Part A: Brief responses (13 points)

1. Consider the following subsets of $\mathbb{C}$:

$$\begin{aligned}
A &= \text{the set of all roots of the polynomial } z^{4023} + z^{2022} + 1,\\
B &= \text{the set of all solutions to the equation } \sin z = 2,\\
C &= \{z \in \mathbb{C} : |z| > 10\},\\
D &= \{z \in \mathbb{C} : |z| \geq 10 \text{ and } \operatorname{Im} z > 0\},\\
E &= \{z \in \mathbb{C} : |z| = 10\}.
\end{aligned}$$

(4 points)

(a) Which of them is/are **open** subset(s) of $\mathbb{C}$? C

(b) Which of them is/are **closed** subset(s) of $\mathbb{C}$? $A,\ B,\ E$

(c) Which of them is/are **compact** subset(s) of $\mathbb{C}$? $A,\ E$

(d) Which of them is/are **connected** subset(s) of $\mathbb{C}$? $C,\ D,\ E$

The classifications of C, D and E are obvious.

- A is a finite set, so it is not open and it is bounded. A is disconnected as it is finite but it consists of more than 1 element. A is closed as it is the inverse image of the closed set $\{0\}$ via the continuous function $f(z) = z^{4023} + z^{2022} + 1$, and so A is also compact.
- B is a countably infinite set, so it is not open and is disconnected. B is unbounded since $\sin$ is a periodic function with period 2π, and so B is not compact. But B is closed as it is the inverse image of the closed set $\{2\}$ via the continuous function $\sin$.

2. Consider the following regions in $\mathbb{C}$:

$$\begin{aligned}
U &= D(0;2),\\
V &= U \setminus \overline{D(0;1)},\\
W &= V \cap \{z \in \mathbb{C} : \operatorname{Re} z < 0\}.
\end{aligned}$$

On which of these regions U, V and W does there exist a holomorphic branch of complex logarithm? Also explain briefly

(i) **why there is** such a branch on each of your choice(s) and

(ii) **why there is no** such branch on each of the other(s).

(3 points)

There is a holomorphic branch of logarithm on W because W is simply connected and $0 \notin W$.
There is no such branch on U because U contains 0.
There is no such branch on V because there exists a simple closed curve in V whose interior contains 0.

3. Let $f: \mathbb{C} \to \mathbb{C}$ be the function

$$f(z) = e^z.$$

What is the **direct image** of the set

$$S = \{z \in \mathbb{C}: 0 \leq \operatorname{Re} z \leq 1 \text{ and } 2 \leq \operatorname{Im} z \leq 3\}$$

via f?

(2 points)

$$\begin{aligned} f(S) &= \{w \in \mathbb{C}: w = e^z \text{ for some } z \in S\} \\ &= \left\{w \in \mathbb{C}: w = e^{x+iy} \text{ for some } x \in [0,1] \text{ and } y \in [2,3]\right\} \\ &= \left\{w \in \mathbb{C}: w = re^{i\theta} \text{ for some } r \in [e^0, e^1] \text{ and } \theta \in [2,3]\right\} \\ &= \{w \in \mathbb{C}: 1 \leq |w| \leq e \text{ and } 2 \leq \operatorname{Arg} w \leq 3\}. \end{aligned}$$

4. Let $\sum_{k=0}^{+\infty} a_k z^k$ be a power series with radius of convergence $R > 0$. What is the **radius of convergence** of the power series

$$\sum_{k=0}^{+\infty} (|a_{3k}| + |a_{3k+1}|) z^{4k}$$

in terms of R?

(2 points)

The answer is $R^{3/4}$. To obtain the answer without rigorous proof, it suffices to just consider an example of sequence of coefficients $\{a_n\}$ such that $\limsup_n |a_n|^{\frac{1}{n}} = \frac{1}{R}$, say

$$a_n = \frac{1}{R^n}.$$

Then

$$\limsup_n (|a_{3n}| + |a_{3n+1}|)^{\frac{1}{4n}} = \limsup_n \left(\frac{1}{R^{3n}} + \frac{1}{R^{3n+1}}\right)^{\frac{1}{4n}} = \limsup_n \frac{1}{R^{3/4}}\left(1 + \frac{1}{R}\right)^{\frac{1}{4n}} = \frac{1}{R^{3/4}},$$

so the radius of convergence of the second power series is $R^{3/4}$.

