Chapter 6 Epilogue

In this final chapter, we explore some **holomorphic or meromorphic functions** which have significance in other branches of mathematics. This chapter is intended to be an **overview** of interesting results only, and much of the details of proofs will be omitted.

1. Möbius transformtions

Definition 6.1 Let $U \subseteq \mathbb{C}$ be a region such that $\mathbb{C} \setminus U$ is bounded, and let $f: U \to \mathbb{C}$ be a function. We say that f is ***holomorphic at*** ∞ if $f(1/z)$ has a removable singularity at 0. We say that f has a ***pole at*** ∞ if $f(1/z)$ has a pole at 0.

In chapter 5 we have been introduced to the notion of **meromorphic functions** on a region $U \subseteq \mathbb{C}$, which are the same as holomorphic functions $f: U \to \mathbb{C} \cup \{\infty\}$ (which takes the value ∞ at poles). We can extend this idea and allow the region U in the domain to include ∞ as well.

Definition 6.1 (restated) Let $U \subseteq \mathbb{C} \cup \{\infty\}$ be a region. A function $f: U \to \mathbb{C} \cup \{\infty\}$ is said to be ***holomorphic*** if either $f \equiv \infty$ or $f(z)$ and $f(1/z)$ are both meromorphic functions. We say that f is ***holomorphic at*** ∞ if $f(\infty) \in \mathbb{C}$, and we say that f has a ***pole at*** ∞ if $f(\infty) = \infty$.

Remark 6.2 Pay attention to the **codomain**. A holomorphic function $f: U \to \mathbb{C}$ has no singularities in U, but a holomorphic function $f: U \to \mathbb{C} \cup \{\infty\}$ is allowed to have poles in U.

Example 6.3 The function $f(z) = \frac{1}{z}$ is holomorphic at ∞. The function $g(z) = z^2$ has a pole at ∞. The function $h(z) = e^z$ is not holomorphic at ∞ and does not have a pole at ∞.

Theorem 6.4 *Let $f: \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be a non-constant holomorphic function. Then f is a rational function.*

Sketch of proof. Since $\mathbb{C} \cup \{\infty\}$ is compact and the zeros and poles of f are isolated, f has finitely many zeros $a_1, \dots, a_m$ in $\mathbb{C} \cup \{\infty\}$ with orders $p_1, \dots, p_m$, and finitely many poles $b_1, \dots, b_n$ in $\mathbb{C} \cup \{\infty\}$ with orders $q_1, \dots, q_n$. Then the function

$$g(z) = f(z) \frac{\prod_{k=1}^{n} (z - b_k)^{q_k}}{\prod_{j=1}^{m} (z - a_j)^{p_j}}$$

is entire, has no zeros, but is meromorphic at ∞. So g must be a constant function, and this implies that f is rational. ■

Definition 6.5 A ***Möbius transformation*** is a function $f: \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ defined by

$$f(z) = \frac{az+b}{cz+d},$$

where a, b, c, d are complex numbers such that $ad - bc \neq 0$.

Lemma 6.6 *A Möbius transformation is holomorphic on* $\mathbb{C} \cup \{\infty\}$.

Lemma 6.7 *If* $f, g: \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ *are Möbius transformations, then* $f \circ g$ *is also a Möbius transformation. More precisely, the set of all Möbius transformations equipped with function composition forms a group.*

Proof. It follows from the commutative diagram below.

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "\begin{pmatrix} a & b \\ c & d \end{pmatrix}"] \arrow[d, "\text{slope}"'] & \mathbb{C}^2 \arrow[d, "\text{slope}"] \\ \mathbb{C} \cup \{\infty\} \arrow[r, "\frac{az+b}{cz+d}"] & \mathbb{C} \cup \{\infty\} \end{tikzcd}$$

To be explicit, the composition of Möbius transformations corresponds to matrix multiplication:

$$f(z) = \frac{az+b}{cz+d} \text{ and } g(z) = \frac{pz+q}{rz+s} \quad \Rightarrow \quad (f \circ g)(z) = \frac{(ap+br)z + (aq+bs)}{(cp+dr)z + (cq+ds)}$$

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \quad \Rightarrow \quad \mathbf{AB} = \begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix}.$$

■

Remark 6.8 The multiplicative group of 2×2 invertible matrices with complex entries is called the ***general linear group*** and is denoted by $\mathrm{GL}(2, \mathbb{C})$. Let G denote the group of Möbius transformations with composition. Then Lemma 6.7 says that the map

$$\begin{array}{ccc} \mathrm{GL}(2, \mathbb{C}) & \longrightarrow & G \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto & \frac{az+b}{cz+d} \end{array}$$

is a surjective group homomorphism. Its kernel is the group of non-zero scalar matrices

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{C} \setminus \{0\} \right\}.$$

So the group of Möbius transformations G is isomorphic to the ***projective linear group***

$$\mathrm{PGL}(2, \mathbb{C}) \coloneqq \mathrm{GL}(2, \mathbb{C}) \,/\, \{a\mathbf{I} : a \in \mathbb{C} \setminus \{0\}\}.$$

Corollary 6.9 *Möbius transformations are invertible.*

Remark 6.10 The video https://www.youtube.com/watch?v=0z1flsUNhO4 gives a very nice visualization of Möbius transformations. Möbius transformations are **geometrically** important because they map a circle or a line to a circle or a line; one may study more about this in MATH4221. Möbius transformations are **analytically** important because of the following theorems concerning invertible holomorphic functions.

