n$*. Show that* s_i *is differentiable and find* $s_i'(t)$*.*

Without writing all the details, recall that Cramer's Rule allows you to write down explicit formulas for the $s_i = \det(B_i)/\det(A)$ where A is the matrix of the coefficients $a_ij(t)$ and the B_i are obtained from A by replacing the i^{th} column with the column of the $b_i(t)$. We can take transposes since the determinant of the transpose is the same as the determinant of the original matrix; this makes the formulas simpler because the formula for derivative of a determinant involved rows and so now you are replacing the i^{th} row rather than the i^{th} column. Anyway, one can use the quotient formula and part (b) to give a formula for the derivative of the $s_i(t)$.

2-17. *Suppose* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is differentiable and has a differentiable inverse* $f^{-1} : \mathbb{R}^n \to \mathbb{R}^n$*. Show that* $(f^{-1})'(a) = (f'(f^{-1}(a)))^{-1}$*.*

This follows immediately by the chain rule applied to $f \cdot f^{-1}(x) = x$.

PARTIAL DERIVATIVES

2-18. *Find the partial derivatives of the following functions:*

(a) $f(x, y, z) = x^y$

$D_1 f(x, y, z) = yx^{y-1}, D_2 f(x, y, z) = \ln(x)x^y, D_3 f(x, y, z) = 0.$

(b) $f(x, y, z) = z$

$D_1 f = D_2 f = 0, D_3 f = 1.$

(c) $f(x, y) = \sin(x \sin(y))$

$D_1 f(x, y) = \sin(y) \cos(x \sin(y)), D_2 f(x, y) = \cos(x \sin(y))(x \cos(y)).$

(d) $f(x, y, z) = \sin(x \sin(y \sin(z)))$

$$D_1 f(x, y, z) = \cos(x \sin(y \sin(z))) \sin(y(\sin(z)),$$
$$D_2 f(x, y, z) = \cos(x \sin(y \sin(z)))(x \cos(y \sin(z))(\sin(z))),$$
$$D_3 f(x, y, z) = \cos(x \sin(y \sin(z)))(x \cos(y \sin(z))(y \cos(z))).$$

(e) $f(x, y, z) = x^{y^z}$

$$D_1 f(x, y, z) = y^z x^{y^z - 1},$$
$$D_2 f(x, y, z) = \ln(x) x^{y^z} (z y^{z-1}),$$
$$d_3 f(x, y, z) = \ln(x) x^{y^z} \ln(y) y^z.$$

(f) $f(x, y, z) = x^{y+z}$

$D_1 f(x, y, z) = (y + z)x^{y+z-1}, D_2 f(x, y, z) = D_3 f(x, y, z) = \ln(x)x^{y+z}.$

(g) $f(x, y, z) = (x + y)^z$

$D_1f(x,y,z) = D_2f(x,y,z) = z(x+y)^{z-1}, D_3f(x,y,z) = \ln(x+y)(x+y)^z.$

(h) $f(x,y) = \sin(xy)$

$D_1f(x,y) = y\cos(xy), D_2f(x,y) = x\cos(xy).$

(i) $f(x,y) = (\sin(xy))^{\cos(3)}$

$$D_1f(x,y) = \cos(3)(\sin(xy))^{\cos(3)-1}(y\cos(xy)),$$
$$D_2f(x,y) = \cos(3)(\sin(xy))^{\cos(3)-1}(x\cos(xy))$$

2-19. *Find the partial derivatives of the following functions (where $g : \mathbb{R} \to \mathbb{R}$ is continuous):*

(a) $f(x,y) = \int_a^{x+y} g$

$D_1f(x,y) = d_2f(x,y) = g(x+y).$

(b) $f(x,y) = \int_y^x g$

$D_ff(x,y) = g(x), D_2f(x,y) = -g(y).$

(c) $f(x,y) = \int_a^{xy} g$

$D_1f(x,y) = yg(xy), D_2f(x,y) = xg(xy).$

(d) $f(x,y) = \int_a^{(\int_b^y g)} g$

$D_1f(x,y) = 0, D_2f(x,y) = g(\int_b^y g)g(y).$

2-20. *If* $f(x,y) = x^{x^{x^{x^y}}} + (\log(x))(\arctan(\arctan(\sin(\cos(xy)) - \log(x+y)))))$ *find* $D_2(f(1,y)$.

Since $f(1,y) = 1$, one has $D_2f(1,y) = D1 = 0.$

2-21. *Find the partial derivatives of f in terms of the derivatives of g and h if*

(a) $f(x,y) = g(x)h(y)$

$D_1f(x,y) = h(y)Dg(x)(1), D_2f(x,y) = g(x)Dh(y)(1).$

(b) $f(x,y) = g(x)^{h(y)}$

$$D_1 f(x,y) = h(y)g(x)^{h(y)-1}Dg(x)(1),$$
$$D_2 f(x,y) = \ln(g(x))g(x)^{h(y)}Dh(y)(1).$$

(c) $f(x,y) = g(x)$

$D_1 f(x,y) = Dg(x)(1), D_2 f(x,y) = 0.$

(d) $f(x,y) = g(y)$

$D_1 f(x,y) = 0, D_2 f(x,y) = Dg(y)(1).$

(e) $f(x,y) = g(x+y)$

$D_1 f(x,y) = D_2 f(x,y) = Dg(x+y)(1).$

2-22. **Let $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by**

$$f(x,y) = \int_0^x g_1(t,0)dt + \int_0^y g_2(x,t)dt.$$

(a) Show that $D_2 f(x,y) = g_2(x,y)$

True since the first term depends only on x.

(b) How should f be defined so that $D_1 f(x,y) = g_1(x,y)$?

One could let $f(x,y) = \int_0^x g_1(t,y)dt + \int_0^y g_2(0,t)dt$.

(c) Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $D_1 f(x,y) = x$ and $D_2 f(x,y) = y$.

One could let $f(x,y) = (x^2+y^2)/2$.

(d) Find one such that $D_1 f(x,y) = y$ and $D_2 f(x,y) = x$.

One could let $f(x,y) = xy$.

2-23. *If $f : \mathbb{R}^2 \to \mathbb{R}$ and $D_2 f = 0$, show that f is independent of the second variable. If $D_1 f = D_2 f = 0$, show that $f is constant$.*

By the mean value theorem, one knows that if a function of one variable has zero derivative on a closed interval $[a,b]$, then it is constant on that interval. Both assertions are immediate consequences of this result.

2-24. *Let* $A = \{(x, y) : x < 0, \ \ or \ \ x \geq 0 \ \ and \ \ y \neq 0\}$.

(a) If $f : A \to \mathbb{R}$ and $D_1 f = D_2 f = 0$, show that f is a constant.

Suppose (a, b) and (c, d) are arbitrary points of A. Then the line segment from $(-1, b)$ to (a, b), from $(-1, b)$ to $(-1, d)$, and from $(-1, d)$ to (c, d) are all contained in A. By the proof of Problem 2-22, it follows that $f(-1, b) = f(a, b)$, $f(-1, b) = f(-1, d)$, and $f(-1, d) = f(c, d)$. So $f(a, b) = f(c, d)$.

(b) Find a function $f : A \to \mathbb{R}$ such that $D_2 f = 0$ but f is not independent of the second variable.

One could let $f(x, y) = \begin{cases} 0, & x < 0 \text{ or } y < 0 \\ x, & \text{otherwise}. \end{cases}$.

2-25. *Define* $f : \mathbb{R}^2 \to \mathbb{R}$ *by*

$$f(x, y) = \begin{cases} xy \frac{x^2 - y^2}{x^2 + y^2} & (x, y) \neq 0, \\ 0 & (x, y) = 0. \end{cases}$$

(a) Show that $D_2 f(x, 0) = x$ for all x and $D_1 f(0, y) = -y$ for all y.

One has $D_2 f(x, y) = x(x^4 - 4x^2y^2 - y^4)/(x^2 + y^2)^2$ when $(x, y) \neq (0, 0)$, and $D_2 f(0, 0) = 0$ since $f(0, y) = 0$. Further, one has $f(x, y) = -f(y, x)$ and so $D_1 f(x, y) = -D_2 f(y, x) = -y(y^4 - 4y^2x^2 - y^4)/(x^2 + y^2)^2$ for all $(x, y) \neq 0$ and $D_1 f(0, 0) = 0$ (because $f(x, 0) = 0$). The assertions follow immediately by substituting $y = 0$ into one formula and $x = 0$ in the other.

(b) Show that $D_{1,2} f(0, 0) \neq D_{2,1} f(0, 0)$.

Using part (a), one has $D_{2,1} f(0, 0) = D(D_2 f(x, 0))(0)(1) = Dx(0)(1) = 1$ and $D_{1,2} f(0, 0) = D(D_1 f(0, y))(0)(1) = D(-y)(0)(1) = -1$.

2-26. *Define* $f : \mathbb{R} \to \mathbb{R}$ *by*

$$f(x) = \begin{cases} e^{-x^{-2}} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

(a) Show that f is a C^∞ function, and $f^{(i)}(0) = 0$ for all i.

Consider the function g defined by

$$g(x) = \begin{cases} e^{-x^{-2}} p(x^{-1}) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

where p is a polynomial. Then, one has $g'(x) = e^{-x^{-2}}(2x^{-3}p(x^{-1}) - x^{-2}p_n'(x^{-1}))$ for $x \neq 0$. By induction on n, define $p_0(x) = 1$ and $p_{n+1}(x) = 2x^3 p_n(x) - x^2 p_n(x))$. Then it follows that $f^{(k)}(x) = e^{-x^{-1}} p_k(x^{-1})$ for all $k \geq 0$. In particular, function f is C^∞ for all $x \neq 0$.

Now, suppose by induction on k, that $f^{(k)}(0) = 0$. We have $f^{(k+1)}(0) = \lim_{h\to 0} \frac{e^{-h^{-2}} p_k(h^{-1})}{h} = \lim_{h\to 0} \frac{h^{-1} p_k(h^{-1})}{e^{h^{-2}}}$. To show that this limit is zero, it suffices to show that $\lim_{h\to 0} \frac{h^{-t}}{e^{h^{-2}}} = 0$ for each integer $t \geq 1$. But this is an easy induction using L'Hôpital's rule: $\lim_{h\to 0} \frac{h^{-t}}{e^{h^{-2}}} = \lim_{h\to 0} \frac{-th^{-t-1}}{e^{h^{-2}}(-2h^{-3})} = \lim_{h\to 0} \frac{(t/2)h^{-t+2}}{e^{h^{-2}}}$.

(b) Let $f(x) = \begin{cases} e^{-(x-1)^{-2}} \cdot e^{-(x+1)^{-2}} & x \in (-1, 1), \\ 0 & x \notin (-1, 1). \end{cases}$

(c) Show that $f : \mathbb{R} \to \mathbb{R}$ is a C^∞ function which is positive on (-1, 1) and 0 elsewhere.

For points other than 1 and -1, the result is obvious. At each of the exceptional points, consider the derivative from the left and from the right, using Problem 2-25 on the side closest to the origin.

(d) Show that there is a C^∞ function $g : \mathbb{R} \to [0, 1]$ such that $f(x) = 0$ for $x \leq 0$ and $g(x) = 1$ for $x \geq \epsilon$.

Following the hint, let f be as in Problem 2-25 and

$$g(x) = \begin{cases} 0 & x < 0, \\ \frac{\int_0^x f(2t/\epsilon - 1)dt}{\int_0^\epsilon f(2t/\epsilon - 1)dt} & 0 \le x \le \epsilon, \\ 1 & x > \epsilon. \end{cases}$$

Now use Problem 2-25 to prove that g works.

(e) If $a \in \mathbb{R}^n$, define $g : \mathbb{R}^n \to \mathbb{R}$ by

$$g(x) = f((x^1 - a^1)/\epsilon) \cdot \ldots \cdot f((x^n - a^n)/\epsilon).$$

Show that g is a C^∞ function which is positive on

$$(a^1 - \epsilon, a^1 + \epsilon) \times \ldots \times (a^n - \epsilon, a^n + \epsilon)$$

and zero elsewhere.

This follows from part (a).

(f) If A is open and $C \subset A$ is compact, show that there is a non-negative C^∞ function $f : A \to \mathbb{R}$ such that $f(x) > 0$ for $x \in C$ and $f = 0$ outside of some closed set contained in A.

Let d be the distance between C and the complement of A, and choose $\epsilon = d/(2\sqrt{n})$. For each $x \in C$, let R_x be the open rectangle centered at x with sides of length 2ϵ. Let g_x be the function defined for this rectangle as in part (c). Since the set of these rectangles is an open cover of the compact set C, a finite number of them also cover C; say the rectangles corresponding to $x_1, x_2, \ldots, x_k$ form a subcover. Finally, let $g = \sum_{i=1}^k g_{x_i}$.

Since we have a subcover of C, we have g positive on C. The choice of ϵ guarantees that the union of the closures of the rectangles in the subcover is contained in A and g is clearly zero outside of this union. This proves the assertion.

(g) Show that we can choose such an f so that $f : A \to [0,1]$ and $f(x) = 1$ for $x \in C$.

Let f be as in part (d). We know that $f(x) > 0$ for all $x \in C$. Since C is compact, one knows that $f(C)$ is attains its minimum $\epsilon > 0$ (Problem 1-29). As suggested in the hint, replace f with $g(f)$ where g is the function of part (b). It is easy to verify that this new f satisfies the required conditions.

2-27. Define $g, h : \{x \in \mathbb{R}^2 : |x| \leq 1\} \to \mathbb{R}$ by

$$g(x,y) = (x, y, \sqrt{1 - x^2 - y^2}),$$

$$h(x,y) = (x, y, -\sqrt{1 - x^2 - y^2}).$$

Show that the maximum of f on $S = \{x \in \mathbb{R}^2 : |x| = 1\}$ is either the maximum of $f \cdot g$ or the maximum of $f \cdot h$ on $B = \{x \in \mathbb{R}^2 : |x| \leq 1\}$.

This is obvious because $S = g(B) \cup h(B)$.

DERIVATIVES

2-28. *Find expressions for the partial derivatives of the following functions:*

(a) $F(x,y) = f(g(x)k(y), g(x) + h(y)$

$$\begin{aligned} D_1F(x,y) &= (D_1f(g(x)k(y), g(x) + h(y)))(k(y)Dg(x)(1)) \\ &\quad + (D_2f(g(x)k(y), g(x) + h(y)))Dg(x)(1), \\ D_2F(x,y) &= (D_1f(g(x)k(y), g(x) + h(y)))(g(x)k'(y)) \\ &\quad + (D_2f(g(x)k(y), g(x) + h(y)))h'(y). \end{aligned}$$

(b) $F(x,y,z) = f(g(x+y), h(y+z))$

$D_1F(x,y,z) = (D_1f(g(x+y), h(y+z)))g'(x+y)$, $D_2F(x,y,z) = (D_1f(g(x+y), h(y+z)))g'(x+y) + (D_2f(g(x+y), h(y+z))))h'(y+z)$, $D_2F(x,y,z) = (D_2f(g(x+y), h(y+z)))h'(y+z)$.

(c) $F(x,y,z) = f(x^y, y^z, z^x)$

$$\begin{aligned} D_1F(x,y,z) &= (D_1f(x^y, y^z, z^x))(yx^{y-1}) + (D_3f(x^y, y^z, z^x))(\ln(z)z^x), \\ D_2F(x,y,z) &= (D_1f(x^y, y^z, z^x))(\ln(x)x^y) + (D_2f(x^y, y^z, z^x))(zy^{z-1}), \\ D_3F(x,y,z) &= (D_2f(x^y, y^z, z^x))(\ln(y)y^z) + (D_3f(x^y, y^z, z^x))(xz^{x-1}). \end{aligned}$$

(d) $F(x,y) = f(x, g(x), h(x,y))$

$$\begin{aligned} D_1F(x,y) &= D_1f(x, g(x), h(x,y)) + g'(x)D_2f(x, g(x), h(x,y)) \\ &\quad + D_1h(x,y)D_3f(x, g(x), h(x,y)) \end{aligned}$$

$$D_2F(x,y) = D_3f(x, g(x), h(x,y))D_2h(x,y).$$

2-29. *Let $f : \mathbf{R}^n \to \mathbf{R}$. For $x \in \mathbf{R}^n$, the limit*

$$\lim_{t\to 0} \frac{f(a+tx) - f(a)}{t},$$

if it exists, is denoted $D_xf(a)$ and called the **directional derivative** *of f at a, in the direction x.*

(a) *Show that $D_{e_i}f(a) = D_if(a)$.*

This is obvious from the definitions.

(b) *Show that $D_{tx}f(a) = tD_xf(a)$.*

$$D_{tx}f(a) = \lim_{s\to 0} \frac{f(a+stx) - f(a)}{s} = \lim_{st\to 0} t\frac{f(a+stx) - f(a)}{st} = tD_f(a)$$

(c) *If f is differentiable at a, show that $D_xf(a) = Df(a)(x)$ and therefore $D_{x+y}f(a) = D_xf(a) + D_yf(a)$.*

One has $0 = \lim_{tx\to 0} \frac{|f(a+tx)-f(a)-Df(a)(tx)|}{|tx|} = \lim_{t\to 0} |\frac{f(a+tx)-f(a)}{t} - Df(a)(x)|/|x|$ which shows the result whenever $x \neq 0$. The case when $x = 0$ is trivially true.

The last assertion follows from the additivity of the function $Df(a)$.

2-30. *Let f be defined as in Problem 2-4. Show that $D_xf(0,0)$ exists for all x, but if $g \neq 0$, then $D_{x+y}f(0,0) = D_xf(0,0) + D_yf(0,0)$ is not true for all x and all y.*

With the notation of Problem 2-4, part (a) of that problem says that $D_xf(0,0)$ exists for all x. Now suppose $g(a,b) \neq 0$. Then $D_{(a,b)}f(0,0) = g(a,b) \neq 0$, But $D_{(a,0)}f(0,0) + D_{(0,b)}f(0,0) = 0 + 0 = 0$.

2-31. *Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined as in Problem 1-26. Show that $D_xf(0,0)$ exists for all x, although f is not even continuous at (0,0).*

By Problem 1-26 (a), $D_xf(0,0) = 0$ for all x.

2-32. (a) *Let $f : \mathbf{R} \to \mathbf{R}$ be defined by*

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Show that f is differentiable at 0 but f' is not continuous at 0.

Clearly, f is differentiable at $x \neq 0$. At $x = 0$, one has $f'(0) = \lim_{h\to 0} \frac{h^2 \sin(1/h)}{h} = \lim_{h\to 0} h \sin(1/h) = 0$ since $|h \sin(1/h)| \leq |h|$.

For $x \neq 0$, one has $f'(x) = 2x \sin(1/x) - \sin(1/x)$. The first term has limit 0 as x approaches 0. But the second term takes on all values between -1 and 1 in every open neighborhood of $x = 0$. So, $\lim_{h\to 0} f'(h)$ does not even exist.

(b) *Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by*

$$f(x, y) = \begin{cases} (x^2 + y^2) \sin \frac{1}{\sqrt{x^2+y^2}} & (x, y) \neq 0, \\ 0 & (x, y) = 0. \end{cases}$$

Show that f is differentiable at (0,0) but that $D_i f$ is not continuous at $(0, 0)$.

The derivative at (0, 0) is the zero linear transformation because

$$\begin{aligned} \lim_{h\to 0} \frac{|h|^2 \sin(1/|h|)}{|h|} &= \lim_{h\to 0} |h| \sin(1/|h|) \\ &= 0 \end{aligned},$$

just as in part (a).

However, $f(x, 0) = f(0, x) = x^2 \sin(1/|x|) = f(|x|)$ for $x \neq 0$ where $f(x)$ is as in part (a). It follows from the differentiability of $f(x)$, that $D_1 f(x, 0)$ and $D_2 f(0, x)$ are defined for $x \neq 0$. (The argument given above also shows that they are defined and 0 at $x = 0$.) Further the partials are equal to $f'(x)$ up to a sign, and so they cannot be continuous at 0.

2-33. **Show that the continuity of $D_1 f^j$ at a may be eliminated from the hypothesis of Theorem 2-8.**

Proceed as in the proof of Theorem 2-8 for all $i > 1$. In the $i = 1$ case, it suffices to note that $\lim_{h\to 0} \frac{|f(a^1+h^1,a^2,\ldots,a^n)-f(a^1,\ldots,a^n)-D_i f(a)h^1|}{|h|} = 0$ follows from the definition of $D_1 f(a)$. This is all that is needed in the rest of the proof.

2-34. *A function $f : \mathbf{R}^n \to \mathbf{R}$ is* **homogeneous** *of degree m if $f(tx) = t^m f(x)$ for all x and t. If f is also differentiable, show that*

$$\sum_{i=1}^{n} x^i D_i f(x) = m f(x).$$

Applying Theorem 2-9 to $g(t) = f(tx)$ gives $g'(t) = \sum_{i=1}^{n} x^i D_i f(tx)$. On the other hand, $g(t) = f(tx) = t^m f(x)$ and so $g'(t) = mt^{m-1} f(x)$. Substituting $t = 1$ in these two formulas show the result.

2-35. *If $f : \mathbf{R}^n \to \mathbf{R}$ is differentiable and $f(0) = 0$, prove that there exist $g_i : \mathbf{R}^n \to \mathbf{R}$ such that*

$$f(x) = \sum_{i=1}^{n} x^i g_i(x).$$

Following the hint, let $h_x(t) = f(tx)$. Then $\int_0^1 h'_x(t)dt = h_x(1) - h_x(0) = f(x) - f(0) = f(x)$. On the other hand, Theorem 2-9 gives

$$\begin{aligned} \int_0^1 h'_x(t)dt &= \int_0^1 (\sum_{i=1}^{n} x^i D_i f(tx))dt \\ &= \sum_{i=1}^{n} x^i \int_0^1 D_i f(tx)dt. \end{aligned}$$

So, we have the result with $g_i(x) = \int_0^1 D_i f(tx)dt$.

INVERSE FUNCTIONS

2-36. *Let $A \subset \mathbf{R}^n$ be an open set and $f : A \to \mathbf{R}^n$ a continusously differentiable 1-1 function such that* $\det f'(x) \neq 0$ *for all x. Show that $f(A)$ is an open set and $f^{-1} : f(A) \to A$ is differentiable. Show also that $f(B)$ is open for any open set $B \subset A$.*

For every $y \in f(A)$, there is an $x \in A$ with $f(x) = y$. By Theorem 2-11, there is an open set $U \subset A$ and an open subset $V \subset \mathbf{R}^n$ such that $x \in U$ and $f(U) = V$. Since clearly $y \in V$, this shows that $f(A)$ is open. Furthermore $f^{-1} : V \to U$ is differentiable. It follows that f^{-1} is differentiable at y. Since y was arbitrary, it follows that $f^{-1} : f(A) \to A$ is differentiable.

By applying the previous results to the set B in place of A, we see that f is open.