Remark: The following is a rigorous proof of the fact that $\limsup_n |a_n|^{\frac{1}{n}} = \frac{1}{R}$ implies

$$\limsup_n (|a_{3n}| + |a_{3n+1}|)^{\frac{1}{4n}} = \frac{1}{R^{3/4}}.$$

Since $\limsup_n |a_n|^{\frac{1}{n}} = \frac{1}{R}$, for each small $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n|^{\frac{1}{n}} < \frac{1}{R} + \frac{\varepsilon}{2}$ for every $n \geq N$, i.e.

$$(|a_{3n}|+|a_{3n+1}|)^{\frac{1}{4n}} < \left(\left(\frac{1}{R}+\frac{\varepsilon}{2}\right)^{3n}+\left(\frac{1}{R}+\frac{\varepsilon}{2}\right)^{3n+1}\right)^{\frac{1}{4n}}$$

$$=\left(\frac{1}{R}+\frac{\varepsilon}{2}\right)^{\frac{3}{4}}\left(1+\left(\frac{1}{R}+\frac{\varepsilon}{2}\right)\right)^{\frac{1}{4n}} < \frac{1}{R^{3/4}}+\varepsilon$$

for every sufficiently large n. Also, there exists infinitely many n's such that $|a_n|^{\frac{1}{n}} > \frac{1}{R}-\frac{\varepsilon}{2}$, so

$$(|a_{3n}|+|a_{3n+1}|)^{\frac{1}{4n}} > \left(\left(\frac{1}{R}-\frac{\varepsilon}{2}\right)^{3n}+\left(\frac{1}{R}-\frac{\varepsilon}{2}\right)^{3n+1}\right)^{\frac{1}{4n}}$$

$$=\left(\frac{1}{R}-\frac{\varepsilon}{2}\right)^{\frac{3}{4}}\left(1+\left(\frac{1}{R}-\frac{\varepsilon}{2}\right)\right)^{\frac{1}{4n}} > \frac{1}{R^{3/4}}-\varepsilon$$

for these infinitely many n's which are sufficiently large.

5. Evaluate the line integral

$$\oint_{\partial D(2;1)} \frac{\cos z}{z(z-2)}\,dz\,.$$

(2 points)

The answer is $(\pi\cos 2)i$. To obtain the answer, we let $U = D(2;2)$. Then

- ⦿ U is a simply connected region,
- ⦿ the function $f(z)=\frac{\cos z}{z}$ is holomorphic on U,
- ⦿ $\partial D(2;1)$ is a simple closed curve in U, and
- ⦿ 2 is in the interior of $\partial D(2;1)$.

So by Cauchy integral formula, we have

$$\oint_{\partial D(2;1)} \frac{\cos z}{z(z-2)}\,dz = \oint_{\partial D(2;1)} \frac{f(z)}{z-2}\,dz = 2\pi i f(2) = 2\pi i\cdot\frac{\cos 2}{2}$$
$$=(\pi\cos 2)i.$$

Part B: Short problems (47 points)

6. Let $z \in \mathbb{C} \setminus \{i, -i\}$. Show that $\frac{z}{1+z^2}$ is a real number **if and only if** either

$$\operatorname{Im} z = 0 \qquad \text{or} \qquad |z| = 1.$$

(4 points)

$$\begin{aligned}
\frac{z}{1+z^2} \in \mathbb{R} \quad &\Leftrightarrow \quad \frac{z}{1+z^2} = \overline{\left(\frac{z}{1+z^2}\right)} \\
&\Leftrightarrow \quad \frac{z}{1+z^2} = \frac{\bar{z}}{1+\bar{z}^2} \\
&\Leftrightarrow \quad z\left(1+\bar{z}^2\right) = \bar{z}(1+z^2) \\
&\Leftrightarrow \quad z + z\bar{z}^2 - \bar{z} - \bar{z}z^2 = 0 \\
&\Leftrightarrow \quad (z-\bar{z})(1-z\bar{z}) = 0 \\
&\Leftrightarrow \quad z = \bar{z} \quad \text{or} \quad |z|^2 = 1 \\
&\Leftrightarrow \quad \operatorname{Im} z = 0 \quad \text{or} \quad |z| = 1
\end{aligned}$$

