Chapter 1 The complex plane

1. Complex numbers

We have learnt in secondary school that the quadratic equation

$$x^2 + 1 = 0$$

has **no real solutions**, because $x^2 + 1 > 0$ for every real number x. In order to invent a solution to such an equation, one comes up with the notion of **complex numbers**. By creating a symbol

$$i \coloneqq \sqrt{-1}$$

so that $i^2 = -1$, we obtain a "solution" to the equation. This motivates the following definition.

Definition 1.1 Consider the real vector space $\mathbb{R}^2$, together with a multiplication defined by

$$(a, b)(c, d) \coloneqq (ac - bd, ad + bc)$$

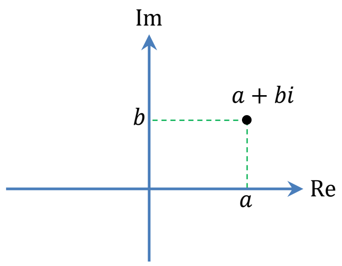
for every $a, b, c, d \in \mathbb{R}$.

- Each element (or point) (a, b) in this space is called a ***complex number***.
- The complex number $(1, 0)$ is alternatively written as 1, while the complex number $(0, 1)$ is alternatively written as i; so the complex number (a, b) is written as $a + bi$ or $a + ib$.
- The ***set of all complex numbers*** is usually denoted by $\mathbb{C}$, i.e.

$$\mathbb{C} \coloneqq \{a + bi : a, b \in \mathbb{R}\}.$$

Remark 1.2

- A real number a is regarded the same as a complex number of the form $(a, 0)$ or $a + 0i$. With this identification we may write $\mathbb{R} \subset \mathbb{C}$ and say that a ***real*** number is also a complex number. The multiplication $(a, b)(c, d) \coloneqq (ac - bd, ad + bc)$ of complex numbers reduces to the usual multiplication when we are dealing with a pair of real numbers.
- A complex number of the form $(0, b)$ or $0 + bi$ is said to be ***purely imaginary***. We can verify that $i^2 = -1$, as expected in the motivation of defining complex numbers.
- A complex number $a + bi$ can be interpreted geometrically as the point (a, b) in the coordinate plane $\mathbb{R}^2$. In this context, the coordinate plane is called the ***complex plane***, the horizontal axis is called the ***real axis*** and the vertical axis is called the ***imaginary axis***.



Definition 1.3 The ***real part*** $\operatorname{Re}: \mathbb{C} \to \mathbb{R}$ and the ***imaginary part*** $\operatorname{Im}: \mathbb{C} \to \mathbb{R}$ are projections onto the first and second coordinates respectively. In other words, given a complex number

$$z = a + bi,$$

- the real number a is called the ***real part*** of z and is denoted by $\operatorname{Re} z \coloneqq a$, and
- the real number b is called the ***imaginary part*** of z and is denoted by $\operatorname{Im} z \coloneqq b$.

As introduced in Definition 1.1, the **arithmetic operations** of complex numbers are done as follows. Given two complex numbers $z = a + bi$ and $w = c + di$, we have

$$z + w = (a + c) + (b + d)i$$
$$z - w = (a - c) + (b - d)i$$
$$zw = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$
$$\frac{z}{w} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i \qquad \text{if } w \neq 0.$$

Lemma 1.4 *$\mathbb{C}$ equipped with the above addition and multiplication is a field.*

Example 1.5 For every $\theta, \phi \in \mathbb{R}$, we have

$$\begin{aligned}(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) &= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi) \\ &= \cos(\theta + \phi) + i\sin(\theta + \phi).\end{aligned}$$

Definition 1.6 The ***complex conjugate*** is the function $\bar{\cdot}: \mathbb{C} \to \mathbb{C}$ defined by

$$\overline{a + bi} := a - bi$$

for every $a, b \in \mathbb{R}$.

Lemma 1.7 *Let $z \in \mathbb{C}$ be a complex number.*

- *If z is real, then $\bar{z} = z$.*
- *If z is purely imaginary, then $\bar{z} = -z$.*

Lemma 1.8 *Given any two complex numbers $z, w \in \mathbb{C}$, we have*

$$\overline{z + w} = \bar{z} + \bar{w}, \qquad \overline{z - w} = \bar{z} - \bar{w}, \qquad \overline{zw} = \bar{z}\,\bar{w} \quad \text{and} \quad \overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}.$$

In other words, the complex conjugate is a field isomorphism.

Lemma 1.9 *Given any complex number $z \in \mathbb{C}$, we have*

$$z + \bar{z} = 2\operatorname{Re} z \qquad \text{and} \qquad z - \bar{z} = 2i\operatorname{Im} z.$$

Lemma 1.10 *For every complex number $z \in \mathbb{C}$,*

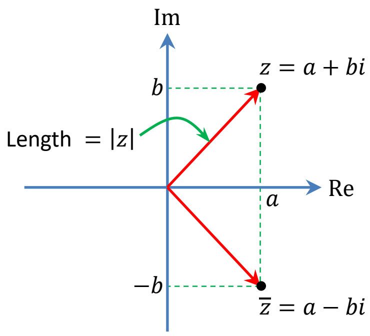
- *$z\bar{z}$ is a non-negative real number, and*
- *$z\bar{z} = 0$ only when $z = 0$.*

Definition 1.11 For each complex number $z = a + bi \in \mathbb{C}$, the ***absolute value*** (or ***modulus***) of z is the non-negative real number defined by

$$|z| := \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}.$$

Remark 1.12 It is important to note that the field $\mathbb{C}$ **does not have a total order** which is compatible with that of $\mathbb{R}$. In other words, if $z, w \in \mathbb{C}$, then it does not make sense to write "inequalities" like $z > w$ or $z < w$. However, inequalities like $|z| > |w|$ or $|z| < |w|$ are still meaningful because these absolute values are (non-negative) real numbers.

The complex conjugate and absolute value can be interpreted **geometrically** in the complex plane as follows.



Lemma 1.13 *For every* $z \in \mathbb{C}$, *we have*

$$|\mathrm{Re}\, z| \leq |z|, \qquad |\mathrm{Im}\, z| \leq |z| \qquad \textit{and} \qquad |\bar{z}| = |z|.$$

Theorem 1.14 (Triangle inequality) *For every complex numbers* $z, w \in \mathbb{C}$, *we have*

$$|z + w| \leq |z| + |w|$$

Proof.

$$\begin{aligned} |z+w|^2 = (z+w)(\bar{z}+\bar{w}) = z\bar{z} + w\bar{w} + z\bar{w} + \bar{z}w &= |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\bar{w}) \\ &\leq |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2. \end{aligned}$$

■

Corollary 1.15 (Triangle inequality) *For every complex numbers* $z, w \in \mathbb{C}$, *we have*

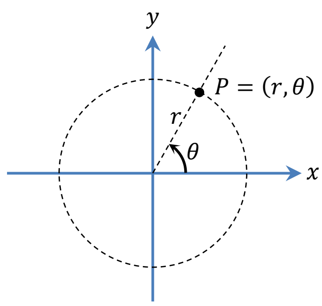
$$\big||z| - |w|\big| \leq |z - w|.$$

Proof. By Theorem 1.14 we have

$$|z| = |(z - w) + w| \leq |z - w| + |w| \qquad \text{and} \qquad |w| = |z - (z - w)| \leq |z| + |z - w|,$$

so $-|z - w| \leq |z| - |w| \leq |z - w|$. ■

Recall that the **polar coordinates** is another useful coordinate system in $\mathbb{R}^2$, apart from the rectangular coordinates. We will see that while addition of complex numbers is easily done using the form $a + bi$ as suggested by the rectangular coordinates, multiplication of complex numbers becomes easy when using another form of complex numbers as suggested by the polar coordinates.



Every point (x, y) in the coordinate plane has polar coordinates given by (r, θ), where

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}.$$

So every complex number $z = x + iy$ can also be written in the **polar form** as

$$z = (r\cos\theta) + i(r\sin\theta) = r(\cos\theta + i\sin\theta).$$

Here the non-negative real number $r = \sqrt{x^2 + y^2}$ is just the modulus of z. If $z \neq 0$, then the real number θ is called an ***argument*** of z. Although θ is not unique (because cos and sin are 2π-periodic functions), θ can still be uniquely chosen on each half-open interval of length 2π.

Lemma 1.16 *For every $z \in \mathbb{C} \setminus \{0\}$, there exists a unique $\theta \in (-\pi, \pi]$ such that*

$$z = |z|(\cos\theta + i\sin\theta).$$

Definition 1.17 Let $z \in \mathbb{C} \setminus \{0\}$. The number $\theta \in (-\pi, \pi]$ such that $z = |z|(\cos\theta + i\sin\theta)$ is called the ***principal argument*** of z, and is denoted as $\operatorname{Arg} z = \theta$. The set of all ***arguments*** of z is denoted as $\arg z = \{\theta + 2n\pi : n \in \mathbb{Z}\}$.

