Honor Analysis

Lecture Notes for MATH 2043 and MATH 3043

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Preface

This is a set of lecture notes written for the courses MATH 2043 (Honor Mathematical Analysis) and MATH 3043 (Honor Real Analysis). The former course covers fundamental notions and techniques in analysis including point-set topology on the real line and on metric spaces, concepts of differentiability for multivariable functions, Riemann integrals, and uniform convergence. The latter course mainly covers measure theory and Lebesgue integrals.

Before reading this lecture notes and taking MATH 2043, it is recommended that students have already acquired basic concepts and skills of rigorous ϵ, δ-approach of limits, at a level equivalent to MATH 1023 or MATH 2033.

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Part 1

MATH 2043 - Honor Mathematical Analysis

Chapter 1

Real Number System

> "The integers, the rationals, and the irrationals, taken together, make up the continuum of real numbers. It's called a continuum because the numbers are packed together along the real number line with no empty spaces between them."
>
> Brian Hayes

1.1. Completeness Axiom

One distinctive feature of the real number system as compared to the rational number system is the **completeness axiom**, which guarantee the existence of the **supremum** for any subset of $\mathbb{R}$ which is bounded above. In this section, we first learn about the notion of supremum and infimum of sets.

1.1.1. Supremum and Infimum. To begin our discussion, let's first explain the difference between **maximum** (resp. minimum) and **supremum** (resp. infimum) of a set. Let S be a non-empty subset of $\mathbb{R}$. We say $x_0 \in \mathbb{R}$ is the **maximum** of S if

(1) $x \leq x_0$ for any $x \in S$; and
(2) $x_0 \in S$.

We denote such an element x_0 (if it exists) by $\max S$. Similarly, we say $y_0 \in \mathbb{R}$ is the **minimum** of S if

(1) $y_0 \leq y$ for any $y \in S$; and
(2) $y_0 \in S$.

We denote such an element y_0 (if it exists) by $\min S$.

Example 1.1. Let $S = (0, 1]$. Then 1 is the maximum of S. However, as $0 \notin S$, we do **not** call 0 to be the minimum of S. For such an S, the minimum of S does not exist.

Example 1.2. Let $S = (0, \infty)$. Then S has no minimum by the same reason as in the previous example. Also, there is no element $x_0 \in \mathbb{R}$ such that $x \leq x_0$ for any $x \in S$, so S has no maximum either.

In the example $S = (0, 1]$ which has no minimum, we could call 0 to be a **lower bound** instead. Precisely, we say $l \in \mathbb{R}$ is a **lower bound** of S if $l \leq x$ for any $x \in S$, and say $L \in \mathbb{R}$ is an **upper bound** of S if $L \geq x$ for any $x \in \mathbb{R}$. Both lower bound and upper bound of a set are not unique. While we can say 0 is a lower bound of $(0, 1]$, we can also call any of the -1, -2, $-\pi$, $-\frac{1}{\pi}$ as lower bounds of $(0, 1]$. Likewise, 1, 2, π, e, etc. are all upper bounds of $(0, 1]$.

Even though there are many upper bounds for the set $(0, 1]$, it should be agreeable that 1 is the *best* upper bound among all others, and 0 is the *best* lower bound among all others. Informally speaking, supremum and infimum are the *best* upper and lower bounds of a set respectively:

Definition 1.3 (Supremum and Infimum)**.** Let S be a subset of $\mathbb{R}$. We say $L \in \mathbb{R}$ is the **supremum** (a.k.a. **least upper bound**) of S if L is an upper bound of S, and any number β such that $\beta < L$ is not an upper bound of S. Likewise, we say $l \in \mathbb{R}$ is the **infimum** (a.k.a. **greatest lower bound**) of S if l is a lower bound of S, and any number α such that $l < \alpha$ is not a lower bound of S. The supremum and infimum of a set S are denoted by $\sup S$ and $\inf S$ respectively.

Example 1.4. Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\} = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then, clearly 0 is a lower bound of S. To argue that it is the infimum, we consider an arbitrary number $\alpha > 0$, one can find $n \in \mathbb{N}$ large enough such that $n > \frac{1}{\alpha}$, or equivalently $\frac{1}{n} < \alpha$. Here we show that any $\alpha > 0$ is not a lower bound, hence by definition 0 is the infimum of S. On the other hand, 1 is the supremum of S since it is obviously an upper bound of S, and any $\beta < 1$ is not an upper bound (it does not bound 1 which is in S).

Exercise 1.1. Explain why that if L is the maximum of a set $S \subset \mathbb{R}$, then $L = \sup S$.

One fundamental axiom of real numbers is the following:

> (Completeness Axiom) Any non-empty subset S of $\mathbb{R}$ which is bounded from above must have a supremum in $\mathbb{R}$.

This axiom is what makes the real number system special when compared to the rational number system. If one replace both $\mathbb{R}$'s in the completeness axiom by $\mathbb{Q}$'s, then the statement is no longer valid. We can find a non-empty subset S of $\mathbb{Q}$ which is bounded from above, but does not have a supremum in $\mathbb{Q}$. The following is such an example:

$$S := (-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}.$$

Clearly, it is bounded from above by, say, 2. Later we will prove a proposition for us to argue that $\sup S = \sqrt{2}$ which is irrational. This shows a subset of $\mathbb{Q}$ could be bounded above but may not have a supremum in $\mathbb{Q}$. As we shall see, it is this completeness axiom that leads to many distinctive properties of $\mathbb{R}$.

While it is fairly easy to find $\sup S$ and $\inf S$ in Example 1.4, it may not be so for more complicated sets. The following result is particularly useful for finding the supremum and infimum of a set.

Proposition 1.5. *Let S be a non-empty set in $\mathbb{R}$ which is bounded from above. Then, $L = \sup S$ if and only if*

(1) L is an upper bound of S; and

(2) there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in S such that $\lim\limits_{n \to \infty} x_n = L$.

A similar result holds for the infimum.

Proof. For the ($\Longrightarrow$)-part, we assume $L = \sup S$. Then, it is by definition of supremum L is an upper bound of S. To prove (2), we consider for each $n \in \mathbb{N}$ the number $\sup S - \frac{1}{n}$ which is strictly less than $\sup S$, so it cannot be an upper bound of S. That means one must be able to find an element $x_n \in S$ such that

$$\sup S - \frac{1}{n} < x_n \leq \sup S.$$

This gives a sequence $\{x_n\}_{n=1}^{\infty}$ in S which converges to L, proving (2).

For the ($\Longleftarrow$)-part, we assume (1) and (2) and want to prove $L = \sup S$. It suffices to argue that any $\beta < L$ is not an upper bound of S. Since it is given by (2) that $x_n \to L$, by $\beta < L$ we have $x_n > \beta$ for sufficiently large n. It implies that β cannot be an upper bound of S since $x_n \in S$. This concludes $L = \sup S$. □

Exercise 1.2. Prove that for a non-empty subset S of $\mathbb{R}$ which is bounded from above, and $\sup S \notin S$, then there exists a **strictly increasing** sequence $\{x_n\}_{n=1}^{\infty}$ in S such that $x_n \to \sup S$. [Hint: modify the proof of Proposition 1.5]

Proposition 1.5 provides a great tool of finding the supremum and infimum, and also proving facts about them.

Example 1.6. Let $S = \left\{\dfrac{1}{n} + \dfrac{1}{2^m} : m, n \in \mathbb{N}\right\}$. Then for any $m, n \in \mathbb{N}$, we have

$$\frac{1}{n} + \frac{1}{2^m} \leq 1 + \frac{1}{2} = \frac{3}{2},$$

so $\frac{3}{2}$ is an upper bound of S. It is in fact the supremum as $\frac{3}{2} \in S$.

For the infimum, we first observe that 0 is a lower bound of S. To prove that it is in fact the infimum, we could find a sequence in S converging to 0. Consider the sequence

$$x_n := \frac{1}{n} + \frac{1}{2^n} \in S,$$

which converges to 0 as $n \to \infty$. Hence $\inf S = 0$.

Example 1.7. In this example we prove that for any bounded subset $S \subset (0, \infty)$, we have

$$\sup S = \frac{1}{\inf S^{-1}}$$

where $S^{-1} := \{x^{-1} : x \in S\}$. To prove this, we first prove that $\frac{1}{\inf S^{-1}}$ is an upper bound of S: for any $x \in S$, we have $x^{-1} \in S^{-1}$, so $x^{-1} \geq \inf S^{-1}$. This proves $x \leq \frac{1}{\inf S^{-1}}$.

Next we argue that there exists a sequence $x_n \in S$ converging to $\frac{1}{\inf S^{-1}}$. For this, we take a sequence $y_n \in S^{-1}$ converging to $\inf S^{-1}$. Then $x_n := \frac{1}{y_n} \to \frac{1}{\inf S^{-1}}$, and more importantly $x_n \in S$ for any n. To conclude, we have $\frac{1}{\inf S^{-1}}$ being an upper bound of S, and there exists a sequence $x_n \in S$ converging to it. By Proposition 1.5, we conclude $\sup S = \frac{1}{\inf S^{-1}}$.

Exercise 1.3. For any non-empty set $S \subset \mathbb{R}$ which is bounded from above, we denote

$$-S := \{-x : x \in S\}.$$

Show that $\sup S = -\inf(-S)$.

Exercise 1.4. Show that any non-empty $S \subset \mathbb{R}$ which is bounded from below must have an infimum in $\mathbb{R}$.

Exercise 1.5. Let S, T be two non-empty set in $\mathbb{R}$ which are bounded from above. Let $S + T := \{s + t : s \in S \text{ and } t \in T\}$. Prove that for any $c > 0$, we have

$$\sup(S + cT) = \sup S + c \sup T.$$

One important consequence of the completeness axiom is the following:

Proposition 1.8 (Archimedean Principle). *For any $x > 0$ and $y \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $nx > y$.*

Proof. We prove by contradiction. Fix the element $x > 0$, and consider the set $S := \{nx : n \in \mathbb{N}\}$. Suppose the opposite that $nx \leq y$ for any $n \in \mathbb{N}$, then S is bounded above by y. The completeness axiom asserts that S has a supremum in $\mathbb{R}$. Denote $L = \sup S$, then $nx \leq L$ for any $n \in \mathbb{N}$.

However, then $L - x$ cannot be an upper bound of S, so there exists $m \in \mathbb{N}$ such that $L - x < mx \leq L$. This shows $L < (m + 1)x$. However, $(m + 1)x \in S$ so it contradicts to the fact that L is an upper bound of S.

To conclude, there must be an $n \in \mathbb{N}$ such that $nx > y$, as desired. $\square$

One immediate consequence of the Archimedean principle is that rational numbers are "densely" distributed over the real line:

Corollary 1.9 (Density of rationals). *For any $a, b \in \mathbb{R}$ such that $a < b$, the interval (a, b) must contain at least one rational number.*

Proof. If $b - a > 1$, then it is clear that (a, b) contains an integer (it can be proved by the well-ordering principle of natural numbers).

Generally, we use Archimedean principle to find an $n \in \mathbb{N}$ such that $n(b - a) > 1$. Then, the interval (na, nb) must contain an integer m. Now that $na < m < nb$, and so we have

$$a < \frac{m}{n} < b.$$

The rational number $\frac{m}{n}$ is then contained in the interval (a, b). $\square$

Exercise 1.6. Show that for any real number $x \in \mathbb{R}$, there exists a sequence $r_n \in \mathbb{Q}$ such that $r_n \to x$.

Exercise 1.7. Show that for any $a, b \in \mathbb{R}$ such that $a < b$, the interval (a, b) must contain at least one irrational number.

Example 1.10. Consider the set:

$$S := \big(-\sqrt{2}, \sqrt{2}\big) \cap \mathbb{Q}.$$

We argue that $\sup S = \sqrt{2}$. It is clear that $\sqrt{2}$ is an upper bound of S. It suffices to construct a sequence $x_n \in S$ converging to $\sqrt{2}$. For any $n \in \mathbb{N}$, there exists $x_n \in \mathbb{Q} \cap \big(\sqrt{2} - \frac{1}{n}, \sqrt{2}\big)$ by the density of $\mathbb{Q}$. Clearly, $x_n \in S$ for any n, and by squeeze principle, $x_n \to \sqrt{2}$. Hence we conclude that $\sup S = \sqrt{2}$.

It is an example of a bounded subset of $\mathbb{Q}$ which does not have a supremum in $\mathbb{Q}$.

Exercise 1.8. Find the supremum and infimum of each of the following sets:

(1) $\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$
(2) $\left\{\frac{(-1)^n n}{n+1} : n \in \mathbb{N}\right\}$
(3) $\{\sqrt{x} + y^2 : x, y \in (0,1] \cap \mathbb{Q}\}$
(4) $\left\{\frac{k}{n!} : k, n \in \mathbb{N} \text{ and } \frac{k}{n!} < \sqrt{2}\right\}$
(5) $\{x^2 + y^3 + z^4 : x \in (-1,0)\backslash\mathbb{Q},\ y \in (0,1) \cap \mathbb{Q},\ z \in (-1,1)\}$
(6) $\{\sqrt{n} - [\sqrt{n}\,] : n \in \mathbb{N}\}$

Remark 1.11. For a non-empty subset $S \subset \mathbb{R}$ which is not bounded from above, we could write $\sup S = \infty$, and similarly for a non-empty subset $S \subset \mathbb{R}$ which is not bounded from below, we could write $\inf S = -\infty$.

Remark 1.12. It is interesting to know that we often regard $\sup \emptyset = -\infty$ whereas $\inf \emptyset = +\infty$. It is because every real number is both an upper bound and a lower bound for $\emptyset$. Therefore, the **least** upper bound, in certain sense, is $-\infty$, whereas the **greatest** lower bound is $+\infty$.

Remark 1.13. When the set S that is defined in a simple way, such as $S = \{\sqrt{n} : n \in \mathbb{N}\}$ where the criterion "$n \in \mathbb{N}$" is very short, we could write its supremum or infimum as

$$\sup_{n\in\mathbb{N}} \sqrt{n} \qquad \inf_{n\in\mathbb{N}} \sqrt{n}.$$

That way we do not need to label the set by the letter S.

1.1.2. Definition of x^y. While it is clear that x^3 means $x \cdot x \cdot x$ and x^{-2} means $\frac{1}{x^2}$, it is unclear what x^y means when both x and y are positive real numbers. Its formal definition in fact requires the use of the completeness axiom. To start with, let's first discuss the definition of $x^{\frac{1}{n}}$ (or equivalently $\sqrt[n]{x}$) where $x > 0$ and $n \in \mathbb{N}$:

Proposition 1.14. *For any $x > 0$, and $n \in \mathbb{N}$, there exists a unique $y > 0$ such that $y^n = x$. We denote such a number y by $x^{\frac{1}{n}}$ or $\sqrt[n]{x}$.*

Proof. Consider the set:

$$S := \{t \in (0,\infty) : t^n < x\}.$$

Such set is non-empty since

$$\left(\frac{x}{1+x}\right)^n < \frac{x}{1+x} < x.$$

It is also bounded from above by $1 + x$. To see this, we suppose $\exists t > 1 + x$ such that $t \in S$. Then we have $1 + x < (1 + x)^n < t^n < x$, which is clearly impossible.

Hence S is a non-empty subset of $\mathbb{R}$ which is bounded from above. The completeness axiom asserts that $\sup S$ exists in $\mathbb{R}$. We claim $(\sup S)^n = x$.

To prove $(\sup S)^n \leq x$, we may take a sequence $t_k \in S \to \sup S$, then we have $t_k^n < x$ for any $k \in \mathbb{N}$. Letting $k \to \infty$, we conclude that $(\sup S)^n \leq x$.

Next we argue by contradiction that it is impossible to have $(\sup S)^n < x$. Suppose $(\sup S)^n < x$, then for any $h \in (0,1)$, we have

$$\begin{aligned}(\sup S + h)^n - (\sup S)^n &= h \sum_{j=0}^{n-1} (\sup S + h)^j (\sup S)^{n-1-j} \\ &\leq h \sum_{j=0}^{n-1} (\sup S + h)^{n-1} \\ &< nh(\sup S + 1)^{n-1}\end{aligned}$$

Picking a positive $h < \min\left\{1, \dfrac{x - (\sup S)^n}{n(\sup S + 1)^{n-1}}\right\}$, we then have

$$(\sup S + h)^n - (\sup S)^n < x - (\sup S)^n \implies (\sup S + h)^n < x.$$

This shows $\sup S + h \in S$ but it is impossible as $h > 0$.

It concludes that $(\sup S)^n = x$. The uniqueness part follows directly from the identity $y_1^n - y_2^n = (y_1 - y_2)(\text{positive terms})$ for any $y_1, y_2 > 0$. $\square$

After $x^{\frac{1}{n}}$ is defined for any $x > 0$ and $n \in \mathbb{N}$, one can then define

$$x^{\frac{m}{n}} := (x^{\frac{1}{n}})^m$$

for any $m, n \in \mathbb{N}$ and $x > 0$. This gives the definition of x^r for any positive rational number r. Yet we need to justify whether it is well-defined as each rational number can be represented in many different ways. We leave it as an exercise for readers:

Exercise 1.9. Suppose $p, q, m, n \in \mathbb{N}$ and $\frac{p}{q} = \frac{m}{n}$. Prove that for any $x > 0$, we have

$$(x^{\frac{1}{q}})^p = (x^{\frac{1}{n}})^m.$$

This justifies that x^r is well-defined for any $x > 0$ and rational number $r > 0$.

Exercise 1.10. Prove that if $x > 0$, and r, s are positive rational numbers, then

$$x^r \cdot x^s = x^{r+s}.$$

Hence, deduce that if $x > 1$ and $r > s$, we have $x^r > x^s$; whereas if $0 < x < 1$ and $r > s$, then $x^r < x^s$.

Next we move on to the definition of x^y for any $x, y > 0$.

Definition 1.15 (x to the power of y)**.** For any $x \geq 1$ and $y > 0$, we consider the set

$$S(x,y) := \{x^r : r \in \mathbb{Q} \text{ and } 0 < r < y\},$$

which is non-empty by the density of rationals and is bounded from above (left as an exercise). Then we define:

$$x^y := \sup S(x,y).$$

When $0 < x < 1$, we then define $x^y := \dfrac{1}{\left(\frac{1}{x}\right)^y} = \dfrac{1}{\sup S(1/x, y)}$.

Exercise 1.11. Justify that $S(x,y)$ in the above definition is bounded from above.

To justify the use of the same notation x^y (since it has already been used in case y is rational), we need to argue that $x^s = \sup S(x,s)$ when $s \in \mathbb{Q} \cap (0,\infty)$. WLOG we assume $x \geq 1$. For any $r \in (0,s) \cap \mathbb{Q}$, then by $x^r < x^s$ we have x^s is an upper bound of $S(x,s)$. To prove x^s is the supremum, we need to find a sequence $r_j \in \mathbb{Q} \cap (0,s)$ such that $x^{r_j} \to x^s$. By density of rationals, take $r_j \in \mathbb{Q}$ to be a strictly increasing sequence converging to s. Then, we have

$$|x^{r_j} - x^s| = |x^s| \left| \frac{1}{x^{s-r_j}} - 1 \right|.$$

It suffices to prove that $x^{s-r_j} \to 1$. For each $j \in \mathbb{N}$, we pick $n_j \in \mathbb{N}$ to be the unique integer such that

$$\frac{1}{n_j+1} < s - r_j < \frac{1}{n_j}.$$

As $s - r_j$ is decreasing, the sequence n_j is increasing. Then, by

$$x^{\frac{1}{n_j+1}} < x^{s-r_j} < x^{\frac{1}{n_j}},$$

and the fact that $\lim_{n\to\infty} x^{\frac{1}{n}} = 1$ (proved in MATH 1023), the subsequences $x^{1/(n_j+1)}$ and x^{1/n_j} both converge to 1, so by squeeze theorem we proved $x^{s-r_j} \to 1$ as well. The case $0 < x < 1$ follows easily from $\frac{1}{\sup S(1/x,r)} = \frac{1}{(1/x)^r} = x^r$.

Exercise 1.12. Prove that for any $x, y, z > 0$, we have $x^{y+z} = x^y \cdot x^z$.

Finally, for $x > 0$ and $y < 0$, then we define $x^y := \frac{1}{x^{-y}}$, and $x^0 := 1$. When $x < 0$, we need to be very careful when handling x^y. While $x^{\frac{1}{3}}$ is well-defined, $x^{\frac{1}{2}}$ is not. When $x < 0$ and y is any real number not of the form $\frac{1}{\text{odd integer}}$, we treat x^y to be undefined.

1.1.3. Consequences of Completeness Axiom. Many theorems about sequences that we have learned in Calculus 1 are in fact consequences of the completeness axiom. They include:

Theorem 1.16 (Monotone and bounded implies convergence)**.** *Any sequence $\{x_n\}$ of real numbers which is monotone increasing and bounded from above must converge.*

Theorem 1.17 (Bolzano-Weiestrass)**.** *Any bounded sequence $\{x_n\}$ of real numbers must have a convergent subsequence $\{x_{n_k}\}$.*

Theorem 1.18 (Cauchy criterion)**.** *Any Cauchy sequence $\{x_n\}$ in $\mathbb{R}$ must converge.*

We will prove the above theorems in this order:

Completeness axiom $\implies$ Theorem 1.16 $\implies$ Theorem 1.17 $\implies$ Theorem 1.18.

Proof of Theorem 1.16. Let $\{x_n\}$ be a monotone increasing sequence which is bounded from above. We argue that x_n converges to $L := \sup\{x_n\}_{n=1}^{\infty}$. To prove that, we consider any $\varepsilon > 0$, then $L - \varepsilon$ is not an upper bound of $\{x_n\}$ (as L is the least upper bound). Hence, one can find some $x_N \in (L - \varepsilon, L]$. By monotonicity, we then have $L - \varepsilon < x_N \leq x_n \leq L < L + \varepsilon$ for any $n \geq N$. In other words, we have

$$|x_n - L| < \varepsilon \ \ \forall n \geq N.$$

Hence, we conclude that $x_n \to L$. □

Proof of Theorem 1.17. Let $\{x_n\}$ be a bounded sequence. We argue that one can always find a monotone subsequence $\{x_{n_k}\}$. To prove this claim, we introduce the concept of *peaks*. We say x_k is a *peak* of the sequence $\{x_n\}$ if $x_k \geq x_{k+1}, x_{k+2}, \ldots$. In other words, a peak is a term that is greater than or equal to all its successive terms in the sequence. For the sequence $\{x_n\}$, two cases will happen:

Case (1): Suppose there are infinitely many peaks in $\{x_n\}$. Let's say the peaks are:

$$x_{n_1}, x_{n_2}, x_{n_3}, \cdots \quad \text{where } n_1 < n_2 < n_3 < \cdots$$

Since x_{n_1} is a peak, we must have $x_{n_1} \geq x_{n_2}$. Similarly, x_{n_2} itself is a peak, so $x_{n_2} \geq x_{n_3}$. Inductively, we have $x_{n_1} \geq x_{n_2} \geq x_{n_3} \geq \cdots$, and so $\{x_{n_k}\}$ is a decreasing subsequence of $\{x_n\}$.

Case (2): Suppose there are finitely many peaks (including the case of no peak). Then, there exists a big integer N such that there is no peak after x_N. Take an arbitrary $n_1 > N$. Since x_{n_1} is not a peak, at least one term after x_{n_1} must be greater than x_{n_1}. Hence, one can always find $n_2 > n_1$ such that $x_{n_1} < x_{n_2}$. Similarly, x_{n_2} is not a peak, one can always find $n_3 > n_2$ such that $x_{n_2} < x_{n_3}$. Inductively, one can find $n_1 < n_2 < n_3 < n_4 < \cdots$ such that

$$x_{n_1} < x_{n_2} < x_{n_3} < x_{n_4} < \cdots$$

so $\{x_{n_k}\}$ is an increasing subsequence of $\{x_n\}$.

It proves the claim that one can always find a monotone subsequence $\{x_{n_k}\}$ of any given sequence $\{x_n\}$. Given that $\{x_n\}$ is bounded, $\{x_{n_k}\}$ is also bounded. By Theorem 1.16, $\{x_{n_k}\}$ converges, so it is exactly the subsequence we look for. This completes the proof. □

Proof of Theorem 1.18. Given a Cauchy sequence $\{x_n\}$, we first show that it is bounded. Take $\varepsilon_0 = 1$, by the definition of Cauchy sequence, there exists $N > 0$ such that when $m, n \geq N$, we have $|x_m - x_n| < \varepsilon_0 = 1$. In particular:

$$|x_n - x_N| < 1 \quad \text{whenever } n \geq N.$$

Then, for any $n \geq N$, we have $-1 < x_n - x_N < 1$, and so $x_N - 1 < x_n < x_N + 1$. It shows $\{x_N, x_{N+1}, x_{N+2}, \cdots\}$ is bounded from above by $x_N + 1$, and bounded from below by $x_N - 1$. Furthermore, $\{x_1, \ldots, x_N\}$ must be bounded since there are only finitely many terms. Consequently, the whole sequence $\{x_1, \ldots, x_N, x_{N+1}, \ldots\}$ is bounded.

Using Bolzano-Weierstrass, one can find a convergent subsequence $\{x_{n_k}\}$. Suppose $\lim_{k\to\infty} x_{n_k} = L$, we next claim that the original sequence $\{x_n\}$ also converges to L:

Given any $\varepsilon > 0$, since x_{n_k} converges to L, one can find $K > 0$ such that when $k \geq K$, we have $|x_{n_k} - L| < \frac{\varepsilon}{2}$.

Moreover, the original sequence $\{x_n\}$ is Cauchy, there exists $N' > 0$ such that whenever $m, n \geq N'$, we have $|x_m - x_n| < \frac{\varepsilon}{2}$. Now we pick m to be a large n_k greater than both n_K and N', then we have:

$$\begin{aligned} |x_{n_k} - L| &< \frac{\varepsilon}{2} \\ |x_{n_k} - x_n| &< \frac{\varepsilon}{2} \qquad\qquad \text{whenever } n \geq N', \end{aligned}$$

which combines to show whenever $n \geq N'$:

$$|x_n - L| = |(x_n - x_{n_k}) + (x_{n_k} - L)| \leq |x_n - x_{n_k}| + |x_{n_k} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

It shows $\{x_n\}$ converges to L, completing the proof. □

Exercise 1.13. Let $\{x_n\}$ be the sequence

$$x_1 = 6 \text{ and } x_{n+1} = \frac{x_n^2 + 4}{2x_n + 3} \; \forall n \in \mathbb{N}.$$

Show that $\{x_n\}$ converges and find its limit.

Exercise 1.14. Let $\{r_n\}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} r_n$ converges. Given $\{x_n\}$ is a sequence of real numbers such that

$$|x_{n+1} - x_n| \leq r_n \ \forall n \in \mathbb{N}.$$

Show that $\{x_n\}$ converges.

Exercise 1.15. Suppose $\{x_n\}$ is a sequence of real numbers, and that there exists $\alpha \in (0, 1)$ such that

$$|x_{n+1} - x_n| \leq \alpha \, |x_n - x_{n-1}| \quad \forall n \geq 2.$$

Show that $\{x_n\}$ converges.

Exercise 1.16. Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences of real numbers. Prove that there exist n_k's such that both subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ converge.

1.2. Limit Superior and Limit Inferior

1.2.1. Limit set of a sequence. In this section we introduce two important concepts: $\limsup$ and $\liminf$. Consider a sequence $\{x_n\}_{n=1}^{\infty} = \{(-1)^n\}_{n=1}^{\infty}$. It clearly does not convergence as we can find two subsequences $\{x_{2n}\}$ and $\{x_{2n-1}\}$ converging to two different limits. While $\lim_{n\to\infty} x_n$ does not exist, one can still talk about the **limit set** of the sequence.

Definition 1.19 (Limit set)**.** Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say $L \in [-\infty, \infty]$ is a **limit point** of the sequence $\{x_n\}$ if there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ approaching L. Also, we define $\mathrm{LIM}\,\{x_n\}$, which is called the **limit set** of $\{x_n\}$, to be the set of all limit points of the sequence $\{x_n\}_{n=1}^{\infty}$.

Remark 1.20. Later we will also talk about limit points of a set, which is slightly different from limit points of a sequence. Some authors used the term *accumulation point* for sequences, and *limit point* for sets.

Example 1.21. Consider $\{x_n\} = \{(-1)^n\}$. It is clear that $-1, 1 \in \mathrm{LIM}\,\{x_n\}$. It is also easy to see that if $L \neq \pm 1$, then $L \notin \mathrm{LIM}\,\{x_n\}$, since one can find a small $\varepsilon > 0$ such that $(L-\varepsilon, L+\varepsilon)$ does not contain 1 or -1. It is impossible to have a subsequence of $(-1)^n$ converging to such an L. Therefore, we have

$$\mathrm{LIM}\,\{(-1)^n\} = \{-1, 1\}.$$

Example 1.22. Consider the sequence $x_n = n$. Then, any subsequence $\{x_{n_k}\}$ of $\{x_n\}$ must also diverge to $+\infty$ as $k \to \infty$. Hence, we have $\mathrm{LIM}\,\{n\} = \{+\infty\}$.

Remark 1.23. For a bounded sequence $\{x_n\}$, the Bolzano-Weiestrass Theorem asserts that it must have a converging subsequence. Therefore $\mathrm{LIM}\,\{x_n\}$ must contain at least one real number.

For more complicated sequences, it may be difficult find out all its limit points, but sometimes the following result could be helpful:

Proposition 1.24. *Let $\{x_n\}$ be a sequence of real numbers, and suppose $L \in \mathbb{R}$. Then the following are equivalent:*

(1) $L \in \mathrm{LIM}\,\{x_n\}$
(2) For any $\varepsilon > 0$, the interval $(L-\varepsilon, L+\varepsilon)$ contains infinitely many x_n's.

Proof. To prove (1) $\implies$ (2), we suppose $L \in \mathrm{LIM}\,\{x_n\}$, then there exist a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to L$. By the definition of limits, there exists $K > 0$ such that $x_{n_k} \in (L-\varepsilon, L+\varepsilon)$ for any $k \geq K$. Therefore, the interval $(L-\varepsilon, L+\varepsilon)$ contains $x_{n_K}, x_{n_{K+1}}, x_{n_{K+2}}, \cdots$. This proves (2).

Now we prove (2) $\implies$ (1). Assume that (2) holds, then we construct a subsequence x_{n_k} in the following way: first pick any x_n in $(L-1, L+1)$, and call it x_{n_1}. It is possible by the assumption (2). Then consider the interval $(L-\frac{1}{2}, L+\frac{1}{2})$. It contains infinitely many x_n's, so we pick one with $n > n_1$, and call it x_{n_2}. We keep going, then one can find a strictly increasing sequence $n_1 < n_2 < n_3 < \cdots$ such that

$$x_{n_k} \in \left(L - \frac{1}{k}, L + \frac{1}{k}\right) \quad \forall k \in \mathbb{N}.$$

Then $\{x_{n_k}\}$ forms a subsequence of $\{x_n\}$, and by squeeze theorem, we have $x_{n_k} \to L$ as $k \to \infty$. □

Exercise 1.17. Let $\{x_n\}$ be a sequence of real numbers. Prove that $+\infty \in \operatorname{LIM}\{x_n\}$ if and only if $\{x_n\}$ is not bounded from above.

Using the proposition, one can easily conclude that $\operatorname{LIM}\{(-1)^n\} = \{-1, 1\}$. For instance, any open interval $(1-\varepsilon, 1+\varepsilon)$ contains infinitely many x_n's, namely $x_2, x_4, x_6, \cdots$ which are all equal to 1.

Exercise 1.18. Suppose $\{x_n\}$ and $\{y_n\}$ are two sequences of real numbers such that $x_n \to L$ and $y_n \to M$. Now, construct a new sequence

$$\{z_n\}_{n=1}^{\infty} = \{x_1, y_1, x_2, y_2, x_3, y_3, \cdots\}$$

Show that $\operatorname{LIM}\{z_n\} = \{L, M\}$.

Exercise 1.19. Find the limit set of the sequence $x_n = \big(1 + (-1)^n\big)\dfrac{n+1}{n} + (-1)^n$.

Exercise 1.20. Let $\mathbb{Q} = \{r_1, r_2, r_3, \cdots\}$ where $n \mapsto r_n$ is injective. Show that $\operatorname{LIM}\{r_n\} = \mathbb{R}$.

1.2.2. Limit set of $\{\sin n\}$. In this short subsection, we will prove that the limit set of $\operatorname{LIM}\{\sin(n)\}_{n=1}^{\infty}$ is the interval $[-1, 1]$. In other words, for any $l \in [-1, 1]$, one can always find a subsequence $\{\sin(n_k)\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} \sin(n_k) = l.$$

By Proposition 1.24, it suffices to show for any $l \in [-1, 1]$ and any $\varepsilon > 0$, there are infinitely many $\sin(n)'s$ in the interval $(l - \varepsilon, l + \varepsilon)$.

We will *complexify* the problem by considering *complex numbers* – note that no knowledge of complex analysis is needed here. Denote the unit circle in the complex plane $\mathbb{C}$ by:

$$\mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} : \theta \in \mathbb{R}\}.$$

If you are not familiar with the notation $e^{i\theta}$, you may simply treat it as $\cos\theta + i\sin\theta$. We will show that for any $\theta_0 \in \mathbb{R}$ and any $\varepsilon > 0$, there exist infinitely many e^{in}'s such that $\left|e^{in} - e^{i\theta_0}\right| < \varepsilon$. Then it would imply there exists infinitely many n's such that

$$\sqrt{(\cos n - \cos\theta_0)^2 + (\sin n - \sin\theta_0)^2} < \varepsilon,$$

and the fact that $|y| \leq \sqrt{x^2 + y^2}$ for any $x, y \in \mathbb{R}$, it will show our desired result that there exist infinitely many n's such that

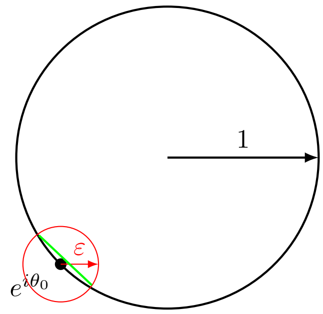
$$|\sin n - \sin\theta_0| < \varepsilon.$$

Since any $l \in [-1, 1]$ can be expressed as $l = \sin\theta_0$ for some $\theta_0 \in \mathbb{R}$, it would show for any $l \in [-1, 1]$ and $\varepsilon > 0$, there are infinitely many $\sin(n)$'s in $(l - \varepsilon, l + \varepsilon)$.

Claim: For any $\theta_0 \in \mathbb{R}$ and $\varepsilon > 0$, there are infinitely many n's such that $\left|e^{in} - e^{i\theta_0}\right| < \varepsilon$.

Proof. Given $\theta_0 \in \mathbb{R}$ and $\varepsilon > 0$, we consider the red ball centered at $e^{i\theta_0}$ with radius $\varepsilon > 0$ (see figure below). Let δ be the length of the green line joining the two intersection points between the two circles (the exact value of δ can be found by elementary geometry – it should be a certain constant multiple of ε).

We cover the unit circle $\mathbb{S}^1$ by **finitely** many open balls $B_k = \{z \in \mathbb{C} : |z - w_k| < \delta/2\}$ with center $w_k \in \mathbb{S}^1$ and radius $\delta/2$ (not shown in the diagram). Since $n \mapsto e^{in}$ is injective, the set $\{e^{in} : n \in \mathbb{N}\}$ is an infinite subset of $\mathbb{S}^1$. As $\{B_k\}$ is a finite cover of $\mathbb{S}^1$, there



must be at least one B_k containing infinitely many e^{in}'s. Pick any pair e^{ip} and e^{iq} (where $q > p$) in this ball B_k, then one must have

$$\left|e^{ip} - e^{iq}\right| < \delta = \text{ diameter of the ball } B_k.$$

Next we consider the sequence

$$z_n := e^{in(q-p)} = \left(e^{i(q-p)}\right)^n.$$

We want to show there are infinitely many n's such that $\left|z_n - e^{i\theta_0}\right| < \varepsilon$.

One important observation is that for any n, we have

$$|z_{n+1} - z_n| = \left|e^{i(q-p)n}\left(e^{i(q-p)} - 1\right)\right| = \left|e^{i(q-p)} - 1\right| = \left|e^{-ip}(e^{iq} - e^{ip})\right| = \left|e^{iq} - e^{ip}\right| < \delta.$$

Note also that z_{n+1} is obtained by rotating z_n counter-clockwisely by a fixed angle. Consider the finite sequence $\{z_1, z_2, \cdots, z_N\}$, all outside the red ball, and that z_N is the closest from the red ball, then z_{N+1} must be inside the red ball otherwise $|z_N - z_{N+1}|$ would be longer than the green line in the figure, i.e. $|z_N - z_{N+1}| > \delta$. Hence we get $\left|z_{N+1} - e^{i\theta_0}\right| < \varepsilon$.

Continue on $\{z_{N+2}, z_{N+3}, \cdots, z_{N'}\}$ outside the red ball and $z_{N'}$ being closest from the red ball, then by the same argument we have $z_{N'+1}$ inside the red ball. Keep going indefinitely, one can produce an infinite set $\{z_{N+1}, z_{N'+1}, z_{N''+1}, \cdots\}$ in the red ball. As all elements in this infinite set is of the form $e^{i(\text{integer})}$, it proves our claim. □

Finally, by the discussion before the claim, we can conclude that there are infinitely many $\{\sin(n)\}'s$ in the interval $(l - \varepsilon, l + \varepsilon)$ for any given $l \in [-1, 1]$ and $\varepsilon > 0$. This shows the limit set of $\sin(n)$ contains $[-1, 1]$. Also, by $|\sin(n)| \leq 1$, all subsequence limits of $\{\sin(n)\}$ must be in $[-1, 1]$. This completes the proof that $\text{LIM}\{\sin(n)\} = [-1, 1]$. Note also that our claim implies $\text{LIM}\{\cos(n)\} = [-1, 1]$ too.

Readers who are not familiar with the use of complex numbers may regard z_n to be a point in $\mathbb{R}^2$, i.e.

$$z_n = \big(\cos(n(q-p)), \sin(n(q-p))\big),$$

and try to use Cauchy-Schwarz's inequality to prove the result $|z_{n+1} - z_n| < \delta$.

1.2.3. Limit superior and inferior of a sequence. Now we introduce two important concepts in real analysis – $\limsup$ and $\liminf$. They are respectively the maximum and minimum limit points of a sequence $\{x_n\}$.

Definition 1.25 (Limit superior and inferior)**.** Let $\{x_n\}$ be a sequence of real numbers. We define the **limit superior** and **limit inferior** of a sequence as follows:

$$\limsup_{n\to\infty} x_n := \sup \text{LIM}\,\{x_n\}$$
$$\liminf_{n\to\infty} x_n := \inf \text{LIM}\,\{x_n\}$$

Example 1.26. According to previous examples, we can easily conclude that

$$\begin{aligned}
\text{LIM}\,\{(-1)^n\} &= \{-1,1\} &&\Longrightarrow \limsup_{n\to\infty}(-1)^n = 1 && \liminf_{n\to\infty}(-1)^n = -1\\
\text{LIM}\,\{n\} &= \{+\infty\} &&\Longrightarrow \limsup_{n\to\infty} n = +\infty && \liminf_{n\to\infty} n = +\infty\\
\text{LIM}\,\{\sin(n)\} &= [-1,1] &&\Longrightarrow \limsup_{n\to\infty}\sin(n) = 1 && \liminf_{n\to\infty}\sin(n) = -1.
\end{aligned}$$

Example 1.27. Define the sequence x_n as follows:

$$x_n = \begin{cases} 1/n & \text{if } n \text{ is a prime} \\ \dfrac{\text{the smallest prime dividing } n}{\text{the largest prime dividing } n} & \text{if } n \text{ is not a prime} \end{cases}.$$

Clearly, $0 < x_n \leq 1$ for any $n \in \mathbb{N}$, so we have

$$0 \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq 1.$$

Let p_n be the n-th prime, i.e. $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, etc. By Prime Number Theorem we know that

$$\lim_{n\to\infty} \frac{p_n}{n\log(n)} = 1.$$

To find its $\liminf$, we observe that $x_{p_n} = \frac{1}{p_n} \to 0$ as $n \to \infty$. Therefore, $0 \in \text{LIM}\,\{x_n\}$, which shows $\liminf_{n\to\infty} x_n = \inf \text{LIM}\,\{x_n\} \leq 0$. This concludes that

$$\liminf_{n\to\infty} x_n = 0.$$

To find its $\limsup$, we consider the subsequence $\{x_{p_n p_{n+1}}\}$, which is equal to

$$x_{p_n p_{n+1}} = \frac{p_n}{p_{n+1}}.$$

Since

$$\lim_{n\to\infty} \frac{p_n}{p_{n+1}} = \lim_{n\to\infty} \frac{n\log(n)}{(n+1)\log(n+1)} = 1.$$

This conclude that $\limsup_{n\to\infty} x_n = 1$.

Exercise 1.21. Find the limit superior and limit inferior of the following sequences:

(a) $x_n = e^{n(-1)^n - 1}$

(b) $x_n = e^{n(-1)^n - 1}\big(n(-1)^n - 1\big)$

(c) $x_n = (\cos n\pi)(\cos \frac{2n\pi}{5})$

(d) $x_n = \dfrac{[n/2]\cos(n\pi)}{n}$

Next we argue that $\limsup$ and $\liminf$ of a sequence are themselves inside the limit set $\text{LIM}\,\{x_n\}$. To establish this, we need to learn about the "demographics" of x_n's in relation to the $\limsup$ and $\liminf$ of the sequence.

Proposition 1.28. *Let $\{x_n\}$ be a sequence of real numbers. Suppose $L := \limsup_{n\to\infty} x_n$ is finite, then we have the following:*

(1) For any $\varepsilon > 0$, the interval $(L+\varepsilon, +\infty)$ contains at most finitely many x_n's.

(2) For any $\varepsilon > 0$, the interval $(L-\varepsilon, +\infty)$ contains infinitely many x_n's.

Similar results hold for $\liminf$ *(readers should try to write down the exact statement).*

Proof. To prove (1), we suppose otherwise that $(L+\varepsilon, +\infty)$ contains infinitely many x_n's, meaning that it has a subsequence $\{x_{n_k}\}$ contained in $(L+\varepsilon, +\infty)$. If this subsequence is not bounded from above, then it has a sub-subsequence $\{x_{n_{k_j}}\}$ diverging to $+\infty$. Since it is also a subsequence of $\{x_n\}$, it implies $+\infty \in \mathrm{LIM}\,\{x_n\}$, but it contradicts to the given fact that $L := \sup \mathrm{LIM}\,\{x_n\} < +\infty$. Hence this subsequence $\{x_{n_k}\}$ must be bounded from above, and hence is contained in $(L+\varepsilon, M]$ for some $M < \infty$. Bolzano-Weiestrass's Theorem then shows it has a sub-subsequence $\{x_{n_{k_i}}\}$ converging to some limit $l \in [L+\varepsilon, M]$. However, it would implies

$$L = \limsup_{n\to\infty} x_n = \sup \mathrm{LIM}\,\{x_n\} \geq l \geq L+\varepsilon,$$

which is clearly impossible. This proves (1).

To prove (2), we suppose there exists $\varepsilon > 0$ such that the interval $(L-\varepsilon, +\infty)$ contains at most finitely many x_n's. Then there exists $N > 0$ such that $x_n \leq L-\varepsilon$ for any $n \geq N$. Hence any converging subsequence x_{n_j} has limit at most $L-\varepsilon$. In other words, $L = \sup \mathrm{LIM}\,\{x_n\} \leq L-\varepsilon$ which is clearly absurd. This proves (2). □

Exercise 1.22. Prove a similar result for $\liminf$.

Corollary 1.29. *Suppose $L := \limsup_{n\to\infty} x_n$ is finite. Then, for any $\varepsilon > 0$, the interval $(L-\varepsilon, L+\varepsilon)$ contains infinitely many x_n's, i.e. $L \in \mathrm{LIM}\,\{x_n\}$. Therefore, we have in fact $\sup \mathrm{LIM}\,\{x_n\} = \max \mathrm{LIM}\,\{x_n\}$. Similar results hold for $\liminf$.*

Proof. From Proposition 1.28, we have for any $\varepsilon > 0$, the interval $(L-\varepsilon, +\infty)$ contains infinitely many x_n's whereas $(L+\frac{\varepsilon}{2}, +\infty)$ contains finitely many. Therefore, the interval $(L-\varepsilon, L+\varepsilon/2]$ must contain infinitely many x_n's, and so does $(L-\varepsilon, L+\varepsilon)$ which is even bigger. □

Exercise 1.23. Prove a similar corollary for $\liminf$.

Exercise 1.24. Prove that if $x_n \leq y_n$ for any $n \in \mathbb{N}$, then

$$\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n \text{ and } \liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} y_n$$

Exercise 1.25. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers. Prove the following properties.

(1) $\limsup_{n\to\infty}(-x_n) = -\liminf_{n\to\infty} x_n$

(2) $\liminf_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \leq \limsup_{n\to\infty}(x_n+y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$

One good use of Proposition 1.28 is to derive the following result:

Example 1.30. Let $\{x_n\}$ be a sequence of real numbers. We are going to prove that

$$\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} \frac{x_1+\cdots+x_n}{n} \leq \limsup_{n\to\infty} \frac{x_1+\cdots+x_n}{n} \leq \limsup_{n\to\infty} x_n.$$

The middle inequality

$$\liminf_{n\to\infty} \frac{x_1+\cdots+x_n}{n} \leq \limsup_{n\to\infty} \frac{x_1+\cdots+x_n}{n}$$

follows directly from the definition of $\limsup$ and $\liminf$. We will only prove

$$\limsup_{n\to\infty} \frac{x_1 + \cdots + x_n}{n} \leq \limsup_{n\to\infty} x_n \tag{1.1}$$

as the left-most inequality is similar.

To show this, we will prove that for any $\varepsilon > 0$, we have

$$\limsup_{n\to\infty} \frac{x_1 + \cdots + x_n}{n} < \limsup_{n\to\infty} x_n + \varepsilon,$$

then letting $\varepsilon \to 0^+$ will yield the desired result (1.1).

Let $L := \limsup_{n\to\infty} x_n$. If $L = +\infty$, then (1.1) trivial holds. Assume $L < +\infty$, then for any $\varepsilon > 0$, the interval $(L+\varepsilon, +\infty)$ contains finitely many x_n's, or in other words, there exists $N = N(\varepsilon) > 0$ such that $x_n \leq L + \varepsilon$ for any $n \geq N$. Then, for any $n > N$, we have

$$\begin{aligned}\frac{x_1 + \cdots + x_n}{n} &= \frac{x_1 + \cdots + x_N}{n} + \frac{x_{N+1} + \cdots + x_n}{n} \\ &\leq \frac{x_1 + \cdots + x_N}{n} + \frac{n-N}{n}(L+\varepsilon).\end{aligned}$$

Note that it is not clear whether $\lim_{n\to\infty} \frac{x_1 + \cdots + x_n}{n}$ exists or not, but its $\limsup$ always exist. Taking $\limsup$ on both sides, we have:

$$\limsup_{n\to\infty} \frac{x_1 + \cdots + x_n}{n} \leq \limsup_{n\to\infty} \left(\frac{x_1 + \cdots + x_N}{n} + \frac{n-N}{n}(L+\varepsilon) \right).$$

As the RHS indeed has a limit as $n \to \infty$, so its $\limsup$ is equal to its limit. We get:

$$\limsup_{n\to\infty} \frac{x_1 + \cdots + x_n}{n} \leq \lim_{n\to\infty} \left(\frac{x_1 + \cdots + x_N}{n} + \frac{n-N}{n}(L+\varepsilon) \right) = L + \varepsilon.$$

Letting $\varepsilon \to 0^+$ we get (1.1).

Exercise 1.26. Let $\{x_n\}$ be a sequence of **positive** numbers. Show that

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} \leq \liminf_{n\to\infty} \sqrt[n]{x_n} \leq \limsup_{n\to\infty} \sqrt[n]{x_n} \leq \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

Find the above $\limsup$'s or $\liminf$'s for the following sequence:

$$x_n = \begin{cases} n/2^n & \text{if } n \text{ is odd} \\ n^2/2^n & \text{if } n \text{ is even} \end{cases}.$$

Exercise 1.27. Let $\{a_n\}$ be a sequence of real numbers, and $\{b_n\}$ be a sequence of positive numbers such that $B_n := \sum_{k=1}^n b_k$ diverges to $+\infty$ as $n \to \infty$. Prove that

$$\liminf_{n\to\infty} \frac{a_n}{b_n} \leq \liminf_{n\to\infty} \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \leq \limsup_{n\to\infty} \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \leq \limsup_{n\to\infty} \frac{a_n}{b_n}.$$

Find an example of a pair of sequences $\{a_n\}$ and $\{b_n\}$ satisfying the given conditions in this exercise while some of the above inequalities is not strict.

1.2.4. Another equivalent form of limit superior and limit inferior. Some textbook defined $\limsup$ and $\liminf$ in a way different from ours, yet we can prove that they are equivalent to our definition.

Proposition 1.31. *Let $\{x_n\}$ be a sequence of real numbers. Then, we have:*

$$\limsup_{n\to\infty} x_n = \inf_{k\in\mathbb{N}} \sup\{x_k, x_{k+1}, x_{k+2}, \cdots\}$$
$$\liminf_{n\to\infty} x_n = \sup_{k\in\mathbb{N}} \inf\{x_k, x_{k+1}, x_{k+2}, \cdots\}$$

Remark: some textbooks use the RHS as the definitions of $\limsup$ *and* $\liminf$.

Proof. We just prove the first result as the second one is similar.

We first prove the case that $\limsup\limits_{n\to\infty} x_n$ is finite. For simplicity we denote $M_k := \sup\{x_k, x_{k+1}, \cdots\}$. The sequence $\{M_k\}$ is decreasing so

$$\inf_{k\in\mathbb{N}} M_k = \lim_{k\to\infty} M_k.$$

From Corollary 1.29, there exists a subsequence of x_{n_k} such that $x_{n_k} \to \limsup\limits_{n\to\infty} x_n$ as $k \to \infty$. Hence, by $M_{n_k} \geq x_{n_k}$ and that M_{n_k} converges (it is a subsequence of M_n which converges), we have

$$\inf_{k\in\mathbb{N}} M_k = \lim_{k\to\infty} M_{n_k} \geq \lim_{k\to\infty} x_{n_k} = \limsup_{n\to\infty} x_n.$$

Now it just suffices to show it is not possible to have $\inf\limits_{k\in\mathbb{N}} M_k > \limsup\limits_{n\to\infty} x_n$. Suppose this holds, then take a small $\varepsilon > 0$ such that

$$\limsup_{n\to\infty} x_n < \inf_{k\in\mathbb{N}} M_k - \varepsilon.$$

Then, Proposition 1.28 shows there exists only finitely many x_n's such that $x_n > \inf\limits_{k\in\mathbb{N}} M_k - \varepsilon$. Then there exists $N > 0$ such that whenever $n \geq N$, we have $x_n \leq \inf\limits_{k\in\mathbb{N}} M_k - \varepsilon$. It implies

$$M_n = \sup\{x_n, x_{n+1}, \cdots\} \leq \inf_{k\in\mathbb{N}} M_k - \varepsilon \text{ for any } n \geq N.$$

It is a contradiction to the definition of $\inf_{k\in\mathbb{N}} M_k$. This shows we must have $\inf\limits_{k\in\mathbb{N}} M_k = \limsup\limits_{n\to\infty} x_n$.

Now consider the case $\limsup\limits_{n\to\infty} x_n = +\infty$. In this case $\{x_n\}$ is not bounded from above, and hence $\sup\{x_k, x_{k+1}, x_{k+2}, \cdots\} \geq x_k$ is not bound from above either. By monotonicity, we have

$$\inf_{k\in\mathbb{N}} \sup\{x_k, x_{k+1}, x_{k+2}, \cdots\} = \lim_{k\to\infty} \sup\{x_k, x_{k+1}, x_{k+2}, \cdots\} = +\infty.$$

Hence $\limsup\limits_{n\to\infty} x_n = \inf\limits_{k\in\mathbb{N}} \sup\{x_k, x_{k+1}, x_{k+2}, \cdots\} = +\infty$.

We leave the case $\limsup\limits_{n\to\infty} x_n = -\infty$ as an exercise for readers. □

Exercise 1.28. Complete the proof of the above proposition, i.e. filling in the missing case for $\limsup$, and write up the proof for $\liminf$.

While the $\limsup$ and $\liminf$ of many sequences can be found by picking two suitably subsequences, there are some examples that may be easier to use the form of $\limsup$ and $\liminf$ in Proposition 1.31.

Example 1.32. Consider binary representations of natural numbers, e.g.

$$3_{(10)} = 11_{(2)} \qquad 5_{(10)} = 100_{(2)} \qquad 30_{(10)} = 11110_{(2)} \qquad 45_{(10)} = 101101_{(2)}.$$

Define the sequences:

$$a_n := \text{number of digits of } n \text{ in binary representation}$$
$$A_n := \frac{1}{n}\sum_{k=1}^{n}(-1)^{a_k}.$$

For example, $a_3 = 2$, $a_5 = 3$, $a_{30} = 5$, and $a_{45} = 6$. The first few terms of $\{a_n\}$ and $\{A_n\}$ are:

$$\{a_n\} = \{1, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4, \cdots\}$$
$$\{A_n\} = \{-1, 0, 1/3, 0, -1/5, -2/6, -3/7, -2/8, -1/9, 0, \cdots\}$$

We want to find $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$.

According to the pattern of $\{A_n\}$ illustrated above. We first claim that $A_n \leq A_{n+1}$ if and only if:

$$n \in \bigcup_{k\in\mathbb{N}}[2^{2k-1}-1, 2^{2k}-2] = [1,2]\cup[7,14]\cup[2^5-1, 2^6-2]\cup\cdots.$$

This can be proved by observing that

$$\begin{aligned}
&A_n \leq A_{n+1}\\
\iff &\frac{1}{n}\sum_{k=1}^{n}(-1)^{a_k} \leq \frac{1}{n+1}\sum_{k=1}^{n+1}(-1)^{a_k} = \frac{1}{n+1}\sum_{k=1}^{n}(-1)^{a_k} + \frac{(-1)^{a_{n+1}}}{n+1}\\
\iff &\frac{1}{n(n+1)}\sum_{k=1}^{n}(-1)^{a_k} \leq \frac{1}{n+1}(-1)^{a_{n+1}}\\
\iff &A_n \leq (-1)^{a_{n+1}}.
\end{aligned}$$

Since $|A_n| \leq 1$ (from the definition of A_n) while $(-1)^{a_{n+1}} = 1$ or -1, we have

$$A_n \leq (-1)^{a_{n+1}} \iff a_{n+1} \text{ is even} \iff n+1 \in \bigcup_{k\in\mathbb{N}}[2^{2k-1}, 2^{2k}-1],$$

as integers in $[2^{2k-1}, 2^{2k}-1]$ have $2k$ digits in binary representations. This proves our claim. Likewise, we have $A_n \geq A_{n+1}$ if and only if:

$$n \in \bigcup_{k\in\mathbb{N}}[2^{2k}-1, 2^{2k+1}-2] = [3,6]\cup[15, 2^5-2]\cup\cdots.$$

Now we consider the $\limsup$ and $\liminf$ of A_n using their equivalent forms in Proposition 1.31. From our claim, we know A_n is a "local maximum" when $n = 2^{2k}-1$ when $k \in \mathbb{N}$. Here "A_n is a local maximum" means $A_{n-1} \leq A_n$ and $A_n \geq A_{n+1}$.

Now consider

$$M_n := \sup\{A_n, A_{n+1}, A_{n+2}, \cdots\}.$$

The supremum of a set S is the same as the supremum of the set S' of local maximums of S. Hence, we have

$$M_{2^{2k}-1} = \sup\{A_{2^{2k}-1}, A_{2^{2k}}, A_{2^{2k}+1}, \cdots\} = \sup\{A_{2^{2k}-1}, A_{2^{2k+2}-1}, A_{2^{2k+4}-1}, \cdots\}.$$

The explicit formula for $A_{2^{2k}-1}$ can be computed easily (by counting how many even and odd a_n's from $n = 1$ to $2^{2k} - 1$

$$\begin{aligned} A_{2^{2k}-1} &= \frac{1}{2^{2k}-1}(1(-1) + 2(1) + 4(-1) + 8(1) + \cdots + 2^{2k-1}(1)) \\ &= \frac{1}{2^{2k}-1} \cdot \frac{1-(-2)^{2k}}{1-(-2)} \\ &= \frac{1}{3}. \end{aligned}$$

Therefore, $M_{2^{2k}-1} = \frac{1}{3}$ for any k. This shows

$$\lim_{k\to\infty} M_{2^{2k}-1} = \frac{1}{3}.$$

As $\{M_n\}$ is a decreasing bounded sequence, it must converge and so all its subsequence converges to the same limit. Hence,

$$\lim_{n\to\infty} M_n = \frac{1}{3}.$$

By Proposition 1.31, we have

$$\limsup_{n\to\infty} A_n = \lim_{n\to\infty} M_n = \frac{1}{3}.$$

The $\liminf$ is similar (consider local minimum instead). We leave it for readers to prove that

$$\liminf_{n\to\infty} A_n = -\frac{1}{3}.$$

Exercise 1.29. Prove that

$$\liminf_{n\to\infty} A_n = -\frac{1}{3}$$

in the above example.

Exercise 1.30. Given a sequence $\{x_n\}$ of real numbers such that $x_n \neq 0$ for any n. Prove that if

$$\limsup_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| < 1,$$

then $x_n \to 0$ as $n \to \infty$.

Prove also that if

$$\liminf_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| > 1,$$

then $|x_n| \to \infty$ as $n \to \infty$.

1.3. Point-Set Topology on $\mathbb{R}$

"Topology" is an important field of mathematics. Roughly speaking, it studies the continuous deformation of geometric objects, and the properties invariant under continuous deformations. It has many branches including algebraic topology, differential topology, etc. The branch *point-set topology* is the most basic and fundamental among other branches. It studies concepts such as open sets, closed sets, compact sets, connected sets, etc. These concepts are fundamental in real analysis because many theorems in this course would require some of these topological conditions.

1.3.1. Countable sets. There are two types of infinities: *countably infinite* and *uncountably infinite*. Roughly speaking, the former means we can assign a unique natural number to each element. Such a distinction is important in real analysis as these two types of infinities may carry drastically different properties.

Definition 1.33 (Countable sets)**.** A non-empty set S is said to be **countable** if there exists an injective function $f : S \to \mathbb{N}$. If S is both countable and infinite, we can say S is **countably infinite**. A non-empty set T is said to be **uncountable** if it is not countable.

Remark 1.34. The empty set $\emptyset$ is considered to be countable.

Example 1.35. $\mathbb{N}$ is clearly countable as the map $\mathrm{id} : \mathbb{N} \to \mathbb{N}$ is injective. $\mathbb{Z}$ is also countable, one can construct an injective map $f : \mathbb{Z} \to \mathbb{N}$ by:

$$f(0) = 1 \qquad f(1) = 2 \qquad f(-1) = 3 \qquad f(2) = 4 \qquad f(-2) = 5 \qquad \cdots$$

and more generally:

$$f(m) = \begin{cases} -2m + 1 & \text{if } m \leq 0 \\ 2m & \text{if } m > 0 \end{cases}$$

To check that f is injective, we suppose $f(m) = f(n)$ for some $m, n \in \mathbb{Z}$. As $f(m)$ is odd when $m \leq 0$ while $f(m)$ is even when $m > 0$, one must have either both $m, n > 0$, or both $m, n \leq 0$. If both $m, n > 0$, then $f(m) = f(n) \implies 2m = 2n \implies m = n$. If both $m, n \leq 0$, then $f(m) = f(n) \implies -2m + 1 = -2n + 1 \implies m = n$. Therefore, f is an injective map, and so $\mathbb{Z}$ is countable.

Any finite set S is countable: write S as $\{a_1, \cdots, a_n\}$ where a_j's are all distinct. Then the map $g : S \to \mathbb{N}$ defined by $g(a_j) = j$ is injective.

Example 1.36. If $A \subset B$ and B is countable, then A is also countable. To prove this, we suppose there exists an injective map $f : B \to \mathbb{N}$ (since B is countable). Then the restricted map $f\big|_A : A \to \mathbb{N}$, defined by $f\big|_A(x) = f(x)$ for any $x \in A$, is also an injective map. Therefore A is also countable.

Example 1.37. $\mathbb{N} \times \mathbb{N}$ is countable, as one can assign a unique natural number in a diagonal way:

$$\begin{array}{llll} (1,1) \mapsto 1 & (1,2) \mapsto 3 & (1,3) \mapsto 6 & (1,4) \mapsto 10 \\ (2,1) \mapsto 2 & (2,2) \mapsto 5 & (2,3) \mapsto 9 & \cdots \\ (3,1) \mapsto 4 & (3,2) \mapsto 8 & \cdots & \\ (4,1) \mapsto 7 & \cdots & & \end{array}$$

Exercise 1.31. Try to write down the precise map in the above example, and prove that it is injective.

Exercise 1.32. Let S be a non-empty finite set. Show that $S \times \mathbb{N}$ is countable.

This diagonal argument can also be applied to prove countable union of countably many sets is countable. Precisely, it means that if $S_1, S_2, S_3, \cdots$ are all countable sets, then $\bigcup_{j=1}^{\infty} S_j$ is also countable. The proof follows by listing the elements of S_1, S_2, S_3, etc. in an array, then count the elements diagonally.

Exercise 1.33. Show that $\mathbb{Q}$ is countable.

Exercise 1.34. Show that the set S of all polynomials with integer coefficients is countable.

For a countably infinite set S, the injective function $f : S \to \mathbb{N}$ can be modified to become a bijective function using the well-ordering axiom. To do so, we will composite f by another map $\varphi : f(S) \to \mathbb{N}$. We first consider the range of f, i.e. $f(S) = \{f(x) : x \in S\}$ which is a non-empty subset of $\mathbb{N}$. The well-ordering axiom shows there exists a minimum (say n_1) in the range of f. We then define $\varphi(n_1) = 1$. Next we take the minimum of $f(S)\backslash\{n_1\}$ which is also a non-empty subset of $\mathbb{N}$. Call this minimum n_2, then we have $n_1 < n_2$, and we define $\varphi(n_2) = 2$. Keep going and consider the minimum of $f(S)\backslash\{n_1, n_2\}$. Denote its minimum by n_3, then $n_1 < n_2 < n_3$ and define $\varphi(n_3) = 3$. Repeat indefinitely (possible as $f(S)$ is infinite), then one can construct a map $\varphi : f(S) \to \mathbb{N}$ such that $\varphi \circ f : S \to \mathbb{N}$ is a bijective map.

To conclude, when S is countably infinite, one may replace the injective map $f : S \to \mathbb{N}$ by a bijective one. Therefore, some authors may define S being countably infinite by saying that there exists a **bijective** map $g : \mathbb{N} \to S$ (swapping the domain and codomain). We will use these two conventions interchangeably.

This little observation can be useful sometimes especially when proving some set is uncountable. Here is an example:

Example 1.38. We argue that the interval $(0, 1)$ is uncountable. Suppose otherwise, then there exists a bijection $g : \mathbb{N} \to (0, 1)$. Then we write each $g(n)$ using decimal representations:

$$\begin{aligned} g(1) &= 0.a_{11}a_{12}a_{13}a_{14}\cdots \\ g(2) &= 0.a_{21}a_{22}a_{23}a_{24}\cdots \\ g(3) &= 0.a_{31}a_{32}a_{33}a_{34}\cdots \\ \vdots &\quad\ \vdots \end{aligned}$$

We then argue that there exists another real number in $(0, 1)$ that is not in the range of g. That will show g is not surjective and hence it leads to a contradiction.

For each n, we pick $b_n \in \{0, 1, \cdots, 9\}$ such that $|b_n - a_{nn}| \geq 1$. For example if $g(5) = 0.14159267\cdots$, then $a_{55} = 9$ so we can pick $b_5 = 8$. Then, create a real number $x = 0.b_1b_2b_3b_4\cdots$. Such a number must then be distinct from any of the $g(n)$'s. One can also choose these b_n's such that they are not all 0's or all 9's. It means that there exists $x \in (0, 1)$ that is not in the range of g. It contradicts to the fact that g is bijective. To conclude, $(0, 1)$ is uncountable.

This also implies $\mathbb{R}$ is uncountable as $(0, 1) \subset \mathbb{R}$. In fact any set such that $(0, 1)$ is its subset must be uncountable too.

Exercise 1.35. Show that any open interval (a, b), where $a < b$, is uncountable. Hint: consider a bijection between $(0, 1)$ and (a, b).

Exercise 1.36. Show that the power set $\mathcal{P}(\mathbb{N})$, which is the set of all subsets of $\mathbb{N}$, is uncountable.

1.3.2. Open and closed sets.

Two of the most important topological concepts in analysis are *open-ness* and *closed-ness*. Note that they are not logically opposite of each other – some sets can be both open and closed, and some could be neither – and we will see why is that from their definitions.

Definition 1.39 (Open sets and closed sets)**.** A set $S \subset \mathbb{R}$ is said to be **open** (or more precisely, **open in** $\mathbb{R}$) if for any $x \in S$, there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset S$.

A set $E \subset \mathbb{R}$ is said to be **closed** (or more precisely, **closed in** $\mathbb{R}$) if its complement, i.e. $E^c := \mathbb{R} - E$, is open in $\mathbb{R}$.

Remark 1.40. In this section we only deal with subsets of $\mathbb{R}$, so we will simply say a set is *open* or *closed* rather than *open in* $\mathbb{R}$ or *closed in* $\mathbb{R}$.

Example 1.41. An open interval (a, b) is open. To see this, pick any $x \in (a, b)$, then one can pick $\varepsilon = \frac{1}{2} \min \{x - a, b - x\}$. Then, $(x - \varepsilon, x + \varepsilon) \subset (a, b)$. This inclusion can be easily "seen" to be true if one sketch a diagram of these intervals. It is be proved by observing

$$\varepsilon \leq \frac{1}{2}(x - a) < x - a \implies x - \varepsilon > x - (x - a) = a.$$

Similar one can also show $x + \varepsilon < b$.

The union of open intervals $\{(a_j, b_j)\}_{j \in \mathcal{I}}$ is also open. Here the number of open intervals can be finite, countably infinite or even uncountable. To prove this, we consider any $x \in \bigcup_{j \in \mathcal{I}} (a_j, b_j)$, then there exists $j_0 \in \mathcal{I}$ such that $x \in (a_{j_0}, b_{j_0})$. Since we have already proved that open intervals are open, there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset (a_{j_0}, b_{j_0})$. In particular, we have

$$(x - \varepsilon, x + \varepsilon) \subset (a_{j_0}, b_{j_0}) \subset \bigcup_{j \in \mathcal{I}} (a_j, b_j).$$

This shows the union $\bigcup_{j \in \mathcal{I}} (a_j, b_j)$ is also open.