Alternative solution:

Let $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$. Then

$$\frac{z}{1+z^2} = \frac{x+iy}{1+(x+iy)^2} = \frac{x+iy}{1+x^2-y^2+2xyi} = \frac{(x+iy)(1+x^2-y^2-2xyi)}{(1+x^2-y^2)^2+(2xy)^2},$$

so

$$\operatorname{Im} \frac{z}{1+z^2} = \frac{y(1+x^2-y^2)-2x^2y}{(1+x^2-y^2)^2+(2xy)^2} = \frac{y(1-x^2-y^2)}{(1+x^2-y^2)^2+(2xy)^2}.$$

Now $\frac{z}{1+z^2}$ is a real number if and only if $\operatorname{Im} \frac{z}{1+z^2} = 0$, which happens if and only if either

$$y = 0 \qquad \text{or} \qquad 1 - x^2 - y^2 = 0,$$

i.e. either $\operatorname{Im} z = 0$ or $|z| = 1$.

7. Let z be a root of the polynomial $p(z) = (z+1)^n + z^n$. Show that

$$\operatorname{Re} z = -\frac{1}{2}.$$

(4 points)

Since z is a root of p, we have $0 = p(z) = (z+1)^n + z^n$, so

$$\left(\frac{z+1}{z}\right)^n = -1.$$

Let $w = \frac{z+1}{z}$. Then $|w| = 1$ and $w \neq 1$, so there exists $\theta \in \mathbb{R}$ such that

$$w = \cos\theta + i\sin\theta$$

and $\cos\theta \neq 1$. Now

$$z = \frac{1}{w-1} = \frac{1}{(\cos\theta - 1) + i\sin\theta} = \frac{(\cos\theta - 1) - i\sin\theta}{(\cos\theta - 1)^2 + \sin^2\theta} = \frac{\cos\theta - 1}{2 - 2\cos\theta} - i\frac{\sin\theta}{2 - 2\cos\theta},$$

so

$$\operatorname{Re} z = \frac{\cos\theta - 1}{2 - 2\cos\theta} = -\frac{1}{2}.$$

Alternative solution:
Since z is a root of p, we have $0 = p(z) = (z+1)^n + z^n$, so

$$|z+1|^n = |z|^n$$

which implies that $|z+1| = |z|$ (note that if $n = 0$ then p has no root). Now let $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$. Then we have

$$|(x+1) + iy| = |x + iy|$$
$$(x+1)^2 + y^2 = x^2 + y^2$$
$$x = -\frac{1}{2}.$$

Therefore $\operatorname{Re} z = -\frac{1}{2}$.

8. Let $b \in \mathbb{C}$ and let $f: \mathbb{C} \to \mathbb{C}$ be defined as

$$f(z) = \begin{cases} \dfrac{\operatorname{Im} z}{z} & \text{if } z \neq 0 \\ b & \text{if } z = 0 \end{cases}.$$

Determine all possible value(s) of b such that f is continuous on $\mathbb{C}$.

(4 points)

Consider the limit $\lim_{z\to 0} f(z)$. Along the real axis, we have

$$\lim_{\substack{(x,y)\to(0,0) \\ y=0}} f(x+iy) = \lim_{x\to 0} \frac{\operatorname{Im}(x+i0)}{x+i0} = \lim_{x\to 0} \frac{0}{x} = 0$$

but along the imaginary axis, we have

$$\lim_{\substack{(x,y)\to(0,0) \\ x=0}} f(x+iy) = \lim_{y\to 0} \frac{\operatorname{Im}(0+iy)}{0+iy} = \lim_{y\to 0} \frac{y}{iy} = -i.$$

This shows that $\lim_{z\to 0} f(z)$ does not exist.

Therefore it is impossible for f to be continuous at 0 for any value of b.

