Math 201A - HW 1

Richard Clark Fitzgerald

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Problem 1. Let (X, d) be a metric space, and let $x, y, w, z \in X$.

a) Prove that
$$d(x,y) \geq |d(x,z) - d(z,y)|.$$

Proof. Applying the triangle inequality two times we have

$$\begin{aligned} d(x,z) \leq d(x,y) + d(y,z) &\implies d(x,z) - d(y,z) \leq d(x,y) \\ d(y,z) \leq d(y,x) + d(x,z) &\implies d(y,z) - d(x,z) \leq d(y,x) = d(x,y). \end{aligned}$$

For $a, b \in \mathbb{R}$ if $a \leq b$ and $-a \leq b$, then $|a| \leq b$. We have shown this, so we have the result:

$$d(x,y) \geq |d(x,z) - d(z,y)|.$$

$\square$

b) Prove that
$$d(x,y) + d(z,w) \geq |d(x,z) - d(y,w)|.$$

Proof. By part a) we have

$$\begin{aligned} d(x,y) &\geq |d(x,z) - d(z,y)| \\ d(z,w) &\geq |d(z,y) - d(y,w)|. \end{aligned}$$

Adding both sides of the above inequalities produces

$$\begin{aligned} d(x,y) + d(z,w) &\geq |d(z,y) - d(y,w)| + |d(x,z) - d(z,y)| \\ &\geq |d(z,y) - d(y,w) + d(x,z) - d(z,y)| \\ &= |d(x,z) - d(y,w)| \end{aligned}$$

In the second step we used that $|a| + |b| \geq |a+b|$ for $a, b \in \mathbb{R}$. $\square$

c) Let (x_n) and (y_n) be converging sequences in X such that $\lim_{n\to\infty}(x_n) = x$ and $\lim_{n\to\infty}(y_n) = y$. Prove that $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$.

Problem 2. Show that the limit of a convergent sequence in a metric space is unique.

Proof. Let $(x_n) \to x, (y_n) \to y$. Above we showed $d(x_n, y_n) \to d(x, y)$. Take $y_n = x_n$. Then $d(x_n, y_n) = 0$ for all n and $d(x_n, y_n) \to 0$, implying $d(x, y) = 0$. Since we're working in a metric space we have $x = y$. $\square$

Problem 3. Let (a_n) be a sequence in $\mathbb{R}$.

a) There exists a subsequence of $(a_{n_k})_{k=1}^{\infty}$ of (a_n) such that $\lim_k a_{n_k} = \liminf_n a_n$.

Proof. Suppose that $\liminf(a_n) = +\infty$. We construct a sequence (b_n) diverging to $+\infty$. Let $b_1 = a_1$. Let b_2 be the first element a_k in a_n where $b_1 + 1 < a_k$. Such an element must exist since the terms in a_n are not bounded above. Let $(\hat{a}_{n_k})$ be the new subsequence of (a_n) formed by only considering those terms in (a_n) occuring after a_k. Repeat the process by choosing b_3 from the new sequence $(\hat{a}_{n_k})$ where $b_3 > b_2 + 1$. Proceeding in this manner we have that (b_n) increases by at least 1 with every element, and hence diverges to $+\infty$. The case for $\liminf(a_n) = -\infty$ is analogous; just choose $b_{i+1} < b_i - 1$.

Now suppose $\liminf a_n \to c$ for some $c \in \mathbb{R}$. We construct a subsequence (b_n) of (a_n) with $(b_n) \to c$. Let $b_1 = a_1$. Let a_k be the first element in (a_n) where $|a_k - c| \leq |\frac{b_1 - c}{2}|$. Such an element exists since $\liminf a_n \to c$ implies there are infinited terms in (a_n) within ϵ of c. Take $b_2 = a_k$. Again take the subsequence of (a_n) formed by only considering those terms in (a_n) occuring after a_k. Repeat the process to create a sequence (b_n) where $|b_n - c| \leq |\frac{b_1 - c}{2^n}|$. Hence $(b_n) \to c = \liminf_n a_n$.

$\square$

b) (a_n) converges to $a \in \mathbb{R}$ if and only if $\liminf_n a_n = \limsup_n a_n = a$.

Problem 4. Let (X, d) be a metric space.

a) The empty set and the set X itself are both open and closed sets in (X, d).

Proof. $\square$

b) The intersection of a finite collection of open sets is open.

Proof. $\square$

c) The union of an arbitrary collection of open sets is open.

Proof. $\square$

d) The union of a finite collection of closed sets is closed.

Proof. $\square$

e) The intersection of an arbitrary collection of closed sets is closed.

Proof. $\square$

Problem 5. Let (X, d_X) and (Y, d_Y) be metric spaces, $f : X \to Y$ a continuous function and $B \subset Y$ a closed set. Then A defined by

$$A = \{x \in X | f(x) \in B\}$$

is a closed set.

Proof. $\square$

Problem 6. Let X be a Banach space and let (x_n) be a sequence in X such that $\sum_{n=1}^{\infty} ||x_n|| = 1$.

a) Prove that the series $\sum_{n=1}^{\infty} x_n$ also converges to a limit $x \in X$.

Proof. $\square$

b) Prove that for an subsequence $(x_{n_k})_{k=1}^{\infty}$ of (x_n), the series $\sum_{n=1}^{\infty} x_{n_k}$ also converges and that the norm of its limit is bounded by 1.

Proof. $\square$