Hence, a set such as

$$\bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \left(\frac{p_n}{p_{n+k}}, \frac{p_{n+k}}{p_n} \right)$$

where p_n is the n-th prime number, can be easily seen to be an open set, even though it may not be easy to sketch it on the real line.

One may ask whether there are other examples of open sets in $\mathbb{R}$ which are not union of open intervals. We can actually prove this using the completeness axiom of $\mathbb{R}$. On the other hand, we will also see later that in higher dimensions (say in $\mathbb{R}^2$), examples of open sets are more diversified.

Proposition 1.42. *Any open set U in $\mathbb{R}$ is a disjoint union of countably open intervals, i.e. for any open set U in $\mathbb{R}$, there exist countably many open intervals (a_j, b_j), where $j \in \mathbb{N}$ and $a_j, b_j \in [-\infty, \infty]$, such that*

$$U = \bigcup_{j=1}^{\infty} (a_j, b_j) \quad \textit{and} \quad (a_j, b_j) \cap (a_k, b_k) = \emptyset \quad \textit{whenever } j \neq k.$$

Proof. Let U be an open set. If it is $\emptyset$, then one can write it as

$$\emptyset = (1,1) \cup (2,2) \cup (3,3) \cup \cdots .$$

Note that "degenerate" open interval $(a,a) = \{x \in \mathbb{R} : a < x < a\}$ is regarded as the empty set.

Now assume U is non-empty. We first show that U is a union of open intervals, then we justify that these open intervals can be chosen to be disjoint. For any $x \in U$, there exists $\varepsilon_x > 0$ depending on x such that $(x - \varepsilon_x, x + \varepsilon_x) \subset U$. Then, we replace extend the interval $(x - \varepsilon_x, x + \varepsilon_x)$ to be the largest open interval containing x, and contained in U. Precisely, we let

$$m_x := \inf\{m \in \mathbb{R} : (m, x] \subset U\}$$
$$M_x := \sup\{M \in \mathbb{R} : [x, M) \subset U\}$$

Then, the open interval $I_x := (m_x, M_x)$ is such an open interval.

We observe that $I_x \subset U$ for any $x \in U$ (think about why). Also, we have $x \in (x - \varepsilon_x, x + \varepsilon_x) \subset I_x$ for any $x \in U$. Therefore,

$$U = \bigcup_{x \in U} \{x\} \subset \bigcup_{x \in U} I_x \subset U \implies U = \bigcup_{x \in U} I_x.$$

This shows U is a union of open intervals – possibly uncountably many. To show that they can be chosen to be disjoint, we consider any pair of $x, y \in U$ with $x \neq y$, and we argue that either $I_x \cap I_y = \emptyset$, or $I_x = I_y$: it is because if $I_x \cap I_y \neq \emptyset$ and $I_x \neq I_y$, then $I_x \cup I_y$ is an open interval which is strictly larger than both I_x and I_y. However, it contradicts to the fact that I_x is the largest open interval containing x and contained in U (since $I_x \cup I_y$ is then a larger such open interval). Therefore, we must have either $I_x \cap I_y = \emptyset$ or $I_x = I_y$ for any $x, y \in U$ and $x \neq y$.

Therefore, the union $U = \bigcup_{x \in U} I_x$ has a lot of repetitions, and one can take a subset $V \subset U$ such that

$$U = \bigcup_{x \in V} I_x \quad \text{such that} \quad I_x \cap I_y = \emptyset \text{ whenever } x, y \in V \text{ and } x \neq y.$$

We are left to argue that such a collection V must be countable. For that we used the density of rationals: for any $x \in V$, the open interval I_x contains at least one $q_x \in \mathbb{Q}$. Consider the map $f : V \to \mathbb{Q}$ defined by $f(x) = q_x$. The mapping is then injective since the intervals $\{I_x\}_{x \in V}$ are pairwise disjoint. This proves V must be countable so it completes our proof. □

Remark 1.43. The empty-set $\emptyset$ is open, we assume it is not open and try to derive a contradiction. By taking negation of the definition, "$\emptyset$ being not open" means the following

"there exists $x \in \emptyset$ such that for any $\varepsilon > 0$, we have $(x - \varepsilon, x + \varepsilon) \not\subset \emptyset$."

However, there is no element in $\emptyset$, so saying "there exists $x \in \emptyset$" is absurd. This proves $\emptyset$ cannot be not open, i.e. it must be open.

To summarize, both $\mathbb{R}$ and $\emptyset$ are open sets. By the definition of closed sets, their complements $\mathbb{R}^c = \emptyset$ and $\emptyset^c = \mathbb{R}$ are then closed. This shows $\mathbb{R}$ and $\emptyset$ are both open and closed.

Next we see some examples on closed sets.

Example 1.44. $[0, \infty)$ is closed because $\mathbb{R} - [0, \infty) = (-\infty, 0)$ is open, even though some people do not call $[0, \infty)$ a closed interval.

Example 1.45. Closed interval $[a,b]$ is closed because $[a,b]^c = (-\infty,a)\cup(b,\infty)$ which is open. However, the union of infinitely many closed intervals may not be closed. Here is one counter-example:

$$\bigcup_{n=1}^{\infty}\left[\frac{1}{n},1\right] = (0,1].$$

The set $(0,1]$ is not closed because $(0,1]^c = (-\infty,0]\cup(1,\infty)$, which is not open by the fact that $0\in(0,1]^c$ but for any $\varepsilon>0$, $(0-\varepsilon,0+\varepsilon)\not\subset(0,1]^c$.

Example 1.46 (Cantor set)**.** The Cantor set $\mathcal{C}$ is constructed by successively removing the middle one-third of closed intervals. Precisely, we first consider the closed interval $\mathcal{C}_0 := [0,1]$, then construct $\mathcal{C}_1$ by removing the middle $(\frac{1}{3},\frac{2}{3})$, i.e.

$$\mathcal{C}_1 := [0,1] - \left(\frac{1}{3},\frac{2}{3}\right) = \left[0,\frac{1}{3}\right]\cup\left[\frac{2}{3},1\right].$$

Then, do the same for both $[0,\frac{1}{3}]$ and $[\frac{2}{3},1]$, and construct $\mathcal{C}_2$. Precisely,

$$\mathcal{C}_2 := \mathcal{C}_1 - \left(\frac{1}{3^2},\frac{2}{3^2}\right)\cup\left(\frac{7}{3^2},\frac{8}{3^2}\right) = \left[0,\frac{1}{3^2}\right]\cup\left[\frac{2}{3^2},\frac{3}{3^2}\right]\cup\left[\frac{6}{3^2},\frac{7}{3^2}\right]\cup\left[\frac{8}{3^2},1\right].$$

Continue indefinitely, one can produce $\mathcal{C}_3$, $\mathcal{C}_4$, etc. Then Cantor set is defined to be

$$\mathcal{C} := \bigcap_{n=1}^{\infty}\mathcal{C}_n.$$

To prove that $\mathcal{C}$ is closed, we consider its complement:

$$\mathcal{C}^c = \mathbb{R} - \bigcap_{n=1}^{\infty}\mathcal{C}_n = \bigcup_{n=1}^{\infty}(\mathbb{R}-\mathcal{C}_n).$$

Each $\mathcal{C}_n$ is a union of finitely many disjoint closed intervals. It is obvious that $\mathbb{R}-\mathcal{C}_n$ is a union of open intervals. This shows $\mathcal{C}^c$, which is the union of all $\mathbb{R}-\mathcal{C}_n$'s, is open as well. Therefore the Cantor set $\mathcal{C}$ is closed.

From the two examples above, we see that finite union of closed intervals is also closed, but infinite union of closed intervals may not be. In fact, we have the following general result:

Proposition 1.47. *On $\mathbb{R}$, we have the following:*

*(1) The union of **any collection** of open sets is open.*

*(2) The intersection of **finitely** many open sets is open.*

*(3) The union of **finitely** many closed sets is closed.*

*(4) The intersection of **any collection** of closed sets is closed.*

Proof. The proof of (1) is the same as the proof that the union of open intervals is open, *mutatis mutandis*. One simply changes the intervals (a_j,b_j)'s to open sets $\mathcal{O}_j$'s.

For (2), we consider two open sets U and V in $\mathbb{R}$, and will show $U\cap V$ is open. Then for any $x\in U\cap V$, we have both $x\in U$ and $x\in V$. Therefore, there exist $\varepsilon_1>0$ and $\varepsilon_2>0$ such that $(x-\varepsilon_1,x-\varepsilon_1)\subset U$ and $(x-\varepsilon_2,x+\varepsilon_2)\subset V$. Then by taking $\varepsilon=\min\{\varepsilon_1,\varepsilon_2\}$, we have both

$$(x-\varepsilon,x+\varepsilon)\subset(x-\varepsilon_1,x-\varepsilon_1)\subset U,$$
$$(x-\varepsilon,x+\varepsilon)\subset(x-\varepsilon_2,x-\varepsilon_2)\subset V.$$

Hence, $(x-\varepsilon,x+\varepsilon)\subset U\cap V$.

By induction (on the number of open sets), we can complete the proof of (2).

(3) and (4) are proved by taking the complements. Consider a finite collection of closed sets $C_1, \cdots, C_n$ in $\mathbb{R}$. To prove that $C_1 \cup \cdots \cup C_n$ is closed, we consider

$$\mathbb{R} - \bigcup_{j=1}^{n} C_j = \bigcap_{j=1}^{n} (\mathbb{R} - C_j).$$

Because each $\mathbb{R}-C_j$ is open (since each C_j is closed), their finite intersection $\bigcap_{j=1}^{n}(\mathbb{R}-C_j)$ is also open, and so $\bigcup_{j=1}^{n} C_j$ is closed. This proves (3), and (4) can be proved in a similar way. $\square$

One can then prove the Cantor set is closed directly by applying this proposition: since $\mathcal{C}_n$ is a finite union of closed intervals, each $\mathcal{C}_n$ is closed. Also, since the Cantor set is

$$\mathcal{C} = \bigcap_{n=1}^{\infty} \mathcal{C}_n,$$

and the intersection of any collection of closed sets is closed, we conclude that $\mathcal{C}$ is also closed.

Exercise 1.37. Determine whether or not each set below is open or closed:

(1) $\mathbb{Z}$

(2) $\mathbb{Q}$

(3) $\{\frac{1}{n} : n \in \mathbb{N}\}$

(4) $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$

(5) $\{\sum_{k=1}^{n} \frac{1}{k^2} : n \in \mathbb{N}\}$

(6) $\{\sum_{k=1}^{n} \frac{1}{k} : n \in \mathbb{N}\}$

Exercise 1.38. Let $\{x_n\}$ be a sequence of real numbers such that $x_n \to L$, and $L \in U$ for some open set U. Show that there exists $N \in \mathbb{N}$ such that $x_n \in U$ for any $n \geq N$.

A set S being closed is also equivalent to another condition in relation to the limit of sequences. To motivate our discussion, we consider two examples: $S_1 = (0, 1]$ and $S_2 = [0, 1]$, and the sequence $\{\frac{1}{n}\}$. It is a sequence converging to 0, and all elements in the sequence are in S_1 and S_2. Viewing it as a sequence in S_1 which is not closed, we see that the limit 0 could be outside S_1. On the other hand, when viewing it as a sequence in S_2 which is closed, we see that the limit is also inside S_2. One equivalent definition for a set S being closed is whether all converging sequences in S have limits inside S. Let's state it formally:

Proposition 1.48. *Let $S \subset \mathbb{R}$. Then the following are equivalent:*

(1) S is closed in $\mathbb{R}$ (i.e. S^c is open in $\mathbb{R}$).

(2) Any converging sequence $\{x_n\}_{n=1}^{\infty}$ in S has its limit $\lim\limits_{n\to\infty} x_n \in S$.

Proof. To the part (1) $\implies$ (2): we let S be closed in $\mathbb{R}$. To show that (2) holds, we pick an arbitrary converging sequence $\{x_n\}$ such that $x_n \in S$ for all n. We need to pick that $L := \lim\limits_{n\to\infty} x_n$ is in S too. Suppose otherwise, then $L \in \mathbb{R} - S$, which is an open set. By definition of open sets, there exists $\varepsilon > 0$ such that $(L - \varepsilon, L + \varepsilon) \subset \mathbb{R} - S$. On the other hand, as $x_n \to L$, there exists $N > 0$ such that $x_n \in (L - \varepsilon, L + \varepsilon)$ for any $n \geq N$. This is certainly a contradiction as then $x_n \in \mathbb{R} - S$ for any $n \geq N$ but we have $x_n \in S$ for all n. Therefore, we must have $L \in S$, completing the proof of (2).

For (2) $\implies$ (1): we consider a set S such that any converging sequence $\{x_n\}$ in S has its limit in S. We then need to prove $\mathbb{R} - S$ is open. Again we suppose it is not the case, i.e. $\mathbb{R} - S$ is not open. Then there exists $x \in \mathbb{R} - S$ such that for any $\varepsilon > 0$, we have $(x - \varepsilon, x + \varepsilon) \not\subset \mathbb{R} - S$. In particular, for any $n \in \mathbb{N}$, we have $(x - \frac{1}{n}, x + \frac{1}{n}) \not\subset \mathbb{R} - S$ meaning that there exists $x_n \in (x - \frac{1}{n}, x + \frac{1}{n})$ such that $x_n \notin \mathbb{R} - S$ (which just means that $x_n \in S$). Clearly $x_n \to x$ as $n \to \infty$, and $x_n \in S$ for all n implies the limit $x \in S$ (by the assumption (2)). This is a contradiction to the fact that $x \in \mathbb{R} - S$. Therefore, we must have $\mathbb{R} - S$ being open, and so S is closed. □

With this proposition, one can see easily that $\{\frac{1}{n} : n \in \mathbb{N}\}$ is not closed, as the sequence $\{\frac{1}{n}\}$ (which is in the set) has limit 0 which is not in the set. By a similar reason, $\mathbb{Q}$ is not closed because for any irrational number $s \notin \mathbb{Q}$, one can always find a sequence $r_n \in \mathbb{Q}$ such that $r_n \to s$.

Exercise 1.39. Show that if S is both closed and bounded (meaning that there exists a real number $M > 0$ such that $|x| \leq M$ for any $x \in S$), then for any sequence $\{x_n\}$ in S, there exists a subsequence $\{x_{n_j}\}$ converging to a limit $\lim\limits_{j \to \infty} x_{n_j} \in S$.

Exercise 1.40. Show that if S is closed and bounded, then $\sup S, \inf S \in S$ (and so $\sup S = \max S$ and $\inf S = \min S$).

Exercise 1.41. Let $\{x_n\}$ be a sequence of real numbers. Show that the LIM $\{x_n\} \cap \mathbb{R}$ is a closed set.

1.3.3. Compact sets.

For a set S on the **real line** (we need to say thrice for important stuff: on the real line, on the real line, and on the real line), we say S is **compact** if it is both **closed** and **bounded**. The formal definition of the term "compact" on general metric spaces (to be introduced in the next chapter) is more complicated.

To formally introduce the concept of compactness, we first consider the example – the closed interval $[0, 1]$. This interval can be covered by open intervals $\left\{(-\frac{1}{n}, 1 - \frac{1}{n})\right\}_{n=1}^{\infty}$ together with $(\frac{1}{2}, \frac{3}{2})$. By "covered" we mean that

$$[0, 1] \subset \bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, 1 - \frac{1}{n}\right) \cup \left(\frac{1}{2}, \frac{3}{2}\right).$$

We call the collection $\{(-\frac{1}{n}, 1 - \frac{1}{n})\}_{n \in \mathbb{N}} \cup \{(\frac{1}{2}, \frac{3}{2}\}$ an **open cover** of $[0, 1]$. It is an infinite collection, but it is easy to observe that we can extract a **finite subcollection**, namely just $\{(-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{3}, \frac{2}{3}), (\frac{1}{2}, \frac{3}{2})\}$, but still they form an open cover of $[0, 1]$, i.e.

$$[0, 1] \subset \left(-\frac{1}{2}, \frac{1}{2}\right) \cup \left(-\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{1}{2}, \frac{3}{2}\right).$$

Mathematically speaking, we may say the open cover $\{(-\frac{1}{n}, 1 - \frac{1}{n})\}_{n \in \mathbb{N}} \cup \{(\frac{1}{2}, \frac{3}{2}\}$ of the interval $[0, 1]$ has a **finite subcover** $\{(-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{3}, \frac{2}{3}), (\frac{1}{2}, \frac{3}{2})\}$. Readers should try to construct another (not-so-complicated) example of an infinite open covers of $[0, 1]$, and you will find that you should be able to extract a finite subcover that still covers $[0, 1]$.

On the other hand, if we consider interval $(0, 1]$ which is not closed (but is bounded), then it can be covered by the collection of open sets $\{(\frac{1}{n}, 2)\}_{n \in \mathbb{N}}$, as

$$\bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 2\right) = (0, 2) \supset (0, 1].$$

However, any subcollection $\{(\frac{1}{n_1}, 2), \cdots, (\frac{1}{n_k}, 2)\}$ of this cover cannot cover the whole interval $(0, 1]$, as

$$\bigcup_{j=1}^{k} \left(\frac{1}{n_j}, 2\right) = \left(\frac{1}{n^*}, 2\right) \not\supset (0, 1]$$

where $n^* = \max\{n_1, \cdots, n_k\}$.

Also, if the interval is closed but not bounded, such as $[0, \infty)$, we can cover it by $\{(n-2, n+2)\}_{n \in \mathbb{N}}$, but any finite subcover of it cannot cover the infinite interval $[0, \infty)$.

In fact, the "finite subcover property" is equivalent to being closed and bounded for sets on the real line:

Theorem 1.49 (Heine-Borel). *Let $S \subset \mathbb{R}$. Then the following are equivalent:*

(1) S is closed and bounded.

(2) Any sequence $\{x_n\}$ in S has a subsequence $\{x_{n_j}\}$ converging to a limit in S.

(3) Any open cover of S has a finite subcover, meaning that if $S \subset \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ where $U_\alpha \subset \mathbb{R}$ is an open set for any α in the index set $\mathcal{A}$, then there exists $\alpha_1, \cdots, \alpha_N \in \mathcal{A}$ such that

$$S \subset U_{\alpha_1} \cup \cdots \cup U_{\alpha_N}.$$

Proof. (1) $\implies$ (2) simply follows from Bolzano-Weiestrass Theorem and Proposition 1.48.

Next we prove (2) $\implies$ (3): Given that S is a set satisfying (2), we need to argue that any open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of S has a finite subcover. First, we prove that there exists a **countable** subcover. By Proposition 1.42, each U_α is a countable union of open intervals. Write

$$U_\alpha = \bigcup_{j=1}^{\infty} (a_{\alpha,j}, b_{\alpha,j}).$$

Recall that each real number is the limit of a monotone sequence of rational numbers, so each open interval (a, b) can be written as

$$(a_{\alpha,j}, b_{\alpha,j}) = \bigcup_{k=1}^{\infty} (q_{\alpha,j,k}, r_{\alpha,j,k})$$

where $q_{\alpha,j,k}$ is a decreasing rational sequence converging to a (as $k \to \infty$), and $r_{\alpha,j,k}$ is an increasing rational sequence converging to b. Therefore, the given open cover U_α can be written as an open cover by open intervals with rational end-points:

$$S \subset \bigcup_{\alpha \in \mathcal{A}} U_\alpha = \bigcup_{\alpha \in \mathcal{A}} \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} (q_{\alpha,j,k}, r_{\alpha,j,k}).$$

Although the collection $\{(q_{\alpha,j,k}, r_{\alpha,j,k})\}$ looks huge in the above union, there are at most countably many (i.e. a lot of repetitions) such intervals as $\mathbb{Q}^2$ (and hence intervals with rational end-points) is countable.

Therefore, the gigantic union $\bigcup_{\alpha \in \mathcal{A}} \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} (q_{\alpha,j,k}, r_{\alpha,j,k})$ is in fact a countable union. For simplicity, we rewrite the union as follows

$$S \subset \bigcup_{i=1}^{\infty} (q_i, r_i)$$

where each (q_i, r_i) is one of the $(q_{\alpha,j,k}, r_{\alpha,j,k})$, and so is contained in U_{α_i} for some $\alpha_i \in \mathcal{A}$. Therefore, we have

$$S \subset \bigcup_{i=1}^{\infty}(q_i, r_i) \subset \bigcup_{i=1}^{\infty} U_{\alpha_i}.$$

It concludes our claim that the open cover $\{U_\alpha\}$ of S has a countable subcover $\{U_{\alpha_i}\}_{i=1}^{\infty}$. Note that up to this point we still haven't used the assumption (2).

Now we proceed to prove that the open cover $\{U_\alpha\}$ of S in fact has a **finite** subcover. Assume the opposite, i.e. the open cover has no finite subcover. Then for any $n \in \mathbb{N}$, the finite union $U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$ cannot cover S, i.e. $\exists x_n \in S$ such that $x_n \notin U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$. Then, $\{x_n\}$ forms a sequence in S. By assumption (2), one can extract a converging subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ that converges to a limit $L \in S$. Since $\{U_{\alpha_i}\}_{i=1}^{\infty}$ is an open cover of S, so it implies $L \in U_{\alpha_{i_0}}$ for some $i_0 \in \mathbb{N}$. Note that $U_{\alpha_{i_0}}$ is open, together with $x_{n_k} \to L$, it implies $x_{n_k} \in U_{\alpha_{i_0}}$ for sufficiently large k. Pick any k such that $n_k \geq i_0$, then we have on one hand $x_{n_k} \in U_{\alpha_{i_0}}$, but on the other hand

$$x_{n_k} \notin U_{\alpha_1} \cup \cdots \cup U_{\alpha_{i_0}} \cup \cdots \cup U_{\alpha_{n_k}}.$$

It leads a contradiction. Therefore, $\{U_\alpha\}$ must have a finite subcover of S. This proves (3).

Finally, we prove (3) $\implies$ (1): Given that S satisfies (3), we need to prove it is closed and bounded. For closedness we need to show that for any converging sequence $\{a_n\}$ in S, the limit L must be in S. Assume on the contrary that $L \notin S$, then for any point $x \in S$, and as then $x \neq L$, so by taking $\varepsilon_x := \frac{1}{2}|x - L|$, we have $L \notin [x - \varepsilon_x, x + \varepsilon_x]$. Then, we consider

$$S = \bigcup_{x \in S}\{x\} \subset \bigcup_{x \in S}(x - \varepsilon_x, x + \varepsilon_x).$$

By assumption (3), the open cover $\{(x - \varepsilon_x, x + \varepsilon_x)\}_{x \in S}$ of S has a finite subcover, i.e. there exists $x_1, \cdots, x_N \in S$ such that

$$S \subset \bigcup_{i=1}^{N}(x_i - \varepsilon_{x_i}, x_i + \varepsilon_{x_i}) \subset \underbrace{\bigcup_{i=1}^{N}[x_i - \varepsilon_{x_i}, x_i + \varepsilon_{x_i}]}_{\text{closed set}}$$

Recall that $a_n \in S$, so that $a_n \in \bigcup_{i=1}^{N}[x_i - \varepsilon_{x_i}, x_i + \varepsilon_{x_i}]$. By closedness, we have that the limit L of $\{a_n\}$ must be in $\bigcup_{i=1}^{N}[x_i - \varepsilon_{x_i}, x_i + \varepsilon_{x_i}]$ as well. However, it contradicts to the fact that $L \notin [x - \varepsilon_x, x + \varepsilon_x]$ for any $x \in S$. This proves S must be closed.

The last thing to show is S must be bounded. We consider the open cover

$$S \subset \bigcup_{x \in S}(x - 1, x + 1).$$

Assumption (3) asserts that there exists $x_1, \cdots, x_N \in S$ such that

$$S \subset \bigcup_{i=1}^{N}(x_i - 1, x_i + 1).$$

Clearly, the finite union $\bigcup_{i=1}^{N}(x_i - 1, x_i + 1)$ of bounded intervals must be bounded. This shows S is bounded too, proving (1). $\square$

Remark 1.50. The whole of Heine-Borel proof may sound a bit daunting when you read through it the first time, yet the proof demonstrates many standard argument in analysis. The best way to learn it is to first understand the overall outline as well as the logical argument of every step, then derive the whole proof by yourself once in closed notes, and also give oral presentation of the whole proof in your study group.

Exercise 1.42. Give an oral presentation of the whole proof of Heine-Borel Theorem to students in your study group.

Exercise 1.43. When S is just a closed and bounded **interval** $[0, 1]$ rather than an arbitrary closed and bounded **set**, it is in fact easier to prove that (3) holds without going through the above "ordeal". Below is the outline. Try to write up the whole proof.

(1) Suppose $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is an open cover of $[0, 1]$. We need to prove that it has a finite subcover, and we suppose not.

(2) Argue that $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ has no finite subcover for one of the $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. Denote this closed subinterval by I_1.

(3) Keep going dividing I_1 into two closed subsubintervals, argue that $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ has no finite subcover for one of the subsubintervals.

(4) Repeat the above process indefinitely, then we get $I_1 \supset I_2 \supset I_3 \supset \cdots$, so that $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ has no finite subcover for each of the I_j's.

(5) Prove that $\bigcap_{j=1}^{\infty} I_j$ is a one-point set. Try to derive a contradiction from here.

Motivated by the Heine-Borel Theorem on $\mathbb{R}$, we define the notion of compactness by the following

Definition 1.51 (Compact, sequentially compact)**.** A set $S \subset \mathbb{R}$ is said to be **compact** if every open cover of S has a finite subcover. A set $S \subset \mathbb{R}$ is said to be **sequentially compact** if every sequence $\{x_n\}$ in S has a converging subsequence $\{x_{n_j}\}$ with limit in S.

Heine-Borel Theorem asserts that for sets on the real line (I repeat: on the real line, on the real line, and on the real line), we have

$$\text{closed and bounded} \iff \text{sequentially compact} \iff \text{compact}$$

However, we will later see that generally on a metric space (to be introduced in the next chapter), we only have sequentially compact $\iff$ compact, but they are not equivalent to being closed and bounded. On a topological space (to be introduced in MATH 4225), all these three notions are not equivalent (and even worse one cannot always make sense of boundedness).

Chapter 2

Topology of Metric Spaces

"Distance between two people is inconsequential when their souls are connected."

Matshona Dhliwayo

2.1. Metric Spaces

2.1.1. Metric spaces: example and definition. Recall that in MATH 1023 (or any rigorous Calculus course), we learned about the rigorous definitions of limits. Given a sequence $\{x_n\}$ in $\mathbb{R}$ and $L \in \mathbb{R}$, we say that $\lim\limits_{n\to\infty} x_n = L$ if $\forall \varepsilon > 0$, there exists $N > 0$ such that if $n \geq N$, then $|x_n - L| < \varepsilon$. It means that x_n can be as close to L as possible provided that n is sufficiently large. For a function $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$, we say $\lim\limits_{x\to a} f(x) = L$, where $L \in \mathbb{R}$, if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. It means that $f(x)$ can be as close to L as possible provided x and a are sufficiently close. The notion of "distance" or "closeness" plays the key role in the above definitions, as $|x - a|$ is essentially the "distance" between x and a, and similarly for $|x_n - L|$ and $|f(x) - L|$. If we write $d(x, y) := |x - y|$, which stands for the "distance" between x and y, then the definitions of limit can be rewritten as:

$$\lim_{n\to\infty} x_n = L \iff \forall \varepsilon > 0, \exists N > 0 \text{ such that if } n \geq N, \text{ then } d(x_n, L) < \varepsilon$$
$$\lim_{x\to a} f(x) = L \iff \forall \varepsilon > 0, \exists \delta > 0 \text{ such that if } 0 < d(x, a) < \delta, \text{ then } d(f(x), L) < \varepsilon$$

Going beyond the real line, say on higher dimensional spaces $\mathbb{R}^2$, $\mathbb{R}^3$, or even more "abstract" spaces, if one can define an appropriate notion of "distance" $d(x, y)$ for any pair of x, y in that space, then one can generalize the concept of limits to that space. That motivates the notion of *metric spaces*:

Definition 2.1 (Metric space)**.** Let X be a non-empty set. We say the two-variable function $d : X \times X \to [0, \infty)$ is a **metric** (or **distance function**) on X if it satisfies all conditions below:

(1) $d(x, y) = 0$ if and only if $x = y$

(2) $d(x, y) = d(y, x)$ for any $x, y \in X$ (symmetric)

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for any $x, y, z \in X$ (i.e. triangle inequality)

The pair (X, d) is then called a **metric space**.

Example 2.2. The "usual" distance $d(x, y) := |x - y|$ is a metric on $\mathbb{R}$. The conditions (1) and (2) are obvious. Condition (3) follows from the triangle inequality on $\mathbb{R}$.

Example 2.3. On $\mathbb{R}^n$, the elements can be represented as $\mathbf{x} = (x_1, x_2, \cdots, x_n)$, where $x_i \in \mathbb{R}$. For any pair of $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we can define $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ as

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Then it is obvious that (1) and (2) hold. Condition (3) is the triangle inequality on $\mathbb{R}^n$, which can be proved using Cauchy-Schwarz's inequality.

Exercise 2.1. Prove the Cauchy-Schwarz inequality:

$$\sum_{i=1}^{n} a_i b_i \leq \sqrt{\sum_{i=1}^{n} a_i^2} \sqrt{\sum_{i=1}^{n} b_i^2}$$

and use it to prove the triangle inequality on $\mathbb{R}^n$.

These above metrics on $\mathbb{R}$ or $\mathbb{R}^n$ are the "standard" or "usual" ones. There are fancier examples. For any real $p \geq 1$, we can define $d_p : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ as

$$d_p(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}.$$

Again, conditions (1) and (2) clearly hold. Condition (3) for d_p is exactly the Minkowski inequality. Therefore $(\mathbb{R}^n, d_p)$ where $p \geq 1$ is a metric space. The Euclidean distance is the special case $p = 2$.

Exercise 2.2. Read the proof of the Minkowski inequality, i.e. given $p \geq 1$, we have

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p \right)^{1/p} \leq \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p \right)^{1/p}.$$

Point out in the proof where the condition $p \geq 1$ is used.

Example 2.4. Here is one trivial example of a metric space. Let X be any non-empty set, and we define $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}.$$

Again (1) and (2) are obvious. To prove (3), we suppose otherwise that there are $x, y, z \in X$ such that

$$d(x, z) > d(x, y) + d(y, z).$$

Note d is either 0 or 1, so we must have $d(x,z) = 1$, and $d(x,y) = d(y,z) = 0$. But it implies $x \neq z$, whereas $x = y$ and $y = z$. It is impossible. This shows (X,d) is a metric space. This particular metric is often called the **discrete metric**.

Example 2.5. Here is a less trivial but easy example of a metric space. Let $X = (0,\infty)$, and define $d : X \times X \to [0,\infty)$ by

$$d(x,y) := |\log x - \log y| .$$

[Remark: In most math courses $\log$ refers to $\log_e$ or $\ln$, which was much more common than the "common logarithm" $\log_{10}$. I never used $\log_{10}$ since my last Chemistry lesson in high school.] Again the conditions (1) and (2) are trivial. For condition (3), we pick arbitrary $x, y, z \in (0,\infty)$, then

$$\begin{aligned} d(x,z) &= |\log x - \log z| \\ &\leq |\log x - \log y| + |\log y - \log z| \\ &= d(x,y) + d(y,z). \end{aligned}$$

This shows (3) holds, so (X,d) is a metric.

Exercise 2.3. Let X and Y be two non-empty sets, and $f : X \to Y$ is an injective function, and (Y,d) be a metric space. Prove that $\rho : X \times X \to [0,\infty)$ defined by

$$\rho(x_1,x_2) := d\big(f(x_1), f(x_2)\big)$$

is a metric on X.

Exercise 2.4. Given an increasing concave function $\varphi : [0,\infty) \to [0,\infty)$ with the property that $\varphi(x) = 0$ if and only if $x = 0$, and a metric space (X,d), prove that $\varphi \circ d : X \times X \to [0,\infty)$ is also a metric on X.

Example 2.6. Let's give an example of a metric space so that X is a bit more complicated. Let $X = C[0,1]$, which is the set of continuous functions $f : [0,1] \to \mathbb{R}$. We define $d : X \to X \to [0,\infty)$ by

$$d(f,g) := \left(\int_0^1 |f(x) - g(x)|^2 \, dx\right)^{1/2}.$$

Condition (2) is trivial. This time condition (1) is less trivial as it seems and the requirement of continuity plays an important role. Without the continuity condition, i.e. let X be the set of all integrable functions on $[0,1]$, we may define g by changing the value of f at one point, then we would still have $d(f,g) = 0$. Condition (3) follows from the integral version of Cauchy-Schwarz.

Exercise 2.5. Prove that for any continuous functions $f, g : [0,1] \to \mathbb{R}$, we have

$$\int_0^1 |f(x)g(x)| \, dx \leq \left(\int_0^1 |f(x)|^2\right)^{1/2} \left(\int_0^1 |g(x)|^2\right)^{1/2}.$$

Hence, prove condition (3) for the d in the above example.

Exercise 2.6. Prove that the following is a metric on $C[0,1]$:

$$d(f,g) := \sup_{x \in [0,1]} |f(x) - g(x)| .$$

There are many other examples of metric spaces in future courses. In graph theory, there is a metric measuring the distance between two vertices by counting how many segments between them. In differential geometry, there is a metric on a surface/manifold measuring the distance between two points by the length of the shortest path joining them. In topology, there is even a metric, known as Gromov-Hausdorff metric, measuring the distance between two metric spaces.

2.1.2. Normed vector space. On a vector space (i.e. a set where addition and scalar multiplication make sense), we construct a metric on it using *norms*:

Definition 2.7 (Normed vector space)**.** Let V be a vector space over $\mathbb{R}$. A function $\| \| : V \to [0,\infty)$ said to be a **norm** on V if it has the following properties:

(1) $\|v\| = 0$ if and only if $v = 0$.
(2) For any $\alpha \in \mathbb{R}$ and $v \in V$, we have $\|\alpha v\| = |\alpha| \|v\|$.
(3) For any $x, y \in V$, we have
$$\|x+y\| \leq \|x\| + \|y\| .$$

The pair $(V, \| \|)$ is called a **normed vector space**.

Example 2.8. For any $p \geq 1$, we can define the following norm denoted by $\| \|_p$ on $\mathbb{R}^n$:
$$\|\mathbf{x}\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} .$$
Conditions (1) and (2) are trivial – that's why we have the $1/p$-exponent so that (2) holds. Condition (3) is the Minkowski inequality.

Example 2.9. One can extend the previous example to sequences. Fix $p \geq 1$, we define the set
$$l_p(\mathbb{R}) := \left\{ \{x_n\}_{n=1}^{\infty} : \sum_{n=1}^{\infty} |x_n|^p \text{ converges} \right\}.$$
It can be checked that $l_p(\mathbb{R})$ is a vector space (exercise for readers). We can define a norm $\| \|_p$ on $l_p(\mathbb{R})$ by:
$$\|\{x_n\}\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} .$$
Again conditions (1) and (2) trivially hold. For condition (3), we use the Minkowski inequality to first argue that
$$\left(\sum_{n=1}^{N} |x_n + y_n|^p\right)^{1/p} \leq \left(\sum_{n=1}^{N} |x_n|^p\right)^{1/p} + \left(\sum_{n=1}^{N} |y_n|^p\right)^{1/p}$$
for any N, then let $N \to \infty$ on both sides (which is valid as we have convergence of the series).

Example 2.10. On $C[0,1]$, then given any $p \geq 1$, we can define
$$\|f\|_p := \left(\int_0^1 |f(x)|^p\right)^{1/p} .$$
It can also be easily checked that it is a norm on $C[0,1]$.

Exercise 2.7. Show that the following is a norm on $C[0,1]$:
$$\|f\|_\infty := \sup_{x\in[0,1]} |f(x)| .$$
Guess why we use such a notation $\|\|_\infty$.

Exercise 2.8. Fix $\alpha \in (0,1)$. Consider the space of α-Hölder continuous functions on $[0,1]$, defined by

$$C^\alpha[0,1] := \{f \in C[0,1] : \exists C > 0 \text{ such that } |f(x) - f(y)| \leq C\,|x-y|^\alpha \;\forall x, y \in [0,1]\}.$$

(1) Prove that $C^\alpha[0,1]$ is a subspace of $C[0,1]$.

(2) For any $f \in C^\alpha[0,1]$, we define the function $[\;]_\alpha : C^\alpha[0,1] \to [0,\infty)$ by:

$$[f]_\alpha := \sup_{x,y\in[0,1] \text{ and } x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

Explain why $[\;]_\alpha$ is NOT a norm of $C^\alpha[0,1]$.

(3) Show that $\|\;\|_\alpha : C^\alpha[0,1] \to [0,\infty)$ defined by

$$\|f\|_\alpha := \|f\|_\infty + [f]_\alpha$$

is a norm on $C^\alpha[0,1]$.

The purpose of introducing normed vector spaces is that it must also be a metric space. Given a norm $\|\;\|$ on a vector space V, one can define a function $d : V \times V \to [0,\infty)$ by

$$d(x,y) := \|x-y\|.$$

Then it can be easily checked that such a function d is a metric on V.

Exercise 2.9. Verify that d above is indeed a metric on V.

Any normed vector space is also a metric space. Certainly, not all metric spaces are normed vector spaces. Some of them are not even vector spaces. For example, on a connected graph, one can define a metric on it by counting the segments of the shortest path. However, vertices in the graph cannot be added together, so the graph does not even have a vector space structure.

With a metric d on a set X, one can then make sense of limits of sequences in X. We may simply replace $|x-y|$ by $d(x,y)$.

Definition 2.11. Let (X,d) be a metric space. Suppose $\{x_n\}$ is a sequence in X. Then we say $x_n \to y$ as $n \to \infty$ with respect to the metric d if $\forall \varepsilon > 0$, there exists $N > 0$ such that $d(x_n, y) < \varepsilon$.

Remark 2.12. From now on we will often write $x_n \to y$ as $n \to \infty$ (or simply $x_n \to y$), instead of $\lim\limits_{n\to\infty} x_n = y$. Similar for limits of functions.

Remark 2.13. By rewriting the definition of limit of sequences on a metric space, we see that $x_n \to y$ with respect to d if and only if $d(x_n, y) \to 0$ as a sequence of real numbers. On a normed vector space $(V, \|\;\|)$, the induced metric d is defined to be $d(v,w) = \|v-w\|$, and hence $v_n \to w$ with respect to $\|\{\|\;\}$ is equivalent to saying $\|v_n - w\| \to 0$ as a sequence of real numbers.

Remark 2.14. Limit of a function is a bit more subtle. Unlike in single-variable calculus where most functions are defined on intervals, there is no issue of saying $f(x) \to L$ as $x \to a$. However, in a metric space which may have complicated "shapes", it may be questionable whether we can make good sense of saying "$x \to a$". We need a notion of *limit points* to be introduced in the next section.

Example 2.15. Consider the standard Euclidean norm $\|\;\|_2$ on $\mathbb{R}^k$. Consider a sequence $\{\mathbf{x}_n\}$ in $\mathbb{R}^k$ and write

$$\mathbf{x}_n = (x_n^1, x_n^2, \cdots, x_n^k).$$

Then we say $\mathbf{x}_n \to \mathbf{y} = (y^1, \cdots, y^k)$ with respect to $\| \; \|_2$ if $\forall \varepsilon > 0$, $\exists N > 0$ such that if $n \geq N$, then $\|\mathbf{x}_n - \mathbf{y}\|_2 < \varepsilon$. Note that for each $i = 1, 2, \cdots, k$, we have

$$\left|x_n^i - y^i\right| \leq \sqrt{\sum_{j=1}^{k} \left|x_n^j - y^j\right|^2} = \|\mathbf{x}_n - \mathbf{y}\|_2 \,.$$

Hence whenever $n \geq N$, we have

$$\left|x_n^i - y^i\right| < \varepsilon$$

for any $i = 1, 2, \cdots, k$ as well. To conclude, if $\mathbf{x}_n \to \mathbf{y}$ with respect to $\| \; \|_2$, then we have $x_n^i \to y^i$ for each $i = 1, 2, \cdots, k$.

The converse is also true. We leave it as an exercise for readers.

Exercise 2.10. Prove that if $x_n^i \to y^i$ for each $i = 1, 2, \cdots, k$, then $\mathbf{x}_n \to \mathbf{y}$ with respect to $\| \; \|_2$.

The above example and exercise tell us that taking limit of sequences in a higher dimensional Euclidean space $\mathbb{R}^k$ is essentially taking limit component-wise.

We have used the standard $\| \; \|_2$-norm on $\mathbb{R}^k$ above. If one uses $\| \; \|_p$ for other $p \geq 1$, we can in fact show a sequence $\{\mathbf{x}_n\}$ converges in $\| \; \|_2$-norm if and only if it converges in other $\| \; \|_p$-norms. It is a consequence of some elementary inequalities (as such Hölder inequality). Try the exercise below:

Exercise 2.11. Let $p, q \geq 1$ and consider $\mathbb{R}^k$. Show that there exist constants $C, c > 0$ depending on p, q, k (and find them explicitly) such that

$$c \|\mathbf{x}\|_p \leq \|\mathbf{x}\|_q \leq C \|\mathbf{x}\|_p$$

for any $\mathbf{x} \in \mathbb{R}^k$. Hence, show that a sequence $\{\mathbf{x}_n\}$ converges to $\mathbf{y}$ in $\| \; \|_p$-norm if and only if it does so in $\| \; \|_q$-norms.

Example 2.16. Consider the space $C[0, 1]$ with the norm $\| \; \|_p$ where $p \geq 1$. Recall that this norm is defined by

$$\|f\|_p := \left(\int_0^1 |f(x)|^p\right)^{1/p}.$$

Consider the sequence of functions $\{f_n\}$ in $C[0, 1]$ defined by $f_n(x) = x^n$, then it is easy to see that

$$\|f_n - 0\|_p = \left(\int_0^1 (x^n)^p \, dx\right)^{1/p} = \frac{1}{(pn+1)^{1/p}} \to 0$$

as $n \to \infty$.

Example 2.17. Now consider the same space $C[0, 1]$ but with the norm $\| \; \|_\infty$. Recall that it is defined by

$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)| \,.$$

Again consider the sequence of functions $\{f_n\}$ in $C[0, 1]$ defined by $f_n(x) = x^n$. We will argue that such a sequence of functions diverge (i.e. not converge) under the norm $\|, \|_\infty$. To argue this, we suppose $f_n \to g \in C[0, 1]$ under the norm $\| \; \|_\infty$. Then for any $x \in [0, 1]$, we have for every $x \in [0, 1]$,

$$|f_n(x) - g(x)| \leq \sup_{x \in [0,1]} |f_n(x) - g(x)| = \|f_n - g\|_\infty \to 0.$$

Therefore, for any $x \in [0, 1]$, we have $f_n(x) \to g(x)$ as a sequence of real numbers.

For this example $f_n(x) = x^n$, we clearly have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}.$$

This says if $f_n \to g$ in $\| \; \|_\infty$-norm, such a limit g must be the function

$$g(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}.$$

But this g is not continuous. Hence f_n has no limit in $C[0,1]$ if $\| \; \|_\infty$ is used as the norm of $C[0,1]$.

Exercise 2.12. Prove that, in contrast, on the space $C[0, \frac{1}{2}]$ under the norm

$$\|f\|_\infty := \sup_{x\in[0,\frac{1}{2}]} |f(x)|,$$

then $f_n(x) := x^n$ converges to 0 under this norm.

Exercise 2.13. Let X be a non-empty set and d is the discrete metric (see Example 2.4). What does it mean by $x_n \to y$ with respect to this metric?

The above examples show that even if the sequence looks that same (say $\{x^n\}$), they may converge in a certain metric space but could diverge if we replace the metric by another one, and change the space. In the notations above, we denote the L^p-norm on $C[0,1]$ by $\| \; \|_p$, making it clear which p we choose to measure the length. If we also need to emphasise the domain $[0,1]$, we may denote these norms by specifying the domain as well, such as $\|f\|_{L^p[0,1]}$ or $\|f\|_{L^\infty[0,1]}$.

2.1.3. Complete metric space. With the notion of distance, we can also make sense of Cauchy sequences. They are sequences whose terms are getting closer and closer to each other. It is different from convergence which means that the terms are getting closer and closer to a *specific* number/element. However, we have seen that they are equivalent on $\mathbb{R}$ using the usual metric.

However on a general metric space or a normed vector space, a sequence being Cauchy is not always equivalent to convergence! Let's first begin with the definition of Cauchy sequences

Definition 2.18 (Cauchy sequence). Let (X, d) be a metric space. We say $\{x_n\}$ is a **Cauchy sequence** with respect to d if $\forall \varepsilon > 0$, $\exists N > 0$ such that if $m, n \geq N$, then $d(x_m, x_n) < \varepsilon$.

Example 2.19. Consider the space $X = C[0,1]$ under the the L^1-norm $\| \; \|_1$. Let $\{f_n\}$ be a sequence in $C[0,1]$ defined by

$$f_n(x) = \begin{cases} 0 & \text{if } t \in [0, \frac{1}{2}] \\ n(x - \frac{1}{2}) & \text{if } t \in (\frac{1}{2}, \frac{1}{2} + \frac{1}{n}) \\ 1 & \text{if } t \in [\frac{1}{2} + \frac{1}{n}, 1] \end{cases}$$

Readers should sketch the diagram of $\{f_n\}$. The function interpolate 0 and 1 linearly in the sub-interval $(\frac{1}{2}, \frac{1}{2} + \frac{1}{n})$. One can check (by finding the area of a triangle) that for any $m > n$, we have

$$\|f_m - f_n\|_1 = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{m}\right) < \frac{1}{2n}.$$

Therefore, given any $\varepsilon > 0$, one can take $N > \frac{1}{2\varepsilon}$, then whenever $m > n \geq N$, we have

$$\|f_m - f_n\|_1 < \frac{1}{2n} \leq \frac{1}{2N} < \varepsilon.$$

Hence $\{f_n\}$ is a Cauchy sequence in $L^1[0,1]$-norm.

However, one can prove that it does not converge to any limit function in $C[0,1]$ under this norm. Suppose otherwise that there exists $g \in C[0,1]$ such that $\|f_n - g\|_1 \to 0$. Note that

$$\|f_n - g\|_1 = \int_0^{1/2} |0 - g(x)|\ dx + \int_{1/2}^{1/2+1/n} |f_n(x) - g(x)|\ dx + \int_{1/2+1/n}^{1} |1 - g(x)|\ dx.$$

Hence, $\|f_n - g\|_1 \to 0$ implies we have

$$\int_0^{1/2} |g(x)|\ dx = 0$$
$$\lim_{n\to\infty} \int_{1/2}^{1/2+1/n} |f_n(x) - g(x)|\ dx = 0$$
$$\lim_{n\to\infty} \int_{1/2+1/n}^{1} |1 - g(x)|\ dx = 0$$

The first result implies $g(x) = 0$ on $[0, \frac{1}{2}]$ by the continuity of g. The third result shows

$$\int_{1/2}^{1} |1 - g(x)|\ dx = 0 \implies g(x) = 1 \text{ on } [1/2, 1],$$

where we have implicitly used the differentiability (and hence continuity) of $G(t) := \int_t^1 |1 - g(x)|\ dx$, which is guaranteed by the continuity of $|1 - g|$.

However, such a function g is not continuous at $1/2$. This shows $\{f_n\}$ does not converge in L^1-norm to any function in $C[0,1]$, even thought $\{f_n\}$ is a Cauchy sequence in L^1-norm.

Hence, the normed vector space $C[0,1]$ with L^1-norm is not complete.

Remark 2.20. You may have heard about an important result in real/functional analysis that L^p-spaces is complete – that will be true if we enlarge the space to (Lebesgue) integrable functions as well, not only continuity functions. We will discuss more on that in MATH 3043.

Example 2.21. Here we talk about an example of a complete metric space. Let $X = C[0,1]$ under the L^∞-norm $\|\ \|_\infty$. Suppose $\{f_n\}$ is a Cauchy sequence in X under such a norm. We need to find a continuous function $g \in C[0,1]$ such that $\|f_n - g\|_\infty \to 0$.

First observe that for any $x \in [0,1]$, we have

$$|f_m(x) - f_n(x)| \leq \|f_m - f_n\|_\infty .$$

Therefore, given that $\{f_n\}$ is a Cauchy sequence in $\big(C[0,1], \|\ \|_\infty\big)$, for each $x \in [0,1]$ the sequence $\{f_n(x)\}$ of real numbers is also a Cauchy sequence $\mathbb{R}$ in the standard metric/norm. Therefore, for each $x \in [0,1]$, the sequence $\{f_n(x)\}$ converges to a real number, call it $g(x)$, in usual sense.

However, it just proves $f_n(x) \to g(x)$ for any x for a certain function g on $[0,1]$. We have yet to show that $\|f_n - g\|_\infty \to 0$ (not just convergence pointwise as a real number sequence) and $g \in C[0,1]$.

To prove that $\|f_n - g\|_\infty \to 0$, we consider an arbitrary $\varepsilon > 0$, then there exists $N > 0$ such that if $m, n \geq N$ then $\|f_m - f_n\|_\infty < \varepsilon/2$. As discussed before, we then have for any $x \in [0,1]$ and $m, n \geq N$,

$$|f_m(x) - f_n(x)| \leq \|f_m - f_n\|_\infty < \frac{\varepsilon}{2}.$$

Letting $n \to \infty$ while fixing m, we have for any $x \in [0,1]$ and $m \geq N$,

$$|f_m(x) - g(x)| \leq \frac{\varepsilon}{2}.$$

Since this holds true for any $x \in [0,1]$, we have for $m \geq N$,

$$\|f_m - g\|_\infty = \sup_{x \in [0,1]} |f_m(x) - g(x)| \leq \frac{\varepsilon}{2} < \varepsilon.$$

This concludes that $\|f_m - g\|_\infty \to 0$ as $m \to \infty$.

To prove continuity of g, we fix an arbitrary $a \in [0,1]$, and will prove that $g(x) \to g(a)$ as $x \to a$. For any $\varepsilon > 0$, by the fact that $\|f_n - g\|_\infty \to 0$ as $n \to \infty$, there exists $N > 0$ such that if $n \geq N$, then $\|f_n - g\|_\infty < \frac{\varepsilon}{3}$. In particular, it also implies

$$|f_n(x) - g(x)| \leq \|f_n - g\|_\infty < \frac{\varepsilon}{3}$$

for any $n \geq N$.

Since f_N is given to be continuous, there exists $\delta > 0$ such that if $|x - a| < \delta$, then $|f_N(x) - f_N(a)| < \frac{\varepsilon}{3}$. Combining all these, we have that if $|x - a| < \delta$, then

$$|g(x) - g(a)| \leq \underbrace{|g(x) - f_N(x)|}_{<\|f_N - g\|_\infty < \frac{\varepsilon}{3}} + \underbrace{|f_N(x) - f_N(a)|}_{<\frac{\varepsilon}{3} \text{ by continuity of } f_N} + \underbrace{|f_N(a) - g(a)|}_{<\|f_N - g\|_\infty < \frac{\varepsilon}{3}} < \varepsilon.$$

This shows g is continuous at $a \in [0,1]$. As a is arbitrary, it means that $g \in C[0,1]$.

In conclusion, we have proved that whenever $\{f_n\}$ is a Cauchy sequence in $C[0,1]$ under the $\| \ \|_\infty$-norm, then it converges in $\| \ \|_\infty$-norm to a function $g \in C[0,1]$. This shows $C[0,1]$ with the norm $\| \ \|_\infty$ is complete.

Exercise 2.14. Show that $\mathbb{R}^n$ with the standard Euclidean norm is complete.

Exercise 2.15. Consider $\mathbb{R}$ equipped with an "unusual" distance function $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ defined by

$$d(x, y) = \left|e^{-x} - e^{-y}\right|.$$

(a) Check that $(\mathbb{R}, d)$ is a metric space.

(b) Determine whether or not $(\mathbb{R}, d)$ is complete.

Exercise 2.16. Let $p \geq 1$ and $l_p(\mathbb{R})$ be the set of all sequences $\{x_j\}$ of real numbers such that $\sum_{j=1}^\infty |x_j|^p < \infty$. Define the l^p-norm by

$$\|\{x_n\}\|_p := \left(\sum_{j=1}^\infty |x_j|^p\right)^{1/p},$$

and denote its induced metric by d_p. Show that the metric space $\left(l^p(\mathbb{R}), d_p\right)$ is complete.

Exercise 2.17. Show that the space $C^\alpha[0,1]$ with the norm $\| \ \|_\alpha$ defined in Exercise 2.8 is complete.

Exercise 2.18. Let $B[0,1]$ be the set of all bounded functions on $[0,1]$, i.e.

$$B[0,1] := \{f : [0,1] \to \mathbb{R} \mid \exists C > 0 \text{ such that } |f(x)| \leq C \ \ \forall x \in [0,1]\}.$$

For each $f \in B[0,1]$, we define $\|f\|_\infty := \sup_{x\in[0,1]} |f(x)|$. Show that $B[0,1]$ is a normed vector space, and that $\| \ \|_\infty$ is complete.

Definition 2.22 (Banach space)**.** A normed vector space $(V, \| \ \|)$ whose induced metric is complete is called a **Banach space**.

In linear algebra, we learned that an inner product space induces a norm via the relation:

$$\|v\| = \sqrt{\langle v, v\rangle}$$

where $\| \ \|$ denotes the norm, and $\langle \cdot, \cdot \rangle$ denotes the inner product. If on vector space its inner product induces a complete norm or metric, then we call them a **Hilbert space**. So far we have seen very few such an example. The typical example of a Hilbert space is $\mathbb{R}^n$, whose inner product is the usual dot product, and so the induced norm is the usual Euclidean magnitude. More examples can be seen in MATH3043.

On $C[0,1]$, although one can define an inner product by

$$\langle f, g\rangle := \int_0^1 f(x)g(x)\,dx,$$

however, its induced norm would be

$$\|f\| = \sqrt{\int_0^1 |f(x)|^2 \ dx}$$

From Example 2.19, we have already known that such a normed vector space is incomplete. Therefore, $C[0,1]$ equipped with the above inner product is not a Hilbert space.

2.2. Open and Closed Sets

2.2.1. Definitions. On a metric space (X, d) with the notion of distance, one can extend the notion of open and closed sets in $\mathbb{R}$ to arbitrary metric spaces. Recall a set $U \subset \mathbb{R}$ is open if for any $x \in U$, there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset U$. In order to extend this notion to metric spaces, we first need to make sense of what are "open intervals" on a metric space. Another way to view an open interval $(x-\varepsilon, x+\varepsilon)$ in $\mathbb{R}$ is the set $\{y \in \mathbb{R} : |y-x| < \varepsilon\}$, i.e. the set of real numbers within ε-distance from x. The natural candidate to use in place of an open interval is a ball, denote and defined to be:

$$B_\varepsilon(x) := \{y \in X : d(y,x) < \varepsilon\}.$$

If we want to emphasise the choice of metric d, then we can write $B_\varepsilon^d(x)$.

In $\mathbb{R}^2$ under the standard metric d_2, the ball $B_\varepsilon^{d_2}(0)$ is a solid circle (without "boundary") centered at the origin with radius ε. If a different metric is used, say d_1, then it can have a different shape:

$$B_\varepsilon^{d_1}(0) = \{(x,y) \in \mathbb{R}^2 : |x| + |y| < \varepsilon\},$$

which is a rotated solid square without "boundary".

Exercise 2.19. Sketch the shape of $B_\varepsilon^{d_\infty}(0)$ in $\mathbb{R}^2$ where d_∞ is the metric:

$$d_\infty\big((x_1,y_1),(x_2,y_2)\big) := \max\{|x_1-x_2|, |y_1-y_2|\}.$$

Exercise 2.20. Consider $C[0,1]$ under the norm $\|\ \|_\infty$, and a given function $g \in C[0,1]$. What is the meaning of $f \in B_\varepsilon(g)$ using the induced metric?

Now we are ready to define what are open and closed sets:

Definition 2.23 (Open sets and closed sets on a metric space). Let (X, d) be a metric space. We say $U \subset X$ is **open** (or more precisely **open in** X) if $\forall x \in X$, $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subset U$. We say $E \subset X$ is closed if $E^c := X - E$ is open in X.

Example 2.24. Consider any metric space (X, d). We want to show that ball $B_\varepsilon(a)$ is open. Take any $x \in B_\varepsilon(a)$, we need to find small enough $\delta > 0$ such that $B_\delta(x) \subset B_\varepsilon(a)$. A quick sketch of a diagram should tell us that the right δ should be

$$\delta := \frac{1}{2}\big(\varepsilon - d(x,a)\big).$$

To argue this, we take any element $y \in B_\delta(x)$, we need to prove $y \in B_\varepsilon(a)$. First we have

$$d(x,y) < \delta < \varepsilon - d(x,a) \implies d(y,a) \le d(x,y) + d(x,a) < \varepsilon.$$

Therefore, $y \in B_\varepsilon(a)$. This shows $B_\delta(x) \subset B_\varepsilon(a)$. Therefore $B_\varepsilon(a)$ is open.

Exercise 2.21. Prove that in $\mathbb{R}^2$ under the d_∞-metric, the set

$$U := \{(x,y) \in \mathbb{R}^2 : x > 1\}$$

is open. How about using d_1-metric?

Exercise 2.22. What are open sets on a metric space X using the discrete metric?

Example 2.25. Let's look at some more "abstract" examples. Consider $C[0,1]$, and two different norms $\|\ \|_1$ and $\|\ \|_\infty$. Consider the open balls $B_1^{d_1}(0) := \{f \in C[0,1] : \|f\|_1 < 1\}$ and $B_1^{d_\infty}(0) := \{f \in C[0,1] : \|f\|_\infty < 1\}$ with respect to each norms. We have already proved that $B_1^{d_1}(0)$ is open under the norm $\|\ \|_1$. Now we discuss whether it is open or not under the norm $\|\ \|_\infty$.

"$B_1^{d_1}(0)$ being open under the norm $\| \ \|_\infty$" means that for any $f \in B_1^{d_1}(0)$, there exists $\varepsilon > 0$ such that $B_\varepsilon^{d_\infty}(f) \subset B_1^{d_1}(0)$. Our geometric intuition tells us that it should be true, because if $\int_0^1 |f(x)| \, dx < 1$, and g is a function in the ε-tubular neighborhood around f, then one should expect that $\int_0^1 |g(x)| \, dx < 1$ too if ε is chosen to be very small. To prove it precisely, for any $f \in B_1^{d_1}(0)$, we pick

$$\varepsilon = \frac{1}{2}\left(1 - \int_0^1 |f(x)| \, dx\right) > 0.$$

Then for any $g \in B_\varepsilon^{d_\infty}(f)$, we have

$$\sup_{x \in [0,1]} |g(x) - f(x)| < \varepsilon.$$

This shows

$$\begin{aligned} \int_0^1 |g(x)| - |f(x)| \, dx &\leq \int_0^1 |g(x) - f(x)| \, dx \\ &\leq \int_0^1 \varepsilon \, dx = \varepsilon \\ &< 1 - \int_0^1 |f(x)| \, dx. \end{aligned}$$

This proves $\int_0^1 |g(x)| \, dx < 1$, and hence $g \in B_1^{d_1}(0)$. This completes the proof that $B_1^{d_1}(0)$ is open in the normed vector space $\big(C[0,1], \| \ \|_\infty\big)$.

However, one can prove that $B_1^{d_\infty}(0)$ is not open under the norm $\| \ \|_1$. To argue this, we will show that even though the zero function $0 \in B_1^{d_\infty}(0)$, but $B_\varepsilon^{d_1}(0) \not\subset B_1^{d_\infty}(0)$ for any $\varepsilon > 0$. In other words, we need to show for any $\varepsilon > 0$, there is a continuous function $f \in B_\varepsilon^{d_1}(0)$ that is not in $B_1^{d_\infty}(0)$. In other words, we need to find $f \in C[0,1]$ such that

$$\int_0^1 |f(x)| \, dx < \varepsilon \ \text{ whereas } \sup_{x \in [0,1]} |f(x)| \geq 1.$$

Such a function should be easy to construct – for example, an f whose graph is a triangle with height 2 and base $< \frac{\varepsilon}{2}$.

Exercise 2.23. Show that if U is an open set in $\big(C[0,1], \| \ \|_1\big)$, then it is also open in $\big(C[0,1], \| \ \|_\infty\big)$.

Exercise 2.24. Suppose d_1 and d_2 are two metrics on X, and there exists a constant $C > 0$ such that

$$d_1(x,y) \leq C d_2(x,y) \ \ \forall x, y \in X.$$

Determine which one of the following is correct:

- U is open in (X, d_1) implies U is open in (X, d_2).
- U is open in (X, d_2) implies U is open in (X, d_1).

Example 2.26. We want to prove that on a metric space (X, d) the set

$$C_\varepsilon(a) := \{x \in X : d(x,a) \leq \varepsilon\}$$

is closed. Consider its complement equals to

$$X - C_\varepsilon(a) = \{x \in X : d(x,a) > \varepsilon\}.$$

We need to show that $X - C_\varepsilon(a)$ is open. For any $x \in X - C_\varepsilon(a)$, we pick $\delta := \frac{1}{2}\big(d(x,a) - \varepsilon\big)$, then we claim $B_\delta(x) \subset X - C_\varepsilon(a)$. To prove the claim, we pick $y \in B_\delta(x)$, then

$$d(x,y) < \delta < d(x,a) - \varepsilon \implies \varepsilon < d(x,a) - d(x,y) \leq d(y,a).$$

This shows $y \in X - C_\varepsilon(a)$, completing the claim.

Hence $X - C_\varepsilon(a)$ is an open set, so by definition $C_\varepsilon(a)$ is closed.

2.2.2. Properties of open and closed sets. We have proved a number of properties for open and closed sets in $\mathbb{R}$. Many of these properties can be carried over to metric spaces with almost the same proof, *mutatis mutandis* – simply replacing $|x - y|$ by $d(x,y)$, and open intervals $(x - \varepsilon, x + \varepsilon)$ by $B_\varepsilon(x)$.

Proposition 2.27. *On a metric space (X,d), we have:*

*(1) The union of **any collection** of open sets is open.*
*(2) The intersection of **finitely** many open sets is open.*
*(3) The union of **finitely** many closed sets is closed.*
*(4) The intersection of **any collection** of closed sets is closed.*

Proposition 2.28. *On a metric space (X,d), the following are equivalent:*

(1) S is closed in X (i.e. S^c is open in X).
(2) Any converging sequence $\{x_n\}_{n=1}^\infty$ in S has its limit $\lim\limits_{n\to\infty} x_n \in S$.

Exercise 2.25. Prove the above propositions.

However, it is generally not true that any open set in a metric space is a disjoint union of open balls. It is a special property for $\mathbb{R}$ only – the proof made use of supremum and infimum, which we do always have in a general metric space. Even in $\mathbb{R}^2$ with the usual metric, one can find many open sets which are not disjoint unions of open balls. It is because the disjoint union of two or more open ball is *disconnected* – a notion we will define later, whereas many open sets in $\mathbb{R}^2$ are *connected* (such as open squares, open triangles, etc).

Proposition 2.28 relates the geometric viewpoint and the analytic viewpoint of closed sets. It also motivates the following notions which are closely related to each other.

Definition 2.29 (Limit point of a set, boundary of a set)**.** Let (X,d) be a metric space, and $S \subset X$. We say $a \in X$ is a **limit point of** S if $\forall \varepsilon > 0$, the punctured ball $B_\varepsilon(a) - \{a\}$ has at least one element in S. The set of all limit points of S is called the **limit set of** S, denoted by S'

We say the $b \in X$ is a **boundary point of** S if $\forall \varepsilon > 0$, the ball $B_\varepsilon(b)$ contains at least one element in S, and at least one element in $X - S$. We denote the set of all boundary points of S by ∂S, and call it the **boundary of** S.

Example 2.30. Consider $\mathbb{R}^2$ with the standard metric, and the set $S = B_1(0) \cup \{(2,0)\}$. Then, for any $\mathbf{x} \in \mathbb{R}^2$ with $|\mathbf{x}| = 1$, it is clearly that any open $B_\varepsilon(\mathbf{x})$ overlaps with both S and S^c, so any such $\mathbf{x}$ is a boundary point of S. Also, any punctured ball $B_\varepsilon(\mathbf{x}) - \{\mathbf{x}\}$ clearly overlaps with S (just draw a picture to see it), so any such $\mathbf{x}$ is also a limit point of S.

However, the point $(2,0)$ is only a boundary point but not a limit point. For any $\varepsilon > 0$, the open ball $B_\varepsilon\big((2,0)\big)$ contains a point in S, namely $(2,0)$, and points outside S.

Hence $(2, 0)$ is a boundary point of S. It is not a limit point of S because the punctured ball $B_{1/2}\big((2, 0)\big) - \{(2, 0)\}$ contains no point in S.

On the other hand, any point $\mathbf{x}$ with $|\mathbf{x}| < 1$ is a limit point – try draw a diagram with a punctured ball $B_\varepsilon(\mathbf{x}) - \{\mathbf{x}\}$ and see. However, such an $\mathbf{x}$ is not a boundary point, we the ball $B_{\frac{1}{2}(1-|\mathbf{x}|)}(\mathbf{x})$ is completely contained in S.

Exercise 2.26. Consider $\mathbb{R}$ under the usual metric. Determine the boundary point(s) and limit point(s) of $\mathbb{Q}$. How about the set below?

$$S := \big(\mathbb{Q} \cap (-1, 1)\big) \cup \mathbb{Z}$$

Exercise 2.27. Let (X, d) be a metric space, and $S \subset X$ is a non-empty set. Prove that $p \in X$ is a limit point of S if and only if there exists a sequence $\{x_n\}$ in $S - \{p\}$ whose elements are mutually distinct (i.e. $x_n \neq x_m$ whenever $m \neq n$) such that $x_n \to p$. [As such, every ball $B_\varepsilon(p)$ contains infinitely many elements inside S if p is a limit point of S]

Remark 2.31. There is a subtle difference between limit points of a *set*, and limit points of a *sequence* LIM $\{x_n\}$ discussed in the previous chapter. For limit points of a sequence, we count repeating elements with different n's as *different* items. For example in the sequence $x_n = (-1)^n$, we count $x_1, x_3, x_5, x_7, \cdots$ as infinitely many items even those they are all -1. Therefore, since $(-1 - \varepsilon, -1 + \varepsilon)$ contains infinitely many "items" in the sequence for any $\varepsilon > 0$, -1 is a limit of the sequence $\{(-1)^n\}$. However, when viewing $\{(-1)^n\}_{n\in\mathbb{N}}$ as a set, i.e. $\{-1, 1\}$, then it has no limit point. These names could sometimes cause confusions, so some authors call a *limit point of a sequence* to be a *cluster point* or an *accumulation point* of a sequence instead, and only use the term *limit points* for a set. We will call both limit points in this course because it is believed that students in MATH 2043 would not be confused over this subtle difference.