Theorem 6.11 *If* $f: D(0;1) \to D(0;1)$ *is holomorphic and invertible, then there exist* $\theta \in \mathbb{R}$ *and* $a \in D(0;1)$ *such that*

$$f(z) = e^{i\theta} \frac{z-a}{1-\overline{a}z}$$

for every $z \in D(0;1)$.

Sketch of proof. Since f is surjective, there exists $a \in D(0;1)$ such that $f(a) = 0$. Recall the function $\varphi_a(z) = \frac{z-a}{1-\overline{a}z}$ introduced in Lemma 4.47. Applying part (i) of Schwarz Lemma to the function $g := f \circ \varphi_{-a}$ and then to g^{-1}, we obtain $|g'(0)| = 1$. Thus by part (ii) of Schwarz Lemma, there exists $\theta \in \mathbb{R}$ such that $g(z) = e^{i\theta} z$. So $f(z) = g\big(\varphi_a(z)\big) = e^{i\theta} \frac{z-a}{1-\overline{a}z}$. ∎

Theorem 6.12 *If* $f: \mathbb{C} \to \mathbb{C}$ *is holomorphic and invertible, then* f *is a linear polynomial, i.e. a polynomial of degree* 1.

Sketch of proof. Since f is entire, there exists a sequence $\{a_n\}$ such that $f(z) = \sum_{k=0}^{+\infty} a_k z^k$ for every $z \in \mathbb{C}$. Since f is injective, $f\left(\frac{1}{z}\right) = \sum_{k=0}^{+\infty} \frac{a_k}{z^k}$ does not have an essential singularity at 0 (otherwise $f(D(0;1) \setminus \{0\})$ is dense in $\mathbb{C}$ by Casorati-Weierstrass and $f\big(D(2;1)\big)$ is open by open mapping theorem, so $f(D(0;1) \setminus \{0\}) \cap f\big(D(2;1)\big) \neq \emptyset$ and f cannot be injective). Thus f must be a polynomial. Since f is injective, its degree has to be 1 by the Fundamental Theorem of Algebra. ∎

Theorem 6.13 *If* $f: \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ *is holomorphic and invertible, then* f *is a Möbius transformation.*

Sketch of proof. Since f is meromorphic on $\mathbb{C} \cup \{\infty\}$, f is a rational function by Theorem 6.4. Since f is injective, both the numerator and denominator of f must have degree at most 1, i.e. there exists $a, b, c, d \in \mathbb{C}$, a and c not both zero, such that $f(z) = \frac{az+b}{cz+d}$ for every $z \in \mathbb{C} \cup \{\infty\}$. It is easy to see that we must have $ad - bc \neq 0$ in order for f^{-1} to exist. ∎

Remark 6.14 Let $U \subseteq \mathbb{C} \cup \{\infty\}$ be a region. An invertible holomorphic function from U to U is called an ***automorphism*** of U. The set of all automorphisms of U equipped with function composition forms a group, which is called the ***automorphism group*** of U and is denoted by $\mathrm{Aut}(U)$. Theorem 6.13 says that $\mathrm{Aut}(\mathbb{C} \cup \{\infty\})$ is the group of Möbius transformations;

$$\mathrm{Aut}(\mathbb{C} \cup \{\infty\}) \cong \mathrm{PGL}(2, \mathbb{C}).$$

Theorems 6.11 and 6.12 say that the automorphisms of $D(0;1)$ and of $\mathbb{C}$ are special kinds of Möbius transformations; or more precisely, that $\mathrm{Aut}\big(D(0;1)\big)$ and $\mathrm{Aut}(\mathbb{C})$ are subgroups of $\mathrm{Aut}(\mathbb{C} \cup \{\infty\})$.

Definition 6.15 Two regions $U, V \subseteq \mathbb{C} \cup \{\infty\}$ are ***biholomorphically equivalent*** (or ***conformally equivalent***, or just ***biholomorphic***) if there exists an invertible holomorphic function $f: U \to V$.

Lemma 6.16 *Biholomorphic equivalence is an equivalence relation on the set of all regions in* $\mathbb{C} \cup \{\infty\}$. *In other words, if* $U, V, W \subseteq \mathbb{C} \cup \{\infty\}$ *are regions, then*

(i) U *and* U *itself are biholomorphic;*

(ii) *If* U *and* V *are biholomorphic, then* V *and* U *are biholomorphic;*

(iii) *If* U *and* V *are biholomorphic and* V *and* W *are biholomorphic, then* U *and* W *are biholomorphic.*

Remark 6.17 When one tries to solve the **Laplace equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in a region $U \subseteq \mathbb{R}^2$, the concept of biholomorphic equivalence is useful. It helps transforming a harmonic function on a possibly complicated domain into a harmonic function on a simple domain, if these domains are biholomorphic to each other.

Theorem 6.18 (Riemann mapping theorem) *If* $U \subset \mathbb{C}$ *is a simply connected region and* $U \neq \mathbb{C}$, *then there exists an invertible holomorphic function* $f: U \to D(0;1)$. *Moreover, this function* f *is uniquely determined if one fix a point* $a \in U$ *and require that* $f(a) = 0$ *and* $f'(a) > 0$.