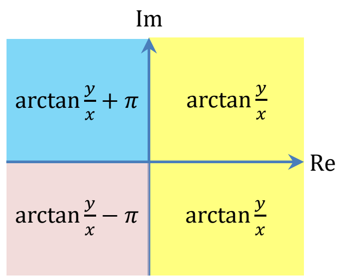
Example 1.18 The complex number $z = -1 + \sqrt{3}i$ is written in polar form as

$$z = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right).$$

Although $2\left(\cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}\right)$, $2\left(\cos\frac{14\pi}{3} + i\sin\frac{14\pi}{3}\right)$, … are also correct polar forms of z, the **principal argument** of z is $\operatorname{Arg} z = 2\pi/3$ as it lies in the interval $(-\pi, \pi]$.

Theorem 1.19 *The principal argument $\operatorname{Arg}: \mathbb{C} \setminus \{0\} \to (-\pi, \pi]$ has the following formula:*

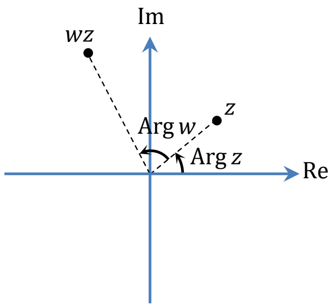
$$\operatorname{Arg}(x + iy) = \begin{cases} \arctan\frac{y}{x} - \pi & \text{if } x < 0 \text{ and } y < 0 \text{ (i.e. Quadrant III)} \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \text{ (i.e. negative imaginary axis)} \\ \arctan\frac{y}{x} & \text{if } x > 0 \text{ (i.e. Quadrants I or IV or positive real axis)} \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \text{ (i.e. positive imaginary axis)} \\ \arctan\frac{y}{x} + \pi & \text{if } x < 0 \text{ and } y \geq 0 \text{ (i.e. Quadrant II or negative real axis)} \end{cases}.$$



Example 1.20 Let w be a non-zero complex number. Then the function $f: \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = wz \qquad \text{for every } z \in \mathbb{C}$$

is interpreted geometrically in the complex plane as the composition of a counterclockwise rotation by the angle $\operatorname{Arg} w$ about the origin (see Example 1.5), and a scaling by the factor $|w|$.



Theorem 1.21 (de Moivre) *Let θ be a real number and n be an integer. Then*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Proof. The equality obviously holds for $n = 1$. Assume that it holds for $n = k$ where k is some positive integer. Then

$$\begin{aligned}(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta) = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)\\ &= \cos(k\theta + \theta) + i\sin(k\theta + \theta) = \cos(k+1)\theta + i\sin(k+1)\theta,\end{aligned}$$

so the equality holds for $n = k + 1$ as well. By induction, it holds for every positive integer n. The equality obviously holds for $n = 0$ also. Finally, if n is a negative integer, then $m := -n$ is a positive integer, so

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= (\cos\theta + i\sin\theta)^{-m} = \frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{\cos m\theta + i\sin m\theta}\\ &= \frac{1}{\cos m\theta + i\sin m\theta}\cdot\frac{\cos m\theta - i\sin m\theta}{\cos m\theta - i\sin m\theta} = \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}\\ &= \cos(-n\theta) - i\sin(-n\theta) = \cos n\theta + i\sin n\theta,\end{aligned}$$

so the equality holds in this case again. ■

Example 1.22 Show that

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$

for every real number θ.

Proof: For every real number θ, de Moivre's Theorem gives

$$\begin{aligned}\cos 4\theta + i\sin 4\theta &= (\cos\theta + i\sin\theta)^4 = \sum_{k=0}^{4}\binom{4}{k}(\cos\theta)^{4-k}(i\sin\theta)^k\\ &= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + i(4\cos^3\theta\sin\theta - 4\sin^3\theta\cos\theta).\end{aligned}$$

Comparing the real parts of both sides, we get

$$\begin{aligned}\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta\\ &= \cos^4\theta - 6\cos^2\theta(1 - \cos^2\theta) + (1 - \cos^2\theta)^2\\ &= 8\cos^4\theta - 8\cos^2\theta + 1\end{aligned}$$

for every real number θ. ■

Remark 1.23 Let n be a positive integer. Then for each integer k, the complex number

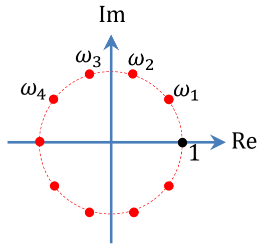
$$\omega_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$$

is a solution to the equation $z^n = 1$ since

$${\omega_k}^n = \left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right)^n = \cos 2k\pi + i\sin 2k\pi = 1$$

by de Moivre's Theorem. We say that the set of all ***n^{th} roots of 1*** is

$$\left\{\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} : k \in \{0, 1, 2, \dots, n-1\}\right\},$$



10^{th} roots of 1

which consists of exactly n elements. Geometrically, the n^{th} roots of 1 are represented in the complex plane as **vertices of a regular n-sided polygon** inscribed in the unit circle centered at the origin, one of these vertices located at 1. More generally, if $a \in \mathbb{C} \setminus \{0\}$, then the set of all ***n^{th} roots of a*** is

$$\left\{\sqrt[n]{|a|}\left(\cos\frac{\operatorname{Arg} a + 2k\pi}{n} + i\sin\frac{\operatorname{Arg} a + 2k\pi}{n}\right) : k \in \{0, 1, 2, \dots, n-1\}\right\},$$

which consists of exactly n elements. At this moment we avoid using the symbol $a^{1/n}$ to denote any particular n^{th} root of a complex number a. We will introduce the exponent notation for a general complex base later in chapter 3.

Example 1.24 Find

(a) all the cube roots of i, and

(b) all the square roots of $1 - \sqrt{3}i$.

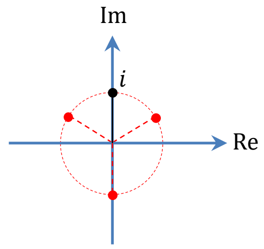
Solution:

(a) In polar form we have $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$. Therefore the cube roots of i is the set

$$\left\{\cos\frac{\frac{\pi}{2} + 2k\pi}{3} + i\sin\frac{\frac{\pi}{2} + 2k\pi}{3} : k \in \{0, 1, 2\}\right\}$$

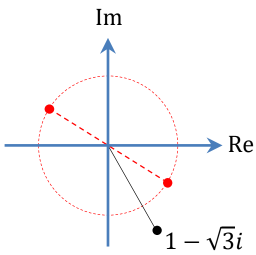
$$= \left\{\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}, \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}, \cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}\right\}$$

$$= \left\{\frac{\sqrt{3} + i}{2}, \frac{-\sqrt{3} + i}{2}, -i\right\}.$$



(b) In polar form we have $1-\sqrt{3}i = 2\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)$. Therefore the square roots of $1-\sqrt{3}i$ is the set

$$\left\{\sqrt{2}\left(\cos\frac{-\frac{\pi}{3}+2k\pi}{2}+i\sin\frac{-\frac{\pi}{3}+2k\pi}{2}\right): k\in\{0,1\}\right\}$$
$$=\left\{\sqrt{2}\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right), \sqrt{2}\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)\right\}$$
$$=\left\{\frac{\sqrt{3}-i}{\sqrt{2}}, \frac{-\sqrt{3}+i}{\sqrt{2}}\right\}.$$



Example 1.25 Factorize the polynomial

$$z^7+z^4+z^3+1$$

completely, into linear factors with complex coefficients.

Solution: [If we were asked to factorize it into factors with real coefficients, then we simply have

$$\begin{aligned} z^7+z^4+z^3+1 &= (z^3+1)(z^4+1) \\ &= (z+1)(z^2-z+1)(z^4+1). \end{aligned}$$

But now we need to factorize it completely with **complex coefficients**, so we need all of its roots.]

We consider the equation $z^7+z^4+z^3+1=0$, i.e. $(z^3+1)(z^4+1)=0$.

(i) If $z^3+1=0$, then $z^3=-1=\cos\pi+i\sin\pi$. So z belongs to the set

$$\left\{\cos\frac{\pi+2k\pi}{3}+i\sin\frac{\pi+2k\pi}{3}: k\in\{0,1,2\}\right\},$$

i.e. $z=\frac{1+\sqrt{3}i}{2}$ or $z=-1$ or $z=\frac{1-\sqrt{3}i}{2}$.

