1 Distributions

1.1 Binomial

Sum of n bernoulli trials with probability of success p.

$X \sim B(n,p)$

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad k = 0, 1, \ldots, n$$

mgf: $M_X(t) = (pe^t + 1 - p)^n$

$\mathrm{E}\,X = np, \quad \mathrm{Var}\,X = np(1-p)$

1.2 Poisson

$X \sim P(\lambda)$

$$p(k) = \frac{e^{-\lambda}\lambda^k}{k!} \qquad k = 0, 1, \ldots$$

mgf: $M_X(t) = e^{\lambda(e^t - 1)}$

$\mathrm{E}\,X = \lambda, \quad \mathrm{Var}\,X = \lambda$

1.3 Multivariate Normal

$X \sim N(\mu, \Sigma)$, Σ positive definite

$$f(x) = \frac{\exp\{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\}}{(2\pi)^{\frac{k}{2}}\sqrt{\det(\Sigma)}}$$

1.4 Beta

$X \sim \mathrm{Beta}(\alpha, \beta)$

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \qquad 0 \le x \le 1$$

using the beta function:

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$$

$\mathrm{E}\,X = \frac{\alpha}{\alpha+\beta}, \quad \mathrm{Var}\,X = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

1.5 Gamma

$X \sim \mathrm{Gamma}(p, \lambda)$

$$f(x) = \frac{\lambda^p x^{p-1} e^{-\lambda x}}{\Gamma(p)} \qquad x > 0$$

mgf: $X \sim \mathrm{Gamma}(p,\lambda) \iff \lambda X \sim \mathrm{Gamma}(p, 1)$

Gamma function: $\Gamma(p) = \int_0^\infty t^{p-1}e^{-t}dt$.

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

$\Gamma(p+1) = p\Gamma(p)$

$\Gamma(k) = (k-1)!$ for k positive integer.

1.6 Chi square

χ^2_n Special case: $\chi^2_n \equiv \mathrm{Gamma}(\frac{n}{2}, \frac{1}{2})$

Let Z_i be iid $N(0,1)$.

$\sum_{i=1}^n Z_i^2 \sim \chi^2_n$

Noncentral χ^2. Let $Y \sim N(\mu, I)$ be an n vector. Then

$$\|Y\|^2 \sim \chi^2_n(\|\mu\|^2)$$

1.7 F

$$F(m,n) \equiv \frac{\frac{\chi^2_m}{m}}{\frac{\chi^2_n}{n}}$$

Where numerator and denominator are independent χ^2.

1.8 T

$$t(n) = \frac{N(0,1)}{\sqrt{\frac{\chi^2_n}{n}}}$$

Where numerator and denominator are independent.

1.9 Studentized range distribution

Let $z_1, \ldots z_K$ be i.i.d $N(0,1)$ and X be an independent χ^2_ν. Then

$$\frac{\max_i z_i - \min_i z_i}{\sqrt{(X/\nu)}} \sim q_{K,\nu}$$

2 Math

Order statistics for sorted sample $X_{(1)}, \ldots, X_{(n)}$ has pdf:

$$n! \prod_{i=1}^{n} f(X_{(i)}) \quad I(X_{(1)} < \cdots < X_{(n)})$$

2.1 Multivariate Normal

$X \sim N(\mu, \Sigma)$, A an $m \times n$ matrix, then

$$AX \sim N(A\mu, A\Sigma A^t)$$

For Σ full rank it's possible to transform between $z \sim N(0, I)$ and X:

$$X = \Sigma^{1/2} z + \mu \qquad z = \Sigma^{-1/2}(X - \mu)$$

Conditional distribution:

2.2 Conditional Distributions

Conditional pdf:

$$f_{x|y}(x, y) \equiv \frac{f_{x,y}(x, y)}{f_y(y)}$$

Iterated expectation:

$$E(Y) = E(E(Y|X))$$

Conditional variance formula:

$$\mathrm{Var}(Y) = \mathrm{Var}(E(Y|X)) + E(\mathrm{Var}(Y|X))$$

3 Applied

Matrix / Vector differentiation

3.1 Linear Models

Least Squares Principle

$$\arg\min_\beta \|Y - X\beta\|^2$$

Normal Model

$$Y = X\beta + \epsilon, \qquad \epsilon \sim N(0, \sigma^2 I)$$

Normal Equations - Any b satistfying this solves the least squares

$$X^T X b = X^T y$$

Gauss Markov Theorem - $\hat{\beta}(X^T X)^{-1} y$ is Best Linear Unbiased Estimator (BLUE) of β.