Alternative solution:

If f is continuous at 0, then the limit $\lim_{z\to 0} f(z)$ exists and equals to b. In particular, there exists $\delta > 0$ such that $\left|\frac{\operatorname{Im} z}{z} - b\right| < \frac{1}{2}$ whenever $0 < |z| < \delta$.

But this implies that both $\left|\frac{\operatorname{Im}(\delta/2)}{\delta/2} - b\right| < \frac{1}{2}$ and $\left|\frac{\operatorname{Im}(i\delta/2)}{i\delta/2} - b\right| < \frac{1}{2}$, so

$$1 = |b - (b+i)| \leq |b| + |b+i| = \left|\frac{\operatorname{Im}(\delta/2)}{\delta/2} - b\right| + \left|\frac{\operatorname{Im}(i\delta/2)}{i\delta/2} - b\right| < \frac{1}{2} + \frac{1}{2} = 1,$$

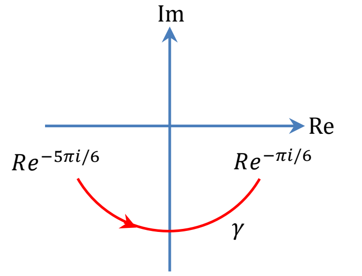
which is a contradiction.

Therefore it is impossible for f to be continuous at 0 for any value of b.

9. For each $R > 1$, let γ be the portion of $\partial D(0;R)$ joining $Re^{-\frac{5\pi i}{6}}$ to $Re^{-\frac{\pi i}{6}}$, oriented counterclockwise about 0 (see the diagram). Show that

$$\lim_{R\to+\infty}\int_\gamma \frac{1}{1+e^{iz}}\,dz = 0.$$

(6 points)



For each $z \in \text{image}\,\gamma$, there exists $t \in \left[-\frac{5\pi}{6}, -\frac{\pi}{6}\right]$ such that $z = Re^{it}$. Since $\sin t \le -\frac{1}{2}$, we have

$$\left|\frac{1}{1+e^{iz}}\right| \le \frac{1}{|e^{iz}|-1} = \frac{1}{e^{-R\sin t}-1} \le \frac{1}{e^{\frac{R}{2}}-1}.$$

Now the arc-length of γ is $\frac{2\pi R}{3}$, so by ML-estimate we have

$$\left|\int_\gamma \frac{1}{1+e^{iz}}\,dz\right| \le \frac{1}{e^{\frac{R}{2}}-1}\cdot\frac{2\pi R}{3}.$$

Since $\lim_{R\to+\infty}\frac{1}{e^{\frac{R}{2}}-1}\cdot\frac{2\pi R}{3} = 0$, we have $\lim_{R\to+\infty}\left|\int_\gamma \frac{1}{1+e^{iz}}\,dz\right| = 0$ by squeeze theorem. Therefore

$$\lim_{R\to+\infty}\int_\gamma \frac{1}{1+e^{iz}}\,dz = 0.$$

10. Let $U \subseteq \mathbb{C}$ be a region and $f: U \to \mathbb{C}$ be a holomorphic function such that for **each** $z \in U$, either $\operatorname{Re} f(z) = 1$ or $\operatorname{Im} f(z) = 1$. Show that f is a constant function.

(7 points)

Identify U as a region in $\mathbb{R}^2$, and let $u, v: U \to \mathbb{R}$ be the functions $u(x,y) = \operatorname{Re} f(x+iy)$ and $v(x,y) = \operatorname{Im} f(x+iy)$. Then

$$[u(x,y)-1][v(x,y)-1] = 0$$

for every $(x,y) \in U$, so it follows that

$$(u-1)v_x + (v-1)u_x = 0 \qquad \text{and} \qquad (u-1)v_y + (v-1)u_y = 0$$

on U. Since f is holomorphic, u and v also satisfy the Cauchy-Riemann equations

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x$$

on U. These imply that

$$(u-1)v_x + (v-1)u_x = 0 \qquad \text{and} \qquad (u-1)u_x - (v-1)v_x = 0$$

on U. Now $(f-1-i)^2$ is a holomorphic function on U with

$$\begin{aligned}((f-1-i)^2)' &= 2(f-1-i)f' \\ &= 2\big((u-1)+i(v-1)\big)(u_x+iv_x) \\ &= 2\big((u-1)u_x-(v-1)v_x\big) + 2i\big((u-1)v_x+(v-1)u_x\big) \\ &= 0\end{aligned}$$

