Lab #1

Fall 2013 R. Beran

DUE in Discussion Section on Tuesday, 8 October 2013.

1. Let A be an arbitrary $m \times n$ matrix of rank r. Show that its pseudoinverse A^+ has the following properties. State clearly the standard results from linear algebra used in your proofs.
   a) If A is square and has an inverse, then $A^+ = A^{-1}$.
   b) If c is any non-zero scalar, then $(cA)^+ = c^{-1}A^+$.
   c) $(A^+)^+ = A$.
   d) $(A^+)' = (A')^+$.
   e) $(A'A)^+ = A^+(A')^+$ and $(AA')^+ = (A')^+A^+$.
   f) A^+A and AA^+ are each symmetric and idempotent matrices.
   g) $A^+ = (A'A)^+A' = A'(AA')^+$.
   h) $A^+ = \lim_{\epsilon\to 0+}(A'A + \epsilon I_n)^{-1}A' = \lim_{\epsilon\to 0+} A'(AA' + \epsilon I_m)^{-1}$.
   i) $\mathrm{rank}(A) = \mathrm{rank}(AA^+) = \mathrm{tr}(AA^+) = \mathrm{tr}(A^+A) = \mathrm{rank}(A^+A) = \mathrm{rank}(A^+)$.
   j) If A is symmetric and positive semidefinite, then we may take $U = V$ in the singular value decomposition $A = ULV'$.
   k) If A is symmetric and idempotent, then $A^+ = A$.
   l) Suppose S is any $s \times m$ matrix, with $s \geq m$, such that $S'S = I_m$. Suppose T is any $t \times n$ matrix, with $t \geq n$,such that $T'T = I_n$. Then $(SAT')^+ = TA^+S'$.
   m) Suppose that P is any $m \times m$ symmetric, idempotent matrix. Suppose that Q is any $n \times n$ symmetric, idempotent matrix. Then $(PAQ)^+ = Q(PAQ)^+P$.
   n) Suppose that A is any $m \times n$ matrix. Let $\mathcal{R}(A) = \{Ax\colon x \in R^n\}$ be the range of A. Then $\mathcal{R}(A) = \mathcal{R}(AA^+)$ and $\mathcal{R}(A^+) = \mathcal{R}(A^+A) = \mathcal{R}(A')$.

2. Let A be an arbitrary $m \times n$ matrix, $m \geq n$, that has mutually orthogonal columns.
   a) Find a singular value decomposition $A = ULV'$, expressing U, L, V explicitly as simple algebraic functions of A.
   b) Find algebraically the pseudoinverse A^+ as a simple function of A. Verify directly that your A^+ has the four properties that characterize a pseudoinverse.