Definition 6.19 Let $U \subseteq \mathbb{C} \cup \{\infty\}$ be a region. We say that U is a ***simply connected region*** if $(\mathbb{C} \cup \{\infty\}) \setminus U$ is connected.

Corollary 6.20 (Uniformization theorem) *There are exactly three types of simply connected regions in* $\mathbb{C} \cup \{\infty\}$ *up to biholomorphic equivalence. They are*

- ⊙ $\mathbb{C} \cup \{\infty\}$,
- ⊙ $\mathbb{C}$, *and*
- ⊙ $D(0;1)$.

2. Gamma function and Riemann zeta function

Let's recall the notion of **analytic continuation**, first mentioned in chapter 4.

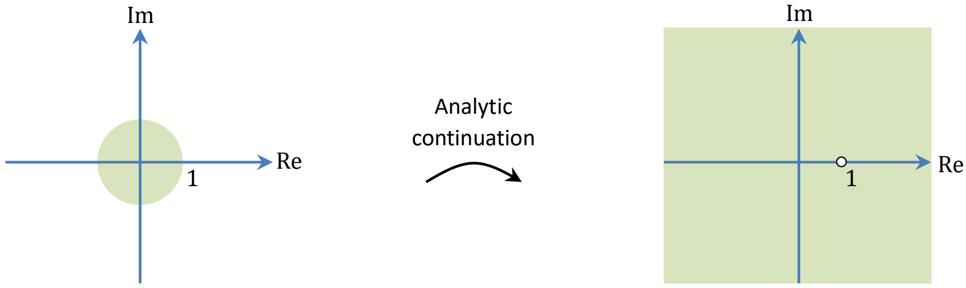
Definition 6.21 Let $U \subseteq \mathbb{C}$ be a region and $f: U \to \mathbb{C}$ be a holomorphic function. If there exist a "bigger" region $V \supset U$ and a holomorphic function $F: V \to \mathbb{C}$ such that

$$F(z) = f(z)$$

for every $z \in U$, then F is called an ***analytic continuation*** of f on V.

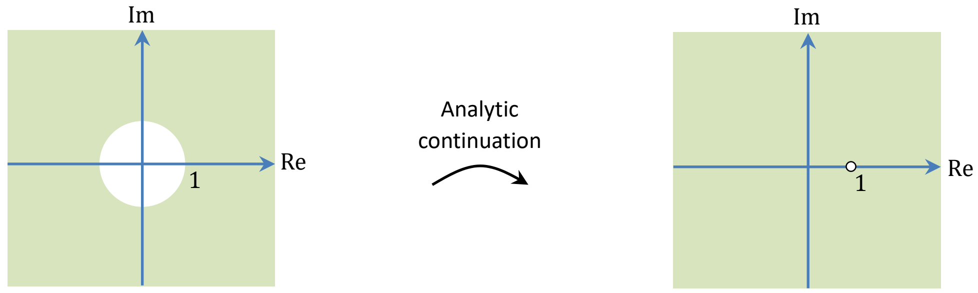
If an analytic continuation of a function exists in a given region, then it must be unique by the **identity theorem** (Corollary 4.33). When a holomorphic function is defined using a series, we can usually obtain its analytic continuation by evaluating the **limit of the series**.

Example 6.22 We have seen in Example 4.36 that the holomorphic function $f: D(0;1) \to \mathbb{C}$ defined by $f(z) = \sum_{k=0}^{+\infty} z^k$ has the analytic continuation $F: \mathbb{C} \setminus \{1\} \to \mathbb{C}$ given by $F(z) = \frac{1}{1-z}$.



$f(z) = \sum_{k=0}^{+\infty} z^k$ $\qquad$ $F(z) = \frac{1}{1-z}$

In a similar way, the holomorphic function $g: \mathbb{C} \setminus \overline{D(0;1)} \to \mathbb{C}$ defined by $g(z) = \sum_{k=0}^{+\infty} \frac{1}{z^k}$ has the analytic continuation $G: \mathbb{C} \setminus \{1\} \to \mathbb{C}$ given by $G(z) = \frac{z}{z-1}$.



$g(z) = \sum_{k=0}^{+\infty} \frac{1}{z^k}$ $\qquad$ $G(z) = \frac{z}{z-1}$

Another way of extending the domain of a holomorphic function is through a **functional equation** that the function satisfies.

Example 6.23 For each $a \in \mathbb{R}$, let $H(a) := \{z \in \mathbb{C}: \operatorname{Re} z > a\}$ denote an ***open right-half plane***. Let $f: H(0) \to \mathbb{C}$ be a holomorphic function which satisfies