on U. Since U is a region, we conclude that $(f-1-i)^2$ is a constant function, i.e. there exists $a \in \mathbb{C}$ such that $[f(z)-1-i]^2 = a^2$ for every $z \in U$. Now since f is continuous and U is connected, the image of f is a non-empty connected subset of $\{a+1+i, -a+1+i\}$, which must be a singleton. Therefore f is a constant function.

11. Let $v: \mathbb{R}^2 \to \mathbb{R}$ be the function

$$v(x, y) = y \sin x \cosh y + x \cos x \sinh y\,.$$

(7 points)

(a) Show that v is harmonic on $\mathbb{R}^2$.

For every $(x, y) \in \mathbb{R}^2$, we have

$$v_x(x, y) = y \cos x \cosh y + \cos x \sinh y - x \sin x \sinh y\,,$$
$$v_{xx}(x, y) = -y \sin x \cosh y - 2 \sin x \sinh y - x \cos x \sinh y\,,$$
$$v_y(x, y) = \sin x \cosh y + y \sin x \sinh y + x \cos x \cosh y\,,$$
$$v_{yy}(x, y) = 2 \sin x \sinh y + y \sin x \cosh y + x \cos x \sinh y\,.$$

Therefore $v_{xx} + v_{yy} = 0$ on $\mathbb{R}^2$.

(b) Find an entire function $f: \mathbb{C} \to \mathbb{C}$ such that

$$\operatorname{Im} f(x + iy) = v(x, y)$$

for every $(x, y) \in \mathbb{R}^2$. Express $f(z)$ in terms of the complex variable z only.

We solve the Cauchy-Riemann equations

$$u_x(x, y) = v_y(x, y) \qquad \text{and} \qquad v_x(x, y) = -u_y(x, y)$$

for the unknown function u. From the first equation we have

$$u_x(x, y) = v_y(x, y) = \sin x \cosh y + y \sin x \sinh y + x \cos x \cosh y\,,$$

so

$$u(x, y) = x \sin x \cosh y - y \cos x \sinh y + C(y),$$

where $C(y)$ is a real differentiable function independent of x. Now

$$u_y(x, y) = x \sin x \sinh y - \cos x \sinh y - y \cos x \cosh y + C'(y),$$

so together with the second equation $u_y(x, y) = -v_x(x, y)$ we have

$$C'(y) = 0,$$

so $C(y) = C$ which is an arbitrary real constant. We choose $C = 0$ to get a solution

$$u(x, y) = x \sin x \cosh y - y \cos x \sinh y\,.$$

Finally, an entire function f with $\operatorname{Im} f(x + iy) = v(x, y)$ is then given by

$$\begin{aligned} f(x + iy) &= (x \sin x \cosh y - y \cos x \sinh y) + i(y \sin x \cosh y + x \cos x \sinh y) \\ &= (x + iy)(\sin x \cosh y + i \cos x \sinh y) \\ &= (x + iy)(\sin x \cos iy + \cos x \sin iy) \\ &= (x + iy) \sin(x + iy)\,, \end{aligned}$$

i.e. $f(z) = z \sin z$.

12. Evaluate the line integral

$$\oint_{\partial D(0;2)} \frac{e^{z^2+|z|^2}}{z^2-z}\,dz\,.$$

Show all your work with explanations in full detail.