Although a limit point of a set may not be a boundary point (or vice versa), they are closely related to each other.

Proposition 2.32. *Let (X, d) be a metric space, and S be a non-empty subset of X. Then, we have*

$$S' \cup S = \partial S \cup S.$$

Proof. We first show $S' \cup S \subset \partial S \cup S$, and for that it suffices to prove $S' \subset \partial S \cup S$. Given any $y \in S'$, we consider two cases: $y \in S$ or $y \notin S$. If $y \in S$, then clearly $y \in \partial S \cup S$. If $y \notin S$, then for any $\varepsilon > 0$, the punctured ball $B_\varepsilon(y) - \{y\}$ must contain a point in S for y being a limit point. Hence $B_\varepsilon(y)$ contains a point in S too. Also, $B_\varepsilon(y)$ contains a point in S^c, namely y itself. This shows $y \in \partial S$. This concludes $S' \subset \partial S \cup S$.

Now we prove $\partial S \cup S \subset S' \cup S$, and it suffices to prove $\partial S \subset S' \cup S$. For any $y \in \partial S$, we again have two cases: $y \in S$ (then trivially $y \in S' \cup S$), or $y \notin S$. In the latter case, $(B_\varepsilon(y) - \{y\}) \cap S = B_\varepsilon(y) \cap S$, which is not empty for any $\varepsilon > 0$ as $y \in \partial S$. This proves $y \in S'$. This concludes that $\partial S \cup S \subset S' \cup S$. $\square$

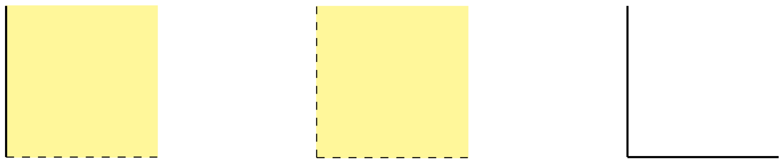
Definition 2.33 (Interior and closure)**.** Let (X, d) be a metric space, and $S \subset X$ be a non-empty set. We denote $\overline{S} := \partial S \cup S$, which is also $S' \cup S$ according to the previous proposition. We call $\overline{S}$ the **closure** of S.

Furthermore, we denote $S^\circ := S - \partial S$, and call it the **interior** of S. Some author denote it by Int(S).

Take the set

$$W := \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } y > 0\}$$

as an example (see Figure 2.1) in $\mathbb{R}^2$ with the standard metric. The interior and boundary of W are the sets shown in the figure. The closure $\overline{W}$ is simply the union of W and ∂W.



(a) W **(b)** W° **(c)** ∂W

Figure 2.1. The set $W = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } y > 0\}$ and its interior and boundary sets

Proposition 2.34. *Let S be a subset in a metric space (X, d), then we have $\overline{S} = S$ if and only if S is closed.*

Proof. For the ($\Longrightarrow$)-part, given that $\overline{S} = S$, then by Proposition 2.32 we know $S' \subset S$, i.e. every limit point of S is contained in S. By the virtue of Proposition 2.28, we consider any converging sequence $\{x_n\}$ in S, and let $y \in X$ be the limit of $\{x_n\}$. We need to argue that $y \in S$ as well. Any sequence $\{x_n\}$ either has a subsequence with all distinct elements, or has a constant subsequence (think about why?). The former case shows $y \in S'$ by Exercise 2.27. In the latter case that constant subsequence must be $\{y, y, y, \cdots\}$ (as it must converge to y), and hence $y \in S$. In both cases, we have $y \in S$ as $S' \subset S$. This proves S is closed.

Now for the ($\Longleftarrow$)-part, given that S is closed, we have S^c being open. We will argue that $\partial S \subset S$, then $\overline{S} = \partial S \cup S = S$. To prove this, pick any $x \in \partial S$. Assume that $x \notin S$, then we have $x \in S^c$. As S^c is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset S^c$, which means $B_\varepsilon(x) \cap S = \emptyset$. It contradicts to the fact that $x \in \partial S$. Therefore, we must have $x \in S$, completing the proof. $\square$

Example 2.35. Under the usual metric of $\mathbb{R}$, we can see $\overline{Q} = \mathbb{R}$. It is because for any $x \in \mathbb{R}$, one can construct a strictly increasing sequence $\{r_n\}$ in $\mathbb{Q}$ such that $r_n \to x$. By Exercise 2.27, we have $x \in \mathbb{Q}'$, so we have $\mathbb{R} \subset \mathbb{Q}'$, which implies $\overline{Q} = \mathbb{Q}' \cup \mathbb{Q} = \mathbb{R}$.

Any subset S of a metric space (X, d) such that $\overline{S} = X$ is called a **dense set in** X.

Example 2.36. You may very well thought that the closure of an open ball $B_\varepsilon(x)$ in a metric space (X, d) is the closed ball $\{y \in X : d(x, y) \leq \varepsilon\}$. Although it is the case when $X = \mathbb{R}^n$ and d is the usual metric, it is in general not true.

Take d as the discrete metric of X. Consider $B_1(x)$ where x is any element in X. By the definition of d, any $y \in X$ such that $d(x, y) < 1$ means that $d(x, y) = 0$ and so $x = y$. This shows $B_1(x) = \{x\}$. Hence, if $B_1(a) = \{a\}$, where $a \in X$, intersects both $B_1(x)$ and $B_1(x)^c$, that would mean $a \in B_1(x)$ and $a \in B_1(x)^c$ which is impossible. Therefore, there is no boundary point for $B_1(x)$, and so $\overline{B_1(x)} = B_1(x) = \{x\}$. However, $C_1(x) = \{y \in X : d(x, y) \leq 1\} = X$.

Exercise 2.28. Show that, however, we always have $\overline{B_\varepsilon(x)} \subset C_\varepsilon(x)$ for any x in a metric space (X, d) and any $\varepsilon > 0$.

Exercise 2.29. Show that the following are equivalent:

(1) $y \in \overline{S}$

(2) There exists a sequence $\{x_n\}$ in S converging to y.

(3) For any $\varepsilon > 0$, $B_\varepsilon(y) \cap S \neq \emptyset$.

Exercise 2.30. Prove that S° is open, and ∂S and $\overline{S}$ are closed.

Exercise 2.31. Show that $x \in S^\circ$ if and only if $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subset S$.

Exercise 2.32. Given any two sets U and V in a metric space (X, d), show that:

(a) $\partial(U \cup V) \subset \partial U \cup \partial V$

(b) $\partial(\partial U) \subset \partial U$.

(c) $(U \cap V)^\circ = U^\circ \cap V^\circ$

(d) If $U \subset V$, then $\overline{U} \subset \overline{V}$ and $U^\circ \subset V^\circ$.

(e) $\partial U = \overline{U} - U^\circ$

Exercise 2.33. Show that on $\mathbb{R}$ with the usual metric, and a bounded set $S \subset \mathbb{R}$, we must have $\sup S \in \overline{S}$.

Exercise 2.34. Let (X, d) be a metric space, and $S \subset X$. Show that the following are equivalent:

(1) $\overline{X - S} = X$

(2) $S^\circ = \emptyset$

Exercise 2.35. Prove that:

$$\overline{S} = \bigcap_{\substack{S \subset E \\ E \text{ is closed}}} E.$$

In other words, $\overline{S}$ is the intersection of all closed sets containing S. It is also the smallest closed set in X containing S.

Exercise 2.36. Suppose (X, d) is a complete metric space. Let $E \subset X$ be a closed set. Show that (E, d) is also a complete metric space.

2.2.3. Continuity of functions. Consider a function $f : X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y). With the notions of distance, one can then make sense of limit and continuity of f. However, unlike in elementary calculus where most functions were defined on intervals, we need to concern about about whether it is still valid to say "$x \to a$" when talking about $\lim\limits_{x \to a} f(x)$, as the domain X could have complicated shapes, and a could be isolated. It may not be possible to make sense of "approaching" a. Therefore, when talking about limit of functions on metric spaces, we often need to assume a is a limit point.

Definition 2.37 (Limit of a function on metric spaces)**.** Consider a function $f : S \to Y$ from a subset S of a metric space (X, d_X) to another metric space (Y, d_Y). Let $p \in X$ be a limit point of S, we say $f(x) \to q$ as $x \to p$ in S, or $\lim_{x \to p, x \in S} f(x) = q$, if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $0 < d_X(x, p) < \delta$ and $x \in S$, then $d_Y(f(x), q) < \varepsilon$.

We may simply say "$f(x) \to q$ as $x \to p$" without the clause "in S", or $\lim_{x \to p} f(x) = q$ without mentioning $x \in S$ if the set S is clear from the context.

In single-variable calculus, we had the notion of left- and right-limits. Now with the above unified notation of limit, we can simply regard $\lim_{x \to a^-} f(x)$ as $\lim_{x \to a, x \in (a-\delta, a)} f(x)$ and $\lim_{x \to a^+} f(x)$ as $\lim_{x \to a, x \in (a, a+\delta)} f(x)$. Here $\delta > 0$ is any small number such that $(a - \delta, a)$ or $(a, a + \delta)$ are in the domain of f.

Example 2.38. Consider the function

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}.$$

We all learned in Calculus 1 that $\lim_{x \to a} \chi_Q(x)$ does not exist for any $a \in \mathbb{R}$. However, we have

$$\lim_{x \to a, x \in \mathbb{Q}} \chi_{\mathbb{Q}}(x) = 1 \quad \text{and} \quad \lim_{x \to a, x \in \mathbb{R} - \mathbb{Q}} \chi_{\mathbb{Q}}(x) = 0.$$

Exercise 2.37. Prove from the definition that

$$\lim_{x \to a, x \in \mathbb{Q}} \chi_{\mathbb{Q}}(x) = 1 \quad \text{and} \quad \lim_{x \to a, x \in \mathbb{R} - \mathbb{Q}} \chi_{\mathbb{Q}}(x) = 0.$$

Suppose $S \subset \mathbb{R}$ has a limit point $a \in \mathbb{R}$ and contains finitely many rational numbers. What is the limit below?

$$\lim_{x \to a, x \in S} \chi_{\mathbb{Q}}(x)$$

Exercise 2.38. As in Calculus 1, we can detect the limit of a function $f(x)$ by substituting a converging sequence $\{x_n\}$ into x. On metric spaces, we have a similar result (after some minor modification). Suppose p is a limit point of S in a metric space (X, d_X), and $f : S \to Y$ is a function from S into a metric space (Y, d_Y). Prove that the following are equivalent:

(1) $\lim_{x \to p, x \in S} f(x) = q$

(2) for any sequence $\{x_n\}$ in S with the property that $x_n \neq p$ for any n and $x_n \to p$, we have $f(x_n) \to q$ as a sequence in (Y, d_Y).

Use this result to do the previous exercise.

Although when dealing with the limit of a function on a metric space one needs to be careful about the limit point issue, such a hassle is not needed when it comes to the *continuity* of a function – we can still talk about continuity at a point p even if p is not a limit point. However, continuity of a function on a metric space, just like limit of a function, is depending on the set S under consideration. Here is the precise definition:

Definition 2.39 (Continuity of a function on metric spaces)**.** Let $f : S \to Y$ be a function from S in a metric space (X, d_X) into another metric space (Y, d_Y). Let $p \in S$, then we say f is **continuous at** p if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $d_X(x, p) < \delta$ and $x \in S$, then $d_Y(f(x), f(p)) < \varepsilon$.

We say f **is continuous on** S if f is continuous at p for any $p \in S$.

Remark 2.40. Here p needs not to be a limit point of S, but if it is, then f being continuous at $p \in S$ is equivalent to saying

$$\lim_{x \to p, x \in S} f(x) = f(p).$$

Remark 2.41. Continuity is now also domain-specific. Take $\chi_{\mathbb{Q}}$ as an example. If one regard it as $\chi_{\mathbb{Q}} : \mathbb{Q} \to \mathbb{R}$, then by

$$\lim_{x \to 1/2, x \in \mathbb{Q}} \chi_{\mathbb{Q}}(x) = 1 = \chi_{\mathbb{Q}}(1/2),$$

we have that $\chi_{\mathbb{Q}}$ is continuous at $\frac{1}{2}$. However, when we view it as $\chi_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$, then $\chi_{\mathbb{Q}}$ is not continuous at $\frac{1}{2}$.

Example 2.42. For functions from $\mathbb{R}^n$ (or its subset) to $\mathbb{R}^m$ (both under usual metrics), we can express them in the form of

$$F(x_1, \cdots, x_n) = \big(f_1(x_1, \cdots, x_n), \cdots, f_m(x_1, \cdots, x_n)\big)$$

where $f_j(x_1, \cdots, x_n)$ are functions from the domain of F to $\mathbb{R}$.

If F is continuous at $(a_1, \cdots, a_n)$, then it means $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if

$$\sqrt{\sum_{j=1}^{n} (x_j - a_j)^2} < \delta,$$

then

$$\sqrt{\sum_{i=1}^{m} |f_i(x_1, \cdots, x_n) - f_i(a_1, \cdots, a_n)|^2} < \varepsilon,$$

which implies for each $i = 1, \cdots, m$, we have

$$|f_i(x_1, \cdots, x_n) - f_i(a_1, \cdots, a_n)| < \varepsilon.$$

Therefore, each f_i is continuous at $(a_1, \cdots, a_n)$. It is also easy to prove that if each f_i is continuous at $(a_1, \cdots, a_n)$, then so does F. As such, functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be handled component-wise.

Examples of multivariable functions from $\mathbb{R}^n$ to $\mathbb{R}$ include polynomials. It is because monomials x_j is continuous: $\forall \varepsilon > 0$, simply choose $\delta = \varepsilon$, then whenever

$$\sqrt{\sum_{j=1}^{n} (x_j - a_j)^2} < \delta \implies |x_j - a_j| < \delta = \varepsilon.$$

Each x_j is a continuous function, so their products and sums are continuous as well (the proof of this fact is identical to the $\mathbb{R} \to \mathbb{R}$ case).

Exercise 2.39. Let (X, d) be a metric, and $f, g : X \to \mathbb{R}$ are continuous at $p \in X$. Prove that $f + g$, $f - g$, fg are all continuous at p, and if furthermore we have $g(p) \neq 0$, then $\frac{f}{g}$ is also continuous at p.

Example 2.43. Here is one less trivial example of a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$. Let

$$f(x,y) = \begin{cases} \frac{x^5+2x^2y+y^4}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

We expect that $f(x,y) \to 0$ as $(x,y) \to (0,0)$, so we start by estimating

$$|f(x,y) - f(0,0)| \leq \frac{|x|^5 + 2\,|x|^2\,|y| + |y|^4}{x^2+y^2}.$$

If $0 < \sqrt{x^2+y^2} < \delta$, then we have $|x| < \delta$, $|y| < \delta$ and hence

$$\begin{aligned} \frac{|x|^5}{x^2+y^2} &\leq |x|^3 \cdot \frac{|x|^2}{x^2+y^2} \leq |x|^3 < \delta^3 \\ \frac{2\,|x|^2\,|y|}{x^2+y^2} &= 2\,|x| \cdot \frac{|x|\,|y|}{x^2+y^2} \leq |x| < \delta \\ \frac{|y|^4}{x^2+y^2} &\leq |y|^2 \cdot \frac{|y|^2}{x^2+y^2} \leq |y|^2 < \delta^2 \end{aligned}$$

To summarize, given any $\varepsilon > 0$, we just take $\delta < \min\{1, \varepsilon/3\}$, then if $0 < \sqrt{x^2+y^2} < \delta$, we have

$$|f(x,y) - f(0,0)| \leq \delta^3 + \delta + \delta^2 < \delta + \delta + \delta \leq \varepsilon.$$

This shows f is continuous at $(0,0)$. It is also continuous at all other points as it is a quotient of polynomials and $x^2+y^2 \neq 0$ when $(x,y) \neq (0,0)$.

Example 2.44. Let (X,d) be a metric space, and $A \subset X$ is a non-empty set. We define $d_A : X \to \mathbb{R}$ as:

$$d_A(x) := \inf\{d(x,a) : a \in A\}.$$

It measures the "shortest" distance from x to the set A. We are going to prove that d_A is continuous on X. We claim that

$$|d_A(x) - d_A(y)| \leq d(x,y) \;\; \forall x, y \in X.$$

To prove this claim, we consider an arbitrary $a \in A$, then by triangle inequality we have

$$d(x,a) \leq d(x,y) + d(y,a).$$

Taking infimum over all $a \in A$, we have

$$\inf_{a\in A} d(x,a) \leq \inf_{a \in A} \big(d(x,y) + d(y,a)\big) = d(x,y) + \inf_{a\in A} d(y,a).$$

This shows $d_A(x) \leq d(x,y) + d_A(y)$. Swapping x and y, we also have $d_A(y) \leq d(y,x) + d_A(x)$. These two inequalities together imply our claim.

With this claim, now we are given any $\varepsilon > 0$, we just choose $\delta = \varepsilon$, then whenever $d(x,x_0) < \delta$, we have

$$|d_A(x) - d_A(x_0)| \leq d(x,x_0) < \delta = \varepsilon.$$

Hence, d_A is continuous at x_0. Since x_0 can be arbitrarily chosen in X, we conclude that d_A is continuous on X.

Exercise 2.40. Suppose (X,d_X), (Y,d_Y) and (Z,d_Z) are metric spaces. Suppose $f : X \to Y$ and $g : Y \to Z$ are functions such that

(1) f is continuous at p, and

(2) g is continuous at $f(p)$,

then $g \circ f : X \to Z$ is continuous at p.

Exercise 2.41. Prove that $d_A(x) = 0$ if and only if $x \in \overline{A}$.

The above exercise shows that if $\gamma(t) : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ is continuous at $t = 0$, and $f : \mathbb{R}^n \to \mathbb{R}$ is continuous at $\gamma(0)$, then $f \circ \gamma$ is continuous at $t = 0$, i.e. $f(\gamma(t)) \to f(\gamma(0))$ as $t \to 0$. In MATH 2023, we often use this fact to detect the discontinuity of a multivariable function. If one can find two continuous curves $\gamma_1(t)$ and $\gamma_2(t)$ such that $\gamma_1(0) = \gamma_2(0) = p$ while $f \circ \gamma_1$ and $f \circ \gamma_2$ approach two different limits as $t \to 0$, then such a function f must be discontinuous at p.

So far, we define continuity of functions using limit. There is an equivalent form of continuity which is related to open sets:

Proposition 2.45. *Let $f : X \to Y$ be a function between metric spaces (X, d_X) and (Y, d_Y). Then the following are equivalent:*

(1) f is continuous on X.

(2) $f^{-1}(U)$ is open in X for any open set U in Y.

Proof. To prove (1) $\implies$ (2), suppose f is continuous on X, and we consider $f^{-1}(U)$ for an open set U in Y. We need to show $f^{-1}(U)$ is open in X. Pick any point $x_0 \in f^{-1}(U)$, we have $f(x_0) \in U$. Since U is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(f(x_0)) \subset U$, which is equivalent to say that for any $y \in Y$ such that $d_Y(y, f(x_0)) < \varepsilon$, we have $y \in U$. Now we consider the continuity of f: there exists $\delta > 0$ such that if $d_X(x, x_0) < \delta$, then $d_Y(f(x), f(x_0)) < \varepsilon$, which implies $f(x) \in U$ (by treating $f(x)$ as y), i.e. $x \in f^{-1}(U)$. In other words, whenever $x \in B_\delta(x_0)$, we have $x \in f^{-1}(U)$. That is equivalent to saying $B_\delta(x_0) \subset f^{-1}(U)$. Since x_0 is an arbitrary point in $f^{-1}(U)$, this shows $f^{-1}(U)$ is open in X.

For (2) $\implies$ (1), we need to prove f is continuous at x_0 for any $x_0 \in X$. For any $\varepsilon > 0$, we consider the ball $B_\varepsilon(f(x_0))$ in Y, which is an open set. Therefore, by (2), $f^{-1}(B_\varepsilon(f(x_0)))$ is open in X. This set certainly contains x_0, so by openness there exists $\delta > 0$ such that $B_\delta(x_0) \subset f^{-1}(B_\varepsilon(f(x_0)))$. That means whenever $d_X(x, x_0) < \delta$, then $x \in f^{-1}(B_\varepsilon(f(x_0)))$, meaning that $d_Y(f(x), f(x_0)) < \varepsilon$. It is exactly the meaning of f being continuous at x_0. As x_0 is arbitrary, we conclude that f is continuous on X. $\square$

Remark 2.46. By the property $f^{-1}(Y - U) = X - f^{-1}(U)$, (2) in Proposition 2.45 is also equivalent to saying $f^{-1}(C)$ is closed in X for any closed set C in Y.

Proposition 2.45 is particularly useful for showing a certain set is open or closed, especially those that are hard to sketch.

Example 2.47. Since $f(x, y) = x^2 + 4xy + y^3 - x^3y^2 : \mathbb{R}^2 \to \mathbb{R}$ is continuous on $\mathbb{R}^2$, and $(0, \infty)$ is open in $\mathbb{R}$, the pre-image

$$f^{-1}\big((0, \infty)\big) = \{(x, y) \in \mathbb{R}^2 : x^2 + 4xy + y^3 - x^3y^2 > 0\}$$

is open in $\mathbb{R}^2$. On the other hand, the single set $\{0\}$ is closed in $\mathbb{R}$, hence

$$f^{-1}(0) = \{(x, y) \in \mathbb{R}^2 : x^2 + 4xy + y^3 - x^3y^2 = 0\}$$

is closed in $\mathbb{R}^2$.

This explains why when we have strict inequalities in the definition of a set, the set is often open.

Example 2.48. Using Proposition 2.45, one can easily prove that if $f : X \to Y$ and $g : Y \to Z$ are continuous everywhere, then so does $g \circ f : X \to Z$: take any open set U in Z, then

$$(g \circ f)^{-1}(U) = f^{-1}\big(g^{-1}(U)\big).$$

Continuity of g shows $g^{-1}(U)$ is open in Y, then continuity of f proves $f^{-1}\big(g^{-1}(U)\big)$ is open in X.

However, it does not fully replace the proof in Exercise 2.40, which proves the continuity of $g \circ f$ at a specific point.

Example 2.49. Recall that the d_A function in Example 2.44 is continuous on X for any subset $A \subset X$. Therefore,

$$N_\varepsilon(A) := d_A^{-1}\big((-\varepsilon, \varepsilon)\big) = \{x \in X : -\varepsilon < d_A(x) < \varepsilon\}$$

is an open set in X. Since d_A is non-negative, the above set is just the same as

$$d_A^{-1}\big([0, \varepsilon)\big) = \{x \in X : d_A(x) < \varepsilon\}.$$

But $[0, \varepsilon)$ is not open in $\mathbb{R}$, so we cannot make use of Proposition 2.45. The set $N_\varepsilon(A)$ is the set of all points in X which is ε-close from A. We usually call it the ε-neighborhood of A.

Exercise 2.42. Let A and B be disjoint non-empty closed sets in a metric space (X, d). Define d_A and d_B as in the previous example. We let:

$$f(x) := \frac{d_A(x)}{d_A(x) + d_B(x)}.$$

Show that f is continuous on X. Hence show that there exist open sets U and V in X such that $U \cap V = \emptyset$, $A \subset U$, and $B \subset V$. [Hint: consider sets of the form f^{-1}(open set).]

Exercise 2.43. Let X be a non-empty set with two metrics d_1 and d_2. Suppose there exists a constant $C > 0$ such that

$$d_1(x, y) \leq C d_2(x, y) \quad \forall x, y \in X.$$

Denote by $\mathrm{id} : X \to X$ the identity map $\mathrm{id}(x) = x$. Which of the following is true?

- $\mathrm{id} : (X, d_1) \to (X, d_2)$ is continuous on X.
- $\mathrm{id} : (X, d_2) \to (X, d_1)$ is continuous on X.

Exercise 2.44. Let $f : X \to Y$ be a function between two metric spaces (X, d_X) and (Y, d_Y). Prove that the following is equivalent:

(1) f is continuous on X.
(2) For any $A \subset X$, we have $f(\overline{A}) \subset \overline{f(A)}$.
(3) For any $B \subset Y$, we have $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$.

Exercise 2.45. Let $f, g : X \to Y$ be two continuous functions between two metric spaces (X, d_X) and (Y, d_Y). Suppose there exists a dense set $D \subset X$ (i.e. $\overline{D} = X$) such that $f(x) = g(x)$ for any $x \in D$. Prove that $f(x) = g(x)$ for any $x \in X$.

2.2.4. Baire Category Theorem. The following Thomae function $f : \mathbb{R} \to \mathbb{R}$ is one that is continuous at every irrational number, and discontinuous at every rational number:

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1] - \mathbb{Q} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in [0,1),\ m, n \in \mathbb{N} \text{ and } \frac{m}{n} \text{ in simplest form} \end{cases}.$$

Then we extend f periodically to the whole real line.

One may ask whether there exists a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at every $r \in \mathbb{Q}$, and discontinuous at every $s \in \mathbb{R} - \mathbb{Q}$? It turns out that such a function does not exist, and it can be proved using an important result called Baire category theorem.

First recall that a complete metric space X means every Cauchy sequence in X converges to a limit in X. The theorem requires the metric space to be complete:

Theorem 2.50 (Baire Category Theorem). *Let (X, d) be a complete metric space. Then, we have:*

(1) If $U_1, U_2, U_3, \cdots$ are open sets in X such that $\overline{U_i} = X$ for any i, then

$$\overline{\bigcap_{i=1}^{\infty} U_i} = X.$$

(2) Given that $\{S_i\}_{i=1}^{\infty}$ is a sequence of sets in X such that $\left(\overline{S_i}\right)^{\circ} = \emptyset$ for any i, then $S := \bigcup_{i=1}^{\infty} S_i$ satisfies

$$\overline{X - S} = X \text{ and } S^{\circ} = \emptyset.$$

Proof. For result (1), we need to prove that for any $x_0 \in X$ and $\delta > 0$, one can find $y \in B_\delta(x_0) \bigcap (\cap_{i=1}^{\infty} U_i)$. Given that $\overline{U_1} = X$, one can find $x_1 \in B_\delta(x_0) \cap U_1$. Since $B_\delta(x_0) \cap U_1$ is open, there exists $\delta_1 \in (0, 1)$ such that $\overline{B_{\delta_1}(x_1)} \subset B_\delta(x_0) \cap U_1$. Inductively repeating this argument, we get a sequence of balls $\{B_{\delta_n}(x_n)\}_{n=1}^{\infty}$ such that

$$\overline{B_{\delta_n}(x_n)} \subset B_{\delta_{n-1}}(x_{n-1}) \cap U_n \quad \forall n \in \mathbb{N},$$

and we have also assume WLOG that $\delta_n < \frac{1}{2^n}$. Note that we have

$$B_\delta(x_0) \supset B_{\delta_1}(x_1) \supset B_{\delta_2}(x_2) \supset \cdots .$$

The sequence $\{x_n\}_{n=0}^{\infty}$ can be easily shown to be Cauchy: it is by the fact that

$$x_n \in B_{\delta_n}(x_n) \subset B_{\delta_{n-1}}(x_{n-1}) \implies d(x_n, x_{n-1}) < \delta_{n-1} < \frac{1}{2^{n-1}}.$$

Hence, by the completeness of (X, d), there exists $y \in X$ such that $x_n \to y$. We claim that $y \in B_\delta(x_0) \bigcap (\cap_{i=1}^{\infty} U_i)$ then we are done with (1). To prove this, we consider an arbitrary $m \in \mathbb{N}$, then $\{x_m, x_{m+1}, x_{m+2}, \cdots\}$ are all in $B_{\delta_m}(x_m)$, and so its limit $y \in \overline{B_{\delta_m}(x_m)}$ which is contained in $B_{\delta_{n-1}}(x_{n-1}) \cap U_n$. Therefore, we have $y \in U_m$. Since m is arbitrary, we conclude that $y \in \cap_{m=1}^{\infty} U_m$. Also by $y \in \overline{B_{\delta_1}(x_1)} \subset B_\delta(x_0) \cap U_1$, we have $y \in B_\delta(x_0)$ too. This completes the proof of (1).

The second result follows from the first. Given the conditions in (2), we consider $U_i := X - \overline{S_i}$ which is open in X. Furthermore, we have

$$\overline{U_i} = \overline{X - \overline{S_i}} = X \quad \text{by } \left(\overline{S_i}\right)^{\circ} = \emptyset \text{ and Exercise 2.34.}$$

Therefore, result from (1) can be applied here, and we conclude that

$$\overline{\bigcap_{i=1}^{\infty} (X - \overline{S_i})} = X \implies \overline{X - \bigcup_{i=1}^{\infty} \overline{S_i}} = X.$$

Since we also have

$$S = \bigcup_{i=1}^{\infty} S_i \subset \bigcup_{i=1}^{\infty} \overline{S_i},$$

this shows

$$X - S \supset X - \bigcup_{i=1}^{\infty} \overline{S_i} \implies \overline{X - S} \supset X.$$

This proves (2), and the result $S^{\circ} = \emptyset$ follows from Exercise 2.34. □

Remark 2.51. If S can be written as a form of $S = \bigcup_{i=1}^{\infty} S_i$ where $\left(\overline{S_i}\right)^{\circ} = \emptyset$, i.e. satisfies the conditions in (2), then we say S is of the **first category**. Otherwise, we say it is of **second category**.

Remark 2.52. After the PhD qualifying exam, the author often forgot which one is the first, which one is the second category.

Example 2.53. One can use Baire Category Theorem to show that there is no function $f : \mathbb{R} \to \mathbb{R}$ which is continuous at any $r \in Q$ and discontinuous at any $s \in \mathbb{R} - \mathbb{Q}$. Suppose such a function exists, we then denote

$$D_f := \{x_0 \in \mathbb{R} : \ f \text{ is discontinuous at } x_0\} = \mathbb{R} - \mathbb{Q}.$$

By Exercise 2.46 below, D_f is a countable union of closed sets (i.e. an F_σ set), we write

$$D_f = \bigcup_{i=1}^{\infty} E_i$$

where E_i's are all closed in $\mathbb{R}$. We claim that $\left(\overline{E_i}\right)^{\circ} = \emptyset$, otherwise we would have $x \in E_i^{\circ}$, so there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset E_i$. However, that would imply

$$(x - \varepsilon, x + \varepsilon) \subset E_i \subset \bigcup_{i=1}^{\infty} E_i = D_f = \mathbb{R} - \mathbb{Q},$$

contradiction to the density of rationals.

Now we have $\left(\overline{E}_i\right)^{\circ} = \emptyset$. Furthermore, write $\mathbb{Q} = \{r_1, r_2, r_3, \cdots\} = \bigcup_{j=1}^{\infty}\{r_j\}$, and noting that $\overline{\{r_j\}}^{\circ} = \emptyset$ for any fixed j, so the set

$$\mathbb{R} = (\mathbb{R} - \mathbb{Q}) \cup \mathbb{Q} = \left(\bigcup_{i=1}^{\infty} E_i\right) \cup \left(\bigcup_{j=1}^{\infty}\{r_j\}\right)$$

is of first category. By Theorem 2.50(2), we then have:

$$\mathbb{R}^{\circ} = \emptyset,$$

which is certainly absurd.

This shows such a function f does not exist.

Exercise 2.46. Here we prove that D_f is a countable union of closed sets. For any bounded, non-empty open interval J, we define:

$$\omega(f, J) := \sup\{|f(x) - f(y)| : x, y \in J\}.$$

For each $x_0 \in \mathbb{R}$, we define

$$\omega(f, x_0) := \inf\{\omega(f, J) : J \text{ is a bounded non-empty open interval containing } x_0\}.$$

Prove that

$$D_f = \bigcup_{i=1}^{\infty}\{x \in \mathbb{R} : \omega(f, x) \geq \frac{1}{i}\}$$

and $\{x \in \mathbb{R} : \omega(f, x) \geq \frac{1}{i}\}$ is closed for any i.

Exercise 2.47. A point p in a metric space (X, d) is called **isolated** if there exists $\varepsilon > 0$ such that $B_\varepsilon(p) = \{p\}$. Show that on complete metric space (X, d), any set closed set S in X such that $S' = S$ must be uncountable.

2.3. Compact Sets

2.3.1. Compactness on metric spaces. Heine-Borel Theorem in $\mathbb{R}$ says that a set $K \subset \mathbb{R}$ is closed and bounded is equivalent to both sequentially compact and compact. It is still true for $\mathbb{R}^n$ with the usual metric. However, we will soon see that it is not generally true on metric spaces. Yet, sequential compactness is still equivalent to compactness.

The definitions of the terms "sequentially compact", "compact", "closed and bounded" on a metric space are similar to those in $\mathbb{R}$.

Definition 2.54 (sequential compact, compact, bounded)**.** Let (X, d) be a metric space, and $S \subset X$. We say:

- S is said to be **bounded** if there exists a ball $B_R(x_0)$, where $x_0 \in X$ and $R \in (0, \infty)$, such that $S \subset B_R(x_0)$.
- S is said to be **sequentially compact** if any sequence $\{x_n\}$ in S has a subsequence $\{x_{n_j}\}$ in S converging to a limit in S.
- S is said to be **compact** if any open cover $\{U_\alpha\}$ of S has a finite subcover.

Exercise 2.48. Show that on $\mathbb{R}^n$ with the usual metric and $K \subset \mathbb{R}^n$, the following are equivalent:

(1) K is closed and bounded.
(2) K is sequentially compact.
(3) K is compact.

[Hint: The proof can be modified from that for $\mathbb{R}$. There is one part we need to use open intervals with rational end-points. Try to replace them with something else.]

Example 2.55. Here is one example of set that is closed and bounded but is not sequentially compact nor compact. Consider the space

$$l_\infty(\mathbb{R}) = \{\{x_n\}_{n=1}^\infty : \{x_n\} \text{ is bounded}\},$$

and define its norm by

$$\|\{x_n\}\|_\infty := \sup_{n \in \mathbb{N}} \{x_n\}.$$

For each $i \in \mathbb{N}$, we define a sequence $e_i = \{0, \cdots, \underbrace{1}_{i\text{-th term}}, 0, 0, \cdots\}$. It is then easy to see that

$$\|e_i - e_j\|_\infty = 1 \text{ whenever } i \neq j.$$

Therefore, for any subsequence $\{e_{i_k}\}$ we consider, it is never a Cauchy sequence, and hence never converges. The set $\{e_i\}_{i=1}^\infty$ is not sequentially compact. We will later show that sequential compactness and compactness are equivalent on a metric space, so this set $\{e_i\}_{i=1}^\infty$ is not compact either.

However, it is obvious that $\{e_i\}_{i=1}^\infty$ is bounded, as $\|e_i\|_\infty = 1$. It is closed because it has no convergent subsequence, so it has no limit point.

Exercise 2.49. Construct a set S in $C[0, 1]$ with sup-norm $\| \ \|_\infty$ that is not sequentially compact, but is closed and bounded.

Although "closed and bounded" is not equivalent to "compact" or "sequentially compact" on metric spaces, we can replace "closed and bounded" by even stronger conditions, namely "complete and totally bounded", then it would be equivalent to "compact" and "sequentially compact". As before, we say $K \subset X$ is complete if every

Cauchy sequence $\{x_n\}$ in K converges to a limit in K. Here is the definition of total boundedness

Definition 2.56 (totally bounded sets)**.** Let (X, d) be a metric space. A set $K \subset X$ is said to be **totally bounded** if for any $\varepsilon > 0$, there exists finitely many points $x_1, \cdots, x_N \in X$ such that

$$K \subset \bigcup_{i=1}^{N} B_\varepsilon(x_i).$$

If a set $K \subset X$ is complete, then it is closed. It simply follows from the definitions: take a convergent sequence $\{x_n\}$ in K, then it must be a Cauchy sequence. The completeness of K implies the limit of $\{x_n\}$ must be in K. This shows K is closed.

For "total boundedness" implies "boundedness": just take $\varepsilon = 1$, then K can be covered by finitely many unit balls, so it must be bounded.

Consider the $\{e_i\}$ set in Example 2.55. It is not totally bounded. To prove this, we take any $\varepsilon \in (0, 1/2)$, then it is impossible to cover $\{e_i\}$ by finitely many ε-balls. By $\|e_i - e_j\|_\infty = 1$ whenever $i \neq j$, each ε-ball $B_\varepsilon(x_k)$ contains at most one e_i's, and so any finite union $\bigcup_{k=1}^{N} B_\varepsilon(x_k)$ must contain at most finitely many e_i's, so it is impossible to cover the infinite set $\{e_i\}$.

Exercise 2.50. Give an example of a metric space (X, d) and a proper subset $S \subset X$ so that S is closed, but not complete.

[Hint: If (X, d) is complete, then any closed set S is complete too.]

Theorem 2.57. *Let (X, d) be a metric space, and $K \subset X$. Then, the following are equivalent:*

(1) K is complete and totally bounded.

(2) K is sequentially compact.

(3) K is compact.

Proof. We will prove (1) $\implies$ (3) $\implies$ (2) $\implies$ (1).

The proof of (1) $\implies$ (3) shares some common idea in the proof of the Baire category theorem. Assume that K is complete and totally bounded. We need to show K is compact, but we assume it is not the case. Then, there is an open cover $\{U_\alpha\}$ of K that has no finite subcover.

Take $\varepsilon = 1/2$, the total boundedness of K shows K can be covered by finitely many balls $B_{1/2}$. Recall that $\{U_\alpha\}$ has no finite subcover, so at least one of the $K \cap B_{1/2}$'s has no finite subcover from $\{U_\alpha\}$ – denote this one to be $K \cap B_{1/2}(x_1)$. Then we define

$$K_1 := K \cap B_{1/2}(x_1).$$

Inductively, we can construct $K_2, K_3, \cdots$ and $x_1, x_2, x_3, \cdots \in X$ such that

$$K_n = K_{n-1} \cap B_{1/2^n}(x_n) \;\; \forall n \in \mathbb{N},$$

and each K_n has no finite subcover from $\{U_\alpha\}$. Here we denote $K_0 := K$.

Each K_n is non-empty (otherwise it has a finite subcover by $\{U_\alpha\}$). Pick any $y_n \in K_n$ for each n, then we can see that $\{y_n\}$ is a Cauchy sequence by the fact that $y_n \in K_n \subset K_{n-1}$ and $y_{n-1} \in K_{n-1}$, so both $y_n, y_{n-1} \in B_{1/2^{n-1}}(x_{n-1})$. This shows $d(y_n, y_{n-1}) < \frac{1}{2^{n-2}}$ for any n. This shows $\{y_n\}$ is Cauchy.

Furthermore, as $y_n \in B_{1/2^n}(x_n)$, we have $d(x_n, y_n) < \frac{1}{2^n}$. The sequence $\{x_n\}$ is also Cauchy. Now that K is complete, and $\{x_n\}$ is a Cauchy sequence in K. It converges to a limit $x_\infty \in K$. Since $\{U_\alpha\}$ covers K, there exists α_0 such that $x_\infty \in U_{\alpha_0}$. By openness of U_{α_0}, there exists $\delta > 0$ such that $B_\delta(x_\infty) \subset U_{\alpha_0}$. However, $x_n \to x_\infty$, so for sufficiently large n, we have $B_{1/2^n}(x_n) \subset B_\delta(x_\infty) \subset U_{\alpha_0}$. However, that would imply $K_n = K_{n-1} \cap B_{1/2^n}(x_n) \subset B_{1/2^n}(x_n) \subset U_{\alpha_0}$, contradicting the fact that K_n has no finite subcover from $\{U_\alpha\}$. This shows K must be compact.

Next we prove (3) $\implies$ (2): given that K is a compact set, we need to show that given any sequence $\{x_n\}$ in K, there is a subsequence converging to a limit in K. We first consider for each $n \in \mathbb{N}$ the set

$$S_n := \overline{\{x_n, x_{n+1}, x_{n+2}, \cdots\}}.$$

We claim that $\bigcap_{n=1}^\infty S_n \cap K$ is non-empty. Suppose otherwise that $\bigcap_{n=1}^\infty S_n \cap K = \emptyset$, then $K \subset \left(\bigcap_{n=1}^\infty S_n\right)^c = \bigcup_{n=1}^\infty S_n^c$. Note that S_n^c is open, by compactness of K there exists $n_1, \cdots, n_k$ such that

$$K \subset S_{n_1}^c \cup \cdots \cup S_{n_k}^c.$$

Note that $S_{n_j}^c \subset \{x_{n_j}, x_{n_j+1}, x_{n_j+2}, \cdots\}^c = X - \{x_{n_j}, x_{n_j+1}, x_{n_j+2}, \cdots\}$, so we have

$$K \subset X - \{x_{n_k}, x_{n_k+1}, x_{n_k+2}, \cdots\}.$$

However, that would mean $x_{n_k}, x_{n_k+1}, x_{n_k+2}, \cdots$ are all not in K.

Now that $\bigcap_{n=1}^\infty S_n \cap K \neq \emptyset$, we pick any $x \in \bigcap_{n=1}^\infty S_n \cap K$, then we are going to show that $\{x_n\}$ has a subsequence converging to x. To start with, we consider

$$x \in S_1 = \overline{\{x_1, x_2, x_3, \cdots\}}.$$

Therefore, the ball $B_1(x)$ must contain at least one x_{n_1}, where $n_1 \in \mathbb{N}$. Then, consider

$$x \in S_{n_1} = \overline{\{x_{n_1}, x_{n_1+1}, x_{n_1+2}, \cdots\}},$$

then there exists $x_{n_2} \in B_{1/2}(x)$ with $n_2 > n_1$. Inductively, one can find $n_1 < n_2 < n_3 < \cdots$ such that

$$x_{n_j} \in B_{1/j}(x)$$

for any j. Then, $\{x_{n_j}\}$ is a subsequence of $\{x_n\}$, and by $d(x_{n_j}, x) < \frac{1}{j}$ for any j, we have $x_{n_j} \to x$.

Finally, we prove (2) $\implies$ (1). Given that K is sequentially compact, we need to prove two things: K is complete, and K is totally bounded. The proof of completeness is exactly the same, *mutatis mutandis*, as the proof of Cauchy criterion of the real line. Instead of using Bolzano-Weiestrass on $\mathbb{R}$, we use the sequential compactness condition of K. We leave it as an exercise for readers.

Now we prove K is totally bounded. Suppose it is not, then $\exists \varepsilon > 0$ such that K cannot be covered by finitely many open balls B_ε. We use this condition to construct a sequence in K that has no convergent subsequence. Pick any $x_1 \in K$ (if $K = \emptyset$ then we are done), as $K \not\subset B_\varepsilon(x_1)$, there exists $x_2 \in K$ but $x_2 \notin B_\varepsilon(x_1)$. In other words, $d(x_1, x_2) \geq \varepsilon$. Likewise, $K \not\subset B_\varepsilon(x_1) \cup B_\varepsilon(x_2)$, so there exists $x_3 \in K$ but $d(x_1, x_3) \geq \varepsilon$ and $d(x_2, x_3) \geq \varepsilon$. Inductively, one can construct $\{x_n\}$ in K such that $d(x_i, x_j) \geq \varepsilon$ whenever $i \not\equiv j$. Such a sequence cannot have any converging subsequence (compared with the sequence $\{e_i\}$ in Example 2.55). It contradicts to the fact that K is sequentially compact. □

Exercise 2.51. Show that if $K \subset X$ is totally bounded, then there exists a countable dense subset $D \subset K$, i.e. D is countable and $\overline{D} \subset K$. With this result, one can give another proof of "sequentially compact implies compact" similar to what we did in Heine-Borel Theorem on $\mathbb{R}$. Try to write up the proof.

2.3.2. Compact set and continuous functions. Extreme value theorem for continuous functions $f : [a,b] \to \mathbb{R}$ assert that $\sup_{x\in[a,b]} f(x)$ and $\inf_{x\in[a,b]} f(x)$ both exist in $\mathbb{R}$ and can be attained (hence they are maximum and minimum). Rolle's Theorem and mean value theorem are consequences of the extreme value theorem.

On metric spaces, extreme value theorem can be further generalized to another form:

Proposition 2.58. *Let K be a compact set in a metric space (X, d_X), and $f : X \to Y$ is a continuous function from (X, d_X) to another metric space (Y, d_Y). Then, the image set $f(K)$ is a compact set in Y.*

Proof. Take an open cover $\{U_\alpha\}$ of $f(K)$, we need to find a finite subcover from it. Note that

$$f(K) \subset \bigcup_\alpha U_\alpha \implies K \subset f^{-1}\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha f^{-1}(U_\alpha).$$

The collection $\{f^{-1}(U_\alpha)\}$ is an open cover of K (openness is from the continuity of f). Since K is compact, there exists $U_{\alpha_1}, \cdots, U_{\alpha_N}$ in the collection $\{U_\alpha\}$ such that

$$K \subset f^{-1}(U_{\alpha_1}) \cup \cdots \cup f^{-1}(U_{\alpha_N}) = f^{-1}\left(\bigcup_{i=1}^N U_{\alpha_i}\right).$$

This implies

$$f(K) \subset \bigcup_{i=1}^N U_{\alpha_i}.$$

Hence there is a finite subcover $\{U_{\alpha_i}\}_{i=1}^N$ of $f(K)$. This proves $f(K)$ is compact. □

Corollary 2.59 (Extreme value theorem)**.** *Any continuous function $f : [a,b] \to \mathbb{R}$ from a closed and bounded interval $[a,b]$ must have a maximum and minimum.*

Proof. Since $[a,b]$ is compact, by the above proposition the image set $f\big([a,b]\big)$ is compact too. Heine-Borel Theorem shows $f\big([a,b]\big)$ is closed and bounded. This shows $\sup_{x\in[a,b]} f(x) = \sup f\big([a,b]\big)$ and $\inf_{x\in[a,b]} f(x) = \inf f\big([a,b]\big)$ exist in $\mathbb{R}$ by completeness axiom of $\mathbb{R}$, and they are attained by the closedness of $f\big([a,b]\big)$. □

Exercise 2.52. It is possible to just use Bolzano-Weiestrass's Theorem to prove the extreme value theorem for continuous functions $f : [a,b] \to \mathbb{R}$ without using Proposition 2.58. Try it. Hint: first prove that such a function is bounded first.

Exercise 2.53. Prove Rolle's Theorem and the mean value theorem using the extreme value theorem for functions $f : [a,b] \to \mathbb{R}$.

In MATH 1024 we have learned about the concept of uniform continuity when we talk about Riemann integrals. Given a function $f : D \subset \mathbb{R} \to \mathbb{R}$ where D is the domain of f, we say f is uniformly continuous on D if $\forall \varepsilon > 0$, there exists $\delta > 0$ which only depends on ε such that whenever $x, x_0 \in D$ and $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| < \varepsilon$.

The major difference between "uniform continuity on D" and "continuity at every point in D" is that the former requires the δ to be universal – once ε is given, the δ can be chosen independent of x_0. One can easily see that e^x is not uniformly continuous on $\mathbb{R}$ because the larger x_0 is, the graph $y = e^x$ near x_0 gets steeper and a smaller value of δ is needed.

We have proved in MATH 1024 (using Bolzano-Weierstrass) that any continuous function $f : [a, b] \to \mathbb{R}$ on a closed and bounded interval must be uniformly continuous on $[a, b]$. This result was used to show that every continuous function on $[a, b]$ is Riemann integrable. We are going to generalize this result to compact metric spaces. We first begin with the definition of uniform continuity on metric spaces.

Definition 2.60 (Uniform continuity). A function $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is said to be **uniformly continuous on** E, where $E \subset X$, if $\forall \varepsilon > 0$, $\exists \delta > 0$ depending on ε but not on points x_0, such that if $x, x_0 \in E$ are points such that $d_X(x, x_0) < \delta$, then $d_Y(f(x), f(x_0)) < \varepsilon$.

Example 2.61. Let $A \subset X$ be a subset in a metric space (X, d), then the function $d_A : X \to \mathbb{R}$ defined by $d_A(x) := \inf\{d(x, a) : a \in A\}$ is uniformly continuous on X. It is given by the inequality that

$$|d_A(x) - d_A(y)| \leq d(x, y) \; \forall x, y \in X.$$

For any $\varepsilon > 0$, one can simply choose $\delta = \varepsilon$, then whenever $d(x, x_0) < \delta = \varepsilon$, we have

$$|d_A(x) - d_A(x_0)| \leq d(x, x_0) < \varepsilon.$$

Exercise 2.54. For any set E in a metric space (X, d), we define

$$\text{diam}E := \sup\{d(x, y) : x, y \in E\}.$$

Now given a function $f : X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y). Show that the following are equivalent:

(1) f is uniformly continuous on X.
(2) $\forall \varepsilon > 0, \exists \delta > 0$ such that if $E \subset X$ satisfies diam$E < \delta$, then diam$f(E) < \varepsilon$.

Exercise 2.55. Suppose $f : (X, d_X) \to (Y, d_Y)$ is uniformly continuous on X, then for any Cauchy sequence $\{x_n\}$ in (X, d_X), the sequence $\{f(x_n)\}$ is a Cauchy sequence in (Y, d_Y).

Proposition 2.62. *If (X, d_X) is a compact metric space, and $f : X \to Y$ is a continuous function from X to another metric space (Y, d_Y), then f is uniformly continuous on X.*

Proof (using compactness). First it is given that f is continuous on X, so for any $x_0 \in X$ and any $\varepsilon > 0$, there exists $\delta(x_0) > 0$ depending on $x_0 \in X$ and ε, such that

$$d_X(x, x_0) < \delta(x_0) \implies d_Y(f(x), f(x_0)) < \varepsilon/2.$$

The collection $\{B_{\delta(x_0)/2}(x_0)\}_{x_0 \in X}$ is an open cover of X, so by compactness of X one can extract a finite subcover: $\exists x_1, \cdots, x_N \in X$ such that

$$X = \bigcup_{i=1}^{N} B_{\delta(x_i)/2}(x_i).$$

Take $\delta = \min\{\delta(x_i)/2\}_{i=1}^N > 0$, then we claim that this is our desired δ. Given any $x_0 \in X$, there exists x_i such that $x_0 \in B_{\delta(x_i)/2}(x_i)$ as these balls cover the whole X. Then whenever $d_X(x, x_0) < \delta \leq \delta(x_i)/2$, we have

$$d_X(x, x_i) \leq d_X(x, x_0) + d_X(x_0, x_i) < \frac{\delta(x_i)}{2} + \frac{\delta(x_i)}{2} = \delta(x_i).$$

By our choice of $\delta(x_i)$, we have

$$d_Y(f(x), f(x_i)) < \frac{\varepsilon}{2}.$$

Since we also have $d_X(x_0, x_i) < \delta(x_i)$ too, we have
$$d_Y(f(x_0), f(x_i)) < \frac{\varepsilon}{2}.$$
Combining these results, we can conclude that whenever $d_X(x, x_0) < \delta$, we have
$$d_Y(f(x), f(x_0)) < \varepsilon,$$
as desired. $\square$

Proof (using sequential compactness). Recall that the metric space X is compact if and only if it is sequentially compact. Suppose f is not uniformly continuous on X, then $\exists \varepsilon_0 > 0$ such that $\forall \delta > 0$, there exists $x, y \in X$ with $d_X(x, y) < \delta$ but $d_Y(f(x), f(y)) \geq \varepsilon_0$. Applying this result with $\delta = \frac{1}{n}$ for each $n \in \mathbb{N}$, one can get a pair of sequences $x_n, y_n \in X$ such that
$$d_X(x_n, y_n) < \frac{1}{n} \text{ whereas } d_Y(f(x_n), f(y_n)) \geq \varepsilon_0.$$
By sequential compactness of X, one can extract convergent subsequences $\{x_{n_j}\}$ and $\{y_{n_j}\}$. Suppose $x_{n_j} \to x_\infty \in X$ and $y_{n_j} \to y_\infty \in X$. Then we have
$$\begin{aligned} d_X(x_\infty, y_\infty) &\leq d_X(x_\infty, x_{n_j}) + d_X(x_{n_j}, y_{n_j}) + d_X(y_{n_j}, y_\infty) \\ &\leq d_X(x_\infty, x_{n_j}) + \frac{1}{n_j} + d_X(y_{n_j}, y_\infty). \end{aligned}$$
Letting $j \to \infty$, the RHS goes to 0, so we conclude that $x_\infty = y_\infty$. Therefore, we have
$$\varepsilon_0 \leq d_Y(f(x_{n_j}), f(y_{n_j})) \leq d_Y(f(x_{n_j}), f(x_\infty)) + d_Y(f(x_\infty), f(y_{n_j})).$$
Recall that f is continuous on X, so $d_Y(f(x_{n_j}), f(x_\infty)) \to 0$ and $d_Y(f(y_{n_j}), f(y_\infty)) \to 0$. However, that would imply
$$0 < \varepsilon_0 \leq \lim_{j \to \infty} \big(d_Y(f(x_{n_j}), f(x_\infty)) + d_Y(f(x_\infty), f(y_{n_j}))\big) = 0,$$
which is impossible. This proves f must be uniformly continuous on X. $\square$

Exercise 2.56. Consider two metric spaces (X, d_X) and (Y, d_Y). A function $f : X \to Y$ is said to be **Lipschitz continuous on** X if there exists $L > 0$ such that
$$d_Y\big(f(x), f(y)\big) \leq L d_X(x, y) \;\; \forall x, y \in X.$$
A function $f : X \to Y$ is said to be **locally Lipschitz continuous on** X if for any $x_0 \in X$, there exists $\varepsilon > 0$ and $L \in (0, \infty)$, which can depend on x_0, such that
$$d_Y\big(f(x), f(y)\big) \leq L d_X(x, y) \;\; \forall x, y \in B_\varepsilon(x_0).$$
Now suppose X is compact. Prove that f is locally Lipschitz continuous on X if and only if f is Lipschitz continuous on X.

Exercise 2.57 (Hausdorff metric). Suppose (X, d) is a metric space. Denote by $\mathcal{K}(X)$ the collection of all compact subsets of X. Let $d_H : \mathcal{K}(X) \times \mathcal{K}(X) \to [0, \infty)$ be the function defined by
$$d_H(A, B) := \max\left\{\sup_{a \in A} d_B(a), \sup_{b \in B} d_A(b)\right\}.$$
Recall that $d_A(b) := \inf\{d(a, b) : a \in A\}$.

Show that d_H is a metric on $\mathcal{K}(X)$. Prove also that if X is compact, then $\big(\mathcal{K}(X), d_H\big)$ is also compact.

Exercise 2.58. Suppose $\{K_i\}_{i=1}^{\infty}$ is a sequence of compact sets in a metric space (X, d) such that

$$K_1 \supset K_2 \supset K_3 \supset \cdots$$

Show that

$$\bigcap_{i=1}^{\infty} K_i \neq \emptyset.$$

2.3.3. Compactness is intrinsic. When we define compactness of a set K, we make reference to its ambient space, i.e. what is the metric space X that K lives. There could be many sets between K and X. Suppose $K \subset Y \subset X$, then Y can also be regarded as a metric space, with the "inherited" metric $d\big|_Y : Y \times Y \to [0, \infty)$ defined as the restriction of d:

$$d\big|_Y(x, y) := d(x, y) \quad \forall x, y \in Y.$$

As K is also a subset of Y, one may wonder whether K is compact in X if and only if K is compact in Y? It is a meaningful question as the collections of open sets in X and in Y are different – even the open balls are defined differently, for any $a \in Y$, we have:

$$B_\varepsilon^X(a) := \{x \in X : d(x, a) < \varepsilon\}$$
$$B_\varepsilon^Y(a) := \{y \in Y : d(y, a) < \varepsilon\} = Y \cap B_\varepsilon^X(a).$$

The openness of a set U in X is not equivalent to the openness of U in Y. For example, if Y is the upper-half plane in $\mathbb{R}^2$ including the x-axis, i.e.

$$Y = \{(x, y) \in \mathbb{R}^2 : y \geq 0\},$$

and $X = \mathbb{R}^2$. Then the hemisphere $\{(x, y) : x^2 + y^2 < 1 \text{ and } y \geq 0\}$ is not open in X, but it is open in Y. Generally, one can show that:

Exercise 2.59. Given (X, d) is a metric space, and $Y \subset X$ is equipped with the inherited metric from X. Show that $U \subset Y$ is open in Y if and only if there exists $\mathcal{O}$ which is open in X such that $U = \mathcal{O} \cap Y$.

To sum up, *openness* is not an intrinsic property as it depends on the space the set lives in. However, one can show that *compactness* is an intrinsic property, even though its definition is not.

Proposition 2.63. *Let (X, d) be a metric space, and $Y \subset X$ be equipped with the inherited metric from X. Given any $K \subset Y$, then K is compact in X if and only if K is compact in Y.*

Proof. For the $(\Longrightarrow)$-direction, we suppose K is compact X. To prove that K is compact in Y, we let $\{U_\alpha\}$ be an open cover of K in Y, i.e. $K \subset \bigcup_\alpha U_\alpha$ where U_α's are open sets in Y. Then each U_α is of the form

$$U_\alpha = Y \cap \mathcal{O}_\alpha$$

for some set $\mathcal{O}_\alpha$ open in X. Therefore, we have

$$K \subset \bigcup_\alpha U_\alpha \subset \bigcup_\alpha \mathcal{O}_\alpha.$$

Since K is compact in X, there exists a finite subcover from $\{\mathcal{O}_\alpha\}$, i.e.

$$K \subset \mathcal{O}_{\alpha_1} \cup \cdots \cup \mathcal{O}_{\alpha_N}$$

for some $\alpha_1, \cdots, \alpha_N$. Taking intersection with Y on both sides, and noting that $K \subset Y$, we get

$$K = K \cap Y \subset (Y \cap \mathcal{O}_{\alpha_1}) \cap \cdots \cap (Y \cap \mathcal{O}_{\alpha_N}) = U_{\alpha_1} \cup \cdots \cup U_{\alpha_N}.$$

Therefore, the open cover $\{U_\alpha\}$ of K in Y has a finite subcover. This shows K is compact in Y.

Conversely, assume K is compact in Y, and we need to prove K is compact in X. Given an open cover $\{\mathcal{O}_\alpha\}$ of K in X, then we have

$$K \subset \bigcup_\alpha \mathcal{O}_\alpha \implies K = K \cap Y \subset \bigcup_\alpha (Y \cap \mathcal{O}_\alpha).$$

Then $\{Y \cap \mathcal{O}_\alpha\}$ is an open cover of K in Y, then by compactness of K in Y we have

$$K \subset (Y \cap \mathcal{O}_{\alpha_1}) \cup \cdots \cup (Y \cap \mathcal{O}_{\alpha_N})$$

for some $\alpha_1, \cdots, \alpha_N$. By $Y \cap \mathcal{O}_{\alpha_i} \subset \mathcal{O}_{\alpha_i}$, we conclude that

$$K \subset \mathcal{O}_{\alpha_1} \cup \cdots \cup \mathcal{O}_{\alpha_N}.$$

Hence, $\{\mathcal{O}_\alpha\}$ has a finite subcover, completing the proof. □

From now on, we will simply say "K is compact" rather than "K is compact in Y" or "K is compact in X".

Exercise 2.60. Suppose (X, d) is a metric space, and $Y \subset X$ be equipped with the inherited metric $d\big|_Y$. Suppose $S \subset Y \subset X$. Prove that if S is totally bounded in X if and only if S is totally bounded in Y.

2.4. Connected Sets

To understand the definition of a *connected set*, it is best to first understand what is *disconnected set*. Note that a set S which can be written as a disjoint union of two other sets A and B is not *disconnected* enough. In fact every set can be expressed in such a way. We need A and B to be even more *separated* than being disjoint:

Definition 2.64 (Connected sets, disconnected sets)**.** Let (X, d) be a metric space. A set $S \subset X$ is said to be **disconnected** if $S = A \cup B$ where $A, B \subset S$ are non-empty such that $A \cap \overline{B} = \overline{A} \cap B = \emptyset$. A set in X is called **connected** if it is not disconnected.

Example 2.65. On $\mathbb{R}^2$ with the usual metric, the set

$$S = B_1\big((-1,0)\big) \cup B_1\big((1,0)\big)$$

is disconnected, since

$$\overline{B_1\big((-1,0)\big)} \cup B_1\big((1,0)\big) = \emptyset$$
$$B_1\big((-1,0)\big) \cup \overline{B_1\big((1,0)\big)} = \emptyset$$

Example 2.66. The Cantor set $\mathcal{C}$ is disconnected, since

$$\mathcal{C} = \underbrace{(\mathcal{C} \cap [0, 1/3])}_{A} \cup \underbrace{(\mathcal{C} \cap [2/3, 1])}_{B}.$$

Proposition 2.67. *A subset $S \subset \mathbb{R}$ is connected if and only if $[x, y] \subset S$ whenever $x < y$ and $x, y \in S$.*

Proof. We first prove $(\Longrightarrow)$: given that S is connected, and we suppose otherwise that there exist $x, y \in S$ with $x < y$ such that $[x, y] \not\subset S$, i.e. $\exists z \in [x, y]$ but $z \notin S$. Then, consider

$$A := S \cap (-\infty, z) \;\text{ and }\; B := S \cap (z, \infty),$$

then $S = A \cup B$. Note that $\overline{A} \subset \overline{S} \cap \overline{(-\infty, z)} = \overline{S} \cap (-\infty, z]$, hence

$$\overline{A} \cap B = \overline{S} \cap (-\infty, z] \cap S \cap (z, \infty) = \emptyset.$$

Similarly $A \cap \overline{B} = \emptyset$ too. This shows S is disconnected, a contradiction.

Conversely, suppose S satisfies the condition $[x, y] \subset S$ whenever $x < y$ and $x, y \in S$. We want to show S is connected, but we suppose otherwise, i.e. $S = A \cup B$ where $A \cap \overline{B} = \overline{A} \cap B = \emptyset$, and A, B are non-empty. Pick $x \in A$ and $y \in B$, and assume WLOG $x < y$. Consider

$$z = \sup(A \cap [x, y]).$$

Then z is the limit of a sequence in $A \cap [x, y]$, and hence $z \in \overline{A \cap [x, y]} \subset \overline{A}$. By our assumption on A, B, we then have $z \notin B$. Note that $x \le z < y$ (since $y \in B$ so $z \neq y$), by the given condition on S we must have $z \in S$. By $S = A \cup B$ and $z \notin B$, we must have $z \in A$. That will imply $z \notin \overline{B}$ as $A \cap \overline{B} = \emptyset$. Note that $\mathbb{R} - \overline{B}$ is open in $\mathbb{R}$ and $z \in \mathbb{R} - \overline{B}$, there exists $\varepsilon > 0$ such that $(z - \varepsilon, z + \varepsilon) \subset \mathbb{R} - \overline{B}$. Pick $w \in (z, z + \varepsilon)$ such that $z < w < y$, then $w \notin \overline{B}$ so $w \in A$. However, this shows $w \in A \cap [z, y] \subset A \cap [x, y]$, which is impossible since $\sup(A \cap [x, y]) = z < w$. It completes the proof that S is connected. $\square$

Corollary 2.68. $\mathbb{R}$ *is connected. Intervals of any kind:* $[a, b]$, $(a, b]$, $[a, b)$, (a, b), $(-\infty, b]$, $(-\infty, b]$, $[a, \infty)$ *and* $[a, \infty)$ *are all connected.*

Exercise 2.61. Show that if S_1 and S_2 are connected sets in a metric space (X, d), and $S_1 \cap S_2 \neq \emptyset$, then $S_1 \cup S_2$ is connected.

[Remark: if your proof is correct, it should be possible to be modified to prove that if $\{S_\alpha\}$ is a collection (possibly uncountable) of connected sets such that $\bigcap_\alpha S_\alpha \neq \emptyset$, then $\bigcup_\alpha S_\alpha$ is connected.]

Exercise 2.62. Suppose S is a connected set in a metric space (X, d). Prove that $\overline{S}$ is also connected.

2.4.1. Continuous functions on connected sets. The intermediate value theorem for continuous function $f : [a, b] \to \mathbb{R}$ from an interval $[a, b]$ saids that if $L, M \in f[a, b]$ where $L < M$, then any value $m \in [L, M]$ can be attained by some $c \in [a, b]$ such that $f(c) = m$, i.e. $m \in f[a, b]$. In other words, the image set $f[a, b]$ is connected according to Proposition 2.67. We can rephrase the statement of intermediate value theorem by saying "f maps the interval $[a, b]$ to a connected set in $\mathbb{R}$".

This reformulation is the version of intermediate value theorem on metric spaces:

Proposition 2.69. *Let $f : (X, d_X) \to (Y, d_Y)$ be a continuous map between two metric spaces. Suppose $S \subset X$ is connected, then $f(S)$ is also connected.*

Proof. Suppose otherwise the $f(S)$ is disconnected, i.e. $f(S) = A \cup B$ with $A, B \neq \emptyset$ and $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. Then,

$$S \subset f^{-1}\big(A \cup B\big) = f^{-1}(A) \cup f^{-1}(B) \implies S = (f^{-1}(A) \cap S) \cup (f^{-1}(B) \cap S).$$

As $A \subset f(S)$ and $B \subset f(S)$, both $f^{-1}(A) \cap S, f^{-1}(B) \cap S$ are non-empty. Furthermore, by continuity of f and Exercise 2.44, we have

$$\overline{f^{-1}(A)} \cap f^{-1}(B) \subset f^{-1}(\overline{A}) \cap f^{-1}(B) = f^{-1}(\overline{A} \cap B) = f^{-1}(\emptyset) = \emptyset.$$

Likewise, we have $f^{-1}(A) \cap \overline{f^{-1}(B)} = \emptyset$. This also shows

$$\overline{f^{-1}(A) \cap S} \cap (f^{-1}(B) \cap S) = \emptyset$$
$$(f^{-1}(A) \cap S) \cap \overline{f^{-1}(B) \cap S} = \emptyset$$

and hence S is disconnected, which leads to a contradiction. Hence $f(S)$ must be connected. □

Corollary 2.70 (Intermediate value theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous map and $M, m \in f[a, b]$ where $m < M$, then given any $l \in [m, M]$, there exists $c \in [a, b]$ such that $f(c) = l$.*

Proof. $[a, b]$ is connected, so $f[a, b]$ is a connected set in $\mathbb{R}$. The result follows directly from Proposition 2.67. □

2.4.2. Path-connected sets. Proving a certain set is connected is typically much harder than proving a set is disconnected. One needs to prove it is impossible to decompose the set S into $A \cup B$ for *any* A and B satisfying the conditions stated in the definition, whereas to prove S is disconnected, one just needs to prove there *exist* such sets A and B. We have seen in the ($\Longleftarrow$)-part of Proposition 2.67 that even for some sets in $\mathbb{R}$ it is not straight-forward to prove that they are connected.