$$f(z+1) = 2f(z)$$

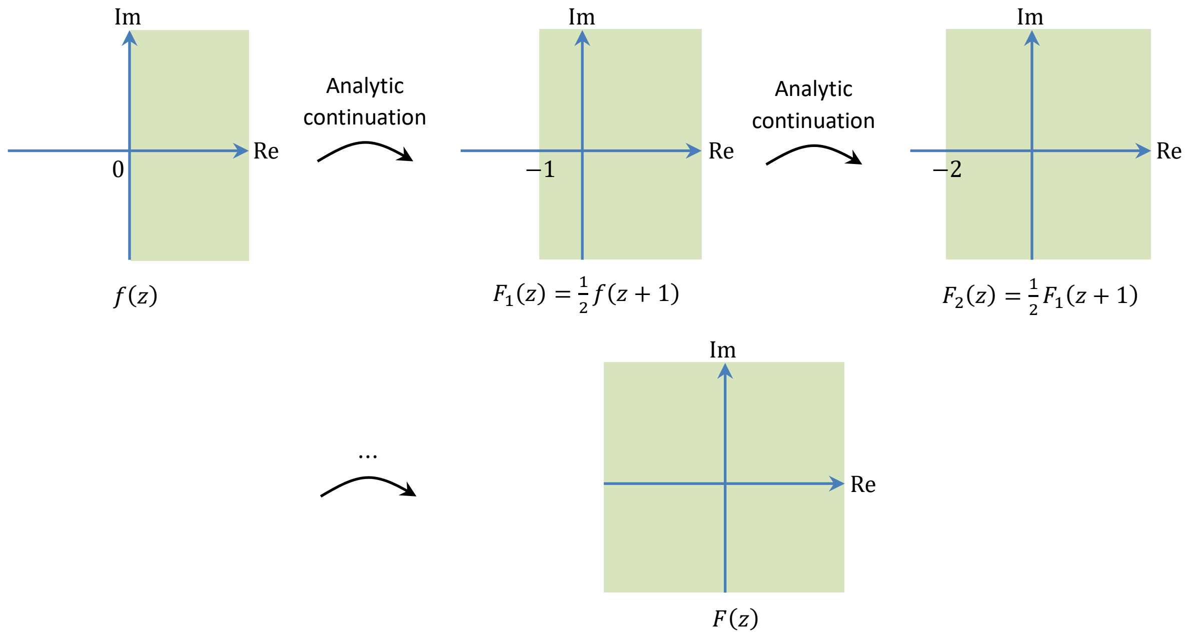
for every $z \in H(0)$. Then we define the function $F_1: H(-1) \to \mathbb{C}$ by

$$F_1(z) = \frac{1}{2} f(z+1).$$

Then F_1 is holomorphic on $H(-1)$ since f is holomorphic on $H(0)$, and $F_1(z) = f(z)$ for every $z \in H(0)$, so F_1 is the analytic continuation of f on the larger region $H(-1)$. In a similar way, we can define the analytic continuation F_2 of f on $H(-2)$ by

$$F_2(z) = \frac{1}{2} F_1(z+1) = \frac{1}{4} f(z+2).$$

Repeating this procedure inductively, we can obtain an analytic continuation of f on $\mathbb{C}$, which is an entire function $F: \mathbb{C} \to \mathbb{C}$.



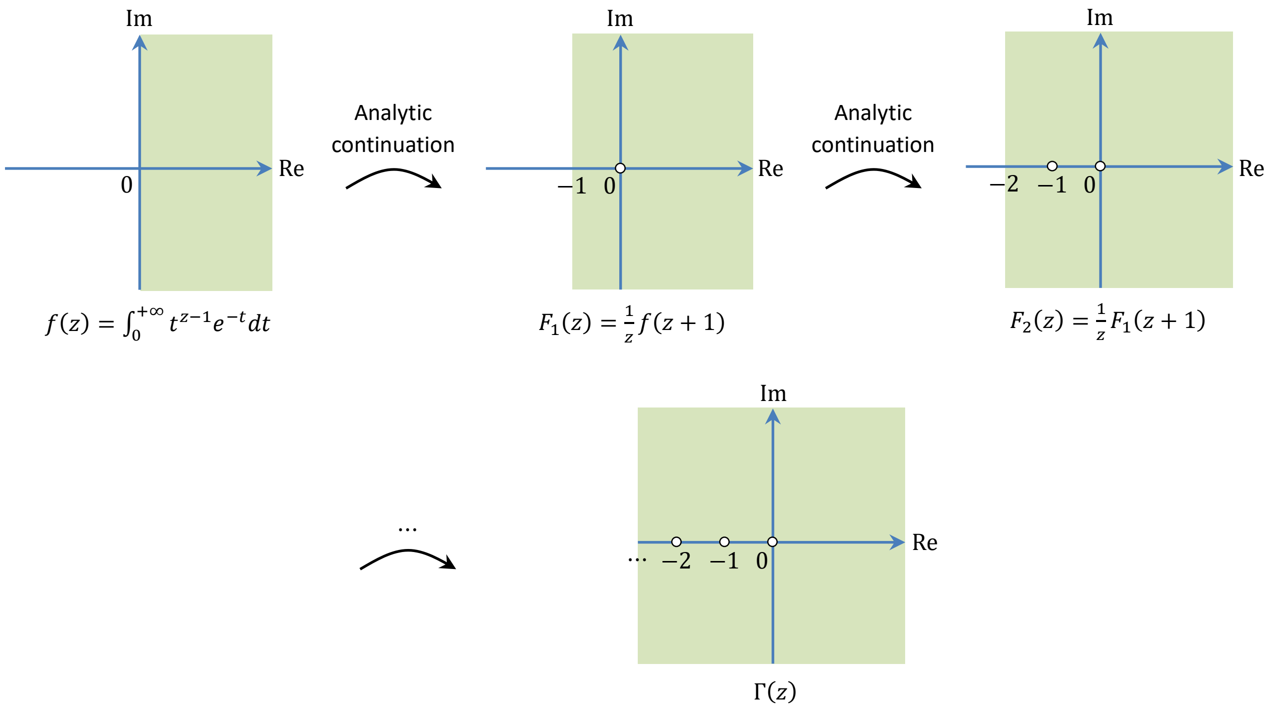
Remark 6.24 Let $f: \{z \in \mathbb{C}: \operatorname{Re} z > 0\} \to \mathbb{C}$ be the function defined by

$$f(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt.$$

It can be proved using Morera's Theorem that f is a holomorphic function. Using integration by parts, one finds that f satisfies a **functional equation**

$$f(z+1) = zf(z)$$

for every $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. Following the same technique as in Example 6.23, one obtains an analytic continuation of f on the region $\mathbb{C} \setminus \{0, -1, -2, -3, \dots\}$.