2-37. (a) **Let $f : \mathbf{R}^n \to \mathbf{R}$ be a continuously differentiable function. Show that f is *not* 1-1.**

We will show the result is true even if f is only defined in a non-empty open subset of $\mathbf{R}^2$. Following the hint, we know that f is not constant in any open set. So, suppose we have $D_1 f(x_0, y_0) \neq 0$ (the case where $D_2 f(x_0, y_0) \neq 0$ is analogous). Then there is an open neighborhood U of (x_0, y_0) with $d_1 f(x, y) \neq 0$ for all $(x, y) \in U$. The function $g : U \to \mathbf{R}^2$ defined by $g(x, y) = (f(x, y), y)$ satisfies $\det g'(x, y) \neq 0$ for all $(x, y) \in A$. Assuming that f and hence g are 1-1, we can apply Problem 2-36. The inverse function is clearly of the form $(h(x,y),y)$ and so $(f(h(x,y),y),y) = (x,y)$ for all $(x,y) \in V = \{(f(x,y),y) : (x,y) \in U\}$. Now V is open but each horizontal line intersects U at most once since f is 1-1. This is a contradiction since U is non-empty and open.

(b) **Generalize this result to tthe case of a continuously differentiable function $f : \mathbf{R}^n \to \mathbf{R}^m$ with $m < n$.**

By replacing y with a vector of variables, the proof of part (a) generalizes to the case where $f : V \to \mathbf{R}$ is a function defined on an open subset V of $\mathbf{R}^n$ where $n > 1$.

For the general case of a map $f : V \to \mathbf{R}^m$ where V is an open subset of $\mathbf{R}^n$ with $n > m$, if f^1 is constant in a non-empty open set $W \subset V$, then we replace V with W and drop out f^1 reducing the value of m by one. On the other hand, if $D_1 f^1(x) \neq 0$ for some x, then consider the function g defined by $g(x) = (f^1(x), x^2, ..., x^n)$. Just as in part (a), this will be invertible on an open subset of V and its inverse will look like $(h(x), x^2, ..., x^n)$. Replace f with $f(h(x), x^2, ..., x^n)$. Note that we have made $f^1 = x^1$. Again, by restricting to an appropriate rectangle, we can simply fix the value of x^1 and get a 1-1 function defined on on a rectangle in one less dimension and mapping into a space of dimension one less. By repeating this process, one eventually gets to the case where m is equal to 1, which we have already taken care of.

2-38. (a) *If $f : \mathbf{R} \to \mathbf{R}$ satisfies $f'(a) \neq 0$ for all $a \in \mathbf{R}$, show that f is 1-1 on all of $\mathbf{R}$.*

Suppose one has $f(x) = f(y)$ for some $x \neq y$. By the mean value theorem, there is a c between x and y such that $0 = f(x) - f(y) = f'(c)(x - y)$. Since both factors on the right are non-zero, this is impossible.

(b) *Define $f : \mathbf{R}^2 \to \mathbf{R}^2$ by $f(x,y) = (e^x \cos(y), e^x \sin(y))$. Show that* $\det f'(x,y) \neq$

0 *for all* (x, y) *but* f *is not 1-1.*

Clearly, $\det f'(x, y) = 1$ for all (x, y). The function is not 1-1 since $f(x, y) = f(x, y + 2\pi)$ for all (x, y).

2-39. *Use the function* $f : \mathbf{R} \to \mathbf{R}$ *defined by*

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin(\frac{1}{x}) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

to show that continuity of the derivative cannot be eliminated from the hypothesis of Theorem 2-11.

Clearly, f is differentiable for $x \neq 0$. At $x = 0$, one has

$$f'(0) = \lim_{h \to 0} \frac{h/2 + h^2 \sin(h/2)}{h} = 1/2.$$

So f satisfies the conditions of Theorem 2-11 at $a = 0$ except that it is not continuously differentiable at 0 since $f'(x) = 1/2 + 2x\sin(1/x) - \sin(1/x)$ for $x \neq 0$.

Now $1/((2n + 1)\pi) < 1/((2n + 0.5)\pi) < 1/(2n\pi)$ and it is straightforward to verify that $f(1/((2n+1)\pi)) < f(1/(2n\pi)) < f(1/((2n+0.5)\pi))$ for all sufficiently large positive integers n. By the intermediate value theorem, there is a c between $1/((2n + 1)\pi$ and $1/((2n + 0.5)\pi)$ where $f(c) = f(1/(2n\pi))$. By taking n larger and larger, we see that f is not 1-1 on any neighborhood of 0.

IMPLICIT FUNCTIONS

2-40. *Use the implicit function theorem to re-do Problem 2-15(c).*

Define $f : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ by $f^i(x, y) = (\sum_{j=1}^n a_{ji}(x)y^j) - b_i(x)$ for $i = 1, ..., n$. One has $\det D_{j+1}f^i(x, y) = \det a_{ji}(x) \neq 0$ for all x. The determinant condition guarantees that for each x, there is exactly one solution of $f(x, y) = 0$, call that solution $(s_1(x), ..., s_n(x))$.

Now, for each x, the Implicit function theorem says that there is a function $g : \mathbf{R} \to \mathbf{R}^n$ defined in an open neighborhood of x and such that $f(x, g(x)) = 0$ and g is differentiable. By the uniqueness of the solutions in the last paragraph, it must be that $g(z) = (s_1(z), ..., s_n(z))$ for all z in the domain of g. In particular, the various functions g all glue together into a single function defined on all of $\mathbf{R}$ and differentiable everywhere.

By differentiating the relation $f(x, g(x)) = 0$, one gets $(\sum_{j=1}^n a_{ji}(x)s_j'(x)) - (b_i'(x) - \sum_{j=1}^n a_{ji}'(x)s_j(x)) = 0$ for $i = 1, ..., n$. Note that this is of the same form as the set of equations for $(s_1, ..., s_n)$ except that the right hand side functions have changed. An explicit formula can be obtained by using Cramer's rule.

2-41. *Let* $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ *be differentiable. For each* $x \in \mathbf{R}$ *defined* $g_x : \mathbf{R} \to \mathbf{R}$ *by* $g_x(y) = f(x, y)$*. Suppose that for each* x *there is a unique* y *with* $g_x'(y) = 0$*; let* $c(x)$ *be this* y*.*

In this problem, it is assumed that f was meant to be continuously differentiable.

(a) *If* $D_{2,2}f(x,y) \neq 0$ *for all* (x,y), *show that* c *is differentiable and*

$$c'(x) = -\frac{D_{2,1}f(x,c(x))}{D_{2,2}f(x,c(x))}.$$

Just as in the last problem, the uniqueness condition guarantees that $c(x)$ is the same as the function provided by the implicit function theorem applied to $g'_x(y) = D_2 f(x,y) = 0$. In particular, c is differentiable and differentiating this last relation gives $D_{2,1}f(x,c(x)) + D_{2,2}f(x,c(x))c'(x) = 0$. Solving for $c'(x)$ gives the result.

(b) *Show that if* $c'(x) = 0$, *then for some* y *we have* $D_{2,1}f(x,y) = 0$ *and* $D_2 f(x,y) = 0$.

This follows immediately from part (a).

(c) *Let* $f(x,y) = x(y\log(y) - y) - y\log(x)$. *Find*

$$\max_{\frac{1}{2}\leq x\leq 2}\left(\min_{\frac{1}{3}\leq y\leq 1} f(x,y)\right).$$

Note that $f(x,y)$ is defined only when x and y are positive. So we need to go back and show that the earlier parts of the problem generalize to this case; there are no difficulties in doing this.

One has $D_2 f(x,y) = x\log(y) - \log(x) = 0$ precisely when $y = x^{1/x}$ and so the hypothesis of part (a) is true and $c(x) = x^{1/x}$. Also, $D_{2,2}f(x,y) = x/y > 0$ (since both x and y are positive), and so for fixed x, the minimum of $f(x,y)$ occurs at $c(x)$ by the second derivative test. Now actually, we are not looking for the minimum over all $y > 0$, but just for those in the interval $[1/3, 1]$. The derivative $c'(x) = x^{1/x-2}(1 - \log(x)) > 0$ for $x < e$. Further $c(x) = 1$ precisely when $x = 1$ and there is a unique $b > 1/2$ where $c(b) = 1/3$. For fixed x, $\min_{1/3\leq y\leq 1} f(x,y)$ is achieved at $y = b$ if $1/2 \leq x \leq b$, at $y = c(x) = x^{1/x}$ if $b \leq x \leq 1$, and at $y = 1$ if $1 \leq x \leq 2$.

We will find where the maximum of the minimum's are located in each of the three cases.

Suppose $1/2 \le x \le b$. Then we need to maximize $f(x, 1/3) = (1/3)(-x(\log(3)+1)-\log(x))$. The derivative of this function is negative throughout the interval; so the maximum occurs at $x = 1/2$. The maximum value is $f(1/2, 1/3) = (1/6)\log(\frac{4}{3e})$.

Suppose $b \le x \le 1$. Then $f(x, x^{1/x}) = -x^{1+1/x}$. The derivative of this function is $h(x) = -x^{-1+1/x}(1 - \log(x) + x)$. This function has no zeros in the interval because $1 - \log(x) + x$ has derivative $-1/x + 1$ which is always negative in the interval and the value of the function is positive at the right end point. Now $h(x) < 0$ on the interval, and so the maximum must occur at the left hand end point. The maximum value is $f(b, 1/3)$. In view of the last paragraph, that means that the maximum over the entire interval $[1/2, 1]$ occurs at $x = 1/2, y = 1/3$.

Suppose $1 \le x \le 2$. Then $f(x, 1) = -x - \log(x)$. This is a decreasing function and so the maximum occurs at the left hand endpoint. By the result of the previous paragraph the maximum over the entire interval $[1/2, 2]$ must therefore occur at $x = 1/2$, $y = 1/3$, and the value of the maximum is $(1/6)\log(\frac{4}{3e})$.

III

Integration

BASIC DEFINITIONS

3-1. *Let* $f : [0,1] \times [0,1] \to \mathbf{R}$ *be defined by*

$$f(x,y) = \begin{cases} 0 & \text{if } 0 \le x < \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Show that f *is integrable and that* $\int_{[0,1]\times[0,1]} f = \frac{1}{2}$.

Apply Theorem 3-3 to the partition $P = (P_1, P_2)$ where $P_1 = P_2 = (0, 0.5, 1)$. For this partition, $U(f,P) = L(f,P) = 1/2$.

3-2. *Let* $f : A \to \mathbf{R}$ *be integrable and let* $g = f$ *except at finitely many points. Show that* g *is integrable and* $\int_A f = \int_A g$.

For any $\epsilon > 0$, there is a partition of A in which every subrectangle has volume less than ϵ. In fact, if you partition A by dividing each side into n equal sized subintervals and $a \subset \mathbf{R}^m$, then the volume of each subrectangle is precisely $v(A)/n^m$ which is less than ϵ as soon as $n > (v(A)/\epsilon)^{1/m}$. Furthermore, if P is any partition, then any common refinement of this partition and P has the same property.

If $A \subset \mathbf{R}^m$ and P is a partition of A, then any point x is an element of at most 2^m of the subrectangles of P. The intuitive iddea of the proof is that the worst case is when the point is in a 'corner'; the real proof is of course an induction on m.

Let $\epsilon > 0$ and P be a partition as in Theorem 3-3 applied to f and $\epsilon/2$. Let P' be a refinement of P such that every subrectangle of P' has volume less than

$\epsilon/(2^{m+1}r(u-l))$ where $A \subset \mathbf{R}^m$, r is the number of points where f and g have values which differ, and u (resp. l) are upper (resp. lower) bounds for the values $g(x)$ for all $x \in A$. Then the hypotheses of Theorem 3-3 are satisfied by g and ϵ, and so g is integrable.

In fact, $U(g, P') \leq U(f, P') + 2^m rvu$ and $L(g, P') \geq L(f, P') - 2^m rvu$ where v is any upper bound for the volume of the subrectangles of P', because the terms of the sum can differ only on those subrectangles which contain at least one of the r points where f and g differ. Taking differences gives $U(g, P') - L(g, P') \leq (U(f, P') - L(f, P') + 2^m rv(u-l) < \epsilon/2 + \epsilon/2 = \epsilon$.

3-3. *Let $f, g : A \to \mathbf{R}$ be integrable.*

(a) *For any partition P of A and any subrectangle S of P, show that $m_S(f) + m_S(g) \leq m_S(f+g)$ and $M_S(f+g) \leq M_S(f) + M_S(g)$ and therefore $L(f, P) + L(g, P) \leq L(f+g, P)$ and $U(f+g, P) \leq U(f, P) + U(g, P)$.*

For each $x \in S$, one has $m_S(f) \leq f(x)$ and $m_S(g) \leq g(x)$ since greatest lower bounds are lower bounds. Adding these inequalities shows that $m_S(f) + m_S(g)$ is a lower bound for $f(x) + g(x)$, and so it is at most equal to the greatest lower bound of these values. A similar argument shows the result for M_S. Since $L(f, P)$, $L(g, P)$, and $L(f+g, P$ are just positively weighted sums of the $m_S(f)$, $m_S(g)$, and $m_S(f+g)$ the result for L can be obtained by summing (with weights) the inequalities for the m_S. A similar argument shows the result for U.

(b) *Show that $f + g$ is integrable and $\int_A f + g = \int_A f + \int_A g$.*

Let P_1 (resp. P_2) be a partition as in Theorem 3-3 applied to f (resp. g) and $\epsilon/2$. Let P be a common refinement of P_1 and P_2. Then by part (a) and Lemma 3-1, $U(f+g, P) - L(f+g, P) \leq (U(f, P) - L(f, P)) + (U(g, P) - L(g, P)) \leq U(f, P_1) - L(f, P_1) + U(g, P_2) - L(g, P_2) \leq \epsilon/2 + \epsilon/2 = \epsilon$. By

Theorem 3-3, $f+g$ is integrable.

Further $\int_A f + \int_A g - \epsilon \le L(f,P_1) + L(g,P_2) \le L(f,P) + L(g,P) \le L(f+g,P) \le \int_A f+g \le U(f+g,P) \le U(f,P)+U(g,P) \le U(f,P_1)+U(g,P_2) \le \int_A f + \int_A g + \epsilon$. By the squeeze principle, one concludes that $\int_A f+g = \int_A f + \int_A g$.

(c) *For any constant c, show that $\int_A cf = c\int_A f$.*

We will show the result in the case where $c<0$; the other case being proved in a similar manner. Let P be a partition as in Theorem 3-3 applied to f and $\epsilon/(2|c|)$. Since $m_S(cf) = cM_S(f)$ and $M_S(cf) = cm_S(f)$ for each subrectangle S of P, we have $c\int_A f - \epsilon/2 = -(|c|(\int_A f + \epsilon/(2|c|))) \le -|c|U(f,P) = L(cf,P) \le U(cf,P) = -|c|L(f,P) \le -|c|(\int_A f - \epsilon/(2|c|)) = c\int_A f + \epsilon/2$. By Theorem 3-3, applied to cf and ϵ, the function cf is integrable; by the squeeze principle, its integral is $c\int_A f$.

3-4. *Let $f: A \to \mathbf{R}$ and P be a partition of A. Show that f is integrable if and only if for each subrectangle S the function $f|_S$, which consists of f restricted to S, is integrble, and that in this case $\int_A f = \sum_S \int_S f|_S$.*

Suppose that f is integrable and $\epsilon > 0$. Let P_1 be a partition of A as in Theorem 3-3 applied to f and ϵ. Let P_2 be a common refinement of P_1 and P. Then there is a partition P_3 of S whose subrectangles are precisely the subrectangles of P_2 which are contained in S. Then $U(f|_S,P_3) - L(f|_S,P_3) \le U(f,P_2) - L(f,P_2) \le U(f,P_1) - L(f,P_1) < \epsilon$. By Theorem 3-3, it follows that $f|_S$ is integrable.

Suppose that all the $f|_S$ are integrable where S is any subrectangle of P. Let P_S be a partition as in Theorem 3-3 applied to $f|_S$ and $\epsilon/(2N)$ where N is the number of rectangles in P. Let P' be the partition of A obtained by taking the union of all the subsequences defining the partitions of the P_S (for each dimension). Then there are refinements P'_S of the P_S whose rectangles are the set of all subrectangles of P' which are contained in S. One has

$\sum_S \int_S f|_S - \epsilon/2 < \sum_S L(f|_S, P_S) \leq \sum_S L(f|_S, P'_S) = L(f, P') \leq U(f, P') = \sum_S U(f|_S, P'_S) \leq \sum_S U(f|_S, P_S) < \sum_S \int_S f|_S + \epsilon/2$. By Theorem 3-3, the function f is integrable, and, by the squeeze principle, it has the desired value.

3-5. *Let $f, g : A \to \mathbf{R}$ be integrable and suppose $f \leq g$. Show that $\int_A f \leq \int_A g$.*

By Problem 3-3, the function $g - f$ is integrable and $\int_A f + \int_A g - f = \int_A g$. Using the trivial partition P_0 in which A is the only rectangle, we have $0 \leq L(g - f, P_0) \leq \int_A g - f$ since $g - f \geq 0$. This proves the result.

3-6. *If $f : A \to \mathbf{R}$ is integrable, show that $|f|$ is integrable and $|\int_A f| \leq \int_A |f|$.*

Consider the function $g(x) = \max(f(x), 0)$. For any rectangle contained in A, we have $M_S(g) = \max(M_S(f), 0)$ and $m_S(g) = \max(m_S(f), 0)$. If $M_S(f) \geq 0$, then $M_S(g) - m_S(g) = M_S(f) - m_S(g) \leq M_S(f) - m_S(f)$. On the other hand, if $M_S(f) < 0$, then $M_S(g) - m_S(g) = 0 - 0 \leq M_S(f) - m_S(f)$. Let P be a partition as in Theorem 3-3 applied to f and ϵ. Then this implies that $U(g, P) - L(g, P) = \sum_S v(S)(M_S(g) - m_S(g)) \leq \sum_S v(S)(M_S(f) - m_S(f)) = U(f, P) - L(f, P) < \epsilon$. So, g is integrable by Theorem 3-3.

Similarly, one can show that $h(x) = \min(f(x), 0)$ is integrable. But then by Problem 3-3, it follows that $|f| = g - h$ is integrable. But then, so if $-|f|$ integrable. Further, since $-|f| \leq f \leq |f|$, Problem 3-5 implies that $\int_A -|f| \int_A f \leq \int_A |f|$. Since $\int_A -|f| = -\int_A |f|$ by Problem 3-3 (c), it follows that $|\int_A f| \leq \int_A |f|$.

3-7. *Let $f : [0, 1] \times [0, 1] \to \mathbf{R}$ be defined by*

$$f(x, y) = \begin{cases} 0 & x \text{ irrational}, \\ 0 & x \text{ rational}, y \text{ irrational}, \\ 1/q & x \text{ rational}, y = p/q \text{ in lowest terms}. \end{cases}$$

Show that f is integrable and $\int_{[0,1]\times[0,1]} f = 0$.

Let $\epsilon > 0$. Choose a positive integer N so that $1/N < \epsilon/2$. Let P be any partition of $A = [0,1] \times [0,1]$ such that every point $(x,y) \in A$ with $y = p/q, (p,q) = 1, N > q > 0$ lies in a rectangle of P of height (in the y direction) at most $\delta = \epsilon/((N+2)(N-1))$. Since there are at most $(H+2)(N-1)/2$ such pairs (x,y), such a P exists and the total volume of all the rectangles containing points of this type is at most $\epsilon/2$. Since $f \leq 1$, the contribution to $U(f,P)$ from these rectangles is also at most $\epsilon/2$. For the remaining rectangles S, the value of $M_S(f) \leq 1/N$ and their total volume is, of course, no larger than 1; so their contribution to $U(f,P)$ is at most $1/N < \epsilon/2$. It follows that $0 \leq L(f,P) \leq U(f,P) < \epsilon/2 + \epsilon/2 = \epsilon$. By Theorem 3-3, f is integrable and the squeeze principle implies that its integral is 0.

MEASURE ZERO AND CONTENT ZERO

3-1. *Prove that $R = [a_1, b_1] \times ... \times [a_n, b_n]$ is not of content 0 if $a_i < b_i$ for $i = 1, ..., n$.*

Suppose U_i for $i = 1, ..., m$ are closed rectangles which form a cover for R. By replacing the U_i with $U_i \cap R$, one can assume that $U_i \subset R$ for all i. Let $U_i = [c_{i1}, d_{i1}] \times ... \times [c_{in}, d_{in}]$. Choose a partition P which refines all of the partitions $P_i = (P_{i1}, ..., P_{in})$ where $P_{ij} = (a_j, c_{ij}, d_{ij}, b_j)$ Note that U_i is a rectangle of the cover P_i. Let S be any rectangle in P with non-empty interior. Since the intersection of any two rectangles of a partition is contained in their boundaries, if S contains an interior point x not in U_i for some i, then U_i contains only boundary points of S. So, if S has non-empty interior, then S is a subset of U_i for some i since the union of the U_i is R. The sum of the volumes of the rectangles of P is the volume of R, which is at most equal to the sum of the volumes of the U_i. So R is not of content 0 as it cannot be covered with rectangles of total area less than the volume of R.

3-2. (a) *Show that an unbounded set A cannot have content 0.*

Suppose $A \subset \cup_{i=1}^{m} U_i$ where U_i are rectangles, say $U_i = [a_{i1}, b_{i1}] \times ... \times [a_{in}, b_{in}]$. Let $R = [a_1, b_1] \times ... \times [a_n, b_n]$ where $a_j = \min(a_{1j}, ..., a_{mj})$ and $b_j = \max(b_{1j}, ..., b_{mj})$. Then R contains all the U_i and hence also A. But then A is bounded, contrary to hypothesis.

(b) *Give an example of a closed set of measure 0 which does not have content 0.*

The set of natural numbers is unbounded, and hence not of content 0 by part (a). On the other hand, it is of measure zero. Indeed, if $\epsilon > 0$, then the union of the open intervals $(i - 2^{-i-1}\epsilon, i + 2^{i+1}\epsilon)$ for $i = 1, 2, ...$ contains all the natural numbers and the total volume of all the intervals is $\sum_{i=1}^{\infty} 2^i \epsilon = \epsilon$.

3-3. (a) *If C is a set of content 0, show that the boundary of C also has content 0.*

Suppose a finite set of open rectangles $U_i = (a_{i1}, b_{i1}) \times ... \times (a_{in}, b_{in})$, $i = 1, ..., m$. cover of C and have total volume less than $\epsilon/2$ where $\epsilon > 0$. Let $V_i = (a_{i1} - \delta, b_{i1} + \delta) \times ... \times (a_{in} - \delta, b_{in} + \delta)$ where $\prod_{j=1}^{n}(b_{ij} - a_{ij} + 2\delta) - \prod_{j=1}^{n} b_{ij} - a_{ij} < \epsilon/(2m)$. Then the union of the V_i cover the boundary of C and have total volume less than ϵ. So the boundary of C is also of content 0.