Fortunately, we have another condition of connectedness, called the *path-connectedness*, which implies connectedness, and it is easier to verify.

Definition 2.71 (Path-connected sets)**.** Let (X,d) be a metric space. A subset $S \subset X$ is said to be **path-connected** if for any $x, y \in S$, there exists a continuous path $\gamma : [0,1] \to S$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Example 2.72. Any open ball $B_r(a)$ in $\mathbb{R}^n$ is path-connected: given $x, y \in B_r(a)$, one can join them by a straight-path:

$$\gamma(t) = (1-t)x + ty.$$

It is obviously continuous, and we leave it for readers to check that $\gamma[0,1] \subset B_r(a)$ whenever $x, y \in B_r(a)$.

Example 2.73. Consider the set S in $\mathbb{R}^2$:

$$S := \{(x, \sin(1/x)) : 0 < x \leq 1\} \cup \{(0,y) : -1 \leq y \leq 1\}.$$

It is not path-connected as it includes the point, say, $(0,0)$ which cannot be connected to any point $(x_0, \sin(1/x_0))$.

Exercise 2.63. Let (X,d) be a metric space, and $S \subset X$. Define a relation on S by the following: $x \sim y$ if and only if there exists a continuous path $\gamma : [0,1] \to S$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Show that $\sim$ is an equivalence relation. If S is path-connected, then what is the quotient set $S/\sim$?

Exercise 2.64. Suppose $\{S_\alpha\}$ is a collection of path-connected sets in a metric space (X,d), and $\bigcap_\alpha S_\alpha \neq \emptyset$. Show that $S := \bigcup_\alpha S_\alpha$ is also path-connected.

Proposition 2.74. *If S is a path-connected set in a metric space (X,d), then it is also connected.*

Proof. Suppose otherwise that S is disconnected, i.e. $S = A \cup B$ with $A, B \neq \emptyset$ and $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. Pick $x \in A$ and $y \in B$, then since S is given to be path-connected, there exists a continuous path $\gamma : [0,1] \to S$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Now consider $[0,1] = \gamma^{-1}(S) = \gamma^{-1}(A) \cup \gamma^{-1}(B)$. By continuity of γ and Exercise 2.44, we have

$$\overline{\gamma^{-1}(A)} \cap \gamma^{-1}(B) \subset \gamma^{-1}(\overline{A}) \cap \gamma^{-1}(B) = \gamma^{-1}(\overline{A} \cap B) = \gamma^{-1}(\emptyset) = \emptyset,$$

and similarly $\gamma^{-1}(A) \cap \overline{\gamma^{-1}(B)} = \emptyset$. Furthermore, $0 \in \gamma^{-1}(A)$ as $\gamma(0) = x \in A$, and $1 \in \gamma^{-1}(B)$ as $\gamma(1) = y \in B$. Both $\gamma^{-1}(A)$ and $\gamma^{-1}(B)$ are non-empty. However, this shows $[0,1]$ is disconnected, which contradicts to Proposition 2.67. To conclude, S must be connected. □

Example 2.75. Any open ball $B_r(a)$ or closed ball $\overline{B_r(a)}$ in $\mathbb{R}^n$ is path-connected, so they are connected too.

However, connected sets may not be path-connected. One such set is the S in Example 2.73. Note that

$$E := \{(x, \sin(1/x)) : x \in (0,1]\}$$

is connected as the map $x \mapsto (x, \sin(1/x))$ from $(0,1]$ to E is continuous, and $(0,1]$ is connected. Note that $S = \overline{E}$ so it is also connected. However, we have seen that S is not path-connected.

Exercise 2.65. Consider the following sets in $\mathbb{R}^2$:

$$S_n := \{(x, x/n) : x \in [0, 1]\}, \quad \text{where } n \in \mathbb{N}.$$

Define

$$S := \{(1, 0)\} \cup \bigcup_{n=1}^{\infty} S_n.$$

Show that S is connected but not path-connected.

Exercise 2.66. Suppose $f : (X, d_X) \to (Y, d_Y)$ is a map between two metric spaces, such that

(1) f is bijective, and

(2) f and f^{-1} are continuous (on X and on Y respectively).

Given a set $S \subset X$, show that

(a) S is open if and only if $f(S)$ is open.

(b) S is closed if and only if $f(S)$ is closed.

(c) S is compact if and only if $f(S)$ is compact.

(d) S is countable if and only if $f(S)$ is countable.

(e) S is connected if and only if $f(S)$ is connected.

(f) S is path-connected if and only if $f(S)$ is path-connected.

How about total boundedness?

Chapter 3

Differentiability

"Most important part of doing physics is the knowledge of approximation."

Lev Landau

3.1. Linear Approximation

This chapter we will focus exclusively on functions $F : \mathbb{R}^m \to \mathbb{R}^n$, and we assume all $\mathbb{R}^n$'s are equipped with the standard Euclidean metric. The goal of this chapter is to extend the theory of differentiations of single-variable functions $f : \mathbb{R} \to \mathbb{R}$ to higher dimensions.

Given a function $f : (a, b) \to \mathbb{R}$, we say f is differentiable at $x_0 \in (a, b)$ if the following limit exists:

$$f'(x_0) := \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

By manipulating the above limit, we also get:

$$\lim_{x\to x_0} \frac{f(x) - \big(f(x_0) + f'(x_0)\,(x - x_0)\big)}{x - x_0} = \lim_{x\to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right) = 0. \tag{3.1}$$

Also, one can easily show that if there exists a constant $A \in \mathbb{R}$ such that

$$\lim_{x\to x_0} \frac{f(x) - \big(f(x_0) + A(x - x_0)\big)}{x - x_0} = 0,$$

then f must be differentiable at x_0, and $f'(x_0) = A$.

The function $T(x) := f(x_0) + f'(x_0)(x - x_0)$ is an affine linear function[1] with $T(x_0) = f(x_0)$, so (3.1) essentially says that when x is very close to x_0, the function $f(x)$ is very close to $T(x)$ in a sense that their gap is much smaller than $|x - x_0|$. We often say "$f(x)$ can be linearly approximated by $T(x)$ near x_0", and using little-o notations, we can write:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(|x - x_0|) \quad \text{as } x \to x_0.$$

The two notions – linear approximations and derivatives – are equivalent for first order differentiation for functions on $\mathbb{R}$. However, in MATH 1023, we have seen that

[1]"Affine" linear simply means linear plus a constant.

the n-th derivative $f^{(n)}(x_0)$ exists would imply (by repeatedly applying L'Hopsital's rule) $f(x)$ can be approximated by an n-th degree polynomial $T_n(x)$, i.e. $f(x) = f(x_0) + T_n(x) + o(|x - x_0|^n)$ as $x \to x_0$, then we must have:

$$T_n(x) = \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

and we call $T_n(x)$ the n-th degree Taylor polynomial for $f(x)$ at x_0.

However, the converse is not true – there are functions that can be approximated by n-th degree polynomial, where $n \geq 2$, near x_0, but $f^{(n)}(x_0)$ does not exist. The functions is of the form $x^n \chi_{\mathbb{Q}}$, where

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}.$$

Now we consider functions $F : \mathbb{R}^n \to \mathbb{R}$ with $n \geq 2$. It turns out the existence of first-order (partial) derivatives does not imply it can be linearly approximated either!

3.1.1. Partial derivatives. Throughout this chapter, we assume the domain $\Omega \subset \mathbb{R}^n$ of a function $F : \Omega \to \mathbb{R}^m$ is a **non-empty open set**, so that whenever $x_0 \in \Omega$, there exists an open ball $B_\varepsilon(x_0) \subset \Omega$, and we can approach x_0 from any direction. Take $\mathbb{R}^2$ as an example, one can measure the rate of change of $f(x, y)$ along x direction leaving y fixed, or along y direction leaving x fixed. Furthermore, we can also measure the rate of change along any unit direction $(u_1, u_2) \in \mathbb{R}^2$ where $\sqrt{u_1^2 + u_2^2} = 1$.

Definition 3.1 (Partial derivatives, directional derivatives)**.** Given $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, $\mathbf{a} = (a_1, \cdots, a_n) \in \Omega$, and a unit vector $\mathbf{u} = (u_1, \cdots, u_n) \in \mathbb{R}^n$, we denote and define the **directional derivative of f along $\mathbf{u}$ at $\mathbf{a}$** by:

$$D_{\mathbf{u}} f(\mathbf{a}) := \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{u}) - f(\mathbf{a})}{h} \quad \text{whenever the limit exists.}$$

Denote $\mathbf{e}_i := (0, \cdots, \underbrace{1}_{i\text{-th slot}}, \cdots, 0)$, then in particular we call:

$$D_{\mathbf{e}_i}(\mathbf{a}) = \lim_{h \to 0} \frac{f(a_1, \cdots, a_i + h, \cdots, a_n) - f(a_1, \cdots, a_i, \cdots, a_n)}{h}$$

the **partial derivative of f with respect to x_i at $\mathbf{a}$**, and is usually denoted by $\dfrac{\partial f}{\partial x_i}(\mathbf{a})$, $\left.\dfrac{\partial f}{\partial x_i}\right|_{\mathbf{a}}$, or simply $\dfrac{\partial f}{\partial x_i}$ if the point $\mathbf{a}$ can be understood in the context or is not relevant.

Calculation of partial derivatives is straight-forward: given a function say $f(x, y)$, one simply treats y as the variable and x as a constant when computing $\frac{\partial f}{\partial y}$; whereas one treats x as the variable and y as a constant when computing $\frac{\partial f}{\partial x}$.

Exercise 3.1. Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ where $f(x, y)$ equals:

$$x^{x^x}, \ x^{x^y}, \ x^{y^x}, \ y^{x^x}, \ x^{y^y}, \ y^{x^y}, \ y^{y^x}, \ y^{y^y}.$$

Example 3.2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}.$$

To find $\frac{\partial f}{\partial x}$ at $(1,1)$, we may simply ignore the value of f at $(0,0)$ since derivatives of f depend only on its value in a small open ball $B_\varepsilon\big((1,1)\big)$ around $(1,1)$. We may simply compute:

$$\frac{\partial f}{\partial x}(1,1) = \frac{\partial}{\partial x}\frac{x^2y}{x^4+y^2}\bigg|_{(1,1)} = \frac{(x^4+y^2)\cdot 2xy - x^2y(4x^3)}{(x^4+y^2)^2}\bigg|_{(1,1)} = 0.$$

Similarly for $\frac{\partial f}{\partial y}$ at $(1,1)$.

Let's demonstrate how to compute the directional derivative from the definition. Suppose $\mathbf{u} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, then

$$\begin{aligned} D_{\mathbf{u}}f(1,1) &= \lim_{h\to 0}\frac{f\big((1,1)+h(1/2,\sqrt{3}/2)\big) - f(1,1)}{h} \\ &= \lim_{h\to 0}\frac{\frac{(1+h/2)^2(1+\sqrt{3}h/2)}{(1+h/2)^4+(1+\sqrt{3}h/2)^2} - \frac{1}{2}}{h} \end{aligned}$$

We leave it as an exercise for reader's MATH 1013 friends to calculate the above limit.

However, when we handle the partial/directional derivatives at $(0,0)$, we then need to be more careful, as $f((0,0)+h\mathbf{u})$ and $f(0,0)$ are defined differently when $h \neq 0$. When letting $h \to 0$, we regard that $h \neq 0$ yet very close to 0, so

$$f((0,0)+h\mathbf{u}) = f(hu_1, hu_2) = \frac{(hu_1)^2(hu_2)}{(hu_1)^4+(hu_2)^2} = \frac{hu_1u_2}{h^2u_1^4+u_2^2}.$$

Therefore, we have

$$D_{\mathbf{u}}f(0,0) = \lim_{h\to 0}\frac{f(\mathbf{0}+h\mathbf{u}) - f(\mathbf{0})}{h} = \lim_{h\to 0}\frac{\frac{hu_1u_2}{h^2u_1^4+u_2^2} - 0}{h} = \frac{u_1}{u_2} \text{ when } u_2 \neq 0.$$

When $u_2 = 0$, we then have $f((0,0)+h(u_1,u_2)) = f(hu_1, 0) = 0$ for any h. Therefore,

$$D_{\mathbf{u}}f(0,0) = \lim_{h\to 0}\frac{f(hu_1,0) - f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0.$$

In other words, it means $\frac{\partial f}{\partial x}(0,0) = 0$.

In the above example, we see that f has directional derivatives (including partial derivatives) along any unit directions $\mathbf{u}$ at the point $(0,0)$. However, it can be shown that f is not even continuous at $(0,0)$.

Along the parabola $y = x^2$, which can be parametrized by $\gamma(t) = (t, t^2)$, we have

$$\lim_{t\to 0} f(\gamma(t)) = \lim_{t\to 0}\frac{t^4}{t^4+t^4} = \frac{1}{2}.$$

However, along the line $y = 0$, we have $f(x,0) = 0$ for any x, and so

$$\lim_{x\to 0} f(x,0) = 0.$$

This shows f is not continuous as $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Therefore, existence of directional derivatives in any unit direction does not imply the function is continuous!

Exercise 3.2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & \text{if } x+y \neq 0 \\ 0 & \text{if } x+y = 0 \end{cases}.$$

Find the directional derivative of f along any unit direction at the points $(0,0)$, $(1,0)$, and $(1,-1)$; or show that it does not exist in some directions.

3.1.2. Differentiable function of several variables. The example in the previous subsection told us that even if a function has directional derivatives along any unit directions, it could still behave badly around that point – such as being discontinuous at that point. The notion of *differentiability* for multivariable functions $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is much stronger than merely existence of partial/directional derivatives. It concerns about the existence of a "good" *linear approximations*:

Definition 3.3 (Differentiable functions)**.** Let Ω be a non-empty open set in $\mathbb{R}^n$, and $\mathbf{a} = (a_1, \cdots, a_n) \in \Omega$. A function $f(x_1, \cdots, x_n) : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is said to be **differentiable at a** if

(1) $\dfrac{\partial f}{\partial x_i}(\mathbf{a})$ exists for any $i = 1, \cdots, n$, and

(2) the following limit exists and equals:

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{f(\mathbf{x}) - \big(f(\mathbf{a}) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{a})(x_i - a_i)\big)}{|\mathbf{x} - \mathbf{a}|} = 0.$$

Condition (2) can be rewritten as:

$$f(\mathbf{x}) = f(\mathbf{a}) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{a})(x_i - a_i) + o\big(\,|\mathbf{x} - \mathbf{a}|\,\big) \quad \text{as } \mathbf{x} \to \mathbf{a}.$$

Therefore, we could call $f(\mathbf{a}) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{a})(x_i - a_i)$ the **linear approximation** of f at $\mathbf{a}$.

For a vector-valued function $F : \Omega \to \mathbb{R}^m$, one can write F as

$$F(\mathbf{x}) = \big(f^1(\mathbf{x}), \cdots, f^n(\mathbf{x})\big),$$

where each $f^i : \Omega \to \mathbb{R}$ is a scalar function. We say F is **differentiable at a** if each component f^i is differentiable at $\mathbf{a}$ is differentiable at $\mathbf{a}$.

Remark 3.4. Recall in MATH 2023 we defined the gradient vector of f as:

$$\nabla f(\mathbf{a}) = \left(\frac{\partial f}{\partial x_1}(\mathbf{a}), \cdots, \frac{\partial f}{\partial x_n}(\mathbf{a})\right).$$

Using this notation, one can express the linear approximation of f by:

$$f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$$

where $\cdot$ is the standard dot product in $\mathbb{R}^n$.

Exercise 3.3. Show that if there exist constants $c_1, \cdots, c_n \in \mathbb{R}$ such that

$$f(\mathbf{x}) = f(\mathbf{a}) + \sum_{i=1}^n c_i(x_i - a_i) + o\big(\,|\mathbf{x} - \mathbf{a}|\,\big) \quad \text{as } \mathbf{x} \to \mathbf{a},$$

then it is necessary that all directional (including partial) derivatives of f exist at $\mathbf{a}$, and $\frac{\partial f}{\partial x_i}(\mathbf{a}) = c_i$ for any $i = 1, \cdots, n$.

[Remark: Therefore, some textbooks define differentiability of f in a more succinct way: f is differentiable at $\mathbf{a}$ if there exists a constant vector $\mathbf{c} \in \mathbb{R}^n$ such that $f(\mathbf{x}) = f(\mathbf{a}) + \mathbf{c} \cdot (\mathbf{x} - \mathbf{a}) + o(|\mathbf{x} - \mathbf{a}|)$ as $\mathbf{x} \to \mathbf{a}$. This exercise shows it is equivalent to our definition.]

Example 3.5. Let $f(x,y) = |xy|$. We will prove that f is differentiable at $(0,0)$. First by observing that $f(x,0) = 0$ and $f(0,y) = 0$ for any $x, y \in \mathbb{R}$, we conclude that

$$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0.$$

Next we work out the limit in Condition (2):

$$\begin{aligned}
&\lim_{\mathbf{x}\to\mathbf{a}} \frac{f(\mathbf{x}) - \left(f(\mathbf{a}) + \sum_{i=1}^{2} \frac{\partial f}{\partial x_i}(\mathbf{a})(x_i - a_i)\right)}{|\mathbf{x} - \mathbf{a}|} \\
&= \lim_{\mathbf{x}\to\mathbf{a}} \frac{|xy| - 0 - 0(x-0) - 0(y-0)}{|\mathbf{x} - \mathbf{0}|} \\
&= \lim_{(x,y)\to(0,0)} \frac{|xy|}{\sqrt{x^2+y^2}}.
\end{aligned}$$

Note that

$$\frac{|xy|}{\sqrt{x^2+y^2}} = \frac{|x|}{\sqrt{x^2+y^2}} \cdot |y| \leq |y| \to 0 \quad \text{as } (x,y) \to (0,0),$$

we conclude that condition (2) holds, and hence f is differentiable at $(0,0)$.

Example 3.6. Let's revisit the function $f : \mathbb{R}^2 \to \mathbb{R}$ we defined earlier:

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

We have already proved that $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$. Let's check if condition (2) holds:

$$\begin{aligned}
&\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - \frac{\partial f}{\partial x}(0,0)(x-0) - \frac{\partial f}{\partial y}(0,0)(y-0)}{\sqrt{x^2+y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{\frac{x^2 y}{x^4+y^2}}{\sqrt{x^2+y^2}}.
\end{aligned}$$

Along the parabolic path $(x,y) = (t, t^2)$, we have

$$\frac{\frac{x^2 y}{x^4+y^2}}{\sqrt{x^2+y^2}} = \frac{\frac{t^2 \cdot t^2}{t^4+t^4}}{\sqrt{2}\,|t|} = \frac{1}{2\sqrt{2}\,|t|} \to +\infty \quad \text{as } t \to 0.$$

Hence the limit does not exist, and so f is not differentiable at $(0,0)$.

Exercise 3.4. Denote the unit circle in $\mathbb{R}^2$ by $\partial B_1(\mathbf{0}) := \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| = 1\}$. Let $g : \partial B_1(\mathbf{0}) \to \mathbb{R}$ be a continuous function on $\partial B_1(\mathbf{0})$ such that $g(0,1) = g(1,0) = 0$, and $g(-\mathbf{x}) = -g(\mathbf{x})$ for any $\mathbf{x} \in \partial B_1(\mathbf{0})$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(\mathbf{x}) = \begin{cases} |\mathbf{x}| \cdot g\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) & \text{if } \mathbf{x} \neq 0 \\ 0 & \text{if } \mathbf{x} = 0 \end{cases}$$

(a) Determine whether (or under what condition) the directional derivative $D_{\mathbf{u}} f(\mathbf{0})$ exists, where $\mathbf{u}$ is a unit vector in $\mathbb{R}^2$.

(b) Determine whether (or under what condition) f is differentiable at $\mathbf{0}$.

Exercise 3.5. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a function satisfying $|f(\mathbf{x})| \leq |\mathbf{x}|^2$ for any $\mathbf{x} \in \mathbb{R}^n$. Show that f is differentiable at $\mathbf{0}$.

3.1.3. Jacobian matrix. Recall that a vector-valued function

$$F(x,y) = \big(u(x,y), v(x,y)\big) : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$$

is differentiable at (a,b) if both u and v are differentiable scalar-valued functions at (a,b), meaning that all partial derivatives of u and v exist at (a,b), and

$$u(x,y) = u(a,b) + \left.\frac{\partial u}{\partial x}\right|_{(x,y)=(a,b)} (x-a) + \left.\frac{\partial u}{\partial y}\right|_{(x,y)=(a,b)} (y-b) + o(|(x,y)-(a,b)|)$$

$$v(x,y) = v(a,b) + \left.\frac{\partial v}{\partial x}\right|_{(x,y)=(a,b)} (x-a) + \left.\frac{\partial v}{\partial y}\right|_{(x,y)=(a,b)} (y-b) + o(|(x,y)-(a,b)|)$$

Express the above in matrix form, we get:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} u(a,b) \\ v(a,b) \end{bmatrix} + \left.\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}\right|_{(x,y)=(a,b)} \begin{bmatrix} x-a \\ y-b \end{bmatrix} + \begin{bmatrix} h_1(x,y) \\ h_2(x,y) \end{bmatrix}$$

where $h_i \in o\big(\,|(x,y)-(a,b)|\,\big)$ for $i = 1,2$. The matrix

$$\left.\frac{\partial(u,v)}{\partial(x,y)}\right|_{(x,y)=(a,b)} = \left.\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}\right|_{(x,y)=(a,b)}$$

is called the **Jacobian matrix** of F at (a,b), which is also commonly denoted by $DF(a,b)$. In higher dimensions, we can rewrite the definition of differentiability for a function $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ as follows: F is differentiable at $\mathbf{a} \in \Omega$ if

$$F(\mathbf{x}) = F(\mathbf{a}) + DF(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \mathbf{h}(\mathbf{x})$$

where $\mathbf{h}(\mathbf{x}) \in o(|\mathbf{x} - \mathbf{a}|)$ as $\mathbf{x} \to \mathbf{a}$. Here $F = (f^1, \cdots, f^m)$, and $DF(\mathbf{a})$ is the Jacobian matrix of F at $\mathbf{a}$ whose (i,j)-th component is $\frac{\partial f^i}{\partial x_j}(\mathbf{a})$. The vector $\mathbf{x} - \mathbf{a}$ is regarded as a column vector, and $\cdot$ means the matrix-vector product. We say a vector-valued function $\mathbf{h}(\mathbf{x})$ is of $o(|\mathbf{x} - \mathbf{a}|)$ if each of its component is of $o(|\mathbf{x} - \mathbf{a}|)$. The (affine) linear map $\mathbf{x} \mapsto F(\mathbf{a}) + DF(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$ is called the **linear approximation** of F at $\mathbf{a}$.

Exercise 3.6. Prove that if $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{a} \in \Omega$, then it is continuous at $\mathbf{a}$.

3.1.4. C^1 functions. Although existence of partial derivatives does not imply differentiability of multivariable functions, if we impose further conditions on the partial derivatives, namely continuity, then it would imply differentiability.

Definition 3.7 (C^1 functions)**.** A function $F = (f^1, \cdots, f^m) : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is said to be C^1 at $\mathbf{a} \in \Omega$ if

(1) there exists $\varepsilon > 0$ such that all partial derivatives $\frac{\partial f^i}{\partial x_j}$ exist on $B_\varepsilon(\mathbf{a})$, and

(2) all partial derivatives $\frac{\partial f^i}{\partial x_j}$ are continuous at $\mathbf{a}$.

If F is C^1 at every point in the open domain Ω, we may simply say F is C^1 on Ω.

Example 3.8. The function $f(x,y) = |xy|$ is not C^1 at $(0,0)$, because $\frac{\partial f}{\partial x}$ does not exist at any $(0,b)$ where $b \neq 0$:

$$\frac{\partial f}{\partial x}(0,b) = \lim_{t \to 0} \frac{f(0+t,b) - f(0,b)}{t - 0} = \lim_{t \to 0} \frac{|t|}{t} \cdot |b|\,.$$

Any open ball $B_\varepsilon(0,0)$ center at $(0,0)$ must include some points $(0,b)$, $b \neq 0$, so condition (1) fails.

Example 3.9. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } x+y \neq 0 \\ 0 & \text{if } x+y=0 \end{cases}.$$

Denote the diagonal $x+y=0$ by $\Delta := \{(a,b) : a+b=0\}$. For any $(a,b) \notin \Delta$, the function f is $\frac{x^2y}{x^2+y^2}$ on a small open ball $B_\varepsilon(a,b)$. Partial derivatives of f at this (a,b) depend only on its values on this ball, and we can simply take partial derivatives as usual:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{(x^2+y^2)(2xy) - x^2y(2y)}{(x^2+y^2)^2} = \frac{2xy(x^2+y^2-xy)}{(x^2+y^2)^2} \\ \frac{\partial f}{\partial y} &= \frac{(x^2+y^2)(x^2) - x^2y(2x)}{(x^2+y^2)^2} = \frac{x^2(x-y)^2}{(x^2+y^2)^2} \end{aligned}$$

As the partial derivatives are rational functions and the denominator is non-zero on $\mathbb{R}^2 - \Delta$, they are continuous on $\mathbb{R}^2 - \Delta$. Therefore, f is C^1 on $\mathbb{R}^2 - \Delta$.

Now we discuss the case when $(a,b) \in \Delta$, i.e. $a=-b$, as f is defined differently at (a,b) and at $(x,y) \notin \Delta$, we need to find its partial derivatives from the definition:

$$\begin{aligned} \frac{\partial f}{\partial x}(a,b) &= \lim_{t\to 0} \frac{f(a+t,b)-f(a,b)}{t-0} = \lim_{t\to 0} \frac{f(a+t,-a)-f(a,-a)}{t-0} \\ &= \lim_{t\to 0} \frac{\frac{(a+t)^2(-a)}{(a+t)^2+a^2} - 0}{t} = \begin{cases} \text{does not exist} & \text{if } a \neq 0 \\ 0 & \text{if } a = 0 \end{cases} \\ \frac{\partial f}{\partial y}(a,b) &= \lim_{t\to 0} \frac{f(a,-a+t)-f(a,-a)}{t-0} \\ &= \lim_{t\to 0} \frac{\frac{a^2(-a+t)}{a^2+(-a+t)^2}}{t-0} = \begin{cases} \text{does not exist} & \text{if } a \neq 0 \\ 0 & \text{if } a = 0 \end{cases} \end{aligned}$$

Therefore, partial derivatives exist at $(a,-a)$ only when $a=0$. In particular, f is not C^1 at any $(a,-a)$ with $a \neq 0$.

We are left to determine whether f is C^1 at $(0,0)$, i.e. whether $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous at $(0,0)$. It can be easily seen that along the two paths $(x,y)=(t,t)$ and $(x,y)=(t,-t)$, the limits of $\frac{\partial f}{\partial y}$ are different as $t \to 0$:

$$\begin{aligned} \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(t,t)} &= \lim_{t\to 0} \frac{t^2(t-t)^2}{(t^2+t^2)^2} = 0 \\ \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(t,-t)} &= \lim_{t\to 0} \frac{t^2(2t)^2}{(t^2+(-t)^2)^2} = 1. \end{aligned}$$

Therefore, $\frac{\partial f}{\partial y}$ is not continuous at $(0,0)$, and so f is not C^1. Also, we can argue that by saying that any open ball $B_\varepsilon\big((0,0)\big)$ must contain some points $(a,-a)$, where $a \neq 0$. Both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ do not exist at those points, so f is not C^1 at $(0,0)$.

The reason why we discuss C^1 functions is because they are also differentiable (but not vice versa).

Proposition 3.10 (C^1 implies differentiability). *If $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is C^1 at $\mathbf{a} \in \Omega$, then it is also differentiable at $\mathbf{a}$.*

Proof. Because both differentiability and continuity conditions can be verified component-wise, it suffices to consider only the case of scalar functions $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. For simplicity, we give the proof when $n=2$. The higher dimensional case is similar – just that we have more variables and terms to write down – we leave it as an exercise for readers.

Consider $f(x,y) : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ and $(a,b) \in \Omega$. If f is C^1 at (a,b), then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist on some open ball $B_\varepsilon\big((a,b)\big)$, and they are continuous at (a,b). The key idea of the proof is to make use of the mean-value theorem in single-variable calculus. First we rewrite:

$$\begin{aligned} f(x,y) - f(a,b) &= f(x,y) - f(x,b) + f(x,b) - f(a,b) \\ &= \left.\frac{\partial f}{\partial y}\right|_{(x,u)} (y-b) + \left.\frac{\partial f}{\partial x}\right|_{(v,b)} (x-a) \end{aligned}$$

where u is between y and b (and it could depend on x too), and v is between x and a. As f is C^1 at (a,b), both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous at (a,b). As $(x,y) \to (a,b)$, by squeeze theorem we have $(x,u) \to (a,b)$ and $(v,b) \to (a,b)$ too. Therefore, by continuity we get

$$\lim_{(x,y)\to(a,b)} \left.\frac{\partial f}{\partial x}\right|_{(x,u)} = \left.\frac{\partial f}{\partial x}\right|_{(a,b)} \quad \text{and} \quad \lim_{(x,y)\to(a,b)} \left.\frac{\partial f}{\partial y}\right|_{(v,b)} = \left.\frac{\partial f}{\partial y}\right|_{(a,b)}.$$

In other words, we have

$$\begin{aligned} \left.\frac{\partial f}{\partial x}\right|_{(x,u)} &= \left.\frac{\partial f}{\partial x}\right|_{(a,b)} + o(1) \\ \left.\frac{\partial f}{\partial y}\right|_{(v,b)} &= \left.\frac{\partial f}{\partial y}\right|_{(a,b)} + o(1) \end{aligned}$$

as $(x,y) \to (a,b)$. Combining the above results, we conclude that

$$\begin{aligned} f(x,y) - f(a,b) &= \left(\left.\frac{\partial f}{\partial x}\right|_{(a,b)} + o(1) \right)(x-a) + \left(\left.\frac{\partial f}{\partial y}\right|_{(a,b)} + o(1) \right)(y-b) \\ &= \left.\frac{\partial f}{\partial x}\right|_{(a,b)} (x-a) + \left.\frac{\partial f}{\partial y}\right|_{(a,b)} (y-b) + o(x-a) + o(y-b). \end{aligned}$$

Since $o(x-a) \in o\big(\,|(x,y)-(a,b)|\,\big)$ as $o(y-b) \in o\big(\,|(x,y)-(a,b)|\,\big)$, we conclude that

$$f(x,y) = f(a,b) + \left.\frac{\partial f}{\partial x}\right|_{(a,b)} (x-a) + \left.\frac{\partial f}{\partial y}\right|_{(a,b)} (y-b) + o\big(\,|(x,y)-(a,b)|\,\big)$$

as desired. □

Exercise 3.7. Write down the proof of the Proposition 3.10 for any $n \in \mathbb{N}$.

Exercise 3.8. Consider a function $F(x_1, \cdots, x_n) = (f^1, \cdots, f^m) : B_R \subset \mathbb{R}^n \to \mathbb{R}^m$ whose domain B_R is an open ball. Suppose the first partial derivatives $\frac{\partial f^j}{\partial x_i}$ are all bounded on B_R, prove that there exists a constant $C > 0$, depending only m, n and the bounds of $\frac{\partial f^j}{\partial x_i}$'s, such that

$$|F(\mathbf{x}) - F(\mathbf{y})| \leq C\,|\mathbf{x} - \mathbf{y}|$$

for any $\mathbf{x}, \mathbf{y} \in B_R$.

Exercise 3.9. Let A be a real $m \times n$ matrix, and we define:

$$\|A\| := \sup\{|A\mathbf{x}| : \mathbf{x} \in \mathbb{R}^n \text{ and } \|\mathbf{x}\| = 1\}.$$

(a) Prove that $\|\cdot\|$ is a norm on the vector space $M_{m\times n}(\mathbb{R})$, the space of $m \times n$ real matrices.

(b) Denote the (i,j)-th entry of A by a_{ij}. Show that $|a_{ij}| \leq \|A\|$ for any i, j.

(c) Show that there exists a constant $C_{m,n} > 0$, depending only on m and n, such that $\|A\| \leq C_{m,n}\sqrt{\sum_{\text{all } i,j} a_{ij}^2}$

(d) Show that for any $\mathbf{x} \in \mathbb{R}^n$, we have $|A\mathbf{x}| \leq \|A\|\,|\mathbf{x}|$.

Remark 3.11. Note that the function $f(x,y) = |xy|$ is differentiable at $(0,0)$, but it is not C^1 at $(0,0)$. Therefore, the converse of Proposition 3.10 does not hold.

3.1.5. Chain rule. From the linear approximation viewpoint on differentiations, one can formulate the multivariable chain rule as matrix products of Jacobian matrices.

Theorem 3.12 (Chain rule). *Consider two functions $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ and $G : U \subset \mathbb{R}^m \to \mathbb{R}^k$ where $F(\Omega) \subset U$ and U is open in $\mathbb{R}^m$. Suppose F is differentiable at $\mathbf{a} \in \Omega$, and G is differentiable at $F(\mathbf{a})$, then $G \circ F : \Omega \to \mathbb{R}^k$ is differentiable at $\mathbf{a}$, and the Jacobian matrices satisfy:*

$$D(G \circ F)(\mathbf{a}) = DG\big(F(\mathbf{a})\big) \cdot DF(\mathbf{a}).$$

Here $\cdot$ denotes the matrix product.

Proof. The key idea is to write F and G into linear approximation forms, i.e. there exist functions $\mathbf{h} : B_\eta(\mathbf{a}) \subset \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{k} : B_\rho\big(F(\mathbf{a})\big) \subset \mathbb{R}^m \to \mathbb{R}^k$ be defined near $\mathbf{a}$ and $F(\mathbf{a})$ respectively, such that:

$$\begin{aligned} F(\mathbf{x}) &= F(\mathbf{a}) + DF(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \mathbf{h}(\mathbf{x}) \\ G(\mathbf{y}) &= G\big(F(\mathbf{a})\big) + DG\big(F(\mathbf{a})\big) \cdot \big(\mathbf{y} - F(\mathbf{a})\big) + \mathbf{k}(\mathbf{y}) \end{aligned}$$

where $\mathbf{h}(\mathbf{x}) \in o\big(|\mathbf{x} - \mathbf{a}|\big)$ as $\mathbf{x} \to \mathbf{a}$, and $\mathbf{k}(\mathbf{y}) \in o\big(|\mathbf{y} - F(\mathbf{a})|\big)$ as $\mathbf{y} \to F(\mathbf{a})$. Since $F(\mathbf{x}) \to F(\mathbf{a})$ as $\mathbf{x} \to \mathbf{a}$, by substituting $F(\mathbf{x})$ into $\mathbf{y}$, we have that when $\mathbf{x}$ is sufficiently close to $\mathbf{a}$,

$$\begin{aligned} G\big(F(\mathbf{x})\big) &= G\big(F(\mathbf{a})\big) + DG\big(F(\mathbf{a})\big) \cdot \big(F(\mathbf{a}) + DF(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \mathbf{h}(\mathbf{x}) - F(\mathbf{a})\big) + \mathbf{k}\big(F(\mathbf{x})\big) \\ &= G\big(F(\mathbf{a})\big) + DG\big(F(\mathbf{a})\big) \cdot DF(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + DG\big(F(\mathbf{a})\big) \cdot \mathbf{h}(\mathbf{x}) + \mathbf{k}\big(F(\mathbf{x})\big). \end{aligned}$$

Our remaining task is to prove that $DG\big(F(\mathbf{a})\big) \cdot \mathbf{h}(\mathbf{x}) + \mathbf{k}\big(F(\mathbf{x})\big) \in o\big(|\mathbf{x} - \mathbf{a}|\big)$ as $\mathbf{x} \to \mathbf{a}$.

Since $DG\big(F(\mathbf{a})\big)$ is a constant matrix, we have $DG\big(F(\mathbf{a})\big) \cdot \mathbf{h}(\mathbf{x}) \in o\big(|\mathbf{x} - \mathbf{a}|\big)$. For the term $\mathbf{k}\big(F(\mathbf{x})\big)$: since $\mathbf{k}(\mathbf{y}) \in o\big(|\mathbf{y} - F(\mathbf{a})|\big)$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $|\mathbf{y} - F(\mathbf{a})| < \delta$, we have

$$|\mathbf{k}(\mathbf{y})| \leq \varepsilon' \,|\mathbf{y} - F(\mathbf{a})|\,.$$

Here $\varepsilon' > 0$ is to be chosen. Furthermore, by $F(\mathbf{x}) \to F(\mathbf{a})$ as $\mathbf{x} \to \mathbf{a}$, there exists $\theta > 0$ sufficiently small such that if $|\mathbf{x} - \mathbf{a}| < \theta$, we have $|F(\mathbf{x}) - F(\mathbf{a})| < \delta$, and so from the above we have

$$\big|\mathbf{k}\big(F(\mathbf{x})\big)\big| \leq \varepsilon' \,|F(\mathbf{x}) - F(\mathbf{a})| = \varepsilon' \,|DF(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \mathbf{h}(\mathbf{x})|$$

Recall that $\mathbf{h} \in o\big(|\mathbf{x} - \mathbf{a}|\big)$, there exists $\theta' > 0$ such that $|\mathbf{h}(\mathbf{x})| \leq |\mathbf{x} - \mathbf{a}|$ when $|\mathbf{x} - \mathbf{a}| < \theta'$.

Combining the above results, when $|\mathbf{x} - \mathbf{a}| < \min\{\theta, \theta'\} =: \theta''$, we have

$$|\mathbf{k}(F(\mathbf{x}))| \leq \varepsilon' \,\|DF(\mathbf{a})\|\, |\mathbf{x} - \mathbf{a}| + \varepsilon' \,|\mathbf{x} - \mathbf{a}|\,.$$

Now picking

$$\varepsilon' = \frac{\varepsilon}{1 + \|DF(\mathbf{a})\|},$$

then we have proven that for any $\varepsilon > 0$, there exists $\theta'' > 0$ such that if $|\mathbf{x} - \mathbf{a}| < \theta''$, then $\big|\mathbf{k}\big(F(\mathbf{x})\big)\big| \leq \varepsilon\,|\mathbf{x} - \mathbf{a}|$. In other words, we conclude that $\mathbf{k}\big(F(\mathbf{x})\big) \in o\big(\,|\mathbf{x} - \mathbf{a}|\,\big)$ as $\mathbf{x} \to \mathbf{a}$, completing the proof. □

In terms of partial derivatives, the chain rule, say in two dimensions, can be written in the following form. Let $F(x, y) = (u(x, y), v(x, y))$, and $G(u, v) = (w(u, v), z(u, v))$, then the chain rule for $(w, z) = G \circ F(x, y)$ is

$$\begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}.$$

In particular, we have

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$

and similarly for other combinations and also in higher dimensions. It is exactly what we have learned in MATH 2023.

Exercise 3.10. Show that the directional derivative $D_{\mathbf{u}}f(\mathbf{a})$ of a differentiable function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ can be written as the single-variable derivative:

$$D_{\mathbf{u}}f(\mathbf{a}) = \left.\frac{d}{dt}\right|_{t=0} f(\mathbf{a} + t\mathbf{u}).$$

Hence, prove using the chain rule that

$$D_{\mathbf{u}}f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{u}.$$

Exercise 3.11. Using the chain rule, generalize the result of Exercise 3.8 so that the domain of F can be any convex open set $\Omega \subset \mathbb{R}^n$, not just open balls. Hint: for any $\mathbf{x}, \mathbf{y} \in \Omega$, construct a straight-path joining $\mathbf{x}$ and $\mathbf{y}$.

3.2. Higher-Order Differentiability

3.2.1. Higher-order partial derivatives. In single-variable calculus, the derivative $f'(x)$ of a function f is denoted by $f''(x) := \frac{d}{dx}f'(x)$ (alternatively $f^{(2)}(x)$). If $f''(x)$ exists, then we say f is twice differentiable, and call f'' the second-order derivative of f. Inductively, we say f is n-times differentiable if f is $(n-1)$-times differentiable and its $(n-1)$-order derivative $f^{(n-1)}$ is differentiable, and we denote $f^{(n)} = \frac{d}{dx}f^{(n-1)}$. In this case we call $f^{(n)}$ the n-th order derivative, and we say f is n-times differentiable.

In MATH 1023, we saw that if $f^{(n)}$ exists at $x = a$, then f can be approximated by an n-th degree polynomial $T_n(x)$ in a sense that

$$f(x) = T_n(x) + o\big((x-a)^n\big) \quad \text{as } x \to a.$$

Note that the the converse is not true (see footnote remark[2]).

Now for multivariable calculus, we could also take second-order partial derivatives by differentiating the first partial derivatives. For instance, for a function $f(x,y) : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, we can define

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &:= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) & \qquad \frac{\partial^2 f}{\partial x \partial y} &:= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \\ \frac{\partial^2 f}{\partial y \partial x} &:= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) & \qquad \frac{\partial^2 f}{\partial y^2} &:= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \end{aligned}$$

whenever they exist. Alternatively, we may denote $\frac{\partial^2 f}{\partial x^2}$ by f_{xx}, and $\frac{\partial^2 f}{\partial x \partial y}$ by f_{yx}, etc. Although f_{yx} and f_{xy} are defined in different ways, we will later prove that they are the same if f_{xy} and f_{yx} are continuous. However, let's assume they are different for now until that result is proved.

Generally speaking, we denote:

$$\frac{\partial^k f}{\partial x_{i_k} \partial x_{i_{k-1}} \cdots \partial x_{i_1}} := \frac{\partial}{\partial x_{i_k}} \frac{\partial}{\partial x_{i_{k-1}}} \cdots \frac{\partial}{\partial x_{i_2}} \frac{\partial f}{\partial x_{i_1}}$$

and call it a k-order partial derivatives. It can also be denoted as $f_{x_{i_1} x_{i_2} \cdots x_{i_k}}$.

Exercise 3.12. For a function $f : \mathbb{R}^n \to \mathbb{R}$, how many different k-th order partial derivatives are there?

Exercise 3.13. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

Find f_x and f_y at any $(x,y) \in \mathbb{R}^2$, and hence find $f_{xy}(0,0)$ and $f_{yx}(0,0)$ (which are not equal for this example).

3.2.2. Higher-order differentiability. Next we give the definition of higher-order differentiability for multivariable functions. Just like for first-order differentiability, the existence of partial derivatives is not sufficient to claim the function is differentiable.

[2] In this course, we define "f is n-times/order differentiable" if the n-order derivative $f^{(n)}$ exists. Some author defines that to mean f can be approximated by an n-th degree polynomial. Note that they are not equivalent.

Definition 3.13 (Higher-order differentiable functions)**.** Consider a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, we say that f is k**-times differentiable** at $\mathbf{a} \in \Omega$ if

(1) all k-th order partial derivatives of f exist at $\mathbf{a}$; and
(2) all $(k-1)$-th order partial derivatives of f are differentiable at $\mathbf{a}$.

Condition (2) means for any $(k-1)$-order partial derivative $f_{x_{i_1}\cdots x_{i_{k-1}}}$, we have

$$f_{x_{i_1}\cdots x_{i_{k-1}}}(\mathbf{x}) = f_{x_{i_1}\cdots x_{i_{k-1}}}(\mathbf{a}) + \sum_{j=1}^{n} \frac{\partial f_{x_{i_1}\cdots x_{i_{k-1}}}}{\partial x_j}(\mathbf{a})(x_i - a_i) + o\big(|\mathbf{x}-\mathbf{a}|\big) \quad \text{as } \mathbf{x} \to \mathbf{a}.$$

We say a vector-valued function $F = (f^1, \cdots, f^m) : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is k-times differentiable at $\mathbf{a} \in \Omega$ if each component $f^i : \Omega \to \mathbb{R}$ is k-times differentiable at $\mathbf{a}$.

Remark 3.14. In order for the k-th order partial derivatives to exist at $\mathbf{a}$, it is necessary that all $(k-1)$-th order partial derivatives exist at an open ball $B_\varepsilon(\mathbf{a})$ centered at $\mathbf{a}$. This condition is implicitly assumed whenever we say k-th order partial derivatives exist at $\mathbf{a}$.

Recall that for first-order derivatives we have a stronger condition called C^1 (which also requires the first partial derivatives are continuous). In higher-order, we also have a stronger notion called C^k:

Definition 3.15 (C^k functions)**.** A function $f : \Omega \to \mathbb{R}$ is said to be C^k at $\mathbf{a} \in \Omega$ if all k-th order partial derivatives of f exist on some ball $B_\varepsilon(\mathbf{a})$, and they are continuous at $\mathbf{a}$.

Recall that C^1 functions are differentiable. Therefore, if $f : \Omega \to \mathbb{R}$ is C^k at $\mathbf{a} \in \Omega$, then all $(k-1)$-th order partial derivatives of are C^1 at $\mathbf{a}$, and so they are differentiable at $\mathbf{a}$. This implies f is k-times differentiable at $\mathbf{a}$. To conclude, we have the following result:

Corollary 3.16 (to Proposition 3.10)**.** *If a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is C^k at $\mathbf{a} \in \Omega$, then it is k-times differentiable at $\mathbf{a}$.*

Example 3.17. Again, the converse to the above corollary is not true. Here is a counterexample:

$$f(x,y) = \left(\int_0^x |s|\ ds\right)\left(\int_0^y |t|\ dt\right).$$

We will show that f is twice differentiable at $(0,0)$, but is not C^2 at $(0,0)$.

By Fundamental Theorem of Calculus (note that the absolute-value function is continuous), we have

$$\frac{\partial f}{\partial x} = |x| \int_0^y |t|\ dt \qquad\qquad \frac{\partial f}{\partial y} = |y| \int_0^x |s|\ ds.$$

As $x \mapsto |x|$ is not differentiable at any $a \neq 0$, the second partial derivative $\dfrac{\partial^2 f}{\partial x^2}$ does not exist at any $(0,b)$ where $b \neq 0$. This already shows f is not C^2 at $(0,0)$, as any small open ball $B_\varepsilon\big((0,0)\big)$ must contain points $(0,b)$ with $b \neq 0$.

However, one can still prove that f is twice differentiable at $(0,0)$. We first compute the second partial derivatives:

$$\frac{\partial^2 f}{\partial x^2}(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial x}(h,0) - \frac{\partial f}{\partial x}(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$
$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial x}(0,h) - \frac{\partial f}{\partial x}(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

Similarly, one can also show $\frac{\partial^2 f}{\partial y^2}(0,0) = \frac{\partial^2 f}{\partial x \partial y}(0,0) = 0$. Now we are left to check that:

$$\lim_{(x,y)\to(0,0)} \frac{\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0) - \frac{\partial^2 f}{\partial x^2}(0,0)\cdot(x-0) - \frac{\partial^2 f}{\partial y \partial x}(0,0)\cdot(y-0)}{\sqrt{x^2+y^2}}$$
$$= \lim_{(x,y)\to(0,0)} \frac{|x| \int_0^y |t|\, dt}{\sqrt{x^2+y^2}} = 0$$

by the facts that $\dfrac{|x|}{\sqrt{x^2+y^2}} \le 1$ and $\displaystyle\lim_{(x,y)\to(0,0)} \int_0^y |t|\, dt = 0$. Hence, we conclude $\frac{\partial f}{\partial x}$ is differentiable at $(0,0)$. Similarly, one can also check that $\frac{\partial f}{\partial y}$ is differentiable at $(0,0)$ too.

This conclude f is twice-differentiable at $(0,0)$.

Exercise 3.14. Determine whether or not the function in Exercise 3.13 is twice-differentiable or not.

The C^k condition of a multivariable function is easier to be verified as it only involves partial derivatives. Therefore, many theorems in analysis in higher dimensions assume the function is C^k instead of being k-times differentiable (even though the former is more restrictive).

One benefit of being C^k is that the partial derivatives up to k-order commute, e.g $f_{xyx} = f_{yxx} = f_{xxy}$ for a C^3 function f.

Theorem 3.18 (Clairaut). *Suppose $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is C^2 at $\mathbf{a} \in \Omega$, then we have*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{a})$$

for any $i, j = 1, 2, \cdots, n$.

Proof. It suffices to prove the case $n = 2$, as the theorem only involves two directions x_i and x_j. Also, for simplicity we denote x_i by x, x_j by y, and $\mathbf{a} = (a,b)$. We need to prove that $\dfrac{\partial^2 f}{\partial x \partial y}(a,b) = \dfrac{\partial^2 f}{\partial y \partial x}(a,b)$.

Denote $\Delta(h,k) := f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a,b)$. We first apply the mean value theorem on Δ in two different ways. Define:

$$g_1(x) := f(x, b+k) - f(x,b)$$
$$g_2(y) := f(a+h, y) - f(a,y)$$

We apply the mean value theorem by grouping the terms as:

$$\begin{aligned}
\Delta(h,k) &= \big(f(a+h,b+k) - f(a+h,b)\big) - \big(f(a,b+k) - f(a,b)\big) \\
&= g_1(a+h) - g_1(a) \\
&= g_1'(c_1)\cdot h \\
&= \big(f_x(c_1, b+k) - f_x(c_1, b)\big)h
\end{aligned}$$

for some c_1 between a and $a+h$. Note that c_1 depends on a, b, h, k, but by squeeze theorem we have $c_1 \to 0$ as $(h,k) \to (0,0)$. Applying mean value theorem on $f_x(c_1, \cdot)$ above, we then get

$$\Delta(h,k) = f_{xy}(c_1, c_2)hk$$

for some c_2 between b and $b+k$, and as $(h,k) \to 0$ we have $c_2 \to b$.

On the other hand, we also have

$$\begin{aligned}\Delta(h,k) &= \big(f(a+h, b+k) - f(a, b+k)\big) - \big(f(a+h, b) - f(a,b)\big)\\ &= g_2(b+k) - g_2(b)\\ &= g_2'(c_2) \cdot k\\ &= \big(f_y(a+h, c_3) - f_y(a, c_3)\big)k\end{aligned}$$

for some c_3 between b and $b+k$, and as $(h,k) \to 0$ we have $c_3 \to b$. Similarly, applying mean value theorem on $f_y(\cdot, c_3)$ we get

$$\Delta(h,k) = f_{yx}(c_4, c_3)hk$$

for some c_4 between a and $a+h$, and as $(h,k) \to 0$ we have $c_4 \to a$

Equating both sides, we then have

$$f_{xy}(c_1, c_2) = f_{yx}(c_4, c_2)$$

for any (h,k) such that $h \neq 0$ and $k \neq 0$.

Since $(0,0)$ is a limit point of the set $\{(h,k) : h \neq 0 \text{ and } k \neq 0\}$, we can let $(h,k) \to (0,0)$ on both sides of the above equation. Noting that f_{xy} and f_{yx} are continuous (as f is C^2), and $(c_1, c_2) \to (a,b)$ and $(c_4, c_3) \to (a,b)$ as $(h,k) \to (0,0)$, we finally conclude that

$$f_{xy}(a,b) = f_{yx}(a,b)$$

as desired. □

By induction, one can then conclude that if f is C^k at $\mathbf{a}$, then all partial derivatives up to and including k-th order commute. For instance, for a C^4 function $f(x,y,z)$ we have

$$f_{xyy} = f_{yyx} = f_{yxy}, \quad f_{xyzx} = f_{zxyx} = f_{yzxx}, \quad \text{etc.}$$

As such, we typically work with C^k multivariable functions instead k-order differentiable functions. In many theorems in analysis, PDE, and related fields, the C^k condition is much more often used as the hypothesis in many theorems.

Exercise 3.15. Modify the proof of Clairaut's Theorem, and prove a stronger result:

Suppose $f(x,y) : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a function such that f_x, f_y and f_{xy} exist on some ball $B_\varepsilon(\mathbf{a})$, and are all continuous at $\mathbf{a}$, then $f_{yx}(\mathbf{a})$ exists and $f_{xy}(\mathbf{a}) = f_{yx}(\mathbf{a})$.

3.2.3. Taylor approximation. Similar to single-variable calculus, we can approximate higher-order differentiable functions by polynomials.

Proposition 3.19. *If $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is k-times differentiable at $\mathbf{a} \in \Omega$, then as $\mathbf{x} \to \mathbf{a}$, we have:*

$$\begin{aligned} f(\mathbf{x}) = f(\mathbf{a}) &+ \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{a})(x_i - a_i) + \frac{1}{2!}\sum_{j,k=1}^n \frac{\partial^2 f}{\partial x_j \partial x_k}(\mathbf{a})(x_j - a_j)(x_k - a_k) \\ &+ \cdots \\ &+ \frac{1}{k!}\sum_{i_1,\cdots,i_k=1}^n \frac{\partial^k f}{\partial x_{i_k}\cdots\partial x_{i_1}}(\mathbf{a})(x_{i_1} - a_{i_1})\cdots(x_{i_k} - a_{i_k}) + o\big(|\mathbf{x}-\mathbf{a}|^k\big). \end{aligned}$$

The k-degree polynomial

$$T_k(\mathbf{x}) := f(\mathbf{a}) + \sum_{j=1}^k \sum_{i_1,\cdots,i_j=1}^n \frac{1}{j!}\frac{\partial^j f}{\partial x_{i_j}\cdots\partial x_{i_1}}(\mathbf{a})(x_{i_1} - a_{i_1})\cdots(x_{i_j} - a_{i_j})$$

is called the k-th degree Taylor approximation of f at $\mathbf{a}$.

Proof. We prove the case $k = 2$ as an example, and leave the general case as an exercise for readers. First consider a small ball $B_\varepsilon(\mathbf{a}) \subset \Omega$. For each $\mathbf{x} \in B_\varepsilon(\mathbf{a})$, we consider the function $g : [0,1] \to \mathbb{R}$ defined by

$$g(t) := f\big((1-t)\mathbf{a} + t\mathbf{x}\big),$$

and the "error" function $E(t) : [0,1] \to \mathbb{R}$ defined by

$$E(t) := g(t) - \left(g(0) + g'(0)t + \frac{g''(0)}{2!}t^2\right).$$

Note that both g and E depends on $\mathbf{x}$ even though we do not explicitly write them as $g(\mathbf{x}, t)$ and $E(\mathbf{x}, t)$.

Using the chain rule, we can easily calculate that

$$\begin{aligned} g'(t) &= \sum_{i=1}^n \left.\frac{\partial f}{\partial x_i}\right|_{(1-t)\mathbf{a}+t\mathbf{x}} (x_i - a_i) \\ g''(t) &= \sum_{i=1}^n \left.\frac{\partial^2 f}{\partial x_j \partial x_i}\right|_{(1-t)\mathbf{a}+t\mathbf{x}} (x_i - a_i)(x_j - a_j). \end{aligned}$$

Therefore, $E(1)$ is given by

$$E(1) = f(\mathbf{x}) - \left(f(\mathbf{a}) + \sum_{i=1}^n \left.\frac{\partial f}{\partial x_i}\right|_{\mathbf{a}} (x_i - a_i) + \frac{1}{2!}\sum_{i=1}^n \left.\frac{\partial^2 f}{\partial x_j \partial x_i}\right|_{\mathbf{a}} (x_i - a_i)(x_j - a_j)\right),$$

which measures the error between $f(\mathbf{x})$ and $T_2(\mathbf{x})$. Therefore, our goal is to prove that $E(1) \in o\big(|\mathbf{x}-\mathbf{a}|^2\big)$ as $\mathbf{x} \to \mathbf{a}$.

Applying the Cauchy's mean value theorem on functions $E(t)$ and t^2, there exists $c \in (0,1)$ such that

$$\frac{E(1) - E(0)}{1^2 - 0^2} = \frac{E'(c)}{2c} \implies E(1) = \frac{g'(c) - g'(0) - g''(0)c}{2c}.$$

This shows

$$\begin{aligned} &E(1) \\ &= \frac{1}{2c}\sum_{i=1}^n \left(\left.\frac{\partial f}{\partial x_i}\right|_{(1-c)\mathbf{a}+c\mathbf{x}} - \left.\frac{\partial f}{\partial x_i}\right|_{\mathbf{a}}\right)(x_i - a_i) - \frac{1}{2}\sum_{i=1}^n \left.\frac{\partial^2 f}{\partial x_j \partial x_i}\right|_{\mathbf{a}} (x_i - a_i)(x_j - a_j). \end{aligned}$$

Next we consider the linear approximation of $\frac{\partial f}{\partial x_i}$ at $\mathbf{a}$ (recall f is twice differentiable at $\mathbf{a}$ so $\frac{\partial f}{\partial x_i}$ is differentiable at $\mathbf{a}$):

$$\left.\frac{\partial f}{\partial x_i}\right|_{(1-c)\mathbf{a}+c\mathbf{x}} = \left.\frac{\partial f}{\partial x_i}\right|_{\mathbf{a}} + \sum_{j=1}^{n} \left.\frac{\partial^2 f}{\partial x_j \partial x_i}\right|_{\mathbf{a}} \big((1-c)a_j + cx_j - a_j\big) + \varepsilon((1-c)\mathbf{a} + c\mathbf{x}),$$

where $\varepsilon(\mathbf{x})$ is a function defined near $\mathbf{a}$ such that $\varepsilon(\mathbf{x}) \in o\big(|\mathbf{x}-\mathbf{a}|\big)$ as $\mathbf{x} \to \mathbf{a}$. Substituting this back in and by cancellation of the second derivative terms, we get:

$$E(1) = \sum_{i=1}^{n} \varepsilon\big((1-c)\mathbf{a} + c\mathbf{x}\big)(x_i - a_i).$$

Since

$$\varepsilon\big((1-c)\mathbf{a} + c\mathbf{x}\big) \in o\big(|(1-c)\mathbf{a} + c\mathbf{x} - \mathbf{a}|\big) = o\big(c\,|\mathbf{x}-\mathbf{a}|\big) \le o\big(|\mathbf{x}-\mathbf{a}|\big),$$

we conclude that $E(1) \in o\big(|\mathbf{x}-\mathbf{a}|^2\big)$, as desired. □

Exercise 3.16. Complete the proof for any $k \geq 3$.

Example 3.20. For a function $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ which is twice differentiable at (a, b), its Taylor approximation at (a, b) is given by

$$\begin{aligned} &f(x,y) \\ &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &\quad + \frac{1}{2!}\big(f_{xx}(a,b)(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) \\ &\qquad + f_{yx}(a,b)(y-b)(x-a) + f_{yy}(a,b)(y-b)^2\big) \\ &\quad + o\big((x-a)^2 + (y-b)^2\big) \end{aligned}$$

as $(x, y) \to (a, b)$. If we assume further that f is C^2 at (a, b), then the quadratic term can be simplified as:

$$\frac{1}{2!}\big(f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\big)$$

Exercise 3.17. Write down the k-order Taylor approximation of a C^k function $f : \mathbb{R}^n \to \mathbb{R}$ at $\mathbf{a} = (a_1, \cdots, a_n)$ when

(a) $n = 3$ and $k = 2$

(b) $n = 2$ and $k = 3$

The Taylor approximation of multivariable functions can be used to explain the *second derivative test* in higher dimensions. For example, when a C^2 function $f : \mathbb{R}^2 \to \mathbb{R}$ achieves an interior local maximum or minimum at (a, b), then we first have $\frac{\partial f}{\partial x}(a, b) = \frac{\partial f}{\partial y}(a, b) = 0$ as f also achieve local min/max at (a, b) when restricted to coordinate directions. Therefore, the second-order Taylor approximation of f at (a, b) is given by only the constant and quadratic terms:

$$\begin{aligned} &f(x,y) \\ &= f(a,b) + \frac{1}{2!}\big(f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\big) \\ &\quad + o\big((x-a)^2 + (y-b)^2\big). \end{aligned}$$

Therefore, whether f achieves local maximum or minimum at (a,b) depends only whether the quadratic form

$$Q(x,y) := f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2$$

is negative-definite or positive-definite respectively. Using the elementary concepts about discriminant, we know that

- if $f_{xx} > 0$ and $f_{xy}^2 - f_{xx}f_{yy} < 0$ at (a,b), then $Q(x,y)$ is positive-definite, and so $f(x,y) \geq f(a,b)$ near (a,b). This implies f achieves a local minimum at (a,b).
- if $f_{xx} < 0$ and $f_{xy}^2 - f_{xx}f_{yy} < 0$ at (a,b), then $Q(x,y)$ is negative-definite, and so $f(x,y) \leq f(a,b)$ near (a,b). This implies f achieves a local maximum at (a,b).
- if $f_{xy}^2 - f_{xx}f_{yy} > 0$ at (a,b), then Q can be positive at some point near (a,b) and be negative at some other points near (a,b). In such case, the graph $z = f(x,y)$ behaves like a saddle near (a,b).

In higher dimensions, the quadratic approximation of a C^2 function $f : \mathbb{R}^2 \to \mathbb{R}$ at $\mathbf{a}$ can be expressed as:

$$f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \frac{1}{2!}(\mathbf{x} - \mathbf{a})^T \cdot \nabla^2 f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$$

where $\nabla^2 f(\mathbf{a})$ is the Hessian matrix[3], which is an $n \times n$ matrix whose (i,j)-th entry is given by $\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a})$. It is a symmetric matrix by the commutativity of partial derivatives. Therefore, the positivity/negativity of the quadratic term

$$(\mathbf{x} - \mathbf{a})^T \cdot \nabla^2 f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$$

can be determined by the eigenvalues of $\nabla^2 f(\mathbf{a})$. This is further discussed in the linear algebra course.

[3] Note about notations: physicists often use $\nabla^2 f$ for the Laplacian of f, which is the trace of the Hessian matrix, whereas mathematicians (especially geometers) often use $\nabla^2 f$ for the Hessian matrix.

3.3. Inverse and Implicit Function Theorems

3.3.1. Motivations. Suppose $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 map on Ω satisfying:

(1) F is injective, and

(2) the image $F(\Omega)$ is open in $\mathbb{R}^n$, and its inverse map $F^{-1} : F(\Omega) \to \Omega$ is C^1 on $F(\Omega)$,

then applying the chain rule on:

$$F \circ F^{-1} = \text{id} \quad \text{and } F^{-1} \circ F = \text{id},$$

we get:

$$DF_{\mathbf{x}} \cdot D(F^{-1})_{F(\mathbf{x})} = I \quad \text{and} \quad D(F^{-1})_{F(\mathbf{x})} \cdot DF_{\mathbf{x}} = I$$

where I is the $n \times n$ identity matrix. Here we write $DF_{\mathbf{x}}$ for $DF(\mathbf{x})$ to avoid having too many confusing brackets. Therefore, the matrices $DF_{\mathbf{x}}$ and $D(F^{-1})_{F(\mathbf{x})}$ are inverses of each other. In particular, $DF_{\mathbf{x}}$ is invertible if F satisfies both (1) and (2) above.

A natural question is that whether the converse is true, i.e. if $DF_{\mathbf{x}}$ is invertible, then whether F satisfies both (1) and (2) above. However, it is not true and here is a counterexample:

Let $\Omega = \{(r, \theta) : r > 0 \text{ and } \theta \in \mathbb{R}\}$ which is an open set in $\mathbb{R}^2$. Let $F(r, \theta) : \Omega \to \mathbb{R}^2$ be defined by

$$F(r, \theta) = (r\cos\theta, r\sin\theta).$$

Then its Jacobian matrix DF is given by:

$$DF(r, \theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix},$$

which has determinant $r > 0$ so it is always invertible. However, it is clear that F is not injective as $F(r, \theta + 2\pi) = F(r, \theta)$.

However, one can still claim that F has a "local inverse": if $DF_{\mathbf{a}}$ is invertible, one can find a small ball $B_\varepsilon(\mathbf{a})$ such that the restricted map $F\big|_{B_\varepsilon(\mathbf{a})} : B_\varepsilon(\mathbf{a})$ is injective and has C^1 inverse. This result is known as the **inverse function theorem**, which is an important result not only in analysis but also in other fields especially in differential geometry and differential topology.

One corollary of the inverse function theorem is the **implicit function theorem**. Consider a curve such that $x^2 + xy^3 + y^4 = 0$ in $\mathbb{R}^2$. In single-variable calculus we probably have learned how to find the slope of a tangent at a point (a, b) on the curve: regard y as a "function" of x, and differentiate both sides of the equation $x^2 + xy^3 + y^4 = 0$ with respect to x. It will yield:

$$2x + y^3 + 3xy^2\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 0.$$

One can then solve for $\frac{dy}{dx}$ by plugging in (a, b) into (x, y). However, this method – called implicit differentiation – needs to be justified. First y may not genuinely a function of x, and in fact it is not the case even for a simple curve like $x^2 + y^2 = 1$. Also, even if y is a function of x, one needs to justify why y is differentiable in order to make sense of $\frac{dy}{dx}$. The implicit function theorem justifies that under certain condition, the method of implicit differentiation is legitimate.

3.3.2. Banach contraction mapping theorem. The proof of the inverse function theorem requires the use of an important tool in analysis, known as Banach contraction mapping theorem. It is so an important tool that not only it is used to prove the inverse function theorem, but also other results such as existence theorem of ODEs.

Theorem 3.21 (Banach contraction mapping)**.** *Let (X, d) be a complete metric space, and $f : X \to X$ be a map such that there exists a constant $\alpha \in (0, 1)$ such that*

$$d\big(f(x), f(y)\big) \leq \alpha d(x, y) \ \ \forall x, y \in X.$$

Then, there exists a unique $x_* \in X$ such that $f(x_*) = x_*$.

Proof. Pick any $x_1 \in X$, and define the sequence recursively by:

$$x_n := f(x_{n-1}) \ \ \forall n \geq 2.$$

We argue that $\{x_n\}$ is a Cauchy sequence in (X, d), and hence it converges to a certain limit $x_* \in X$. We then claim argue the limit x_* is the desired fixed point of f.

Observe that for any $n \geq 2$, we have

$$d(x_{n+1}, x_n) = d\big(f(x_n), f(x_{n-1})\big) \leq \alpha d(x_n, x_{n-1}).$$

By induction, one can see that

$$d(x_{n+1}, x_n) \leq \alpha^{n-1} d(x_2, x_1).$$

Hence, by triangle inequality and sum of geometric series, $\{x_n\}$ is a Cauchy sequence in (X, d) – we leave it as an exercise for readers. Note that we need to use the condition $0 < \alpha < 1$ here.

Given that (X, d) is complete, $\{x_n\}$ converges to a limit $x_* \in X$. Note that f is continuous by the given condition

$$d\big(f(x), f(y)\big) \leq \alpha d(x, y) \ \ \forall x, y \in X,$$

so $f(x_{n-1}) \to f(x_*)$ as $n \to \infty$. Also we have $x_n = f(x_{n-1})$ for any $n \geq 2$, letting $n \to \infty$ on both sides we conclude that $x_* = f(x_*)$ as desired.