Definition 6.25 The ***Gamma function*** $\Gamma: \mathbb{C} \setminus \{0, -1, -2, -3, \dots\} \to \mathbb{C}$ is the analytic continuation of the holomorphic function $f: \{z \in \mathbb{C}: \operatorname{Re} z > 0\} \to \mathbb{C}$ defined by

$$f(z) = \int_0^{+\infty} t^{z-1}e^{-t}dt.$$

Corollary 6.26 For every $z \in \mathbb{C} \setminus \{0, -1, -2, -3, \dots\}$, we have

$$\Gamma(z+1) = z\Gamma(z).$$

Corollary 6.27 For every $n \in \mathbb{N}$, we have

$$\Gamma(n) = (n-1)!.$$

Corollary 6.28 For every $n \in \mathbb{N} \cup \{0\}$, Γ has a simple pole at $-n$ and

$$\operatorname{Res}(\Gamma; -n) = \frac{(-1)^n}{n!}.$$

Proof. Note that Γ is holomorphic at 1 with $\Gamma(1) = 1$. For every $n \in \mathbb{N} \cup \{0\}$, we have

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)} = \cdots = \frac{\Gamma(z+n+1)}{z(z+1)(z+2)\cdots(z+n-1)(z+n)}.$$

So Γ has a simple pole at $-n$ by Factor Theorem (Corollary 5.24), and

$$\operatorname{Res}(\Gamma; -n) = \lim_{z \to -n}(z+n)\Gamma(z) = \frac{\Gamma(1)}{(-n)(-n+1)(-n+2)\cdots(-1)} = \frac{(-1)^n}{n!}.$$

■

Let $f: \{z \in \mathbb{C}: \operatorname{Re} z > 1\} \to \mathbb{C}$ be the function defined by

$$f(z) = \sum_{k=1}^{+\infty} \frac{1}{k^z}.$$

We have seen in Example 4.21 that f is a holomorphic function. In the following, we aim to find a **functional equation** satisfied by f so as to construct its analytic continuation on a region as large as possible.

Lemma 6.29 *Let* $f: \{z \in \mathbb{C}: \operatorname{Re} z > 1\} \to \mathbb{C}$ *be the function defined by* $f(z) = \sum_{k=1}^{+\infty} \frac{1}{k^z}$. *Then*

$$f(z)\Gamma(z) = \int_0^{+\infty} \frac{t^{z-1}}{e^t - 1} dt$$

for every $z \in \mathbb{C}$ *with* $\operatorname{Re} z > 1$.

Lemma 6.30 *Let* $I_1: \{z \in \mathbb{C}: \operatorname{Re} z > 1\} \to \mathbb{C}$ *and* $I_2: \mathbb{C} \to \mathbb{C}$ *be the functions defined by*

$$I_1(z) = \int_0^1 \frac{t^{z-1}}{e^t - 1} dt \qquad \text{and} \qquad I_2(z) = \int_1^{+\infty} \frac{t^{z-1}}{e^t - 1} dt.$$

Then I_1 *has an analytic continuation on* $\mathbb{C} \setminus \{1, 0, -1, -2, \dots\}$ *and* I_2 *is entire.*

Sketch of proof. I_2 is entire by an easy application of Morera's Theorem. To analyze I_1, we let g be the function $g(w) = \frac{1}{e^w - 1} - \frac{1}{w}$. Then 0 is a removable singularity of g, so g extends to be holomorphic at 0, with a Maclaurin series $g(w) = \sum_{k=0}^{+\infty} \frac{g^{(k)}(0)}{k!} w^k$. This series converges uniformly on every compact subset of $D(0; 2\pi)$, so for each fixed $z \in \mathbb{C}$ with $\operatorname{Re} z > 1$, the series

$$t^{z-1} g(t) = \sum_{k=0}^{+\infty} \frac{g^{(k)}(0)}{k!} t^{z+k-1}$$

(as a series of functions of t) also converges uniformly on $[0, 1]$ in particular. Therefore

$$I_1(z) = \int_0^1 \frac{t^{z-1}}{e^t - 1} dt = \int_0^1 t^{z-1} \left(\frac{1}{t} + g(t) \right) dt = \int_0^1 \left(t^{z-2} + \sum_{k=0}^{+\infty} \frac{g^{(k)}(0)}{k!} t^{z+k-1} \right) dt$$

$$= \int_0^1 t^{z-2} dt + \sum_{k=0}^{+\infty} \frac{g^{(k)}(0)}{k!} \int_0^1 t^{z+k-1} dt = \frac{1}{z-1} + \sum_{k=0}^{+\infty} \frac{g^{(k)}(0)}{k!} \frac{1}{z+k}.$$