(b) *Give an example of a bounded set C of measure 0 such that the boundry of C does not have measure 0.*

The set of rational numbers in the interval $[0, 1]$ is of measure 0 (cf Proof of Problem 3-9 (b)), but its boundary $[0, 1]$ is not of measure 0 (by Theorem 3-6 and Problem 3-8).

3-4. *Let A be the set of Problem 1-18. If $T = \sum_{i=1}^{\infty}(b_i - a_)$ < 1, show that the boundary of A does not have measure 0.*

The set $[0, 1] - A$ closed and bounded, and hence compact. If it were also of measure 0, then it would be of content 0 by Theorem 3-6. But then there is a finite collection of open intervals which cover the set and have total volume less than $1 - T$. Since the set these open intervals together with the set of (a_i, b_i) form an open cover of $[0, 1]$, there is a finite subcover of $[0, 1]$. But then the sum of the lengths of the intervals in this finite subcover would be less than 1, contrary to Theorem 3-5.

3-5. *Let $f : [a, b] \to \mathbf{R}$ be an increasing function. Show that* $\{x : f$ is discontinuous at $x\}$

is a set of measure 0.

Using the hint, we know by Problem 1-30 that the set of x where $o(f,x) > 1/n$ if finite for every $n > 0$. Hence the set of discontinuities of f is a countable union of finite sets, and hence has measure 0 by Theorem 3-4.

3-6. (a) *Show that the set of all rectangles $[a_1, b_1] \times ... \times [a_n, b_n]$ where each a_i and each b_i are rational can be arranged into a sequence (i.e. form a countable set).*

Since the set of rational numbers is countable, and cartesian products of countable sets are countable, so is the set of all $2n$-tuples of rational numbers. Since the set of these intervals is just a subset of this set, it must be countable too.

(b) *If $A \subset \mathbf{R}^n$ is any set and $\mathcal{O}$ is an open cover of A, show that there is a sequence $U_1, U_2, U_3, ...$ of members of $\mathcal{O}$ which also cover A.*

Following the hint, for each x in A, there is a rectangle B of the type in part (a) such that B has non-zero volume, contains x and is contained in some U in $\mathcal{O}$. In fact, we can even assume that x is in the interior of the rectangle B. In particular, the union of the interiors of the rectangles B (where x is allowed to range throughout A) is a cover of A. By part (a), the set of these B are countable, and hence so are the set of corresponding U's; this set of corresponding U's cover A.

INTEGRABLE FUNCTIONS

3-14. *Show that if* $f, g : A \to \mathbf{R}$ *are integrable, so is* $f \cdot g$.

The set of $x \in A$ where $f \cdot g$ is not continuous is contained in the union of the sets where f and g are not continuous. These last two sets are of measure 0 by Theorem 3-8; so theee first set is also of measure 0. But then $f \cdot g$ is integrable by Theorem 3-8.

3-15. *Show that if* C *has content 0, then* $C \subset A$ *for some closed rectangle* A *and* C *is Jordan-measurable and* $\int_A \chi_C = 0$.

If C has content 0, then it is bounded by Problem 3-9 (a); so it is a subset of an closed rectangle A. Since C has content 0, one has $C \subset \cup_{i=1}^m U_i$ for some open rectangles U_i the sum of whose volumes can be made as small as desired. But then the boundary of C is contained in the closure of C, which is contained in the union of the closures of the U_i (since this union is closed). But then the boundary of C must be of content 0, and so C is Jordan measurable by Theorem 3-9. Further, by Problem 3-5, one has $0 \leq \int_A \chi_C \leq \sum_{i=1}^m \int_A \chi_{U_i} \leq \sum_{i=1}^m v(U_i)$ which can be made as small as desired; so $\int_A \chi_C = 0$.

3-16. *Give an example of a bounded set* C *of measure 0 such that* $\int_A \chi_C$ *does not exist.*

Let C be the set of rational numbers in $[0, 1]$. Then the boundary of C is $[0, 1]$, which is not of measure 0. So $\int_A \chi_C$ does not exist by Theorem 3-9.

3-17. *If* C *is a bounded set of measure 0 and* $\int_A \chi_C$ *exists, show that* $\int_A \chi_C = 0$.

Using the hint, let P be a partition of A where A is a closed rectangle containing C. Then let S be a rectangle of P of positive volume. Then S is not of measure 0 by Problem 3-8, and so $S \not\subset C$. But then there is a point of S outside of C; so $m_S(\chi_C) = 0$. Since this is true of all S, one has $L(\chi_C, P) = 0$. Since this holds for all partitions P of A, it follows that $\int_A \chi_C = 0$ if the integral exists.

3-18. *If $f : A \to \mathbf{R}$ is non-negative and $\int_A f = 0$, show that $B = \{x : f(x) \neq 0\}$ has measure 0.*

Following the hint, let n be a positive integer and $A_n = \{x : f(x) > 1/n\}$. Let $\epsilon > 0$. Let P be a partition of A such that $U(f, P) < \epsilon/(2n)$. Then if S is a rectangle of P which intersects A_n, we have $M_S(f) > 1/n$. So $\sum_{S \cap A \neq \emptyset} v(S)/n \leq U(f, P) < \epsilon/(2n)$. By replacing the closed rectangles S with slightly larger open rectangles, one gets an open rectangular cover of A_n with sets, the sum of whose volumes is at most ϵ. So A_n has content 0. Now apply Theorem 3-4 to conclude that $\cup A_n = B$ has measure 0.

3-19. *Let U be the open set of Problem 3-11. Show that if $f = \chi_U$ except on a set of measure 0, then f is not integrable on* $[0, 1]$.

The set of x where χ_U is not continuous is $[0, 1] - A$ which is not of measure 0. If the set where f is not continuous is not of measure 0, then f is not integrable by Theorem 3-8. On the other hand, if it is of measure 0, then taking the union of this set with the set of measure 0 consisting of the points where f and χ_C differ gives a set of measure 0 which contains the set of points where f is not continuous. So this set is also of measure 0, which is a contradiction.

3-20. *Show that an increeasing function $f : [a, b] \to \mathbf{R}$ is integrable on* $[a, b]$.

This is an immediate consequence of Problem 3-12 and Theorem 3-8.

3-21. *If A is a closed rectangle, show that $C \subset A$ is Jordan measurable if and only if for* *every $\epsilon > 0$ there is a partition P of A such that $\sum_{S\in\mathcal{S}_1} v(S) - \sum_{S\in\mathcal{S}_2} v(S) < \epsilon$, where $\mathcal{S}_1$ consists of all subrectangles intersecting C and $\mathcal{S}_2$ consists of allsub-rectangles contained in C.*

Suppose C is Jordan measurable. Then its boundary is of content 0 by Theorem 3-9. Let $\epsilon > 0$ and choose a finite set S_i for $i = 1, ..., m$ of open rectangles the sum of whose volumes is less than ϵ and such that the S_i form a cover of the boundary of C. Let P be a partition of A such that every subrectangle of P is either contained within each S_i or does not intersect it. This P satisfies the condition in the statement of the problem.

Suppose for every $\epsilon/2 > 0$, there is a partition P as in the statement. Then by replacing the rectangles with slightly larger ones, one can obtain the same result except now one will have ϵ in place of $\epsilon/2$ and the S will be open rectangles. This shows that the boundary of C is of content 0; hence C is Jordan measurable by Theorem 3-9.

3-22. *If A is a Jordan measurable set and $\epsilon > 0$, show that there is a compact Jordan measurable set $C \subset A$ such that $\int_{A-C} 1 < \epsilon$.*

Let B be a closed rectangle containing A. Apply Problem 3-21 with A as the Jordan measurable set. Let P be the partition as in Problem 3-21. Define $C = \cup_{S\in\mathcal{S}_2} S$. Then $C \subset A$ and clearly C is Jordan measurable by Theorem 3-9. Further $\int_{A-C} 1 < \sum_{S\in\mathcal{S}_1} v(S) - \sum_{S\in\mathcal{S}_2} v(S) < \epsilon$.

FUBINI'S THEOREM

3-23. *Let $C \subset A \times B$ be a set of content 0. Let $A' \subset A$ be the set of all $x \in A$ such that $\{y \in B : (x, y) \in C\}$ is* not *of content 0. Show that A' is a set of measure 0.*

Following the hint, χ_C is integrable with $\int_{A\times B} \chi_C = \int_A \mathcal{U} = \int_A \mathcal{L} = 0$ by Problem 3-15 and Fubini's Theorem. We have $\int_A \mathcal{U} - \mathcal{L} = 0$. Now $x \in A'$ is equivalent to the condition that either $\mathcal{U}(x) - \mathcal{L}(x) \neq 0$ or $\mathcal{L}(x) \neq 0$. Both of these having integral 0 implies by Problem 3-18 that the sets where their integrand is non-zero are of measure 0, and so A' is also of measure 0.

3-24. *Let $C \subset [0, 1] \times [0, 1]$ be the union of all $\{p/q\} \times [0, 1/q]$ where p/q is a rational number in $[0, 1]$ written in lowest terms. Use C to show that the word "measure" in Problem 3-23 cannot be replaced with "content".*

The set A' is the set of rational numbers in $[0, 1]$ which is of measure 0, but not of content 0, because the integral of its characteristic function does not exist. To see that the set C has content 0, let $\epsilon > 0$. Let $N > 0$ be such that $1/N < \epsilon/2$. Then the set C can be covered by the rectangles $[0, 1] \times [0, 1/N]$ and for each p/q in lowest terms with $q \leq N$, the rectangle $[p/q - \delta, p/q + \delta \times [0, 1]$ where $\delta = 1/(2N^2(N + 1))$. The sum of the areas of these rectangles is less than ϵ.

3-25. *Show by induction on n that $R = [a_1, b_1] \times ... \times [a_n, b_n]$ is not a set of measure 0 (or content 0) if $a_i < b_i$ for each i*

This follows from Problem 3-8 and Theorem 3-6, but that is not an induction.

Fubini's Theorem and induction on n show that $\int_R 1 \neq 0$ and so R does not have content 0, and hence is not of measure 0.

3-26. *Let $f : [a,b] \to \mathbf{R}$ be integrable and non-negative, and let $A_f = \{(x,y) : a \leq x \leq b$ and $0 \leq y \leq f(x)\}$. Show that A_f is Jordan measurable and has area $\int_a^b f$.*

One has $\mathcal{L}(x) = \int_0^{f(x)} 1 = f(x)$ and so by Fubini, $\int_{[a,b]\times[0,M]} \chi_{A_f} = \int_{[a,b]} \mathcal{L} = \int_{\dashv}^{\lfloor} \{$ where M is an upper bound on the image of f.

3-27. *If $f : [a,b] \times [a,b] \to \mathbf{R}$ is continuous, show that*

$$\int_a^b \int_a^y f(x,y)dxdy = \int_a \int_x f(x,y)dydx$$

where the upper bounds need to be determined.

By Fubini, the left hand iterated integral is just $\int_C f$ where $C = \{(x,y) : a \leq y \leq b$ and $a \leq x \leq y\} = \{(x,y) : a \leq x \leq b$ and $x \leq y \leq b\}$. Applying Fubini again, shows that this integral is equal to $\int_a^b \int_x^b f(x,y)dxdy$.

3-28. *Use Fubini's Theorem to give an easy proof that $D_{1,2}f = D_{2,1}f$ if these are continuous.*

Following the hint, if $D_{1,2}f - D_{2,1}f$ is not zero for some point a, then we may assume (by replacing f with $-f$ if necessary that it is positive at a. But then continuity implies that it is positive on a rectangle A containing a. But then its integral over A is also positive.

On the other hand, using Fubini on $A = [a,b] \times [c,d]$ gives:

$$\begin{aligned}\int_A D_{2,1}f &= \int_c^d \int_a^b D_{2,1}f dxdy \\ &= \int_c^d D_2 f(b,y) - D_2 f(a,y)dy \\ &= f(b,d) - f(a,d) - f(b,c) + f(a,c).\end{aligned}$$

Similarly, one has $\int_A D_{1,2}f = \int_a^b \int_c^d D_{1,2}f dydx = \int_a^b D_1 f(x,d) - D_1 f(x,c)dx =$

$f(b,d) - f(b,c) - f(a,d) + f(a,c)$. Subtracting gives: $\int_A D_{1,2}f - D_{2,1}f = 0$ which is a contradiction.

3-29. *Use Fubini's Theorem to derive an expression for the volume of a set in* $\mathbf{R}^3$ *obtained by revolving a Jordan measurable set in the yz-plane about the z-axis.*

To avoid overlap, it is convenient to keep the set in the positive y half plane. To do this, let R be the original Jordan measurable set in the yz-plane, and replace it with $S = \{(y,z) : y \geq 0 \text{ and } (y,z) \text{ or } (-y,z) \in R\}$. Theorem 3-9 can be used to show that S is Jordan measurable if R is.

The problem appears to be premature since we really want to be able to do a change of variables to cylindrical coordinates. Assuming that we know how to do that, the result becomes $\int_S 2\pi y$.

3-30. *Let* C *be the set in Problem 1-17. Show that*

$$\int_{[0,1]} \left(\int_{[0,1]} \chi_C(x,y)dx \right) dy = \int_{[0,1]} \left(\int_{[0,1]} \chi_C(y,x)dy \right) dx = 0.$$

but that $\int_{[0,1]\times[0,1]} \chi_C$ *does not exist.*

The problem has a typo in it; the author should not have switched the order of the arguments of χ_C as that trivializes the assertion.

The iterated integrals are zero because the inside integral is the zero function. The last integral cannot exist by Theorem 3-9 and Problem 1-17.

3-31. *If* $A = [a_1, b_1] \times \ldots \times [a_n, b_n]$ *and* $f : A \to \mathbf{R}$ *is continuous, define* $F : A \to \mathbf{R}$ *by*

$$F(x) = \int_{[a_1,x^1]\times\ldots\times[a_n,x^n]} f.$$

What is $D_iF(x)$*, for* x *in the interior of* A*?*

Let c be in the interior of A, fix i. We have

$$D_iF(c) = \lim_{h\to 0} \frac{F(c^1,...,c^{i-1},c^i+h,c^{i+1},...,c^n) - F(c^1,...,c^n)}{h}$$
$$= \lim_{h\to 0} \frac{\int_{c^i}^{c^i+h} \int_{[a^1,c^1]\times...\times[a^{i-1},c^{i-1}]\times[a^{i+1},c^{i+1}]\times...\times[a^n,c^n]} f}{h}$$
$$= \int_{[a^1,c^1]\times...\times[a^{i-1},c^{i-1}]\times[a^{i+1},c^{i+1}]\times...\times[a^n,c^n]} f(x^1,...,x^{i-1},c^i,x^{i+1},...x^n)$$

by Fubini's Theorem.

3-32. *Let $f : [a,b]\times[c,d] \to \mathbf{R}$ be continuous and suppose D_2f is continuous. Define $F(y) = \int_a^b f(x,y)dx$. Prove* Leibnitz' Rule $F'(y) = \int_a^b D_2f(x,y)dx$.

Using the hint, we have $F(y) = \int_a^b f(x,y)dx = \int_a^b(\int_c^y D_2f(x,y)dy + f(x,c))dx$. One has

$$F'(y)(1) = \lim_{h\to 0} \frac{F(y+h)-F(y)}{h}$$
$$= \lim_{h\to 0} \frac{\int_a^b \int_y^{y+h} D_2f(x,y)dydx}{h}$$
$$= \lim_{h\to 0} \frac{\int_y^{y+h} \int_a^b D_2f(x,y)dxdy}{h}$$
$$= \lim_{h\to 0} \frac{\int_c^{y+h} \int_a^b D_2f(x,y)dxdy - \int_c^y \int_a^b D_2f(x,y)dxdy}{h}$$
$$= \int_a^b D_2f(x,y)dx.$$

3-33. *If $f : [a,b]\times[c,d] \to \mathbf{R}$ is continuous and D_2f is continuous, define $F(x,y) = \int_a^x f(t,y)dt$.*

(a) *Find D_1F and D_2F.*

One has $D_1F = f(x,y)$ and $D_2F = \int_a^x D_2f(t,y)dt$ where the second assertion used Problem 3-32.

(b) *If $G(x) = \int_a^{g(x)} f(t,x)dt$, find $G'(x)$.*

We have $G(x) = F(g(x),x)$ and so by the chain rule one has $G'(x) = D_1F(g(x),x)g'(x) + D_2F(g(x),x) = f(g(x),x)g'(x) + \int_a^g(x)D_2f(t,x)dt$.

3-34. *Let $g_1, g_2 : \mathbf{R}^2 \to \mathbf{R}$ be continuously differentiable and suppose $D_1 g_2 = D_2 g_1$. As in Problem 2-21, let*

$$f(x,y) = \int_0^x g_1(t,0)dt + \int_0^y g_2(x,t)dt.$$

Show that $D_1 f(x,y) = g_1(x,y)$.

One has

$$\begin{aligned} D_1 f(x,y) &= \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \\ &= \lim_{h\to 0} \frac{\int_x^{x+h} g_1(t,0)dt}{h} + \frac{\int_0^y g_2(x+h,t)dt - \int_0^y g_2(x,t)dt}{h} \\ &= g_1(x,0) + \int_0^y D_1 g_2(x,t)dt = g_1(x,0) + \int_0^y D_2 g_1(x,t) \\ &= g_1(x,0) + (g_1(x,y) - g_1(x,0)) = g_1(x,y). \end{aligned}$$

3-35. (a) *Let $g : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation of one of the following types:*

$$\begin{cases} g(e_i) = e_i & i \neq j \\ g(e_j) = ae_j \end{cases}$$

$$\begin{cases} g(e_i) = e_i & i \neq j \\ g(e_j) = e_j + e_k \end{cases}$$

$$\begin{cases} g(e_k) = e_k & k \neq i,j \\ g(e_i) = e_j \\ g(e_j) = e_i \end{cases}$$

If U is a rectangle, show that the volume of $g(U)$ is $|\det g| \cdot v(U)$.

In the three cases, $|\det g|$ is $|a|$, 1, and 1 respectively. If the original rectangle $U = [a_1,b_1] \times ... \times [a_n,b_n]$, then $g(U)$ is

$$\begin{cases} [a_1,b_1] \times ... \times [aa_j, ab_j] \times ... \times [a_n,b_n] & a \geq 0 \\ [a_1,b_1] \times ... \times [ab_j, aa_j] \times ... \times [a_n,b_n] & a < 0 \end{cases}$$

in the first case, is a cylinder with a parallelogram base in the second case, and is the same rectangle except that the intervals in the i^{th} and j^{th} places are swapped in the third case. In the second case, the parallelogram base is in the j^{th} and k^{th} directions and has corners $[a_j, a_j + a_k], [b_j, b_j + a_k], [a_j, a_j + b_k], [b_j, b_j + b_k]$. So the volumes do not change in the second and third case and get multiplied by $|a|$ in the first case. This shows the result.

(b) *Prove that* $|\det g|v(U)$ *is the volume of* $g(U)$ *for any linear transformation* $g : \mathbf{R}^n \to \mathbf{R}^n$.

If g is non-singular, then it is a composition of linear transformations of the types in part (a) of the problem. Since det is multiplicative, the result follows in this case.

If g is singular, then $g(\mathbf{R}^n)$ is a proper subspace of $\mathbf{R}^n$ and $g(U)$ is a compact set in this proper subspace. In particular, $g(U)$ is contained in a hyperplane. By choosing the coordinate properly, the hyperplane is the image of a linear transformation from $\mathbf{R}^{n-1}$ into $\mathbf{R}^n$ made up of a composition of maps of the first two types. This shows that the compact portion of the hyperplane is of volume 0. Since the determinant is also 0, this shows the result in this case too.

3-36. *(Cavalieri's principle). Let* A *and* B *be Jordan measurable subsets of* $\mathbf{R}^3$*. Let* $A_c = \{(x, y) : (x, y, c) \in A\}$ *and define* B_c *similarly. Suppose that each* A_c *and* B_c *are Jordan measurable and have the same area. Show that* A *and* B *have the same volume.*

This is an immediate consequence of Fubini's Theorem since the inside integrals are equal.

PARTITIONS OF UNITY

3-37. (a) *Suppose that* $f : (0,1) \to \mathbf{R}$ *is a non-negative continuous function. Show that* $\int_{(0,1)} f$ *exists if and only if* $\lim_{\epsilon \to 0} \int_{\epsilon}^{1-\epsilon} f$ *exists.*

For n a natural number, define $U_n = (1/(n+2), 1/n) \cup (1-1/n, 1-1/(n+2)$ and $U_0 = (1/3, 2/3)$. Consider a partition of unity Φ subordinate to the cover $\{U_n : n = 0, 1, 2, ...\}$. By summing the $\phi \in \Phi$ with the same U in condition (4) of Theorem 3-11, one can assume that there is only one function for each U_n, let it be ϕ_n. Now $\int_{(0,1)} f$ exists if and only $\sum_{n=0}^{\infty} \int_{(0,1)} \phi_n \cdot f$ converges. But $\int_{1/M}^{1-1/M} f \leq \sum_{n=0}^{M} \int_{(0,1)} \phi_n \cdot f \leq \int_{1/(M+2)}^{1-1/(M+2)} f$. So the sum converges if and only if $\lim_{\epsilon \to 0} \int_{\epsilon}^{1-\epsilon} f$ exists.

(b) *Let* $A_n = [1 - 1/2^n, 1 - 1/2^{n+1}]$. *Suppose that* $f : (0,1) \to \mathbf{R}$ *satisfies* $\int_{A_n} f = (-1)^n/n$ *and* $f(x) = 0$ *for all* $x \notin \cup_n A_n$. *Show that* $\int_{(0,1)} f$ *does not exist, but* $\lim_{\epsilon \to 0} \int_{(\epsilon, 1-\epsilon)} f = \log 2$.

Take a partition of unity Φ subordinate to the cover $\{U_n\}$ where $U_k = (1 - 1/(2^k - 1), 1 - 1/(2^{k+1} + 1))$ for $k > 0$. As in part (a), we can assume there is only one $\phi_k \in \Phi$ as in condition (4) of Theorem 3-11. Consider the convergence of $\sum_n \int_{(0,1)} \phi_n \cdot |f|$. One has $\sum_{k=1}^{n-1} b_k \leq \sum_k = 1^n \int_{(0,1)} \phi_k \cdot |f| \leq \sum_{k=1}^{n} b_k$ where $b_k = \int_{A_n} |f| \geq 1/n$. It follows that the sum in the middle does not converge as $n \to \infty$ and so $\int_{(0,1)} f$ does not exist.

The assertion that $\lim_{\epsilon \to 0} \int_{(\epsilon, 1-\epsilon)} f = \log 2$. If not necessrily true. From the hypothesis, we only know the values of the integral of f on the sets A_n, but don't know how f behaves on other intervals – so it could be that $\int_{\epsilon, 1-\epsilon} f$

may not even exist for all $\epsilon > 0$ To correct the situation, let us assume that f is of constant sign and bounded on each set A_n. Then f is bounded on each interval $(\epsilon, 1-\epsilon)$ and so by Theorem 3-12, the integral in the extended sense is same as the that in the old sense. Clearly, the integral in the old sense is monotone on each interval of $\epsilon \in [1/2^{n+1}, 1/2^n]$, and the limit is just $\sum_n \int_{A_n} f = \log(2)$.