The uniqueness of x_* follows directly from the contraction condition. Suppose $x_{**} = f(x_{**})$ for some $x_{**} \in X$ as well. By the contraction condition, we get

$$d\big(f(x_*), f(x_{**})\big) \leq \alpha d(x_*, x_{**}) \implies d(x_*, x_{**}) \leq \alpha d(x_*, x_{**}).$$

Since $0 < \alpha < 1$, the above can only happen when $d(x_*, x_{**}) = 0$, hence it concludes that $x_* = x_{**}$. □

3.3.3. Inverse Function Theorem.

For simplicity, from now on we will denote a vector in $\mathbb{R}^n$ by simply x, y, a etc. whenever it is clear from the context that they are vectors. Now we are ready to state the inverse function theorem, and will prove it using Banach contraction mapping.

Theorem 3.22 (Inverse function theorem)**.** *Let $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a $C^{k\geq 1}$ function on Ω. If $\det DF(a) \neq 0$ at some $a \in \Omega$, then there exists an open set $U \subset \Omega$ containing a such that $\widetilde{F} := F\big|_U : U \to F(U)$ is bijective, $F(U)$ is open in $\mathbb{R}^n$, and the inverse map $\widetilde{F}^{-1} : F(U) \to U$ is C^k.*

Before we proceed to the proof, let's first prove an analogue of "mean value theorem" for vector-valued function, which can be helpful for estimating $|F(x) - F(y)|$ for any two points x, y. Given an $m \times n$ matrix A whose (i, j)-th entry is denoted by a_{ij}, one can define the "Euclidean norm" of A by

$$\|A\|_2 := \sqrt{\sum_{1 \leq i, j \leq n} (a_{ij})^2}.$$

One can check easily that $\| \ \|_2$ is a norm on the vector space $M_{m\times n}(\mathbb{R})$ of $m \times n$ matrices. Similar to $\mathbb{R}^n$, one can also define the sup-norm of the matrix A by:

$$\|A\|_\infty := \max_{1\le i,j\le n} |a_{ij}| .$$

We leave it as an exercise to prove that the following:

Exercise 3.18. For any $n \times n$ real matrix A, and a vector $x \in \mathbb{R}^n$, we have

$$|Ax| \le \|A\|_2 \, |x| .$$

Hint: Cauchy-Schwarz inequality.

Exercise 3.19. Prove that for any $m \times n$ real matrix A, we have

$$\|A\|_\infty \le \|A\|_2 \le \sqrt{mn}\, \|A\|_\infty .$$

Remark 3.23. By the above exercise result, one can use either norm interchangeably. If we have a sequence of matrices $\{A_j\}_{j=1}^\infty$, then A_j converges to a matrix B with respect to $\| \ \|_\infty$ if and only if it does so with respect to $\| \ \|_2$-norm. Note that the latter is also equivalent to saying each entry of A_j converges as a real number sequence to the corresponding entry of B.

Lemma 3.24. *Let $G : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ be a C^1 function on a convex domain Ω in $\mathbb{R}^n$. Then, for any $x, y \in \Omega$, we have*

$$|G(x) - G(y)| \le \sqrt{mn} \sup_{z\in\Omega} \|DG(z)\|_\infty \cdot |x - y| ,$$

Proof. Consider the straight path $\gamma(t) = (1-t)x + ty$ which connects x and y. The path is contained in Ω by convexity. Write $G(x) = (g_1(x), \cdots, g_m(x))$, $x = (x_1, \cdots, x_n)$, and $y = (y_1, \cdots, y_n)$, then by the single-variable mean-value theorem, we have for each j:

$$|g_j(x) - g_j(y)| = |g_j \circ \gamma(0) - g_j \circ \gamma(1)| = \left| \frac{d(g_j \circ \gamma)}{dt}(s) \right| |1 - 0|$$

for some $s \in (0,1)$. The chain rule applied to $g_j \circ \gamma$ shows

$$\frac{d(g_j \circ \gamma)}{dt} = \sum_{i=1}^n \frac{\partial g_j}{\partial x_i} \frac{d((1-t)x_i + ty_i)}{dt} = \sum_{i=1}^n \frac{\partial g_j}{\partial x_i} \cdot (y_i - x_i).$$

By Cauchy-Schwarz, we have

$$\left| \frac{d(g_j \circ \gamma)}{dt}(s) \right| \le \sqrt{\sum_{i=1}^n \left| \frac{\partial g_j}{\partial x_i}(\gamma(s)) \right|^2} \sqrt{\sum_{i=1}^n |y_i - x_i|^2} \le \sqrt{n}\, \|DG(\gamma(s))\|_\infty \, |x - y| .$$

This proves for each j, we have

$$|g_j(x) - g_j(y)| \le \sqrt{n}\, \|DG(\gamma(s))\|_\infty \, |x - y| \le \sqrt{n} \sup_{z\in\Omega} \|DG(z)\|_\infty \, |x - y| ,$$

which implies

$$|G(x) - G(y)| = \sqrt{\sum_{j=1}^m |g_j(x) - g_j(y)|^2} \le \sqrt{mn} \sup_{z\in\Omega} \|DG(z)\|_\infty \, |x - y| .$$

$\square$

Proof of Inverse Function Theorem. Given that $DF(a)$ is invertible, we will pick ε, δ sufficiently small such that any $y \in B_\varepsilon(F(a))$ has a unique $x \in \overline{B_{\delta/2}(a)}$ so that $F(x) = y$. That will show that F is injective when restricted on $U := F^{-1}\big(B_\varepsilon(F(a))\big) \cap B_{\delta/2}(a)$.

Step 1: Define a contraction map.

In order to apply the contraction mapping theorem, we consider for each $y \in B_\varepsilon(F(a))$ the following map:

$$T_y(x) := x + DF(a)^{-1}(y - F(x)).$$

We will show that by choosing ε and δ sufficiently small, such a map T_y is a contraction map from $\overline{B_\delta(a)}$ to itself. Then by Banach's contraction mapping theorem, such T_y has a unique fixed point $\bar{x}(y) \in \overline{B_\delta(a)}$, i.e. $T_y(\bar{x}(y)) = \bar{x}(y)$. One can easily check that it implies $y = F(\bar{x}(y))$ as desired.

To verify that T_y is a contraction, we consider

$$\begin{aligned} |T_y(x_1) - T_y(x_2)| &= \left|x_1 - DF(a)^{-1}F(x_1) - x_2 + DF(a)^{-1}F(x_2)\right| \\ &= \left|DF(a)^{-1}\left(DF(a)x_1 - F(x_1)\right) - \left(DF(a)x_2 - F(x_2)\right)\right| \\ &\leq \left\|DF(a)^{-1}\right\|_2 |G(x_1) - G(x_2)| \end{aligned}$$

where $G(x) := DF(a)\cdot x - F(x)$. Note that $DG(x) = DF(a)I - DF(x) = DF(a) - DF(x)$. Suppose $x_1, x_2 \in B_\delta(a)$, then we have by Lemma 3.24 that

$$|G(x_1) - G(x_2)| \leq n \sup_{z \in B_\delta(a)} \|DG(z)\|_\infty |x_1 - x_2|.$$

Substitute this back in, we get:

$$|T_y(x_1) - T_y(x_2)| \leq n \left\|DF(a)^{-1}\right\|_2 \sup_{z \in B_\delta(a)} \|DF(a) - DF(z)\|_\infty |x_1 - x_2|$$

for any $x_1, x_2 \in B_\delta(a)$. By continuity of the map $T_y(x)$, the above inequality also holds for any $x_1, x_2 \in \overline{B_\delta(a)}$.

Since F is C^1, each entry of $DF(z)$ approaches to the corresponding entry of $DF(a)$ as $z \to a$. By Remark 3.23, $\|DF(z) - DF(a)\|_\infty \to 0$ as $z \to a$. Hence, one can choose $\delta > 0$ small so that

$$\|DF(a) - DF(z)\|_\infty \leq \frac{1}{2n \left\|DF(a)^{-1}\right\|_2}$$

whenever $z \in \overline{B_\delta(a)}$, and consequently we have

$$|T_y(x_1) - T_y(x_2)| \leq \frac{1}{2} |x_1 - x_2|. \tag{3.2}$$

for any $x_1, x_2 \in \overline{B_\delta(a)}$. This proves $T_y : \overline{B_\delta(a)} \to \mathbb{R}^n$ is a contraction map.

In order to use the Banach's Contraction Mapping Theorem, we will show T_y maps $\overline{B_{\delta/2}(a)}$ to itself. For that we need y being sufficiently close to $F(a)$: for any $x \in \overline{B_{\delta/2}(a)}$, we estimate that:

$$\begin{aligned} |T_y(x) - a| &\leq |T_y(x) - T_y(a)| + |T_y(a) - a| \\ &\leq \frac{1}{2}|x - a| + \left|DF(a)^{-1}(y - F(a))\right| \\ &\leq \frac{\delta}{4} + \left\|DF(a)^{-1}\right\|_2 |y - F(a)|. \end{aligned}$$

Let $\varepsilon < \frac{\delta}{4\|DF(a)^{-1}\|}$, then when $y \in B_\varepsilon(F(a))$, we have

$$|T_y(x) - a| \leq \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2}.$$

In other words, we have $T_y(x) \in \overline{B_{\delta/2}(a)}$ whenever $y \in B_\varepsilon\big(F(a)\big)$ and $x \in \overline{B_{\delta/2}(a)}$.

Now by the Banach's contraction mapping applied to $T_y : \overline{B_{\delta/2}(a)} \to \overline{B_{\delta/2}(a)}$ where $y \in B_\varepsilon(F(a))$, there exists a unique fixed point $\bar{x} \in \overline{B_{\delta/2}(a)}$ such that $T_y(\bar{x}) = \bar{x}$, or equivalently, $y = F(\bar{x})$. To summarize, we have proved that by taking $U := F^{-1}(B_\varepsilon(F(a))) \cap B_{\delta/2}(a)$, then

$$\widetilde{F} := F\big|_U : U \to F(U)$$

is injective, and one can define an inverse $\widetilde{F}^{-1} : F(U) \to U$.

Step 2: Prove that $F(U)$ is open in $\mathbb{R}^n$.

For any $y_0 \in F(U) \subset B_\varepsilon(F(a))$, one can write it as $y_0 = F(x_0)$ for some $x_0 \in U$. As U is open, there exists $\theta > 0$ such that $\overline{B_\theta(x_0)} \subset U \subset B_{\delta/2}(a)$. Also, as $y_0 \in B_\varepsilon(F(a))$, there exists $\eta > 0$ such that $B_\eta(y_0) \subset B_\varepsilon(F(a))$.

By (3.2), we have for any $x \in \overline{B_\theta(x_0)}$ and $y \in B_{\eta'}(y_0)$ where $\eta' \in (0, \eta)$ is to be chosen,

$$\begin{aligned}
& |T_y(x) - T_y(x_0)| \le \frac{1}{2}|x - x_0| \\
\Longrightarrow \quad & |T_y(x) - T_{y_0}(x_0)| \le |T_y(x) - T_y(x_0)| + |T_y(x_0) - T_{y_0}(x_0)| \\
\Longrightarrow \quad & |T_y(x) - x_0| \le \frac{1}{2}|x - x_0| + |T_y(x_0) - x_0| \\
& \qquad = \frac{1}{2}|x - x_0| + \left|DF(a)^{-1}\big(y - F(x_0)\big)\right| \\
& \qquad \le \frac{1}{2}|x - x_0| + \left\|DF(a)^{-1}\right\|_2 |y - y_0| \\
& \qquad \le \frac{\theta}{2} + \left\|DF(a)^{-1}\right\|_2 \eta'.
\end{aligned}$$

Therefore, if one chooses $\eta' < \min\left\{\frac{\theta}{2\|DF(a)^{-1}\|_2 + 1}, \eta\right\}$, then we have $T_y(x) \in \overline{B_\theta(x_0)}$ for any $x \in \overline{B_\theta(x_0)}$ and $y \in B_{\eta'}(y_0)$. Then, $T_y : \overline{B_\theta(x_0)} \to \overline{B_\theta(x_0)}$ is a contraction map whenever $y \in B_{\eta'}(y_0)$, and hence it has a fixed point $x_* \in \overline{B_\theta(x_0)}$, which is a point such that $y = F(x_*)$. Note that $x_* \in \overline{B_\theta(x_0)} \subset U$, it shows $y \in F(U)$ whenever $y \in B_{\eta'}(y_0)$. In other words, $B_{\eta'}(y_0) \subset F(U)$. This concludes that $F(U)$ is open.

Step 3: Prove that $\widetilde{F}^{-1}$ is continuous on $F(U)$.

We will see why it is needed later, although we will prove it is even differentiable on $F(U)$. For that we use the contraction inequality (3.2) again. For any $y_1, y_2 \in F(U)$, we write $y_i = F(x_i)$ for some $x_i \in U$. Then, by (3.2), we get

$$\begin{aligned}
& \frac{1}{2}|x_1 - x_2| \\
& \ge |T_{y_1}(x_1) - T_{y_1}(x_2)| \\
& = \left|x_1 - x_2 + DF(a)^{-1}(y_1 - F(x_2))\right| \\
& \ge |x_1 - x_2| - \left\|DF(a)^{-1}\right\|_2 |y_1 - y_2|.
\end{aligned}$$

Therefore, we have

$$|x_1 - x_2| \le 2\left\|DF(a)^{-1}\right\|_2 |y_1 - y_2|,$$

or in other words,

$$\left|\widetilde{F}^{-1}(y_1) - \widetilde{F}^{-1}(y_2)\right| \le 2\left\|DF(a)^{-1}\right\|_2 |y_1 - y_2|.$$

This shows $\widetilde{F}^{-1}$ is continuous on $F(U)$.

Step 4: Prove that $\widetilde{F}^{-1}$ is differentiable on $F(U)$ with Jacobian matrix given by $(DF)^{-1}$.

Consider $y_0 = F(x_0) \in F(U)$, and by bijectivity of $\widetilde{F}$ we have write every $y \in F(U)$ as $F(x)$, and by continuity of F and $\widetilde{F}^{-1}$ we have $x \to x_0$ if and only if $y \to y_0$. Then one can check that as $y \to y_0$ (and hence as $x \to x_0$), we have:

$$\begin{aligned}
&\widetilde{F}^{-1}(y) - \widetilde{F}^{-1}(y_0) - DF(x_0)^{-1}(y - y_0) \\
&= \widetilde{F}^{-1}(F(x)) - \widetilde{F}^{-1}(F(x_0)) - DF(x_0)^{-1}(F(x) - F(x_0)) \\
&= x - x_0 - DF(x_0)^{-1}(DF(x_0)(x - x_0) + E(x)) \quad \text{where } E(x) \in o\big(|x - x_0|\big) \\
&= DF(x_0)^{-1}E(x) \in o\big(|x - x_0|\big).
\end{aligned}$$

Hence $\widetilde{F}^{-1}$ is differentiable at any $y_0 \in F(U)$ where $D(\widetilde{F}^{-1})(y_0) = (DF(x_0))^{-1}$.

Step 5: Prove that $\widetilde{F}^{-1}$ is C^k on $F(U)$.

The inverse function $\widetilde{F}^{-1}$ is C^k because its partial derivatives are entries of $(DF)^{-1}$ by the previous step. By Crammer's rule, each entry of $(DF)^{-1}$ is a rational function of partial derivatives of F (which are C^{k-1} as F is C^k). Therefore, each entry of $(DF)^{-1}$ is C^{k-1} too, so all the first-order partial derivatives of $\widetilde{F}^{-1}$ are all C^{k-1}, completing the proof of that $\widetilde{F}^{-1}$ is C^k. □

Example 3.25. Consider the system of equations

$$\begin{aligned}
x - y^2 &= a \\
x^2 + y + y^3 &= b
\end{aligned}$$

Unlike in linear algebra, it is rather difficult to determine for what values of a and b the system has solution. However, the inverse function theorem can be used to give a partial answer to this question. For instance, one can prove that when $\sqrt{a^2 + b^2}$ is sufficiently small, then the system always has a solution.

Define $F(x, y) = (x - y^2, x^2 + y + y^3)$, then $F(0, 0) = (0, 0)$. The Jacobian matrix $DF(0, 0)$ is given by

$$DF(0,0) = \begin{bmatrix} 1 & -2y \\ 2x & 1 + 3y^2 \end{bmatrix}\Bigg|_{(x,y)=(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is clearly invertible.

Inverse function theorem asserts that there exist open sets U containing $(0, 0)$ and V containing $F(0, 0) = (0, 0,$ such that $F\big|_U : U \to V$ is bijective. In other words, for any $(a, b) \in V$, there exists $(x_0, y_0) \in U$ such that $F(x_0, y_0) = (a, b)$. In other words, the system is solvable when $(a, b) \in V$.

Exercise 3.20. Find another pair (a_0, b_0), apart from $(0, 0)$, such that the system in the above example is solvable when (a, b) is sufficiently close to (a_0, b_0). Justify your claim.

Example 3.26. Another typical use of inverse function theorem (especially in differential geometry) is to prove that the local inverse is C^k or smooth without actually finding the inverse explicitly. For instance, consider the spherical coordinate map $F(\rho, \theta, \phi) : (0, \infty) \times (0, \pi) \times (0, 2\pi) \to \mathbb{R}^3$ defined by:

$$F(\rho, \theta, \phi) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi).$$

Using elementary trigonometry, it is not hard to prove that F is injective on the specific domain. However, it is rather difficult to write down F^{-1} explicitly – at least it takes different form in different octant, and checking continuity and differentiability of F^{-1} can be tedious. However, one can easily check that

$$\det DF(\rho, \theta, \phi) = \rho^2 \sin\phi \neq 0$$

for any (ρ, θ, ϕ) in our domain of F. Inverse function theorem not only proves that a local inverse exists (which is a redundant fact since we already know F^{-1} exists by the injectivity of F), but it also shows that the inverse F^{-1} is C^∞ (meaning that it is C^k for any $k \in \mathbb{N}$) as F is C^∞.

3.3.4. Lagrange's Multiplier. The inverse function theorem can also be used to rigorously explain why the method of Lagrange's multiplier in MATH 2023 works. Given two C^1 functions $f(x, y)$ and $g(x, y)$, we want to maximize and minimize f subject to the constraint $g(x, y) = 0$. In MATH 2023, we learn that we need to solve for (x, y, λ) in the system:

$$\begin{aligned} \nabla f(x,y) &= \lambda \nabla g(x,y) \qquad\qquad \text{or} \quad \nabla g(x,y) = (0,0) \\ g(x,y) &= 0 \end{aligned}$$

Then, those solutions for (x, y) are candidate points of this optimization problem. One could plug each of them into f to see which give the largest/smallest value. They are then the maximum and minimum of f subject to the constraint $g = 0$.

In MATH 2023, you probably learned that the rational behind this method: at the point (x_0, y_0) at which f achieves its maximum or minimum under the the constraint $g = 0$, the level curves of f and g passing through (x_0, y_0) are parallel to each other. Since $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ are perpendicular to the level sets of f and g respectively[4], the two gradient vectors must be parallel to each other, which is equivalent to saying that $\nabla f = \lambda \nabla g$ (or $\nabla g = 0$).

The above the rather an "intuitive" explanation of the Lagrange's multiplier method. To rigorously justify that ∇f and ∇g are parallel at those candidate points, we need to use the inverse function theorem too. We will argue that if ∇f and ∇g are linearly independent at (x_0, y_0), then f cannot achieve maximum or minimum under the constraint $g(x, y) = 0$. To argue this, we define a map $F(x, y) : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x,y) = \big(f(x,y), g(x,y)\big).$$

Note that F maps the level curves $f = c$, $g = c$ into coordinate curves in the target $\mathbb{R}^2$. Suppose (x_0, y_0) is a point on the level curve $g = 0$ such that $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ are linearly independent, then

$$DF(x_0,y_0) = \begin{bmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{bmatrix}$$

has linearly independent rows, and so is invertible. Inverse function theorem shows there exists an open set U containing (x_0, y_0), and an open set V containing $F(x_0, y_0) = \big(f(x_0, y_0), 0\big)$ such that $F\big|_U : U \to V$ is bijective.

Therefore, as $(f(x_0, y_0), 0) \in V$ which is open, the line segment $(f(x_0, y_0) + t, 0) \in V$ too when $-\varepsilon < t < \varepsilon$ for some $\varepsilon > 0$. The line segment in the set V correspond to the curve $(x(t), y(t))$ in U such that

$$F(x(t),y(t)) = (f(x_0,y_0)+t, 0) \implies \big(f(x(t),y(t)), g(x(t),y(t))\big) = (f(x_0,y_0)+t, 0).$$

Therefore, the curve $\gamma(t) := (x(t), y(t))$ lies in the constraint $g = 0$, and when $t \in (0, \varepsilon)$, we have $f(\gamma(t)) = f(x_0, y_0)+t > f(x_0, y_0)$; whereas when $t \in (-\varepsilon, 0)$, we have $f(\gamma(t)) < f(x_0, y_0)$. This shows f achieves neither the maximum nor the minimum at (x_0, y_0) even along the curve $\gamma(t) \subset \{g = 0\}$.

[4] Remark: this can be justified by the implicit function theorem later.

3.3.5. Implicit Function Theorem. Now we address the issue about the method of implicit differentiation as discussed in the beginning paragraphs of this section. Given a C^1 function $f(x,y)$, and consider the level set $f(x,y)=0$. Typically, it is a curve in $\mathbb{R}^2$ and we want to determine the slope of tangent at a point (x_0,y_0) to this curve, without explicitly solving y in terms of x.

However, we need to justify whether y can really be regarded as a C^1 function of x – at least locally near (x_0,y_0), as the slope of tangent only depends on the part of the curve locally near (x_0,y_0). This can be justified by the inverse function theorem, and the result is often called the implicit function theorem. We will discuss and prove the two variable version, then we will state the general dimension version and leave the proof as an exercise for readers.

We will prove that if $f(x_0,y_0)=0$ and $\frac{\partial f}{\partial y}(x_0,y_0)\neq 0$, then the level curve $f=0$ can be locally regarded as a C^1 function of x near (x_0,y_0). Precisely, there exists a C^1 function $g(x):(x_0-\varepsilon,x_0+\varepsilon)\to\mathbb{R}$ such that $g(x_0)=y_0$ and $f(x,g(x))=0$ for any $x\in(x_0-\varepsilon,x_0+\varepsilon)$.

We define the map $F(x,y):\mathbb{R}^2\to\mathbb{R}^2$ by

$$F(x,y)=\big(x,f(x,y)\big).$$

This function maps the family of level curves $\{f=c\}_{c\in\mathbb{R}}$ into horizontal lines in the target $\mathbb{R}^2$. By checking its Jacobian, we get that:

$$DF(x_0,y_0)=\begin{bmatrix}1 & 0\\ \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0)\end{bmatrix}$$

whose determinant is given by $\frac{\partial f}{\partial y}(x_0,y_0)\neq 0$. Inverse function theorem asserts that there exists an open set U containing (x_0,y_0) and an open set V containing $F(x_0,y_0)=(x_0,0)$ such that $F|_U:U\to V$ is bijective, and $F^{-1}:V\to U$ is C^1. Then, each point $(x,y)\in U\cap\{f=0\}$ must have a unique x-coordinate. To argue this, we suppose there are points (x,y_1) and (x,y_2) in $U\cap\{f=0\}$. Then $F(x,y_i)=(x,0)$ as $f(x,y_i)=0$ for $i=1,2$. Since F is injective on U, we must have $(x,y_1)=(x,y_2)$. This proves that the level set $\{f=0\}$ is a graph of x inside U.

Next we construct the $g(x)$ function so that $f(x,y)=0$ is locally the graph $y=g(x)$ near (x_0,y_0). Take $\varepsilon>0$ small so that $B_\varepsilon(x_0,0)\subset V$, then for any $x\in(x_0-\varepsilon,x_0+\varepsilon)$, then point $(x,0)\in B_\varepsilon(x_0,0)\subset V$, so it is in the domain of F^{-1}. The point $F^{-1}(x,0)$ must be of the form $(x,*)$ for some unique value $*$ (uniqueness follows from the above paragraph). We denote this unique value by $g(x)$, then $\forall x\in(x_0-\varepsilon,x_0+\varepsilon)$, we have

$$F^{-1}(x,0)=(x,g(x))\implies (x,0)=\big(x,f(x,g(x))\big)\implies f(x,g(x))=0.$$

Therefore the part of the level set $U\cap\{f=0\}$ can be written as the graph $\{(x,g(x)):x\in(x_0-\varepsilon,x_0+\varepsilon)\}$. As F^{-1} is C^1, and $(x,g(x))=F^{-1}(x,0)$, so g is C^1 as well. See Figure **??** as a reference (to be added).

Remark 3.27. If we have $\frac{\partial f}{\partial x}(x_0,y_0)\neq 0$ instead, then one can argue that the level set $f=0$ is locally a graph of y near (x_0,y_0). One may consider the map $F(x,y)=(f(x,y),y)$ instead and almost repeat the above argument.

Exercise 3.21. Prove the three-variable version of the implicit function theorem: given a C^1 function $f(x,y,z):\mathbb{R}^3\to\mathbb{R}$. Suppose (x_0,y_0,z_0) is a point on the level set $\{f=0\}$ and $\frac{\partial f}{\partial z}(x_0,y_0,z_0)\neq 0$. Prove that the level set $\{f=0\}$ is locally a graph $z=g(x,y)$ near (x_0,y_0,z_0) for some C^1 function $g(x,y)$ defined on a ball $B_\varepsilon(x_0,y_0)$, i.e. $f\big(x,y,g(x,y)\big)=0$.

From the implicit function theorem, one can prove that $\nabla f(p)$, if it is non-zero, is perpendicular to the level set $\{f = c\}$ at p. Consider a C^1 function $f : \mathbb{R}^3 \to \mathbb{R}$, and p is a point on the level set $\{f = c\}$. Suppose $\nabla f(p) \neq \mathbf{0}$, then at least one of $\frac{\partial f}{\partial x}(p)$, $\frac{\partial f}{\partial y}(p)$ and $\frac{\partial f}{\partial z}(p)$ is non-zero. Without loss of generality, assume it is $\frac{\partial f}{\partial z}(p)$, then by Exercise 3.21, the level set $f = c$ can be locally regarded as a graph $z = g(x, y)$ near p for some C^1 function $g(x, y)$, i.e. $f\big(x, y, g(x, y)\big) = 0$ for (x, y) near (x_0, y_0). Here we let $p = (x_0, y_0, z_0)$.

As f and g are C^1, by differentiating $f\big(x, y, g(x, y)\big)$ we get:

$$\begin{aligned} 0 &= \frac{\partial}{\partial x} f\big(x, y, g(x, y)\big) = \frac{\partial f}{\partial x}(p) + \frac{\partial f}{\partial z}(p)\frac{\partial g}{\partial x}(x_0, y_0) \\ 0 &= \frac{\partial}{\partial y} f\big(x, y, g(x, y)\big) = \frac{\partial f}{\partial y}(p) + \frac{\partial f}{\partial z}(p)\frac{\partial g}{\partial y}(x_0, y_0) \end{aligned}$$

Rewrite these equations in vector form, we get:

$$\begin{aligned} \nabla f(p) \cdot \left(1, 0, \frac{\partial g}{\partial x}(x_0, y_0)\right) &= 0 \\ \nabla f(p) \cdot \left(0, 1, \frac{\partial g}{\partial y}(x_0, y_0)\right) &= 0. \end{aligned}$$

Note that $\left(1, 0, \frac{\partial g}{\partial x}(x_0, y_0)\right)$ and $\left(0, 1, \frac{\partial g}{\partial y}(x_0, y_0)\right)$ are tangent vectors of the graph $z = g(x, y)$ at p, which is locally coincide with the level set $f = c$ near p. Therefore, $\nabla f(p)$ is perpendicular to two of basis tangent vectors of $f = c$ at p, or equivalently it is perpendicular to the level set $f = c$ at p.

Here we state the general (i.e. higher dimensions) versions of implicit function theorem for vector-valued functions $F : \mathbb{R}^n \to \mathbb{R}^m$. Its proof is slightly modified from the case $f : \mathbb{R}^3 \to \mathbb{R}$. We again leave the proof as an exercise. The higher dimensional case $F : \mathbb{R}^n \to \mathbb{R}^m$ may not be of uses in this course, but it will become a handy tool in differentiable geometry – such as for showing the intersection of two surfaces is a C^1 curve.

Theorem 3.28 (Implicit Function Theorem). *Let $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ be a C^k function with $n > m$. Express the points in $\mathbb{R}^n$ by $x = (x_1, \cdots, x_m, x_{m+1}, \cdots, x_n) \in \mathbb{R}^n$, and the components of F by $F(x) = (f^1(x), \cdots, f^m(x))$. Suppose that there exists $p = (a_1, \cdots, a_m, a_{m+1}, \cdots, a_n) \in \mathbb{R}^n$ such that $F(p) = 0 \in \mathbb{R}^m$, and*

$$\det \frac{\partial(f^1, \cdots, f^m)}{\partial(x_1, \cdots, x_m)}(p) \neq 0$$

Then, the level set $\{F = 0\}$ is locally a graph of $(x_{m+1}, \cdots, x_n)$ near p. Precisely, there exists a C^k function $g : U \subset \mathbb{R}^{n-m} \to \mathbb{R}^m$ defined on an open set U in $\mathbb{R}^{n-m}$ containing $(a_{m+1}, \cdots, a_n)$ such that

$$F\big(g(x_{m+1}, \cdots, x_n), x_{m+1}, \cdots, x_n\big) = 0$$

for any $(x_1, \cdots, x_m) \in U$.

Exercise 3.22. Prove Theorem 3.28.

Chapter 4

Uniform Convergence

> "A man becomes the creature of his uniform."
>
> Napoleon Bonaparte

This chapter is mainly about sequences of functions and their convergence. As discussed in Chapter 2, the convergence of a sequence of continuous functions $f_n : [a, b] \to \mathbb{R}$ depends on the choice of metric d or norm $\| \ \|$ on the space $C[a, b]$. The sequence $\{f_n\}$ may converge in one norm say $\| \ \|_p$ but not others like $\| \ \|_\infty$, i.e. the sup-norm defined by

$$\|f\|_\infty := \sup_{x \in [a,b]} |f(x)| .$$

In this chapter, we will further discuss the convergence of sequences of functions under the sup-norm $\| \ \|_\infty$. This kind of convergence is commonly called **uniform convergence**.

4.1. Definition and Properties of Uniform Convergence

4.1.1. Pointwise vs uniform.

Uniform convergence is a stronger type of convergence than pointwise convergence. Let's state the definitions of both of them, and we will explain their differences.

Definition 4.1 (Pointwise convergence and uniform convergence)**.** Let $E \subset D \subset \mathbb{R}$ be non-empty sets. Consider a sequence of functions $f_n : D \to \mathbb{R}$.

- We say that f_n **converges pointwise on** E to a function $f : E \to \mathbb{R}$ if for each $x \in E$, the sequence of real number $\{f_n(x)\}_{n=1}^\infty$ converges to $f(x)$.
- We say that f_n **converges uniformly on** E to a function $f : E \to \mathbb{R}$ if

$$\sup_{x \in E} |f_n(x) - f(x)| \to 0 \quad \text{as } n \to \infty.$$

  Under the norm notation $\|f\|_E := \sup_{x \in E} |f(x)|$, it is equivalent to saying that $\|f_n - f\|_E \to 0$ as $n \to \infty$.

Remark 4.2. Both pointwise convergence and uniform convergence depend on the set E. If E_1, E_2 are two different non-empty subsets of D, then a sequence of functions

$f_n : D \to \mathbb{R}$ which converges pointwise (resp. uniformly) on E_1 may not imply it does so on E_2. For pointwise convergence, one good example is

$$f_n(x) := \sum_{k=1}^{n} \frac{1}{n^x}$$

which converges pointwise on $(1, \infty)$, but not so on $(0, 1]$. We will provide another example for uniform convergence soon.

Remark 4.3. Using the fact that for any $x \in E$

$$|f_n(x) - f(x)| \leq \|f_n - f\|_E \,,$$

it is clear that uniform convergence to f on E implies pointwise convergence to f on E. The converse is not true.

Example 4.4. Consider $f_n(x) = x^n$. It is straight-forward to see that f_n converges pointwise on $(-1, 1]$ to the function

$$f(x) = \begin{cases} 0 & \text{if } x \in (-1, 1) \\ 1 & \text{if } x = 1 \end{cases} .$$

To see if it converges uniformly on $(-1, 1]$, the limit function f must coincide with the pointwise limit (from the above remark), so we try to see if $\|f_n - f\|_{(-1,1]} \to 0$ or not.

$$\|f_n - f\|_{(-1,1]} = \sup_{x \in (-1,1]} |x^n - f(x)| \,.$$

It is clear that $|x^n - f(x)|$ is bounded above by 1 when $x \in (-1, 1]$. Furthermore, by taking a sequence $\{x_k\}_{k=1}^{\infty}$ where $x_k = 1 - \frac{1}{k} \in (-1, 1)$, we have

$$|x_k^n - f(x_k)| = \left| \left(1 - \frac{1}{k}\right)^n - 0 \right| \to 1 \text{ as } k \to \infty.$$

Therefore, 1 is in fact the supremum of $|x^n - f(x)|$, and so

$$\|f_n - f\|_{(-1,1]} = 1,$$

which does not converge to 0 as $n \to \infty$. Therefore, f_n does not converge uniformly on $(-1, 1]$.

However, it is easy to see that

$$\|f_n - f\|_{[0,1/2]} = \sup_{x \in [0,1/2]} |x^n - f(x)| = \sup_{x \in [0,1/2]} x^n = \frac{1}{2^n} \to 0.$$

Therefore, f_n converges uniformly on $[0, 1/2]$ to f. Similar argument can show f_n converges uniformly on $[0, r]$ to f where $0 < r < 1$ is any constant.

This example shows that uniform convergence depends on the domain, and the dependence is very sensitive. From the above we see that f_n could converge uniformly on $[0, r]$ for any $r \in (0, 1)$, but it does not converge uniformly on $[0, 1]$ even though $[0, 1]$ is (in some sense) the limit of $[0, r]$ as $r \to 1$. Therefore, we should always mention the **domain**, i.e. “uniformly converge on a **set**?” instead of simply “uniforml converge”, whenever we talk about uniform convergence.

Exercise 4.1. Consider the sequence of functions

$$f_n(x) = \begin{cases} 0 & \text{if } t \in [0, \frac{1}{2}] \\ n(x - \frac{1}{2}) & \text{if } t \in (\frac{1}{2}, \frac{1}{2} + \frac{1}{n}) \\ 1 & \text{if } t \in [\frac{1}{2} + \frac{1}{n}, 1] \end{cases} .$$

Show that f_n converges pointwise on $[0,1]$, and find out the limit function $f(x)$ explicitly. Determine whether or not f_n converges uniformly on $[0,1]$.

How about uniform convergence on $[1/2+\varepsilon, 1]$ for any fixed $\varepsilon > 0$?

Exercise 4.2. Suppose $E_1 \subset E_2 \subset D \subset \mathbb{R}^n$. Consider a sequence of functions $f_n : D \to \mathbb{R}$. Which one of the following is correct? Explain why.

- f_n converges uniformly on E_1 implies f_n converges uniformly on E_2
- f_n converges uniformly on E_2 implies f_n converges uniformly on E_1
- None of the above is true.

Remark 4.5. Do not mix up uniform convergence with uniform continuity! The former is about convergence of a sequence of functions, whereas the latter is the continuity of ONE function.

Remark 4.6. The name "uniform" in the term uniform convergence comes from the following uniformity: if f_n converges uniformly on E to a function f, then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $\sup_{x\in E} |f_n(x) - f(x)| < \varepsilon$, and hence $|f_n(x) - f(x)| < \varepsilon$ whenever $n \geq N$. Such a convergence is "uniform" in a sense that N depends only on ε but not on the point x.

Exercise 4.3. Prove the following Cauchy criterion for uniform convergence: Given a sequence of functions $f_n : D \subset \mathbb{R} \to \mathbb{R}$, and suppose for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ depending only on ε such that whenever $m, n \geq N$ we have $\|f_m - f_n\|_D < \varepsilon$, then there exists a function $f : D \to \mathbb{R}$ such that f_n converges uniformly to f on D.

[Remark: The proof is largely similar (after removing unnecessary parts) to the proof that $C[0,1]$ with sup-norm is complete.]

4.1.2. Uniform convergence tests for series. We have seen examples of how to determine a sequence of functions f_n converges uniformly (on a certain set E). The first step requires us to identify the pointwise limit f, followed by estimating $\|f_n - f\|_E$.

Now we are given an infinite series of functions such as $\sum_{k=1}^{\infty} g_k(x)$, which is the limit of the partial sum sequence $\sum_{k=1}^{n} g_k(x)$, then we say $\sum_{k=1}^{n} g_k(x)$ converges uniformly on E to $\sum_{k=1}^{\infty} g_k(x)$ (or simply say $\sum_{k=1}^{\infty} g_k$ converges uniformly on E) if

$$\sup_{x\in E} \left| \sum_{k=1}^{n} g_k(x) - \sum_{k=1}^{\infty} g_k(x) \right| \to 0 \quad \text{as } n \to \infty.$$

However, to prove uniform convergence of a series using the above definition is often impractical. It is usually very challenging to determine the exact pointwise limit $\sum_{k=1}^{\infty} g_k$.

Fortunately, we have the following result that is a very handy tool for proving uniform convergence of a series.

Proposition 4.7 (Weiestrass' M-test). *Consider $E \subset \mathbb{R}$ be a non-empty subset, and a series of functions $\sum_{k=1}^{\infty} g_k$ where each g_k is a real-valued function on E. Suppose there exists a sequence of real numbers $\{M_k\}_{k=1}^{\infty}$, independent of x, such that*

(1) $|g_k(x)| \leq M_k$ for all $x \in E$, and

(2) $\sum_{k=1}^{\infty} M_k$ converges as a series of real numbers,

then $\sum_{k=1}^{\infty} g_k$ converges uniformly on E.

Proof. We first show that $\sum_{k=1}^{\infty} g_k(x)$ converges pointwise for any $x \in E$ using Cauchy criterion on $\mathbb{R}$. For any positive integers $m > n$ and $x \in E$, we have

$$\left|\sum_{k=1}^{m} g_k(x) - \sum_{k=1}^{n} g_k(x)\right| = \left|\sum_{k=n+1}^{m} g_k(x)\right| \leq \sum_{k=n+1}^{m} |g_k(x)| \leq \sum_{k=n+1}^{m} M_k \leq \sum_{k=n+1}^{\infty} M_k.$$

As $\sum_{k=1}^{\infty} M_k$ converges, $\sum_{k=n+1}^{\infty} M_k$ can be arbitrarily small provided that n is sufficiently large. This shows $\sum_{k=1}^{n} g_k(x)$ is a Cauchy sequence of real numbers, and hence it converges as $n \to \infty$.

Next we check that $\sum_{k=1}^{n} g_k$ converges uniformly to $\sum_{k=1}^{\infty} g_k$ as $n \to \infty$. For any $x \in E$,

$$\begin{aligned} &\left|\sum_{k=1}^{n} g_k(x) - \sum_{k=1}^{\infty} g_k(x)\right| \\ &= \left|\sum_{k=n+1}^{\infty} g_k(x)\right| \leq \sum_{k=n+1}^{\infty} |g_k(x)| \\ &\leq \sum_{k=n+1}^{\infty} M_k. \end{aligned}$$

Since the inequality holds true for any $x \in E$, taking supremum on both sides yields

$$\left\|\sum_{k=1}^{n} g_k - \sum_{k=1}^{\infty} g_k\right\|_E = \sup_{x \in E}\left|\sum_{k=1}^{n} g_k(x) - \sum_{k=1}^{\infty} g_k(x)\right| \leq \sum_{k=n+1}^{\infty} M_k.$$

Letting $n \to \infty$, we have $\sum_{k=n+1}^{\infty} M_k \to 0$ by the convergence of $\sum_{k=1}^{\infty} M_k$. This proves

$$\left\|\sum_{k=1}^{n} g_k - \sum_{k=1}^{\infty} g_k\right\|_E \to 0,$$

concluding that $\sum_{k=1}^{\infty} g_k$ converges uniformly on E. $\square$

Example 4.8. Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(k^{|x|}x^2)}{2^k}$, where $x \in \mathbb{R}$. It is impossible to find out the exact value of the series, so proving its uniform convergence on $\mathbb{R}$ using the definition is not possible. But using the Weierstrass' M-test, we consider

$$\left|\frac{\sin(k^{|x|}x^2)}{2^k}\right| \leq \frac{1}{2^k} \quad \forall x \in \mathbb{R}, k \in \mathbb{N}.$$

Since $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges, we conclude that $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(k^{|x|}x^2)}{2^k}$ converges uniformly on $\mathbb{R}$.

Example 4.9. Consider the series $\displaystyle\sum_{k=1}^{\infty}(x \log x)^k$ where $x \in (0, 1]$. A quick sketch of the graph $y = x \log x$ reveals that $x \log x \leq 0$ and is bounded from below on $(0, 1]$. To find its minimum point, we use the first derivative test: $(x \log x)' = 1 + \log x$ which is zero when $x = \frac{1}{e}$. Also $x \log x \to 0$ as $x \to 0^+$ or $x \to 1$. Therefore, for any $x \in (0, 1]$.

$$x \log x \geq \frac{1}{e} \log \frac{1}{e} = -\frac{1}{e}.$$

This concludes that

$$\left|(x \log x)^k\right| = |x \log x|^k \leq \frac{1}{e^k} \quad \forall x \in (0, 1], k \in \mathbb{N}.$$

Since $\sum_{k=1}^{\infty} \frac{1}{e^k}$ converges, we conclude that $\displaystyle\sum_{k=1}^{\infty}(x \log x)^k$ converges uniformly on $(0,1]$.

Exercise 4.4. Show that each of the following series converges uniformly on the set indicated.

(a) $\sum_{k=1}^{\infty} ke^{-kx}$ on $[1, \infty)$

(b) $\sum_{k=1}^{\infty} \dfrac{kx}{e^{kx}}$ on $[\varepsilon, \infty)$ where $\varepsilon > 0$.

(c) $\sum_{k=2}^{\infty} \dfrac{x^k}{k(\log k)^2}$ on $[-1, 1]$.

Note that the Weiestrass' M-test could only be used for *proving* uniform convergence, it cannot be used to disprove it. For showing a series of functions does not converge uniformly, we should consider the definition. For example, consider $\sum_{k=1}^{\infty} x^k$ on $x \in [0,1)$. By the sum of geometric series formula, we know

$$\left|\sum_{k=1}^{n} x^k - \sum_{k=1}^{\infty} x^k\right| = \left|\sum_{k=n+1}^{\infty} x^k\right| = \left|\frac{x^{n+1}}{1-x}\right|.$$

It can be easily shown that

$$\sup_{x \in [0,1)} \left|\frac{x^{n+1}}{1-x}\right| = \infty.$$

Hence,

$$\left\|\sum_{k=1}^{n} x^k - \sum_{k=1}^{\infty} x^k\right\|_{[0,1)} = \sup_{x \in [0,1)} \left|\frac{x^{n+1}}{1-x}\right|_{[0,1)} = \infty.$$

Certainly, the convergence is not uniform on $[0,1)$.

Remark 4.10. I have seen a number of MATH3033/4023 students who wrote something like "$\left|x^k\right| \leq 1$ on $[0,1)$ and $\sum_{k=1}^{\infty} 1 = \infty$ diverges, so $\sum_{k=1}^{\infty} x^k$ diverges uniformly". LOL!

Exercise 4.5. Show that the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges uniformly (to e^x) on any closed and bounded interval $[a,b]$, but it does not converge uniformly on $\mathbb{R}$.

Exercise 4.6 (Dirichlet's test for uniform convergence)**.** There is a uniform version of Dirichlet test, which says the following: Let $f_n(x)$ and $g_n(x)$ be two sequences of real-valued functions defined on a domain $E \subset \mathbb{R}$. Suppose they satisfy the following conditions

(1) there exists a constant $C > 0$ such that

$$\left|\sum_{n=1}^{N} f_n(x)\right| \leq C$$

for any $N \in \mathbb{N}$ and $x \in E$,

(2) $g_n(x) \geq g_{n+1}(x)$ for any $n \in \mathbb{N}$ and $x \in E$, and

(3) g_n converges uniformly on E to 0.

Then, $\sum_{n=1}^{\infty} f_n g_n$ converges uniformly on E.

Prove this test, and use it to show that $\displaystyle\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ converges uniformly on $[\varepsilon, 2\pi - \varepsilon]$ where $\varepsilon > 0$.

Exercise 4.7 (Abel's test for uniform convergence)**.** There is also a uniform version of Abel's test: Let $f_n(x)$ and $g_n(x)$ be two sequences of real-valued functions defined on $E \subset \mathbb{R}$. Suppose:

(1) $\displaystyle\sum_{n=1}^{\infty} f_n$ converges uniformly on E,

(2) $g_n(x) \geq g_{n+1}(x)$ for any $n \in \mathbb{N}$ and $x \in E$, and

(3) there exists a constant $C > 0$ such that $|g_n(x)| \leq C$ for any $n \in \mathbb{N}$ and $x \in E$.

Then, $\displaystyle\sum_{n=1}^{\infty} f_n g_n$ converges uniformly on E.

Prove this test, and use it to show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-x)^n}{n}$ converges uniformly on $[0, 1]$.

Exercise 4.8 (Abel)**.** Show that if the power series

$$\sum_{n=0}^{\infty} a_n x^n,$$

converges when $x = \pm R$ for some $R \in (0, \infty)$, then it in fact converges uniformly on $[-R, R]$.

4.1.3. Properties of uniform convergence. The goal of this subsection is to discuss under what condition would the following rules hold:

$$\lim_{n\to\infty} \lim_{x\to a} f_n(x) = \lim_{x\to a} \lim_{n\to\infty} f_n(x)$$
$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx$$
$$\frac{d}{dx} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{d}{dx} f_n(x)$$

They are not generally true. For instance,

$$\lim_{n\to\infty} \lim_{x\to 1^-} x^n = 1 \quad \text{whereas} \quad \lim_{x\to 1^-} \lim_{n\to\infty} x^n = 0.$$

Consider the sequence of functions

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \leq x < 1/n \\ -n^2 x + 2n & \text{if } 1/n \leq x < 2/n \\ 0 & \text{if } 2/n \leq x \leq 1 \end{cases}$$

whose graph is a triangle with base $2/n$ and height n. Then we have

$$\lim_{n\to\infty} f_n(x) = 0$$

for any fixed $x \in [0, 1]$, so

$$\int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 0\,dx = 0.$$

However, we have

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \lim_{n\to\infty}\frac{1}{2}\cdot\frac{2}{n}\cdot n = 1.$$

For the interchange between differentiation and limit, the following is a counterexample.

$$g_n(x) := \frac{x}{1+nx^2}.$$

It can be easily checked that

$$\lim_{n\to\infty}\frac{d}{dx}\bigg|_{x=0} g_n(x) = 1 \quad\text{whereas}\quad \frac{d}{dx}\bigg|_{x=0}\lim_{n\to\infty} g_n(x) = 0.$$

Interchange of limit, integral and differentiation signs is not generally valid, but fortunately they become valid with the additional assumptions of uniform convergence.

Proposition 4.11. *Suppose $f_n : D \subset \mathbb{R} \to \mathbb{R}$ is a sequence of functions which converges uniformly on D to a function $f : D \to \mathbb{R}$. Then, given any limit point a of D such that $\lim_{x\to a} f_n(x)$ exist for any $n \in \mathbb{N}$, then $\lim_{x\to a} f(x)$ exists, and we have*

$$\lim_{n\to\infty}\lim_{x\to a} f_n(x) = \lim_{x\to a}\lim_{n\to\infty} f_n(x) = \lim_{x\to a} f(x).$$

Remark: here we denote $\lim_{x\to a, x\in D}$ simply by $\lim_{x\to a}$.

Proof. By the definition of uniform convergence, we have $\sup_{x\in D}|f_n(x)-f(x)| \to 0$ as $n\to\infty$. Therefore, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$, depending only on ε, such that if $n \geq N$ and $x \in D$, then

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}.$$

We first argue using the above inequality that $\lim_{x\to a} f(x)$ exists. Consider the function f_N, since we are given that $\lim_{x\to a} f_N(x)$ exists, there exists $\delta > 0$ depending on N (which ultimately depends on $\varepsilon > 0$), such that if $x, y \in B_\delta(a)\backslash\{a\}$ and $x, y \in D$, then

$$|f_N(x) - f_N(y)| < \frac{\varepsilon}{3}.$$

Combining with the previous inequality, we conclude that for any $x, y \in B_\delta(a)\backslash\{a\}$ and $x, y \in D$, we have

$$|f(x)-f(y)| \leq \underbrace{|f(x)-f_N(x)|}_{<\frac{\varepsilon}{3}} + \underbrace{|f_N(x)-f_N(y)|}_{<\frac{\varepsilon}{3}} + \underbrace{|f_N(y)-f(y)|}_{<\frac{\varepsilon}{3}} < \varepsilon.$$

To sum up, we have proved that for any $\varepsilon > 0$, there exists $\delta > 0$ which depends on ε such that whenever $x, y \in B_\delta(a)\backslash\{a\}$ and $x, y \in D$, we have $|f(x) - f(y)| < \varepsilon$. By Cauchy criterion, we conclude that $\lim_{x\to a} f(x)$ exists.

To complete our proof, we recall that for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ depending only on ε, such that for any $n \geq N$, the inequality below holds for any $x \in D$:

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{3}.$$

Note that N is independent of x, so one can let $x \to a$ on both sides of the above inequality, and get:

$$\left|\lim_{x\to a} f_n(x) - \lim_{x\to a} f(x)\right| = \lim_{x\to a}|f_n(x) - f(x)| \leq \frac{\varepsilon}{3}$$

whenever $n \geq N$. Equivalently, it means

$$\lim_{n\to\infty}\lim_{x\to a} f_n(x) = \lim_{x\to a} f(x).$$

□

Corollary 4.12. *Suppose $f_n : D \subset \mathbb{R} \to \mathbb{R}$ is a sequence of continuous functions which converges uniformly on D to a function $f : D \to \mathbb{R}$. Then f is also continuous on D.*

Proof. Fix any $a \in D$. Recall that we have proved the result that

$$\lim_{n\to\infty} \lim_{x\to a} f_n(x) = \lim_{x\to a} f(x).$$

By the continuity of f_n, we then have

$$\lim_{n\to\infty} f_n(a) = \lim_{x\to a} f(x).$$

Note that f_n converges to f (uniformly on D, and hence pointwise on D too), so $f_n(a) \to f(a)$ as $n \to \infty$. It concludes that

$$f(a) = \lim_{x\to a} f(x)$$

as desired. □

Exercise 4.9. Prove that if $f_n : D \subset \mathbb{R} \to \mathbb{R}$ is a sequence of bounded functions which uniformly converges on D to a function f, then f is also bounded.

Example 4.13. Uniform convergence is commonly used to prove the continuity of a function defined using an infinite series, such as the zeta's function:

$$\zeta(x) := \sum_{n=1}^{\infty} \frac{1}{n^x}, \quad \text{where } x > 1.$$

Although $\frac{1}{n^x}$ is continuous on $x \in (1, \infty)$ for any fixed n, the infinite sum of continuous functions is not necessary continuous. To prove the continuity of $\zeta(x)$, it would be best if $\sum_{n=1}^{\infty} \frac{1}{n^x}$ converges uniformly on $(1, \infty)$. However, it is not really the case.

Fortunately, one can still prove that $\sum_{n=1}^{\infty} \frac{1}{n^x}$ converges uniformly on $[1 + \varepsilon, \infty)$ for any fixed $\varepsilon > 0$. It does not imply uniform convergence on $(1, \infty)$, but by the previous corollary, it would imply $\zeta(x)$ is continuous on $[1 + \varepsilon, \infty)$ for any $\varepsilon > 0$, which would imply $\zeta(x)$ is continuous on $(1, \infty)$.

To prove this, we consider a fixed domain $[1 + \varepsilon, \infty)$ where $\varepsilon > 0$, then we have

$$\left|\frac{1}{n^x}\right| \le \frac{1}{n^{1+\varepsilon}} \quad \forall x \in [1 + \varepsilon, \infty).$$

Since the series of real numbers $\sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}}$ converges (by p-test), the series $\sum_{n=1}^{\infty} \frac{1}{n^x}$ converges uniformly on $[1 + \varepsilon, \infty)$.

Exercise 4.10. Pretend that you do not know $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to e^x. Show that the series defines a continuous function on $\mathbb{R}$.

Exercise 4.11. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ is a continuous function on $\mathbb{R}$.

A sequence of functions which uniformly converges also allows the interchange of the integral and limit signs:

Proposition 4.14. *Suppose $f_n : [a,b] \to \mathbb{R}$ is a sequence of continuous*[1] *functions which converges uniformly to f on a closed and bounded interval $[a,b]$, then we have*

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Proof. The result immediately follows from

$$\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| \leq \int_a^b \left|f_n(x) - f(x)\right|\,dx \leq (b-a)\left\|f_n - f\right\|_{[a,b]}.$$

□

Remark 4.15. It is clear from the above proof that $b-a$ needs to be finite in order for the result to hold. Therefore, one can only apply the above result when a, b are finite. When either a or b is infinite, we will learn in MATH 3043 another tool called Lebesgue's dominated convergence theorem (LDCT) to justify the interchange.

Example 4.16. Here we prove a (symbolically) interesting result:

$$\int_0^1 \frac{1}{x^x}\,dx = \sum_{n=1}^\infty \frac{1}{n^n}.$$

We first express $\frac{1}{x^x}$ as an infinite series:

$$\frac{1}{x^x} = x^{-x} = e^{-x\log x} = \sum_{n=0}^\infty \frac{(-x\log x)^n}{n!}.$$

[Remark: we can treat x^x as 1, and $x\log x$ as 0 at $x=0$.]

As $|x\log x|$ is bounded from above by $\frac{1}{e}$ on $[0,1]$, we have

$$\left|\frac{(-x\log x)^n}{n!}\right| \leq \frac{1}{e^n n!} \quad \forall x \in [0,1].$$

By ratio test, $\sum_{n=0}^\infty \frac{1}{e^n n!}$ converges, hence Weierstrass' M-test shows the series $\sum_{n=0}^\infty \frac{(-x\log x)^n}{n!}$ converges uniformly on $[0,1]$. Therefore, we have

$$\int_0^1 \frac{1}{x^x}\,dx = \int_0^1 \sum_{n=0}^\infty \frac{(-x\log x)^n}{n!}\,dx = \sum_{n=0}^\infty \int_0^1 \frac{(-x\log x)^n}{n!}\,dx.$$

By straight-forward computations (say by successive integration by parts – a good exercise for MATH1014/1024), we get

$$\int_0^1 \frac{(-x\log x)^n}{n!}\,dx = \frac{1}{(n+1)^{n+1}},$$

which yields our desired result after relabelling the summation index.

Exercise 4.12. Express the value of each integral below as the sum of an infinite series:

$$\int_0^1 (\sin x)(\log x)\,dx \qquad \int_0^1 \frac{e^{x^2}-1}{x}\,dx \qquad \int_{1/3}^{2/3} \frac{\log(1-x^2)}{x^2}\,dx.$$

Regarding the interchange of differentiation and limit signs, it turns out that uniform convergence of the sequence/series is not enough. Here is a counter-example:

$$g_n(x) := \frac{x}{1+nx^2}.$$

It can be easily shown using elementary calculus that $|g_n(x)| \leq \frac{1}{2\sqrt{n}}$ for any $x \in [0,1]$. It converges uniformly to 0 on $[0,1]$. However, it can be easily checked that

$$\lim_{n\to\infty} \frac{d}{dx}\bigg|_{x=0} g_n(x) = 1 \quad \text{whereas} \quad \frac{d}{dx}\bigg|_{x=0} \lim_{n\to\infty} g_n(x) = 0.$$

In fact, we need the uniform convergence of the *derivative* of the sequence/series, and just pointwise convergence of the sequence/series:

Proposition 4.17. *Suppose each $f_n : (a,b) \to \mathbb{R}$ is differentiable on (a,b) and given that*

- *f_n converges pointwise to a function f on (a,b), and*
- *f_n' converges uniformly on (a,b) to a function g.*

Then, f is differentiable with $f' = g$ on (a,b).

Note that $f'(x_0) = g(x_0)$ where $x_0 \in (a,b)$ is equivalent to:

$$\frac{d}{dx}\bigg|_{x=x_0} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{d}{dx}\bigg|_{x=x_0} f_n(x).$$

Proof. For each $x_0 \in (a,b)$, we define functions $q_n, q : (a,b)\backslash\{x_0\} \to \mathbb{R}$ by:

$$q_n(x) := \frac{f_n(x) - f_n(x_0)}{x - x_0} \qquad\qquad q(x) := \frac{f(x) - f(x_0)}{x - x_0}.$$

We are going to justify that $\lim\limits_{x\to x_0} q(x)$ exists, and

$$\lim_{x\to x_0} \lim_{n\to\infty} q_n(x) = \lim_{n\to\infty} \lim_{x\to x_0} q_n(x).$$

Noting that x_0 is a limit point of the domain $(a,b)\backslash\{x_0\}$ of q_n's and q, one can justify (using Proposition 4.11) the above interchange of limit by showing q_n converges uniformly on $(a,b)\backslash\{x_0\}$ to q.

To prove such a convergence is uniform, we estimate that for any $x \in (a,b)\backslash\{x_0\}$

$$\begin{aligned}
&|q_n(x) - q_m(x)| \\
&= \left| \frac{f_n(x) - f_n(x_0)}{x - x_0} - \frac{f_m(x) - f_m(x_0)}{x - x_0} \right| \\
&= \left| \frac{\big(f_n(x) - f_m(x)\big) - \big(f_n(x_0) - f_m(x_0)\big)}{x - x_0} \right| \\
&= \left|(f_n - f_m)'(c)\right| \quad \exists c, \in (x_0, x) \text{ or } (x, x_0) \\
&\leq \|f_n' - f_m'\|_{(a,b)}
\end{aligned}$$

where we have applied the mean value theorem on the function $x \mapsto f_n(x) - f_m(x)$. This shows

$$\|q_n - q_m\|_{(a,b)\backslash\{x_0\}} \leq \|f_n' - f_m'\|_{(a,b)}.$$

As f_n' converges uniformly on (a,b), the sequence $\{f_n'\}$ is Cauchy with respect to the sup-norm $\|\ \|_{(a,b)}$. The above estimate implies $\{q_n\}$ is Cauchy with respect to the sup-norm $\|\ \|_{(a,b)\backslash\{x_0\}}$. By the result from Exercise 4.3, $\{q_n\}$ converges uniformly to some function on $(a,b)\backslash\{x_0\}$. As f_n converges to f pointwise, the pointwise limit of q_n is:

$$\lim_{n\to\infty} q_n(x) = \lim_{n\to\infty} \frac{f_n(x) - f_n(x_0)}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0} = q(x).$$

which must also be the limit of uniform convergence as well. Therefore, q_n converges uniformly to q on $(a,b)\backslash\{x_0\}$.

By Proposition 4.11, we conclude that $\lim_{x\to x_0} q(x)$ exists, meaning that f is differentiable at x_0, and

$$\begin{aligned} &\lim_{x\to x_0}\lim_{n\to\infty} q_n(x) = \lim_{n\to\infty}\lim_{x\to x_0} q_n(x) \\ \implies &\lim_{x\to x_0} q(x) = \lim_{n\to\infty} f_n'(x_0) \\ \implies &f'(x_0) = g(x_0) \end{aligned}$$

as desired. □

Exercise 4.13. Show that the assumptions in Proposition 4.17 implies f_n converges uniformly to f on (a, b).

Example 4.18. We will show that $\zeta(x) := \sum_{n=1}^{\infty} \frac{1}{n^x}$ is differentiable on $(1, \infty)$. Recall that we have proved in Example 4.16 that $\sum_{n=1}^{\infty} \frac{1}{n^x}$ converges uniformly on $(1+\varepsilon, \infty)$ for any $\varepsilon > 0$, but it is not enough to justify the interchange between $\sum$ and $\frac{d}{dx}$. We need the uniform convergence of the derivative series.

$$\frac{d}{dx}\frac{1}{n^x} = -\frac{\log n}{n^x}.$$

Therefore, we need to argue that $\sum_{n=1}^{\infty} \frac{\log n}{n^x}$ converges uniformly on some appropriate domain.

For any $\varepsilon > 0$, we consider the domain $(1+\varepsilon, \infty)$, and we see that

$$\left|\frac{\log n}{n^x}\right| \le \frac{\log n}{n^{1+\varepsilon}} \quad \forall x \in (1+\varepsilon, \infty).$$

Note that $\log n - n^{\varepsilon/2} \to -\infty$ as $n \to \infty$, so $\log n < n^{\varepsilon/2}$ for sufficiently large n, which implies $\frac{\log n}{n^{1+\varepsilon}} \le \frac{1}{n^{1+\varepsilon/2}}$ when n is sufficiently large.

As $\sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon/2}}$ converges, the series $\sum_{n=1}^{\infty} \frac{\log n}{n^{1+\varepsilon}}$ converges too, proving that

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^x}\right)' = -\sum_{n=1}^{\infty}\frac{\log n}{n^x}$$

converges uniformly on $(1+\varepsilon, \infty)$. Together with the fact that $\sum_{n=1}^{\infty} \frac{1}{n^x}$ converges pointwise on $(1+\varepsilon, \infty)$, Proposition 4.17 shows that $\zeta(x)$ is differentiable on $(1+\varepsilon, \infty)$ with $\zeta'(x) = -\sum_{n=1}^{\infty} \frac{\log n}{n^x}$. Since $\varepsilon > 0$ is arbitrary, $\zeta(x)$ is differentiable on $(1, \infty)$.

Exercise 4.14. Prove that $\zeta(x)$ is C^∞ on $(1, \infty)$, and

$$\zeta^{(k)}(x) = \sum_{n=1}^{\infty} \frac{(-\log n)^k}{n^x} \quad \forall x \in (1, \infty).$$

Combining with the results from Exercise 4.8 and Proposition 4.17, one can justify that we can do term-by-term differentiation for a power series:

Proposition 4.19. *Suppose* $f(x) := \sum_{n=1}^{\infty} a_n(x-c)^n$ *converges pointwise on* $(c-R, c+R)$ *where* $R \in (0, \infty)$*, then* f *is differentiable on* $(c-R, c+R)$ *with*

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}.$$

Proof. Consider an arbitrary $\varepsilon \in (0, R/3)$ and consider the slightly smaller interval $[c-R+\varepsilon, c+R-\varepsilon] \subset (c-R, c+R)$. As the series $\sum_{n=1}^{\infty} a_n(x-c)^n$ converges pointwise on $[c-R+\varepsilon, c+R-\varepsilon]$, we have (by the root test – see the stronger version in the exercise below),

$$\limsup_{n\to\infty} \sqrt[n]{|a_n(x-c)^n|} \leq 1 \implies \limsup_{n\to\infty} \sqrt[n]{|a_n|}\,|x-c| \leq 1$$

for any $x \in [c-R+\varepsilon, c+R-\varepsilon]$. Unless $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 0$, the equality $(= 1)$ case (if achieved) must be attained when $x = c \pm (R-\varepsilon)$, therefore it would imply for any $x \in [c-R+2\varepsilon, c+R-2\varepsilon]$, we have

$$\limsup_{n\to\infty} \sqrt[n]{|a_n(x-c)^n|} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}\,|x-c| < 1.$$

This also implies for any $x \in [c-R+2\varepsilon, c+R-2\varepsilon]$, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \sqrt[n]{|na_n|\,(x-c)^{n-1}} \\
&= \limsup_{n\to\infty} n^{1/n} \sqrt[n]{|a_n|}\,|x-c|^{\frac{n-1}{n}} \\
&= 1 \cdot \sqrt[n]{|a_n|} \cdot |x-c|^1 < 1.
\end{aligned}$$

This shows

$$\sum_{n=1}^{\infty} na_n(x-c)^{n-1} = \sum_{n=1}^{\infty} \frac{d}{dx} a_n(x-c)^n$$

converges pointwise on $[c-R+2\varepsilon, c+R-2\varepsilon]$ by root test. From Exercise 4.8, the series converges uniformly on $[c-R+2\varepsilon, c+R-2\varepsilon]$

By Proposition 4.17, the series

$$f(x) = \lim_{N\to\infty} \underbrace{\sum_{n=1}^{N} a_n(x-c)^n}_{=:g_N(x)}$$

is differentiable on $(c-R+2\varepsilon, c+R-2\varepsilon)$ with

$$f'(x) = \lim_{N\to\infty} \underbrace{\frac{d}{dx} \sum_{n=1}^{N} a_n(x-c)^n = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}}_{=g'_N(x)}.$$

Since f is differentiable on $(c-R+2\varepsilon, c+R-2\varepsilon)$ for any $\varepsilon > 0$, it is in fact differentiable on $(c-R, c+R)$, and its derivative is given as above. $\square$

Corollary 4.20. *Suppose the power series* $f(x) := \sum_{n=1}^{\infty} a_n(x-c)^n$ *converges pointwise on* $(c-R, c+R)$*, where* $R \in (0, \infty]$ *(note that when* $R = \infty$ *we treat* $(c-R, c+R)$ *as* $\mathbb{R}$*), then* f *is* C^∞ *on* $(c-R, c+R)$*.*

Proof. If $0 < R < \infty$, then the result just follows from induction. If $R = \infty$, then consider the interval $(c-r, c+r)$ for any $r > 0$. Apply the proposition to show f is C^∞ on $(c-r, c+r)$ for any $r > 0$. This shows f is C^∞ on $\mathbb{R}$. $\square$

Exercise 4.15. Given a sequence of real numbers $\{a_n\}_{n=1}^{\infty}$, show that

- If $\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|} > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Exercise 4.16. Show that the coefficients of a power series are unique, in a sense that if

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

on $(c-R, c+R)$, then we have

$$a_n = \frac{f^{(n)}(c)}{n!} \quad \forall n = 0, 1, 2, 3, \cdots.$$

Exercise 4.17. Show that any real analytic solution $f(x)$ (recall: "real analytic" means it can be expressed as a power series) to the ODE $f^{(4)} = f$ on $\mathbb{R}$ must be of the form

$$ae^x + be^{-x} + c\cos x + d\sin x$$

for some constants $a, b, c, d \in \mathbb{R}$.

4.2. Special Topics on Uniform Convergence

4.2.1. Weiestrass's Function. We all know that differentiability implies continuity, but not conversely. One quick counter-example to the converse is the absolute-value function $x \mapsto |x|$, which is continuous everywhere on $\mathbb{R}$, but not differentiable at 0.

