

# 1 Definitions

## 1.0.1 Metric Space

A **metric** on  $X$  is a function

$$d : X \times X \rightarrow \mathbb{R}$$

such that

1. Nonnegative  $d(x, y) \geq 0$  for all  $x, y \in X$ .
2.  $d(x, y) = 0 \iff x = y$ .
3. Symmetric  $d(x, y) = d(y, x)$ .
4. Triangle Inequality  $d(x, y) \leq d(x, z) + d(z, y)$

## 1.0.2 Linear Space

A linear space aka vector space defines a **norm** on  $X$

$$\|\cdot\| : X \rightarrow \mathbb{R}$$

such that

1. Nonnegative  $\|x\| \geq 0$  for all  $x \in X$ .
2.  $\|x\| = 0 \iff x = 0$ .
3. Scalar multiplication  $\|\lambda x\| = |\lambda| \cdot \|x\|$
4. Triangle Inequality  $\|x + y\| \leq \|x\| + \|y\|$

There is a corresponding metric induced by the norm.

A sequence  $(x_n)$  **converges** to  $x \in X$  if for all  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that  $d(x_n, x) < \epsilon$  for all  $n \geq N$ .

A sequence  $(x_n)$  is **Cauchy** if for all  $\epsilon > 0$  there is  $N \in \mathbb{N}$  such that  $d(x_m, x_n) < \epsilon$  for all  $m, n \geq N$ .

A metric space  $(X, d)$  is **complete** if every Cauchy sequence in  $X$  converges to a limit in  $X$ .

$G \subset X$  is **open** if for all  $x \in G$  there exists  $r > 0$  such that  $B_r(x) \subset G$ .  $F \subset G$  is **closed** if the complement  $F^C$  is open.

$\bar{A}$ , the **closure** of  $A$  is the smallest closed set containing  $A$ .

$K \subset X$  is **sequentially compact** if every sequence in  $K$  contains a subsequence that converges in  $K$ .

$K \subset X$  is **compact** if every open cover of  $K$  has a finite subcover.

A subset of  $X$  is **totally bounded** if there exists a finite  $\epsilon$  net for every  $\epsilon > 0$ .

$A \subset X$  is **dense** in  $X$  if  $\bar{A} = X$ .

A subset of  $X$  is **separable** if it has a countable dense subset. Compact metric spaces are separable.

A subset of  $X$  is **precompact** if its closure in  $X$  is compact.  $\iff$  Subset is totally bounded.

### 1.0.3 Theorems

A subset of a metric space is complete if and only if it is closed.

A subset of a metric space is compact if and only if it is complete and totally bounded.

Continuous functions on a compact set are uniformly continuous.

The image of a compact set under a continuous function is compact.

## 1.1 Functions

A function  $f : X \rightarrow Y$  is **continuous** at  $x_0 \in X$  if for all  $\epsilon > 0$  there exists  $\delta > 0$  such that  $d_X(x, x_0) < \delta$  implies  $d_Y(f(x), f(x_0)) < \epsilon$ . Equivalently:

1. There is a ball of positive radius  $\delta$  around a point  $x_0$  that is mapped into the ball of radius  $\epsilon$  around  $f(x_0)$ .
2.  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$

A function  $f : X \rightarrow Y$  is **uniformly continuous** on  $X$  if for all  $\epsilon > 0$  there exists  $\delta > 0$  such that  $d_X(x, y) < \delta$  implies  $d_Y(f(x), f(y)) < \epsilon$  for all  $x, y \in X$ .

The **sup norm** or  $L^\infty$  norm between functions on  $\mathbb{R}$  is defined as

$$d(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|$$

A sequence of bounded, real valued functions  $(f_n)$  on a metric space **converges uniformly** to a function  $f$  if

$$\lim_{n \rightarrow \infty} \|f_n - f\| = 0$$

$C(K)$ , the space of continuous functions on a compact metric space  $K$  is complete.

**Weierstrass Approximation** The set of polynomials is dense in  $C([a, b])$ .