Now although I_1 is defined only on $\{z \in \mathbb{C}: \operatorname{Re} z > 1\}$, this final series on the right-hand side converges uniformly on every compact subset of $\mathbb{C} \setminus \{1, 0, -1, -2, \dots\}$. So I_1 has an analytic continuation on $\mathbb{C} \setminus \{1, 0, -1, -2, \dots\}$. ∎

Remark 6.31 Using the functional equation in Lemma 6.29 together with Lemma 6.30, we obtain the analytic continuation $F: \mathbb{C} \setminus \{1, 0, -1, -2, \dots\} \to \mathbb{C}$ of $f(z) = \sum_{k=1}^{+\infty} \frac{1}{k^z}$ which is defined by

$$F(z) = \underbrace{\frac{1}{\Gamma(z)}}_{\text{entire}} \left(\underbrace{\frac{1}{z-1} + \sum_{k=0}^{+\infty} \frac{g^{(k)}(0)}{k!} \frac{1}{z+k}}_{\text{holomorphic on } \mathbb{C}\setminus\{1,0,-1,-2,\dots\}} + \underbrace{\int_1^{+\infty} \frac{t^{z-1}}{e^t - 1} dt}_{\text{entire}} \right)$$

$$= \frac{1}{(z-1)\Gamma(z)} + \sum_{k=0}^{+\infty} \frac{g^{(k)}(0)}{k!} \frac{1}{(z+k)\Gamma(z)} + \frac{1}{\Gamma(z)} \int_1^{+\infty} \frac{t^{z-1}}{e^t - 1} dt,$$

where $g(z) = \frac{1}{e^z-1} - \frac{1}{z}$. But for each $n \in \mathbb{N} \cup \{0\}$, since Γ has a simple pole at $-n$, it turns out that $1/(z+n)\Gamma(z)$ has a **removable singularity** at $-n$. So in fact one can even obtain the analytic continuation of f on $\mathbb{C} \setminus \{1\}$ by extending F continuously at $0, -1, -2$, etc.

Definition 6.32 The ***Riemann zeta function*** $\zeta: \mathbb{C} \setminus \{1\} \to \mathbb{C}$ is the analytic continuation of the holomorphic function $f: \{z \in \mathbb{C}: \operatorname{Re} z > 1\} \to \mathbb{C}$ defined by

$$f(z) = \sum_{k=1}^{+\infty} \frac{1}{k^z}.$$

From Remark 6.31, we see that ζ has a simple pole at 1, with residue $\operatorname{Res}(\zeta; 1) = 1$. Moreover, for each $n \in \mathbb{N} \cup \{0\}$, we have

$$\zeta(-n) = \frac{g^{(n)}(0)}{n!} \frac{1}{\operatorname{Res}(\Gamma; -n)} = (-1)^n g^{(n)}(0),$$

where $g(z) = \frac{1}{e^z-1} - \frac{1}{z}$. The following are some **special values** of the Riemann zeta function.

Theorem 6.33 *For every positive integer n, we have*

$$\zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n}}{2 \cdot (2n)!} B_{2n} \qquad \text{and} \qquad \zeta(1-n) = (-1)^{n-1} \frac{B_n}{n},$$

*where the B_n's are the **Bernoulli numbers** defined by the coefficient of the Maclaurin series*

$$\frac{z}{e^z - 1} = \sum_{k=0}^{+\infty} \frac{B_k}{k!} z^k.$$

On the other hand, the values of ζ at the odd positive integers are still mysterious. It was proved in 1978 that $\zeta(3)$ is irrational, but whether $\zeta(5), \zeta(7), \dots$ are rational or irrational is still unknown.

Corollary 6.34 *The Riemann zeta function has the following special values:*

$$\zeta(0) = -\frac{1}{2}, \qquad \zeta(-1) = -\frac{1}{12}, \qquad \zeta(-2) = 0, \qquad \zeta(-3) = \frac{1}{120}, \qquad \dots$$

and $\zeta(-2n) = 0$ *for every positive integer* n.

Remark 6.35 The value $\zeta(-1) = -\frac{1}{12}$ is used in string theory in theoretical physics. However, many people misinterpret the value of ζ at the non-positive integers using its series representation $\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}$ which is supposed to be valid only for those $z \in \mathbb{C}$ with $\operatorname{Re} z > 1$. These misinterpretations lead to weird and mathematically **incorrect "formulas"** such as

$$1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}, \qquad 1 + 1 + 1 + 1 + \cdots = -\frac{1}{2}, \qquad 1^2 + 2^2 + 3^2 + 4^2 + \cdots = 0,$$

etc.

Definition 6.36 The negative even integers are called ***trivial zeros*** of the Riemann zeta function. A number $z \in \mathbb{C} \setminus \{-2, -4, \dots\}$ such that $\zeta(z) = 0$ is called a ***non-trivial zero*** of the Riemann zeta function.

Conjecture 6.37 (Riemann hypothesis) *Let* z *be a non-trivial zero of the Riemann zeta function. Then*

$$\operatorname{Re} z = \frac{1}{2}.$$

Proof. Insert your proof here. As of year 2022, anyone who gives a correct proof or counter-example to this conjecture will be awarded US$1,000,000 from the Clay Mathematics Institute. ■

Remark 6.38 The zeros of the Riemann zeta function are important in **analytic number theory** because they have deep connections with the distributions of prime numbers. The **prime number theorem** asserts that

$$\lim_{x \to +\infty} \frac{\pi(x)}{x / \ln x} = 1,$$

where $\pi(x)$ denotes the number of positive prime numbers that are less than or equal to x. This implies that the n^{th} prime number is approximately equal to $n \ln n$. The proof of the prime number theorem relies surprisingly on the fact that the Riemann zeta function has no zero with real part equal to 1. If the Riemann hypothesis is proven to be true, then one can substantially improve the prime number theorem, and will reveal many mysteries about the distribution of prime numbers.