In this subsection, we will study a much more intriguing counter-example to the converse, that is continous *everywhere* on R, but is not differentiable at any point! This example of function was constructed by Karl Weiestrass in 1872, published in his paper *Über continuirliche Functionen eines reellen Arguments, die für keinen Werth des letzeren einen bestimmten Differentialquotienten besitzen*.

Theorem 4.21 (Weiestrass (1872)). *For any $a \in (0,1)$ and odd integer b such that $ab > 1 + \frac{3\pi}{2}$, the following series*

$$f(x) := \sum_{n=0}^{\infty} a^n \cos(b^n n x)$$

defines a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that f is not differentiable at any $x_0 \in \mathbb{R}$.

Here we only outline the proof, and leave it as an exercise for readers to fill in the technical detail. Complete proof can be found in the above-mentioned paper (written in German), and many other reference textbooks.

(1) The uniform convergence follows directly from Weiestrass's M-test, as

$$|a^n \cos(b^n n x)| \leq a^n \quad \forall x \in \mathbb{R}, n = 0, 1, 2, \cdots$$

and that $0 < a < 1$ so $\sum_n a^n$ converges. Because $a^n \cos(b^n n x)$'s are all continuous functions on $\mathbb{R}$, the uniform limit $f(x)$ is also continuous on $\mathbb{R}$.

(2) Our next goal is to show that f is not differentiable at any $x_0 \in \mathbb{R}$, meaning that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

does not exist. For any integer $m \geq 0$, there exists $k_m \in \mathbb{Z}$ such that

$$b^m x_0 - k_m \in \left(-\frac{1}{2}, \frac{1}{2}\right],$$

because the set $\{b^m x_0 - k : k \in \mathbb{Z}\}$ is a set of points separated by distance of 1. Then, we denote

$$x_m := b^m x_0 - k_m \qquad y_m := \frac{k_m - 1}{b^m}$$

Try to verify that $y_m \to x_0$.

(3) Write the quotient as

$$\begin{aligned} &\frac{f(y_m) - f(x_0)}{y_m - x_0} \\ &= \underbrace{\sum_{n=0}^{m-1} (ab)^n \frac{\cos(b^n \pi y_m) - \cos(b^n \pi x_0)}{b^n (y_m - x_0)}}_{=:F_m} - \underbrace{\sum_{n=m}^{\infty} (ab)^n \frac{\cos(b^n \pi y_m) - \cos(b^n \pi x_0)}{b^n (y_m - x_0)}}_{=:S_m}. \end{aligned}$$

Try to derive that

$$|F_m| < \pi \frac{(ab)^m}{ab - 1}.$$

(4) Using fact that $b^n y_m \in \mathbb{Z}$ when $n \geq m$ (and that's why we split the infinite sum as $F_m + S_m$), try to show that
$$|S_m| \geq \frac{2}{3}(ab)^m.$$

(5) Combine the above results to prove that
$$\left|\frac{f(y_m) - f(x_0)}{y_m - x_0}\right| \geq (ab)^m \left(\frac{2}{3} - \frac{\pi}{ab - 1}\right)$$
for any $m \in \mathbb{N}$. This would imply
$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$
does not exist, completing the proof.

Exercise 4.18. Fill in the omitted detail in the above proof.

You may very well think that such a "continuous everywhere, differentiable nowhere" function is very rare. In contrast, there are many such functions, so many that any continuous function can be approximated by such a "pathological" function. Precisely, the set:
$$E := \{f \in C[0,1] : f'(x_0) \text{ does not exists for any } x_0 \in [0,1]\}$$
is dense in $C[0,1]$ under the sup-norm $\| \ \|_{[0,1]}$. We outline the proof in the exercise below, and let readers to fill in the detail.

Exercise 4.19. Let
$$\begin{aligned} E &:= \{f \in C[0,1] : f'(x_0) \text{ does not exists for any } x_0 \in [0,1]\} \\ S_n &:= \{f \in C[0,1] : \exists x_0 \in [0,1] \text{ such that } |f(x) - f(x_0)| \leq n\,|x - x_0| \quad \forall x \in [0,1]\} \end{aligned}$$

(a) Show that
$$\bigcup_{n=1}^{\infty} S_n = C[0,1] - E.$$

(b) Show that S_n is closed for any $n \in \mathbb{N}$.

(c) Show that $S_n^\circ = \emptyset$ for any $n \in \mathbb{N}$. Hint: drawing diagrams will help.

Then, as $C[0,1]$ with sup-norm is complete, Baire category theorem shows
$$E = C[0,1] - \bigcup_{n=1}^{\infty} S_n$$
is dense in $C[0,1]$.

4.2.2. Arzela-Ascoli's Theorem.

Recall that on a closed and bounded subset of $C[a,b]$ (with sup-norm) may not be sequentially compact. Therefore, even if we are given a *uniformly bounded* sequence $\{f_n\}$ in $C[a,b]$ (see definition below), we may not always be able to extract a converging subsequence f_{n_k}. However, if we impose one more condition on the sequence of functions, namely *equicontinuity*, then one can extract a convergent subsequence from it. Let's first state the definitions.

Definition 4.22 (Uniform boundedness)**.** A sequence of functions $\{f_n\}_{n=1}^\infty$ in $C[a,b]$ is said to be **uniformly bounded on** $[a,b]$ if there exists a constant $C > 0$ independent of n and x, such that
$$|f_n(x)| \leq C \quad \forall n \in \mathbb{N}, x \in [a,b].$$

Definition 4.23 (Equicontinuity)**.** A sequence of functions $\{f_n\}_{n=1}^{\infty}$ in $C[a,b]$ is said to be **equicontinuous on** $[a,b]$ if for any $\varepsilon > 0$, there exists $\delta > 0$ which only depends on ε (not on n, x or y) such that whenever $x, y \in [a,b]$ and $|x-y| < \delta$, then

$$|f_n(x) - f_n(y)| < \varepsilon \ \ \forall n \in \mathbb{N}.$$

Example 4.24. Let $x_n(t) = \frac{t}{n}$ on $I = [0,1]$. Then $\{x_n(t)\}$ is an equicontinuous sequence on I because for any $\varepsilon > 0$, one can find $\delta = \varepsilon > 0$ (of course depends only on ε), such that whenever $t,\ s \in [0,1]$ and $|t-s| < \delta$, we have:

$$|x_n(t) - x_n(s)| = \left|\frac{t}{n} - \frac{s}{n}\right| = \frac{1}{n}|t-s| \leq |t-s| < \delta = \varepsilon.$$

However, $y_n(t) = nt$ on $I = [0,1]$ is not an equicontinuous sequence on I. Take $\varepsilon_0 = 1$, for any $\delta > 0$, one can take $t = 0$ and $s = \frac{\delta}{2}$, and we have $t,\ s \in I$ and $|t-s| < \delta$. However, then

$$|y_n(t) - y_n(s)| = n|t-s| = \frac{n\delta}{2} \geq \varepsilon$$

when $n \geq \frac{2\varepsilon}{\delta}$.

Nonetheless, for each n, the function $y_n(t)$ is continuous on I. □

Exercise 4.20. Let $I = [0, 2\pi]$. Show that the sequence of functions $x_n(t) := \sin\frac{t}{n}$ is equicontinuous on I, but $y_n(t) := \sin(nt)$ is not.

Exercise 4.21. Suppose $\{f_n(t)\}_{n=1}^{\infty}$ is a sequence of differentiable functions on $t \in [a,b]$ such that $f_n'(t)$ is uniformly bounded on $[a,b]$. Show that $\{f_n(t)\}$ is equicontinuous on $[a,b]$.

Here we state and prove a compactness theorem for a uniformly bounded and equicontinuous sequence of functions:

Theorem 4.25 (Arzelà-Ascoli's Theorem)**.** *Let $I = [a,b]$ be a closed and bounded interval. Let $f_n(t) : I \to \mathbb{R}$ be a sequence of functions which is uniformly bounded and equicontinuous on I. Then, there exists a subsequence $f_{n_k}(t)$ that converges uniformly on I to a limit function $f(t)$ as $k \to \infty$.*

Proof. The proof is divided into two major steps. The first step is to consider the set $\mathbb{Q} \cap I = \{r_1, r_2, r_3, \cdots\}$, which is countable and dense in I. We first show using the uniform boundedness condition that f_n has a subsequence that converges at every $r_j \in \mathbb{Q} \cap I$. The second step is to use the equicontinuous condition to extend the convergence result to other points in $\mathbb{Q} \cap I$.

The first step is known as the *diagonalization argument*. Consider the sequence of real numbers $\{f_n(r_1)\}_{n=1}^{\infty}$, which is bounded by the uniform boundedness condition. Bolzano-Weierstrass's Theorem shows that there exists a convergent subsequence $\{f_{n_k}(r_1)\}$. As we will repeatedly taking subsequences, let's denote the subsequences in a more system way. We denote f_{n_k} by f_k^1, where the superscript 1 means that we have taken a subsequence the first time. Similarly, we consider $\{f_k^1(r_2)\}_{k=1}^{\infty}$ which is also bounded, so it has a convergent subsequence $\{f_k^2(r_2)\}$. Inductively, there is a decreasing chain of subsequences

$$\{f_n\}_{n=1}^{\infty} \supset \{f_n^1\}_{n=1}^{\infty} \supset \{f_n^2\}_{n=1}^{\infty} \supset \{f_n^3\}_{n=1}^{\infty} \supset \cdots$$

such that for each $j \in \mathbb{N}$, $\{f_n^j(r_j)\}_{n=1}^{\infty}$ converges as $n \to \infty$.

Now we construct, by diagonalization, a new subsequence $\{f_n^n\}_{n=1}^\infty$ of $\{f_n\}$. Then, we claim that for any $j \in \mathbb{N}$, the sequence of real numbers $\{f_n^n(r_j)\}_{n=1}^\infty$ converges. Fix $j \in \mathbb{N}$, and consider the tail part:

$$\{f_n^n(r_j)\}_{n=j}^\infty = \left\{f_j^j(r_j), f_{j+1}^{j+1}(r_j), f_{j+2}^{j+2}(r_j), \cdots\right\}.$$

Recall that f_{j+1}^{j+1}, which is the $(j+1)$-th term of the $(j+1)$-th subsequence, equals to a certain term of its parent sequence $\{f_n^j\}_{n=1}^\infty$. Similarly, for f_{j+2}^{j+2}, f_{j+3}^{j+3}, etc., they are all belong to the parent sequence $\{f_n^j\}_{n=1}^\infty$. Therefore, $\{f_n^n\}_{n=j}^\infty$ is a subsequence of $\{f_n^j\}_{n=j}^\infty$. Since $\{f_n^j(r_j)\}_{n=j}^\infty$ converges as $n \to \infty$, its subsequence $\{f_n^n(r_j)\}_{n=j}^\infty$ (and hence $\{f_n^n(r_j)\}_{n=1}^\infty$) also converges as desired.

The second step is to extend the convergent result of $\{f_n^n\}_{n=1}^\infty$ to uniform convergence on I. Suppose otherwise that $\{f_n^n\}$ does not converge uniformly on I to any function, then there exists $\varepsilon_0 > 0$ such that for any $k \in \mathbb{N}$, there exist integers $m_k, n_k \geq k$ such that $\sup_{x\in I}\left|f_{n_k}^{n_k}(x) - f_{m_k}^{m_k}(x)\right| \geq \varepsilon_0$. We may choose m_k's and n_k's such that $m_1 < m_2 < m_3 < \cdots$ and $n_1 < n_2 < n_3 < \cdots$, so that $\{f_{m_k}^{m_k}\}$ and $\{f_{n_k}^{n_k}\}$ are subsequences of $\{f_n^n\}$. By the above supremum condition, for each $k \in \mathbb{N}$, there exists $x_k \in I$ such that

$$\left|f_{m_k}^{m_k}(x_k) - f_{n_k}^{n_k}(x_k)\right| \geq \frac{\varepsilon_0}{2}.$$

By sequential compactness of I, $\{x_k\}_{k=1}^\infty$ has a convergent subsequence $\{x_{k_i}\}_{i=1}^\infty$. Suppose $x_{k_i} \to L$ as $i \to \infty$.

By equicontinuity, there exists $\delta_0 > 0$ depending only on ε_0 such that whenever $x, y \in I$ and $|x-y| < \delta_0$, we have

$$|f_n^n(x) - f_n^n(y)| < \frac{\varepsilon_0}{8} \quad \forall n \in \mathbb{N}.$$

By density of rationals, we can pick a rational number $r \in \mathbb{Q}\cap I$ such that $|r - L| < \delta_0$. The above equicontinuity condition asserts that

$$\left|f_{m_{k_i}}^{m_{k_i}}(x_{k_i}) - f_{m_{k_i}}^{m_{k_i}}(r)\right| < \frac{\varepsilon_0}{8} \quad \forall i \in \mathbb{N}.$$

By the convergence of $\{f_n^n(r)\}_{n=1}^\infty$, we have

$$\left|f_{m_{k_i}}^{m_{k_i}}(r) - f_{n_{k_i}}^{n_{k_i}}(r)\right| < \frac{\varepsilon_0}{8}$$

for sufficiently large i.

Combining the above estimates together, we arrive at

$$\begin{aligned}
\frac{\varepsilon_0}{2} &\leq \left|f_{m_{k_i}}^{m_{k_i}}(x_{k_i}) - f_{n_{k_i}}^{n_{k_i}}(x_{k_i})\right| \\
&\leq \left|f_{m_{k_i}}^{m_{k_i}}(x_{k_i}) - f_{m_{k_i}}^{m_{k_i}}(r)\right| + \left|f_{m_{k_i}}^{m_{k_i}}(r) - f_{n_{k_i}}^{n_{k_i}}(r)\right| + \left|f_{n_{k_i}}^{n_{k_i}}(r) - f_{n_{k_i}}^{n_{k_i}}(x_{k_i})\right| \\
&< \frac{\varepsilon_0}{8} + \frac{\varepsilon_0}{8} + \frac{\varepsilon_0}{8} = \frac{3}{8}\varepsilon_0
\end{aligned}$$

which is a contradiction. □

Remark 4.26. If $\mathbb{R}$ and $[a,b]$ are replaced by a metric space (X,d) and a compact set $K \subset X$ respectively, then the Arzelá-Ascoli's theorem still holds. The proof is almost the same, *mutatis mutandis*. The set $\mathbb{Q}\cap I$ is replaced by a countable dense subset $D \subset K$.

Example 4.27. Let $\{f_n\}_{n=1}^\infty$ be continuous function on $[a,b]$ such that there exists a constant $M > 0$ such that $\|f_n\|_{[a,b]} \leq M$ for any n. Then, define

$$F_n(x) := \int_a^x f_n(t)\,dt.$$

Then, one can use Arzelà-Ascoli to prove that F_n has a uniform convergent subsequence. We simply need to verify the uniform boundedness and equicontinuous conditions.

For uniform boundedness:

$$|F_n(x)| \leq \int_a^x |f_n(t)|\, dt \leq \int_a^x M\, dt = M(x-a) \leq M(b-a) \;\; \forall x \in [a,b].$$

For equicontinuity:

$$|F_n(x) - F_n(y)| = \left|\int_x^y |f_n(t)|\, dt\right| \leq \left|\int_x^y M\, dt\right| = M\,|x-y|\,.$$

Clearly $\{F_n\}$ is equicontinuous on $[a,b]$. Arzelà-Ascoli shows $\{F_n\}$ has a convergent subsequence $\{F_{n_j}\}$ converging uniformly on $[a,b]$.

Exercise 4.22. Let $\{f_n\}_{n=1}^{\infty}$ be continuous function on $[a,b]$ such that there exist constants $p > 1$ and $M > 0$ such that

$$\int_a^b |f_n(t)|^p\, dt \leq M.$$

Then, define

$$F_n(x) := \int_a^x f_n(t)\, dt.$$

Prove that F_n has a uniform convergent subsequence.

Exercise 4.23. Recall that we have defined $C^{\alpha}[0,1]$ and $\|\;\|_{\alpha}$-norm in Exercise 2.8. Show that the closed ball

$$\overline{B} := \{f \in C^{\alpha}[0,1] : \|f\|_{\alpha} \leq 1\}$$

is compact in $C[0,1]$ with respect to the sup-norm $\|f\|_{\infty} := \sup_{x\in[0,1]} |f(x)|$.

The first step in the proof of Arzelà-Ascoli was known “diagonlization technique” in analysis. The technique itself can be useful when we need to take subsequences for infinitely many times. The following is a good example to demonstrate this technique.

Example 4.28. Suppose $f_n : [a,b] \to \mathbb{R}$ is a sequence of C^{∞} functions. Suppose for each $k \in \mathbb{N} \cup \{0\}$, there exist constants $C_k > 0$ independent of n but may depend on k, such that

$$\left\|f^{(k)}\right\|_{[a,b]} := \sup_{x\in[a,b]} \left|f^{(k)}(x)\right| \leq C_k.$$

We will show that there exists a subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that $f_{n_j}^{(k)}$ converges uniformly on $[a,b]$ as $j \to \infty$ for all $k \in \mathbb{N} \cup \{0\}$.

By $\|f_n\|_{[a,b]} \leq C_0$, we know that $\{f_n\}$ is uniformly bounded on $[a,b]$, and by $\|f_n'\|_{[a,b]}$ and the mean value theorem, $\{f_n\}$ is equicontinuous on $[a,b]$. Arzelà-Ascoli's Theorem shows there exists a subsequence $\{f_{1,n}\}$ of $\{f_n\}$ converging uniformly on $[a,b]$.

Certainly, if one replaces f_n by $f_n^{(k)}$, and f_n' by $f_n^{(k+1)}$ in the above argument, then Arzelà-Ascoli's Theorem can be applied in a similar to show there exists a subsequence $\{f_{n_j}\}$ so that $\{f_{n_j}^{(k)}\}$ converges uniformly as $j \to \infty$. However, in doing so, we would have *different* subsequences for different k's, while we hope to find ONE subsequence $\{f_{n_j}\}$ such that $\{f_{n_j}^{(k)}\}$ converges uniformly for all k.

To fix this issue, we will use diagonalization argument. After getting the subsequence $\{f_{1,n}\}$ that converges uniformly on $[a,b]$, we will apply Arzelà-Ascoli again to handle the case $k = 1$. Using the given facts that

$$\|f_{1,n}'\|_{[a,b]} \leq C_1 \quad \text{and} \quad \|f_{1,n}''\|_{[a,b]} \leq C_2,$$

we know that $\{f'_{1,n}\}$ is uniformly bounded and equicontinuous on $[a, b]$, Arzelà-Ascoli implies there exists a subsequence $\{f_{2,n}\}$ of $\{f_{1,n}\}$ such that $\{f'_{2,n}\}_{n=1}^{\infty}$ converges uniformly on $[a, b]$.

Inductively, one can then construct a decreasing chain of subsequences

$$\{f_n\}_{n=1}^{\infty} \supset \{f_{1,n}\}_{n=1}^{\infty} \supset \{f_{2,n}\}_{n=1}^{\infty} \supset \{f_{3,n}\}_{n=1}^{\infty} \supset \cdots$$

such that for each $k \in \mathbb{N}$, the subsequence $\{f_{k,n}^{(k-1)}\}_{n=1}^{\infty}$ converges uniformly on $[a, b]$ as $n \to \infty$.

Finally, the diagonal subsequence $\{f_{n,n}\}_{n=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ will be the desired subsequence. Using the $\{f_{n_j}\}$-notations, we take $f_{n_j} := f_{j,j}$. For each $k \in \mathbb{N} \cup \{0\}$, the sequence $\{f_{n,n}\}_{n=k+1}^{\infty}$ is a subsequence of $\{f_{k+1,n}\}_{n=1}^{\infty}$. Since $\{f_{k+1,n}^{(k)}\}_{n=1}^{\infty}$ converges uniformly on $[a, b]$ as $n \to \infty$, so does $\{f_{n,n}^{(k)}\}_{n=k+1}^{\infty}$ and hence $\{f_{n,n}^{(k)}\}_{n=1}^{\infty}$.

Exercise 4.24. Suppose $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of differentiable functions such that for any $R > 0$, there exists a constant $C_R > 0$ which could depend on R such that

$$\|f_n\|_{[-R,R]} + \|f'_n\|_{[-R,R]} \leq C_R.$$

Show that there exists a subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ such that for any $R > 0$, we have f_{n_j} converges uniformly on $[-R, R]$.

We may wonder are there any "practical use" of Arzelà-Ascoli's Theorem, apart from being able to extract uniformly convergent subsequences. One common use of the theorem in the context of Calculus of Variations is to justify the minimizer of a functional exists. For example, given a "functional" $E : C[a, b] \to \mathbb{R}$, which maps a function $f \in C[a, b]$ to a real number, we want to see if there exists $f_0 \in C[a, b]$ such that

$$E(f_0) = \inf\{E(f) : f \in C[a, b]\}.$$

For example in physics, E could represent the total energy, and f could mean the temperature distribution, etc. Suppose we have already proved that E is bounded from below (say by 0) so that $\inf\{E(f) : f \in C[a, b]\}$ exists. We may need Arzelà-Ascoli to justify that the infimum can be achieved by a certain function $f_0 \in C[a, b]$. In order to do so, one typical way is to let $\{f_n\}$ be a sequence in $C[a, b]$ such that

$$E(f_n) \to \inf\{E(f) : f \in C[a, b]\} \quad \text{as } n \to \infty.$$

This follows from the standard $[\inf, \inf + \frac{1}{n})$-argument). If one can justify that f_n has a uniformly convergent subsequence $\{f_{n_j}\}$ (e.g. using Arzelà-Ascoli) and that

$$\lim_{j\to\infty} E(f_{n_j}) = E\left(\lim_{j\to\infty} f_{n_j}\right),$$

then $\inf\{E(f) : f \in C[a, b]\} = E(f_0)$ where $f_0 := \lim\limits_{j\to\infty} f_{n_j}$. This will show a minimizer of E exists, and then we can try to derive its Euler-Lagrange's equation to solve for f_0.

We do not plan to dive into specific example, because usually showing $E(f_{n_j}) \to E(f_0)$ may not be easy as it may require some knowledge from PDE and functional analysis.

In the next subsection, we will give one concrete example of the use of Arzelà-Ascoli's Theorem, proving that solutions to the ODE:

$$x'(t) = F\big(x(t)\big), \quad x(0) = x_0$$

always exist on a short interval $t \in [0, \varepsilon]$ provided that F is continuous.

4.2.3. Peano's Existence Theorem of ODE. As mentioned in the previous subsection, we are going to use the Arzelà-Ascoli to prove an existence theorem of ODE.

Theorem 4.29 (Peano's Existence Theorem). *Let Ω be a non-empty open set in $\mathbb{R}$ and $I = [-T, T]$ be a closed and bounded time interval. Suppose $F(x,t) : \Omega \times I \to \mathbb{R}$ is a continuous vector field on $\Omega \times I$. Then for any $x_0 \in \Omega$, the initial-value problem (IVP)*

$$x' = F(x,t), \quad x(0) = x_0$$

has a solution $x(t)$ defined on $t \in [-\varepsilon, \varepsilon]$ for some small $\varepsilon > 0$.

Proof. We will only prove that there is a solution defined on $t \in [0, \varepsilon]$ for some $\varepsilon > 0$, since to show there is a solution on $[-\varepsilon, 0]$ is similar.

Let $B_r(x_0)$ be an open ball in Ω such that $\overline{B_r(x_0)} \subset \Omega$. By continuity of the vector field F, there exists $M > 0$ such that $|F(x,t)| \leq M$ for any $(x,t) \in B_r(x_0) \times I$. Let $\varepsilon < \min\{\frac{r}{M}, T\} > 0$. We define a sequence of functions $x_n(t) : [0, \varepsilon] \to \mathbb{R}^d$ in the following way:

Denote $J = [0, \varepsilon]$. For each n, divide J into n-subintervals (each has width $\frac{\varepsilon}{n}$):

$$\begin{aligned} J_n^1 &= [0,\ \varepsilon/n] \\ J_n^2 &= [\varepsilon/n,\ 2(\varepsilon/n)] \\ &\ \ \vdots \\ J_n^n &= [(n-1)(\varepsilon/n),\ \varepsilon], \end{aligned}$$

or in short, $J_n^k = [(k-1)(\varepsilon/n),\ k(\varepsilon/n)]$ for each $1 \leq k \leq n$.

Unlike the Picard's iteration sequence whose n-th term is defined by the previous $(n-1)$-th term, we define $x_n(t)$ on each subinterval J_n^k successively by its previous values on J_n^{k-1}. We first define:

$$x_n(t) := x_0 \quad \text{for } t \in J_n^1.$$

Then on the next subinterval J_n^2, we define:

$$x_n(t) := x_0 + \int_0^{t-\frac{\varepsilon}{n}} F(\underbrace{x_n(s)}_{\text{not a typo!}}, s)ds \quad \text{for } t \in J_n^2.$$

When $t \in J_n^2$ which is an interval of width $\frac{\varepsilon}{n}$, then $t - \frac{\varepsilon}{n}$ is in the previous subinterval J_n^1. Therefore, the integral

$$\int_0^{t-\frac{\varepsilon}{n}} F(x_n(s),\ s)ds$$

is well-defined since $s \in [0,\ t - \frac{\varepsilon}{n}] \subset J_n^1$ on which we have already defined x_n.

Now that we have already defined $x_n(t)$ for $t \in J_n^1 \cup J_n^2$, next we move on to J_n^3 in a same fashion. Define:

$$x_n(t) := x_0 + \int_0^{t-\frac{\varepsilon}{n}} F(x_n(s),\ s)ds \quad \text{for } t \in J_n^3.$$

It is again well-defined since $t \in J_n^3$ implies $t - \frac{\varepsilon}{n} \in J_n^2$, and $x_n(s)$ is already defined for $s \in J_n^1 \cup J_n^2$.

By successive definition of $x_n(t)$ on each J_n^k via the relation:

$$x_n(t) = x_0 + \int_0^{t-\frac{\varepsilon}{n}} F(x_n(s),\ s)ds \quad \text{for } t \in J_n^2 \cup \ldots \cup J_n^n, \tag{4.1}$$

we get a sequence of functions $\{x_n(t)\}_{n=1}^{\infty}$ defined on $J := [0, \varepsilon]$.

Next we show that $x_n(t) \in B_r(x_0)$ for $t \in J$. Obviously, this is true when $t \in J_n^1$. If this is true for $t \in J_n^1 \cup \ldots \cup J_n^{k-1}$, then for $t \in J_n^k$, the relation (4.1) implies:

$$|x_n(t) - x_0| \leq \int_0^{t-\frac{\varepsilon}{n}} |F(\underbrace{x_n(s)}_{\in B_r(x_0)}, s)|ds \leq M\left|t - \frac{\varepsilon}{n}\right| \leq M \cdot \varepsilon < M \cdot \frac{r}{M} = r.$$

By induction, we have $x_n(t) \in B_r(x_0)$ for any $t \in J$. In particular, the sequence of functions $x_n(t)$ is uniformly bounded on J since by triangle inequality, we have:

$$|x_n(t)| = |x_n(t) - x_0 + x_0| \leq |x_n(t) - x_0| + |x_0| \leq \underbrace{r + |x_0|}_{\text{finite constant}}$$

for any $t \in J$ and $n \in \mathbb{N}$. It verifies the first condition of the Arzelà-Ascoli's Theorem.

Next we argue that the sequence $x_n(t)$ is equicontinuous on J:

We first show that for any $t, s \in J$, we have $|x_n(t) - x_n(s)| \leq M|t - s|$, then the equicontinuity will follow easily.

When $t, s \in J_n^2 \cup \ldots \cup J_n^n$, by (4.1), we have (without loss of generality assuming $s < t$):

$$\begin{aligned}
|x_n(t) - x_n(s)| &= \left|\int_0^{t-\frac{\varepsilon}{n}} F(x_n(\tau), \tau)d\tau - \int_0^{s-\frac{\varepsilon}{n}} F(x_n(\tau), \tau)d\tau\right| \\
&= \left|\int_{s-\frac{\varepsilon}{n}}^{t-\frac{\varepsilon}{n}} F(x_n(\tau), \tau)d\tau\right| \\
&\leq \int_{s-\frac{\varepsilon}{n}}^{t-\frac{\varepsilon}{n}} |F(x_n(\tau), \tau)|\, d\tau \\
&\leq M\left|\left(t - \frac{\varepsilon}{n}\right) - \left(s - \frac{\varepsilon}{n}\right)\right| = M|t - s|.
\end{aligned}$$

We leave it as an exercise for readers to verify this is also true if at least one of t, s is in J_n^1.

For any $\varepsilon' > 0$ (we use ε' here because ε was already used to denote the width of the interval), let $\delta < \frac{\varepsilon'}{M}$, then given any $t, s \in J$ and $|t - s| < \delta$, we have:

$$|x_n(t) - x_n(s)| \leq M|t - s| \leq M\delta < \varepsilon'.$$

Therefore, the sequence $x_n(t)$ is equicontinuous on J.

Now the sequence $x_n(t)$ is both uniformly bounded and equicontinuous on J. By Arzelà-Ascoli's Theorem, there exists a subsequence $x_{n_i}(t)$ which converges uniformly on J to a limit function $x_\infty(t)$ as $n_i \to \infty$.

Finally, we are going to show that this $x_\infty(t)$ is a solution to the given IVP. Clearly, $x_n(0) = x_0$ for all $n \in \mathbb{N}$, and so $x_\infty(0) = x_0$, which verifies the initial condition. For any $t > 0$, we and choose n_i sufficiently large so that $t \notin J_{n_i}^1$ which has width $\frac{\varepsilon}{n_i} \to 0$ as $n_i \to \infty$. By (4.1), we have:

$$\begin{aligned}
x_{n_i}(t) &= x_0 + \int_0^{t-\frac{\varepsilon}{n_i}} F(x_{n_i}(s), s)ds \\
&= x_0 + \int_0^t F(x_{n_i}(s), s)ds - \underbrace{\int_{t-\frac{\varepsilon}{n_i}}^t F(x_{n_i}(s), s)ds}_{\leq M\cdot\frac{\varepsilon}{n_i} \to 0 \text{ as } n_i \to \infty}.
\end{aligned}$$

As $n_i \to \infty$, we have:

$$x_\infty(t) = x_0 + \lim_{n_i \to \infty} \int_0^t F(x_{n_i}(s),\, s)ds.$$

Since F is continuous on the compact set $\overline{B_r(x_0)} \times J$, it is also uniformly continuous on $\overline{B_r(x_0)} \times J$. For any $\varepsilon' > 0$, there exists $\delta > 0$ depending only on ε', such that whenever $y, z \in B_r(x_0)$ and $|y - z| < \delta$, we have

$$|F(y,t) - F(z,t)| < \varepsilon'$$

for any $t \in J$.

As x_{n_i} converges uniformly on J to x_∞ as $n_i \to \infty$, there exists $N > 0$ such that whenever $n_i > N$, we have $\|x_{n_i} - x_\infty\|_J < \delta$, which implies

$$|F(x_{n_i}(t)) - F(x_\infty(t))| < \varepsilon'$$

for any $t \in J$. Therefore, $F(x_{n_i}, \cdot)$ converges uniformly on J to $F(x_\infty, \cdot)$, thus allow switching between limit and integral signs, i.e.

$$\lim_{n_i \to \infty} \int_0^t F(x_{n_i}(s),\, s)ds = \int_0^t \lim_{n_i \to \infty} F(x_{n_i}(s),\, s)ds = \int_0^t F(x_\infty(s),\, s)ds.$$

Therefore, the continuous function $x_\infty(t)$ on J satisfies the integral equation:

$$x_\infty(t) = x_0 + \int_0^t F(x_\infty(s),\, s)ds$$

and so it is a solution to the given IVP. □

Remark 4.30. If F is further assumed to be Lipschitz continuous, then one can also prove that the IVP has a *unique* solution. Also, with the Lipschitz continuous, the existence of solution can also be proved in a much simpler way using Banach contraction mapping theorem. Theorem 4.29 is proved in a much complicated way, but it only requires F to be continuous.

Exercise 4.25. Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous function. Fix $x_0 \in \mathbb{R}$ and $r > 0$. We denote $M := \sup_{x \in \overline{B_r(x_0)}} |F(x)|$, and let $\delta := \frac{r}{M+1}$.

For each $n \in \mathbb{N}$, we define successively on each subinterval

$$[0, \frac{\delta}{n}], [\frac{\delta}{n}, \frac{2\delta}{n}], \cdots, [\frac{(n-1)\delta}{n}, \delta]$$

the following sequence of functions $x_n : [0, \delta] \to \mathbb{R}$

$$x_n(t) = \begin{cases} x_0 + tF(x_0) & \text{if } t \in [0, \frac{\delta}{n}] \\ x_n(\frac{j\delta}{n}) + (t - \frac{j\delta}{n})F\big(x_n(\frac{j\delta}{n})\big) & \text{if } t \in (\frac{j\delta}{n}, \frac{(j+1)\delta}{n}], \text{ where } j = 1, 2, \cdots, n-1 \end{cases}$$

(a) Define a sequence of step functions $\{y_n(t)\}_{n=1}^\infty$ on $[0, \delta]$ by:

$$y_n(t) := \begin{cases} x_0 & \text{if } t \in [0, \frac{\delta}{n}] \\ x_n(\frac{j\delta}{n}) & \text{if } t \in (\frac{j\delta}{n}, \frac{(j+1)\delta}{n}], \text{ where } j = 1, 2, \cdots, n-1. \end{cases}$$

Prove that for any $n \in \mathbb{N}$ and $t \in [0, \delta]$, we have:

$$x_n(t) = x_0 + \int_0^t F\big(y_n(s)\big)\, ds.$$

(b) Hence, or otherwise, prove that $\{x_n(t)\}_{n=1}^\infty$ has a subsequence $\{x_{n_i}(t)\}_{n=1}^\infty$ converging uniformly on $[0, \delta]$ to a limit function $x_\infty(t)$.

4.2.4. Stone-Weierstrass's Theorem. The last special topic about uniform convergence we are going to discuss is an approximation theorem of continuous functions by polynomials: given any $f \in C[0,1]$, one can find a sequence of polynomials P_n such that P_n converges uniformly f on $[0,1]$. In contrast to so many "existence" type of results proved before, such a sequence P_n can be explicitly written down. Namely,

$$P_n(x) := \sum_{k=0}^{n} f(k/n) C_k^n x^k (1-x)^{n-k}.$$

Theorem 4.31 (Stone-Weierstrass). *For any $f \in C[0,1]$, the sequence of polynomials P_n defined above converges uniformly to f on $[0,1]$. In other words, the set of polynomials is dense in $C[0,1]$ under the sup-norm.*

Proof. One good observation we are going to use is the fact that:

$$\sum_{k=0}^{n} C_k^n x^k (1-x)^{n-k} = \big(x + (1-x)\big)^n = 1$$

using the binomial theorem.

Using this fact, we estimate that for any $x \in [0,1]$:

$$\begin{aligned}
&|P_n(x) - f(x)| \\
&= \left| \sum_{k=0}^{n} f(k/n) C_k^n x^k (1-x)^{n-k} - f(x) \sum_{k=0}^{n} C_k^n x^k (1-x)^{n-k} \right| \\
&= \left| \sum_{k=0}^{n} \left(f(k/n) - f(x) \right) C_k^n x^k (1-x)^{n-k} \right| \\
&\leq \sum_{k=0}^{n} |f(k/n) - f(x)| \, C_k^n x^k (1-x)^{n-k}
\end{aligned}$$

By continuity (hence uniform continuity) of f on $[0,1]$, the factor $|f(k/n) - f(x)|$ is small when $|k/n - x|$ is small. Given any $\varepsilon > 0$, there exists $\delta > 0$ depending only on ε such that whenever $|x-y| < \delta$ and $x, y \in [0,1]$, we have $|f(x) - f(y)| < \varepsilon$. Inspired by this, we split the above sum into two parts:

$$\begin{aligned}
&|P_n(x) - f(x)| \\
&\leq \sum_{k=0}^{n} |f(k/n) - f(x)| \, C_k^n x^k (1-x)^{n-k} \\
&= \sum_{\{k : 0 \leq k \leq n, \left|\frac{k}{n} - x\right| < \delta\}} |f(k/n) - f(x)| \, C_k^n x^k (1-x)^{n-k} \\
&\qquad + \sum_{\{k : 0 \leq k \leq n, \left|\frac{k}{n} - x\right| \geq \delta\}} |f(k/n) - f(x)| \, C_k^n x^k (1-x)^{n-k} \\
&\leq \sum_{\{k : 0 \leq k \leq n, \left|\frac{k}{n} - x\right| < \delta\}} \varepsilon C_k^n x^k (1-x)^{n-k} \\
&\qquad + \sum_{\{k : 0 \leq k \leq n, \left|\frac{k}{n} - x\right| \geq \delta\}} 2 \|f\|_{[0,1]} \, C_k^n x^k (1-x)^{n-k}
\end{aligned}$$

For the first sum, we observe that

$$\sum_{\{k:0\le k\le n,\left|\frac{k}{n}-x\right|<\delta\}} \varepsilon C_k^n x^k(1-x)^{n-k} \le \sum_{k=0}^{n} \varepsilon C_k^n x^k(1-x)^{n-k} = \varepsilon.$$

The second sum is a bit trickier. Using binomial theorem only would only bound it above by $2\|f\|_{[0,1]}$ which is not arbitrarily small. However, the condition $\left|\frac{k}{n}-x\right| \ge \delta$, which is equivalent to $\frac{1}{\delta^2}\left(\frac{k}{n}-x\right)^2 \ge 1$, is helpful:

$$\begin{aligned}
&\sum_{\{k:0\le k\le n,\left|\frac{k}{n}-x\right|\ge\delta\}} C_k^n x^k(1-x)^{n-k} \\
&\le \sum_{\{k:0\le k\le n,\left|\frac{k}{n}-x\right|\ge\delta\}} \frac{1}{\delta^2}\left(\frac{k}{n}-x\right)^2 C_k^n x^k(1-x)^{n-k} \\
&\le \sum_{k=0}^{n} \frac{1}{\delta^2}\left(\frac{k}{n}-x\right)^2 C_k^n x^k(1-x)^{n-k} \\
&= \frac{1}{\delta^2}\frac{x-x^2}{n}.
\end{aligned}$$

We leave the last step (high school stuff) as an exercise for readers.

Combining all results proven, we get that for $x \in [0,1]$

$$|P_n(x)-f(x)| \le \varepsilon + \frac{2\|f\|_{[0,1]}}{\delta^2}\frac{x-x^2}{n} \le \varepsilon + \frac{2\|f\|_{[0,1]}}{\delta^2 n}.$$

Therefore, whenever $n > \frac{2\|f\|_{[0,1]}}{\varepsilon\delta^2}$, we have

$$\|P_n - f\|_{[0,1]} \le 2\varepsilon.$$

Hence P_n converges uniformly to f on $[0,1]$. □

Exercise 4.26. Now given $f \in C[a,b]$. By modifying P_n in Theorem 4.31, find a sequence of polynomials which uniformly converges to f on $[a,b]$.

Exercise 4.27. Let $f \in C[0,1]$ be a function such that

$$\int_0^1 f(x)x^n\,dx = 0$$

for any $n \in \mathbb{N}\cup\{0\}$. Prove that $f(x)=0$ for any $x\in[0,1]$.

Chapter 5

Riemann Integrals

> "Does anyone believe that the difference between the Lebesgue and Riemann integrals can have physical significance, and that whether say, an airplane would or would not fly could depend on this difference? If such were claimed, I should not care to fly in that plane."
>
> Richard Hamming

This chapter formally introduces the definition of Riemann integrals and discusses some criterion under which a function is Riemann integrable. For the definition, we will mainly present Darboux's version which is easier to work with. Riemann's original definition (which turns out to be equivalent to Darbous'x one) will also be mentioned. In the last part of the course (MATH 2043), we will present a very neat theorem due to Lebesgue, which characterizes Riemann integrable functions using Lebesgue measures (which is a the main dish for MATH 3043).

5.1. Definition of Riemann Integral: Darboux's approach

In this chapter, we will mainly discuss functions $f : [a, b] \to \mathbb{R}$ whose domain $[a, b]$ is a closed and bounded interval, and the function is bounded on $[a, b]$. We know that the definite integral

$$\int_a^b f(x)\, dx$$

represents the "area" under the graph $y = f(x)$ over $[a, b]$, but then what is the definition of "area"? In this chapter we aim to give a rigorous meaning of definite integrals.

A **partition**, say denoted by P, of the interval $[a, b]$ is a **finite** set of points $[a, b]$:

$$P : a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Then P divides the interval $[a, b]$ into n sub-intervals

$$[a, x_1] \cup [x_1, x_2] \cup [x_2, x_3] \cup \cdots \cup [x_{n-1}, x_n].$$

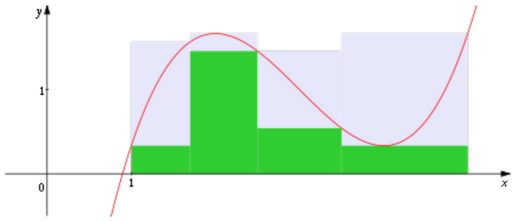


Figure 5.1. Darboux sums

Note that here we do **not** require the above sub-intervals to have equal length – we will see later that it allows more flexibility.

Definition 5.1 (Darboux's sums). Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and $P = \{x_i\}_{i=0}^n$ is a partition of $[a, b]$, then we denote

$$U(f, P) := \sum_{i=1}^{n} \sup_{x \in [x_{i-1}, x_i]} f(x) \cdot (x_i - x_{i-1})$$

$$L(f, P) := \sum_{i=1}^{n} \inf_{x \in [x_{i-1}, x_i]} f(x) \cdot (x_i - x_{i-1})$$

They are called **Darboux's upper and lower sums** respectively.

Remark 5.2. The geometric meaning of $U(f, P)$ is the total area of the *smallest* "histogram", *containing* the graph $y = f(x)$, and with sub-divisions given by P, whereas the meaning of $L(f, P)$ is the the total area of the *largest* histogram, *contained in* the graph $y = f(x)$, and with sub-divisions given by P. See Figure 5.1 for example: the purple histogram has area given by $U(f, P)$, and the green histogram has area $L(f, P)$.

Now we are ready to state the definition of Riemann integrals.

Definition 5.3 (Riemann integral). Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Suppose there exists a sequence of partitions $\{P_k\}_{k=1}^\infty$ of $[a, b]$ such that the following limits exist and

$$\lim_{k\to\infty} U(f, P_k) = \lim_{k\to\infty} L(f, P_k),$$

then we say f is **Riemann integrable on** $[a, b]$, and we define

$$\int_a^b f(x)\, dx := \lim_{k\to\infty} U(f, P_k)$$

and call it the **Riemann integral** of f over $[a, b]$.

One fact needs to be checked in order to justify the above *definition* of Riemann integral. What if we can find two sequences of partitions $\{P_k\}_{k=1}^\infty$ and $\{Q_k\}_{k=1}^\infty$ of $[a, b]$ such that

$$\lim_{k\to\infty} U(f, P_k) = \lim_{k\to\infty} L(f, P_k), \qquad \text{and} \qquad \lim_{k\to\infty} U(f, Q_k) = \lim_{k\to\infty} L(f, Q_k),$$

then whether we must have

$$\lim_{k\to\infty} U(f, P_k) = \lim_{k\to\infty} U(f, Q_k).$$

If the limit was not equal, then there would be ambiguity when using it to define the value of $\displaystyle\int_a^b f(x)\, dx$.

To justify this, we first observe one fact that $U(f,\cdot)$ and $L(f,\cdot)$ are monotone when one *refines* the partition. For any two partitions P and Q of $[a,b]$, we say Q is a **refinement** of P, denoted by $P \subset Q$. In other words, Q is obtained by adding finitely many points to P. It can be easily checked that if $P \subset Q$, then
$$L(f,P) \le L(f,Q) \qquad \text{and} \qquad U(f,Q) \le U(f,P).$$

Exercise 5.1. Prove the above inequalities.

Since $P_k \subset P_k \cup Q_k$ and $Q_k \subset P_k \cup Q_k$, we have
$$L(f,P_k) \le L(f,P_k \cup Q_k) \le U(f,P_k \cup Q_k) \le U(f,Q_k).$$
Letting $k \to \infty$, we have
$$\lim_{k\to\infty} L(f,P_k) \le \lim_{k\to\infty} U(f,Q_k).$$
Combining with $\lim\limits_{k\to\infty} U(f,P_k) = \lim\limits_{k\to\infty} L(f,P_k)$, we conclude that
$$\lim_{k\to\infty} U(f,P_k) \le \lim_{k\to\infty} U(f,Q_k).$$
By swapping P_k and Q_k in all the above arguments, we have
$$\lim_{k\to\infty} U(f,Q_k) \le \lim_{k\to\infty} U(f,P_k),$$
proving that they are actually equal.

Remark 5.4. Historically, one should call the above **Darboux integrable/integral** instead of **Riemann integrable/integral**, as this version of definition is due to Darboux in 1875, not due to Riemann. The original definition due to Riemann is more complicated than that, but can be shown to be equivalent to Darboux's one. We will still use the term Riemann integrable/integral, with abuse of its historical background.

Example 5.5. Consider $f(x) = x^2 : [0,1] \to \mathbb{R}$. Consider the uniform partitions
$$P_n : x_0 := 0 < \underbrace{\frac{1}{n}}_{x_1} < \underbrace{\frac{2}{n}}_{x_2} < \cdots < \underbrace{\frac{n-1}{n}}_{x_{n-1}} < 1 =: x_n.$$
Then, because f is increasing, so we have
$$\sup_{x\in[\frac{i-1}{n},\frac{i}{n}]} x^2 = \frac{i^2}{n^2},$$
$$\inf_{x\in[\frac{i-1}{n},\frac{i}{n}]} x^2 = \frac{(i-1)^2}{n^2}.$$
Hence,
$$U(f,P_n) = \sum_{i=1}^{n} \frac{i^2}{n^2}\cdot\frac{1}{n} = \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6},$$
$$L(f,P_n) = \sum_{i=0}^{n-1} \frac{i^2}{n^3} = \frac{1}{n^3}\cdot\frac{(n-1)n(2n-1)}{6}.$$
It is easy to see that
$$\lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n) = \frac{1}{3}.$$
Therefore, $f(x) = x^2$ is Riemann integrable on $[0,1]$, and
$$\int_0^1 f(x)\,dx = \frac{1}{3}.$$

Exercise 5.2. Show that the following functions are Riemann integrable on the specified interval:

(a) $f(x) = \sin x$ on $[0, \pi]$

(b) $f(x) = e^x$ on $[a, b]$

While the integrals in the above examples and exercises can be found explicitly, it is often (for mathematicians) that we only care about whether or not the function is integrable. Below are several equivalent criteria that can be used to check whether or not a function is integrable, although they cannot give us the exact value of the integral.

Proposition 5.6. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then, the following are equivalent:*

(i) f is Riemann integrable on $[a, b]$ (in the sense of Definition 5.3).

(ii) There exists a sequence of partitions $\{P_k\}_{k=1}^{\infty}$ of $[a, b]$ such that

$$\lim_{k\to\infty} (U(f, P_k) - L(f, P_k)) = 0.$$

(iii) For any $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

(iv) $\sup_P L(f, P) = \inf_P U(f, P)$, where $\sup$ and $\inf$ are taken over all partitions P of $[a, b]$.

Proof. (i) $\implies$ (ii) is trivial. Also, (ii) $\implies$ (iii) simply follows from the ε, N-definition of limit of sequence. Next, (iii) $\implies$ (iv) follows easily from

$$0 \leq \inf_P U(f, P) - \sup_P L(f, P) \leq U(f, P) - L(f, P)$$

for any partition P of $[a, b]$. If (iii) holds, it would imply

$$0 \leq \inf_P U(f, P) - \sup_P L(f, P) < \varepsilon$$

for any $\varepsilon > 0$. Letting $\varepsilon \to 0^+$, we get (iv).

Now we prove (iv) $\implies$ (i): First note that since f is bounded, both $\sup_P L(f, P)$ and $\inf_P U(f, P)$ both exist in $\mathbb{R}$, so there exist sequences of partitions $\{P_k\}$ and $\{Q_k\}$ of $[a, b]$ such that

$$\lim_{k\to\infty} L(f, P_k) = \sup_P L(f, P) = \inf_P U(f, P) = \lim_{k\to\infty} U(f, Q_k).$$

Note that to prove (i) we need to have the **same** sequence of partitions $\{R_k\}$ such that $L(f, R_k)$ and $U(f, R_k)$ converging to the same limit. To achieve this, we let $R_k := P_k \cup Q_k$ which is a refinement of both P_k and Q_k, then by monotonicity of refinement, we have

$$L(f, P_k) \leq L(f, R_k) \leq \sup_P L(f, P) = \inf_P U(f, P) \leq U(f, R_k) \leq U(f, Q_k).$$

Letting $k \to \infty$, we can see easily that

$$\lim_{k\to\infty} L(f, R_k) = \sup_P L(f, P) = \inf_P U(f, P) = \lim_{k\to\infty} U(f, R_k),$$

proving (i). □

Remark 5.7. We often denote

$$\underline{\int_a^b} f(x)\, dx := \sup_P L(f, P) \qquad \overline{\int_a^b} f(x)\, dx := \inf_P U(f, P).$$

They represent the "best" Darboux lower sum and the "best" Darboux upper sum respectively. They are called **Darboux lower and upper integrals** of f respectively. From the proof of (iv) $\implies$ (i), we can also see that if f is Riemann integrable on $[a,b]$, then

$$\underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

Example 5.8. Consider the function $f(x) = -x\log x : [0,1] \to \mathbb{R}$. As $f(x) \to 0$ as $x \to 0^+$, we can treat $x\log x$ as 0 when $x = 0$. It can be easily shown that f achieves its maximum at $x = 1/e$, is increasing on $[0, 1/e]$, and decreasing on $[1/e, 1]$.

Therefore, we will consider partitions of $[0,1]$, it may not be good idea to use uniform partitions (meaning that sub-intervals have equal length), because it is not straightforward to find out the $\sup$ and $\inf$ of f over the sub-interval containing $1/e$. Therefore, we instead consider the following sequence of partitions

$$P_k : 0 < \frac{1}{k}\cdot\frac{1}{e} < \frac{2}{k}\cdot\frac{1}{e} < \cdots < \frac{1}{e} < \frac{1}{e} + \frac{1}{k}\left(1-\frac{1}{e}\right) < \frac{1}{e} + \frac{2}{k}\left(1-\frac{1}{e}\right) < \cdots < \left(1-\frac{1}{e}\right).$$

Each P_k is uniform on $[1, 1/e]$ and $[1/e, 1]$, but not overall uniform on $[0,1]$. However, finding the Darboux sums with respect to this partition is easy:

$$\begin{aligned}
U(f,P_k) &= \sum_{i=1}^{k} f\left(\frac{i}{ke}\right)\frac{1}{ke} + \sum_{i=0}^{k-1} f\left(\frac{1}{e} + \frac{i}{k}\left(1-\frac{1}{e}\right)\right)\cdot\frac{1}{k}\left(1-\frac{1}{e}\right)\\
L(f,P_k) &= \sum_{i=1}^{k} f\left(\frac{i-1}{ke}\right)\frac{1}{ke} + \sum_{i=0}^{k-1} f\left(\frac{1}{e} + \frac{i+1}{k}\left(1-\frac{1}{e}\right)\right)\cdot\frac{1}{k}\left(1-\frac{1}{e}\right)\\
&= \sum_{i=0}^{k-1} f\left(\frac{i}{ke}\right)\frac{1}{ke} + \sum_{i=1}^{k} f\left(\frac{1}{e} + \frac{i}{k}\left(1-\frac{1}{e}\right)\right)\cdot\frac{1}{k}\left(1-\frac{1}{e}\right)
\end{aligned}$$

After cancellation of common terms, we get

$$\begin{aligned}
&U(f,P_k) - L(f,P_k)\\
&= \underbrace{f\left(\frac{1}{e}\right)\frac{1}{ke}}_{i=k} - \underbrace{f(0)\frac{1}{ke}}_{i=0} + \underbrace{f\left(\frac{1}{e}\right)\cdot\frac{1}{k}\left(1-\frac{1}{e}\right)}_{i=0} - \underbrace{f\left(\frac{1}{e}+1-\frac{1}{e}\right)\cdot\frac{1}{k}\left(1-\frac{1}{e}\right)}_{i=k}\\
&= \left(\frac{1}{e^2} + \frac{1}{e}\left(1-\frac{1}{e}\right)\right)\cdot\frac{1}{k}
\end{aligned}$$

which tends to 0 as $k \to \infty$. By (ii) of Proposition 5.6, f is Riemann integrable on $[0,1]$.

Example 5.9. Dirichlet function, defined as:

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not Riemann integrable on any closed and bounded interval $[a,b]$, where $a < b$. To see this, we use criterion (iv) in Proposition 5.6.

For any partition $P = \{x_i\}_{i=0}^n$ of $[a,b]$ (we may assume $x_{i-1} < x_i$ after removing redundancy), each sub-interval $[x_{i-1}, x_i]$ must contain at least one rational number, and one irrational number. Therefore, one must have

$$\begin{aligned}
\sup_{[x_{i-1},x_i]} \chi_{\mathbb{Q}} = 1 \implies &\qquad U(\chi_{\mathbb{Q}}, P) = \sum_{i=1}^{n} 1\cdot(x_i - x_{i-1}) = b-a\\
\inf_{[x_{i-1},x_i]} \chi_{\mathbb{Q}} = 0 \implies &\qquad L(\chi_{\mathbb{Q}}, P) = 0.
\end{aligned}$$

Since the above hold for all partitions P of $[a, b]$, the Darboux upper and lower integrals equal to

$$\overline{\int_a^b} \chi_{\mathbb{Q}}(x)\, dx = \inf_P U(\chi_{\mathbb{Q}}, P) = b - a$$
$$\underline{\int_a^b} \chi_{\mathbb{Q}}(x)\, dx = \sup_P L(\chi_{\mathbb{Q}}, P) = 0$$

By (iv) of Proposition 5.6, $\chi_{\mathbb{Q}}$ is not Riemann integrable on $[a, b]$.

Example 5.10. Any continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on a closed and bounded interval $[a, b]$. By compactness, such a continuous function is also uniformly continuous on $[a, b]$. Given any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $x, y \in [a, b]$ and $|x - y| < \delta$, we have $|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)}$. It implies that if an interval $[x_{i-1}, x_i]$ has length $< \delta$, then

$$\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f \leq \frac{\varepsilon}{2(b-a)}.$$

Therefore, let $P = \{x_i\}_{i=0}^n$ be any partition (say uniform one) such that $x_i - x_{i-1} < \delta$ for each i, then we have

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^n \left(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f \right) (x_i - x_{i-1}) \\ &\leq \sum_{i=1}^n \frac{\varepsilon}{2(b-a)} \cdot (x_i - x_{i-1}) = \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

By criterion (iii) of Proposition 5.6, f is Riemann integrable on $[a, b]$.

Exercise 5.3. Let $c \in (a, b)$. Show that if $f : [a, b] \to \mathbb{R}$ is a bounded function that is Riemann integrable, and $g : [a, b] \to \mathbb{R}$ is another bounded function such that $f(x) = g(x)$ for any $x \in [a, b] \backslash \{c\}$, then g is also Riemann integrable on $[a, b]$, and we have

$$\int_a^b f(x)\, dx = \int_a^b g(x)\, dx.$$

Hint: For any partition P of $[a, b]$, consider a finer partition $P' := P \cup \{c - \delta, c, c + \delta\}$ where δ is sufficiently small. What can you say about the following?

$$U(f, P') - U(g, P') \qquad \text{and} \qquad L(f, P') - L(g, P').$$

Exercise 5.4. Show that any monotone bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

Exercise 5.5. Suppose $c \in (a, b)$. Show that if a bounded function f is Riemann integrable on $[a, c]$ and on $[c, b]$, then it is Riemann integrable on $[a, b]$.

Exercise 5.6. Show that if $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, then it is Riemann integrable on any $[c, d] \subset [a, b]$.

Below are some properties of Riemann integrals that we have been using. Here we provide their proofs using rigorous definitions of integrals.

Proposition 5.11. *Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions that are Riemann integrable. Then, we have*

(i) $f + g$ *is also Riemann integrable on* $[a, b]$*, and we have*
$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$
(ii) cf *is also Riemann integrable on* $[a, b]$ *for any constant* $c \in \mathbb{R}$*, and we have*
$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$
(iii) If $f(x) \le g(x)$ *for any* $x \in [a, b]$*, then*
$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$
(iv) $|f|$ *is also Riemann integrable on* $[a, b]$*, and the following holds*
$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$
(v) For any $c \in [a, b]$*, we have*
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Proof. We will prove some of them, while outline the others and leave as exercises for readers.

For (i), the key observation is that
$$L(f, P) + L(g, P) \le L(f + g, P) \qquad \text{and} \qquad U(f + g, P) \le U(f, P) + U(g, P)$$
for any partition P of $[a, b]$. They simply follow from $\sup(f + g) \le \sup f + \sup g$ and $\inf f + \inf g \le \inf(f + g)$. First we let $\{P_k\}$ and $\{Q_k\}$ be sequences of partitions of $[a, b]$ such that
$$\begin{aligned}\lim_{k\to\infty} U(f, P_k) &= \int_a^b f(x)\,dx = \lim_{k\to\infty} L(f, P_k)\\ \lim_{k\to\infty} U(g, Q_k) &= \int_a^b g(x)\,dx = \lim_{k\to\infty} L(g, Q_k).\end{aligned}$$
Then, we consider
$$\begin{aligned}&L(f, P_k) + L(g, Q_k)\\ &\le L(f, P_k \cup Q_k) + L(g, P_k \cup Q_k)\\ &\le L(f + g, P_k \cup Q_k) \le U(f + g, P_k \cup Q_k)\\ &\le U(f, P_k \cup Q_k) + U(g, P_k \cup Q_k) \le U(f, P_k) + U(g, Q_k).\end{aligned}$$
Letting $k \to \infty$, we conclude by squeeze theorem that
$$\lim_{k\to\infty} L(f + g, P_k \cup Q_k) = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \lim_{k\to\infty} U(f + g, P_k \cup Q_k).$$
This proves (i).

(ii) is easy so it is left as an exercise for reader.

(iii) follows easily from $L(f, P) \le L(g, P)$ for any partition P.

To prove (iv): for any interval I, let $x_n \in I$ and $y_n \in I$ be sequences such that $|f(x_n)| \to \sup_I |f|$ and $|f(y_n)| \to \inf_I |f|$. Then by the inequality

$$\big| |f(x_n)| - |f(y_n)| \big| \leq |f(x_n) - f(y_n)| \leq \sup_{x \in I} f(x) - \inf_{y \in I} f(y).$$

Letting $n \to \infty$, we get

$$\sup_{x \in I} |f(x)| - \inf_{y \in I} |f(y)| \leq \sup_{x \in I} f(x) - \inf_{y \in I} f(y).$$

This would imply

$$U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P)$$

for any partition P. This shows $|f|$ is also Riemann integrable by (iii) of Proposition 5.6. By $-|f| \leq f \leq |f|$, we conclude (iv) using the result from (iii) of this proposition.

For (v): for any partition P of $[a, b]$, we let $P' := P \cup \{c\}$. Since $c \in P'$, it is easy to see that

$$\begin{aligned} U(f, P' \cap [a, c]) + U(f, P' \cap [c, b]) &= U(f, P') \\ L(f, P' \cap [a, c]) + L(f, P' \cap [c, b]) &= L(f, P'). \end{aligned}$$

This implies

$$\begin{aligned} L(f, P) &\leq L(f, P') \\ &= L(f, P' \cap [a, c]) + L(f, P' \cap [c, b]) \\ &\leq \int_a^c f(x)\, dx + \int_c^b f(x)\, dx. \end{aligned}$$

Taking $\sup$ over all P, we get

$$\int_a^b f(x)\, dx = \sup_P L(f, P) \leq \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

Similarly, we have

$$\begin{aligned} &\int_a^c f(x)\, dx + \int_c^b f(x)\, dx \\ &\leq U(f, P' \cap [a, c]) + U(f, P' \cap [c, b]) \\ &= U(f, P') \leq U(f, P). \end{aligned}$$

Taking $\inf$ over all P, we get

$$\int_a^c f(x)\, dx + \int_c^b f(x)\, dx \leq \inf_P U(f, P) = \int_a^b f(x)\, dx.$$

This proves (v). □

Exercise 5.7. Prove that if $f : [a, b] \to \mathbb{R}$ is a bounded function that is Riemann integrable, then

$$F(x) := \int_a^x f(t)\, dt$$

is uniformly continuous on $[a, b]$. Furthermore, if f is continuous on $[a, b]$, then F is differentiable on $[a, b]$ with $F' = f$.

Exercise 5.8. Show that if $f : [a, b] \to \mathbb{R}$ is a bounded function that is Riemann integrable, and $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous function, then $\varphi \circ f$ is Riemann integrable on $[a, b]$ too. [Hint: make good use of the fact that f is bounded.]

5.2. Lebesgue's Theorem on Riemann Integrability

As the last chapter of MATH 2043, we conclude the course by a topic that has some interplay with the next course – MATH 3043. We discuss one measure-theoretic criterion for Riemann integrability, which turns out to be equivalent to any of the four criteria in Proposition 5.6.

5.2.1. Sets with zero Lebesgue measure. The new measure-theoretic criterion is related to Lebesgue measures, which is a set function $\mathcal{L}^*$ mapping a subset E in $\mathbb{R}$ to a number $\mathcal{L}^*(E)$ in $[0, \infty]$ which presents the "length" of the subset E.

Definition 5.12. We say $E \subset \mathbb{R}$ **zero Lebesgue (outer) measure**, if

$$\mathcal{L}^*(E) := \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) : E \subset \bigcup_{i=1}^{\infty} (a_i, b_i), a_i \leq b_i \, \forall i \right\} = 0.$$

Note that the countable collection of open intervals $\{(a_i, b_i)\}$ does NOT need to be disjoint.

Here we explain the geometric meaning of $\mathcal{L}^*(E)$: we use open intervals to cover E from outside. The infinite sum $\sum_{i=1}^{\infty}(b_i - a_i)$ is (essentially) the total "length" of the cover $\bigcup_{i=1}^{\infty}(a_i, b_i)$. Therefore, each countable open cover $\{(a_i, b_i)\}$ gives an upper estimate of the length of E. Taking infimum, $\mathcal{L}^*(E)$ can be interpreted as the "best" upper estimate of the length E by the total lengths of open intervals.

$\mathcal{L}^*(E)$ is often called the **Lebesgue outer measure of** E. There is also Lebesgue *inner* measure of E, denoted by $\mathcal{L}_*(E)$, which is the "best" lower estimate of the length of E by sets from inside. We will not talk about it in this course, since if the Lebesgue outer measure of E is zero, then so is its Lebesgue inner measure.

In order to show that $\mathcal{L}^*(E) = 0$ of a given set $E \subset \mathbb{R}$, one can prove that for any $\varepsilon > 0$, there exist countably many open intervals $\{(a_i, b_i)\}_{i=1}^{\infty}$ such that

$$E \subset \bigcup_{i=1}^{\infty} (a_i, b_i) \qquad \text{and} \qquad \sum_{i=1}^{\infty} (b_i - a_i) \leq \varepsilon.$$

Then, it would imply for any $\varepsilon > 0$, we have

$$\mathcal{L}^*(E) := \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) : E \subset \bigcup_{i=1}^{\infty} (a_i, b_i) \right\} \leq \varepsilon.$$

Since $\mathcal{L}^*(E) \geq 0$ by definition, letting $\varepsilon \to 0^+$ gives $\mathcal{L}^*(E)$.

Example 5.13. We have $\mathcal{L}^*(\emptyset) = 0$: for any $\varepsilon > 0$, consider the intervals $(a_1, b_1) := (-\varepsilon/2, \varepsilon/2)$, and $(a_2, b_2) = (a_3, b_3) = \cdots = (1, 1) = \emptyset$. Then obviously

$$\emptyset \subset \bigcup_{i=1}^{\infty} (a_i, b_i) = (-\varepsilon/2, \varepsilon/2)$$

and $\sum_{i=1}^{\infty}(b_i - a_i) = \varepsilon$. This shows $0 \leq \mathcal{L}^*(\emptyset) \leq \varepsilon \;\; \forall \varepsilon > 0$. Letting $\varepsilon \to 0^+$, we proved $\mathcal{L}^*(E) = 0$.

Exercise 5.9. Show that any finite set $E \subset \mathbb{R}$ has zero Lebesgue measure.

Lebesgue outer measure $\mathcal{L}^*$ is monotone in a sense that if $D \subset E \subset \mathbb{R}$, then $\mathcal{L}^*(D) \leq \mathcal{L}^*(E)$. To argue this, we observe that any countable open interval cover $\{(a_i, b_i)\}_{i=1}^{\infty}$ of

E is also a countable open interval cover of D because $D \subset E \subset \bigcup_{i=1}^{\infty}(a_i, b_i)$, we have:

$$\left\{\sum_{i=1}^{\infty}(b_i - a_i) : E \subset \bigcup_{i=1}^{\infty}(a_i, b_i)\right\} \subset \left\{\sum_{i=1}^{\infty}(b_i - a_i) : D \subset \bigcup_{i=1}^{\infty}(a_i, b_i)\right\}.$$

Taking infimum on both sides (keep in mind that smaller set has larger infimum), we get

$$\mathcal{L}^*(E) \geq \mathcal{L}^*(D).$$

As a result, if $\mathcal{L}^*(E) = 0$, then all its subsets $D \subset E$ has $\mathcal{L}^*(D) = 0$.

Another very useful properties of Lebesgue outer measure is the *countable subadditivity*, meaning that if $\{E_i\}_{i=1}^{\infty}$ is a countable collection of sets in $\mathbb{R}$, then we have

$$\mathcal{L}^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mathcal{L}^*(E_i). \tag{5.1}$$

Equality does not generally hold because the sets E_i's may have overlap. Moreover, even if E_i's are pairwise disjoint, equality still may not hold. It motivates the definition of Lebesgue *measurable* sets – to be discussed in MATH 3043 – which satisfy the equality case of (5.1) if the sets are pairwise disjoint.