3-38. *Let A_n be a closed set contained in $(n, n+1)$. Suppose that $f : \mathbf{R} \to \mathbf{R}$ satisfies $\int_{A_n} f = (-1)^n/n$ and $f = 0$ outside $\cup_n A_n$. Find two partitions of unity Φ and Ψ such that $\sum_{\phi\in\Phi} \int_{\mathbf{R}} \phi \cdot f$ and $\sum_{\psi\in\Psi} \int_{\mathbf{R}} \psi \cdot f$ converge absolutely to different values.*

The sums $\sum_{n \text{ odd}} \int_{A_n} f$ and $\sum_{n \text{ even}} \int_{A_n} f$ have terms of the same sign and are each divergent. So, by re-ordering the terms of $\sum_n \int_{A_n} f$, one can make the sum approach any number we want; further this can be done so that there are sequences of partial sums which converge monotonically to the limit value. By forming open covers each set of which consists of intervals $(n, n+1)$ for the sum of terms added to each of these partial sums, one gets covers of $\cup_n (n, n+1)$. Because f is zero outside $\cup_n A_n$, one can 'fatten' up the covering sets so that they are a cover of the real numbers no smaller than 1 without adding any points where f is non-zero. Finally, one can take a partition of unity subordinate to this cover. By using arrangements with different limiting values, one gets the result.

CHANGE OF VARIABLE

3-39. *Use Theorem 3-14 to prove Theorem 3-13 without the assumption that* $g'(x) \neq 0$.

Let $B = \{x \in A : \det g'(x) = 0\}$ Then $A' = A - B$ is open and Theorem 3-13 applies with A' in place of the A in its statement. Let Φ be a partition of unity subordinate to an admissible cover $\mathcal{O}$ of A. Then $\Phi' = \{\phi|_{A'} : \phi \in \Phi\}$ is a partion of unity subordinate to the cover $\mathcal{O}' = \{U \cap A' : U \in \mathcal{O}\}$. Now $\sum_{\phi\in\Phi} \int_A \phi|f(g) \det g'|$ is absolutely convergent, and so $\sum_{\phi'\in\Phi'} \int_A' \phi'|f(g) \det g'|$ also converges since the terms are identical. So, $\int_A f(g) \det g' = \int_A' f(g) \det g'$. By Theorem 3-14, we know that $\int_{g(A)} f = \int_{g(A)-g(B)} f = \int_{g(A')} f$. Combining results, we get Theorem 3-13.

3-40. *If* $g : \mathbf{R}^n \to \mathbf{R}^n$ *and* $\det g'(x) \neq 0$, *prove that in some open set containing* x *we can write* $g = T \circ g_n \circ \cdots \circ g_1$, *where* g_i *is of the form* $g_i(x) = (x^1, \ldots, F_i(x), \ldots, X^n)$, *and* T *is a linear transformation. Show that we can write* $g = g_n \circ \cdots \circ g_1$ *if and only if* $g'(x)$ *is a diagonal matrix.*

We use the same idea as in the proof of Theorem 3-13. Let a be a point where $\det g'(a) \neq 0$. Let $T = g'(a)$, and $k = T^{-1} \circ g$. Then $k'(a) = (g'(a))^{-1} \circ g'(a) = Id$. Define for $i = n, n-1, ..., 0$, $h_i = (k^1, ..., k^i, x^{i+1}, ..., x^n)$. Then $h_i'(a) = Id$. So we can define on successively smaller open neighborhoods of a, inverses h_{i-1}^{-1} of h_{i-1} and $g_i = (x^1, ..., x^{i-1}, k^i(h_{i-1}^{-1}), x^{i+1}, ..., x^n)$. One then can verify that $g_i \circ h_{i-1} = h_i$. Combining results gives $T^{-1} \circ g = k = h_n = g_n \circ h_{n-1} = g_n \circ g_{n-1} \circ h_{n-2} = \cdots = g_n \circ \cdots \circ g_1 \circ h_0 = g_n \circ cdots \circ g_1$ and so $g = T \circ g_n \circ \cdots \circ g_1$.

Now, if T is a diagonal matrix, then replace g_i with $r_i = T \circ g_i \circ T^{-1}$. for $i > 1$ and $r_1 = T \circ g_1$. Then the r_i have the same form as the g_i and $g = r_n \circ \cdots \circ r_1$. On the other hand, the converse is false. For example, consider the function $g(x, y) = (x + y, y) = (x + y, y) \circ (x, y)$. Since g is linear, $g'(a) = g$; so $g'(a)$ is not a diagonal matrix.

3-41. *Define* $f : \{r : r > 0\} \times (0, 2\pi) \to \mathbf{R}^2$ *by* $f(r, \theta) = (r \cos(\theta), r \sin(\theta))$.

(a) *Show that f is 1-1, compute $f'(r, \theta)$, and show that* $\det f'(r, \theta) \neq 0$ *for all (r, θ). Show that $f(\{r : r > 0\} \times (0, 2\pi))$ is the set A of Problem 2-23.*

Since $r = \sqrt{(f^1)^2 + (f^2)^2}$, to show that the function f is 1-1, it suffices to show that $\cos(\theta_1) = \cos(\theta_2)$ and $\sin(\theta_1) = \sin(\theta_2)$ imply $\theta_1 = \theta_2$. Suppose $\theta_1 \neq \theta_2$. Then $\cos(\theta_1) = \cos(\theta_2)$ implies that $\theta_1 = 2\pi - \theta_2$ (or $\theta_2 = 2\pi - \theta_2$). If $\sin(\theta_1) = \sin(\theta_2)$, it follows that $\sin(\theta_2) = -\sin(\theta_2)$. But then $\theta_2 = \pi$ and θ_2 has the same value, contrary to hypothesis. So, f is 1-1.

One has $f'(r, \theta) = \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & -r\cos(\theta) \end{pmatrix}$. So, $\det f'(r, \theta) = r \neq 0$ for all (r, θ) in the domain of f.

Suppose $(x, y) \notin A$, i.e. $x \geq 0$ and $y = 0$. If $f(r, \theta) = (x, y)$, then $r > 0$ implies $\sin(\theta) = 0$ and so $\theta = \pi$. But then $x = r\cos(\theta) = -r < 0$ contrary to hypothesis. On the other hand, if $(x, y) \in A$, then let $r = \sqrt{x^2 + y^2} > 0$ and let θ be the angle between the positive x-axis and the ray from (0,0) through (x, y). Then $f(r, \theta) = (x, y)$.

(b) *If $P = f^{-1}$, show that $P(x, y) = (r(x, y), \theta(x, y))$, where*

$$r(x, y) = sqrtx^2 + y^2$$

$$\theta(x,y) = \begin{cases} \arctan y/x & x>0, y>0, \\ \pi + \arctan y/x & x<0, \\ 2\pi + \arctan y/x & x>0, y<0, \\ \pi/2 & x=0, y>0, \\ 3\pi/2 & x=0, y<0. \end{cases}$$

(Here arctan *denotes the inverse of the function* $\tan : (-\pi/2, \pi/2) \to \mathbf{R}$.*) Find P'(x,y). The function P is called the* **polar coordinate system** *on A.*

The formulas for $r(x,y)$ and $\theta(x,y)$ follow from the last paragraph of the solution of part (a). One has $P'(x,y) = \begin{pmatrix} x/\sqrt{x^2+y^2} & y/\sqrt{x^2+y^2} \\ -y/\sqrt{x^2+y^2} & x/\sqrt{x^2+y^2} \end{pmatrix}$. This is trivial from the formulas except in case $x=0$. Clearly, $D_2\theta(0,y) = 0$. Further, L'Hôpital's Rule allows one to calculate $D_1\theta$ when $x=0$ by checking separately for the limit from the left and the limit from the right. For example, $\lim_{x\to 0^+} \frac{\arctan(y/x)-\pi/2}{x} = \lim_{x\to 0^+} \frac{\frac{-y/x^2}{1+(y/x)^2}}{1} = -1/y$.

(c) *Let $C \subset A$ be the region between the circles of radii r_1 and r_2 and the half-lines through 0 which make angles of θ_1 and θ_2 with the x-axis. If $h : C \to \mathbf{R}$ is integrable and $h(x,y) = g(r(x,y), \theta(x,y))$, show that*

$$\int_C h = \int_{r_1}^{r_2} \int_{\theta_1}^{\theta_2} rg(r,\theta)d\theta dr.$$

If $B_r = \{(x,y) : x^2+y^2 \le r^2\}$, show that

$$\int_{B_r} h = \int_0^r \int_0^{2\pi} rg(r,\theta)d\theta dr.$$

Assume that $r_1 \le r_2$ and $\theta_1 \le \theta_2$. Apply Theorem 3-13 to the map $c : [r_1, r_2] \times [\theta_1, \theta_2] \to C$ by $c(r,\theta) = (r\cos(\theta), r\sin(\theta))$. One has $|\det c'| = r$ and $h \circ c = g$. So the first identity holds. The second identity is a special case of the first.

(d) *If* $C_r = [-r, r] \times [-r, r]$, *show that*

$$\int_{B_r} e^{-(x^2+y^2)} dxdy = \pi(1 - e^{-r^2})$$

and

$$\int_{C_r} e^{-(x^2+y^2)} dxdy = \left(\int_{-r}^{r} e^{-x^2} dx \right)^2.$$

For the first assertion, apply part (c) with $g(r, \theta) = e^{-r^2}$. Then $h(x, y) = e^{-(x^2+y^2)}$. Applying (c) gives $\int_{B_r} h = \int_0^r \int_0^{2\pi} re^{-r^2} d\theta dr = \int_0^r 2\pi re^{-r^2} dr = \pi(1 - e^{-r^2})$.

The second assertion follows from Fubini's Theorem.

(e) *Prove that*

$$\lim_{r\to\infty} \int_{B_r} e^{-(x^2+y^2)} dxdy = \lim_{r\to\infty} \int_{C_r} e^{-x^2+y^2)} dxdy$$

and conclude that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

One has $B_r \subset C_r \subset B_{\sqrt{2}r}$ and the integrands are everywhere positive. So $\int_{B_r} e^{-(x^2+y^2)} dxdy \leq \int_{C_r} e^{-(x^2+y^2)} dxdy \leq \int_{B_{\sqrt{2}r}} e^{-(x^2+y^2)} dxdy$. Since part (d) implies that $\lim_{r\to\infty} \int_{B_r} e^{-(x^2+y^2)} dxdy = \lim_{r\to\infty} \pi(1-e^{-r^2}) = \pi$, the squeeze principle implies that $\lim_{r\to\infty} \int_{C_r} e^{-(x^2+y^2)} dxdy = \pi$ also.

But using part (d) again, we get $\lim_{r\to\infty} \int_{-r}^{r} e^{-x^2} dx$ also exists and is $\sqrt{\pi}$ (since the square root function is continuous).

IV

Integration on Chains

ALGEBRAIC PRELIMINARIES

4-1. *Let $e_1, \ldots, e_n$ be the usual basis of $\mathbf{R}^n$ and let $\phi_1, \ldots, \phi_n$ be the dual basis.*

(a) *Show that $\phi_{i_1} \wedge \ldots \wedge \phi_{i_k}(e_{i_1}, \ldots, e_{i_k}) = 1$. What would the right hand side be if the factor $(k+l)!/(k!l!)$ did not appear in the definition of $\wedge$?*

The result is false if the i_j are not distinct; in that case, the value is zero. Assume therefore that the i_j are distinct. One has using Theorem 4-1(3): $\phi_{i_1} \wedge \ldots \wedge \phi_{i_k}(e_{i_1}, \ldots, e_{i_k}) = k!Alt(\phi_{i_1} \otimes \cdots \otimes \phi_{i-1})(e_{i_1}, \ldots, e_{i_k}) = k!(1/k!)\sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\phi_{i_1}(e_{\sigma(i_1)}) \cdot \ldots \cdot \phi_{i_k}(e_{\sigma(i_k)}) = 1$ because all the summands except that corresponding to the identity permutation are zero. If the factor were not in the definition of $\wedge$, then the right hand side would have been $1/k!$.

(b) *Show that $\phi_{i_1} \wedge \cdots \wedge \phi_{i_k}(v_1, \ldots, v_k)$ is the determinant of thee $k \times k$ minor of $\begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}$ obtained by selecting columns $i_1, \ldots, i_k$.*

Assume as in part (a) that the i_s are all distinct.

A computation similar to that of part (a) shows that $\phi_{i_1} \wedge \cdots \wedge \phi_{i_k}(e_{j_1}, \ldots, e_{j_k}) = 0$ if some $j_r \neq i_s$ for all $s = 1, \ldots, k$. By multilinearity, it follows that we need only verify the result when the v_i are in the subspace generated by the e_{i_s} for $s = 1, \ldots, k$.

Consider the linear map $f : \mathbf{R}^k \to \mathbf{R}^n$ defined by $f(e_i) = e_{i_k}$. Then $f^*\phi_{i_s} = \phi_s$. One has for all w_i: $\phi_{i_1} \wedge \cdots \wedge \phi_{i_k}(f(w_1), \ldots, f(w_k)) = f^*(\phi_{i_1} \wedge \cdots \wedge$

$\phi_{i_k})(w_1,\dots,w_k) = f^*\phi_{i_1}\wedge\cdots\wedge f^*\phi_{i_k}(w_1,\dots,w_k) = \phi_1\wedge\cdots\wedge\phi_k(w_1,\dots,w_k) =$ $\det\begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix}$. This shows the result.

4-2. *If $f: V \to V$ is a linear transformation and $\dim V = n$, then $f^*: \Lambda^n(V) \to \Lambda^n(V)$ must be multiplication by some constant c. Show that $c = \det f$.*

Let $A = (f(e_1),\dots,f(e_n))$. Then by Theorem 4-6, one has for $\omega \in \Lambda^n$,

$$\omega(f(e_1),\dots,f(e_n)) = \det A^T\omega(e_1,\dots,e_n).$$

So $c = \det f$.

4-3. *If $\omega \in \Lambda^n(v)$ is the volume element determined by T and μ, and $w_1,\dots,w_n \in V$, show that*

$$|\omega(w_1,\dots,w_n)| = \sqrt{\det(g_{ij}},$$

where $g_{ij} = T(w_iw_j)$.

Let $v_1,\dots,w_n$ be an orthonormal basis for V with respect to T, and let $A = (a_{ij})$ where $w_i = \sum_j a_{ij}v_j$. Then we have by blinearity: $g_{ij} = T(w_i,w_j) = \sum_k a_{ik}a_{jk}$; the right hand sides are just the entries of AA^T and so $\det(g_{ij}) = \det(AA^T) = (\det(A))^2$. By Theorem 4-6, $\omega(w_1,\dots,w_n) = \det A\omega(v_1,\dots,v_n) = \det A$. Taking absolute values and substituting gives the result.

4-4. *If ω is the volume element of V determined by T and μ, and $f: \mathbf{R}^n \to V$ is an isomorphism such that $f^*T = <,>$ and such that $[f(e_1),\dots,f(e_n)] = \mu$, show that $f^*\omega = \det$.*

One has $f^*\omega(e_1,\dots,e_n) = \omega(f(e_1),\dots,f(e_n)) = 1$ by the definition of f^* and the fact that ω is the volume element with respect to T and μ. Further, $f^*\omega = \lambda\det$ for some λ because $\Lambda^n\mathbf{R}^n$ is of dimension 1. Combining, we have

$\lambda \det(e_1, \ldots, e_n) = f^*\omega(e_1, \ldots, e_n) = 1 = \det(e_1, \ldots, e_n)$, and so $\lambda = 1$ as desired.

4-5. *If $c : [0,1] \to \mathbf{R}^{nn}$ is continuous and each $(c_1(t), \ldots, c_n(t))$ is a basis for $\mathbf{R}^n$, show that $[c_1(0), \ldots, c_n(0)] = [c_1(1), \ldots, c_n(1)]$.*

The function $f = \det \circ c : [0,1] \to \mathbf{R}$ is a continuous function, whose image does not contain 0 since $c(t)$ is a basis for every t. By the intermediate value theorem, it follows that the image of f consists of numbers all of the same sign. So all the $c(t)$ have the same orientation.

4-6. (a) *If $v \in \mathbf{R}^2$, what is $v\times$?* $v\times$ is the cross product of a single vector, i.e. it is the vector z such that $< w, z >= \det \begin{pmatrix} v \\ w \end{pmatrix}$ for every w. Substitution shows that $z = (-v^2, v^1)$ works.

(b) *If $v_1, \ldots, v_{n-1} \in \mathbf{R}^n$ are linearly independent, show that $[v_1, \ldots, v_{n-1}, v_1 \times \cdots \times v_n]$is the usual orientation of $\mathbf{R}^n$.*

By the definition, we have $\det \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ v_1 \times \cdots \times v_{n-1} \end{pmatrix} =< v_1 \times \cdots \times v_{n-1}, v_1 \times \cdots \times v_{n-1} >\geq 0$. Since the v_i are linearly independent, the definition of cross product with w completing the basis shows that the cross product is not zero. So in fact, the determinant is positive, which shows the result.

4-7. *Show that every non-zero $\omega \in \Lambda^n(V)$ is the volume element determined by some inner product T and orientation μ for V.*

Let ω_0 be the volume element determined by some inner product T_0 and orientation μ_0, and let $v_1, \ldots, v_n$ be an orthornormal basis (with respect to T_0) such that $[v_1, \ldots, v_n] = \mu_0$. There is a scalar $\lambda \neq 0$ such that $\omega = \lambda\omega_0$. Let

$\mu = \text{sgn}(\lambda)\mu_0$, $c = |\lambda|^{1/n}$, $T = c^2 T_0$, and $w_i = v_i/c$ for $i = 1, \ldots, n$. Then $w_1, \ldots, w_n$ are an orthonormal basis of V with respect to T, and $\omega(w_1, \ldots, w_n) = \lambda\omega_0(v_1/c, \ldots, v_n/c) = (\lambda/|\lambda|)\omega_0(v_1, \ldots, v_n) = \lambda/|\lambda|$. This shows that ω is the volume element of V determined by T and μ.

4-8. *If $\omega \in \Lambda^n(V)$ is a volume element, define a "cross product" $v_1 \times \cdots \times v_n$ in terms of ω.*

The cross product is the $z \in V$ such that $T(w, z) = \omega(v_1, \ldots, v_{n-1}, w)$ for all $w \in V$.

4-9. *Deduce the following properties of the cross product in $\mathbf{R}^3$:*

(a) $e_1 \times e_1 = 0$ $\quad e_2 \times e_1 = -e_3$ $\quad e_3 \times e_1 = e_2$

$e_1 \times e_2 = e_3$ $\quad e_2 \times e_2 = 0$ $\quad e_3 \times e_2 = -e_1$

$e_1 \times e_3 = -e_2$ $\quad e_2 \times e_3 = e_1$ $\quad e_3 \times e_3 = 0.$

All of these follow immediately from the definition, e.g. To show that $e_1 \times e_3 = -e_2$, note that $< w, -e2 >= -w^2 = \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ w^1 & w^2 & w^3 \end{pmatrix}$ for all w.

(b) $v \times w = (v^2w^3 - v^3w^2)e_1 + (v^3w^1 - v^1w^3)e_2 + (v^1w^2 - v^2w^1)e_3.$

Expanding out the determinant shows that: $\det \begin{pmatrix} v^1 & v^2 & v^3 \\ w^1 & w^2 & w^3 \\ x^1 & x^2 & x^3 \end{pmatrix} = (v^2w^3 - v^3w^2)x_1 + (v^3w^1 - v^1w^3)x_2 + (v^1w^2 - v^2w^1)x_3 =< x, v^2w^3 - v^3w^2)e_1 + (v^3w^1 - v^1w^3)e_2 + (v^1w^2 - v^2w^1)e_3 >.$

(c) *$|v \times w| = |v| \cdot |w| \cdot |\sin\theta|$, where $\theta = \angle(v, w)$, and $< v \times w, v >=< v \times w, w >= 0$.*

The result is true if either v or w is zero. Suppose that v and w are both non-zero. By Problem 1-8, $\angle(v, w) = \arccos(< v, w > /(|v||w|))$ and since

$\sin(\arccos(\alpha)) = \sqrt{1-\alpha^2}$, the first identity is just $< v, w >^2 + |v \times w|^2 = |v|^2|w|^2$. This is easily verified by substitution using part (b).

The second assertion follows from the definition since the determinant of a square matrix with two identical rows is zero.

(d) $< v, w \times z > = < w, z \times v > = < z, v \times w >$
$v \times (w \times z) = < v, z > w - < v, w > z$
$(v \times w) \times z = < v, z > w - < w, z > v.$

For the first assertion, one has $< v, w \times z > = \det \begin{pmatrix} w \\ z \\ v \end{pmatrix} = -\det \begin{pmatrix} v \\ w \\ z \end{pmatrix} = \det \begin{pmatrix} z \\ v \\ w \end{pmatrix} = < w, z \times v >$ and $< z, v \times w > = \det \begin{pmatrix} v \\ w \\ z \end{pmatrix} = -\det \begin{pmatrix} w \\ v \\ z \end{pmatrix} = \det \begin{pmatrix} w \\ z \\ v \end{pmatrix} = < v, w \times z >$.

For the second assertion, one has: $< x, v \times (w \times z) > = \det \begin{pmatrix} v \\ w \times z \\ x \end{pmatrix} = -\det \begin{pmatrix} v \\ x \\ w \times z \end{pmatrix} = - < w \times z, v \times x > = < w \times z, x \times v >$. So, one needs to show that $< v, z >< x, w > - < v, w >< x, z > = < w \times z, x \times v >$ for all z. But this can be easily verified by expanding everything out using the formula in part (b).

The third assertion follows from the second: $(v \times w) \times z = -z \times (v \times w) =$

$-(< z, w > v- < z, v > w) =< v, z > w- < w, z > v.$

(e) $|v \times w| = \sqrt{(v, v) \cdot (w, w) - (v, w)^2}$.

See the proof of part (c).

4-10. *If* $w_1, \ldots, w_{n-} \in \mathbf{R}^n$, *show that*

$$|w_1 \times \cdots \times w_{n-1}| = \sqrt{\det(g_{ij})},$$

where $g_{ij} =< w_i, w_j >$.

Using the definition of cross product and Problem 4-3, one has: $|w_1 \times \cdots \times w_{n-1}|^2 = \det \begin{pmatrix} w_1 \\ \vdots \\ w_{n-1} \\ w_1 \times \cdots \times w_{n-1} \end{pmatrix} = \sqrt{\det(g_{ij})|w_1 \times \cdots \times w_{n-1}|^2}$ since the matrix from Problem 4-3 has the form $\begin{pmatrix} (g_{ij}) & 0 \\ 0 & |w_1 \times \cdots \times w_{n-1}|^2 \end{pmatrix}$. This proves the result in the case where $w_1 \times \cdots \times w_{n-1}$ is not zero. When it is zero, the w_i are linearly dependent, and the bilinearity of inner product imply that $\det(g_{ij}) = 0$ too.

