1 Definitions

1.0.1 Metric Space

A **metric** on X is a function

$$d : X \times X \to \mathbb{R}$$

such that

1. Nonnegative $d(x, y) \geq 0$ for all $x, y \in X$.
2. $d(x, y) = 0 \iff x = y$.
3. Symmetric $d(x, y) = d(y, x)$.
4. Triangle Inequality $d(x, y) \leq d(x, z) + d(z, y)$

1.0.2 Linear Space

A linear space aka vector space defines a **norm** on X

$$|| \cdot || : X \to \mathbb{R}$$

such that

1. Nonnegative $||x|| \geq 0$ for all $x \in X$.
2. $||x|| = 0 \iff x = 0$.
3. Scalar multiplication $||\lambda x|| = |\lambda| \cdot ||x||$
4. Triangle Inequality $||x + y|| \leq ||x|| + ||y||$

There is a corresponding metric induced by the norm.

A sequence (x_n) **converges** to $x \in X$ if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$.

A sequence (x_n) is **Cauchy** if for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for all $m, n \geq N$.

A metric space (X, d) is **complete** if every Cauchy sequence in X converges to a limit in X.

$G \subset X$ is **open** if for all $x \in G$ there exists $r > 0$ such that $B_r(x) \subset G$. $F \subset G$ is **closed** if the complement F^C is open.

$\bar{A}$, the **closure** of A is the smallest closed set containing A.

$K \subset X$ is **sequentially compact** if every sequence in K contains a subsequence that converges in K.

$K \subset X$ is **compact** if every open cover of K has a finite subcover.

A subset of X is **totally bounded** if there exists a finite ϵ net for every $\epsilon > 0$.

$A \subset X$ is **dense** in X if $\bar{A} = X$.

A subset of X is **separable** if it has a countable dense subset. Compact metric spaces are separable.

A subset of X is **precompact** if its closure in X is compact. $\iff$ Subset is totally bounded.

1.0.3 Theorems

A subset of a metric space is complete if and only if it is closed.

A subset of a metric space is compact if and only if it is complete and totally bounded.

Continuous functions on a compact set are uniformly continuous and bounded, attaining their max and min.

The image of a compact set under a continous function is compact.

1.1 Functions

A function $f : X \to Y$ is **continuous** at $x_0 \in X$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that $d_X(x, x_0) < \delta$ implies $d_Y(f(x), f(x_0)) < \epsilon$. Equivalently:

1. There is a ball of positive radius δ around a point x_0 that is mapped into the ball of radius ϵ around $f(x_0)$.
2. f continuous $\iff$ if $x_n \to x_0$ then $f(x_n) \to f(x_))$.

A function $f : X \to Y$ is **uniformly continuous** on X if for all $\epsilon > 0$ there exists $\delta > 0$ such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$ for all $x, y \in X$.

The **sup norm** or L^∞ norm between functions on $\mathbb{R}$ is defined as

$$d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

A sequence of bounded, real valued functions (f_n) on a metric space **converges uniformly** to a function f if

$$\lim_{n = \to \infty} ||f_n - f|| = 0$$

$C(K)$, the space of continuous functions on a compact metric space K is complete.

Weierstrass Approximation The set of polynomials is dense in $C([a, b])$.