To prove (5.1), we use a common "$\frac{\varepsilon}{2^i}$-trick" that will appear frequently in measure theory. We may assume $\mathcal{L}^*(E_i) < \infty$ for any i, otherwise (5.1) trivially holds. For any $\varepsilon > 0$ and $i \in \mathbb{N}$, there exists a countable collection of open intervals $\{(a_{i,j}, b_{i,j}\}_{j=1}^{\infty}$ such that

$$E_i \subset \bigcup_{j=1}^{\infty}(a_{i,j}, b_{i,j}) \quad \text{and} \quad \mathcal{L}^*(E_i) \leq \sum_{j=1}^{\infty}(b_{i,j} - a_{i,j}) < \mathcal{L}^*(E_i) + \frac{\varepsilon}{2^i}.$$

Then, the double-index collection $\{(a_{i,j}, b_{i,j})\}_{i,j=1}^{\infty}$, still countable, is a cover of $\bigcup_{i=1}^{\infty} E_i$ since

$$\bigcup_{i=1}^{\infty} E_i \subset \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{\infty}(a_{i,j}, b_{i,j}).$$

Therefore, by the infimum definition of $\mathcal{L}^{(}\bigcup_{i=1}^{\infty} E_i)$, we have

$$\mathcal{L}^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(b_{i,j} - a_{i,j}) < \sum_{i=1}^{\infty}\left(\mathcal{L}^*(E_i) + \frac{\varepsilon}{2^i}\right) = \sum_{i=1}^{\infty}\mathcal{L}^*(E_i) + \varepsilon.$$

Letting $\varepsilon \to 0^+$, we proved (5.1).

One immediately corollary of (5.1) is that if $\{E_i\}_{i=1}^{\infty}$ is a countable collection of sets in $\mathbb{R}$ with $\mathcal{L}^*(E_i) = 0$, then $\mathcal{L}^*\left(\bigcup_{i=1}^{\infty} E_i\right) = 0$ as well. Combining this with the result in Exercise 5.9, we conclude that any countable set in $\mathbb{R}$, such as $\mathbb{Q}$, has zero Lebesgue measure.

5.2.2. Riemann integrability and discontinuity set. We conclude the course (MATH 2043) with a very beautiful theorem that relates Riemann integrals and Lebesgue measures.

Theorem 5.14 (Lebesgue). *Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and denote*

$$D_f := \{x_0 \in [a, b] : f \text{ is not continuous at } x_0\}.$$

Then, f is Riemann integrable on $[a, b]$ if and only if D_f has zero Lebesgue measure.

We will later outline the proof of the theorem in an exercise, and let readers to fill in the detail. Here we present the applications of the theorem.

Example 5.15. Any continuous function $f : [a, b] \to \mathbb{R}$ has $D_f = \emptyset$, so $\mathcal{L}^*(D_f) = 0$. By Lebesgue's Theorem, f is Riemann integrable on $[a, b]$.

Example 5.16. To prove that any monotone function $f : [a, b] \to \mathbb{R}$ is Riemann integrable, we first argue that D_f is countable. Assume WLOG that f is increasing on $[a, b]$. Then, for any $x_0 \in D_f$, we must have

$$\lim_{x \to x_0^-} f(x) < \lim_{x \to x_0^+} f(x).$$

By density of rationals, we can pick $q_{x_0} \in \mathbb{Q}$ such that

$$\lim_{x \to x_0^-} f(x) < q_{x_0} < \lim_{x \to x_0^+} f(x).$$

Consider the map $\varphi : D_f \to \mathbb{Q}$ defined as $\varphi(x_0) := q_{x_0}$. This map is injective as f is increasing. This shows D_f is countable, and hence $\mathcal{L}^*(D_f) = 0$. By Lebesgue's Theorem, f is Riemann integrable on $[a, b]$.

Example 5.17. $\chi_{\mathbb{Q}} : [a, b] \to \mathbb{R}$ is not Riemann integrable, as it is discontinuous at all $x_0 \in [a, b]$. Hence, $D_{\chi_{\mathbb{Q}}} = [a, b]$. It can be shown that $\mathcal{L}^*([a, b]) = b - a > 0$ (detail will be discussed in MATH 3043) and so $\chi_{\mathbb{Q}}$ is not Riemann integrable on $[a, b]$.

Example 5.18. With the Lebesgue's Theorem, it is very straight-forward to prove that if $f, g : [a, b] \to \mathbb{R}$ are bounded Riemann integrable functions, then so are $f \pm g$ and fg. To argue this, we recall that if f and g are both continuous at $x_0 \in [a, b]$, then so is $f \pm g$ and fg. This means

$$D_f^c \cap D_g^c \subset D_{f \pm g}^c$$
$$D_f^c \cap D_g^c \subset D_{fg}^c$$

This implies $D_{f \pm g} \subset \left(D_f^c \cap D_g^c\right)^c = D_f \cup D_g$ and $D_{fg} \subset D_f \cup D_g$. Since f and g are Riemann integrable, we have $\mathcal{L}^*(D_f) = \mathcal{L}^*(D_g) = 0$, this shows $\mathcal{L}^*(D_f \cup D_g) = 0$ and hence $\mathcal{L}^*(D_{f \pm g}) = \mathcal{L}^*(D_{fg}) = 0$, proving that $f \pm g$ and fg are Riemann integrable.

Example 5.19. With the Lebesgue's Theorem, Exercise 5.7 can be done in a much easier way. Given that $f : [a, b] \to \mathbb{R}$ is Riemann integrable and $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous, then if f is continuous at $x_0 \in [a, b]$, then $\varphi \circ f$ is also continuous at x_0. This means

$$D_f^c \subset D_{\varphi \circ f}^c \implies D_{\varphi \circ f} \subset D_f.$$

Since $\mathcal{L}^*(D_f) =)$, we have $\mathcal{L}^*(D_{\varphi \circ f}) = 0$ too, proving that $\varphi \circ f$ is Riemann integrable.

Exercise 5.10. Suppose $f_n : [a, b] \to \mathbb{R}$ is a sequence of bounded Riemann integrable functions such that f_n converges uniformly on $[a, b]$ to a bounded function $f : [a, b] \to \mathbb{R}$. Show that f is also Riemann integrable on $[a, b]$.

Exercise 5.11. Let $f, g : [0, 2] \to \mathbb{R}$ be bounded Riemann integrable functions. Prove that $h : [0, 2] \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} \max\{f(x), g(x)\} & \text{if } x \in [0, 1] \\ \min\{f(x), g(x)\} & \text{if } x \in (1, 2] \end{cases}$$

is also Riemann integrable on $[0, 2]$.

Exercise 5.12. Prove that if $E \subset \mathbb{R}$ has zero Lebesgue measure, then so is $E^2 := \{x^2 : x \in E\}$. Hence, prove that if $f : [0, 1] \to [0, 1]$ is Riemann integrable, then $g : [0, 1] \to [0, 1]$ defined by $g(x) := f(\sqrt{x})$ is also Riemann integrable.

The exercise below is an outline of the proof of Theorem 5.14. Try to complete it.

(a) Suppose f is Riemann integrable on $[a,b]$, we want to show $\mathcal{L}^*(D_f)=0$. Recall from Q7 of Problem Set 2 that

$$D_f=\bigcup_{k=1}^{\infty}\underbrace{\{x_0\in[a,b]:\omega(f,x_0)\geq 1/k\}}_{=:\Omega_k}.$$

Try to show Ω_k has zero Lebesgue measure for each $k\in\mathbb{N}$. Take a partition $P=\{x_i\}_{i=0}^n$ of $[a,b]$ such that $U(f,P)-L(f,P)<\varepsilon'$, where ε' is a suitably chosen small number. Think about what can you say about $\sup_{[x_{i-1},x_i]}f-\inf_{[x_{i-1},x_i]}f$ when $\Omega_k\cap(x_{i-1},x_i)\neq\emptyset$? Consider the following set inclusion:

$$\Omega_k\backslash\{x_0,\cdots,x_n\}\subset\bigcup_{\{i:\Omega_k\cap(x_{i-1},x_i)\neq\emptyset\}}(x_{i-1},x_i),$$

try to bound the total length of open intervals on RHS from above.]

(b) Before proving the converse, we first prove a modified version of "continuous on $[a,b]$ implies uniform continuity": let $g:[a,b]\to\mathbb{R}$ be continuous on a subset $K:=[a,b]\backslash\bigcup_{n=1}^{\infty}(\alpha_n,\beta_n)$ where $\{(\alpha_n,\beta_n)\}_{n=1}^{\infty}$ is a countable collection of open intervals. Show that for any $\varepsilon>0$, there exists $\delta>0$ (independent of of x,y) such that

$$x\in K, y\in[a,b], \text{ and } |x-y|<\delta \implies |g(x)-g(y)|<\varepsilon.$$

[Remark: The improvement here is that K needs not be a closed and bounded interval (just a closed and bounded *set*), and we allow y to be outside K.]

(c) Now prove the converse of (a): suppose D_f has Lebesgue measure zero, show that f is Riemann integrable on $[a,b]$. Try to cover D_f by a small open cover $\bigcup_{n=1}^{\infty}(\alpha_n,\beta_n)$, then consider $K:=[a,b]\backslash\bigcup_{n=1}^{\infty}(\alpha_n,\beta_n)$. Pick a small enough partition $\{x_i\}_{i=0}^k$ of $[a,b]$, estimate $U(f,P)-L(f,P)$ by splitting the summation into two: one with $[x_{i-1},x_i]\cap K=\emptyset$, another with $[x_{i-1},x_i]\cap K\neq\emptyset$.

Exercise 5.13. Complete the proof of Lebesgue's Theorem by following the above outline.

Remark 5.20. As a final remark about Riemann integrals, we have been restricting ourselves on functions which are bounded on a closed and bounded interval in $\mathbb{R}$.

- We need the domain to be an interval because intervals have a natural definition of "lengths". Without the appropriate concept of "length" of an arbitrary set, it is rather difficult to define integrals on functions with an arbitrary domain. Lebesgue measure on $\mathbb{R}$ will be useful in this respect, and will be introduced in MATH 3043.
- Even when the domain is restricted to be an interval, we cannot always define Riemann integrals naturally if f is unbounded. In some special cases, such as when $|f|\to\infty$ when x approaches to a finite set of points, we can still define improper integrals using limits:

$$\begin{aligned}&\int_0^2\frac{1}{x^2(x-1)^2}\,dx\\&=\lim_{a\to0^+}\int_a^{1/2}\frac{1}{x^2(x-1)^2}\,dx+\lim_{b\to1^-}\int_{1/2}^{b}\frac{1}{x^2(x-1)^2}\,dx+\lim_{c\to1^+}\int_c^2\frac{1}{x^2(x-1)^2}\,dx.\end{aligned}$$

However, if an unbounded function $f:[a,b]\to\mathbb{R}$ blows up at too many points, say uncountably many, the above definition does not work.

- For bounded functions f, but whose domain is a non-compact interval say $[0,\infty)$, we can define improper integrals of f by:

$$\int_0^{+\infty} f(x)\,dx := \lim_{a\to+\infty}\int_0^a f(x)\,dx.$$

  However, what if the domain is just a non-compact set $K \subset \mathbb{R}$?

All these technical issues would be resolved by introducing Lebesgue measures and integrals. Interested readers may continue on the journey in the next course: MATH 3043[1].

[1] Rumors said it is the hardest undergraduate math course in HKUST.

Part 2

MATH 3043 - Honor Real Analysis

Chapter 6

Lebesgue Measures

"My disappointment is immeasurable, and my day is ruined."

YouTuber *Reviewbrah*

This chapter introduces Lebesgue measures on both the real line and $\mathbb{R}^{n\geq 2}$. We then give the general definition of measure spaces, whose axioms are motivated by some properties of Lebesgue measures.

6.1. Lebesuge measures on the real line

As remarked in MATH 2043, the definition of Riemann integrals of functions $f : [a, b] \to \mathbb{R}$ relies on the notion of "length" of intervals, which can be naturally defined. When the domain of the function is an arbitrary, we need a more general notion of "length" of arbitrary sets.

6.1.1. Lebesgue outer measure. We have already discussed the definition of Lebesgue measure on the real line in MATH 2043. As a reminder, it is defined as:

Definition 6.1 (Lebesgue outer measure on $\mathbb{R}$)**.** The **Lebesuge outer measure on** $\mathbb{R}$ is a set function $\mathcal{L}^* : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ mapping any subset $E \subset \mathbb{R}$ to an extended real number $\mathcal{L}^*(E)$ defined as

$$\mathcal{L}^*(E) := \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) : E \subset \bigcup_{i=1}^{\infty} (a_i, b_i), a_i \leq b_i \, \forall i \right\}.$$

We have already seen examples of sets that have zero Lebesgue outer measure, such as countable sets in $\mathbb{R}$. However, we still haven't proved that $\mathcal{L}^*(\langle a, b\rangle) = b - a$ where $\langle$ could mean (or [, and $\rangle$ could mean) or]. It is actually not as trivial as it seems, and one needs to invoke the compactness property of $[a, b]$.

Let's first show $\mathcal{L}^*([a, b]) = b - a$. Given any $\varepsilon > 0$, the open interval $(a - \varepsilon, b + \varepsilon) \supset [a, b]$, so $(b+\varepsilon)-(a-\varepsilon) \geq \mathcal{L}^*([a, b])$ from the definition of outer measure. Letting $\varepsilon \to 0^+$, we get

$$\mathcal{L}^*([a, b]) \leq b - a.$$

To prove the reverse inequality, we consider an arbitrary open interval cover

$$\bigcup_{i=1}^{\infty}(a_i,b_i)\supset[a,b].$$

By compactness of $[a,b]$, there exists a finite subcover

$$[a,b]\subset\bigcup_{k=1}^{N}(a_{i_k},b_{i_k}).$$

In particular, we have

$$(a,b)\subset\bigcup_{k=1}^{N}(a_{i_k},b_{i_k}).$$

Here, we claim that the length function of open intervals satisfies **finite subadditivity**, meaning that the above inclusion implies

$$b-a\le\sum_{k=1}^{N}(b_{i_k}-a_{i_k}).$$

It is not hard to prove if $N=1$, of course. The general result can be proved by induction on N, which is left as an exercise for readers.

Exercise 6.1. Suppose $m,n\in\mathbb{N}$ and $\{(a_i,b_i)\}_{i=1}^m$ is a finite collection of disjoint open intervals, and $\{(c_j,d_j)\}_{j=1}^n$ is a finite collection of open intervals (not necessarily disjoint) such that

$$\bigsqcup_{i=1}^{m}(a_i,b_i)\subset\bigcup_{j=1}^{n}(c_j,d_j),$$

then we have

$$\sum_{i=1}^{m}(b_i-a_i)\le\sum_{j=1}^{n}(d_j-c_j).$$

Using this, we get

$$b-a\le\sum_{k=1}^{N}(b_{i_k}-a_{i_k})\le\sum_{i=1}^{\infty}(b_i-a_i)$$

as $\{a_{i_k},b_{i_k}\}_{k=1}^N$ is a subcollection of $\{(a_i,b_i)\}_{i=1}^\infty$. Since the open interval cover $\bigcup_{i=1}^\infty(a_i,b_i)$ of $[a,b]$ is arbitrary, taking infimum over all such covering of $[a,b]$, we conclude that

$$\mathcal{L}^*([a,b])=\inf\left\{\sum_{i=1}^{\infty}(b_i-a_i):[a,b]\subset\bigcup_{i=1}^{\infty}(a_i,b_i)\right\}\ge b-a$$

as desired.

Remark 6.2. It often happens in measure theory that we prove an equation about measures by splitting it into $\le$ and $\ge$. Make sure you do not prove the same direction twice instead. LOL!

For intervals of other type, we recall that Lebesgue measure is monotone, so that

$$(a,b)\subset[a,b)\subset[a,b]\implies\mathcal{L}^*\big((a,b)\big)\le\mathcal{L}^*\big([a,b)\big)\le\mathcal{L}^*\big([a,b]\big).$$

Recall also that Lebesgue measure subadditive as proved by the previous chapter, we have

$$\mathcal{L}^*\big([a,b]\big)\le\mathcal{L}^*\big((a,b)\big)+\mathcal{L}^*(\{a\})+\mathcal{L}^*(\{b\})=\mathcal{L}^*\big((a,b)\big).$$

Since we also have $\mathcal{L}^*\big((a,b)\big) \leq \mathcal{L}^*\big([a,b]\big)$, we have equality and so $\mathcal{L}^*\big((a,b)\big) = b-a$ as well. It also implies $\mathcal{L}^*\big([a,b)\big) = \mathcal{L}^*\big((a,b]\big) = b-a$.

Exercise 6.2. Since $\mathcal{L}^*\big((0,1)\big) = 1$ whereas $\mathcal{L}^*(S) = 0$ for any countable set $S \subset \mathbb{R}$, it proves $(0,1)$ is not countable. Is it a valid proof without circular reasoning?

Although the definition of Lebesgue measure makes use of open intervals cover to estimate the length of a set, we can use closed intervals instead and the outcome will be the same. For any $E \subset \mathbb{R}$, we let

$$L^*(E) := \inf\left\{\sum_{i=1}^{\infty}(b_i - a_i) : E \subset \bigcup_{i=1}^{\infty}[a_i, b_i]\right\}.$$

We will show that $L^* = \mathcal{L}^*$.

First of all, given any open interval cover $\{(a_i,b_i)\}_{i=1}^{\infty}$ of E, the closed interval collection $\{[a_i,b_i]\}_{i=1}^{\infty}$ is also a cover of E. Therefore, we have

$$\left\{\sum_{i=1}^{\infty}(b_i - a_i) : E \subset \bigcup_{i=1}^{\infty}(a_i, b_i)\right\} \subset \left\{\sum_{i=1}^{\infty}(b_i - a_i) : E \subset \bigcup_{i=1}^{\infty}[a_i, b_i]\right\}.$$

Infimum of a smaller set is larger, so we have

$$\mathcal{L}^*(E) \geq L^*(E).$$

We then prove the reverse inequality. We use the "ε-trick" that will be very common in this course. For any $\varepsilon > 0$, we hope to show $\mathcal{L}^*(E) < L^*(E) + \varepsilon$, then letting $\varepsilon \to 0^+$ will yield the desired inequality. Given $\varepsilon > 0$, by the definition of infimum, there exists a closed interval cover $\{[a_i,b_i]\}_{i=1}^{\infty}$ of E such that

$$L^*(E) \leq \sum_{i=1}^{\infty}(b_i - a_i) < L^*(E) + \varepsilon.$$

The "fatten" collection $\{(a_i - \varepsilon/2^i, b_i + \varepsilon/2^i)\}_{i=1}^{\infty}$ is then an open interval cover of E. Therefore, we have

$$\mathcal{L}^*(E) \leq \sum_{i=1}^{\infty}\big((b_i + \varepsilon/2^i) - (a_i - \varepsilon/2^i)\big) = \sum_{i=1}^{\infty}(b_i - a_i) + 2\varepsilon.$$

Combining the above result, we get

$$\mathcal{L}^*(E) \leq L^*(E) + 3\varepsilon.$$

Letting $\varepsilon \to 0^+$ yields $\mathcal{L}^*(E) \leq L^*(E)$ as desired.

Exercise 6.3. Prove that for any $E \subset \mathbb{R}$, all of the following are equal:

$$\begin{aligned}
&\mathcal{L}^*(E)\\
&= \inf\left\{\sum_{i=1}^{\infty}(b_i - a_i) : E \subset \bigcup_{i=1}^{\infty}[a_i, b_i)\right\}\\
&= \inf\left\{\sum_{i=1}^{\infty}(b_i - a_i) : E \subset \bigcup_{i=1}^{\infty}(a_i, b_i]\right\}\\
&= \inf\left\{\sum_{i=1}^{\infty}(b_i - a_i) : E \subset \bigcup_{i=1}^{\infty}(a_i, b_i) \quad \text{where } 0 \leq b_i - a_i < 1 \ \forall i\right\}
\end{aligned}$$

6.1.2. Lebesgue inner measure. The Lebesgue inner measure is to estimate the length of a set $E \subset \mathbb{R}$ from inside using compact sets.

To begin with, we need to define what it means by the *length* of an open set $U \subset \mathbb{R}$ and a compact set $K \subset \mathbb{R}$. Since every open set $U \subset \mathbb{R}$ is a disjoint union of countably many open intervals, say

$$U = \bigsqcup_{i=1}^{\infty}(a_i, b_i),$$

naturally one would define

$$l(U) := \sum_{i=1}^{\infty}(b_i - a_i).$$

Just like $\mathcal{L}^*$, the length of open sets also satisfy countable subadditivity:

Lemma 6.3. *Suppose $\{(a_i, b_i)\}_{i=1}^{\infty}$ is a countable collection disjoint open intervals, and given that*

$$V := \bigsqcup_{i=1}^{\infty}(a_i, b_i) \subset \bigcup_{j=1}^{\infty} U_j$$

for some open sets U_j's. Then, we have

$$l(V) = \sum_{i=1}^{\infty}(b_i - a_i) \leq \sum_{j=1}^{\infty} l(U_j).$$

Proof. The result is clear if any a_i or b_i is infinite. We assume that $a_i, b_i \in \mathbb{R}$ for any i.

We may also assume that each U_j is an open interval. First we show that for any $n \in \mathbb{N}$, we have $\sum_{i=1}^{n}(b_i - a_i) \leq \sum_{j=1}^{\infty} l(U_j)$. Given any $\varepsilon > 0$, we consider the compact set

$$K := \bigsqcup_{i=1}^{n}[a_i + \varepsilon, b_i - \varepsilon].$$

Since $K \subset V \subset \bigcup_{j=1}^{\infty} U_j$, by compactness there exists $j_1, \cdots, j_N$ such that

$$K \subset \bigcup_{k=1}^{N} U_{j_k}.$$

It implies

$$\bigsqcup_{i=1}^{n}(a_i + \varepsilon, b_i - \varepsilon) \subset \bigcup_{k=1}^{N} U_{j_k},$$

and by the result of Exercise 6.1, we have

$$\sum_{i=1}^{n}(b_i - a_i) - 2n\varepsilon \leq \sum_{k=1}^{N} l(U_{j_k}) \leq \sum_{j=1}^{\infty} l(U_j).$$

Fixing $n \in \mathbb{N}$ while letting $\varepsilon \to 0^+$, we concluded that

$$\sum_{i=1}^{n}(b_i - a_i) \leq \sum_{j=1}^{\infty} l(U_j).$$

Then, letting $n \to \infty$, we get

$$\sum_{i=1}^{\infty}(b_i - a_i) \leq \sum_{j=1}^{\infty} l(U_j),$$

as desired.

In case that U_j's are generally open sets, we can express

$$U_j = \bigsqcup_{k=1}^{\infty} I_{j,k}$$

where $I_{j,k}$'s are open intervals. Then,

$$\bigsqcup_{i=1}^{\infty} (a_i, b_i) \subset \bigcup_{j=1}^{\infty} U_j = \bigcup_{j=1}^{\infty} \bigsqcup_{k=1}^{\infty} I_{j,k}.$$

The RHS is a countable union of open intervals, so using what we have proven we conclude that

$$\sum_{i=1}^{\infty} (b_i - a_i) \leq \sum_{j=1}^{\infty} \underbrace{\sum_{k=1}^{\infty} l(I_{j,k})}_{=l(U_j)} = \sum_{j=1}^{\infty} l(U_j).$$

□

Exercise 6.4. Show that for any $E \subset \mathbb{R}$, we have

$$\mathcal{L}^*(E) = \inf\{l(U) : E \subset U \text{ and } U \text{ is open}\}.$$

Note that some books use the RHS as the definition of Lebesgue outer measure. Hint: Use Lemma 6.3.

We next extend the notion of lengths to compact sets in $\mathbb{R}$. For a compact set $K \subset \mathbb{R}$, since K must be bounded, one can find a big open interval $(a, b) \supset K$. Also K is closed, so $(a, b) - K$ is open. Having defined the length of all open sets, we can naturally define the length of K by:

$$l(K) := l\big((a, b)\big) - l\big((a, b) - K\big).$$

We leave it as an exercise for readers to check that the above definition is independent of the choice of the open interval (a, b) containing K.

Exercise 6.5. Show that the definition of $l(K)$ is independent of the choice of the open interval (a, b) containing K.

Example 6.4. Consider $K = [a, b]$. It is contained inside the open interval $(a - 1, b + 1)$, with

$$(a - 1, b + 1) - K = (a - 1, a) \cup (b, b + 1).$$

Hence, $l([a, b]) = \big((b + 1) - (a - 1)\big) - \big(a - (a - 1) + (b + 1) - b\big) = b - a$.

Example 6.5. Recall that the Cantor set $\mathcal{C}$ is a compact set written in the form of:

$$\mathcal{C} = [0, 1] - \big(I_{1/3} \sqcup 2I_{1/3^2} \sqcup 2^2 I_{1/3^3} \sqcup \cdots\big)$$

where $2^j I_{1/3^{j+1}}$ denotes the union of 2^j disjoint open intervals of length $\frac{1}{3^{j-1}}$. One can bound $\mathcal{C}$ by the open interval $(-1, 2)$, then

$$(-1, 2) - \mathcal{C} = (-1, 0) \sqcup (1, 2) \sqcup \big(I_{1/3} \sqcup 2I_{1/3^2} \sqcup 2^2 I_{1/3^3} \sqcup \cdots\big).$$

This shows

$$l(\mathcal{C}) = 3 - \left(1 + 1 + \sum_{j=0}^{\infty} 2^j \cdot \frac{1}{3^{j+1}}\right) = 3 - \left(2 + \frac{\frac{1}{3}}{1 - \frac{2}{3}}\right) = 0.$$

Exercise 6.6. Suppose $\{[a_i, b_i]\}_{i=1}^n$ is a finite collection of disjoint closed intervals. Show that

$$l\left(\bigsqcup_{i=1}^n [a_i, b_i]\right) = \sum_{i=1}^n (b_i - a_i).$$

Exercise 6.7. Show that if U is an open set in $\mathbb{R}$, and $K \subset U$ is a compact set, then we have

$$l(U) = l(K) + l(U - K).$$

Here is the outline: first write

$$U = \bigsqcup_{i=1}^\infty (a_i, b_i).$$

By compactness of K, we can assume WLOG that $K \subset (a_i, b_i)$ for $i = 1, \cdots, n$, whereas $K \cap (a_i, b_i) = \emptyset$ when $i > n$. We can also assume

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n.$$

Try to find out $l(U - K)$, $l(U)$, and $l(K)$.

Exercise 6.8. Prove that if U is an open set in $\mathbb{R}$, and K is a compact set in $\mathbb{R}$ (but we do not assume $K \subset U$), prove that

$$l(K) \leq l(K - U) + l(U),$$
$$l(U) \leq l(U - K) + l(K).$$

Hint: first draw a Venn diagram

Remark 6.6. Note that at this moment, we still do not know whether $l(K) = \mathcal{L}^*(K)$ and $l(U) = \mathcal{L}^*(U)$ for any compact set K and any open set U, but we will prove that they are indeed true later.

With the notion of length on compact sets, we can introduce the Lebesgue inner measure:

Definition 6.7 (Lebesgue inner measure on $\mathbb{R}$)**.** The **Lebesgue inner measure on** $\mathbb{R}$ is a set function $\mathcal{L}_* : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ such that for any $E \subset \mathbb{R}$, we have

$$\mathcal{L}_*(E) := \sup\{l(K) : K \subset E \text{ and } K \text{ is compact}\}.$$

Remark 6.8. Stein's book uses m_* for outer measure, and m^* for inner measure.

Given any $E \subset \mathbb{R}$, one can easily show $\mathcal{L}_*(E) \leq \mathcal{L}^*(E)$. For any compact set K and open intervals $\{(a_i, b_i)\}_{i=1}^\infty$ such that $K \subset E \subset \bigcup_{i=1}^\infty (a_i, b_i) =: U$, by Exercise 6.7 we have

$$l(K) = l(U) - l(U - K) \leq l(U).$$

Express $U = \bigsqcup_{j=1}^\infty (c_j, d_j)$, then Lemma 6.3 shows

$$l(U) := \sum_{j=1}^\infty (d_j - c_j) \leq \sum_{i=1}^\infty (b_i - a_i).$$

This shows

$$l(K) \leq \sum_{i=1}^\infty (b_i - a_i).$$

Taking supremum over all $K \supset E$, we get

$$\mathcal{L}_*(E) \le \sum_{i=1}^{\infty}(b_i - a_i),$$

and taking infimum over all open interval covers of E, we get

$$\mathcal{L}_*(E) \le \mathcal{L}^*(E),$$

as desired.

Naturally one would ask whether $\mathcal{L}_*(E) = \mathcal{L}^*(E)$ for any $E \subset \mathbb{R}$. Unfortunately the answer is no, but counterexamples are very “rare” and not easy to construct. We will give one counterexample at the end of this chapter. Let’s learn some “good” examples first.

Example 6.9. We claim that $\mathcal{L}_*(U) = l(U) = \mathcal{L}^*(U)$ for any open set $U \subset \mathbb{R}$. First we write

$$U = \bigsqcup_{i=1}^{\infty}(a_i, b_i).$$

We may assume a_i’s and b_i’s are all finite, since if not then it is to show $\mathcal{L}_*(U) = l(U) = \mathcal{L}^*(U) = +\infty$.

As $\{(a_i, b_i)\}$ is an open interval cover of U, we immediately conclude that

$$\mathcal{L}^*(U) \le \sum_{i=1}^{\infty}(b_i - a_i) = l(U).$$

Then, for each $n \in \mathbb{N}$ and $\varepsilon > 0$, we consider the compact set

$$K := \bigsqcup_{i=1}^{n}[a_i + \varepsilon, b_i - \varepsilon] \subset U.$$

Hence, $l(K) \le \mathcal{L}_*(U)$. By Exercise 6.6, we know

$$l(K) = \sum_{i=1}^{n}(b_i - a_i - 2\varepsilon).$$

Combining all results above, we get

$$\sum_{i=1}^{n}(b_i - a_i) - 2n\varepsilon = l(K) \le \mathcal{L}_*(U) \le \mathcal{L}^*(U) \le l(U) = \sum_{i=1}^{n}(b_i - a_i).$$

Letting $\varepsilon \to 0^+$ first, then $n \to \infty$, we conclude that

$$\sum_{i=1}^{\infty}(b_i - a_i) \le \mathcal{L}_*(U) \le \mathcal{L}^*(U) \le \sum_{i=1}^{\infty}(b_i - a_i),$$

and therefore, $\mathcal{L}_*(U) = l(U) = \mathcal{L}^*(U)$.

Recall that $\mathcal{L}^*$ is monotone, in a sense that if $E \subset F$, then $\mathcal{L}^*(E) \le \mathcal{L}^*(F)$, the above result shows l is also monotone on open sets.

Exercise 6.9. Show that if $C \subset K$ are two compact sets, then $l(C) \le l(K)$.

Exercise 6.10. Prove that if $E \subset F \subset \mathbb{R}$, then $\mathcal{L}_*(E) \le \mathcal{L}_*(F)$.

Example 6.10. We prove that $\mathcal{L}_*(K) = l(K)$ for any compact set $K \subset \mathbb{R}$. From the definition of $\mathcal{L}_*$, we clearly have

$$\mathcal{L}_*(K) = \sup\{l(C) : C \subset K \text{ and } C \text{ is compact}\} \ge l(K)$$

as $K \subset K$.

The reverse inequality follows from the monotonicity of length on compact sets: for any compact set $C \subset K$, we have $l(C) \leq l(K)$. Taking supremum over all compact sets $C \subset K$, we get $\mathcal{L}_*(K) \leq l(K)$ as desired.

In fact, we have $\mathcal{L}^*(K) = l(K)$ for any compact set $K \subset \mathbb{R}$ as well, but we will prove it later after introducing a famous result by Caratheodory.

Exercise 6.11. Show that $\mathcal{L}_*\big((-\infty, a)\big) = \mathcal{L}^*\big((-\infty, a)\big) = +\infty$ for any $a \in \mathbb{R}$. A similar result holds for intervals of the form $(a, +\infty)$. It implies $\mathcal{L}_*(U) = \mathcal{L}^*(U) = l(U) = +\infty$ for any open set $U = \bigsqcup_{i=1}^{\infty}(a_i, b_i)$ even if some of the a_i's and b_i's are infinity.

For **bounded** sets whose Lebesgue outer and inner measures are equal are called **Lebesgue measurable sets**. Generally, we have

Definition 6.11 (Lebesgue measurable sets in $\mathbb{R}$)**.** A set $E \subset \mathbb{R}$ is said to be **Lebesgue measurable** if there exists $R_0 > 0$ such that, we have

$$\mathcal{L}_*(E \cap [-R, R]) = \mathcal{L}^*(E \cap [-R, R]) \ \ \forall R > R_0.$$

In particular, if E is bounded, then E is Lebesgue measurable if and only if $\mathcal{L}_*(E) = \mathcal{L}^*(E)$.

So far, we have proved that any bounded open set is Lebesgue measurable, and after proving $\mathcal{L}^*(K) = l(K)$ later, we can also conclude that any compact set is Lebesgue measurable. Certainly, there are many more such sets – including those "generated" by open sets and compact sets by taking countable union, intersections, and complements.

6.1.3. Caratheodory measurable sets.

One technical difficulty of proving Lebesgue measurability by checking $\mathcal{L}_* = \mathcal{L}^*$ is that both $\mathcal{L}_*$ and $\mathcal{L}^*$ are not easy to be computed. So far, we have only computed $\mathcal{L}_*(U)$ and $\mathcal{L}^*(U)$ for open sets U, and $\mathcal{L}_*(K)$ for compact sets K.

Here we introduce another definition of measurability, although a bit abstract, is in fact more important than Definition 6.11. We will prove later that these two definitions of measurability are equivalent.

Definition 6.12 (Caratheodory measurable sets in $\mathbb{R}$)**.** Let $E \subset \mathbb{R}$. We say E is **Caratheodory measurable** if for any $X \subset \mathbb{R}$, we have

$$\mathcal{L}^*(X) = \mathcal{L}^*(X \cap E) + \mathcal{L}^*(X - E). \tag{6.1}$$

The term $\mathcal{L}^*(X \cap E)$ is the Lebesgue outer measure of the part of X inside E, and the second term $\mathcal{L}^*(X - E)$ is the Lebesgue outer measure of the part of X outside E. Equation (6.1) means that the measures of the parts of X inside E and outside E add up to the measure of X. Roughly speaking, if E is a set that such phenomenon holds for ALL sets $X \subset \mathbb{R}$, then such a set E is said to be Caratheodory measurable.

Remark 6.13. To verify (6.1), it suffices to prove

$$\mathcal{L}^*(X) \geq \mathcal{L}^*(X \cap E) + \mathcal{L}^*(X - E),$$

as the reverse inequality always holds by the countable (hence finite) sub-additivity. Also, we only need to consider those $X \subset \mathbb{R}$ with $\mathcal{L}^*(X) < \infty$, otherwise the above inequality trivially holds.

It is quite surprising that Definitions 6.11 and 6.12 are in fact equivalent. We will give a proof for this later. Meanwhile, let's explore more about what sets are Caratheodory measurable, and what properties they have. Denote

$$\mathcal{M} := \{E \subset \mathbb{R} : E \text{ is Caratheodory measurable}\}.$$

Example 6.14. Any set $E \subset \mathbb{R}$ with $\mathcal{L}^*(E) = 0$ is Caratheodory measurable. Since for any $X \subset \mathbb{R}$, we have

$$\mathcal{L}^*(X \cap E) + \mathcal{L}^*(X - E) \leq \underbrace{\mathcal{L}^*(E)}_{X \cap E \subset E} + \underbrace{\mathcal{L}^*(X)}_{X - E \subset X} = 0 + \mathcal{L}^*(X) = \mathcal{L}^*(X).$$

Example 6.15. We first show that open intervals are all Caratheodory measurable. Consider an interval (c, d), where $-\infty < c < d < \infty$. Then, given any $X \subset \mathbb{R}$ with $\mathcal{L}^*(X) < \infty$, we need to show

$$\mathcal{L}^*(X) \geq \mathcal{L}^*(X \cap (c, d)) + \mathcal{L}^*(X - (c, d)).$$

Although it is generally hard to compute $\mathcal{L}^*$ of an arbitrary set, we know that $\mathcal{L}^*(U) = l(U)$ for any open set $U \subset \mathbb{R}$. This gives us the idea that one could "approximate" the arbitrary set X by open sets. Given any $\varepsilon > 0$, we take an open set $U \supset X$ such that

$$\mathcal{L}^*(X) + \varepsilon > l(U) \geq \mathcal{L}^*(X).$$

Such an open set U exists by the definition of infimum. Write

$$U = \bigsqcup_{i=1}^{\infty} (a_i, b_i),$$

then we decompose U into a disjoint union of two sets: one inside (c, d), and the other outside it.

$$\begin{aligned} U &= \big(U \cap (c, d)\big) \sqcup \big(U - (c, d)\big) \\ &= \bigsqcup_{i=1}^{\infty} (a_i, b_i) \cap (c, d) \sqcup \bigsqcup_{i=1}^{\infty} \big((a_i, b_i) - (c, d)\big) \\ &\supset \bigsqcup_{i=1}^{\infty} \underbrace{(a_i, b_i) \cap (c, d)}_{=: V_i} \sqcup \bigsqcup_{i=1}^{\infty} \underbrace{\big((a_i, b_i) - [c, d]\big)}_{=: W_i}. \end{aligned}$$

All V_i's and W_i's defined above are open sets and they are disjoint, so we have

$$l(U) \geq l\left(\bigsqcup_i V_i \sqcup \bigsqcup_i W_i\right) = \sum_{i=1}^{\infty} l(V_i) + \sum_{i=1}^{\infty} l(W_i).$$

Recall that $U \supset X$, so $\bigsqcup_i V_i = U \cap (c, d) \supset X \cap (c, d)$. This shows

$$\mathcal{L}^*\big(X \cap (c, d)\big) \leq l\big(\sqcup_i V_i\big) = \sum_{i=1}^{\infty} l(V_i).$$

Also, we have $\sqcup_i W_i = U - [c, d] \supset X - [c, d]$, we have

$$\mathcal{L}^*\big(X - [c, d]\big) \leq \sum_{i=1}^{\infty} l(W_i).$$

Since $X - [c, d] \subset X - (c, d) \subset \big(X - [c, d]\big) \cup \{c, d\}$, we have

$$\mathcal{L}^*\big(X - [c, d]\big) \leq \mathcal{L}^*\big(X - (c, d)\big) \leq \mathcal{L}^*\big(X - [c, d]\big) + \underbrace{\mathcal{L}^*(\{c, d\})}_{=0}.$$

This shows $\mathcal{L}^*\big(X - [c, d]\big) = \mathcal{L}^*\big(X - (c, d)\big)$.

To summarize, we have

$$\mathcal{L}^*(X)+\varepsilon > l(U) \geq \sum_{i=1}^{\infty} l(V_i) + \sum_{i=1}^{\infty} l(W_i) \geq \mathcal{L}^*\big(X\cap(c,d)\big)+\mathcal{L}^*\big(X-(c,d)\big).$$

Letting $\varepsilon \to 0^+$, we proved $\mathcal{L}^*(X) \geq \mathcal{L}^*(X\cap(c,d)) + \mathcal{L}^*(X-(c,d))$ as desired.

We leave it as an exercise for reader to prove the case when at least one of the end-point of the interval is infinite.

Exercise 6.12. Prove that (a,∞), $(-\infty,b)$ and $(-\infty,\infty)$ are all Caratheodory measurable.

So far, we have proved that open intervals are all belong to $\mathcal{M}$. Next we argue that $\mathcal{M}$ is closed under complement and countable union. These facts together will imply a lot of sets are Caratheodory measurable.

Proposition 6.16. *$\mathcal{M}$ satisfies the following properties:*

(1) $\emptyset \in \mathcal{M}$

(2) For any $E \in \mathcal{M}$, *we have* $E^c := \mathbb{R} - E \in \mathcal{M}$

(3) For any countable collection $\{E_j\}_{j=1}^{\infty}$ *in* $\mathcal{M}$, *we have*

$$\bigcup_{j=1}^{\infty} E_j \in \mathcal{M}.$$

As a corollary of (2) and (3), we have that if $E_j \in \mathcal{M}$ *for any* j, *then* $\displaystyle\bigcap_{j=1}^{\infty} E_j \in \mathcal{M}$.

Proof. (1) is trivial as $X\cap\emptyset=\emptyset$ and $X-\emptyset=X$. Recall that $\mathcal{L}^*(\emptyset)=0$. For (2), we simply note that $X\cap E^c = X-E$ and $X-E^c = X\cap E$, so that if (6.1) holds for E, it also holds for E^c.

For (3), we break it down into two lemmas, some of which will also be useful later on.

Lemma 6.17. *Whenever* $E_1, E_2 \in \mathcal{M}$, *we have* $E_1 \cup E_2 \in \mathcal{M}$.

Proof of Lemma. For any $X\subset\mathbb{R}$, we apply (6.1) on E_1 and then E_2:

$$\begin{aligned}
\mathcal{L}^*(X) &= \mathcal{L}^*(X\cap E_1) + \mathcal{L}^*(X-E_1) && (E_1\in\mathcal{M})\\
&= \mathcal{L}^*(X\cap E_1) + \mathcal{L}^*((X-E_1)\cap E_2) + \mathcal{L}^*((X-E_1)-E_2) && (E_2\in\mathcal{M})\\
&= \mathcal{L}^*(X\cap E_1) + \mathcal{L}^*((X\cap E_2)-E_1) + \mathcal{L}^*(X-(E_1\cup E_2))\\
&\geq \mathcal{L}^*\Big(\underbrace{(X\cap E_1)\cup((X\cap E_2)-E_1)}_{=X\cap(E_1\cup E_2)}\Big) + \mathcal{L}^*(X-(E_1\cup E_2)) && \text{(subadditivity)}\\
&= \mathcal{L}^*(X\cap(E_1\cup E_2)) + \mathcal{L}^*(X-(E_1\cup E_2)).
\end{aligned}$$

This proves (6.1) holds for $E_1\cup E_2$, so $E_1\cup E_2\in\mathcal{M}$. □

Corollary 6.18. *For any* $E_1, E_2 \in \mathcal{M}$, *we have* $E_1\cap E_2, E_1 - E_2 \in \mathcal{M}$.

Proof of Corollary. From (2), we have $E_1^c, E_2^c \in \mathcal{M}$, so from the above lemma, we know $E_1^c\cup E_2^c \in \mathcal{M}$ too. By (2) again we have $\mathbb{R}-(E_1^c\cup E_2^c)\in\mathcal{M}$. Note that $E_1\cap E_2 = \mathbb{R}-(E_1\cap E_2)^c = \mathbb{R}-(E_1^c\cup E_2^c)$, we have proved that $E_1\cap E_2\in\mathcal{M}$. For

$E_1 - E_2$, we simply note that $E_1 - E_2 = E_1 \cap E_2^c$, and that $E_2^c \in \mathcal{M}$, and hence by the intersection result just proved, we have $E_1 - E_2 \in \mathcal{M}$. □

Lemma 6.19 (Finite additivity). *Let $E_1, E_2 \in \mathcal{M}$ and $E_1 \cap E_2 = \emptyset$, we have*

$$\mathcal{L}^*(E_1 \cup E_2) = \mathcal{L}^*(E_1) + \mathcal{L}^*(E_2).$$

Proof of Lemma. Simply apply (6.1) on the set E_1 by taking $X = E_1 \cup E_2$, and noting that E_1 and E_2 are disjoint (so that $(E_1 \cup E_2) \cap E_1 = E_1$ and $(E_1 \cup E_2) - E_1 = E_2$:

$$\mathcal{L}(E_1 \cup E_2) = \mathcal{L}((E_1 \cup E_2) \cap E_1) + \mathcal{L}((E_1 \cup E_2) - E_1) = \mathcal{L}(E_1) + \mathcal{L}(E_2).$$

□

Exercise 6.13. Let $E_1, E_2 \in \mathcal{M}$ and $E_1 \cap E_2 = \emptyset$. Prove that for any $X \subset \mathbb{R}$ (not necessarily in $\mathcal{M}$), we have

$$\mathcal{L}^*(X \cap (E_1 \cup E_2)) = \mathcal{L}^*(X \cap E_1) + \mathcal{L}^*(X \cap E_2).$$

Exercise 6.14. Show that if $A \subset B$ are both Lebesgue measurable sets in $\mathbb{R}$, then we have

$$\mathcal{L}^*(A - B) = \mathcal{L}^*(A) - \mathcal{L}^*(B).$$

Now we proceed to prove (3) to complete the proof of Proposition 6.16. Given $\{E_j\}_{j=1}^\infty$ which may not be disjoint, we can create a new collection $\{E_j'\}_{j=1}^\infty$ by taking $E_1' = E_1$, and $E_j' = E_j - \bigcup_{i=1}^{j-1} E_i$. One can easily check that $\{E_j'\}_{j=1}^\infty$ is still a collection of sets in $\mathcal{M}$ by Lemma 6.17 and Corollary 6.18, and that E_j''s are pairwise disjoint, and that

$$\bigcup_{j=1}^\infty E_j = \bigcup_{j=1}^\infty E_j'.$$

Applying (6.1) on $\bigcup_{j=1}^n E_j' \in \mathcal{M}$, we have for any $X \subset \mathbb{R}$,

$$\begin{aligned}
\mathcal{L}^*(X) &= \mathcal{L}^*\left(X \cap \bigcup_{j=1}^n E_j'\right) + \mathcal{L}^*\left(X - \bigcup_{j=1}^n E_j'\right) \\
&\geq \underbrace{\sum_{j=1}^n \mathcal{L}^*(X \cap E_j')}_{\text{by Exercise 6.13}} + \underbrace{\mathcal{L}^*\left(X - \bigcup_{j=1}^\infty E_j'\right)}_{\text{by monotonicity}} \\
&= \sum_{j=1}^n \mathcal{L}^*(X \cap E_j') + \mathcal{L}^*\left(X - \bigcup_{j=1}^\infty E_j\right).
\end{aligned}$$

Letting $n \to \infty$, we get

$$\mathcal{L}^*(X) \geq \sum_{j=1}^\infty \mathcal{L}^*(X \cap E_j') + \mathcal{L}^*\left(X - \bigcup_{j=1}^\infty E_j\right) \geq \mathcal{L}^*\left(\bigcup_{j=1}^\infty (X \cap E_j')\right) + \mathcal{L}^*\left(X - \bigcup_{j=1}^\infty E_j\right)$$

by the countable sub-additivity of Lebesgue outer measures. Noting that

$$\bigcup_{j=1}^\infty (X \cap E_j') = X \cap \bigcup_{j=1}^\infty E_j' = X \cap \bigcup_{j=1}^\infty E_j,$$

we have proved that (6.1) holds for $\bigcup_{j=1}^\infty E_j$. It completes the proof of (3). □

Corollary 6.20. *Open sets and closed sets in $\mathbb{R}$ are Caratheodory measurable.*

Proof. We have formerly proven that all open intervals are Caratheodory measurable. Together with (3) of Proposition 6.16, it shows all open sets in $\mathbb{R}$ (which are countable union of open intervals) are all Caratheodory measurable.

Closed sets are complements of open sets, so by (2) of Proposition 6.16 they are all Caratheodory measurable. □

Example 6.21. The set of irrational numbers $\mathbb{R} - \mathbb{Q}$ is Caratheodory measurable, since $\mathbb{Q}$ is so. For any Caratheodory measurable $E \subset \mathbb{R}$, the set $E \cap (\mathbb{R} - \mathbb{Q})$, i.e. the irrational numbers in E, is Caratheodory measurable too by Corollary 6.18.

Example 6.22. Let $\{A_n\}$ be a sequence of Caratheodory measurable sets in $\mathbb{R}$. Denote

$$\begin{aligned} S_1 &:= \text{ set of } x \in \mathbb{R} \text{ such that } x \in A_n \text{ for infinitely many } n\text{'s} \\ S_2 &:= \text{ set of } x \in \mathbb{R} \text{ such that there exists } m \in \mathbb{N} \text{ so that } x \in A_n \text{ for all } n \geq m \end{aligned}$$

Then both S_1 and S_2 are Caratheodory measurable. In fact, it can be shown easily that

$$S_1 = \bigcap_{k=1}^{\infty} \bigcup_{n=k+1}^{\infty} A_n$$

$$S_2 = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n.$$

Exercise 6.15. Let E be the set of all $x \in [0,1]$ whose decimal representation does not contain the digit 7. Show that E is Caratheodory measurable.

Exercise 6.16. Suppose $A, B \subset \mathbb{R}$ are sets such that A and $A \Delta B := (A-B) \cup (B-A)$ are Caratheodory measurable. Show that B is also Caratheodory measurable.

We showed in Lemma 6.19 that $\mathcal{L}^*$ is finite additive on Caratheodory measurable sets, i.e. if $E_1, \cdots, E_n$ are disjoint Caratheodory measurable sets, then

$$\mathcal{L}^* \left(\bigsqcup_{i=1}^{n} E_i \right) = \sum_{i=1}^{n} \mathcal{L}^*(E_i).$$

Note that using the method of induction *alone* could only prove the above holds when $n \in \mathbb{N}$, but with some further argument, one can extend the above result to *countably* many disjoint measurable sets.

Proposition 6.23 (Countable additivity). *Let $\{E_j\}_{j=1}^{\infty}$, where $E_j \subset \mathbb{R}$, be a countable disjoint collection of Caratheodory measurable sets in $\mathbb{R}$. Then we have*

$$\mathcal{L}^* \left(\bigsqcup_{j=1}^{\infty} E_j \right) = \sum_{j=1}^{\infty} \mathcal{L}^*(E_j).$$

Proof. We have already proved the countable sub-additivity:

$$\mathcal{L}^* \left(\bigsqcup_{j=1}^{\infty} E_j \right) \leq \sum_{j=1}^{\infty} \mathcal{L}^*(E_j),$$

so it suffices to prove the reverse inequality. By monotonicity of measures and the finite additivity (Lemma 6.19 and induction), we get that

$$\mathcal{L}^*\left(\bigsqcup_{j=1}^{\infty} E_j\right) \geq \mathcal{L}^*\left(\bigsqcup_{j=1}^{n} E_j\right) = \sum_{j=1}^{n} \mathcal{L}^*(E_j)$$

for any $n \in \mathbb{N}$. Letting $n \to +\infty$, we get that

$$\mathcal{L}^*\left(\bigsqcup_{j=1}^{\infty} E_j\right) \geq \sum_{j=1}^{\infty} \mathcal{L}^*(E_j).$$

□

Example 6.24. Let's modify the construction of the Cantor set a bit. Fix a constant $\lambda \in (0,1)$. Start with $[0,1]$, then removing the middle interval of length λ. In the remaining two interval of length $\frac{1-\lambda}{2}$, we each remove the middle interval of length $\lambda \cdot \frac{1-\lambda}{2}$, and continue indefinitely. Here, by "interval" we include any of the type $[a,b]$, $(a,b]$, $[a,b)$ or (a,b). We denote the resulting set to be $\mathcal{C}_\lambda$.

It is not hard to see that $\mathcal{C}_\lambda$ can be written as

$$\mathcal{C}_\lambda = [0,1] - \left(I_\lambda \sqcup 2I_{\frac{\lambda(1-\lambda)}{2}} \sqcup 2^2 I_{\frac{\lambda(1-\lambda)^2}{2^2}} \sqcup \cdots\right),$$

where $2^j I_{\frac{\lambda(1-\lambda)^j}{2^j}}$ is the union of 2^j disjoint closed intervals of length $\frac{\lambda(1-\lambda)^j}{2^j}$, all of which are subsets of $[0,1]$.

Then, by Exercise 6.14 and using Proposition 6.23, we have

$$\begin{aligned}
\mathcal{L}^*(\mathcal{C}_\lambda) &= \mathcal{L}^*\big([0,1]\big) - \mathcal{L}^*\big(I_\lambda \sqcup 2I_{\frac{\lambda(1-\lambda)}{2}} \sqcup 2^2 I_{\frac{\lambda(1-\lambda)^2}{2^2}} \sqcup \cdots\big) \\
&= 1 - \sum_{j=0}^{\infty} 2^j \mathcal{L}^*\big(I_{\frac{\lambda(1-\lambda)^j}{2^j}}\big) \\
&= 1 - \sum_{j=0}^{\infty} 2^j \cdot \frac{\lambda(1-\lambda)^j}{2^j} \\
&= 1 - \frac{\lambda}{1-(1-\lambda)} = 0.
\end{aligned}$$

Exercise 6.17. Suppose $E_1 \subset E_2 \subset \cdots$ is an increasing sequence of Caratheodory measurable sets in $\mathbb{R}$. Show that

$$\mathcal{L}^*\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{n\to\infty} \mathcal{L}^*(E_n).$$

Hint: consider $F_j := E_j - E_{j-1}$. What can you say about the collection $\{F_j\}_{j=1}^{\infty}$?

Exercise 6.18. Suppose $E_1 \supset E_2 \supset \cdots$ is a decreasing sequence of Caratheodory measurable sets in $\mathbb{R}$, and $\mathcal{L}^*(E_1) < \infty$. Show that

$$\mathcal{L}^*\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{n\to\infty} \mathcal{L}^*(E_n).$$

Show that the condition $\mathcal{L}^*(E_1) < \infty$ is necessary for the above to hold by finding a counterexample.

6.1.4. Caratheodory's criterion.

Now we have introduced two definitions of measurability, namely in the sense of Lebesgue and Caratheodory. Although they do not seem to be related, they are actually equivalent!

Theorem 6.25 (Caratheodory's criterion). *Let $E \subset \mathbb{R}$. Then, E is Lebesgue measurable if and only if E is Caratheodory measurable.*

Before we prove the theorem, we derive the following lemma which appears in the proof quite often.

Lemma 6.26. *Given any $A, B \subset \mathbb{R}$ such that $A \cap B = \emptyset$, then*
$$\mathcal{L}_*(A \sqcup B) \leq \mathcal{L}_*(A) + \mathcal{L}^*(B) \leq \mathcal{L}^*(A \sqcup B).$$

Proof. For any open set $U \supset A \sqcup B$, and any compact set $K \subset A$, we must have $K \subset U$ so by Exercise 6.7 we have
$$l(U) = l(K) + l(U - K).$$
Note that $U - K$ is open and $B \subset U - K$, so by Exercise 6.4 we have
$$l(U - K) \geq \mathcal{L}^*(B).$$
This shows
$$l(U) \geq l(K) + \mathcal{L}^*(B).$$
Taking infimum over all $U \supset A \sqcup B$, and supremum over all $K \subset A$, we proved
$$\mathcal{L}^*(A \sqcup B) \geq \mathcal{L}_*(A) + \mathcal{L}^*(B),$$
proving the right inequality.

For the left inequality, we consider an arbitrary compact set $C \subset A \sqcup B$ and open set $V \supset B$. Then $C - V$ is a compact set contained in A. We then have
$$l(C - V) \leq \mathcal{L}_*(A).$$
Furthermore, by Exercise 6.8, we have
$$l(C) \leq l(C - V) + l(V) \leq \mathcal{L}_*(A) + l(V).$$
Taking supremum over all $C \subset A \sqcup B$, and infimum over all $V \supset B$, we conclude that
$$\mathcal{L}_*(A \sqcup B) \leq \mathcal{L}_*(A) + \mathcal{L}^*(B),$$
proving the left inequality. □

Now we are ready to prove the Caratheodory's criterion:

Proof of Theorem 6.25. ($\Longleftarrow$)-part: Assume $E \subset \mathbb{R}$ is Caratheodory measurable. Then, given any $R > 0$, by taking $X = [-R, R]$ in (6.1), we have:
$$\mathcal{L}^*([-R,R] \cap E) + \mathcal{L}^*([-R,R] - E) = \mathcal{L}^*([-R,R]) = 2R = \mathcal{L}_*([-R,R]).$$
On the other hand, using Lemma 6.26 by taking $A = [-R,R] \cap E$ and $B = [-R,R] - E$, then they are disjoint and so
$$\mathcal{L}_*([-R,R]) = \mathcal{L}_*(A \sqcup B) \leq \mathcal{L}_*([-R,R] \cap E) + \mathcal{L}^*([-R,R] - E).$$
Combining the two result and by cancellations (note that $\mathcal{L}^*([-R,R] - E) < \infty$), we have
$$\mathcal{L}^*([-R,R] \cap E) \leq \mathcal{L}_*([-R,R] \cap E).$$
Since $\mathcal{L}^* \geq \mathcal{L}_*$, the above is in fact an equality. This shows E is Lebesgue measurable (in the sense of Definition 6.11).

For the ($\Longrightarrow$)-part, we are given a set $E \subset \mathbb{R}$ that is Lebesgue measurable, i.e. there exists $R_0 > 0$ such that

$$\mathcal{L}_*(E \cap [-R, R]) = \mathcal{L}^*(E \cap [-R, R]) \ \ \forall R \geq R_0.$$

We need to show E is Caratheodory measurable.

For simplicity, we denote $E_R := E \cap [-R, R]$. We first prove that $E_R \cap U$ is Lebesgue measurable for any open set U and $R \geq R_0$. Recall that any open set U is Caratheodory measurable, so by taking $X = E_R$ in (6.1) we get:

$$\mathcal{L}^*(E_R \cap U) + \mathcal{L}^*(E_R - U) = \mathcal{L}^*(E_R) = \mathcal{L}_*(E_R).$$

The last equality follows from E being Lebesgue measurable.

Consider $E_R = (E_R \cap U) \sqcup (E_R - U)$. Lemma 6.26 shows

$$\mathcal{L}_*(E_R) \leq \mathcal{L}_*(E_R \cap U) + \mathcal{L}^*(E_R - U).$$

Combining the above results, we get

$$\mathcal{L}^*(E_R \cap U) + \mathcal{L}^*(E_R - U) \leq \mathcal{L}_*(E_R \cap U) + \mathcal{L}^*(E_R - U).$$

Since $\mathcal{L}^*(E_R - U) < \infty$, we conclude that

$$\mathcal{L}^*(E_R \cap U) \leq \mathcal{L}_*(E_R \cap U).$$

This shows $E_R \cap U$ is Lebesgue measurable.

The second step is to show that E_R is Caratheodory measurable for any $R \geq R_0$. Consider an arbitrary set $X \subset \mathbb{R}$ with $\mathcal{L}^*(X) < \infty$. Given any $\varepsilon > 0$, we take an open set $U \supset X$ such that

$$\mathcal{L}^*(X) + \varepsilon > l(U) = \mathcal{L}^*(U).$$

Consider $U = (U \cap E_R) \sqcup (U - E_R)$ and apply Lemma 6.26, we get

$$\mathcal{L}^*(U) \geq \mathcal{L}_*(U \cap E_R) + \mathcal{L}^*(U - E_R) = \mathcal{L}^*(U \cap E_R) + \mathcal{L}^*(U - E_R).$$

The last equality follows from the Lebesgue measurability of $U \cap E_R$. Recall that $U \supset X$, we have

$$\mathcal{L}^*(U \cap E_R) \geq \mathcal{L}^*(X \cap E_R) \ \text{ and } \ \mathcal{L}^*(U - E_R) \geq \mathcal{L}^*(X - E_R).$$

Combining the above results, we finally get:

$$\mathcal{L}^*(X) + \varepsilon > \mathcal{L}^*(X \cap E_R) + \mathcal{L}^*(X - E_R).$$

Letting $\varepsilon \to 0^+$, we conclude that

$$\mathcal{L}^*(X) \geq \mathcal{L}^*(X \cap E_R) + \mathcal{L}^*(X - E_R),$$

proving that E_R is Caratheodory measurable.

Finally, by writing

$$E = \bigcup_{n=0}^{\infty} \big(E \cap [-R_0 - n, R_0 + n]\big),$$

and the result that $E \cap [-R_0 - n, R_0 + n]$ is Caratheodory measurable for any integer $n \geq 0$, part (3) of Proposition 6.16 shows E is also Caratheodory measurable. □

Now that we know Lebesgue measurability and Caratheodory measurability are equivalent, so from now on we could apply both definition interchangeably. For simplicity, from now on we would only use the term "Lebesgue measurable" (a bit unfair to Caratheodory, though).

We have proved that open sets are Lebesgue measurable (using both definitions). Proposition 6.16 shows their complements are Lebesgue measurable too. Therefore, for any compact set K, we have

$$\mathcal{L}_*(K) = l(K) = \mathcal{L}^*(K).$$

6.2. Lebesgue measures on $\mathbb{R}^{n\geq 2}$

Next we move on to Lebesgue measures on $\mathbb{R}^n$ where $n \geq 2$. One reason that we single out the $n = 1$ case is because the fact that open set in $\mathbb{R}$ is a countable union of disjoint open intervals, but on $\mathbb{R}^{n\geq 2}$ open sets are not necessarily disjoint union of open balls. However, a replacement fact is that any open set in $\mathbb{R}^{n\geq 2}$ is a union of countably many "almost" disjoint cubes (see Stein's p7). Therefore, we will use cubes to estimate the volume of a set $E \subset \mathbb{R}^n$.

By a closed n-rectangle R we mean a set of the form

$$R = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n,$$

and the volume of R is defined to be:

$$|R| = (b_1 - a_1) \cdots (b_n - a_n).$$

When $b_1 - a_1 = \cdots = b_n - a_n$, we call the n-rectangle to be an n-cube.

Definition 6.27 (Lebesgue outer measure on $\mathbb{R}^n$). Given $E \subset \mathbb{R}^n$, the **n-dimensional Lebesgue outer measure of** E, denoted by $(\mathcal{L}^n)^*(E)$, is defined to be:

$$(\mathcal{L}^n)^*(E) := \inf \left\{ \sum_{j=1}^{\infty} |Q_j| : E \subset \bigcup_{j=1}^{\infty} Q_j \text{ where } Q_j\text{'s are closed } n\text{-cubes in } \mathbb{R}^n \right\}.$$

When the dimension n is clear from the context, we may simply write $\mathcal{L}^*(E)$.

Remark 6.28. When $n = 1$, the above definition is consistent with Definition 6.11, since closed "cubes" in $\mathbb{R}$ are simply closed intervals. Recall that on $\mathbb{R}$, the Lebesgue outer measure can also be equivalently defined as

$$\mathcal{L}^*(E) = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) : E \subset \bigcup_{i=1}^{\infty} [a_i, b_i] \right\}.$$

Remark 6.29. Lebesgue outer measures depend on the dimensions. The set $[0, 1]^2$ in $\mathbb{R}^2$ has 2-dimensional Lebesgue outer measure equal to 1 (it sounds trivially true but we will prove it later), but when we view it as $[0, 1] \times [0, 1] \times \{0\}$ in $\mathbb{R}^3$, it has 3-dimensional Lebesgue outer measure being 0.

Example 6.30. A one-point set $\{p\}$ in $\mathbb{R}^n$ has n-dimensional Lebesgue outer measure 0. To see this, given any $\varepsilon > 0$, we let Q_ε be the closed L^∞-ball centered at p with radius ε, then each Q_ε is a closed cube in $\mathbb{R}^n$ with $|Q_\varepsilon| = (2\varepsilon)^n$. By the definition of $\mathcal{L}^*$ we then have

$$\mathcal{L}^*(\{p\}) \leq |Q_\varepsilon| = 2^n \varepsilon^n \quad \forall \varepsilon > 0.$$

Letting $\varepsilon \to 0^*$, we get $\mathcal{L}^*(\{p\}) = 0$. A similar argument also shows $\mathcal{L}^*(\emptyset) = 0$.

□

Example 6.31. Any closed n-cube Q has $(\mathcal{L}^n)^* = |Q|$. It is obvious from the definition of $\mathcal{L}^*$ that $\mathcal{L}^*(Q) \leq |Q|$ as $Q \subset Q$. To prove the reverse inequality, we consider an arbitrary countable cover $\{Q_j\}_{j=1}^{\infty}$ of Q by closed n-cubes Q_j's. Given any $\varepsilon > 0$, we let $\mathcal{O}_j(\varepsilon)$ be the open L^∞-neighborhood of Q_j with a sufficiently small radius such that $|\mathcal{O}_j(\varepsilon)| < (1 + \varepsilon) |Q_j|$. Then, we have

$$Q \subset \bigcup_{j=1}^{\infty} Q_j \subset \bigcup_{j=1}^{\infty} \mathcal{O}_j(\varepsilon),$$

and by compactness of the closed cube Q, there exists $j_1, \cdots, j_k \in \mathbb{N}$ such that

$$Q \subset \mathcal{O}_{j_1}(\varepsilon) \cup \cdots \cup \mathcal{O}_{j_k}(\varepsilon) \subset \overline{\mathcal{O}_{j_1}(\varepsilon)} \cup \cdots \cup \overline{\mathcal{O}_{j_k}(\varepsilon)}.$$

Obviously, $\overline{\mathcal{O}_{j_i}(\varepsilon)}$'s are closed n-cubes with $\left|\overline{\mathcal{O}_{j_i}(\varepsilon)}\right| \leq (1+\varepsilon)\left|Q_{j_i}\right|$. By the finite sub-additivity of volume of rectangles (see Lemma 1.2 of Stein), we have

$$|Q| \leq \sum_{i=1}^{k} \left|\overline{\mathcal{O}_{j_i}(\varepsilon)}\right| \leq (1+\varepsilon)\sum_{i=1}^{k}\left|Q_{j_i}\right| \leq (1+\varepsilon)\sum_{j=1}^{\infty}\left|Q_j\right|.$$

Letting $\varepsilon \to 0^+$, we get

$$|Q| \leq \sum_{j=1}^{\infty}|Q_j|$$

Since $\{Q_j\}_{j=1}^{\infty}$ is taken to be any closed cube cover of Q, taking infimum over such covers we get:

$$|Q| \leq \inf\left\{\sum_{j=1}^{\infty}|Q_j| : Q \subset \bigcup_{j=1}^{\infty} Q_j \text{ where } Q_j\text{'s are closed } n\text{-cubes in } \mathbb{R}^n\right\} = \mathcal{L}^*(Q).$$

It completes the proof that $\mathcal{L}^*(Q) = |Q|$. □

Example 6.32. We have $(\mathcal{L}^n)^*(\mathbb{R}^n) = \infty$. To prove this, we take an arbitrary countable cover of $\mathbb{R}^n$ by closed cubes $\{Q_j\}_{j=1}^{\infty}$. Then we have for any $N \in \mathbb{N}$,

$$[0,N]^n \subset \mathbb{R}^n \subset \bigcup_{j=1}^{\infty} Q_j.$$

Hence $\{Q_j\}_{j=1}^{n}\infty$ is also a countable cover of $[0,N]^n$ by closed cubes. Note that $[0,N]^n$ is a closed cube so by Example 6.31 we have $\mathcal{L}^*\left([0,N]^n\right) = |[0,N]^n| = N^n$. By definition of Lebesgue measure, we then have

$$N^n = \mathcal{L}^*\left([0,N]^n\right) \leq \sum_{j=1}^{\infty}|Q_j|.$$

Since the cover $\{Q_j\}_{j=1}^{\infty}$ of $\mathbb{R}^n$ is arbitrary, taking infimum on both sides yields

$$N^n \leq \mathcal{L}^*(\mathbb{R}^n) \ \ \forall N \in \mathbb{N}.$$

Letting $N \to \infty$, we proved $\mathcal{L}^*(\mathbb{R}^n) = +\infty$. □

Although we used cube-coverings to define Lebesgue outer measures, we could use rectangle-coverings instead and get the same measure. For any $E \subset \mathbb{R}^n$, we denote:

$$\mu^*(E) := \inf\left\{\sum_{j=1}^{\infty}|R_j| : E \subset \bigcup_{j=1}^{\infty} R_j \text{ where } R_j\text{'s are closed } n\text{-rectangles}\right\}.$$

As cubes are also rectangles, it is obvious that $\mu^*(E) \leq (\mathcal{L}^n)^*(E)$. We need to show the reverse inequality. Let $\{R_j\}_{j=1}^{\infty}$ be an arbitrary rectangle-cover of E, we need to show $\sum_{j=1}^{\infty}|R_j|$ is bounded below by $\mathcal{L}^*(E)$, then taking infimum on both sides will give the desired result.