4-11. *If T is an inner product on V, a linear transformation $f : V \to V$ is called* **self-adjoint** *(with respect to T) if $T(x, f(y) = T(f(x), y)$ for all $x, y \in V$. If $v_1, \ldots, v_n$ is an orthogonal basis and $A = (a_{ij})$ is the matrix of f with respect to this basis, show that $a_{ij} = a_{ji}$.*

One has $f(v_j) = \sum_k a_{kj} v_i$ for each j. Using the orthonormality of the basis, one has: $T(v_i, f(v_j) = T(v_i, \sum_k a_{kj} v_k) = a_{ij}$ But $T(v_i, f(v_j)) = T(f(v_i), v_j) = T(\sum_k a_{ki} v_k, v_j) = a_{ji}$, which shows the result.

4-12. *If $f_1, \ldots, f_{n-1} : \mathbf{R}^m \to \mathbf{R}^n$, define $f_1 \times \cdots \times f_{n-1} : \mathbf{R}^m \to \mathbf{R}^n$ by $f_1 \times \cdots \times f_{n-1}(p) = f_1(p) \times \cdots \times f_{n-1}(p)$. Use Problem 2-14 to derive a formula for $D(f_1 \times \cdots \times f_{n-1})$ when $f_1, \ldots, f_{n-1}$ are differentiable.*

Since the cross product is multilinear, one can apply Theorem 2-14 (b) and the chain rule to get: $D(f_1 \times \cdots \times f_{n-1})(a) = \sum_i f_1(a) \times \cdots \times f_{i-1}(a) \times Df_i(a) \times Fi + 1(a) \times \cdots \times f_{n-1}(a)$.

4-13. (a) *If $f : \mathbf{R}^n \to \mathbf{R}^m$ and $g : \mathbf{R}^m \to \mathbf{R}^p$, show that $(g \circ f)_* = g_* \circ f_*$ and $(g \circ f)^* = f^* \circ g^*$.*

The notation does not fully elucidate the meaning of the assertion. Here is the interpretation: $(g \circ f)_*(v_p) = (D(g \circ f)(p)(v))_{g(f(p))} = (Dg(f(p)) \circ Df(p)(v))_{g(f(p))} = g_*(f(p))(f_*(p)(v)) = ((g_*(f(p)) \circ (f_*(p)))(v)$.

The second assertion follows from:

$$\begin{aligned}(g \circ f)^*\omega(p)(v_1, \ldots, v_k) &= (g \circ f)^*(\omega(g(f(p)))(v_1, \ldots, v_k) \\ &= \omega(g(f(p))(g_*(f(p))(f_*(p)(v_1), \ldots, g_*(f(p))(f_*(p)(v_k)) \\ &= (g^*\omega)(f(p)(f_*(p)(v_1), \ldots, f_*(p)(v_k)) \\ &= (f^* \circ g^*)\omega(p)(v_1, \ldots, v_k).\end{aligned}$$

(b) *If $f, g : \mathbf{R}^n \to \mathbf{R}$, show that $d(f \cdot g) = f \cdot dg + g \cdot df$.*

One has by the definition and the product rule: $d(f \cdot g)(p)(v_p) = D(f \cdot g)(p)(v) = f(p)Dg(p)(v) + g(p)Df(p)(v) = f(p)dg(p)(v_p) + g(p)df(p)(v_p)$.

4-14. *Let c be a differentiable curve in $\mathbf{R}^n$, that is, a differentiable function $c : [0, 1] \to \mathbf{R}^n$. Define the* **tangent vector** *v of c at t as $c_*((e_1)_t) = ((c^1)'(t), \ldots, (c^n)'(t))_{c(t)}$. If $f : \mathbf{R}^n \to \mathbf{R}^m$, show that the tangent vector to $f \circ c$ at t is $f_*(v)$.*

This is an immediate consequence of Problem 4-13 (a).

4-15. *Let $f : \mathbf{R} \to \mathbf{R}$ and define $c : \mathbf{R} \to \mathbf{R}^2$ by $c(t) = (t, f(t))$. Show that the end point of the tangent vector of c at t lies on the tangent line to the graph of f at $(f, f(t))$.*

The tangent vector of c at t is $v = (1, f'(t))$. The end point of the tangent vector of c at t is $(t+1, f(t)+f'(t))$ which is certainly on the tangent line $y - f(t) = f'(t)(x-t)$ to the graph of f at $(t, f(t))$.

4-16. *Let $d : [0,1] \to \mathbf{R}^n$ be a curve such that $|c(t)| = 1$ for all t. Show that $c(t)_{c(t)}$ and the tangent vector to c at t are perpendicular.*

Differentiating $\sum_i c^i(t)^2 = 1$, gives $\sum_i 2c^i(t)c'(t) = 0$, i.e. $< c(t), v >= 0$ where v is the tangent vector to c at t.

4-17. *If $f : \mathbf{R}^n \to \mathbf{R}$, define a vector field $\mathbf{f}$ by $\mathbf{f}(p) = f(p)_p \in \mathbf{R}^n_p$.*

(a) *Show that every vector field f on $\mathbf{R}^n$ is of the form $\mathbf{f}$ for some f.*

A vector field is just a function F which assigns to each $x \in \mathbf{R}$ an element $F(x) \in \mathbf{R}^n_p = \{(p, y) : y \in \mathbf{R}^n\}$. Given such an F, define $f : \mathbf{R}^n \to \mathbf{R}^n$ by $f(x) = \pi^2(F(x))$. Then $\mathbf{f} = F$.

(b) *Show that* div $\mathbf{f} = \text{trace} f'$.

One has div $\mathbf{f} = \sum_i D_i f^i = \text{trace}(D_i f^j) = \text{trce} f'$.

4-18. *If $f : \mathbf{R}^n \to \mathbf{R}$, define a vector field* grad f *by*

$$(\text{grad } f)(p) = D_1 f(p) \cdot (e_1)_p + \cdots + D_n f(p) \cdot (e_n)_p.$$

For obvious reasons we also write grad $f = \nabla f$. *If $\nabla f(p) = w_p$, prove that $D_v f(p) =< v, w >$ and conclude that $\nabla f(p)$ is the direction in which f is changing fastest at p.*

By Problem 2-29, $< v, w >=< v, D_1 f(p)e_1 + \cdots + D_n f(p)e_n >= \sum_i v^i D_i f(p) = D_v f(p)$. The direction in which f is changing fastest is the direction given by a unit vector v such thatt $D_v f(p) =< v, w >$ is largest possible. Since $< v, w >= |v||w| \cos(\theta)$ where $\theta = \angle(v, w)$, this is clearly when $\theta = 0$, i.e. in the direction of w.

4-19. *If f is a vector field on $\mathbf{R}^3$, define the forms*

$$\omega_F^1 = F^1 dx + F^2 dy + F^3 dz, \quad (1)$$

$$\omega_F^2 = F^1 dy \wedge dz + F^2 dz \wedge dx + F^3 dx \wedge dy. \quad (2)$$

(a) *Prove that*

$$df = \omega^1_{\text{grad } f}, \quad (3)$$

$$d(\omega_F^1) = \omega^2_{\text{curl } F}, \quad (4)$$

$$d(\omega_F^2) = (\text{div } F) dx \wedge dy \wedge dz. \quad (5)$$

The first equation is just Theorem 4-7.

For the second equation, one has: $d(\omega_F^1) = dF^1 \wedge dx + dF^2 \wedge dy + dF^3 \wedge dz = D_2F^1 dy \wedge dx + D_3F^1 dz \wedge dx + D_1F^2 dx \wedge dy + D_3F^2 dz \wedge dy + D_1F^3 dx \wedge dz + D_2F^3 dy \wedge dz = (D_2F^3 - D_3F^2) dy \wedge dz + (D_3F^1 - D_1F^3) dz \wedge dx + (D_1F^2 - D_2F^1) dy \wedge dz = \omega^2_{\text{curl } f}$.

For the third assertion: $d(\omega_F^2) = D_1F^1 dx \wedge dy \wedge dz + D_2F^2 dy \wedge dz \wedge dx + D_3F^3 dz \wedge dx \wedge dy = (\text{div } F) dx \wedge dy \wedge dz$.

(b) *Use (a) to prove that*

$$\text{curl grad } f = 0, \quad (6)$$

$$\text{div curl } F = 0. \quad (7)$$

One has $\omega^2_{\text{curl grad } f} = d(w^1_{\text{grad}}\ f) = d(df) = 0$ by part (a) and Theorem 4-10 (3); so curl grad $f = 0$.

Also, $(\text{div curl } F) dx \wedge dy \wedge dz = d(\omega^2_{\text{curl } F}) = d(d(\omega_F^1)) = 0$ by part (a) and Theorem 4-10 (3); so the second assertion is also true.

(c) *If F is a vector field on a star-shaped open set A and* curl $F = 0$, *show that* $F = \text{grad } f$ *for some function* $f : A \to \mathbf{R}$. *Similarly, if* div $F = 0$, *show that* $F = \text{curl } G$ *for some vector field G on A.*

By part (a), if curl $F = 0$, then $d(\omega_F^1) = \omega_{\text{curl } F}^2 = 0$. By the Theorem 4-11, ω_F^1 is exact, i.e. $\omega_F^1 = df = \omega_{\text{grad } f}^1$. So $F = \text{grad } f$.

Similarly, if div $F = 0$, then $d(\omega_F^2) = (\text{div } F) dx \wedge dy \wedge dz = 0$ and so ω_F^2 is closed. By Theorem 4-11, it must then be exact, i.e. $\omega_F^2 = d(\omega_G^1) = \omega_{\text{curl } G}^2$ for some G. So $F = \text{curl } G$ as desired.

4-20. *Let $f : U \to \mathbf{R}^n$ be a differentiable function with a differentiable inverse $f^{-1} : f(U) \to \mathbf{R}^n$. If every closed form on U is exact, show that the same is true of $f(U)$.*

Suppose that the form ω on $f(U)$ is closed, i.e. $d\omega = 0$. Then $df^*\omega = f^*d\omega = f^*0 = 0$ and so there is a form ω' on U such that $f^*\omega = d\omega'$. But then $\omega = (f^{-1})^* f^*\omega = (f^{-1})^* d\omega' = d((f^{-1})^*\omega')$ and so ω is also exact, as desired.

4-21. *Prove that on the set where θ is defined, we have*

$$d\theta = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy.$$

Except when $x = 0$, the assertion is immediate from the definition of θ in Problem 2-41. In case $x = 0$, one has trivially $D_2\theta(0, y) = 0$ because θ is constant when $x = 0$ and $y > 0$ (or $y < 0$). Further, L'Hôpital's Rule allows one to calculate $D_1\theta$ when $x = 0$ by checking separately for the limit from the left and the limit from the right. For example, $\lim_{x \to 0^+} \frac{\arctan(y/x) - \pi/2}{x} = \lim_{x \to 0^+} \frac{\frac{-y/x^2}{1+(y/x)^2}}{1} = -1/y$.

4-22. *Let $\mathcal{S}$ be the set of all singular n-cubes, and $\mathbf{Z}$ the integers. An n-**chain** is a function $f : \mathcal{S} \to \mathbf{Z}$ such that $f(c) = 0$ for all but finitely many c. Define $f + g$ and nf by $(f + g)(c) = f(c) + g(c)$ and $nf(c) = n \cdot f(c)$. Show that $f + g$ and nf are n-chains if f and g are. If $c \in \mathcal{S}$, let c also denote the function f such that $f(c) = 1$ and $f(c') = 0$ for $c' \neq c$. Show that every n-chain f can be written $a_1c_1 + \cdots + a_kc_k$ for some integers $a_1, \ldots, a_k$ and singular n-cubes $c_1, \ldots, c_k$.*

Since $\{c : (f + g)(c) \neq 0\} \subset \{c : f(c) \neq 0\} \cup \{c : g(c) \neq 0\}$ and $\{c : (nf)(c) \neq 0\} \subset \{c : f(c) \neq 0\}$, the functions $f + g$ and nf are n-chains if f and g are.

The second assertion is obvious since $f = \sum_{c\in\{c:f(c)\neq 0\}} f(c)c$.

4-23. *For $R > 0$ and n an integer, define the singular 1-cube $c_{R,n} : [0, 1] \to \mathbf{R}^2 - 0$ by $c_{R,n}(t) = (R \cos 2\pi nt, R \sin 2\pi nt)$. Show that there is a singular 2-cube $c : [0, 1]^2 \to \mathbf{R}^2 - 0$ such that $c_{R_1,n} - c_{R_2,n} = \partial c$.*

Define $c : [0, 1]^2 \to \mathbf{R}^2 - 0$ by $c(t_1, t_2) = (t_1R_1 + (1 - t_1)R_2)(\cos 2\pi nt_2, \sin 2\pi nt_2)$ where R_1 and R_2 are positive real numbers. The boundary of c is easily seen to be $c_{R_1,n} - c_{R_2,n}$.

4-24. *If c is a singular 1-cube in $\mathbf{R}^2 - 0$ with $c(0) = c(1)$, show that there is an integer n such that $c - c_{1,n} = \partial c^2$ for some 2-chain c^2.*

Given c, let $\theta_0 = \theta(c(0))$ where θ is the function of Problem 3-41 extended so that it is 0 on the positive x-axis. Let $L = \int_0^1 c^*d\theta$ so that $n = L/(2\pi)$ is an integer because $c(0) = c(1)$. Define $c^2(s, t) = c_{t|c(s)|+(1-t),1}((t(\theta_0 + \int_0^s c^*d\theta) + (1 -$

$t)sL)/(2\pi))$. One has $c^2(s,0) = c_{1,n}(s)$ and $c^2(s,1) = c_{|c(s)|,1}((\theta_0 + \int_0^1 c^* d\theta)/(2\pi)) = c(s)$. On the other boundaries, $c^2(0,t) = c_{t|c(0)|+1-t}((t\theta_0)/(2\pi))$ and $c^2(1,t) = c_{t|c(1)|+1-t,1}((t(\theta_0 + L) + (1-t)L)/(2\pi)) = c(0,t)$. So $\partial c^2 = c - c_{1,n}$, as desired.

THE FUNDAMENTAL THEOREM OF CALCULUS

4-25. Independence of parametrization). *Let c be a singular k-cube and $p : [0,1]^k \to [0,1]^k$ a 1-1 function such that $p([0,1]^k) = [0,1]^k$ and $\det p'(x) \geq 0$ for $x \in [0,1]^k$. If ω is a k-form, show that*

$$\int_c \omega = \int_{c \circ p} \omega.$$

Suppose $c^*\omega = f dx^1 \wedge \cdots \wedge dx^k$. Using the definition of the integral, Theorem 4-9, the chain rule, and Theorem 3-13 augmented by Problem 3-39: $\int_{c \circ p} \omega = \int_{[0,1]^k} (c \circ p)^*\omega = \int_{[0,1]^k} p^*(c^*\omega) = \int_{[0,1]^k} f(p) detp' dx^1 \wedge \cdots \wedge dx^k = \int_{[0,1]^k} f(p)|detp'| dx^1 \ldots dx^k = \int_{[0,1]^k} f dx^1 \ldots dx^k = \int_c \omega$

4-26. *Show that $\int_{c_{R,n}} d\theta = 2\pi n$, and use Stokes Theorem to conclude that $c_{R,n} \neq \partial c$ for any 2-chain c in $\mathbf{R}^2 - 0$ (recall the definition of $c_{R,n}$ in Problem 4-23).*

One has $\int_{c_{R,n}} d\theta = \int_{c_{R,n}} \frac{-ydx}{x^2+y^2} + \frac{xdy}{x^2+y^2} = \int_0^{2\pi} 2\pi n dt = 2\pi n$. If $c_{R,n} = \partial c$, then Stokes Theorem gives $2\pi n == \int_{c_{R,n}} d\theta = \int_{\partial c} d\theta = \int_c d(d\theta) = \int_c 0 = 0$, because $d\theta$ is closed. So $n = 0$.

Note however that no curve is the boundary of any two chain – as the sum of the coefficients of a boundary is always 0.

4-27. *Show that the integer n of Problem 4-24 is unique. This integer is called the* **winding number** *of c around 0.*

If $c - c_{1,n} = \partial c_1$ and $c - c_{1,m} = \partial c_2$ where $n \neq m$ and c_1 and c_2 are 2-chains, the letting $c = c_2 - c_1$, one has $c_{1,n} - c_{1,m} = \partial c$. Using Stokes Theorem, one gets

$2\pi(n-m) = \int_{c_{1,n}-c1,m} d\theta = \int_{\partial c} d\theta = \int_c d(d\theta) = \int_c 0 = 0$, which is a contradiction.

4-28. *Recall that the set of complex numbers* $\mathbf{C}$ *is simply* $\mathbf{R}^2$ *with* $(a,b) = a+bi$. *If* $a_1, \dots, a_n \in \mathbf{C}$ *let* $f : \mathbf{C} \to \mathbf{C}$ *be* $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$. *Define thee singular 1-cube* $c_{R,f} : [0,1] \to \mathbf{C} - 0$ *by* $c_{R,f} = f \circ c_{R,1}$, *and the singular 2-cube* c *by* $c(s,t) = t \cdot c_{R,n}(s) + (1-t)c_{R,f}(s)$.

(a) *Show that* $\partial c = c_{R,f} - c_{R,n}$, *and that* $c([0,1] \times [0,1]) \subset \mathbf{C} - 0$ *if* R *is large enough.*

The problem statement is flawed: the author wants c to be defined to be $c(s,t) = t \cdot c_{R^n,n} + (1-t)c_{R,f}$. This would make the boundary $c_{R,f} - c_{R^n,n}$. We assume these changes have been made.

When $s = 0$ or $s = 1$, c is the curve $t + (1-t)f(1)$. When $t = 0$, it is the curve $c_{R,f}(s)$, and when $t = 1$, it is the curve $c_{R^n,n}(s)$. So $\partial c = c_{R,f} - c_{R^n,n}$. Let $g(z) = f(z) - z^n$. Then, if $R > \max(n \max_i |a_i|, 1)$, we have $|(1-t)g(z)| \le n(\max_i |a_i|)R^{n-1} < R^n$ for all $t \in [0,1]$ and all z with $|z| = R$. Since $c(s,t) = z^n + (1-t)g(z)$ where $z = c_{R,1}(t)$, we see that $c(s,t)$ cannot be zero since it is the sum of a number of length R^n and one which is smaller in absolute value.

(b) *Using Problem 4-26, prove the* Fundamental Theorem of Algebra*: Every polynomial* $z^n + a_1 z^{n-1} + \cdots + a_n$ *with* $a_i \in \mathbf{C}$ *has a root in* $\mathbf{C}$.

Suppose that f as above has no complex root. Letting R be sufficiently large, we see by part (a) and Stokes' Theorem that $\int_{c_{R,f}-c_{R^n,n}} d\theta = \int_{\partial c} d\theta = \int_c d(d\theta) = \int_c 0 = 0$, and so $\int_{R,f} d\theta = 2\pi n$.

Now consider the 2-chain c' defined by

$$c'(s,t) = c_{tR,f}(s) = f(tR\cos(2\pi s), tR\sin(2\pi s)).$$

Now, when $t = 0$, we get the constant curve with value a_n; when $t = 1$, we get the curve $c_{R,f}$; and when $s = 0$ or $s = 1$, we get the curve $f(tR)$.

So the boundary of c' is $a_n - c_{R,f}$. Further, we have assumed that f has no complex root, and so c' is a 2-chain with values in $\mathbf{C} - 0$. Again, applying Stokes' Theorem, we get $\int_{a_n - c_{R,f}} d\theta = \int_{\partial c'} d\theta = \int_c d(d\theta) = \int_c 0 = 0$, and so $\int_{c_{R,f}} d\theta = \int_{a_n} d\theta = 0$. This contradicts the result of the last paragraph.

4-29. *If ω is a 1-form $f dx$ on $[0,1]$ with $f(0) = f(1)$, show that theere is a unique number λ such that $\omega - \lambda dx = dg$ for some function g with $g(0) = g(1)$.*

Following the hint, $\int_0^1 f dx - \lambda dx = \int_0^1 dg = g(1) - g(0) = 0$ implies $\lambda = \int_0^1 f dx$ and so λ is unique. On the other hand, if we let λ be this value and $g(t) = \int_0^t f - \lambda dt$, then $dg = f dx - \lambda dx$ and $g(0) = g(1) = 0$.

4-30. *If ω is a 1-form on $\mathbf{R}^2 - 0$ such that $d\omega = 0$, prove that*

$$\omega = \lambda d\theta + dg$$

for some $\lambda \in \mathbf{R}$ and $g : \mathbf{R}^2 - 0 \to \mathbf{R}$.

The differential $c_{R,1}^* \omega$ is of the type considered in the last problem. So there is a unique λ_R for which there is a g_R such that $c_{R,1}^* \omega = \lambda_R (c_{R,1}^* d\theta) + dg_R$.

For positive R_1 and R_2, define the singular 2-cube $c : [0,1]^2 \to \mathbf{R}^2$ by $c(s,t) = c_{tR_1 + (1-t)R_2, 1}(s)$. By Stokes' Theorem, we have $\int_{c_{R_2,1} - c_{R_1,1}} \omega = \int_{\partial c} \omega = \int_c d\omega = \int_c 0 = 0$. So $\int_{c_{R_1,1}} \omega = \int_{c_{R_2,1}} \omega$. By the proof of the last problem, it follows that $\lambda_{R_1} = \lambda_{R_2}$. Henceforth, let λ denote this common value. Note that $\int_{c_{R,1}} \omega - \lambda d\theta = 0$; and in particular, $\int_{c_{R,n}} \omega - \lambda d\theta = n \int_{c_{R,1}} \omega - \lambda d\theta = 0$.

Let c be a singular 1-cube with $c(0) = c(1) = (R, 0)$. By Problem 4-24, there is a 2-chain c' and an n such that $c - c_{R,n} = \partial c'$. By Stokes' Theorem, $\int_{c - c_{R,n}} \omega - \lambda d\theta = \int_{\partial c'} \omega - \lambda d\theta = \int_{c'} d(\omega - \lambda d\theta) = \int_{c'} 0 = 0$. So $\int_c \omega - \lambda d\theta = \int_{c_{R,n}} \omega - \lambda d\theta = 0$.

From the result of the last paragraph, integrating $\omega - \lambda d\theta$ is independent of path. In fact, if you have two singular 1-cubes c_1 and c_2 with $c_1(0) = c_2(0)$ and $c_1(1) = c_2(1)$, then prepend a curve from (1,0) to $c_1(0)$ and postpend a path from $c_1(1)$ to (1,0) to get two paths as in the last paragraph. The two integrals are both 0, and so the integrals over c_1 and c_2 are equal.

Now the result follows from Problem 4-32 below.