(7 points)

The circle $\partial D(0;2)$ has a parametrization $\gamma:[0,2\pi]\to\mathbb{C}$ given by $\gamma(t)=2e^{it}$. So

$$\oint_{\partial D(0;2)} \frac{e^{z^2+|z|^2}}{z^2-z}\,dz = \int_0^{2\pi} \frac{e^{\left(2e^{it}\right)^2+2^2}}{(2e^{it})^2-(2e^{it})}\cdot 2ie^{it}\,dt = \oint_{\partial D(0;2)} \frac{e^{z^2+4}}{z^2-z}\,dz\,.$$

In the last line integral, the integrand $\frac{e^{z^2+4}}{z^2-z}=\frac{e^{z^2+4}}{z(z-1)}$ is holomorphic away from 0 and 1. Since $\partial D\left(0;\frac{1}{3}\right)$ and $\partial D\left(1;\frac{1}{3}\right)$ are simple closed curves in the interior of $\partial D(0;2)$, whose interiors are mutually disjoint, we have

$$\oint_{\partial D(0;2)} \frac{e^{z^2+4}}{z^2-z}\,dz = \oint_{\partial D\left(0;\frac{1}{3}\right)} \frac{e^{z^2+4}}{z(z-1)}\,dz + \oint_{\partial D\left(1;\frac{1}{3}\right)} \frac{e^{z^2+4}}{z(z-1)}\,dz$$

by the general version of Cauchy-Goursat Theorem (Theorem 3.45). Now applying Cauchy integral formula, we have

$$\oint_{\partial D\left(0;\frac{1}{3}\right)} \frac{e^{z^2+4}}{z(z-1)}\,dz = \oint_{\partial D\left(0;\frac{1}{3}\right)} \frac{e^{z^2+4}/(z-1)}{z}\,dz = 2\pi i\cdot\frac{e^{0^2+4}}{0-1} = -2\pi i e^4$$

and

$$\oint_{\partial D\left(1;\frac{1}{3}\right)} \frac{e^{z^2+4}}{z(z-1)}\,dz = \oint_{\partial D\left(1;\frac{1}{3}\right)} \frac{e^{z^2+4}/z}{z-1}\,dz = 2\pi i\cdot\frac{e^{1^2+4}}{1} = 2\pi i e^5.$$

Therefore

$$\oint_{\partial D(0;2)} \frac{e^{z^2+|z|^2}}{z^2-z}\,dz = 2\pi i(e^5-e^4).$$

13. Let $f(z) = \sum_{k=0}^{+\infty} a_k z^k$ be a power series such that

$$f\left(e^{\frac{2\pi i}{4023}}\frac{1}{n}\right) = e^{\frac{2\pi i}{4023}} f\left(\frac{1}{n}\right)$$

for every $n \in \mathbb{N}$. Show that $a_k = 0$ whenever $k-1$ is not divisible by 4023.

(8 points)

Let g and h be the power series centered at 0 defined by

$$g(z) = \sum_{k=0}^{+\infty} a_k e^{\frac{2\pi k i}{4023}} z^k \qquad \text{and} \qquad h(z) = \sum_{k=0}^{+\infty} a_k e^{\frac{2\pi i}{4023}} z^k.$$

Since the radius of convergence of f is at least 1, the radii of convergence of g and of h are also at least 1. For every $n \in \mathbb{N}$, we have

$$g\left(\frac{1}{n}\right) = \sum_{k=0}^{+\infty} a_k e^{\frac{2\pi k i}{4023}} \left(\frac{1}{n}\right)^k = f\left(e^{\frac{2\pi i}{4023}}\frac{1}{n}\right) = e^{\frac{2\pi i}{4023}} f\left(\frac{1}{n}\right) = \sum_{k=0}^{+\infty} a_k e^{\frac{2\pi i}{4023}} \left(\frac{1}{n}\right)^k = h\left(\frac{1}{n}\right).$$

Since the sequence $\left\{\frac{1}{n}\right\}$ converges to the center 0 of the power series g and h, and no term of this sequence is 0, by the uniqueness of power series we must have $g = h$, i.e.

$$a_k e^{\frac{2\pi k i}{4023}} = a_k e^{\frac{2\pi i}{4023}} \qquad \text{for every } k \in \mathbb{N} \cup \{0\}$$

or in other words,

$$a_k \left(e^{\frac{2\pi k i}{4023}} - e^{\frac{2\pi i}{4023}}\right) = 0 \qquad \text{for every } k \in \mathbb{N} \cup \{0\}.$$

If $k-1$ is not divisible by 4023, then

$$e^{\frac{2\pi k i}{4023}} \neq e^{\frac{2\pi i}{4023}}$$

so the above equality implies that $a_k = 0$.