3. Elliptic functions

Definition 6.39 Two complex numbers ω_1, ω_2 are said to be $\mathbb{R}$-***independent*** if $\omega_1/\omega_2 \notin \mathbb{R}$. In particular, ω_1 and ω_2 are both non-zero.

Definition 6.40 Let $\omega_1, \omega_2 \in \mathbb{C}$ be $\mathbb{R}$-independent. An ***elliptic function*** on $\mathbb{C}$ with periods ω_1 and ω_2 is a meromorphic function $f: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ such that

$$f(z+\omega_1) = f(z+\omega_2) = f(z)$$

for every $z \in \mathbb{C}$. For each $a \in \mathbb{C}$, the parallelogram region with vertices a, $a+\omega_1$, $a+\omega_1+\omega_2$ and $a+\omega_2$ is called a ***fundamental parallelogram*** of the elliptic function f.

Remark 6.41 The name "elliptic functions" comes from the fact that these functions are related to **elliptic integrals**, which represent the arc-length of an ellipse. More precisely, elliptic functions can be obtained as analytic continuations of inverses of functions defined as elliptic integrals.

Elliptic functions have the following properties, some of which have already been covered as exercise problems in the Problem Sets.

Theorem 6.42 (Liouville) *If $f: \mathbb{C} \to \mathbb{C}$ is an entire elliptic function, then f is constant.*

Proof. The proof is similar to Example 4.54. Let $U \subset \mathbb{C}$ be a fundamental parallelogram of f. Since $\overline{U}$ is compact and f is continuous, the direct image $f(\overline{U})$ is bounded. By the periodicity of f, $f(\mathbb{C}) = f(\overline{U})$ is also bounded. Now f is a bounded entire function, so f is constant by Liouville's Theorem (Theorem 4.51). ■

Given an elliptic function f, there always exist fundamental parallelograms $U \subset \mathbb{C}$ of f such that f has no zeros and no poles on ∂U, because of the isolated zeros theorem.

Theorem 6.43 *Let $f: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ be a non-constant elliptic function. Then the sum of residues of f is zero, i.e.*

$$\sum_{a \in \mathbb{C}} \operatorname{Res}(f; a) = 0.$$

Sketch of proof (cf. Q19 of Problem Set 6). Let $U \subset \mathbb{C}$ be a fundamental parallelogram of f such that f has no zeros and no poles on ∂U. By Cauchy's residue theorem,

$$\sum_{a \in U} \operatorname{Res}(f; a) = \frac{1}{2\pi i} \oint_{\partial U} f(z)\,dz = 0$$

by the periodicity of f. Therefore $\sum_{a \in \mathbb{C}} \operatorname{Res}(f; a) = 0$. ■

Theorem 6.42 and Theorem 6.43 immediately imply the following.

Corollary 6.44 *Let $f: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ be a non-constant elliptic function and let $U \subset \mathbb{C}$ be a fundamental parallelogram of f such that f has no zeros and no poles on ∂U. Then f has at least two poles in U, counting multiplicities.*

Theorem 6.45 *Let $f: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ be a non-constant elliptic function with $\mathbb{R}$-independent periods $\omega_1, \omega_2 \in \mathbb{C}$, and let U be a fundamental parallelogram of f such that f has no zeros and no poles on ∂U. If in U, f has a zero of order m_j at a_j for $j \in \{1, 2, \dots, m\}$ and has a pole of order n_k at b_k for $k \in \{1, 2, \dots, n\}$, then*

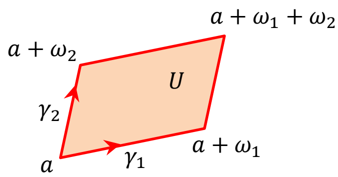
$$\sum_{j=1}^{m} m_j a_j - \sum_{k=1}^{n} n_k b_k = p\omega_1 + q\omega_2$$

for some integers p and q.

Proof (cf. Q14 of Problem Set 7). By the generalized argument principle, we have

$$\sum_{j=1}^{m} m_j a_j - \sum_{k=1}^{n} n_k b_k = \frac{1}{2\pi i} \oint_{\partial U} \frac{z f'(z)}{f(z)} dz.$$

Suppose that the vertices of ∂U are a, $a + \omega_1$, $a + \omega_1 + \omega_2$ and $a + \omega_2$. Let γ_1 be the line segment joining a and $a + \omega_1$, and γ_2 be the line segment joining a and $a + \omega_2$. Then by the periodicity of f, we have



$$\begin{aligned}
&\frac{1}{2\pi i} \oint_{\partial U} \frac{z f'(z)}{f(z)} dz \\
&= \frac{1}{2\pi i} \left[\oint_{\gamma_1} \frac{z f'(z)}{f(z)} dz + \oint_{\gamma_2} \frac{(z + \omega_1) f'(z)}{f(z)} dz + \oint_{-\gamma_1} \frac{(z + \omega_2) f'(z)}{f(z)} dz + \oint_{-\gamma_2} \frac{z f'(z)}{f(z)} dz \right] \\
&= \left(\frac{1}{2\pi i} \oint_{\gamma_2} \frac{f'(z)}{f(z)} dz \right) \omega_1 + \left(-\frac{1}{2\pi i} \oint_{\gamma_1} \frac{f'(z)}{f(z)} dz \right) \omega_2.
\end{aligned}$$

