Solution sketches

# Midterm—Statistics 232A

**NAME:** ______________________________ 25 October 2011

**Instructions:** Solve the problems in the blank space below each, continuing on the back of the sheet as needed. **Closed book**. You may consult two, double-sided, letter-size sheets of personal notes and cite results from class.

1. Consider the Gaussian linear model

$$y_i = \beta x_i^2 + e_i, \qquad 1 \le i \le n,$$

where the $\{x_i\}$ are distinct known values and the $\{e_i\}$ are independent, identically distributed $N(0, \sigma^2)$ random variables. Here $\beta$ and $\sigma^2 > 0$ are unknown real-valued parameters.

a) (4 points) Find and simplify the least squares estimator $\hat{\beta}_{LS}$ of $\beta$. Find its distribution.

b) (4 points) Consider the alternative estimator of $\beta$ given by $\hat{\beta}_A = \sum_{i=1}^n y_i / \sum_{i=1}^n x_i^2$. Find its distribution.

c) (2 points) Evaluate the quadratic risks $\mathrm{E}(\hat{\beta}_{LS} - \beta)^2$ and $\mathrm{E}(\hat{\beta}_A - \beta)^2$.

d) (4 points) Which of the estimators $\hat{\beta}_{LS}$ and $\hat{\beta}_A$ has smaller quadratic risk? Prove your answer by using standard theorems in statistics or mathematics.

a) $y = X\beta + e$ with $\underset{n\times 1}{X} = (x_1^2, x_2^2, \ldots, x_n^2)'$ and $y = (y_1, y_2, \ldots, y_n)'$

$\hat{\beta}_{LS} = (X'X)^{-1}X'y = \sum_{i=1}^n x_i^2 y_i / \sum_{i=1}^n x_i^4$

$E(\hat{\beta}_{LS}) = \beta$, $Var(\hat{\beta}_{LS}) = \sigma^2 (X'X)^{-1} = \sigma^2 / \sum_{i=1}^n x_i^4$, $\hat{\beta}_{LS} \sim N(\beta, \sigma^2 / \sum_{i=1}^n x_i^4)$

b) $E(\hat{\beta}_A) = \beta$, $Var(\hat{\beta}_A) = n\sigma^2 / (\sum_{i=1}^n x_i^2)^2$, $\hat{\beta}_A \sim N(\beta, n\sigma^2/(\sum_{i=1}^n x_i^2)^2)$

c) $E(\hat{\beta}_{LS} - \beta)^2 = Var(\hat{\beta}_{LS}) = \sigma^2 / \sum_{i=1}^n x_i^4$

$E(\hat{\beta}_A - \beta)^2 = Var(\hat{\beta}_A) = n\sigma^2 / (\sum_{i=1}^n x_i^2)^2$

d) $\hat{\beta}_{LS}$ has smaller risk when the $\{x_i\}$ are distinct

Argument 1: Gauss-Markov theorem because $\hat{\beta}_{LS}$ and $\hat{\beta}_A$ are both linear unbiased estimators

Argument 2: Cauchy-Schwarz inequality yields

$\frac{1}{n}(\sum_{i=1}^n x_i^2)^2 \le \frac{1}{n}(\sum_{i=1}^n x_i^4)(\sum_{i=1}^n 1^2) = \sum_{i=1}^n x_i^4$. Hence $\frac{n}{(\sum_{i=1}^n x_i^2)^2} \ge \frac{1}{\sum_{i=1}^n x_i^4}$

Strict inequality when the $\{x_i\}$ are distinct

2. $A$ and $B$ are $m \times m$ **symmetric** matrices such that $AB = 0$ (i.e. the matrix whose elements are all 0).

   a) (5 points) Show that $A^+B = B^+A = AB^+ = BA^+ = 0$.

   b) (5 points) Hence show that $(A+B)^+ = A^+ + B^+$.

a) $BA = B'A' = (AB)' = 0$. Then

$A^+B = (A'A)^+A'B = (A'A)^+AB = 0$

$B^+A = (B'B)^+B'A = (B'B)^+BA = 0$

$AB^+ = AB'(BB')^+ = AB(BB')^+ = 0$

$BA^+ = BA'(AA')^+ = BA(AA')^+ = 0$

b) Verify the 4 properties that characterize $(A+B)^+$:

(i) $(A^+ + B^+)(A+B) = A^+A + B^+B$ by part a). Symmetric

(ii) $(A+B)(A^+ + B^+) = AA^+ + BB^+$ by part a) Symmetric

(iii) $(A+B)(A^+ + B^+)(A+B) = (A+B)(A^+A + B^+B)$ by (i)

$= AA^+A + BB^+B$ using part a)

$= A + B$

(iv) $(A^+ + B^+)(A+B)(A^+ + B^+) = (A^+ + B^+)(AA^+ + BB^+)$ by (ii)

$= A^+AA^+ + B^+BB^+$ using part a)

$= A^+ + B^+$

3. Consider the Gaussian linear model in which $y = (y_1, y_2, y_3, y_4)'$ has a $N(\eta, \sigma^2 I_4)$ distribution and $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)'$ has the structure

$$\eta_1 = m + a_1, \qquad \eta_2 = m + a_1$$
$$\eta_3 = m + a_2, \qquad \eta_4 = m + a_2.$$

In this model, $m$, $a_1$, $a_2$ and $\sigma^2 > 0$ are unknown real-valued parameters.

a) (4 points) Let $\beta = (m, a_1, a_2)'$. Write out the matrix $X$ such that $\eta = X\beta$. Find the rank of $X$, justifying your answer.

b) (4 points) Show that $m$ is **not** linearly estimable.

c) (4 points) Show that $\hat{\beta} = (1/2)(0, y_1 + y_2, y_3 + y_4)'$ is a least squares estimator of $\beta$. Is it an unbiased estimator of $\beta$? Explain your finding.

d) (4 points) Show that $\psi = a_1 - a_2$ is linearly estimable. Find the minimum variance unbiased linear estimator of $\psi$ as a function of $y$.

a) $\underset{4\times 3}{X} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ rank$(X) \geq 2$ because cols 2, 3 are orthogonal
$\leq 2$ because col 2 + col 3 = col 1
Hence rank$(X) = 2$

b) $m = \lambda'\beta$ with $\lambda' = (1, 0, 0)$. $m$ is linearly estimable iff $\lambda' = a'X$ for some $\underset{4\times 1}{a}$

Such $a = (a_1, a_2, a_3, a_4)'$ must satisfy
$$\left.\begin{aligned} a_1 + a_2 + a_3 + a_4 &= 1 \\ a_1 + a_2 &= 0 \\ a_3 + a_4 &= 0 \end{aligned}\right\} \text{Inconsistent equations!}$$

c) Normal equation $X'X\beta = X'y$ says

$$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{pmatrix} = \begin{pmatrix} y_1 + y_2 + y_3 + y_4 \\ y_1 + y_2 \\ y_3 + y_4 \end{pmatrix}$$

The given $\hat{\beta}$ solves this.

$E(\hat{\beta}) = (0, m + a_1, m + a_2)' \neq \beta$. Because of part b), $\beta$ is <u>not</u> linearly estimable

d) $\psi = a_1 - a_2 = \eta_1 - \eta_3 = E(y_1 - y_3)$ is therefore linearly estimable. Because $\psi = \lambda'\beta$ with $\lambda' = (0, 1, -1)$, its unique minimum variance unbiased linear estimator is $\hat{\psi} = \lambda'\hat{\beta} = \hat{\beta}_2 - \hat{\beta}_3 = \frac{1}{2}(y_1 + y_2 - y_3 - y_4)$