(ii) If $z^4+1=0$, then $z^4=-1=\cos\pi+i\sin\pi$. So z belongs to the set

$$\left\{\cos\frac{\pi+2k\pi}{4}+i\sin\frac{\pi+2k\pi}{4}: k\in\{0,1,2,3\}\right\},$$

i.e. $z=\frac{1+i}{\sqrt{2}}$ or $z=\frac{-1+i}{\sqrt{2}}$ or $z=\frac{-1-i}{\sqrt{2}}$ or $z=\frac{1-i}{\sqrt{2}}$.

Consequently we obtain the factorization

$$\begin{aligned} & z^7+z^4+z^3+1 \\ &= (z+1)\underbrace{\left(z-\frac{1+\sqrt{3}i}{2}\right)\left(z-\frac{1-\sqrt{3}i}{2}\right)}_{z^2-z+1}\underbrace{\left(z-\frac{1+i}{\sqrt{2}}\right)\left(z-\frac{-1+i}{\sqrt{2}}\right)\left(z-\frac{-1-i}{\sqrt{2}}\right)\left(z-\frac{1-i}{\sqrt{2}}\right)}_{z^4+1}. \end{aligned}$$

Example 1.26 Let $n \geq 2$ be an integer. Show that

$$\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \cdots + \cos\frac{2(n-1)\pi}{n} = -1.$$

Proof:
Let p be the polynomial $p(z) = z^n - 1$. Then the roots of p are

$$\omega_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$$

where $k \in \{0, 1, \dots, n-1\}$. Since $n \geq 2$, the coefficient of z^{n-1} in p is zero. So the sum of roots of p is given by

$$\omega_0 + \omega_1 + \cdots + \omega_{n-1} = -0/1 = 0.$$

Therefore we have

$$\sum_{k=1}^{n-1}\cos\frac{2k\pi}{n} + i\sum_{k=1}^{n-1}\sin\frac{2k\pi}{n} = \sum_{k=1}^{n-1}\left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right) = \omega_1 + \cdots + \omega_{n-1} = -\omega_0 = -1.$$

Comparing the real parts of both sides, we get

$$\sum_{k=1}^{n-1}\cos\frac{2k\pi}{n} = -1$$

as required. Note that if we compare the imaginary parts of both sides instead, then we also get another (less useful) equality $\sum_{k=1}^{n-1}\sin\frac{2k\pi}{n} = 0$. ■

2. Sequence and series of complex numbers

The notion of **sequences** can be generalized to the context of complex numbers easily and naturally.

Definition 1.27 Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers and let $L \in \mathbb{C}$. We say that the sequence $\{a_n\}$ ***converges to L*** if for each $\varepsilon > 0$, there exists a positive integer N such that

$$|a_n - L| < \varepsilon \qquad \text{whenever } n \geq N.$$

Remark 1.28 It is easy to prove that the complex number L in Definition 1.27 is unique if it exists. Therefore in this situation we say that L is the ***limit*** of the sequence $\{a_n\}$, and we write

$$\lim_{n\to+\infty} a_n = L.$$

Note that this is also equivalent to the following limit of sequence of real numbers:

$$\lim_{n\to+\infty}|a_n - L| = 0.$$

Lemma 1.29 *If $\{a_n\}$ and $\{b_n\}$ are* ***convergent*** *sequences of complex numbers such that $\lim_{n\to+\infty} a_n = L$ and $\lim_{n\to+\infty} b_n = M$, then*

- $\lim_{n\to+\infty} (a_n \pm b_n) = L \pm M$,
- $\lim_{n\to+\infty} a_n b_n = LM$,
- $\lim_{n\to+\infty} \frac{a_n}{b_n} = \frac{L}{M}$ *if* $M \neq 0$.

Lemma 1.30 *Let $\{a_n\}$ be a sequence of complex numbers and let $L \in \mathbb{C}$. Then $\lim_{n\to+\infty} a_n = L$ if and only if $\lim_{n\to+\infty} \operatorname{Re} a_n = \operatorname{Re} L$ and $\lim_{n\to+\infty} \operatorname{Im} a_n = \operatorname{Im} L$ as sequences of real numbers.*

Example 1.31 (Geometric sequence) Let z be a fixed complex number and let $\{a_n\}_{n\in\mathbb{N}}$ be the geometric sequence defined by

$$a_n = z^n.$$

Show that

(i) $\lim_{n\to+\infty} a_n = 0$ if $|z| < 1$

(ii) $\lim_{n\to+\infty} a_n = 1$ if $z = 1$, and

(iii) $\{a_n\}$ diverges if $|z| \geq 1$ and $z \neq 1$.

Proof:

(i) If $z = 0$, then $a_n = 0$ for all n, so it is trivial that $\lim_{n\to+\infty} a_n = 0$. Suppose that $0 < |z| < 1$.

For each $\varepsilon > 0$, let N be a positive integer greater than $\frac{\ln \varepsilon}{\ln|z|}$. So whenever $n \geq N$, we have

$$|a_n - 0| = |z^n| = |z|^n \leq |z|^N < |z|^{\frac{\ln \varepsilon}{\ln|z|}} = \varepsilon.$$

This shows that $\lim_{n\to+\infty} a_n = 0$ also.

(ii) If $z = 1$, then $a_n = 1$ for all n, so it is trivial that $\lim_{n\to+\infty} a_n = 1$.

(iii) Now suppose that $|z| \geq 1$ and $z \neq 1$. We let $r = |z|$ and $\theta = \operatorname{Arg} z$, so that

$$a_n = z^n = r^n(\cos n\theta + i\sin n\theta) = r^n \cos n\theta + ir^n \sin n\theta\,.$$

by de Moivre. Then we must have either ($r > 1$ and $\theta = 0$) or ($r \geq 1$ and $\theta \neq 0$).

- In the former case ($r > 1$ and $\theta = 0$), the sequence $\{\operatorname{Re} a_n\} = \{r^n\}$ diverges.
- In the latter case ($r \geq 1$ and $\theta \neq 0$), it is well-known from real analysis that at least one of $\{\cos n\theta\}$ and $\{\sin n\theta\}$ diverges, so at least one of $\{\operatorname{Re} a_n\} = \{r^n \cos n\theta\}$ and $\{\operatorname{Im} a_n\} = \{r^n \sin n\theta\}$ diverges.

Therefore in both cases, $\{a_n\}$ diverges by Lemma 1.30. ■

Recall that in MATH2033/2043 we have learnt that $\mathbb{R}$ is complete, i.e. every **Cauchy sequence** of real numbers converges. In $\mathbb{C}$ we have a similar result.

Definition 1.32 Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers. $\{a_n\}$ is called a ***Cauchy sequence*** if for each $\varepsilon > 0$, there exists a positive integer N such that

$$|a_n - a_m| < \varepsilon \qquad \text{whenever } m, n \geq N.$$

Theorem 1.33 (Completeness of $\mathbb{C}$) *Each Cauchy sequence of complex numbers converges to some complex number.*

Proof. Let $\{a_n\}$ be a sequence of complex numbers. Then for every $m, n \in \mathbb{N}$, we have

$$|\operatorname{Re} a_n - \operatorname{Re} a_m| = |\operatorname{Re}(a_n - a_m)| \leq |a_n - a_m| \qquad \text{and}$$
$$|\operatorname{Im} a_n - \operatorname{Im} a_m| = |\operatorname{Im}(a_n - a_m)| \leq |a_n - a_m|.$$

Now if $\{a_n\}$ is Cauchy, then the above inequalities imply that $\{\operatorname{Re} a_n\}$ and $\{\operatorname{Im} a_n\}$ are both Cauchy sequences of real numbers, which converge to some real numbers x and y respectively by the completeness of $\mathbb{R}$. Therefore by Lemma 1.30, $\{a_n\}$ converges to $x + iy$. ■

We can also generalize the notion of **series** to the context of complex numbers. Recall that a series is nothing but a sequence of "partial sums".

Theorem 1.34 (Geometric series) *Let a and z be a pair of complex numbers with $a \neq 0$. Let $a_n = az^{n-1}$ for all n, so that $\{a_n\}$ is a geometric sequence. Then the **geometric series***

$$\sum_{k=1}^{+\infty} a_k = a\sum_{k=0}^{+\infty} z^k$$

converges if and only if $|z| < 1$. In this case we have

$$\sum_{k=1}^{+\infty} a_k = \frac{a}{1-z}.$$

Lemma 1.35 (Term test for divergence) *If a series of complex numbers* $\sum_{k=1}^{+\infty} a_k$ *converges, then* $\lim_{n\to+\infty} a_n = 0$. *Consequently, if the sequence of complex numbers* $\{a_n\}$ *does not converge to* 0, *then the series* $\sum_{k=1}^{+\infty} a_k$ *diverges.*

Definition 1.36 We say that a series of complex numbers $\sum_{k=1}^{+\infty} a_k$ ***converges absolutely*** if the series of (non-negative) real numbers $\sum_{k=1}^{+\infty} |a_k|$ converges. We say that $\sum_{k=1}^{+\infty} a_k$ ***converges conditionally*** if it converges but does not converge absolutely.