Estimating the variance: $\frac{\|y - X\hat{\beta}\|^2}{\sigma^2} \sim \chi^2_{n-p}$.

$$\hat{\sigma}^2 = \frac{\left\|y - X\hat{\beta}\right\|^2}{n - p}$$

Use t test for hypothesis testing and confidence intervals for the value of a particular β_j coefficient.

$$\frac{\beta_j - \beta_j^*}{\hat{\sigma}\sqrt{w_{ii}}} \sim t_{n-p}$$

General linear tests. Partition $\beta = (\beta_1, \beta_2)$ where β_1 is an r vector and β_2 is $p - r$. Null hypothesis $H_0 : \beta_2 = \beta_2^*$ (often 0), and $H_a : \beta_2 \neq \beta_2^*$. Then $SSE_r = \left\|y - X_2\beta_2^* - X_1\tilde{\beta}_1\right\|^2$ is the sum of squared error for the reduced model and $SSE_f = \left\|y - X\hat{\beta}\right\|^2$ is the squared sum of error for the full model. Under H_0:

$$\frac{\frac{SSE_r - SSE_f}{p-r}}{\frac{SSE_f}{n-p}} \sim F_{p-r,n-p}$$

Alternate forms of linear test, and testing a linear combination if $R\beta = r$.

$$\frac{(R\hat{\beta} - r)^T (R(X^T X)^{-1} R^T)^{-1} (R\beta - r)/s^2}{\hat{\sigma}} \sim F_{s,n-p}$$

Confidence intervals for new observation Y_h and $E[Y_h]$.

Simultaneous confidence bands for all $f(x)$.

If the model has an intercept then $SSTO = SSR + SSE$.

$$\sum (y_i - \bar{y})^2 = \sum (\hat{y_i} = \bar{y})^2 + (yi - \hat{yi})^2$$

R^2 and adjusted R^2

3.2 ANOVA

Three principles of experimental design: 1) Replication 2) Randomization 3) Blocking

One way ANOVA with n total observations, K groups:

SS		DF
SSTR	$\sum_{j=1}^{K} n_j(\bar{y_j.} - \bar{y..})^2$	K - 1
SSE	$\sum_{i=1}^{n} (y_{ij} - \bar{y_j.})^2$	n - K
SSTO	$\sum_{i=1}^{n} (y_{ij} - \bar{y..})^2$	n - 1

Contrasts are sums of the form $\Phi = \sum_{i=1}^{K} c_i \mu_i$ with $\sum_{i=1}^{K} c_i = 0$. Tukey's works for all pairwise contrasts. Scheffe's and extended Tukey works for all contrasts. Bonferroni's is for a limited number of pre specified contrasts.

3.3 General Techniques

Singular Value Decompostion (SVD) Any matrix X can be written

$$X = UDV^T$$

with U, V orthogonal, and D diagonal.

Moore Penrose Psuedoinverse A^+ exists uniquely for every matrix A.

Projection matrix P are symmetric and idempotent. They have eigenvalues either 0 or 1.

$$P = P^T \qquad P^2 = P$$

Covariance of linear transformations

$$Cov(Ay, Bx) = ACov(y, x)B^T$$

4 Lecture topics

4.1 Applied

4 Nov - Interaction plots for two way ANOVA with balanced design, Linear models with random X

2 Nov - Midterm

28 Oct - Kronecker product formulae for two way ANOVA

26 Oct - Kronecker product 1 way ANOVA, decomposition of two way ANOVA, noncentral χ^2 distributions for ANOVA table SSA, SSB, SSAB

21 Oct - Tukey's method for pairwise contrasts, Bonferroni's method, definition and properties of Kronecker product

4.2 Math

4 Nov - Midterm

2 Nov - Location-scale families, invariance, ancillary and sufficient statistics, order statistics, multinomial distribution

28 Oct - Transformation of discrete and continuous random variables, Jacobian, examples with beta distributions, Dirichlet distribution

26 Oct - Jensen's and Holder's inequality, convex functions and sets, products of normal random variables

21 Oct - Convolution formula, examples with Uniform, Gamma, Poisson, marginal and conditional distributions for multivariate normal

5 Miscellaneous

Integration by parts:

$$\int uv' = uv - \int u'v$$