4-31. *If $\omega \neq 0$, show that there is a chain c such that $\int_c \omega \neq 0$. Use this fact, Stokes' theorem and $\partial^2 = 0$ to prove $d^2 = 0$.*

One has $\omega = \sum_{i_1<\cdots<i_k} f_{i_1,\ldots,i_k}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. Suppose $f_{i_1,\ldots,i_k}(x_0) \neq 0$ for some x_0 and some choice of $i_1, ldots, i_k$. Then $f(x) \neq 0$ in a closed rectangle of positive volume centered at x_0. Take for c the k-cube defined in an obvious way so that its image is the part of the closed rectangle with $x^j = x_0^j$ for all j different from the i_s for $s = 1, ldots, k$. Then $\int_c \omega = \int_c f_{i_1,\ldots,i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k} = \int_{[0,1]^k} c^* f_{i_1,\ldots,i_k} dx^{i_1} \cdots dx^{i_k} \neq 0$ since the integrand is continuous and of the same sign throughout the region of integration.

Suppose $d^2\omega \neq 0$. Let c be a chain such that $\int_c d^2\omega \neq 0$. By Stokes' Theorem, we would have: $0 \neq \int_c d(d\omega) = \int_{\partial c} d\omega = \int_{\partial\partial c} \omega = 0$ because $\partial(\partial(\omega)) = 0$. This is a contradiction.

4-32. (a) *Let c_1, c_2 be singular 1-cubes in $\mathbf{R}^2$ with $c_1(0) = c_2(0)$ and $c_1(1) = c_2(1)$. Show that there is a singular 2-cube c such that $\partial c = c_1 - c_2 + c_3 - c_4$, where c_3 and c_4 are* degenerate*, that is, $c_3([0,1])$ and $c_4([0,1])$ are points. Conclude that $\int_{c_1} \omega = \int_{c_2} \omega$ if ω is exact. Give a counter-example on $\mathbf{R}^2 - 0$ if ω is merely closed.*

Let $c : [0,1]^2 \to \mathbf{R}^2$ be defined by $c(s,t) = tc_2(s) + (1-t)c_1(s)$. Then $\partial c = c_1 - c_2 + c_1(1) - c_1(0)$ where $c_1(1)$ is the curve with constant value $c_1(1)$ and similarly for $c_1(0)$.

Suppose ω is exact, and hence closed. Then by Stokes' Theorem, we have $\int_{c_1-c_2} \omega = \int_{\partial c} \omega = \int_c d\omega = \int_c 0 = 0$ (since ω is closed), and so $\int_{c_1} \omega = \int_{c_2} \omega$.

The example: $\omega = d\theta$, $c_1 = c_{R,1}$, and $c_2 = c_{R,-1}$ shows that there is no independence of path in $\mathbf{R}^2 - 0$ for closed forms.

(b) *If ω is a 1-form on a subset of $\mathbf{R}^2$ and $\int_{c_1} \omega = \int_{c_2} \omega$ for all c_1 and c_2 with $c_1(0) = c_2(0)$ and $c_1(1) = c_2(1)$, show that ω is exact.*

Although it is not stated, we assume that the subset is open. Further, by treating each component separately, we assume that the subset is pathwise connected.

Fix a point x_0 in the subset. For every x in the set, let c be any curve from x_0 to x, and set $g(x) = \int_c \omega$. Because of independence of path, g is well defined. Now, if $\omega = Pdx^1 + Qdx^2$, then because x is in the interior of the subset, we can assume that $g(x)$ is calculated with a path that ends in a segment with x^2 constant. Clearly, then $D_1 g = P$. Similarly, $D_2 g = Q$. Note that because P and Q are continuously differentible, it follows that ω is closed since $D_2 P - D_1 Q = D_{1,2} g - D_{2,1} g = 0$.

We want to check differentiability of g. One has $g(x+h, y+k) - g(x, y) - P(x, y)h - Q(x, y)k = (\int_x^{x+h} P(t, y)dt - P(x, y)h) + (\int_y^{y+k} Q(x, t)dt - Q(x, y)k) + \int_y^{y+k} (Q(x+h, t) - Q(x, t))dt$. The first pair of terms is $o(h)$ because $P = D_1 g$; similarly the second pair of terms is $o(k)$. Finally, continuity of Q implies that the integrand is $o(1)$, and so the last integral is also $o(|(h, k)|)$. So g is differentiable at g. This establishes the assertion.

4-33. (A first course in complex variables.) *If $f : \mathbf{C} \to \mathbf{C}$, define f to be* **differentiable** *at $z_0 \in \mathbf{C}$ if the limit*

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. (This quotient involves two complex numbers and this definition is completely different from the one in Chapter 2.) If f is differeentiable at every point z in an open set A and f' is continuous on a, then f is called **analytic** *on A.*

(a) *Show that $f(z) = z$ is analytic and $f(z) = \bar{z}$ is not (where $\overline{x+iy} = x - iy$). Show that the sum, product, and quotient of analytic functions are analytic.*

$\lim_{z\to z_0} \frac{z-z_0}{z-z_0} = \lim_{z\to z_0} 1 = 1$ and so $z' = 1$. On the other hand, $\frac{\bar{z}-\bar{z}_0}{z-z_0} = \frac{|z-z_0|^2}{(z-z_0)^2}$ does not have a limit as $z \to z_0$ because $\lim_{x\to 0} |x|^2/x^2 = 1$, but $\lim_{y\to 0} |yi|^2/(yi)^2 = -1$.

It is straightforward to check that the complex addition, subtraction, multiplication, and division operations are continuous (except when the quotient is zero). The assertion that being analytic is preserved under these operations as well as the formulas for the derivatives are then obvious, if you use the identities:

$$\frac{(f+g)(z) - (f+g)(z_0)}{z - z_0} = \frac{f(z) - f(z_0}{z - z_0} + \frac{g(z) - g(z_0)}{z - z_0} \tag{8}$$

$$\frac{(fg)(z) - (fg)(z_0)}{z - z_0} = g(z)\frac{f(z) - f(z_0)}{z - z_0} + f(z_0)\frac{g(z) - g(z_0)}{z - z_0} \tag{9}$$

$$\frac{1/f(z) - 1/f(z_0)}{z - z_0} = \frac{1}{f(z)f(z_0)}\frac{f(z) - f(z_0}{z - z_0} \tag{10}$$

(b) *If $f = u+iv$ is analytic on A, show that u and v satisfy the* Cauchy-Riemann *equations:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

(The converse is also true, if u and v are continuously differentiable; this is more difficult to prove.)

Following the hint, we must have:

$$\begin{aligned}\lim_{x\to 0} \frac{u(z_0+x) - u(z_0)}{x} + i\frac{v(z_0+x) - v(z_0)}{x} &= \lim_{x\to 0} \frac{f(z_0+x) - f(z_0)}{x} \\ &= \lim_{y\to 0} \frac{f(z_0+iy) - f(z_0)}{iy} \\ &= \lim_{y\to 0}(\frac{u(z_0+iy) - u(z_0)}{iy} \\ &+ i\frac{v(z_0+iy) - v(z_0)}{iy})\end{aligned}$$

Comparing the real and imaginary parts gives the Cauchy-Riemann equations.

(c) *Let $T : \mathbf{C} \to \mathbf{C}$ be a linear transformation (where $\mathbf{C}$ is considered as a vector space over $\mathbf{R}$). If the matrix of T with respect to the basis $(1, i)$ is $\begin{pmatrix} a, b \\ c, d \end{pmatrix}$, show that T is multiplication by a complex number if and only if $a = d$ and $b = -c$. Part (b) shows that an analytic function $f : \mathbf{C} \to \mathbf{C}$, considered as a function $f : \mathbf{R} \to \mathbf{R}$, has a derivative $Df(z_0)$ which is multiplication by a complex number. What complex number is this?*

Comparing $T(x + iy) = ax + by + (cx + dy)i$ and $(r + si)(x + iy) = rx - sy + (ry + sx)i$ gives $a = r$, $b = -s$, $c = r$, and $d = s$. So, r and s exist if and only if $a = c$ and $b = -d$.

From the last paragraph, the complex number is $r + is$ where $r = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x}$ and $s = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y}$.

(d) *Define*

$$
\begin{aligned}
d(\omega + i\eta) &= d\omega + id\eta, && (11)\\
\int_c \omega + i\eta &= \int_c \omega + i \int_c \eta, && (12)\\
(\omega + i\eta) \wedge (\theta + i\lambda) &= \omega \wedge \theta - \eta \wedge \lambda + i(\omega \wedge \lambda + \eta \wedge \theta), && (13)\\
& && (14)
\end{aligned}
$$

and

$$dz = dx + idy.$$

Show that $d(fdz) = 0$ if and only if f satisfies the Cauchy-Riemann equations.

One has for $f = u + iv$ that $d(fdz) = d((u + iv)(dx + idy)) = d((udx - vdy) + i(vdx + udy)) = du \wedge dx - dv \wedge dy + i(dv \wedge dx + du \wedge dy) = -(D_2u +$

$D_1v)dx \wedge dy + i(-D_2v + D_1u)dx \wedge dy$. Clearly this is zero if and only if the Cauchy-Riemann equations hold true for f.

(e) *Prove the* Cauchy Integral Theorem*: If f is analytic in A, then $\int_c f dz = 0$ for every closed curve c (singular 1-cube with $c(0) = c(1)$) such that $c = \partial c'$ for some 2-chain c' in A.*

By parts (b) and (d), the 1-form $f dz$ is closed. By Stokes' Theorem, it follows that $\int_c f dz = \int_{\partial c'} f dz = \int_{c'} d(f dz) = \int_{c'} 0 = 0$.

(f) *Show that if $g(z) = 1/z$, then $g \cdot dz$ (or $(1/z)dz$ in classical notation) equals $id\theta + dh$ for some function $h : \mathbf{C} - 0 \to \mathbf{R}$. Conclude that $\int_{c_{R,n}} (1/z)dz = 2\pi in$.*

One has $(1/z)dz = \frac{1}{x+iy}(dx + idy) = \frac{(x-iy)(dx+idy)}{x^2+y^2} = \frac{xdx+ydy}{x^2+y^2} + i\frac{xdy-ydx}{x^2+y^2} = dh + id\theta$ if h is defined by $h(x, y) = \ln\sqrt{x^2 + y^2} = \ln|z|$.

This then gives $\int_{c_{R,n}} (1/z)dz = \int_{c_{R,n}} dh + i\int_{c_{R,n}} d\theta = 0 + i(2\pi n)$.

(g) *If f is analytic on $\{z : |z| < 1\}$, use the fact that $g(z) = f(z)/z$ is analytic in $\{z : 0 < |z| < 1\}$ to show that*

$$\int_{c_{R_1,n}} \frac{f(z)}{z} dz = \int_{c_{R_2,n}} \frac{f(z)}{z} dz$$

if $0 < R_i < 1$ for $i = 1, 2$. Use (f) to evaluate $\lim_{R\to 0} \int_{c_{R,n}} (f(z)/z)dz$ and conclude:

Cauchy Integral Formula: *If f is analytic on $\{z : |z| < 1\}$ and c is a closed curve in $\{z : 0 < |z| < 1\}$ with winding number n around 0, then*

$$n \cdot f(0) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z} dz.$$

The first assertion follows from part (e) applied to the singular 2-cube c' defined by $c'(s, t) = c_{tR_1+(1-t)R_2,n}(s)$.

By a trivial modification of Problem 4-24 (to use $c_{1/2,n}$) and Stokes' Theorem, $\int_c \frac{f(z)}{z}dz = \int_{c_{R,n}} \frac{f(z)}{z}dz$ for R with $0 < R < 1$.

Further,

$$\begin{aligned}\left|\int_{c_{r,n}} \frac{f(z)-f(0)}{z}dz\right| &= \left|\int_{c_{R,n}} ((u(z)-u(0))+i(v(z)-v(0))(dh+id\theta)\right| \\ &= |\int_{c_{R,n}} ((u(z)-u(0))dh-(v(z)-v(0)d\theta)+ \\ &+ i\int_{c_{R,n}} ((u(z)-u(0))d\theta+(v(z)-v(0))dh)| \\ &\leq 2\epsilon(\int_{c_{R,n}} dh + \int_{c_{R,n}} d\theta) = 2\pi n\epsilon.\end{aligned}$$

if ϵ is chosen so that $|f(z)-f(0)| < \epsilon$ for all z with $|z| = R$. It follows that $\lim_{R\to 0}\int_{c_{R,n}} \frac{f(z)-f(0)}{z}dz = 0$. Using part (f), we conclude that $\lim_{R\to 0}\int_{c_{R,n}} \frac{f(z)}{z}dz = f(0)\lim_{R\to 0}\int_{c_{R,n}} \frac{1}{z}dz = 2\pi n i f(0)$. The Cauchy integral formula follows from this and the result of the last paragraph.

4-34. *If $F : [0,1]^2 \to \mathbf{R}^3$ and $s \in [0,1]$, define $F_s : [0,1] \to \mathbf{R}^3$ by $F_s(t) = F(s,t)$. If each F_s is a closed curve, F is called a homotopy between the closed curve F_0 and the closed curve F_1. Suppose F and G are homotopies of closed curves; if for each s the closed curves F_s and G_s do not intersect, the pair (F,G) is called a homotopy between the non-intersecting closed curves F_0, G_0 and F_1, G_1. It is intuitively obvious that there is no such homotopy with F_0, G_0 the pair of curves shown in Figure 4-6 (a), and F_1, G_1 the pair of (b) or (c). The present problem, and Problem 5-33 prove this for (b) but the proof for (c) requires different techniques.*

(a) *If $f, g : [0,1] \to \mathbf{R}^3$ are nonintersecting closed curves, define $c_{f,g} : [0,1] \to \mathbf{R}^3 - 0$ by*

$$c_{f,g}(u,v) = f(u) - g(v).$$

If (F,G) is a homotopy of nonintersecting closed curves define $C_{F,G} : [0,1]^3 \to$

$\mathbf{R}^3 - 0$ *by*

$$C_{F,G}(s,u,v) = c_{F_s,G_s}(u,v) = F(s,u) - G(s,v).$$

Show that

$$\partial C_{F,G} = c_{F_0,G_0} - c_{F_1,G_1}.$$

When $u = 0, 1$, one gets the same singular 2-cube $F(x,0) - G(s,v)$; similarly, when $v = 0, 1$, one gets the same singular 2-cube $F(s,u) - G(s,0$. When $s = 0$ (respectively $s = 1$), one gets the singular 2-cube c_{F_0,G_0} (respectively c_{F_1,G_1}). So $\partial C_{F,G} = c_{F_1,G_1} - c_{F_0,G_0}$ which agrees with the assertion only up to a sign.

(b) *If* ω *is a closed 2-form on* $\mathbf{R}^3 - 0$*, show that*

$$\int_{c_{F_0,G_0}} \omega = \int_{c_{F_1,G_1}} \omega.$$

By Stokes' Theorem and part (a), one has $\int_{c_{F_1,G_1} - c_{F_0,G_0}} \omega = \int_{\partial C_{F,G}} \omega = \int_{C_{F,G}} d\omega = \int_{C_{F,G}} 0 = 0.$

V

Integration on Manifolds

MANIFOLDS

5-1. *If M is a k-dimensional manifold with boundary, prove that ∂M is a $(k-1)$-dimensional manifold and $M - \partial M$ is a k=dimensional manifold.*

The boundary of M is the set of points $x \in M$ which satisfy condition (M'). Let $h : U \to V$ be as in condition (M'); then the same h works for every point in $z \in U \cap M$ such that $h(z) = 0$. In particular, each such z is in ∂M. Further, h also is map which shows that condition (M) is satisfied for each such z. So ∂M is a manifold of dimension $k-1$, and because those points which don't satisfy (M') must satisfy (M), it follows that $M - \partial M$ is a manifold of dimension k.

5-2. *Find a counter-example to Theorem 5-2 if condition (3) is omitted.*

Following the hint, consider $f : (-2\pi, 2\pi) \to \mathbf{R}^2$ defined by

$$f(t) = \begin{cases} (t, -1) & 2\pi < t < 0, \\ (\cos(t - \pi/2), \sin(t - \pi/2)) & 0 \le t < 2\pi. \end{cases}$$

Let $k = 2$, $W = (-2\pi, 2\pi)$, $U = \mathbf{R}^2$, $M = f(W)$. Then condition (C) holds except for part (3) since $f'(t) = \begin{cases} (1, 0) & 2\pi < t < 0, \\ (-\sin(t - \pi/2), \cos(t - \pi/2)) & 0 \le t < 2\pi. \end{cases}$

5-3. (a) *Let $A \subset \mathbf{R}^n$ be an open set such that boundary A is an $(n-1)$-dimensional manifold. Show that $N = A \cup$ boundary A is an n-dimensional manifold with boundary. (It is well to bear in mind the following example: if $A = \{x \in \mathbf{R}^n : |x| < 1 \text{ or } 1 < |x| < 2\}$, then $N = A \cup$ boundary A is a manifold with boundary, but $\partial N \neq$ boundary A.)*

Since A is open, each of its points satisfies condition (M) with $k = n$. Let $x \in \partial A$. Then x satisfies (M) with $k = n-1$, say with the function $h : U \to V$. Let H be one of the half-planes $\{y : y^n < 0\}$ or $\{y : y^n > 0\}$ Suppose there is a sequence $x_1, x_2, \ldots$ of points of A such that the $h(x_i)$ all lie in H and converge to $h(x)$. If there is no open neighborhood W of $h(x)$ such that $H \cap W \subset h(U)$, then there is a sequence $y_1, y_2, \ldots$ of points of $H - h(U)$ such that the sequence converges to $h(x)$. But then the line segments from x_m to y_m must contain a point o the boundary of A, which is absurd since the points of U in the boundary of A all map to points with last coordinate 0. It follows that h restricted to an appropriately small open subset of U either satisfies condition (M) or condition (M'). This proves the assertion.

(b) *Prove a similar assertion for an open subset of an n-dimensional manifold.*

The generalization to manifolds is proved in the same way, except you need to restrict attention to a coordinate system around x. By working in the set W of condition (C), one gets back into the case where one is contained within $\mathbf{R}^n$, and the same argument applies.

5-4. *Prove a partial converse to Theorem 5-1: If $M \subset \mathbf{R}^n$ is a k-dimensional manifold and $x \in M$, then there is an open set $A \subset \mathbf{R}^n$ containing x and a differentiable function $g : A \to \mathbf{R}^{n-k}$ such that $A \cap M = g^{-1}(0)$ and $g'(y)$ has rank $n-k$ when $g(y) = 0$.*

Let $h : U \to V$ be as in condition (M) applied to $x \in M$, $A = U$, and $\pi : \mathbf{R}^n \to \mathbf{R}^{n-k}$ be defined by $\pi(x) = (x^{k+1}, \ldots, x^n)$. Then the function $g = \pi \circ h$ satisfies all the desired conditions.

5-5. *Prove that a k-dimensional (vector) subspace of $\mathbf{R}^n$ is a k-dimensional manifold.*

Let $v_1, \ldots, v_k$ be a basis for the subspace, and choose $v_{k+1}, \ldots, v_n$ so that all the

v_i together form a basis for $\mathbf{R}^n$. Define a map $h : \mathbf{R}^n \to \mathbf{R}^n$ by $h(\sum_{i=1}^n a_i v_i) = (a_1, \ldots, a_n)$. One can verify that h satisfies the condition (M).

5-6. *If $f : \mathbf{R}^n \to \mathbf{R}^n$, the* **graph** *of f is $\{(x, y) : y = f(x)\}$. Show that the graph of f is an n-dimensional manifold if and only if f is differentiable.*

If f is differentiable, the map $k : \mathbf{R}^n \to \mathbf{R}^{2n}$ defined by $k(x) = (x, f(x))$ is easily verified to be a coordinate system around all points of the graph of f; so the graph is a manifold of dimension n.

Conversely, suppose $h : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ is as in condition (M) for some point $(x, f(x)$ in the graph. Let π be the projection on the last n coordinates. Then apply the Implicit function theorem to $\pi \circ h : \mathbf{R}^{2n} \to \mathbf{R}^n$. The differentiable function g obtained from this theorem must be none other than f since the graph is the set of points which map to zero by $\pi \circ h$.

5-7. *Let $\mathbf{K}^n = \{x \in \mathbf{R}^n : x^1 = 0$ and $x^2, \ldots, x^{n-1} > 0\}$. If $M \subset \mathbf{K}^n$ is a k-dimensional manifold and N is obtained by revolving M around the axis $x^1 = \cdots = x^{n-1} = 0$, show that N is a $(k+1)$-dimensional manifold. Example: the torus (Figure 5-4).*

Consider the case where n $= 3$. If M is defined in some open set by $g(y, z) = 0$, then N is defined by $g(\sqrt{x^2 + y^2}, z) = 0$. The Jacobian is

$$\begin{pmatrix} D_1 g & D_2 g \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since either x or y is non-zero, it is easy to see that the Jacobian has the proper rank.

In the case where n > 3, it is not obvious what one means by “rotate".

5-8. (a) *If M is a k-dimensional manifold in $\mathbf{R}^n$ and $k < n$, show that M has measure 0.*

For each $x \in M$, one has condition (M) holding for some function $h : U \to V$. Let U_x be the domain of one of these functions, where we can choose U_x to be a ball with center at rational coordinates and rational radius. Then $\mathcal{O} = \{U_x : x \in M\}$ is a countable cover of M. Now each h maps points of M in U_x to points with the last $n-k$ coordinates 0. Take a thin plate including the image of $M \cap U_x$; its inverse image has volume which can be bounded by $(\sup deth'(x))v$ where v is the volume of the plate (by the change of variables formula). By choosing the thickness of the plate sufficiently small, we can guarantee that this value is no more than $\epsilon/2^n$ for the n^{th} element of the cover. This shows the result.

(b) *If M is a closed n-dimensional manifold with boundary in $\mathbf{R}^n$, show that the boundary of M is ∂M. Give a counter-example if M is not closed.*

Clearly, every element of ∂M is in the boundary of M by the condition (M'). If x is in the boundary of M, then $x \in M$ since M is closed. So if $x \notin \partial M$, it must satisfy condition (M). But then x is in the interior of M because the dimension of M is n.

The open unit interval in $\mathbf{R}$ is a counter-example if we do not require M to be closed.

(c) *If M is a compact n-dimensional manifold with boundary in $\mathbf{R}^n$, show that M is Jordan-measurable.*

By part (b), the boundary of M is ∂M. By Problem 5-1, ∂M is an $(n-1)$-dimensional manifold contained in $\mathbf{R}^n$. By part (a), it follows that ∂M is of measure 0. Finally, since M is bounded, the definition of Jordan measurable is satisfied.

5-9. *Show that M_x consists of the tangent vectors at t of curves c in M with $c(t) = x$.*

Let $f : W \to \mathbf{R}^n$ be a coordinate system around $x = f(a)$ in M; by replace W with a subset, one can assume that W is a rectangle centered at a. For $w \in W$ and $r \in \mathbf{R}$, let $c : [0, 1] \to \mathbf{R}^n$ be the curve $c(t) = f((t - 1/2)rw + a))$. Then $c'(1/2) = Df(a)rw$ ranges through out M_x as r and w vary.