Theorem 1.37 (Absolute convergence test) *If a series of complex numbers* $\sum_{k=1}^{+\infty} a_k$ *converges absolutely, then it converges.*

Proof. Suppose that $\sum_{k=1}^{+\infty} a_k$ converges absolutely. Then $\sum_{k=1}^{+\infty} |a_k|$ converges, so in particular $\{\sum_{k=1}^{n} |a_k|\}_{n\in\mathbb{N}}$ is a Cauchy sequence of real numbers. Now for each $\varepsilon > 0$, there exists a positive integer N such that whenever $m > n \geq N$, we have $\sum_{k=n+1}^{m} |a_k| < \varepsilon$ and so

$$\left|\sum_{k=n+1}^{m} a_k\right| \leq \sum_{k=n+1}^{m} |a_k| < \varepsilon,$$

which implies that $\{\sum_{k=1}^{n} a_k\}_{n\in\mathbb{N}}$ is a Cauchy sequence of complex numbers. Therefore $\sum_{k=1}^{+\infty} a_k$ converges by the completeness of $\mathbb{C}$. ■

Example 1.38 Discuss the convergence of each of the following series.

(a) $\sum_{k=1}^{\infty} \frac{i^k}{k}$ (b) $\sum_{k=1}^{\infty} \frac{i^k}{k^2}$

Solution:

(a) Note that $\sum_{k=1}^{\infty} \left|\frac{i^k}{k}\right| = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges by p-test. Next for each positive integer n,

$$\sum_{k=1}^{n} \frac{i^k}{k} = \begin{cases} \left(-\frac{1}{2} + \frac{1}{4} - \cdots + \frac{(-1)^m}{2m}\right) + i\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{m-1}}{2m-1}\right) & \text{if } n = 2m \\ \left(-\frac{1}{2} + \frac{1}{4} - \cdots + \frac{(-1)^m}{2m}\right) + i\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{m+1}}{2m+1}\right) & \text{if } n = 2m+1 \end{cases}.$$

In both cases, the real and imaginary parts both converge by alternating series test. So the series $\sum_{k=1}^{\infty} \frac{i^k}{k}$ converges by Lemma 1.30. Therefore it converges conditionally.

(b) Since $\sum_{k=1}^{\infty} \left|\frac{i^k}{k^2}\right| = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by p-test, the series $\sum_{k=1}^{\infty} \frac{i^k}{k}$ converges absolutely.

Remark 1.39 One cares about absolute convergence of a series not only because it is a convenient test of convergence, but more importantly, because **the terms of an absolutely convergent series can be "regrouped" or "rearranged" without changing the limit of the series**. More precisely, if a series $\sum_{k=1}^{+\infty} a_k$ converges absolutely and $\sum_{k=1}^{+\infty} a_k = L$, then given any bijection $\sigma: \mathbb{N} \to \mathbb{N}$, the "rearranged series" $\sum_{k=1}^{+\infty} a_{\sigma(k)}$ also converges absolutely and $\sum_{k=1}^{+\infty} a_{\sigma(k)} = L$.

Like in real analysis, we have the following two convergence tests as **consequences of the geometric series** and the absolute convergence test. The proofs are similar to the real case.

Theorem 1.40 (Root test) *Let* $\{a_n\}$ *be a sequence of complex numbers.*

(i) *If* $\lim\limits_{n\to+\infty} \sqrt[n]{|a_n|} < 1$, *then* $\sum_{k=1}^{+\infty} a_k$ *converges absolutely.*

(ii) *If* $\lim\limits_{n\to+\infty} \sqrt[n]{|a_n|} > 1$, *then* $\sum_{k=1}^{+\infty} a_k$ *diverges.*

Theorem 1.41 (Ratio test) *Suppose that* $a_n \neq 0$ *for every sufficiently large* n.

(i) *If* $\lim\limits_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, *then* $\sum_{k=1}^{+\infty} a_k$ *converges absolutely.*

(ii) *If* $\lim\limits_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, *then* $\sum_{k=1}^{+\infty} a_k$ *diverges.*

Example 1.42 Let z be any fixed complex number. Show that the series of complex numbers

$$\sum_{k=0}^{+\infty} \frac{z^k}{k!}$$

converges absolutely.

Proof: Since

$$\lim_{n\to+\infty} \left|\frac{z^{n+1}/(n+1)!}{z^n/n!}\right| = \lim_{n\to+\infty} \left|\frac{z^{n+1}}{z^n} \frac{n!}{(n+1)!}\right| = \lim_{n\to+\infty} \frac{|z|}{n+1} = 0 < 1,$$

the series $\sum_{k=0}^{+\infty} \frac{z^k}{k!}$ converges absolutely by ratio test. ∎

Definition 1.43 (Complex exponential) Let z be a complex number. The ***exponential*** of z, denoted as e^z or $\exp z$, is defined by

$$e^z := \sum_{k=0}^{+\infty} \frac{z^k}{k!}$$

Theorem 1.44 (Euler's formula) *Let* $\theta \in \mathbb{R}$. *Then*
$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Proof. Since the series $e^{i\theta} = \sum_{k=0}^{+\infty} \frac{(i\theta)^k}{k!}$ converges absolutely according to Example 1.42, we can regroup the terms of this series without affecting its limit. So in particular by grouping the terms with odd indices and those with even indices, we get

$$e^{i\theta} = \sum_{k=0}^{+\infty} \frac{(i\theta)^k}{k!} = \lim_{n\to+\infty} \sum_{k=0}^{2n+1} \frac{(i\theta)^k}{k!} = \lim_{n\to+\infty} \left(\sum_{k=0}^{n} \frac{i^{2k}\theta^{2k}}{(2k)!} + \sum_{k=0}^{n} \frac{i^{2k+1}\theta^{2k+1}}{(2k+1)!} \right)$$

$$= \lim_{n\to+\infty} \sum_{k=0}^{n} \frac{(-1)^k \theta^{2k}}{(2k)!} + i \lim_{n\to+\infty} \sum_{k=0}^{n} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} = \cos\theta + i\sin\theta\,.$$

■

From Theorem 1.44 we obtain the following beautiful identity which connects the five most important constants in mathematics.

Corollary 1.45 (Euler's identity) $e^{i\pi} + 1 = 0$.

Remark 1.46 Thanks to Euler's formula, the **polar form** of a complex number
$$z = r(\cos\theta + i\sin\theta)$$
can now be written much more compactly as
$$z = re^{i\theta}.$$
In particular, every $z \in \mathbb{C}$ with $|z| = 1$ can be written as $z = e^{i\theta}$ for some $\theta \in \mathbb{R}$.

Theorem 1.47 *Let* z *and* w *be complex numbers. Then*
$$e^{z+w} = e^z e^w.$$

Sloppy proof. For complex numbers z and w, we have

$$e^z e^w = \sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} \frac{z^k}{k!} \frac{w^j}{j!} \overset{(?)}{=} \sum_{m=0}^{+\infty} \sum_{j+k=m} \frac{z^k}{k!} \frac{w^j}{j!} = \sum_{m=0}^{+\infty} \sum_{k=0}^{m} \frac{z^k w^{m-k}}{k!\,(m-k)!}$$

$$= \sum_{m=0}^{+\infty} \frac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} z^k w^{m-k} = \sum_{m=0}^{+\infty} \frac{(z+w)^m}{m!} = e^{z+w},$$

where the second last step follows from binomial theorem. Refer to Supplementary Note A for a justification of the rearrangement of the double infinite series at the second step. ■

Corollary 1.48 *Let* z *be a complex number. Then*

(i) $(e^z)^n = e^{nz}$ *for every integer* n.

(ii) $e^z = e^{\operatorname{Re} z + i \operatorname{Im} z} = e^{\operatorname{Re} z}(\cos \operatorname{Im} z + i \sin \operatorname{Im} z)$, *i.e.*

$$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y) \qquad \textit{for every } x, y \in \mathbb{R}.$$

(iii) $e^{z+2\pi i} = e^z$, *i.e.* $\exp$ *is a periodic function with period* $2\pi i$.

(iv) $|e^z| = e^{\operatorname{Re} z}$.

(v) $e^z \neq 0$.

Definition 1.49 Let $a > 0$ be a positive real number and z be a complex number. We define

$$a^z := e^{(\ln a)z}.$$

Corollary 1.50 *Let* $a, b > 0$ *be positive real numbers and* z, w *be complex numbers. Then*

(i) $(a^z)^n = a^{nz}$ *for every integer* n.

