CS6033 – Design and Analysis of Algorithms I

Homework 7

Manuel — NYU (Spring 2021)

Reminders

- Write in a neat and legible handwriting or use LaTeX
- Clearly explain the reasoning process
- Write in a complete style (subject, verb, and object)
- Be critical on your results

*Questions preceded by a * are optional. Although they can be skipped without any deduction, it is important to know and understand the results they contain.*

Ex. 1 — *Karger-Stein's Algorithm*

In the lectures, although Karger-Stein's Algorithm was presented (5.28|5.247), only a sketch of proof was provided (5.29|5.248). In this exercise we want to prove the missing part, i.e. solve the recurrence relation

$$P(t) = 1 - \left(1 - \frac{1}{2}P\left(\frac{t}{\sqrt{2}}\right)\right)^2.$$

1. Prove that the probability of a cut to survive when $n < 6$, at least $1/15$.
2. Using an appropriate change of variable, show that

$$\begin{cases} p_{k+1} &= p_k - \frac{1}{4}p_k^2, \\ p_0 &= 1/15. \end{cases}$$

3. Let $z_k = 4/p_k - 1$.

   a) Prove that

$$\begin{cases} z_{k+1} &= z_k + 1 + \frac{1}{z_k}, \\ z_0 &= 59. \end{cases}$$

   * b) Show that for all $k \geq 1$, $k < z_k < 59 + 2k$.

4. Recalling that $t = n/\sqrt{2}$ and noting that the depth of the recursion is $2\log_2 n + \mathcal{O}(1)$, conclude that $P(n) = \Omega(1/\log n)$.

Ex. 2 — *Critical thinking*

1. Is it possible to design a stack supporting `push`, `pop`, and retrieving the minimum element in constant time? Explain.
2. A total of n ants are walking at speed 1 m/s on a thin 1 m long cable; if they collide they instantly reverse direction, and if an ant reaches the end of the cable it falls. How long will it take before all the ants have fallen?

* **Ex. 3 —** *Interview problem*

Given an array A of size n, split it into as few subarrays as possible, with the property that for each of them the gcd of its first and last elements must be larger than 1.