Conversely, suppose that c is a curve in M with $c(t) = x$. Then let $h : U \to \mathbf{R}^n$ be as in condition (M) for the point x. We know by the proof of Theorem 5-2, that $h^{-1} : W \to \mathbf{R}^n$ is a coordinate system about x where $W = \{b \in \underline{\mathrm{R}}^k : (b, 0) \in h(M)\}$. Since $c = h^{-1} \circ h \circ c$, it follows that the tangent vector of c is in M_x.

5-10. *Suppose $\mathcal{C}$ is a collection of coordinate systems for M such that (1) For each $x \in M$ there is $f \in \mathcal{C}$ which is a coordinate system around x; (2) if $f, g \in \mathcal{C}$, then* $\det(f^{-1} \circ g)' > 0$. *Show that there is a unique orientation of M such that f is orientation-preserving for all $f \in \mathcal{C}$.*

Define the orientation to be the $\mu_x = [f_*((e_1)_a), \ldots, f_*((e_k)_a)]$ for every $x \in M$, $f\mathcal{C}$, and a with $f(a) = x$. In order for this to be well defined, we must show that we get the same orientation if we use use $f, g \in \mathcal{C}$ and $f(a) = x = g(b)$. But analogous to the author's observation of p. 119, we know that $\det(f^{-1} \circ g)' > 0$ implies that $\det((f^{-1} \circ g)_*(e_1)_b, ldots, (f^{-1} \circ g)_*(e_k)_b) = \alpha = \alpha \det((e_1)_a, \ldots, (e_k)_a)$ where $\alpha > 0$. Let ω be such that $\det = g^*\omega$. Then we have

$\omega(f(a))(f_*(f^{-1}\circ g)_*(e_1)_b, \ldots, f_*(f^{-1}\circ g)_*(e_k)_b) = \alpha\omega(f(a))(f_*(e_1)_a, \ldots, f_*(e_k)_b)$, i.e. $[g_*(e_1)_b, \ldots, g_*(e_k)_b] = [f_*(e_1)_a, \ldots, f_*(e_k)_b]$ as desired.

Clearly, the definition makes f orientation preserving for all $f \in \mathcal{C}$, and this is only orientation which could satisfy this condition.

5-11. *If M is an n-dimensional manifold-with-boundary in $\mathbf{R}^n$, define μ_x as the usual orientation of $M_x = \mathbf{R}^n_x$ (the orientation μ so defined is the* **usual orientation of** *M. If $x \in \partial M$, show that the two definitions of $n(x)$ given above agree.*

Let $x \in \partial M$ and $f : w \to \mathbf{R}^n$ be a coordinate system about x with $f(W) \subset \mathbf{H}^n$ and $f(0) = x$. Let $n(x) = f_*(v_0)$ where $v_0^n < 0$, $|n(x)| = 1$ and $n(x)$ is perpendicular to $(\partial M)_x$. Note that $[n(x), (-1)^n f_*e_1, f_*e_2, \ldots, f_*e_{n-1}]$ is the usual orientation of M_x, and so, by definition, $[(-1)^n f_*e_1, f_*e_2, \ldots, f_*e_{n-1}]$ is the induced orientation on $(\partial M)_x$. But then $n(x)$ is the unit normal in the second sense.

5-12. (a) *If F is a differentiable vector field on $M \subset \mathbf{R}^n$, show that there is an open set $A \supset M$ and a differentiable vector field $\tilde{F}$ on A with $\tilde{F}(x) = F(x)$ for $x \in M$.*

Let $\pi : \mathbf{R}^n \to \mathbf{R}^k$ be the projection on the first k coordinates, where k is the dimension of M. For every $x \in M$, there is a diffeomorphism $h_x : U_x \to \mathbf{R}^n$ satisfying condition (M). For $y \in U_x$, define $k_x(y) = (F(z))_y \in M_y$ where $z = h_x^{-1}(\pi \circ h_x(y))$. Then k is a differentiable vector field on U_x which extends the restriction of F to $U_x \cap M$.

Let $\mathcal{O} = \{U_x : x \in M\}$ and Phi be a partition of unity subordinate to $\mathcal{O}$. For $\varphi \in \Phi$, choose a $x_\varphi \in M$ with φ non-zero only for elements of U_{x_φ}. Define $K_\varphi(y) = \begin{cases} \varphi(y)k_{x_\varphi}(y) & y \in U_{x_\varphi} \\ 0 & \text{otherwise} \end{cases}$. Finally, let $\tilde{F} = \sum_{\varphi\in\Phi} K_\varphi$. Then $\tilde{F}$ is a differentiable extension of F to $A = \cup_{x\in M} U_x$.

(b) *If M is closed, show that we can choose $A = \mathbf{R}^n$.*

In the construction of part (a), one can assume that the U_x are open rectangles with sides at most 1. Let $A_n = \{x \in \mathbf{R}^n : n-1 \leq x \leq n+1\}$. Since M is closed, $M \cap A_n$ is compact, and so we can choose a finite subcover of $\mathcal{O}$. We can then replace $\mathcal{O}$ with the union of all these finite subcovers for all n. This assures that there are at most finitely many $U \in \mathcal{O}$ which intersect any given bounded set. But now we see that the resulting $\tilde{F}$ is a differentiable extension of F to all of $\mathbf{R}^n$. In fact, we have now assured that in a neighborhood of any point, $\tilde{F}$ is a sum of finitely many differentiable vector fields K_φ.

Note that the condition that M was needed as points x on the boundary of the set A of part (a) could have infinitely many U_x intersecting every open neighborhood of x. For example, one might have a vector field defined on $\mathbf{R}^n - 0$ by $F(x) = (x/|x|)_x$. This is a vector field of outward pointing unit vectors, and clearly it cannot be extended to the point 0 in a differentiable manner.

5-13. *Let $f : A \to \mathbf{R}^p$ be as in Theorem 5-1.*

(a) *If $x \in M = g^{-1}(0)$, let $h : U \to \mathbf{R}^n$ be the essentially unique diffeomorphism such that $g \circ h(y) = (y^{n-p+1}, \dots, y^n)$ and $h(0) = x$. Define $f : \mathbf{R}^{n-p} \to \mathbf{R}^n$ by $f(a) = h(0, a)$. Show that f_* is 1-1 so that the $n-p$ vectors $f_*((e_1)_0), \dots, f_*((e_{n-p})_0)$ are linearly independent.*

The notation will be changed. Let $(a, b) \in M = g^{-1}(0)$, and $G : A \to \mathbf{R}^{n-p} \times \mathbf{R}^p$ be defined, as in the proof of the implicit function theorem, by $G(x, y) = (x, g(x, y))$; let $H = G^{-1}$ and $h(x, y) = H(x + a, y)$. Then $g \circ H(x, y) = y$ and so $g \circ h(x, y) = y$. Also, $h(0, 0) = H(a, 0) = (G^{-1} \circ G)(a, b) = (a, b)$. Let $f : \mathbf{R}^{n-p} \to \mathbf{R}^n$ be defined by $f(x) = h(x, 0)$. We have changed the order of the arguments to correct an apparent typographical error in the problem statement.

Now $f_*(v) = Df(x)(v) = DH(x+a,0)(v+a,0) = (G'(G^{-1}(x+a,0)))^{-1}(v+a,0)^T$ which is 1-1 because G is a diffeomorphism. Since it is 1-1, it maps its domain onto a space of dimension $n-p$ and so the vectors, being a basis, must map to linearly independent vectors.

(b) *Show that the orientations μ_x can be defined consistently, so that M is orientable.*

ince f is a coordinate system about every point of M, this follows from Problem 5-10 with $\mathcal{C} = \{f\}$.

(c) *If $p = 1$, show that the components of the outward normal at x are some multiple of $D_1g(x), \ldots, D_ng(x)$.*

We have $G'(G^{-1}(x+a,0))(f_*(v)) = (v+a,0)^T$ and so by considering the components, we get $Dg(G^{-1}(x+a,0))(f_*(v)) = 0$ This shows that $(D_1(x), \ldots, D_ng(x))$ is perpendicular to M_x as desired.

5-14. *If $M \subset \mathbf{R}^n$ is an orientable $(n-1)$-dimensional manifold, show that there is an open set $A \subset \mathbf{R}^n$ and a differentiable $g : A \to \mathbf{R}^1$ so that $M = g^{-1}(0)$ and $g'(x)$ has rank 1 for $x \in M$.*

Choose an orientation μ for M. As the hint says, Problem 5-4 does the problem locally. Further, using Problem 5-13, we can assume locally that the orientation imposed by g is the given orientation μ. By replacing g with its square, we can assume that g takes on non-negative values. So for each $x \in M$, we have a g_x defined in an open neighborhood U_x of x. Let $A = \cup_{x \in M} U_x$, $\mathcal{O} = \{U_x : x \in M\}$, and Φ be a partion of unity subordinate to $\mathcal{O}$. Each $\varphi \in \Phi$ is non-zero only inside some U_{x_φ}, and we can assume by replacing the φ with sums of the φ, that the x_φ are distinct for distinct φ. Let g be defined by $g = \sum_\varphi \varphi \cdot g_{x_\varphi}$. Then g

satisfies the desired conditions.

5-15. *Let M be an $(n-1)$-dimensional manifold in $\mathbf{R}^n$. Let $M(\epsilon)$ be the set of end-points of normal vectors (in both directions) of length ϵ and suppose ϵ is small enough so that $M(\epsilon)$ is also an $(n-1)$-dimensional manifold. Show that $M(\epsilon)$ is orientable (even if M is not). What is $M(\epsilon)$ if M is the Möbius strip?*

Let $a \in M$, and g be as in Problem 5-4 in a neighborhood of a. Let h be as in Problem 5-13. Then we have a coordinate systems of the form $f_\epsilon(x) = (h + (\epsilon/|\nabla g|)\nabla g)(x, 0)$ and of the form $f_{-\epsilon}$. Choose an orientation on each piece so that adding $\nabla g(x, 0)$ (respectively $-\nabla g(x, 0)$) gives the usual orientation on $\mathbf{R}^n$. This is an orientation for $M(\epsilon)$.

In the case of the Möbius strip, the $M(\epsilon)$ is equivalent to a single ring $\{(x, y) : x^2 + y^2 = 1\} \times [0, 1]$.

5-16. *Let $g : A \to \mathbf{R}^p$ be as in Theorem 5-1. If $f : \mathbf{R}^n \to \mathbf{R}$ is differentiable and the maximum (or minimum) of f on $g^{-1}(0)$ occurs at a, show that there are $\lambda_1, \ldots, \lambda_p \in \mathbf{R}$, such that*

$$D_j f(a) = \textstyle\sum_{i=1}^p \lambda_i D_j g^i(a) \quad j = 1, \ldots, n.$$

The maximum on f on $g^{-1}(0)$ is sometimes called the maximum of f subject to the **constraints** *$g^i = 0$. One can attempt to find a by solving the system of equations. In particular, if $g : A \to \mathbf{R}$, we must solve $n+1$ equations*

$$\begin{aligned} D_j f(a) &= \lambda D_j g(a), & (15) \\ g(a) &= 0, & (16) \end{aligned}$$

in $n+1$ unknowns $a^1, \ldots, a^n, \lambda$, which is often very simple if we leave the equation $g(a) = 0$ for last. This is **Lagrange's method**, *and the useful but irrelevant λ is called a* **Lagrangian multiplier**. *The following problem gives a nice theoretical use for Lagrangian multipliers.*

Let F be a coordinate system in a neighborhood of the extremum at $F(x) = a$. Then $Df \circ F(x) = 0$ and so $Df(F(x)) \circ DF(x) = 0$. Now the image of $DF(x)$ is just the tangent space M_a, and so the row of $Df(F(x)) = Df(a)$ is perpendicular to the tangent space M_a. But we also have $g(F(y)) = 0$ for all y near x, and so $Dg(F(y))DF(y) = 0$. In particular, this is true at $y = x$, and so the rows of $Dg(a)$ are also perpendicular to M_a. But, $Dg(a)$ is of rank p and M_a is of dimension $n-p$, and so the rows of $Dg(a)$ generate the entire subspace of vectors perpendicular to M_a. In particular, $Df(a)$ is in the subspace generated by the $Dg(a)$, which is precisely the condition to be proved.

5-17. (a) *Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be self-adjoint with matrix $A = (a_{ij})$, so that $a_{ij} = a_{ji}$. If $f(x) =< Tx, x >= \sum a_{ij}x^i x^j$, show that $D_k f(x) = 2\sum_{j=1}^n a_{kj}x^j$. By considering the maximum of $< Tx, x >$ on S^{n-1} show that there is $x \in S^{n-1}$ and $\lambda \in \mathbf{R}$ with $Tx = \lambda x$.*

One has $D_k f(x) = D_k(\sum_{i=1}^n a_{ik}x_i x_k + \sum_{j=1}^n a_{kj}x_k x_j - a_{kk}x_k^2) = 2\sum_{j=1}^n a_{kj}x^j$. Apply Problem 5-16 with $g(x) =< x, x > -1$, so that the manifold is S^{n-1}. In this case, λ is a Lagrangian multiplier precisely when $Tx = \lambda x$. Since S^{n-1} is compact, f takes on a maximum on S^{n-1}, and so the maximum has a λ for which the Lagrangian multiplier equations are true. This shows the result.

(b) *If $V = \{y \in \mathbf{R}^n :< x, y >= 0\}$, show that $T(V) \subset V$ and $T : V \to V$ is self-adjoint.*

Suppose $y \in V$. Then $< x, T(y) >=< T(x), y >=< \lambda x, y >= \lambda < x, y >= \lambda \cdot 0 = 0$ and so $T(y) \in V$. This shows that $T(V) \subset V$. Since T as a map of $\mathbf{R}^n$ is self-adjoint and $V \subset \mathbf{R}^n$, it is clear that T as a map of V is also self-adjoint (cf p. 89 for the definition).

(c) *Show that T has a basis of eigenvectors.*

Proceed by induction on n; the case $n = 1$ has already been shown. Suppose It is true for dimension $n - 1$. Then apply part (a) to find the eigenvector v_1 with eigenvalue λ_1. Now, V is of dimension $n - 1$. So, V has a basis of eigenvectors $v_2, \ldots, v_n$ with eigenvalues $\lambda_n, \ldots, \lambda_n$ respectively. All the v_i together is the basis of eigenvectors for $\mathbf{R}^n$.

STOKES' THEOREM ON MANIFOLDS

5-18. *If M is an n-dimensional manifold (or manifold-with-boundary) in $\mathbf{R}^n$, with the usual orientation, show that $\int_M f dx^1 \wedge \cdots \wedge dx^n$, as defined in this section, is the same as $\int_M f$, as defined in Chapter 3.*

We can assume in the situation of Chapter 3 that $\mathbf{R}^n$ has the usual orientation. The singular n-cubes with $U \subset c([0,1]^n)$ can be taken to be linear maps $c(x) = ax + b$ where $a > 0$ and b are scalar constants. One has with $g = \varphi \cdot f$, that $\int_c g(x) dx^1 \wedge \cdots \wedge dx^n = \int_{[0,1]^n} a^n g(ax+b) = \int_{b^1}^{b^1+a} \cdots \int_{b^n}^{b^n+a} g(x)$. So, the two integrals give the same value.

5-19. (a) *Show that Theorem 5-5 is false if M is not required to be compact.*

For example, if we let M be the open interval $(0,1)$, one has $\int_{(0,1)} dx = 1$ but $\int_{\partial M} dx = 0$. One can also let $M = \mathbf{R}$ and $\omega = e^{-x^2} dx$.

(b) *Show that Theorem 5-5 holds for noncompact M provided that ω vanishes outside of a compact subset of M.*

The compactness was used to guarantee that the sums in the proof were finite; it also works under this assumption because all but finitely many summands are zero if ω vanishes outside of a compact subset of M.

5-20. *If ω is a $(k-1)$-form on a compact k-dimensional manifold M, prove that $\int_M d\omega = 0$. Give a counter-example if M is not compact.*

One has $\int_M d\omega = \int_{\partial M} \omega = 0$ as ∂M is empty. With M the set of positive real numbers, one has with $\omega = e^{-x}$ that $\int_M d\omega = -1$.

5-21. *An* **absolute k-tensor** *on V is a function $\eta : V^k \to \mathbf{R}$ of the form $|\omega|$ for $\omega \in \Lambda^k(V)$. An* **absolute k-form** *on M is a function η such that $\eta(x)$ is an absolute k-tensor on M_x. Show that $\int_M \eta$ can be defined, even if M is not orientable.*

Make the definition the same as done in the section, except don't require the manifold be orientable, nor that the singular k-cubes be orientation preserving. In order for this to work, we need to have the argument of Theorem 5-4 work, and there the crucial step was to replace $\det g'$ with its absolute value so that Theorem 3-13 could be applied. In our case, this is automatic because Theorem 4-9 gives $f^*|hdx^1 \wedge \cdots \wedge dx^n| = |h \circ f||\det f'||dx^1 \wedge \cdots \wedge dx^n|$.

5-22. *If $M_1 \subset \mathbf{R}^n$ is an n-dimensional manifold-with-boundary and $M_2 \subset M_1 - \partial M_1$ is an n-dimensional manifold with boundary, and M_1, M_2 are compact, prove that*

$$\int_{\partial M_1} \omega = \int_{\partial M_2} \omega$$

where ω is an $(n-1)$-form on M_1, and ∂M_1 and ∂M_2 have the orientations induced by the usual orieentations of M_1 and M_2.

Following the hint, let $M = (M1 - M2) \cup \partial M_2$. Then M is an n-dimensional manifold-with-boundary and its boundary is the union of ∂M_1 and ∂M_2. Because the outward directed normals at points of ∂M_2 are in opposite directions for M and M_2, the orientation of ∂M_2 are opposite in the two cases. By Stokes' Theorem, we have $\int_{\partial M_1} \omega - \int_{\partial M_2} \omega = \int_{\partial M} \omega = \int_M d\omega$. So the result is equivalent to $\int_M d\omega = 0$. So, the result, as stated, is not correct; but, for example, it would be true if ω were closed.

THE VOLUME ELEMENT

5-22. *If M is an oriented one-dimensional manifold in $\mathbf{R}^n$ and $c : [0,1] \to M$ is orientation-preserving, show that*

$$\int_{[0,1]} c^*(ds) = \int_{[0,1]} \sqrt{[(c^1)']^2 + \cdots + [(c^n)']^2}.$$

Consider the 1-form defined by $\omega(p)(v) = |v|$. This is the form which matches the proposed solution since

$$\begin{aligned}(c^*\omega)(t)((\alpha e_1)_t) &= \omega(c(t))(c_*(\alpha e_1))\\ &= \omega(c(t))((Dc(t)(\alpha e_1))_{c(t)})\\ &= |\alpha|\sqrt{[(c^1)'(t)]^2 + \cdots + [(c^n)'(t)]^2}.\end{aligned}$$

Furthermore, it is the volume element. To see this, choose an orthonormal basis $\{v\}$ where $v = c_*(\alpha e_1)$ where $< v, v >= 1$. Then $v = (\alpha(c^1)'(t), \ldots, \alpha(c^n)'(t))$ and $< v, v >= 1$ if and only if $\omega(c(t))(v) = 1$, as desired.

5-23. *If M is an n-dimensional manifold in $\mathbf{R}^n$, with the usual orientation, show that $dV = dx^1 \wedge \cdots \wedge dx^n$, so that the volume of M, as defined in this section, is the volume as defined in Chapter 3. (Note that this depends on the numerical factor in the definition of $\omega \wedge \eta$.)*

Let c be an orientation-preserving n-cube, i.e. $\det(c_*(e_1), \ldots, c_*(e_n)) > 0$. Let $c_*(v_1), \ldots, c_*(v_n)$ be an orthonormal basis with the usual orientation. By Theorem 4-9, we have $c^*(dx^1 \wedge \cdots \wedge dx^n) = (\det c')(dx^1 \wedge \cdots \wedge dx^n)$ and so applying this to $e_1, \ldots, e_n$ gives $\det(c_*e_1, \ldots, c_*e_n) = (dx^1 \wedge \cdots \wedge dx^n)(c_*e_1, \ldots, c_*e_n) = c^*(dx^1 \wedge \cdots \wedge dx^n)(e_1, \ldots, e_n) = \det c'$. Since c is orientation preserving, it must be that $\det c' > 0$.

Now, $c^*(dx^1 \wedge \cdots \wedge dx^n)(v_1, \ldots, v_n) = (\det c')(dx^1 \wedge \cdots \wedge dx^n)(v_1, \ldots, v_n) = (\det c')(\det(v_1, \ldots, v_n))$. Now, c is orientation preserving, so $v_1, \ldots, v_n$ must have the usual orientation, i.e. $\det(v_1, \ldots, v_n) > 0$. But then $c^*(dx^1 \wedge \cdots \wedge dx^n)(v_1, \ldots, v_n) > 0$. Since this is also equal to $(dx^1 \wedge \cdots \wedge dx^n)(c_*v_1, \ldots, c_*v_n) = \det(c_*v_1, \ldots, c_*v_n) = \pm 1$ by orthonormality, it follows that the value is precisely 1, and so it is the volume element in the sense of this section.

5-24. *Generalize Theorem 5-6 to the case of an oriented $(n-1)$-dimensional manifold in $\mathbf{R}^n$.*

The generalization is $\omega \in \Lambda^{n-1}(M_x)$ defined by

$$\omega(v_1, \ldots, v_{n-1}) = (-1)^{n-1} \det \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ n(x) \end{pmatrix}$$

where $n(x)$ is the unit outward normal at $x \in M$. As in the 2-dimensional case, $\omega(v_1, \ldots, v_{n-1}) = 1$ if the v_i are an orthonormal basis with orientation μ (where μ was the orientation used to determine the outward normal). So ω is the volume element dV.

Expanding in terms of cofactors of the last row gives:

$$dV = \sum_{i=1}^{n} (-1)^{i-1} n^i dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n.$$

As in the 2-dimensional case, $v_1 \times \cdots v_{n-1} = \alpha n(x)$ for some scalar α, and so $< z, n(x) > \cdot < v_1 \times \cdots \times v_{n-1}, n(x) >=< z, n(x) > \alpha =< z, \alpha n(x) >=< z, v_1 \times \cdots \times v_{n-1} >$ for all $z \in \mathbf{R}^n_x$. Letting $z = e_i$, we get $n^i dV =< e_i, v_1 \times \cdots \times v_{n-1} >=$

$$\det\begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ e_i \end{pmatrix} = (-1)^{i-1}dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n.$$

5-25. (a) *If $f : [a,b] \to \mathbf{R}$ is non-negative and the graph of f in the xy-plane is revolved around the x-axis in $\mathbf{R}^3$ to yield a surface M, show that the area of M is*

$$\int_a^b 2\pi f \sqrt{1 + (f')^2}.$$

One can use singular 2-cubes of the form $c(x,\theta) = (x, f(x)\cos(\theta), f(x)\sin(\theta))$. The quantities E, F, and G calculate out to $E = 1+(f'(x))^2$, $F = 0$, and $G = f(x)^2$. So the surface area is $S = \int_a^b \int_0^{2\pi} \sqrt{EG - F^2} = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}dx$.