(ii) $a^{z+w} = a^z a^w$

(iii) $|a^z| = a^{\operatorname{Re} z}$.

(iv) $a^z \neq 0$.

(v) $(ab)^z = a^z b^z$.

Recall that in real analysis we saw that the series of real numbers $\sum_{k=1}^{+\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$. With the complex exponential introduced, we may now consider an analogue of this $\boldsymbol{p}$**-test** for series of complex numbers.

Example 1.51 ($\boldsymbol{p}$-test) Let z be a fixed complex number with $\operatorname{Re} z > 1$. Show that the series of complex numbers

$$\sum_{k=1}^{+\infty} \frac{1}{k^z}$$

converges absolutely.

Solution:

Note that

$$\left|\frac{1}{k^z}\right| = \left|\frac{1}{e^{(\ln k)z}}\right| = \frac{1}{e^{(\ln k)\operatorname{Re} z}} = \frac{1}{k^{\operatorname{Re} z}}.$$

Since $\operatorname{Re} z > 1$, the series of real numbers $\sum_{k=1}^{+\infty} \frac{1}{k^{\operatorname{Re} z}}$ converges by p-test, and so the series $\sum_{k=1}^{+\infty} \frac{1}{k^z}$ converges absolutely.

Remark 1.52 The following is a quick summary on the useful tests for convergence and divergence of a series of complex numbers.

- **Term test**: If $\{a_n\}$ does not converge to 0, then $\sum_{k=1}^{+\infty} a_k$ diverges.
- **Geometric series test**: $\sum_{k=1}^{+\infty} z^k$ converges if $|z| < 1$ and diverges if $|z| \geq 1$.
- **Telescoping series**: $\sum_{k=1}^{+\infty}(a_{k+1} - a_k)$ converges if and only if $\{a_n\}$ converges.
- **Absolute convergence test**: If $\sum_{k=1}^{+\infty}|a_k|$ converges, then $\sum_{k=1}^{+\infty} a_k$ also converges.
- **Root test**: Let $L = \lim_{n\to+\infty} \sqrt[n]{|a_n|}$. Then $\sum_{k=1}^{+\infty} a_k \begin{cases} \text{converges absolutely} & \text{if } L < 1 \\ \text{diverges} & \text{if } L > 1 \end{cases}$.
- **Ratio test**: Let $L = \lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right|$. Then $\sum_{k=1}^{+\infty} a_k \begin{cases} \text{converges absolutely} & \text{if } L < 1 \\ \text{diverges} & \text{if } L > 1 \end{cases}$.
- ***p*-test**: $\sum_{k=1}^{+\infty} \frac{1}{k^{p+iq}}$ converges absolutely if $p > 1$ and $q \in \mathbb{R}$.

Don't forget other convergence tests which are still useful for **series of real numbers**, e.g. integral test, comparison test, limit comparison test and alternating series test.

Example 1.53 Discuss the convergence of each of the following series.

(a) $\sum_{k=0}^{+\infty} \frac{(1+2i)^k}{2^{2k}}$ (b) $\sum_{k=1}^{+\infty} \frac{i^k}{k^2+i}$ (c) $\sum_{k=0}^{+\infty} \frac{(2+i)^k}{(2-i)^k - \pi i}$

Solution:

(a) Since $\left|\frac{1+2i}{4}\right| = \frac{\sqrt{5}}{4} < 1$, the series $\sum_{k=0}^{+\infty} \left|\frac{(1+2i)^k}{2^{2k}}\right| = \sum_{k=0}^{+\infty} \left|\frac{1+2i}{4}\right|^k$ converges by geometric series test. Therefore the series $\sum_{k=0}^{+\infty} \frac{(1+2i)^k}{2^{2k}}$ converges absolutely.

(b) For each positive integer n, we have

$$\left|\frac{i^n}{n^2+i}\right| = \frac{1^n}{\sqrt{(n^2)^2 + 1^2}} = \frac{1}{\sqrt{n^4+1}} \leq \frac{1}{\sqrt{n^4}} = \frac{1}{n^2}.$$

Now the series (of real numbers) $\sum_{k=1}^{+\infty} \frac{1}{k^2}$ converges by p-test, so the series (of real numbers) $\sum_{k=1}^{+\infty} \left|\frac{i^k}{k^2+i}\right|$ converges by comparison test. Thus the series $\sum_{k=1}^{+\infty} \frac{i^k}{k^2+i}$ converges absolutely.

(c) Triangle inequality gives

$$\left|\frac{(2+i)^n}{(2-i)^n - \pi i}\right| \geq \frac{|2+i|^n}{|2-i|^n + |\pi i|} = \frac{\sqrt{5}^n}{\sqrt{5}^n + \pi} = \frac{1}{1 + \frac{\pi}{\sqrt{5}^n}} > \frac{1}{2}$$

for every sufficiently large n (namely for every $n \geq 2$). So in particular, the n^{th} term $\frac{(2+i)^n}{(2-i)^n - \pi i}$ does not converge to 0. Therefore the series $\sum_{k=0}^{+\infty} \frac{(2+i)^k}{(2-i)^k - \pi i}$ diverges by term test.

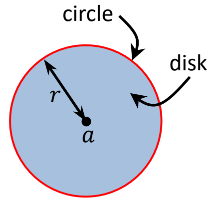
3. Point-set topology of $\mathbb{C}$

In MATH1013/1023 we learnt that **closed intervals** include their boundary points, while **open intervals** do not. Subsets of $\mathbb{C}$ also have similar behaviors.

Definition 1.54 Let $a \in \mathbb{C}$ and $r > 0$. The sets denoted by

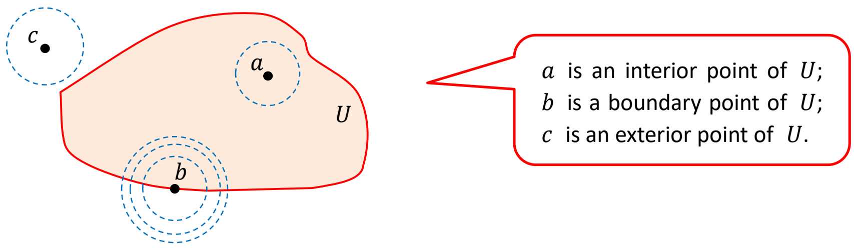
$$D(a;r) := \{z \in \mathbb{C} : |z-a| < r\},$$
$$\overline{D(a;r)} := \{z \in \mathbb{C} : |z-a| \le r\}, \quad \text{and}$$
$$\partial D(a;r) := \{z \in \mathbb{C} : |z-a| = r\}$$

are the ***open disk***, ***closed disk*** and ***circle*** centered at a with radius r, respectively.



Definition 1.55 Let U be a subset of $\mathbb{C}$ and $z \in \mathbb{C}$ be a complex number.

- z is called an ***interior point*** of U if it is the center of a disk which contains entirely in U, i.e. there exists $\varepsilon > 0$ such that $D(z;\varepsilon) \subseteq U$. The set of all interior points of U is called the ***interior*** of U and is denoted as U°.
- z is called a ***boundary point*** of U if every disk centered at z contains some point in U and some point not in U, i.e. for any $\varepsilon > 0$, both $D(z;\varepsilon) \cap U$ and $D(z;\varepsilon) \cap (\mathbb{C} \setminus U)$ are non-empty. The set of all boundary points of U is called the ***boundary*** of U and is denoted as ∂U.
- z is called an ***exterior point*** of U if it is an interior point of $\mathbb{C} \setminus U$, i.e. there exists $\varepsilon > 0$ such that $D(z;\varepsilon) \subseteq \mathbb{C} \setminus U$. The set of all exterior points of U is called the ***exterior*** of U and is denoted as $(\mathbb{C} \setminus U)^\circ$.



Remark 1.56 Let U be a subset of $\mathbb{C}$. Intuitively speaking,

- an interior point of U is a point whose "neighbourhood" is completely contained in U;
- a boundary point of U is a point whose "neighbourhood" is partly in U, partly outside U;
- an exterior point of U is a point whose "neighbourhood" is completely outside U.

According to Definition 1.55, it is easy to deduce that

- if z is an interior point of U, then $z \in U$ (i.e. $U^\circ \subseteq U$);
- if z is an exterior point of U, then $z \notin U$;
- if z is a boundary point of U, then z may or may not belong to U.

Moreover, the interior of U, the boundary of U and the exterior of U together form a (disjoint) partition of the whole complex plane $\mathbb{C}$.

Example 1.57 Let

$$U = \{z \in \mathbb{C}: \operatorname{Im} z \geq 0\}.$$

Find all the interior points, boundary points and exterior points of U.