Since f has no zeros and no poles on image γ_1, there exists a small region V containing image γ_1 such that f has no zeros and no poles on V. By Theorem 3.66, there exists a holomorphic function $h: V \to \mathbb{C}$ such that $e^h = f$ on V. Hence

$$\frac{1}{2\pi i} \oint_{\gamma_1} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \oint_{\gamma_1} h'(z) dz = \frac{1}{2\pi i} [h(a + \omega_1) - h(a)],$$

which is an integer since $e^{h(a+\omega_1)} = f(a + \omega_1) = f(a) = e^{h(a)}$. Similarly

$$\frac{1}{2\pi i} \oint_{\gamma_2} \frac{f'(z)}{f(z)} dz$$

is also an integer. ■

An important pair of examples of elliptic functions is the **Weierstrass' $\wp$-function** and its derivative.

Definition 6.46 Let $\omega_1, \omega_2 \in \mathbb{C}$ be $\mathbb{R}$-independent. The function $\wp: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{(p,q)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left(\frac{1}{(z - p\omega_1 - q\omega_2)^2} - \frac{1}{(p\omega_1 + q\omega_2)^2}\right)$$

is called ***Weierstrass' $\wp$-function*** with periods ω_1 and ω_2.

The constant terms $-1/(p\omega_1 + q\omega_2)^2$ in the series are included so that the double two-sided series is uniformly absolutely-convergent on every compact subset of $\mathbb{C} \setminus \{p\omega_1 + q\omega_2 : p, q \in \mathbb{Z}\}$.

Lemma 6.47 *$\wp$ is an even elliptic function on $\mathbb{C}$, which has double poles at $p\omega_1 + q\omega_2$ for each $p, q \in \mathbb{Z}$ and no other singularities. Its derivative $\wp'$ is an odd elliptic function on $\mathbb{C}$ which has triple poles at $p\omega_1 + q\omega_2$ for each $p, q \in \mathbb{Z}$ and no other singularities. $\wp'$ has simple zeros at*

$$\left(p + \frac{1}{2}\right)\omega_1 + q\omega_2, \quad p\omega_1 + \left(q + \frac{1}{2}\right)\omega_2 \quad \text{and} \quad \left(p + \frac{1}{2}\right)\omega_1 + \left(q + \frac{1}{2}\right)\omega_2$$

for each $p, q \in \mathbb{Z}$ and no other zeros.

Theorem 6.48 *$\wp$ and $\wp'$ are algebraically dependent. More precisely, we have*

$$(\wp')^2 = 4\wp^3 + a\wp + b$$

for some constants $a, b \in \mathbb{C}$ which have explicit expressions in terms of ω_1 and ω_2.

A remarkable importance of the Weierstrass' $\wp$-function is that it connects complex analysis with the study of **elliptic curves**. This connection is explicitly described in the following two theorems.

Theorem 6.49 *Every elliptic function on $\mathbb{C}$ with given periods can be expressed as a rational function in terms of $\wp$ and $\wp'$. More precisely, if f is an elliptic function on $\mathbb{C}$ with $\mathbb{R}$-independent periods $\omega_1, \omega_2 \in \mathbb{C}$, then there exist polynomials g_1, g_2, h_1, h_2 such that*

$$f = \frac{g_1(\wp)}{g_2(\wp)}\wp' + \frac{h_1(\wp)}{h_2(\wp)}.$$

Theorem 6.50 *Let $\omega_1, \omega_2 \in \mathbb{C}$ be $\mathbb{R}$-independent, and let E be the elliptic curve*

$$y^2 = 4x^3 + ax + b$$

in $\mathbb{C}^2 \cup \{\infty\}$, where $a, b \in \mathbb{C}$ are related to ω_1 and ω_2 as in Theorem 6.48. Then the map

$$\begin{aligned} \mathbb{C} / (\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) &\longrightarrow E \\ z &\mapsto (\wp(z), \wp'(z)) \end{aligned}$$

is a bijective holomorphic map and a group isomorphism.

Summary of Chapter 6

The following are what you need to know in this chapter in order to get a pass (a distinction) in this course:

- ✓ **Möbius transformations**
  - ⦿ Meromorphic functions on $\mathbb{C} \cup \{\infty\}$
  - ⦿ Residue at ∞
  - ⦿ **Möbius transformations** and the **Möbius group** $\mathrm{Aut}(\mathbb{C} \cup \{\infty\})$
  - ⦿ **Biholomorphic equivalence**
  - ⦿ **Riemann mapping theorem**, classification of simply connected regions in $\mathbb{C} \cup \{\infty\}$

- ✓ **Gamma function** and **Riemann zeta function**
  - ⦿ Analytic continuation
  - ⦿ **Gamma function** and its properties
  - ⦿ **Riemann zeta function** and its properties
  - ⦿ **Riemann hypothesis**

- ✓ **Elliptic functions**
  - ⦿ Analytic properties of elliptic functions
  - ⦿ **Weierstrass' $\wp$-function**
  - ⦿ **Field of elliptic functions** with given periods