(b) *Compute the area of S^2.*

Apply part (a) with $f(x) = \sqrt{1-x^2}$ and $[a,b] = [-1,1]$. One has $f'(x) = \frac{-2x}{\sqrt{1-x^2}}$ and $f(x)^2(1 + f'(x)^2) = 1$. So $V = \lim_{\epsilon \to 0^+} \int_{-1+\epsilon}^{1-\epsilon} 2\pi = 4\pi$.

5-26. *If $T : \mathbf{R}^n \to \mathbf{R}^n$ is a norm preserving linear transformation and M is a k-dimensional manifold in $\mathbf{R}^n$, show that M has the same volume as $T(M)$.*

Although it is not stated, it is assumed that M is orientable.

By Problem 1-7, T is inner product preserving and so it maps orthonormal bases to orthonormal bases. Further, if c is a singular k-cube which is a coordinate system for M in a neighborhood of x, then $T \circ c$ is a singular k-cube which is a coordinate system for $T(M)$ in a neighborhood of $T(x)$. Depending on the sign of $\det T$, the new k-cube is either orientation preserving or reversing. In particular, $\operatorname{sgn}(\det T)T^*\omega$ is the volume element of M if ω is the volume element of $T(M)$ (which is also orientable). Since the volume is just the integral of the volume element and the integral is calculated via the k-cubes, it follows that the volumes of M and $T(M)$ are equal.

5-27. (a) *If M is a k-dimensional manifold, show that an absolute k-tensor $|dV|$ can be defined, even if M is not orientable, so that the volume of M can be defined as $\int_M |dV|$.*

This was already done in Problem 5-21.

(b) *If $c : [0, 2\pi] \times [-1, 1] \to \mathbf{R}^3$ is defined by*

$$c(u, v) = (2\cos(u)+v\sin(u/2)\cos(u), 2\sin(u)+v\sin(u/2)\sin(u), v\cos(u/2)),$$

show that $c([0, 2\pi] \times (-1, 1))$ is a Möbius strip and find its area.

To see that it is a Möbius strip, note that in cylindrical coordinates, the equations are: $r = 2 + v\sin(u/2), \theta = u, z = v\cos(u/2)$. In particular, for fixed θ, we have u fixed and the path is a line segment traversed from $v = -1$ to $v = 1$. Calculating the length of the line, one gets that is a line segment of length 2. Again, for fixed θ, the line segment in the rz-plane has slope $\cot(u/2)$. Note that this varies from ∞ down to $-\infty$ as u ranges from 0 to 2π, i.e. the line segment starts vertically at $u = 0$ and reduces in slope until it becomes vertical again at $u = 2\pi$. This corresponds to twisting the paper 180 degrees as it goes around the ring, which is the Möbius strip.

To find the area, one can actually, just use the formulas for an orientable surface, since one can just remove the line at $\theta = 0$. In thatt case one can verify, preferably with machine help, that $E = (2 + v\sin(u/2))^2 + (v/2)^2$, $F = 0$, and $G = 1$. So the area is $\int_{-1}^{1}\int_0^{2\pi} \sqrt{(2 + v\sin(u/2))^2 + (v/2)^2}dudv$. Numerical evaluation of the integral yields the approximation $25.41308559 \approx 8.09\pi$, which is just slightly larger than 8π, the area of a circular ring of radius 2 and height 2.

5-28. *If there is a nowhere-zero k-form on a k-dimensional manifold M, show that M is orientable*

Suppose ω is the nowhere-zero k-form on M. If c is a singular k-cube, then for every $p \in [0,1]^k$, we have $c^*\omega(p)(e_1, \dots, e_k) \neq 0$ because the space $\Lambda^k(\mathbf{R}^k)$ is of dimension 1. Choose a k-cube so that the value is positive for some p. Then if it were negative at another point $q \in [0,1]^k$, then because this is a continuous function from $[0,1]^k$ into $\mathbf{R}$, the intermediate value theorem would guarantee a point r where the function were zero, which is absurd. So the value is positive for all $p \in [0,1]^k$.

For every $x \in M$, choose a k-cube of this type with x in its image. Define $\mu_x = [c_*e_1, \dots, c_*e_k]$. This is well defined. Indeed, if we had two such k-cubes, say c_1 and c_2, then the $c_i^*\omega(p_1)(e_1, \dots, e_k) = \omega(x)(c_*e_1, \dots, c_*e_k)$ (where $c_i(p_i) = x$) are positive for $i = 1, 2$. But then the values are of the same sign regardless of which non-zero k-form in M_x is used. So, both maps define the same orientation on M_x.

5-29. (a) *If $f : [0,1] \to \mathbf{R}$ is differentiable and $c : [0,1] \to \mathbf{R}^2$ is defined by $c(x) = (x, f(x))$, show that $c([0,1])$ has length $\int_0^1 \sqrt{1 + (f')^2}$.*

This is an immediate consequence of Problem 5-23.

(b) *Show that this length is the least upper bound of lengths of inscribed broken lines.*

We will need f to be continuously differentiable, not just differentiable.

Following the hint, if $0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1$, then by the mean value theorem, $|c(t_i) - c(t_{i-1})| = \sqrt{(t_i - t_{i-1})^2 + (f(t_i) - f(t_{i-1}))^2} = (t_i - t_{i-1})\sqrt{1 + f'(s_i)^2}$ for some $s_i \in [t_{i-1}, t_i]$. Summing over $i = 1, \dots, n$ gives a Riemann sum for $\int_0^1 \sqrt{1 + (f')^2}$. Taking the limit as the mesh approaches 0, shows that these approach the integral. Starting from any partition, and taking successively finer partitions of the interval with the mesh approaching zero, we get an increasing sequence of values with limit the value of the integral; so the integral is the least upper bound of all these lengths.

5-30. *Consider the 2-form ω defined on $\mathbf{R}^3 - 0$ by*

$$\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}.$$

(a) *Show that ω is closed.*

This is a straightforward calculation using the definition and Theorem 4-10. For example, $d(\frac{xdy\wedge dz}{(x^2+y^2+z^2)^{3/2}} = \frac{(-2x^2+y^2+z^2)dx\wedge dy\wedge dz}{(x^2+y^2+z^2)^(5/2)}$. The other two terms give similar results, and the sum is zero.

(b) *Show that*

$$\omega(p)(v_p, w_p) = \frac{< v \times w, p >}{|p|^3}.$$

For $r > 0$ let $S^2(r) = \{x \in \mathbf{R}^3 : |x| = r\}$. Show that ω restricted to the tangent space of $S^2(r)$ is $1/r^2$ times the volume element and that $\int_{S^2(r)} \omega = 4\pi$. Conclude that ω is not exact. Nevertheless, we denote ω by $d\Theta$ since, as we shall see, $d\Theta$ is the analogue of the 1-form $d\theta$ on $\mathbf{R}^2 - 0$.

As in the proof of Theorem 5-6 (or Problem 5-25), the value of $\omega(p)(v_p, w_p)$ can be evaluated by expanding $< v \times w, p >= \det \begin{pmatrix} v \\ w \\ (x, y, z) \end{pmatrix}$ using cofactors of the third row.

The second assertion follows from $dA =< v \times w, n(x) >$ and the fact that the outward directed normal can be taken to be $(x, y, z)/r$ be an appropriate choice of orientation. One has $\int_{S^2(r)} \omega = v(S^2(r))/r^2 = 4\pi$ by Problem 5-26. If $\omega = d\omega_1$, then Stokes' Theorem would imply that $\int_{S^2(r)} \omega = \int_{S^2(r)} d\omega_1 = \int_{\partial S^2(r)} \omega_1 = 0$. Since the value is 4π, we conclude that ω is not exact.

(c) *If v_p is a tangent vector such that $v = \lambda p$ for some $\lambda \in \mathbf{R}$, show that $d\Theta(p)(v_p, w_p) = 0$ for all w_p. If a two-dimensional manifold M in $\mathbf{R}^3$ is part of a generalized cone, that is, M is the union of segments of rays through the origin, show that $\int_M d\Theta = 0$.*

If $v_p = \lambda p$, then using part (b), one has $d\Theta(p)(v_p, w_p) = \frac{<(\lambda p)\times w, p>}{|p|^3} = 0$. By Problem 5-9, for any point $x \neq 0$ on the generalized cone, the line through x and the origin lies on the surface and so its tangent line (the same line) is in M_x. But then $d\Theta(x)$ is identically 0 for all points $x \neq 0$. But then $\int_M d\Theta = 0$.

(d) *Let $M \subset \mathbf{R}^3 - 0$ be a compact two-dimensionaal manifold-with-boundary such that every ray through 0 intersects M at most once (Figure 5-10). The union of those rays through 0 which intersect M, is a solid cone $C(M)$. The* **solid angle** *subtended by M is defined as the area of $C(M) \cap S^2$, or equivalently as $1/r^2$ times the area of $c(M) \cap S^2(r)$ for $r > 0$. Prove that the solid angle subtended by M is $|\int_M d\Theta|$.*

We take the orientations induced from the usual orientation of $\mathbf{R}^3$.

Following the hint, choose r small enough so that there is a three dimensional manifold-with-boundary N (as in Figure 5-10) such that ∂N is the union of M and $C(M) \cap S^2(r)$, and a part of a generalized cone. (Actually, N will be a manifold-with-corners; see the remarks at the end of the next section.) Note that this is essentially the same situation as in Problem 5-22. Applying Stokes' Theorem gives $\int_{\partial N} d\Theta = \int_N d(d\Theta) = 0$ because $d\Theta$ is closed by part (a). By part (c), the integral over the part of the boundary making up part of a generalized cone is zero. The orientation of the part of the boundary on $S^2(r)$ is opposite to that of the orientation of the same set as a part of $S^2(r)$. So, we have $\int_M d\Theta = \int_{N \cap S^2(r)} d\Theta$ and the last integral is the solid angle subtended by M by the interpretation of $d\Theta$ of part (b).

5-31. *Let* $f, g : [0,1] \to \mathbf{R}^3$ *be nonintersecting closed curves. Define the* **linking number l(f,g)** *of* f *and* g *by (cf. Problem 4-34)*

$$l(f,g) = \frac{-1}{4\pi} \int_{c_{f,g}} d\Theta.$$

(a) *Show that if (F,G) is a homotopy of nonintersecting closed curves, then* $l(F_0, G_0) = l(F_1, G_1)$.

This follows immediately from Problem 4-34 (b) and Problem 5-31 (a).

(b) *If* $r(u,v) = |f(u) - g(v)|$ *show that*

$$l(f,g) = \frac{-1}{4\pi} \int_0^1 \int_0^1 \frac{1}{[r(u,v)]^3} \cdot A(u,v) dudv$$

where

$$A(u,v) = \det \begin{pmatrix} (f^1)'(u) & (f^2)'(u) & (f^3)'(u) \\ (g^1)'(v) & (g^2)'(v) & (g^3)'(v) \\ (f^1)(u) - (g^1)(v) & (f^2)(u) - (g^2)(v) & (f^3)(u) - (g^3)(v) \end{pmatrix}.$$

This follows by direct substitution using the expression for $d\Theta$ in the preamble to Problem 31.

(c) *Show that* $l(f,g) = 0$ *if* f *and* g *both lie in the xy-plane.*

This follows from the formula in part (b) since the third column of the determinant defining $A(u,v)$ is zero.

The curves of Figure 4-5(b) are given by $f(u) = (\cos(u), \sin(u), 0)$ *and* $g(v) = (1 + \cos(v), 0, \sin(v))$. *You may easily convince yourself that calculating* $l(f,g)$ *by the above integral is hopeless in this case. The following problem shows how to find* $l(f,g)$ *without explicit calculations.*

5-32. (a) *If* $(a,b,c) \in \mathbf{R}^3$ *define*

$$d\Theta_{(a,b,c)} = \frac{(x-a)dy \wedge dz + (y-b)dz \wedge dx + (z-c)dx \wedge dy}{[(x-a)^2 + (y-b)^2 + (z-c)^2]^{\frac{3}{2}}}$$

If M *is a compact two-dimensional manifold-with-boundary in* $\mathbf{R}^3$ *and* $(a,b,c) \notin M$ *define*

$$\Omega(a,b,c) = \int_M d\Theta_{(a,b,c)}.$$

Let (a,b,c) *be a point on the same side of* M *as the outward normal and* (a',b',c') *be a point on the opposite side. Show that by choosing* (a,b,c) *sufficiently close to* (a',b',c') *we can make* $\Omega(a,b,c) - \Omega(a',b',c')$ *as close to* -4π *as desired.*

Following the hint, suppose that $M = \partial N$ where N is a compact manifold-with-boundary of dimension 3. Suppose $x = (a,b,c) \in N - \partial M$. Removing a ball centered at x from the interior of N gives another manifold-with-boundary N' with boundary $\partial M \cup S^2(r)$, where the orientation on $S^2(r)$ is opposite to that of the induced orientation. So by Stokes' Theorem and Problem 5-31 (b), $\Omega(a,b,c) = \int_M d\Theta_{(a,b,c)} = \int_{S^2(r)} d\Theta = 4\pi$. (Note the discrepancy in sign between this and the hint.) On the other hand, if $x = (a,b,c) \notin N$, then by Stokes' Theorem, $\Omega(a,b,c) = \int_M d\Theta_{(a,b,c)} = \int_N d(d\Theta)) = 0$.

The rest of the proof will be valid only in the case where there is a 3-dimensional compact oriented manifold-with-boundary N such that $\partial N = M \cup M'$ where M' is a two-dimensional manifold. Then $x = (a,b,c) \notin N$, and and we can take $x' = (a',b',c') \in N - \partial N$. So, $\Omega(a,b,c) + \int_{M'} d\Theta(a,b,c)$ and $\Omega(a',b',c') + \int_{M'} d\Theta(a',b',c') = 4\pi$ by the last paragraph. Subtracting gives $\Omega(a,b,c) - \Omega(a',b',c') = \int_{M'}(d\Theta(a',b',c') - d\Theta(a,b,c)) - 4\pi$. The first term can be made as small as we like by making $|x - x'|$ sufficiently small.

(b) *Suppose* $f([0,1]) = \partial M$ *for some compact oriented two-dimensional manifold-with-boundary* M. *(If* f *does not intersect itself such an* M *always exists,* *even if f is knotted, see [6], page 138.) Suppose that whenever g intersects M at x, the tangent vector v of g is not in M_x. Let n^+ be the number of intersections where v points in the same direction as the outward normal, and n^- the number of other intersections. If $n = n^+ - n^-$, show that*

$$n = \frac{-1}{4\pi}\int_g d\Omega.$$

In the statement, what one means is that the component of v in the outward normal direction is either in the same or opposite direction as the outward normal.

Parameterize g with $c : [0,1] \to \mathbf{R}^3$ where $c(0) = c(1)$ is not in M. Let $0 < x_1 < x_2 < \ldots < x_n < 1$ be the values where g intersects M. To complete the proof, we will need to assume that n is finite. Let $x_{n+1} = x_1$. Choose $\epsilon > 0$ small enough so that $2\epsilon < x_{i+1} - x_i$ for all i (where by $x_{n+1} - x_n$ we mean $1 - x_n + x_1$. One has $\int_g d\Omega = \lim_{\epsilon\to 0^+}\sum_{i=1}^n \Omega_{(c(x_{i+1}-\epsilon))} - \Omega_{(c(x_i+\epsilon))} = \lim_{\epsilon\to 0^+}\sum_{i=1}^n \Omega_{(c(x_i-\epsilon))} - \Omega_{(c(x_i+\epsilon))}$.

By part (a), if the tangent vector to c at $c(x_i)$ has a component in the outward normal direction of M is positive, then $\lim_{\epsilon\to 0^+}\Omega_{(c(x_i-\epsilon))} - \Omega_{(c(x_i+\epsilon))}$ is 4π; if it is negative, then it is -4π. So, the last paragraph has $\int_g d\Omega = 4\pi(n^+ - n^-) = 4\pi n$.

Note that this result differs from the problem statement by a sign.

(c) *Prove that*

$$D_1\Omega(a,b,c) = \int_f \frac{(y-b)dz - (z-c)dy}{r^3} \tag{17}$$

$$D_2\Omega(a,b,c) = \int_f \frac{(z-c)dx - (x-a)dx}{r^3} \tag{18}$$

$$D_3\Omega(a,b,c) = \int_f \frac{(x-a)dy - (y-b)dx}{r^3} \tag{19}$$

where $r(x, y, z) = |(x, y, z)|$.

The definition of r *should be* $r(x, y, z) = |(x - a, y - b, z - c)|$.

The proofs are analogous; we will show the first result. Start with

$$\begin{aligned} D_1\Omega(a, b, c) &= \lim_{h\to 0} \frac{\Omega(a + h, b, c) - \Omega(a, b, c)}{h} \\ &= \lim_{h\to 0} \frac{\int_M d\Theta_{(a+h,b,c)} - d\Theta_{a,b,c}}{h} \end{aligned}$$

$$= \int\limits_M \lim_{h\to 0} h^{-1} \left(\frac{x - a - h}{((x - a - h)^2 + (y - b)^2 + (z - c)^2)^{3/2}} - \frac{x - a}{r^3} \right) dy \wedge dz + \dots$$

$$= \int\limits_M -D_1 \frac{x - a}{r^3} dy \wedge dz - D_1 \frac{y - b}{r^3} dz \wedge dx - D_1 \frac{z - c}{r^3} dx \wedge dy.$$

where we have used Problem 3-32 to interchange the order of the limit and the integral.

On the other hand, we have by Stokes' Theorem that $\int_f \frac{(y-b)dz-(z-c)dy}{r^3} = \int_M -D_1(\frac{y-b}{r^3})dz \wedge dx + D_2(\frac{y-b}{r^3})dy \wedge dz - D_1(\frac{z-c}{r^3})dx \wedge dy + D_3(\frac{z-c}{r^3})dy \wedge dz$. Comparing the two expressions, we see that two of the terms in the first expression match up with corresponding terms in the second expression. It remains to check that the remaining terms are equal. But a straightforward expansion gives: $-D_1(\frac{x-a}{r^3} = \frac{2(x-a)^2-(y-b)^2-(z-c)^2}{r^5} = D_2(\frac{y-b}{r^3}) + D_3(\frac{z-c}{r^3})$ as desired.

(d) *Show that the integer* n *of (b) equals the integral of Problem 5-32(b), and use this result to show that* $l(f, g) = 1$ *if* f *and* g *are the curves of Figure 4-6 (b), while* $l(f, g) = 0$ *if* f *and* g *are the curves of Figure 4-6 (c). (These results were known to Gauss [7]. The proofs outlined here are from [4] pp. 409-411; see also [13], Volume 2, pp. 41-43.)*

By part (c), one has $D_1\Omega(a, b, c) = \int_0^1 \frac{(f^2(u)-b)(f^3)'(u)-(f^3(u)-c)(f^2)'(u)}{((f^1(u)-a)^2+(f^2(u)-b)^2+(f^3(u)-c)^2)^{3/2}} du$ and so $\int_g D_1\Omega dx = \int_0^1 \int_0^1 \frac{(f^2(u)-g^2(v)(f^3)'(u)(g^1)'(v)-(f^3(u)-g^3(v))(f^2)'(u)(g^1)'(v)}{((f^1(u)-g^1(v))^2+(f^2(u)-g^2(v))^2(f^3(u)-g^3(v))^2)^{3/2}} dudv$. Similar results hold for $\int_g D_2\Omega$ and $\int_g D_3\Omega$. One then substitutes into: $n =$

$\frac{-1}{4\pi}\int_g d\Omega = \int_g D_1\Omega dx + \int_g D_2\Omega dy + \int_g D_3\Omega dz$. After collecting terms, we see that this is equal to the expression for $l(f,g)$ in Problem 5-32 (b). So, $l(f,g) = n$.

By inspection in Figure 4-6, one has $n = 1$ and $n = 0$ in parts (b) and (c) of the figure. So, these are also the values of $l(f,g)$.

5-34. *Generalize the divergence theorem to the case of an n-manifold with boundary in $\mathbf{R}^n$.*

The generalization: Let $M \subset \mathbf{R}^n$ be a compact n-dimensional manifold-with-boundary and n the unit outward normal on ∂M. Let F be a differentiable vector fieldd on M. Then

$$\int_M \text{div } F dV = \int_{\partial M} < F, n > dS.$$

As in the proof of the divergence theorem, let $\omega = \sum_{i=1}^n (-1)^{i-1} F^i dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n$. Then $d\omega = \text{ div } F$. By Problem 5-25, on ∂M, we have $n^i dS = (-1)^{i-1} dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n$ for $i = 1, \cdots, n$. So, $< F, n > dS = \sum_{i=1}^n F^i n^i dS = \sum_{i=1}^n (-1)^{i-1} F^i dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n = \omega$. By Stokes' Theorem, it follows that

$$\int_M \text{div } F dV = \int_M d\omega = \int_{\partial M} \omega = \int_{\partial M} < F, n > dS.$$

5-35. *Applying the generalized divergence theorem to the set $M = \{x \in \mathbf{R}^n : |x| \leq a\}$ and $F(x) = x_x$, find the volume of $S^{n-1} = \{x \in \mathbf{R}^n : |x| = 1\}$ in terms of the n-dimensional volume of $B_n = \{x \in \mathbf{R}^n : |x| \leq 1\}$. (This volume is $\pi^{n/2}/(n/2)!$ if n is even and $2^{(n+1)/2}\pi^{(n-1)/2}/1 \cdot 3 \cdot 5 \cdot \ldots \cdot n$ if n is odd.)*

One has div $F = n$ and $< F, n >=< x, x/a >= a$ since the outward normal is in the radial direction. So $nV = \int_M \text{ div } F dV = \int_{\partial M} < F, n > dS = aS$. In particular, if $a = 1$, this says the surface area of S^{n-1} is n times the volume of B_n.

5-36. *Define F on $\mathbf{R}^3$ by $F(x) = (0, 0, cx^3)_x$ and let M be a compact three-dimensional manifold-with-boundary with $M \subset \{x : x^3 \leq 0\}$. The vector field F may be thought of as the downward pressure of a fluid of density c in $\{x : x^3 \leq 0\}$. Since a fluid exerts equal pressures in all directions, we define the* buoyaant force *on M, due to the fluid, as $-\int_{\partial M} < F, n > dA$. Prove the following theorem.*

Theorem (Archimedes). *The buoyant force on M is equal to the weight of the fluid displaced by M.*

The definition of buoyant force is off by a sign.

The divergence theorem gives $cV = \int_M c dV = \int_M \text{ div } F dv = \int_{\partial M} < F, n > dA$. Now cV is the weight of the fluid displaced by M. So the right hand side should be the buoyant force. So one has the result if we define the buoyant force to be $\int_{\partial M} < F, n > dA$. (This would make sense otherwise the buoyant force would be negative.)

♡♢♣♠