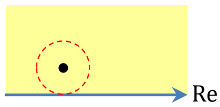
Solution:

We claim that

$$U^\circ = \{z \in \mathbb{C}: \operatorname{Im} z > 0\},$$
$$\partial U = \{z \in \mathbb{C}: \operatorname{Im} z = 0\} \quad \text{and}$$
$$(\mathbb{C} \setminus U)^\circ = \{z \in \mathbb{C}: \operatorname{Im} z < 0\}.$$

Here is the proof.

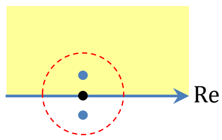
- Let $z \in \mathbb{C}$ satisfy $\operatorname{Im} z > 0$. Then we choose $\varepsilon = \operatorname{Im} z > 0$, so that for every $w \in D(z, \varepsilon)$, we have



$$\operatorname{Im} w = \operatorname{Im} z - \operatorname{Im}(z - w) \geq \operatorname{Im} z - |z - w| > \operatorname{Im} z - \varepsilon = 0,$$

so $w \in U$. This shows that $D(z, \varepsilon) \subseteq U$ and so z is an interior point of U.

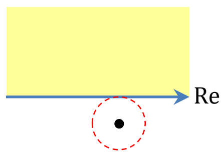
- Let $z \in \mathbb{C}$ satisfy $\operatorname{Im} z = 0$. Then for each $\varepsilon > 0$, we consider the points $z + \frac{\varepsilon}{2}i$ and $z - \frac{\varepsilon}{2}i$, both of which lie inside $D(z, \varepsilon)$. Since



$$\operatorname{Im}\left(z + \frac{\varepsilon}{2}i\right) = \operatorname{Im} z + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} > 0 \quad \text{and} \quad \operatorname{Im}\left(z - \frac{\varepsilon}{2}i\right) = \operatorname{Im} z - \frac{\varepsilon}{2} = -\frac{\varepsilon}{2} < 0,$$

it follows that $z + \frac{\varepsilon}{2}i \in U$ and $z - \frac{\varepsilon}{2}i \in \mathbb{C} \setminus U$. Now both $D(z; \varepsilon) \cap U$ and $D(z; \varepsilon) \cap (\mathbb{C} \setminus U)$ are non-empty, so z is a boundary point of U.

- Let $z \in \mathbb{C}$ satisfy $\operatorname{Im} z < 0$. Then we choose $\varepsilon = -\operatorname{Im} z > 0$, so that for every $w \in D(z, \varepsilon)$, we have



$$\operatorname{Im} w = \operatorname{Im} z + \operatorname{Im}(w - z) \leq \operatorname{Im} z + |w - z| < \operatorname{Im} z + \varepsilon = 0,$$

so $w \in \mathbb{C} \setminus U$. This shows that $D(z, \varepsilon) \subseteq \mathbb{C} \setminus U$ and so z is an exterior point of U. ■

The following types of subsets of $\mathbb{C}$ are the focus of the study of **point-set topology in** $\mathbb{C}$. These sets will be used throughout the whole course.

- **Open** sets
- **Closed** sets
- **Compact** sets
- **Connected** sets

Definition 1.58 Let U be a subset of $\mathbb{C}$.

- U is ***open*** if every point in U is an interior point of U (i.e. $U \subseteq U^\circ$).
- U is ***closed*** if it contains all its boundary points (i.e. $\partial U \subseteq U$).

Example 1.59 Let $a \in \mathbb{C}$ and $r > 0$. Show that

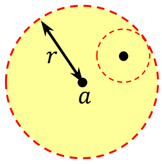
(a) $D(a;r)$ is open; and

(b) $\overline{D(a;r)}$ is closed.

This explains why we call them "open" disks and "closed" disks, to begin with.

Proof:

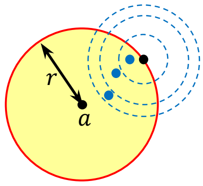
(a) Let $z \in D(a;r)$, we aim to show that z is an interior point of $D(a;r)$. To do this, we choose $\varepsilon = r - |z-a| > 0$, so that for every $w \in D(z;\varepsilon)$, we have



$$|w-a| \le |w-z| + |z-a| < \varepsilon + |z-a| = r,$$

so $w \in D(a;r)$. This shows that $D(z;\varepsilon) \subseteq D(a;r)$ and so z is an interior point of $D(a;r)$. Therefore $D(a;r)$ is open.

(b) Let z be a boundary point of $\overline{D(a;r)}$, we aim to show that $z \in \overline{D(a;r)}$. Since z is a boundary point of $\overline{D(a;r)}$, $D(z;\varepsilon) \cap \overline{D(a;r)}$ is non-empty for any $\varepsilon > 0$. In other words, there exists a sequence of complex numbers $\{a_n\}_{n\in\mathbb{N}}$ converging to z such that $a_n \in \overline{D(a;r)}$ for every n. Now



$$|z-a| = \left|\lim_{n\to+\infty} a_n - a\right| = \lim_{n\to+\infty} |a_n - a| \le r,$$

(why does the second equality hold?) so $z \in \overline{D(a;r)}$. Therefore $\overline{D(a;r)}$ is closed.

Remark 1.60 "Openness" and "closedness" of sets are **not opposite concepts**. Consider the following subsets of $\mathbb{C}$:

$$U = \{z \in \mathbb{C} : \operatorname{Im} z \ge 0\} \qquad \text{and} \qquad V = \{z \in \mathbb{C} : \operatorname{Re} z > 1\}.$$

Then

- U is closed but not open;
- V is open but not closed;
- $U \cap V = \{z \in \mathbb{C} : \operatorname{Re} z > 1 \text{ and } \operatorname{Im} z \ge 0\}$ is neither open nor closed.
- $\emptyset$ and $\mathbb{C}$ are (the only two subsets of $\mathbb{C}$ that are) both open and closed (some people call them "clopen sets").

Take a look at the interesting video https://www.youtube.com/watch?v=SyD4p8_y8Kw.

In fact, the concepts "openness" and "closedness" of sets are related by **complementation**, not by negation.

Theorem 1.61 *Let U be a subset of $\mathbb{C}$. U is closed if and only if $\mathbb{C} \setminus U$ is open.*

Proof.

($\Rightarrow$) Suppose that U is closed, i.e. U contains all its boundary points. Let $z \in \mathbb{C} \setminus U$, the aim is to show that z is an interior point of $\mathbb{C} \setminus U$. Now z cannot be a boundary point of U, so there exists $\varepsilon > 0$ such that either $D(z;\varepsilon) \cap U$ is empty or $D(z;\varepsilon) \cap (\mathbb{C} \setminus U)$ is empty. But the second set $D(z;\varepsilon) \cap (\mathbb{C} \setminus U)$ cannot be empty because it must contain z. Thus the first set $D(z;\varepsilon) \cap U$ is empty, or in other words $D(z;\varepsilon) \subseteq \mathbb{C} \setminus U$. Therefore z is an interior point of $\mathbb{C} \setminus U$, and so $\mathbb{C} \setminus U$ is open.

($\Leftarrow$) Suppose that $\mathbb{C} \setminus U$ is open, i.e. every point in $\mathbb{C} \setminus U$ is an interior point of $\mathbb{C} \setminus U$. Let z be a boundary point of U, and the aim is to show that $z \in U$. If this were not the case, i.e. $z \in \mathbb{C} \setminus U$, then z would be an interior point of $\mathbb{C} \setminus U$, so there exists $\varepsilon > 0$ such that $D(z;\varepsilon) \subseteq \mathbb{C} \setminus U$, or in other words $D(z;\varepsilon) \cap U$ is empty. This contradicts the assumption that z is a boundary point of U. Therefore we must have $z \in U$, and so U is closed. ∎

Theorem 1.62 *The following are properties of unions and intersections of open subsets of $\mathbb{C}$.*

(i) *The union of (finitely or infinitely many) open sets is open.*

(ii) *The intersection of finitely many open sets is open.*

Proof.

(i) Let $\{U_\alpha\}_{\alpha \in I}$ be a family of open subsets of $\mathbb{C}$ indexed by I. Let $z \in \bigcup_{\alpha \in I} U_\alpha$, the aim is to show that z is an interior point of $\bigcup_{\alpha \in I} U_\alpha$. Now there exists $\beta \in I$ such that $z \in U_\beta$ by definition of union. Since U_β is open, there exists $\varepsilon > 0$ such that $D(z;\varepsilon) \subseteq U_\beta$. In particular, we have $D(z;\varepsilon) \subseteq \bigcup_{\alpha \in I} U_\alpha$, so z is an interior point of $\bigcup_{\alpha \in I} U_\alpha$. This shows that $\bigcup_{\alpha \in I} U_\alpha$ is open.

(ii) By induction, it suffices to show that the intersection of two open sets is open. Let U and V be two open subsets of $\mathbb{C}$. Let $z \in U \cap V$, the aim is to show that z is an interior point of $U \cap V$. Since U and V are both open, there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $D(z;\varepsilon_1) \subseteq U$ and $D(z;\varepsilon_2) \subseteq V$. Now we choose $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\} > 0$. Then

$$D(z;\varepsilon) \subseteq D(z;\varepsilon_1) \subseteq U \qquad \text{and} \qquad D(z;\varepsilon) \subseteq D(z;\varepsilon_2) \subseteq V,$$

so $D(z;\varepsilon) \subseteq U \cap V$. This shows that z is an interior point of $U \cap V$, so $U \cap V$ is open. ∎

Corollary 1.63 *The following are properties of unions and intersections of closed subsets of $\mathbb{C}$.*

(i) *The intersection of (finitely or infinitely many) closed sets is closed.*

(ii) *The union of finitely many closed sets is closed.*

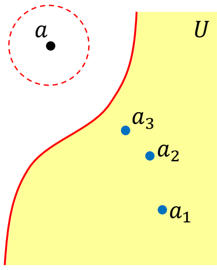
Closed subsets of $\mathbb{C}$ have the following nice characterization using **sequences**.

Lemma 1.64 *Let U be a subset of $\mathbb{C}$. U is closed if and only if for every convergent sequence $\{a_n\}$ of complex numbers in U, the limit a is also in U.*

Proof. ($\Rightarrow$) Suppose that $a \in \mathbb{C} \setminus U$. Since $\mathbb{C} \setminus U$ is open, there exists $\varepsilon > 0$ such that $D(a;\varepsilon) \subseteq \mathbb{C} \setminus U$. Now if $\{a_n\}$ converges to a, then with this particular ε there exists a positive integer N such that

$$|a_n - a| < \varepsilon$$

whenever $n \geq N$. This implies that $a_N \in D(a;\varepsilon) \subseteq \mathbb{C} \setminus U$, which is a contradiction to the fact that $a_n \in U$ for all n.



($\Leftarrow$) Let z be a boundary point of U. Then for each $n \in \mathbb{N}$, there exists $z_n \in D(z; 1/n) \cap U$. Now $\{z_n\}$ becomes a sequence of numbers in U which converges to z, so $z \in U$ according to the assumption. Therefore U contains all its boundary points. ■

Definition 1.65 Let U be a subset of $\mathbb{C}$. The ***closure*** of U is the set

$$\overline{U} := U \cup \partial U.$$

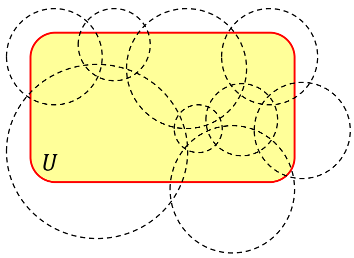
Remark 1.66 Let U be a subset of $\mathbb{C}$. Then its closure $\overline{U}$ is the intersection of all closed sets containing U. Therefore $\overline{U}$ is the **smallest closed set** containing U.

Example 1.67 Let $a \in \mathbb{C}$ and $r > 0$. In Q2, Problem Set 2, we will show that the boundary of the open disk $D(a;r)$ is indeed the circle $\{z \in \mathbb{C}: |z-a| = r\}$. So the closure of $D(a;r)$ is the set $D(a;r) \cup \{z \in \mathbb{C}: |z-a| = r\}$, which is indeed the closed disk $\{z \in \mathbb{C}: |z-a| \leq r\}$. This shows that the notations $\partial D(a;r)$ for the circle and $\overline{D(a;r)}$ for the closed disk introduced in Definition 1.54 are compatible with the notations of closure and boundary.

Apart from openness and closedness, **compactness** is another important concept in topology. The definition of compactness may sound abstract for beginners. If you find it too difficult, you may skip the following two definitions and directly proceed to **Definition 1.70** and **Remark 1.73**, which provide enough understanding for the purpose of this course.

Definition 1.68 Let U be a subset of $\mathbb{C}$.

- An ***open cover*** of U is a collection of open sets $\{U_\alpha\}_{\alpha \in I}$ indexed by a set I, such that $U \subseteq \bigcup_{\alpha \in I} U_\alpha$.
- A ***subcover*** of this open cover is a subcollection $\{U_\alpha\}_{\alpha \in J}$ where $J \subseteq I$, such that $U \subseteq \bigcup_{\alpha \in J} U_\alpha$.
- If the index set is finite, we say that the open cover is ***finite***.



A finite open cover of U
(which consists of open disks)

Definition 1.69 Let U be a subset of $\mathbb{C}$. U is ***compact*** if every (infinite) open cover of U has a finite subcover.

Definition 1.70 Let U be a subset of $\mathbb{C}$. U is ***bounded*** if it is completely contained in some disk, i.e. there exists $r > 0$ such that $U \subseteq D(0; r)$.

Theorem 1.71 *Let U be a subset of $\mathbb{C}$. Then the following statements are equivalent.*

(i) *U is compact.*

(ii) *U is closed and bounded.*

(iii) *Every sequence in U has a subsequence which converges to some limit in U.*

Proof. We first show that (i) implies (ii). Suppose that U is compact.

- If U were not bounded, then $\{D(0; n)\}_{n\in\mathbb{N}}$ is an open cover of U which has no finite subcover, a contradiction.
- If U were not closed, then there exists $z \in \mathbb{C} \setminus U$ which is a boundary point of U. Then $\left\{\mathbb{C} \setminus \overline{D\left(z, \frac{1}{n}\right)}\right\}_{n\in\mathbb{N}}$ is an open cover of U which has no finite subcover, a contradiction.

Next we show that (ii) implies (iii). Let $\{a_n\}$ be a sequence such that $a_n \in U$ for every n. Since U is bounded, the real part of those points in U is also bounded, i.e. the set

$$\{x \in \mathbb{R}: x = \operatorname{Re} z \text{ for some } z \in U\}$$

is a bounded subset of $\mathbb{R}$. So $\{a_n\}$ has a subsequence $\left\{a_{n_m}\right\}_m$ whose real parts $\left\{\operatorname{Re} a_{n_m}\right\}_m$ converges in $\mathbb{R}$ by Bolzano-Weierstrass Theorem (in $\mathbb{R}$). Similarly, $\left\{a_{n_m}\right\}_m$ has a subsequence $\left\{a_{n_{m_k}}\right\}_k$ whose imaginary parts $\left\{\operatorname{Im} a_{n_{m_k}}\right\}_k$ converges in $\mathbb{R}$ by Bolzano-Weierstrass (in $\mathbb{R}$) again. Now $\left\{a_{n_{m_k}}\right\}_k$ is a subsequence of $\{a_n\}$ which converges to a limit by Lemma 1.30, and this limit is in U by Lemma 1.64, since U is closed.

Finally we show that (iii) implies (i), or equivalently we show that the negation of (i) implies the negation of (iii). Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of U which does not have a finite subcover. We construct a sequence $\{a_n\}$ inductively as follows:

- Take $a_1 \in U$. Then $a_1 \in U_{\beta_1}$ for some $\beta_1 \in I$.
- Suppose that $a_1, a_2, \ldots, a_k$ are taken so that $a_1 \in U_{\beta_1}$, $a_2 \in U_{\beta_2}$, …, $a_k \in U_{\beta_k}$. By assumption, the finite union does not cover U, i.e. $U \nsubseteq U_{\beta_1} \cup \cdots \cup U_{\beta_k}$, so there must exist $a_{k+1} \in U \setminus \left(U_{\beta_1} \cup \cdots \cup U_{\beta_k}\right)$ and $a_{k+1} \in U_{\beta_{k+1}}$ for some $\beta_{k+1} \in I$.

By induction we obtain an infinite sequence $\{a_n\}$ such that different terms a_n belong to different open sets U_{β_n}, and now we aim to show that $\{a_n\}$ has no subsequence which converges to a limit in U. If not, say $\{a_{n_m}\}_m$ is a subsequence that converges to some $a \in U$, then $a \in U_\beta$ for some $\beta \in I$. Since U_β is open, there exists $\varepsilon > 0$ such that $D(a, \varepsilon) \subseteq U_\beta$. But since $\{a_{n_m}\}_m$ converges to a, it follows that $a_{n_m} \in D(a, \varepsilon) \subseteq U_\beta$ for every sufficiently large m, which contradicts to the construction of the sequence. ∎

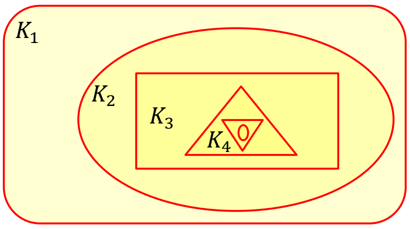
Remark 1.72 The part (ii) ⇒ (i) in Theorem 1.71 is known as the **Heine-Borel Theorem**. The part (ii) ⇒ (iii) in Theorem 1.71 is known as the **Bolzano-Weierstrass Theorem in** $\mathbb{C}$.

Remark 1.73 Thanks to Theorem 1.71, we know that a subset of $\mathbb{C}$ is **compact** if and only if it is **closed and bounded**. We may therefore take this as the definition of a compact set instead, as the notion of "open covers" will seldom be used in the rest of this course.

The following is a two-dimensional analogue of the **nested interval theorem** we have learnt in MATH2033/2043. It says that the intersection of **nested non-empty compact sets** is non-empty.

Theorem 1.74 (Cantor intersection theorem) *Let $K_1, K_2, \ldots$ be non-empty compact subsets of $\mathbb{C}$ such that $K_{n+1} \subseteq K_n$ for every $n \in \mathbb{N}$. Then*

$$\bigcap_{n=1}^{+\infty} K_n \neq \emptyset.$$



Proof. Suppose on the contrary that the intersection is empty. Then $\bigcup_{n=1}^{+\infty}(\mathbb{C} \setminus K_n) = \mathbb{C} \supset K_1$. Now K_n is closed for each $n \in \mathbb{N}$, so $\{\mathbb{C} \setminus K_n\}_{n\in\mathbb{N}}$ is an open cover of K_1. Since K_1 is compact, this open cover has a finite subcover, say

$$K_1 \subseteq (\mathbb{C} \setminus K_{n_1}) \cup (\mathbb{C} \setminus K_{n_2}) \cup \cdots \cup (\mathbb{C} \setminus K_{n_m})$$

where $1 < n_1 < n_2 < \cdots < n_m$. But by the "nested" assumption, the above finite union just equals to $\mathbb{C} \setminus K_{n_m}$. Thus

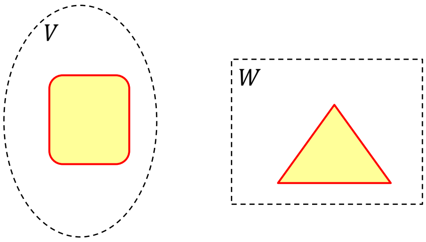
$$K_{n_m} \subseteq K_1 \subseteq \mathbb{C} \setminus K_{n_m},$$

which implies that $K_{n_m} = \emptyset$, a contradiction. ∎

Finally we study **connectedness**. Intuitively speaking, a connected subset of $\mathbb{C}$ is a set which consists of only **one "piece"**. To put this intuition into rigorous mathematics, we need to first define what a disconnected set is. Again, if the following topological definition is too difficult for you, you may skip it and proceed directly to **Definition 1.76 – 1.78**, as we will mostly be interested in **connected sets that are also open** for the rest of the course.

Definition 1.75 Let U be a subset of $\mathbb{C}$.

- We say that U is ***disconnected*** if it is contained separately in two disjoint open sets, i.e. there exist two disjoint open sets V and W ($V \cap W = \emptyset$) such that
$$U \cap V \neq \emptyset, \qquad U \cap W \neq \emptyset \qquad \text{and} \qquad U \subseteq V \cup W.$$
- We say that U is ***connected*** if U is not disconnected.



U (the shaded set) is disconnected

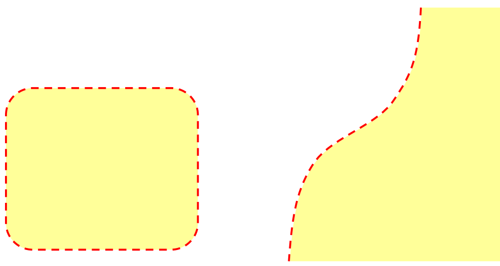
Definition 1.76 Let U be a subset of $\mathbb{C}$. We say that U is ***polygonally path-connected*** if each pair of points in U can be joined by a "polygonal path" in U, i.e. a path consisting of finitely many line segments contained in U.

Theorem 1.77 *Let U be an* ***open*** *subset of $\mathbb{C}$. Then U is connected if and only if U is polygonally path-connected.*

Proof. Omitted. ■

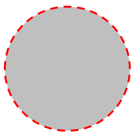
Definition 1.78 A non-empty subset of $\mathbb{C}$ is called a ***region*** if it is open and connected.

Some authors use the word "domain" instead of "region". We do not use this terminology as it may be easily mixed up with the "domain" of a function.

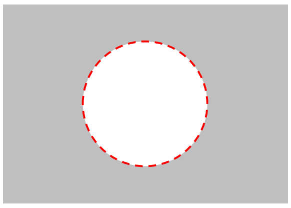


A bounded region An unbounded region

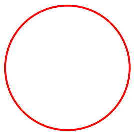
Example 1.79 The open disk $D(0;1)$ is a bounded region. $\mathbb{C} \setminus \overline{D(0;1)}$ is an unbounded region. The circle $\partial D(0;1)$ is not a region because it is not open. $\mathbb{C} \setminus \partial D(0;1)$ is not a region because it is disconnected.



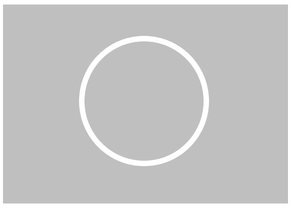
$D(0;1)$



$\mathbb{C} \setminus \overline{D(0;1)}$



$\partial D(0;1)$



$\mathbb{C} \setminus \partial D(0;1)$

For disconnected sets, we are still interested in each "piece" of it at a time.

Definition 1.80 Let U be a subset of $\mathbb{C}$. A non-empty subset $V \subseteq U$ is called a ***connected component*** of U if V is a "maximal" connected subset of U. In other words,

- V is connected, and
- whenever W is a connected set such that $V \subseteq W \subseteq U$, we must have $V = W$.

Lemma 1.81 *Let U be a subset of $\mathbb{C}$. If V and W are connected components of U, then either $V = W$ or $V \cap W = \emptyset$.*

Lemma 1.82 *Every subset of $\mathbb{C}$ is the union of its connected components.*

U (the whole shaded set) is disconnected;
its connected components are U_1 and U_2, which are disjoint.
$U = U_1 \cup U_2$.

Summary of Chapter 1

The following are what you need to know in this chapter in order to get a pass (a distinction) in this course:

✓ **Complex numbers** and their **arithmetics**
- **Addition** and **multiplication** of complex numbers, $i^2 = -1$, $\mathbb{C}$ is a field
- The **real part** $\operatorname{Re} z$, the **imaginary part** $\operatorname{Im} z$, the **modulus** $|z|$, the **arguments** $\arg z$, the **principal argument** $\operatorname{Arg} z$, and the **complex conjugate** $\bar{z}$ of a complex number z
- To convert any given complex number into its **standard form** $(x + iy)$ and **polar form** $(r(\cos\theta + i\sin\theta)$ or $re^{i\theta})$
- The **triangle inequalities** $\big||z| - |w|\big| \le |z + w| \le |z| + |w|$
- **de Moivre's Theorem**, n^{th} roots of 1, n^{th} roots of a complex number
- **Geometric interpretation** of complex numbers in the **complex plane**: $\mathbb{C} \cong \mathbb{R}^2$

✓ **Sequence** and **series** of complex numbers
- $\boldsymbol{\varepsilon}$-$\boldsymbol{N}$ **definition of limit** of a sequence of complex number
- **Cauchy sequences**, completeness of $\mathbb{C}$
- **Series** and sequence of its partial sums
- **Absolute convergence** of a series of complex numbers, rearrangement theorem
- Tests for convergence / divergence of a series of complex numbers

✓ **Complex exponential function**
- Definition: $e^z := \sum_{k=0}^{+\infty} \frac{z^k}{k!}$
- Properties of the complex exponential function
- Exponential functions with **positive real** bases

✓ **Point-set topology** in $\mathbb{C}$
- To determine the **interior points** and **boundary points** of a given subset of $\mathbb{C}$
- **Open** sets and **closed** sets; **interior**, **exterior**, **boundary** and **closure**
- **Properties** of open and closed sets regarding **union**, **intersection** and **complements**
- Property of **closed sets** regarding **sequences**
- **Compact** sets: Every open cover has a finite subcover
- A set is **compact** if and only if it is **closed** and **bounded**
- Intersection of **nested non-empty compact sets** is non-empty
- **Connected** sets; connected components of a set
- **Regions**: (Non-empty) **open** and **connected** sets