Given any $\varepsilon > 0$, and for each R_j, we let $\{C_{j,k}\}_{k=1}^{N_j}$ be a finite cover of R_j by cubes $C_{j,k}$, i.e. $R_j \subset \bigcup_{k=1}^{N_j} C_{j,k}$, such that

$$\sum_{k=1}^{N_j}|C_{j,k}| < |R_j| + \frac{\varepsilon}{2^j}.$$

It can be made possible by choosing sufficiently small size of cubes $C_{j,k}$'s. Then, we have

$$\sum_{j=1}^{\infty}\sum_{k=1}^{N_j}|C_{j,k}| \leq \sum_{j=1}^{\infty}\left(|R_j| + \frac{\varepsilon}{2^j}\right) = \sum_{j=1}^{\infty}|R_j| + \varepsilon.$$

Since the entire collection of $\{C_{j,k}\}$'s form a cube-covering of E, as

$$E \subset \bigcup_{j=1}^{\infty} R_j \subset \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{N_j} C_{j,k},$$

by the definition of Lebesgue measures we have

$$\mathcal{L}^*(E) \leq \sum_{j=1}^{\infty} \sum_{k=1}^{N_j} |C_{j,k}| \leq \sum_{j=1}^{\infty} |R_j| + \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, letting $\varepsilon \to 0^+$ and then taking infimum over all possible rectangle-coverings $\{R_j\}$ of E give $\mathcal{L}^*(E) \leq \mu^*(E)$.

From now on, when we consider estimate the measure of a given set, we may not restrict ourselves to cube-coverings, and rectangle-coverings can also be used.

Example 6.33. Nonetheless, although $\mathbb{R}^n \times \{0\}$ is "essentially" quite the same set as $\mathbb{R}^n$, its $(n+1)$-dimensional Lebesgue measure is 0. To prove this, we first cover $\mathbb{R}^n$ by countably many unit n-cubes $\{Q_j\}_{j=1}^{n}$. For any $\varepsilon > 0$ and $j \in \mathbb{N}$, we consider the $(n+1)$-rectangles $Q_j \times [-\frac{\varepsilon}{2^j}, \frac{\varepsilon}{2^j}]$, then $\{Q_j \times [-\frac{\varepsilon}{2^j}, \frac{\varepsilon}{2^j}]\}_{j=1}^{\infty}$ covers $\mathbb{R}^n \times \{0\}$, and so

$$\left(\mathcal{L}^{n+1}\right)^* (\mathbb{R}^n \times \{0\}) \leq \sum_{j=1}^{\infty} \left| Q_j \times \left[-\frac{\varepsilon}{2^j}, \frac{\varepsilon}{2^j}\right] \right| = \sum_{j=1}^{\infty} \underbrace{|Q_j|}_{=1} \cdot \frac{\varepsilon}{2^{j-1}} = 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\left(\mathcal{L}^{n+1}\right)^* (\mathbb{R}^n \times \{0\}) = 0$. □

Exercise 6.19. Suppose $E \subset \mathbb{R}^m$ is a set such that $(\mathcal{L}^m)^*(E) = 0$, show that $(\mathcal{L}^{n+m})^*(\mathbb{R}^n \times E) = 0$.

Exercise 6.20. Show that for any **open** n-cube Q in $\mathbb{R}^n$, we have $(\mathcal{L}^n)^*(Q) = \left|\overline{Q}\right|$. Note that not all argument in the closed cube case can be directly adopted. (Solution: see Stein p11)

Exercise 6.21. Show that for any closed n-rectangle R in $\mathbb{R}^n$, we have $(\mathcal{L}^n)^*(R) = |R|$. (Solution: see Stein p12).

Exercise 6.22. Show that if we replace closed n-cubes by open n-cubes in the definition of $(\mathcal{L}^n)^*$, and define for any $E \subset \mathbb{R}^n$ by

$$\mu^*(E) := \inf \left\{ \sum_{j=1}^{\infty} \left|\overline{Q_j}\right| : E \subset \bigcup_{j=1}^{\infty} Q_j \text{ where } Q_j\text{'s are open } n\text{-cubes in } \mathbb{R}^n \right\}.$$

Show that $\mu^* = (\mathcal{L}^n)^*$.

Proposition 6.34 (Monotoncity)**.** *Suppose $E_1 \subset E_2 \subset \mathbb{R}^n$, we have $\mathcal{L}^*(E_1) \leq \mathcal{L}^*(E_2)$.*

Proof. An immediate consequence of the property of infimum. □

Proposition 6.35 (Countable subadditivity)**.** *Suppose $E \subset \bigcup_{j=1}^{\infty} E_j$ where $E_j \subset \mathbb{R}^n$, then we have*

$$\mathcal{L}^*(E) \leq \sum_{j=1}^{\infty} \mathcal{L}^*(E_j).$$

Proof. The proof is almost identical to the $\mathbb{R}$ case, except that we use closed cubes instead of open intervals here.

Given any $\varepsilon > 0$, we want to show

$$\mathcal{L}^*(E) \leq \sum_{j=1}^{\infty} \mathcal{L}^*(E_j) + \varepsilon.$$

Then the result follows from letting $\varepsilon \to 0^+$.

To prove this, we consider any $j \in \mathbb{N}$, take a countable cover $\{Q_{j,k}\}_{k=1}^{\infty}$ of E_j by closed n-cubes such that

$$\mathcal{L}^*(E_j) \leq \sum_{k=1}^{\infty} |Q_{j,k}| < \mathcal{L}^*(E_j) + \frac{\varepsilon}{2^j}.$$

Then, $\{Q_{k,j}\}_{j,k=1}^{\infty}$ is a countable cover of $\bigcup_{j=1}^{\infty} E_j$ by closed n-cubes, and hence also a cover of E. By definition of Lebegues outer measures, we get

$$\mathcal{L}^*(E) \leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |Q_{j,k}| < \sum_{j=1}^{\infty} \left(\mathcal{L}^*(E_j) + \frac{\varepsilon}{2^j}\right) = \sum_{j=1}^{\infty} \mathcal{L}^*(E_j) + \varepsilon.$$

Letting $\varepsilon \to 0^+$, we complete the proof. □

Corollary 6.36. *Countable sets have Lebesgue outer measure* 0.

Exercise 6.23. Show that for any $E \subset \mathbb{R}^n$, we have

$$\mathcal{L}^*(E) = \inf\{\mathcal{L}^*(\mathcal{O}) : E \subset \mathcal{O} \text{ and } \mathcal{O} \text{ is open in } \mathbb{R}^n\}.$$

Hint: The $\frac{\varepsilon}{2^j}$-trick could be useful. For solution, see Stein p13.

In the previous section, we also introduced the Lebesgue inner measure on $\mathbb{R}$. Then a set $E \subset \mathbb{R}$ to said be Lebesgue measurable if $\mathcal{L}_*(E \cap [-R, R]) = \mathcal{L}^*(E \cap [-R, R])$ for sufficiently large R. We then proved that this definition is equivalent to the Caratheodory definition, that is

$$\mathcal{L}^*(X) = \mathcal{L}^*(X \cap E) + \mathcal{L}^*(X - E)$$

for any set $X \subset \mathbb{R}$. Although the Lebesgue's definition sounds more intuitive, the Caratheodory's definition was more useful since many results such as Proposition 6.16 were derived using the Caratheodory definition of measurable sets. From now on, we will seldom talk about Lebesgue inner measures, and we even define Lebesgue measurable set using Caratheodory's definition:

Definition 6.37 (Lebesgue measurable set)**.** A set $E \subset \mathbb{R}^n$ is said to be n**-dimensional Lebesgue measurable** if for any $X \subset \mathbb{R}^n$, we have

$$\mathcal{L}^*(X) = \mathcal{L}^*(X \cap E) + \mathcal{L}^*(X - E). \tag{6.2}$$

Here $\mathcal{L}^* := (\mathcal{L}^n)^*$, and when n is clear from the context, we simply set E is Lebesgue measurable. For such a set E, we may simply denote $\mathcal{L}^*(E)$ by $\mathcal{L}(E)$, or $\mathcal{L}^n(E)$ if the dimension n is important.

Remark 6.38. By countable sub-additivity, it suffices to argue that

$$\mathcal{L}^*(X) \geq \mathcal{L}^*(X \cap E) + \mathcal{L}^*(X - E)$$

when proving (6.2), since the reverse inequality always holds.

Example 6.39. Any set $E \subset \mathbb{R}^n$ with $\mathcal{L}^*(E) = 0$ is Lebesgue measurable: take any $X \subset \mathbb{R}^n$, we have

$$\begin{cases} X \cap E \subset E \\ X - E \subset X \end{cases} \implies \begin{cases} \mathcal{L}^*(X \cap E) \leq \mathcal{L}^*(E) = 0 \\ \mathcal{L}^*(X - E) \leq \mathcal{L}^*(X) \end{cases}.$$

These shows

$$\mathcal{L}^*(X \cap E) + \mathcal{L}^*(X - E) \leq \mathcal{L}^*(X).$$

This shows E is Lebesgue measurable. In particular, this shows any countable set is Lebesgue measurable. □

Exercise 6.24. Show that $\mathbb{R}^n$ is n-dimensional Lebesgue measurable.

Example 6.40. Let's look at a less-trivial example (although the result sounds trivial) – that rectangles are Lebesgue measurable sets. Note that we have only shown that $(\mathcal{L}^n)^*(R) = |R|$. This result does not imply R is Lebesgue measurable, as $X \cap R$ and $X - \mathbb{R}$ are not necessarily rectangles, so it is not trivial to see the Carathéodory criterion holds.

Let R be a closed n-rectangle, and $X \subset \mathbb{R}^n$ be any arbitrary subset and suppose $\mathcal{L}^*(X) < \infty$ (the infinite case is trivial). We need to prove

$$\mathcal{L}^*(X) \geq \mathcal{L}^*(X \cap R) + \mathcal{L}^*(X - R).$$

X may not be a rectangle, but we can approximate it by a rectangle cover. For any arbitrary $\varepsilon > 0$, we consider a rectangle cover $\bigcup_{j=1}^\infty R_j \supset X$ such that

$$\mathcal{L}^*(X) \leq \sum_{j=1}^\infty |R_j| < \mathcal{L}^*(X) + \varepsilon.$$

For each R_j, we consider $R_j \cap R$ and $R_j - R$. Obviously, $R'_j := R_j \cap R$ is a rectangle, and $R_j - R$ is a finite union of almost disjoint rectangles. Write $R_j - R = \bigcup_{k=1}^{N_j} R''_{j,k}$ where $R''_{j,k}$'s are rectangles. Since these rectangles are almost disjoint, we have

$$|R_j| = |R'_j| + \sum_{k=1}^{N_j} |R''_{j,k}|.$$

For a proof of this fact, see Stein Lemma 1.1 in p4. Then we have:

$$\begin{aligned}\mathcal{L}^*(X) + \varepsilon > \sum_{j=1}^\infty |R_j| &= \sum_{j=1}^\infty \left(|R'_j| + \sum_{k=1}^{N_j} |R''_{j,k}| \right) \\ &= \sum_{j=1}^\infty |R'_j| + \sum_{j=1}^\infty \sum_{k=1}^{N_j} |R''_{j,k}|.\end{aligned}$$

Since $\bigcup_{j=1}^\infty R'_j \supset X \cap R$, and $\bigcup_{j=1}^\infty \bigcup_{k=1}^{N_j} R''_{j,k} \supset X - R$, we have from the definition of Lebesgue measures:

$$\sum_{j=1}^\infty |R'_j| \geq \mathcal{L}^*(X \cap R) \text{ and } \sum_{j=1}^\infty \sum_{k=1}^{N_j} |R''_{j,k}| \geq \mathcal{L}^*(X - R).$$

This shows $\mathcal{L}^*(X) + \varepsilon > \mathcal{L}^*(X \cap R) + \mathcal{L}^*(X - R)$. Letting $\varepsilon \to 0^+$, we completes the proof. □

6.3. σ-algebra

Denote

$$\mathcal{M}_{\mathcal{L}^n} := \{E \subset \mathbb{R}^n : E \text{ is } n\text{-dimensional Lebesgue measurable}\}.$$

When the dimension n is clear from the context, we will simply write $\mathcal{M}$. We have just shown that $\emptyset, \mathbb{R}^n \in \mathcal{M}$. Just like on $\mathbb{R}$, we could argue that $\mathcal{M}$ is closed under complement and countable union. The proof is exactly the same as in the $\mathbb{R}$ case (Proposition 6.16), hence we omit it and leave it as an exercise for readers:

Proposition 6.41. *$\mathcal{M} := \mathcal{M}_{\mathcal{L}^n}$ satisfies the following properties:*

(1) $\emptyset \in \mathcal{M}$

(2) For any $E \in \mathcal{M}$, we have $E^c := \mathbb{R}^n - E \in \mathcal{M}$

(3) For any countable collection $\{E_j\}_{j=1}^\infty$ in $\mathcal{M}$, we have

$$\bigcup_{j=1}^\infty E_j \in \mathcal{M}.$$

Furthermore, for disjoint countable collection $\{E_j\}_{j=1}^\infty$ of Lebesgue measurable sets in $\mathbb{R}^n$, one can prove in exactly the same way as in Proposition 6.35 that they satisfy:

$$\mathcal{L}^n\left(\bigsqcup_{j=1}^\infty E_j\right) = \sum_{j=1}^\infty \mathcal{L}^n(E_j).$$

Properties (1), (2) and (3) in Proposition 6.41 motivates the definition of σ-algbera in measure theory:

Definition 6.42 (σ-algebra)**.** Given a set X (not necessarily $\mathbb{R}^n$), and a collection $\mathcal{A}$ of subsets of X, we call $\mathcal{A}$ is a **σ-algebra** if it satisfies all properties below:

(1) $\emptyset \in \mathcal{A}$

(2) $E \in \mathcal{A} \implies E^c := X - E \in \mathcal{A}$

(3) $E_1, E_2, \ldots \in \mathcal{A} \implies \bigcup_{j=1}^\infty E_j \in \mathcal{A}$

Although we only list three properties in the definition (for minimality), a number of consequences can be deduced by these three properties. We leave their proofs as an exercise for readers:

Exercise 6.25. Let X be a set and $\mathcal{A}$ be a σ-algebra of X. Show that

(1) $E_1, E_2, \ldots \in \mathcal{A} \implies \bigcap_{j=1}^\infty E_j \in \mathcal{A}$

(2) $E_1, E_2 \in \mathcal{A} \implies E_1 - E_2 \in \mathcal{A}$

$\mathcal{M}_{\mathcal{L}^n}$ is certainly a σ-algebra of $\mathbb{R}^n$. There are other examples too. Trivial ones are $\mathcal{A} = \mathcal{P}_X$ (the power set of X, which means the set of all subsets of X), and $\mathcal{A} = \{\emptyset, X\}$. A less trivial one is the **Borel algebra** of $\mathbb{R}^n$, which is the smallest σ-algebra containing all open sets in $\mathbb{R}^n$, i.e.

$$\mathcal{B} = \bigcap_{\mathcal{T}_{\mathbb{R}^n} \subset \mathcal{A}} \mathcal{A}$$

where $\mathcal{T}_{\mathbb{R}^n}$ denotes the standard topology of open sets in $\mathbb{R}^n$. The countable union and intersection of open sets in $\mathbb{R}^n$ are contained in $\mathcal{B}$, and so do the countable union and intersection of closed sets. The collection $\mathcal{B}$ contains many other sets as well, such as the collection $F_{\sigma\delta}$, which means the countable intersection of countable unions of closed sets, and $G_{\delta\sigma}$, the countable union of countable intersections of open sets, etc. It can be shown later using the Cantor function that $\mathcal{B}$ is a proper subset of $\mathcal{M}$. Sets in $\mathcal{B}$ are called **Borel sets**.

Proposition 6.43. *Given any $\varepsilon > 0$, and any $E \in \mathcal{M}_{\mathcal{L}^n}$ in $\mathbb{R}^n$, there exists an open set $\mathcal{O}$ containing E such that $\mathcal{L}^n(\mathcal{O} - E) < \varepsilon$.*

Proof. From Exericse 6.23, we have

$$\mathcal{L}^n(E) = \inf\{\mathcal{L}^n(\mathcal{O}) : E \subset \mathcal{O} \text{ where } \mathcal{O} \text{ is open}\}.$$

Therefore for any $\varepsilon > 0$, one can find $\mathcal{O}_\varepsilon$ such that

$$\mathcal{L}^n(E) \leq \mathcal{L}^n(\mathcal{O}) < \mathcal{L}^n(E) + \varepsilon.$$

By finite additivity (Lemma 6.19), we know

$$\mathcal{L}^n(\mathcal{O}) = \mathcal{L}^n((\mathcal{O} - E) \sqcup E) = \mathcal{L}^n(\mathcal{O} - E) + \mathcal{L}^n(E).$$

Therefore, we have $\mathcal{L}^n(\mathcal{O} - E) = \mathcal{L}^n(\mathcal{O}) - \mathcal{L}^n(\mathcal{E}) < \varepsilon$. □

Exercise 6.26. In Stein's book, Lebesgue measurable sets are defined as follows:

> "$E \subset \mathbb{R}^n$ is said to be n-dimensional Lebesgue measurable if for any $\varepsilon > 0$, there exists an open set $\mathcal{O} \subset \mathbb{R}^n$ containing E such that $(\mathcal{L}^n)^*(\mathcal{O} - E) \leq \varepsilon$."

It turns out that it is equivalent to our definition. Denote by $\mathcal{M}_S$ the collection of Lebesgue measurable sets in the sense of Stein's book. We have already proved in the above proposition that $\mathcal{M}_{\mathcal{L}^n} \subset \mathcal{M}_S$. Complete the proof of $\mathcal{M}_S \subset \mathcal{M}_{\mathcal{L}^n}$.

6.4. Abstract measure spaces

One important property about the Lebesgue measure is the **countable additivity**, which means that the Lebesgue measure of a countable **disjoint** union of measurable sets is simply the sum of the measures of the sets. In Lemma 6.19, we have only proved **finite** additivity, whereas in Proposition 6.35 we have only proved countable **sub**-additivity (which also holds for non-measurable sets).

We want to come up with a general notion of "measures" on X that share some of the good properties of Lebesgue measures such as countable additivity, monotonicity, etc. This leads to the definition of (abstract) measure spaces:

Definition 6.44 (Measure space)**.** Let X be a set, $\mathcal{A}$ be a σ-algebra of X, and $\mu : \mathcal{A} \to [0, \infty]$ be a function. The triple $(X, \mathcal{A}, \mu)$ is said to be a **measure space** if they satisfy

(1) $\mu(\emptyset) = 0$, and

(2) for any disjoint countable collection of sets $\{E_j\}_{j=1}^{\infty}$, where $E_j \in \mathcal{A} \ \ \forall j \in \mathbb{N}$, we have

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j).$$

Remark 6.45. (1) and (2) together imply a number of consequences, including:

(i) finite additivity of μ

(ii) $\mu(B - A) = \mu(B) - \mu(A)$ whenever $A \subset B$ and $A, B \in \mathcal{A}$.

(iii) monotonicity of μ

(iv) countable sub-additivity

Exercise 6.27. Prove all listed properties in the above remark using only (1) and (2) in the definition of a measure space.

$(\mathbb{R}^n, \mathcal{M}_{\mathcal{L}^n}, \mathcal{L}^n)$ is an example of a measure space. Let's also look at some other examples.

Example 6.46 (Counting measure)**.** Let X be any set, and for any subset $E \subset X$, we denote $\#(E)$ be the cardinality of E, which is equal to ∞ when E is an infinite set. Then $(X, \mathcal{P}_X, \#)$ is a measure space. □

Example 6.47 (Dirac delta measure)**.** Let X be a non-empty set, and fix $a \in X$. For any $E \subset X$, we define

$$\delta_a(E) = \begin{cases} 1 & \text{if } a \in E \\ 0 & \text{if } a \notin E \end{cases}.$$

Then, $(X, \mathcal{P}_X, \delta_a)$ is a measure space. This is trivial to verify (1) $\delta_a(\emptyset) = 0$. For (2), note that if $\{E_j\}_{j=1}^{\infty}$'s are pairwise disjoint subsets of X, then a is in at most one of the E_j's. If a is in exactly one of the E_j's, then both sides of (2) equal 1. If a is not in any E_j's, then both sides of (2) are 0. This shows (2) holds. □

In the next chapter, we will introduce Hausdorff measure on any metric space (X, d). It would give a less trivial example of a measure space. This measure is a fundamental notion in geometric measure theory. In MATH 1024, we introduced the "Jordan measure" on $\mathbb{R}^2$ as a motivational concept to Riemann integrals. The difference between Jordan measure and Lebesgue measure is that the former requires the covering to be **finite** union of rectangles, whereas the latter allows a **countable** union of rectangles/cubes. Note that the Jordan measure is **not** countable additive (even worse, $\mathbb{Q}^2$ is not Jordan measurable). Therefore, the space $\mathbb{R}^2$ equipped with the collection $\mathcal{J}$ of Jordan measurable sets, and the Jordan measure μ do not form a measure space. The collection $\mathcal{J}$ is not even an σ-algebra. Although we call it Jordan "measure", it is not formally a measure (c.f. jellyfish is not a fish).

The following results are known to be hold for the Lebesgue measure on $\mathbb{R}$ (see Exercises 6.17 and 6.18. The proof only uses the countable additivity of Lebesgue measures. Now that the countable additivity holds for any measure space, we can generalize the results below to any measure space with the same proof. We leave the proof as an exercise for readers.

Proposition 6.48. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Then,*

(a) If $\{E_j\}_{j=1}^\infty$ are sets in $\mathcal{A}$ such that $E_1 \subset E_2 \subset E_3 \subset \cdots$, then we have

$$\mu\left(\bigcup_{j=1}^\infty E_j\right) = \lim_{j\to\infty} \mu(E_j).$$

(b) If $\{E_j\}_{j=1}^\infty$ are sets in $\mathcal{A}$ such that $E_1 \supset E_2 \supset E_3 \supset \cdots$ and $\mu(E_1) < \infty$, then

$$\mu\left(\bigcap_{j=1}^\infty E_j\right) = \lim_{j\to\infty} \mu(E_j).$$

Exercise 6.28. Prove Proposition 6.48. Hint: replace E_j's by E_j''s such that $\{E_j'\}_{j=1}^\infty$ are disjoint and that $\cup_j E_j = \cup_j E_j'$ for (a). Similar for (b).

An immediate application of Proposition 6.48 is the Borel-Cantelli's trick (or Borel-Cantelli's Lemma in many probability textbooks):

Corollary 6.49 (Borel-Cantelli's trick)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space. Suppose $\{E_j\}_{j=1}^\infty$ are subsets of X in $\mathcal{A}$ such that*

$$\sum_{j=1}^\infty \mu(E_j) < \infty,$$

then we have

$$\mu\left(\bigcap_{i=1}^\infty \bigcup_{j=i}^\infty E_j\right) = 0.$$

Proof. Denote $F_i := \bigcup_{j=i}^\infty E_j$, then clearly we have $F_1 \supset F_2 \supset F_3 \supset \cdots$. Also,

$$\mu(F_1) = \mu\left(\bigcup_{j=1}^\infty E_j\right) \leq \sum_{j=1}^\infty \mu(E_j) < \infty$$

where we have used the countable sub-additivity of μ. Hence, one can invoke Proposition 6.48 and show

$$\mu\left(\bigcap_{i=1}^\infty \bigcup_{j=i}^\infty E_j\right) = \mu\left(\bigcap_{i=1}^\infty F_i\right) = \lim_{i\to\infty} \mu(F_i).$$

By countable sub-additivity again, one can also deduce that

$$\mu(F_i) \leq \sum_{j=i}^{\infty} \mu(E_j).$$

By the finiteness of $\sum_{j=1}^{\infty} \mu(E_j)$, we have $\lim\limits_{i\to\infty} \sum\limits_{j=i}^{\infty} \mu(E_j) = 0$, and hence $\lim\limits_{i\to\infty} \mu(F_i) = 0$, completing the proof. □

6.5. Non-measurable sets

Most of the sets in $\mathbb{R}^n$ we encounter in "daily" life are Lebesgue measurable. It is in fact very hard to construct one that isn't, and the construction uses the Axiom of Choice.

First define an equivalent relation $\sim$ on $[0,1]$ by declaring that $x \sim y$ if and only if $x - y \in \mathbb{Q}$. Then, $[0,1]$ can be partitioned according to uncountably many equivalent classes $[x] := \{y \in [0,1] : y - x \in \mathbb{Q}\}$:

$$[0,1] = \bigsqcup_{x \in \mathcal{I}} [x].$$

Here $\mathcal{I}$ is the index set. For each equivalent class $[x]$, we pick a representative $\alpha_x \in [x]$, and form a new set $\mathcal{N}$ by putting in all these α_x, i.e.

$$\mathcal{N} := \{\alpha_x : x \in \mathcal{I}\}.$$

This is where we have used the Axiom of Choice. Next, we enumerate the rational numbers in $[-1,1]$ by $\mathbb{Q} \cap [-1,1] = \{r_1, r_2, \ldots\}$, we construct a new set

$$\mathcal{S} := \bigcup_{k=1}^{\infty} (\mathcal{N} + r_k)$$

and we claim that at least one of the set $\mathcal{N} + r_k$ is not Lebesgue measurable.

We prove by contradiction. Suppose $\mathcal{N} + r_k \in \mathcal{M}_{\mathcal{L}^1}$ for all $k \in \mathbb{N}$. It is easy to see that the collection $\{\mathcal{N} + r_k\}_{k=1}^{\infty}$ is disjoint, so by countable additivity, we have

$$\mathcal{L}(\mathcal{S}) = \sum_{k=1}^{\infty} \mathcal{L}(\mathcal{N} + r_k) = \sum_{k=1}^{\infty} \mathcal{L}(\mathcal{N}).$$

Note that we have used the fact that Lebesgue measures are translational invariant because the volume functional of cubes is so. This shows $\mathcal{L}(S)$ is either 0 if $\mathcal{L}(\mathcal{N}) = 0$, or $\mathcal{L}(\mathcal{S}) = \infty$ if $\mathcal{L}(\mathcal{N}) > 0$. However, we argue that either case will not happen because we have

$$[0,1] \subset \mathcal{S} \subset [-1,2] \implies 1 \leq \mathcal{L}(\mathcal{S}) \leq 3.$$

To see this, we consider any $\beta \in [0,1]$, there exists $x_\beta \in \mathcal{I}$ such that $\beta \in [x_\beta] = [\alpha_{x_\beta}]$, i.e. there exists $r \in \mathbb{Q}$ such that $\beta = \alpha_{x_\beta} + r$. Such an r must be in the range $[-1,1]$ as $\beta \in [0,1]$ and $\alpha_{x_\beta} \in [0,1]$, and hence it must be one of the r_k's. This shows $\beta \in \mathcal{S}$, proving $[0,1] \subset \mathcal{S}$. The other inclusion $\mathcal{S} \subset [-1,2]$ easily follows from $\mathcal{N} + r_k \subset [0,1] + [-1,1]$.

Exercise 6.29. Construct a Lebesgue non-measurable set in $\mathbb{R}^{n \geq 2}$.

Exercise 6.30. Show that there exist disjoint sets E_1 and E_2 in $\mathbb{R}$ such that

$$\mathcal{L}^*(E_1 \cup E_2) \neq \mathcal{L}^*(E_1) + \mathcal{L}^*(E_2).$$

Hint: Proof by contradiction. Suppose it is not true, then show $\mathcal{L}^*$ would satisfy countable additivity on any sets in $\mathbb{R}$, then derive a contradiction.

Chapter 7

Hausdorff Measures

> "Measure what is measurable, and make measurable what is not so."
>
> Galileo Galilei

7.1. Definition of Hausdorff measure

The Hausdroff measure is an important measure in geometric measure theory and fractal analysis. It is a more general type of measure that allows the **dimension** to be any positive real number. It can also be defined on any metric space.

Definition 7.1 (Hausdorff measure). Let (X, d) be a metric space, and $s > 0$. The s**-dimensional Hausdorff measure** of any given subset $E \subset X$ is defined to be:

$$\mathcal{H}^s(E) := \sup_{\delta > 0} \underbrace{\inf \left\{ \sum_{j=1}^{\infty} (\operatorname{diam} C_j)^s : E \subset \bigcup_{j=1}^{\infty} C_j, \text{ where } C_j \subset X \text{ and } \operatorname{diam} C_j < \delta \ \ \forall j \right\}}_{=: \mathcal{H}^s_\delta(E)}.$$

Remark 7.2. Note that $\mathcal{H}^s_\delta(E)$ increases as δ decreases. Therefore we also have

$$\mathcal{H}^s(E) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(E).$$

Recall that we use n-cubes as the covering sets for the n-dimensional Lebesgue measure. Now for Hausdorff measures, we are allowed to use any countable cover of *any* sets in X. Now $(\operatorname{diam} C_j)^s$ plays the role of volume of n-cubes. When $s = n \in \mathbb{N}$ and C_j's are all n-cubes with volume V_j, then

$$(\operatorname{diam} C_j)^s = (\sqrt{n} V_j^{\frac{1}{n}})^n = n^{\frac{1}{2n}} V_j.$$

Hence, the quantity $\sum_j (\operatorname{diam} C_j)^s$ is essentially the generalization of $\sum_j |Q_j|$ modulo the dimensional factor $n^{1/2n}$. There are some technical reasons (to be discussed later) why we do not define the Hausdorff measure simply as:

$$\inf \left\{ \sum_{j=1}^{\infty} (\operatorname{diam} C_j)^s : E \subset \bigcup_{j=1}^{\infty} C_j, \text{ where } C_j \subset X \text{ for all } j \right\}.$$

Example 7.3. We are going to prove that $\mathcal{H}^1(E) = (\mathcal{L}^1)^*(E)$ for any $E \subset \mathbb{R}$. The result is also true for $\mathcal{H}^n$ and $(\mathcal{L}^n)^*$ where $n \geq 2$, but the proof is more technical and we will postpone it. We consider

$$\begin{aligned}
\mathcal{L}^1(E) &= \inf\left\{\sum_{j=1}^{\infty}(b_j - a_j) : E \subset \bigcup_{j=1}^{\infty}(a_j, b_j)\right\} \\
&\leq \inf\left\{\sum_{j=1}^{\infty}(b_j - a_j) : E \subset \bigcup_{j=1}^{\infty}(a_j, b_j) \text{ and } b_j - a_j < \delta \ \ \forall j\right\} \\
&= \inf\left\{\sum_{j=1}^{\infty} \operatorname{diam} U_j : E \subset \bigcup_{j=1}^{\infty} U_j,\ U_j \text{ is open and } \operatorname{diam} U_j < \delta \ \ \forall j\right\}
\end{aligned}$$

The last equality follows from the fact that each open set U_j in $\mathbb{R}$ is a countable disjoint union of open intervals with diameters less than that of U_j. To proceed, we consider any countable covering $\{C_j\}_{j=1}^{\infty}$ of E with $\operatorname{diam} C_j < \frac{\delta}{2}$ for all j. For any $\varepsilon > 0$ and $j \in \mathbb{N}$, we consider the $\frac{\varepsilon}{2^j}$-neighborhoods of C_j defined by:

$$V_j := \left\{x \in \mathbb{R} : |x - y| < \frac{\varepsilon}{2^j} \text{ for some } y \in C_j\right\},$$

then we have $E \subset \bigcup_{j=1}^{\infty} C_j \subset \bigcup_{j=1}^{\infty} V_j$, V_j is open. Also, since C_j's are subsets of $\mathbb{R}$, so $\operatorname{diam} C_j = \sup C_j - \inf C_j$ and $\operatorname{diam} V_j = \operatorname{diam} C_j + 2 \cdot \frac{\varepsilon}{2^j}$, which is less than δ whenever $\varepsilon < \frac{\delta}{2}$. Continuing on the above inequality, we have

$$\begin{aligned}
\mathcal{L}^1(E) &\leq \inf\left\{\sum_{j=1}^{\infty} \operatorname{diam} U_j : E \subset \bigcup_{j=1}^{\infty} U_j,\ U_j \text{ is open and } \operatorname{diam} U_j < \delta \ \ \forall j\right\} \\
&\leq \sum_{j=1}^{\infty} \operatorname{diam} V_j = \sum_{j=1}^{\infty}\left(\operatorname{diam} C_j + \frac{\varepsilon}{2^{j-1}}\right) = \sum_{j=1}^{\infty} \operatorname{diam} C_j + 2\varepsilon
\end{aligned}$$

for any $\varepsilon \in (0, \frac{\delta}{2})$. Letting $\varepsilon 0^{\cdot} \quad +$, we proved

$$\mathcal{L}^1(E) \leq \sum_{j=1}^{\infty} \operatorname{diam} C_j.$$

As $\{C_j\}_{j=1}^{\infty}$ is an arbitrary countable cover with diameter less than $\frac{\delta}{2}$, taking infimum over all such covers gives:

$$\mathcal{L}^1(E) \leq \mathcal{H}^1_{\frac{\delta}{2}}(E) \ \ \forall \delta > 0.$$

Letting $\delta \to 0^+$ we proved $\mathcal{L}^1(E) \leq \mathcal{H}^1(E)$.

To prove the reverse inequality, we note that any finite open interval (a, b) can be divided into finitely many disjoint intervals $\{I_j\}$ with (not necessarily open) of diameter less than δ and $\sum_j \text{diam} I_j = b - a$. Therefore, we have for any $\delta > 0$,

$$\begin{aligned}
\mathcal{L}^1(E) &= \inf \left\{ \sum_{j=1}^{\infty} (b_j - a_j) : E \subset \bigcup_{j=1}^{\infty} (a_j, b_j) \right\} \\
&= \inf \left\{ \sum_{j=1}^{\infty} \text{diam}\, I_j : E \subset \bigcup_{j=1}^{\infty} I_j \text{ where } I_j\text{'s are intervals with } \text{diam}\, I_j < \delta \right\} \\
&\geq \inf \left\{ \sum_{j=1}^{\infty} \text{diam}\, C_j : E \subset \bigcup_{j=1}^{\infty} C_j \text{ where } I_j\text{'s are subsets of } \mathbb{R} \text{ with } \text{diam}\, C_j < \delta \right\} \\
&= \mathcal{H}^1_\delta(E).
\end{aligned}$$

Letting $\delta \to 0^+$, we get $\mathcal{L}^1(E) \geq \mathcal{H}^1(E)$, completing the proof. □

The following properties can be easily deduced from the definition of $\mathcal{H}^s$. We leave the proofs as exercises for readers:

Proposition 7.4. *Let $\mathcal{H}^s$, $s > 0$, be the Hausdorff measure $\mathcal{H}^s$ on $\mathbb{R}^n$. Then,*

(1) For any $\lambda > 0$ and $E \subset \mathbb{R}^n$, we have $\mathcal{H}^s(\lambda E) = \lambda^s \mathcal{H}^s(E)$.

(2) For any distance-preserving map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, i.e. $|\Phi(x) - \Phi(y)| = |x - y|$ for any $x, y \in \mathbb{R}^n$, we have $\mathcal{H}^s(E) = \mathcal{H}^s(\Phi(E))$ for any $E \subset \mathbb{R}^n$.

(3) Suppose $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is α-Hölder continuous, i.e. there exists $M > 0$ such that

$$|\Phi(x) - \Phi(y)| \leq M\, |x - y|^\alpha \quad \forall x, y \in \mathbb{R}^n,$$

then $\mathcal{H}^{\frac{s}{\alpha}}(\Phi(E)) \leq M^{\frac{s}{\alpha}} \mathcal{H}^s(E)$ for any $E \subset \mathbb{R}^n$.

Next we show that Hausdorff measures share some similar properties as Lebesgue outer measures.

Proposition 7.5. *Let (X, d) be a metric space. For any $s > 0$, the Hausdorff measure $\mathcal{H}^s$ satisfies the following properties:*

(1) $\mathcal{H}^s(\emptyset) = 0$,

(2) (monotonicity) whenever $E_1 \subset E_2 \subset X$, we have $\mathcal{H}^s(E_1) \leq \mathcal{H}^s(E_2)$, and

(3) (countable sub-additivity) for any $E_1, E_2, \ldots \subset X$, we have

$$\mathcal{H}^s\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mathcal{H}^s(E_j).$$

Proof. (1) is trivial by the fact that $\text{diam}\,(\emptyset) = 0$, so one could simply use $\{\emptyset\}$ as a cover of $\emptyset$ and one could immediately conclude that $\mathcal{H}^s_\delta(\emptyset) = 0$ for any $\delta > 0$, and so $\mathcal{H}^s(\emptyset) = \sup_{\delta > 0} \mathcal{H}^s_\delta(\emptyset) = 0$.

For (2), it is easy to see that $\mathcal{H}^s_\delta(E_1) \leq \mathcal{H}^s_\delta(E_2)$ from the definition of infimum. Letting $\delta \to 0^+$ yields the desired result.

(3) is again proved by the $\frac{\varepsilon}{2^j}$-trick. We first prove that $\mathcal{H}^s_\delta$ satisfies the countable sub-additivity for any $\delta > 0$. The proof is similar to that of Lebesgue measure. Given any $\varepsilon > 0$ and $j \in \mathbb{N}$, we consider a collection of sets $\{C_{j,k}\}_{k=1}^{\infty}$ in X with $\text{diam}\, C_{j,k} < \delta$ such

that $E_j \subset \bigcup_{k=1}^{\infty} C_{j,k}$, and

$$\mathcal{H}_\delta^s(E_j) \leq \sum_{k=1}^{\infty} (\operatorname{diam} C_{j,k})^s < \mathcal{H}_\delta^s(E_j) + \frac{\varepsilon}{2^j}.$$

Then, $\{C_{j,k}\}_{j,k=1}^{\infty}$ is a collection of sets in X that covers $\bigcup_{j=1}^{\infty} E_j$, and so by the definition of $\mathcal{H}_\delta^s$, we have

$$\mathcal{H}_\delta^s\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} (\operatorname{diam} C_{j,k})^s < \sum_{j=1}^{\infty}\left(\mathcal{H}_\delta^s(E_j) + \frac{\varepsilon}{2^j}\right) = \sum_{j=1}^{\infty} H_\delta^s(E_j) + 2\varepsilon.$$

Letting $\varepsilon \to 0^+$, we conclude that $\mathcal{H}_\delta^s$ satisfies countable sub-additivity for any $\delta > 0$. To extend the result to $\mathcal{H}^s$, we note that for any $\delta > 0$,

$$\mathcal{H}_\delta^s\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mathcal{H}_\delta^s(E_j) \leq \sum_{j=1}^{\infty} \sup_{\delta>0} \mathcal{H}_\delta^s(E_j) = \sum_{j=1}^{\infty} \mathcal{H}^s(E_j).$$

Letting $\delta \to 0^+$, we conclude that $\mathcal{H}^s$ satisfies countable sub-additivity too. $\square$

Similar to Lebesgue measures, we define Hausdorff measurable sets using the Carathéodory's criterion.

Definition 7.6 (Hausdorff measurable sets)**.** Let (X, d) be a metric space, and $\mathcal{H}^s$ be its s-dimensional Hausdorff measure (where $s > 0$). We say a set $E \subset X$ is **s-dimensional Hausdorff measurable** (or simply **Hausdorff measurable** when the dimension is clear from the context) if for any subset $S \subset X$, we have

$$\mathcal{H}^s(S) = \mathcal{H}^s(S \cap E) + \mathcal{H}^s(S - E).$$

We denote the collection of s-dimensional Hausdorff measurable sets by $\mathcal{M}_{\mathcal{H}^s}$.

7.2. Outer measures

Before we proceed to prove more properties about the Hausdorff measure, we define a general notion called **outer measure**, which is not necessarily referring to a specific type of measures like Lebesgue outer measure, but a general class of measures satisfying the three properties (1), (2) and (3) in Proposition 7.5.

Definition 7.7 (Outer measure)**.** Let X be a set, and $\mu : \mathcal{P}_X \to [0, \infty]$ is a function of subsets in X. We say that μ^* is an **(abstract) outer measure of** X if it satisfies

(1) $\mu^*(\emptyset) = 0$

(2) (monotonicity) whenever $E_1 \subset E_2 \subset X$, we have $\mu^*(E_1) \leq \mu^*(E_2)$, and

(3) (countable sub-additivity) for any $E_1, E_2, \ldots \subset X$, we have

$$\mu^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu^*(E_j).$$

We call E to be μ**-measurable** if for any $S \subset X$, we have

$$\mu^*(S) = \mu^*(S \cap E) + \mu^*(S - E).$$

We denote the collection of μ-measurable sets by $\mathcal{M}_\mu$, and if $E \in \mathcal{M}_\mu$, we may simply write $\mu^*(E)$ as $\mu(E)$. In other words, $\mu := \mu^*|_{\mathcal{M}_\mu}$.

Example 7.8. n-dimensional Lebesgue outer measure $(\mathcal{L}^n)^*$ where $n \in \mathbb{N}$, the s-dimensional Hausdorff measure $\mathcal{H}^s$ where $s > 0$, and also $\mathcal{H}^s_\delta$ for any $s, \delta > 0$ are examples of outer measures. □

Outer measures can be used to generate new measure spaces, as we have the following results whose proof is almost identical to the corresponding results for Lebesgue measures.

Proposition 7.9. *Suppose μ^* is an outer measure on a set X, and $\mathcal{M}_\mu$ is the collection of μ-measurable sets. Then, the triple $(X, \mathcal{M}_\mu, \mu)$ is a measure space.*

Idea of proof. To prove that $\mathcal{M}_\mu$ forms a σ-algebra and that μ is countable additivity, we could simply modify the proofs of Proposition 6.41 and Proposition 6.23 by relabelling all $\mathcal{L}^n$ by μ. The proofs of Proposition 6.41 and Proposition 6.23 used monotonicity and countable sub-additivity of $(\mathcal{L}^n)^*$, and also the Carathéodory's criterion for sets in $\mathcal{M}_{\mathcal{L}^n}$. All of these hold for μ^* under our assumptions. □

Corollary 7.10. *For any metric space (X, d) and $s, \delta > 0$, the spaces $(X, \mathcal{M}_{\mathcal{H}^s}, \mathcal{H}^s)$ and $(X, \mathcal{M}_{\mathcal{H}^s_\delta}, \mathcal{H}^s_\delta)$ are measure spaces.*

Next we introduce a very useful criterion that can be used to show all Borel sets are μ-measurable, where μ^* is an outer measure on a metric space (X, d), by checking an additivity condition.

Proposition 7.11. *Let (X, d) be a metric space and μ^* is an outer measure on X. Suppose for any $A, B \subset X$ with $d(A, B) > 0$, we have*

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B).$$

Then, all Borel sets in (X, d) are μ-measurable.

Proof. It suffices to show $\mathcal{M}_\mu$ contains all closed sets in (X,d), since $\mathcal{M}_\mu$ is a σ-algebra. Given a closed set E in (X,d), we need to prove that for any $S \subset X$, we have

$$\mu^*(S) \geq \mu^*(S \cap E) + \mu^*(S - E). \tag{7.1}$$

When $\mu^*(S) = \infty$, (7.1) trivially holds. From now we assume that $\mu^*(S) < \infty$. Note that $S \cap E$ and $S - E$ are disjoint, but they may not have positive distance and so the given condition cannot be applied. To tackle this issue, we introduce some "spacing" between them. We let

$$E_j := \left\{ x \in X : d(x,E) \leq \frac{1}{j} \right\}, \quad j \in \mathbb{N}$$

and consider $S \cap E$ and $S - E_j$ instead. Note that $d(S \cap E, S - E_j) \geq \frac{1}{j} > 0$. By the given condition we have

$$\mu^*((S \cap E) \cup (S - E_j)) = \mu^*(S \cap E) + \mu^*(S - E_j).$$

Observing that $S = (S \cap E) \cup (S - E) \supset (S \cap E) \cup (S - E_j)$, we have

$$\mu^*(S) \geq \mu^*(S \cap E) + \mu^*(S - E_j). \tag{7.2}$$

It remains to show that $\mu^*(S - E_j) \to \mu^*(S - E)$ as $j \to \infty$. To prove this, we decompose the "gap" between $S - E$ and $S - E_j$ by a sequence of "ring" regions. Let

$$R_j := \left\{ x \in X : \frac{1}{j+1} < d(x,E) \leq \frac{1}{j} \right\}, \quad j \in \mathbb{N}.$$

Then, we have for any $j \in \mathbb{N}$,

$$S - E_j \subset S - E = (S - E_j) \cup \bigcup_{k=j}^{\infty} (S \cap R_k),$$

and so by countable sub-additivity, we have

$$\mu^*(S - E_j) \leq \mu^*(S - E) \leq \mu^*(S - E_j) + \sum_{k=j}^{\infty} \mu^*(S \cap R_k). \tag{7.3}$$

We are left to prove that $\sum_{k=1}^{\infty} \mu^*(S \cap R_k) < \infty$.

Since $d(S \cap R_{2k}, S \cap R_{2k+2}) > 0$ and $d(S \cap R_{2k-1}, S \cap R_{2k+1}) > 0$, we have for any $N \in \mathbb{N}$

$$\sum_{k=1}^{N} \mu^*(S \cap R_{2k}) = \mu^* \left(\bigcup_{k=1}^{N} S \cap R_{2k} \right) \leq \mu^*(S) < \infty,$$

and similarly, $\displaystyle\sum_{k=1}^{N} \mu^*(S \cap R_{2k-1}) \leq \mu^*(S) < \infty$. This shows $\sum_{k=1}^{\infty} \mu^*(S \cap R_k)$ converges , and so

$$\lim_{j \to \infty} \sum_{k=j}^{\infty} \mu^*(S \cap R_k).$$

Letting $j \to \infty$ in (7.3), we show $\mu^*(S - E_j) \to \mu^*(S - E)$ as $j \to \infty$. By applying this on (7.2) shows (7.1), completing the proof. $\square$

Back to our discussion of Hausdorff measures. One can prove that $\mathcal{H}^s$ satisfies the condition in Proposition 7.11, and hence all Borel sets are Hausdorff measurable.

Remark 7.12. Some authors call the condition in Proposition 7.11 to be **Carathéodory's criterion**. Try to avoid mixing it up with another Carathéodory's criterion (6.2).

Proposition 7.13. *Let (X,d) be a metric space and $s > 0$. All Borel sets in (X,d) are $\mathcal{H}^s$-measurable.*

Proof. From Proposition 7.11, it suffices to show that for any $A, B \subset X$ with $d(A,B) > 0$, we have

$$\mathcal{H}^s(A \cup B) \geq \mathcal{H}^s(A) + \mathcal{H}^s(B).$$

Note that the reverse inequality holds by the sub-additivity of $\mathcal{H}^s$. We assume that $\mathcal{H}^s(A \cup B) < \infty$ otherwise we have nothing to prove.

Let $\delta_0 := \frac{1}{3}d(A,B) > 0$. For any $\delta \in (0, \delta_0)$, we consider an arbitrary countable cover $\{C_j\}_{j=1}^\infty$ of $A \cup B$ such that $\operatorname{diam}(C_j) < \delta < \frac{1}{3}d(A,B)$ for any j. Note that each C_j intersects at most one of A and B, so we have

$$\sum_{C_j \cap A \neq \emptyset} (\operatorname{diam} C_j)^s + \sum_{C_j \cap B \neq \emptyset} (\operatorname{diam} C_j)^s \leq \sum_{j=1}^\infty (\operatorname{diam} C_j)^s.$$

We use "$\leq$" above because there may be some C_j's not intersecting either A or B. Since $\{C_j : C_j \cap A \neq \emptyset\}$ is a countable cover of A and $\operatorname{diam} C_j < \delta$, we have

$$\mathcal{H}^s_\delta(A) \leq \sum_{C_j \cap A \neq \emptyset} (\operatorname{diam} C_j)^s,$$

and similarly for $\mathcal{H}^s_\delta(B)$. These show

$$\mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B) \leq \sum_{j=1}^\infty (\operatorname{diam} C_j)^s.$$

Since $\{C_j\}_{j=1}^\infty$ is any countable cover of $A \cup B$ with $\operatorname{diam} C_j < \delta$, taking infimum on both sides yields

$$\mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B) \leq \mathcal{H}^s_\delta(A \cup B),$$

which holds for any $\delta \in (0, \delta_0)$. Letting $\delta \to 0^+$, we proved

$$\mathcal{H}^s(A \cup B) \geq \mathcal{H}^s(A) + \mathcal{H}^s(B)$$

as desired. $\square$

Remark 7.14. Note that although we proved $\mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B) \leq \mathcal{H}^s_\delta(A \cup B)$ in the above proof, it holds only under the condition $\delta < \frac{1}{3}d(A,B)$ but not any $A, B \subset X$ with $d(A,B) > 0$. It does not conclude that all Borel sets are $\mathcal{H}^s_\delta$-measurable. In fact, one can easily show that some open sets are not $\mathcal{H}^1_\delta$-measurable. For example, let

$$E = \{(x,y) : y > 0\} \subset \mathbb{R}^2.$$

Consider the double-line set $S_\varepsilon := ([0,1] \times \{0\}) \cup ([0,1] \times \{\varepsilon\})$ where $\varepsilon > 0$. It can be covered the N many rectangles

$$C_j = \left[\frac{j-1}{N}, \frac{j}{N}\right] \times [0, \varepsilon], \quad j = 1, \cdots, N.$$

Note that $\operatorname{diam} C_j = \sqrt{\frac{1}{N^2} + \varepsilon^2}$. Given any $\delta > 0$, we choose N sufficiently large and ε sufficiently small such that $\operatorname{diam} C_j < \delta$ and $N\varepsilon < 1$, then we have

$$\mathcal{H}^1_\delta(S_\varepsilon) \leq \sum_{j=1}^N \operatorname{diam} C_j = \sqrt{1 + N^2\varepsilon^2} < \sqrt{2}.$$

However, $S_\varepsilon \cap E = [0,1] \times \{\varepsilon\}$ and $S_\varepsilon - E = [0,1] \times \{0\}$. Both have $\mathcal{H}^1_\delta$ equal to 1, so

$$\mathcal{H}^1_\delta(S_\varepsilon) < \mathcal{H}^1_\delta(S_\varepsilon \cap E) + \mathcal{H}^1_\delta(S_\varepsilon - E).$$

That shows E is not $\mathcal{H}^1_\delta$-measurable, despite E being open.

For the same reason, we do not define Hausdorff measure simply as

$$\inf\left\{\sum_{j=1}^{\infty}(\operatorname{diam} C_j)^s : E \subset \bigcup_{j=1}^{\infty} C_j,\ \text{where } C_j \subset X \text{ for all } j\right\},$$

because $E = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ would not be measurable either. $\square$

7.3. Rectifiable curves and $\mathcal{H}^1$

Given a curve $\gamma(t) : [a, b] \to \mathbb{R}^n$, we define its **length** to be the total length of the "best" approximated line segments. That is, take a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ and consider the sum:

$$l_P(\gamma) := \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})| .$$

It is the total length of the line segments joining points $\gamma(t_0), \gamma(t_1), \cdots, \gamma(t_n)$. As we are taking more and more refined partitions P, we expect l_P gets larger by the triangle inequality. Therefore, the "best" approximation of the length of the curve is naturally defined as the supremum of l_P among all partitions P.

Definition 7.15 (Rectifiable curve and arc length)**.** Let $\gamma(t) : [a, b] \to \mathbb{R}^n$ be a curve in $\mathbb{R}^n$. We call $\gamma(t)$ a **rectifiable curve** if $l_P(\gamma) \le C$ for some constant $C \in (0, \infty)$ independent of partitions P of $[a, b]$. In such case, we define the **arc length** of $\{\gamma(t)\}_{t\in[a,b]}$ to be:

$$L(\gamma) := \sup_P l_P(\gamma) = \sup \left\{ \sum_{i=1}^{k} |\gamma(t_i) - \gamma(t_{i-1})| : a = t_0 < t_1 < \cdots < t_k = b \right\} .$$

Next we will show that for a continuous, simple rectifiable curve γ in $\mathbb{R}^n$, its 1-dimensional Hausdorff measure is exactly its length.

Proposition 7.16. *Let $\gamma : [a, b] \to \mathbb{R}^n$ be a continuous simple rectifiable curve with arc length L. Then, we have $\mathcal{H}^1\big(\gamma([a, b])\big) = L$.*

We first prove a useful lemma relating $\mathcal{H}^1$ and the diameter:

Lemma 7.17. *Let (X, d) be a metric space, then for any connected set $E \subset X$, we have*

$$\operatorname{diam} E \le \mathcal{H}^1(E).$$

Proof. For each a point $p \in \overline{E}$, we define a function $f_p : X \to \mathbb{R}$ as

$$f_p(x) := d(x, p).$$

Then, we have $|f_p(x) - f_p(y)| \le d(x, y)$ for any $x, y \in X$. This shows (c.f. (3) of Proposition 7.4)

$$\mathcal{H}^1(f_p(E)) \le \mathcal{H}^1(E).$$

By the connectedness of E, the image $f_p(E)$ is an interval with $\sup f_p(E)$ equal to the furthest distance in $\overline{E}$ from p, and $\inf f_p(E) = 0$. Also $f_p(E) \subset \mathbb{R}$, so by the result from Example 7.3, we have $\mathcal{H}^1(f_p(E)) = \mathcal{L}^1(f_p(E)) = \sup f_p(E) - \inf f_p(E) = \sup\{d(x, p) : x \in \overline{E}\}$. Combining the above results, we proved

$$\mathcal{H}^1(E) \ge \sup\{d(x, p) : x \in \overline{E}\} \ \ \forall p \in \overline{E},$$

and so $\mathcal{H}^1(E) \ge \sup\{d(x, p) : x, p \in \overline{E}\} = \operatorname{diam} \overline{E} = \operatorname{diam} E$. □

Proof of Proposition 7.16. For simplicity we write $\mathcal{H}^1\big(\gamma([a, b])\big)$ as $\mathcal{H}^1(\gamma)$. We first prove $\mathcal{H}^1(\gamma) \ge L$.

For any partition $P = \{t_j\}_{j=0}^k$ of $[a, b]$, Lemma 7.17 shows for any j,

$$\mathcal{H}^1\big(\gamma([t_{j-1}, t_j])\big) \ge \operatorname{diam} \gamma([t_{j-1}, t_j]) \ge |\gamma(t_j) - \gamma(t_{j-1})| .$$

Summing all up, we get

$$\sum_{j=1}^{k} \mathcal{H}^1\big(\gamma([t_{j-1}, t_j])\big) \geq \sum_{j=1}^{k} |\gamma(t_j) - \gamma(t_{j-1})| = l_P(\gamma).$$

On the other hand, $\{\gamma([t_{j-1}, t_j])\}_{j=1}^{k}$'s are "almost" disjoint as they only intersect at end-points, which have $\mathcal{H}^1$ being 0. We conclude that

$$\mathcal{H}^1(\gamma) = \mathcal{H}^1\left(\bigcup_{j=1}^{k} \gamma([t_{j-1}, t_j])\right) = \sum_{j=1}^{k} \mathcal{H}^1\big(\gamma([t_{j-1}, t_j])\big) \geq l_P(\gamma).$$

Taking supremum over all partitions P of $[a, b]$, we get $\mathcal{H}^1(\gamma) \geq L$.

Now we prove the reverse inequality. Given any $\varepsilon > 0$, we choose a partition $P = \{t_j\}_{j=0}^{k}$ of $[a, b]$ such that $L - \varepsilon < l_P(\gamma) \leq L$. As refinement of P increases l_P, and γ is in fact uniformly continuous on the compact interval $[a, b]$, we may assume without loss of generality that $|\gamma(t_j) - \gamma(t_{j-1})| < \varepsilon$ for any j.

One good observation is that $\operatorname{diam} \gamma([t_{j-1}, t_j]) < 2\varepsilon$ for any j. Suppose otherwise that $\operatorname{diam} \gamma([t_{j-1}, t_j]) \geq 2\varepsilon$ for some j, then pick two points $\gamma(s_1), \gamma(s_2) \in \gamma([t_{j-1}, t_j])$ such that $|\gamma(s_1) - \gamma(s_2)| = \operatorname{diam} \gamma([t_{j-1}, t_j])$, then we have for this j,

$$|\gamma(t_{j-1}) - \gamma(s_1)| + |\gamma(s_1) - \gamma(s_2)| + |\gamma(s_2) - \gamma(t_j)| \geq |\gamma(t_{j-1}) - \gamma(t_j)| + 2\varepsilon$$

and so $L \geq l_{P \cup \{s_1, s_2\}} \geq l_P(\gamma) + 2\varepsilon > L + \varepsilon$, leading to a contradiction.

Now that $\operatorname{diam} \gamma([t_{j-1}, t_j]) < 2\varepsilon$ for all j, and they cover $\gamma([a, b])$, so we have

$$\mathcal{H}^1_{2\varepsilon}(\gamma([a, b])) \leq \sum_{j=1}^{k} \operatorname{diam} \gamma([t_{j-1}, t_j]).$$

For each $j = 1, \cdots, k$, we pick $t_j', t_j'' \in [t_{j-1}, t_j]$ such that

$$\big|\gamma(t_j') - \gamma(t_j'')\big| > \operatorname{diam} \gamma([t_{j-1}, t_j]) - \frac{\varepsilon}{k}.$$

Then, we consider the refinement $P' := P \cup \{t_j', t_j''\}_{j=1}^{k}$, we have

$$L \geq l_{P'}(\gamma) \geq \sum_{j=1}^{k} \big|\gamma(t_j') - \gamma(t_j'')\big| > \sum_{j=1}^{k} \operatorname{diam} \gamma([t_{j-1}, t_j]) - \varepsilon.$$

Combining the above results, we get

$$\mathcal{H}^1_{2\varepsilon}\big(\gamma([a, b])\big) < L + \varepsilon$$

for any $\varepsilon > 0$. Letting $\varepsilon \to 0^+$, we get $\mathcal{H}^1\big(\gamma([a, b])\big) \leq L$. We complete the proof. $\square$

7.4. Hausdorff measures on $\mathbb{R}^n$

We have proved in Example 7.3 that $\mathcal{H}^1$ coincides with $(\mathcal{L}^1)^*$ on $\mathbb{R}$. In this section we will prove that $\mathcal{H}^n$ also coincides (up to a multiple of a dimensional constant) with $(\mathcal{L}^n)^*$ on $\mathbb{R}^n$ for any integer $n \geq 2$. The proof is relatively easier for $\mathbb{R}^1$ because the diameter of a subset $E \subset \mathbb{R}$ is simply $\sup E - \inf E$. The proof for $\mathbb{R}^{n\geq 2}$ relies on a geometric-analytic result called the **isodiametric inequality** (not confuse it with *isoperimetric inequality*).

Define for each $s > 0$ the constant

$$\omega(s) := \frac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)}$$

where $\Gamma(s) := \displaystyle\int_0^\infty e^{-x}x^{s-1}\,dx$ is the Gamma function. Note that $\omega(s)$ is defined in such a way so that for any $n \in \mathbb{N}$, $\omega(n)$ is the n-th dimensional Lebesgue measure of the unit ball in $\mathbb{R}^n$.

Lemma 7.18 (Isodiametric inequality). *For any set $E \subset \mathbb{R}^n$, we have*

$$(\mathcal{L}^n)^*(E) \leq \omega_n \left(\frac{\operatorname{diam} E}{2}\right)^n.$$

Proof. We may assume E is compact (since if E were not bounded then $\operatorname{diam} E = \infty$ and we are done; or if E were not closed, one could note that $(\mathcal{L}^n)^*(E) \leq \mathcal{L}^n(\overline{E})$ and $\operatorname{diam} E = \operatorname{diam} \overline{E}$, so if the isodiametric inequality holds for $\overline{E}$, it also holds for E). The key idea of the proof is the use of **Steiner symmetrization**. For each $j = 1, \cdots, n$, we define the Steiner symmetrization $S_j(E)$ about the coordinate plane $\{x_j = 0\}$ in $\mathbb{R}^n$ by the following procedures

(1) For each $p \in \mathbb{R}^n$, we consider the set $E \cap \{p + te_j : t \in \mathbb{R}\}$, which is the intersection of E and the j-th coordinate line passing through p.
(2) By projecting the set $E \cap \{p + te_j : t \in \mathbb{R}\}$ onto the j-th axis, one can regard it as a subset of $\mathbb{R}$, and hence it makes sense to define:
$$\delta_j(E, p) := \frac{1}{2}\mathcal{L}^1\big(E \cap \{p + te_j : t \in \mathbb{R}\}\big).$$
(3) The line segment $\{\pi_j(p) + te_j : |t| \leq \delta_j(E,p)\}$, where π_j is the projection onto the plane $\{x_j = 0\}$, is then symmetric about $\{x_j = 0\}$ and has the same $\mathcal{L}^1$ as $E \cap \{p + te_j : t \in \mathbb{R}\}$. We construct $S_j(E)$ as
$$S_j(E) = \bigcup_{\{p : E\cap\{p+te_j : t\in\mathbb{R}\}\neq\emptyset\}} \{\pi_j(p) + te_j : |t| \leq \delta_j(E,p)\}.$$

Note that by Fubini's Theorem for $\mathcal{L}^n$, we have $\mathcal{L}^n(S_j(E)) = S_j(E)$. It can also be shown (see Exercise 7.2) by elementary geometry that

$$\operatorname{diam}(S_j(E)) \leq \operatorname{diam} E.$$

Applying the Steiner symmetrization successively about each coordinate plane, we get a set

$$S(E) := S_n \circ \cdots \circ S_1(E)$$

which is symmetric about the antipodal map $p \mapsto -p$, has the same $\mathcal{L}^n$ as E, and $\operatorname{diam} S(E) \leq \operatorname{diam} E$. The resulting set $S(E)$ is contained inside the closed n-ball with diameter equal to that of $S(E)$. Combining all the above observations, we conclude that

$$\mathcal{L}^n(E) = \mathcal{L}^n(S(E)) \leq \mathcal{L}^n(B_{\operatorname{diam} S(E)/2}(0)) = \omega_n \left(\frac{\operatorname{diam} S(E)}{2}\right)^n \leq \omega_n \left(\frac{\operatorname{diam} E}{2}\right)^n.$$

$\square$

Exercise 7.1. In the above proof, we have in fact used the fact that $S_j(E)$ is compact, and hence is $\mathcal{L}^n$-measurable. Prove this fact.

Exercise 7.2. Prove that $\operatorname{diam} S_j(E) \leq \operatorname{diam} E$ for any compact set $E \subset \mathbb{R}^n$.

Now we are ready to prove:

Theorem 7.19. *For any $\delta > 0$ and $n \in \mathbb{N}$, we have*
$$\frac{\omega_n}{2^n}\mathcal{H}^n(E) = \frac{\omega_n}{2^n}\mathcal{H}^n_\delta(E) = (\mathcal{L}^n)^*(E)$$
for any $E \subset \mathbb{R}^n$.

Proof. First we argue that $(\mathcal{L}^n)^*(E) \leq \frac{\omega_n}{2^n}\mathcal{H}^n_\delta(E)$ for any $\delta > 0$ and $E \subset \mathbb{R}^n$. Suppose $\{C_j\}_{j=1}^n$ is any countable cover of E with $\operatorname{diam} C_j < \delta$, then by countable sub-additivity, we have
$$(\mathcal{L}^n)^*(E) \leq \mathcal{L}^n\left(\bigcup_{j=1}^\infty C_j\right) \leq \sum_{j=1}^n \mathcal{L}^n(C_j) \leq \sum_{j=1}^\infty \omega_n \left(\frac{\operatorname{diam} C_j}{2}\right)^n.$$
We have used the isodiametric inequality in the last step. Taking the infimum over all such covers $\{C_j\}_{j=1}^\infty$, we proved $(\mathcal{L}^n)^*(E) \leq \frac{\omega_n}{2^n}\mathcal{H}^n_\delta(E)$.

To prove the reverse inequality, we first need a lemma concerning the Lebesgue measure:

Lemma 7.20. *For any bounded open set $U \subset \mathbb{R}^n$ and any $\delta > 0$, there exists a pairwise disjoint family of closed balls $\{B_j\}_{j=1}^\infty$ satisfying*
$$\bigcup_{j=1}^\infty B_j \subset U, \quad \operatorname{diam} B_j < \delta, \quad \textit{and} \quad \mathcal{L}^n\left(U - \bigcup_{j=1}^\infty B_j\right) = 0.$$

Proof of Lemma. By Stein p7, one can express $U = \bigcup_{j=1}^\infty C_j$ where C_j are closed n-cubes with $\operatorname{diam} C_j < \delta$, and the interiors $\{C_j^\circ\}_{j=1}^\infty$ are pairwise disjoint. Pick any closed ball B_j inside C_j° with radius at least $\frac{1}{4}$ of the edge-length of C_j. These balls are certainly disjoint. Then by elementary geometry, we have
$$\mathcal{L}^n(C_j - B_j) = \mathcal{L}^n(C_j) - \mathcal{L}^n(B_j) < \mathcal{L}^n(C_j) - \omega_n \cdot \left(\frac{1}{4}\mathcal{L}^n(C_j)^{1/n}\right)^n = \left(1 - \frac{\omega_n}{4^n}\right)\mathcal{L}^n(C_j).$$
By countable additivity (noting that intersections of any pair C_i and C_j must have $\mathcal{L}^n$ equal 0), we get
$$\mathcal{L}^n\left(U - \bigcup_{j=1}^\infty B_j\right) = \mathcal{L}^n\left(\bigcup_{j=1}^\infty (C_j - B_j)\right) \leq \left(1 - \frac{\omega_n}{4^n}\right)\sum_{j=1}^\infty \mathcal{L}^n(C_j) = \left(1 - \frac{\omega_n}{4^n}\right)\mathcal{L}^n(U).$$
Consider $U^{(1)} := U - \bigcup_{j=1}^\infty B_j$ which is also open. One can repeat the above procedure and find disjoint closed n-balls $\{B_k^{(1)}\}_{k=1}^\infty$ with $\operatorname{diam} B_k^{(1)} < \delta$ such that
$$\mathcal{L}^n\left(U^{(1)} - \bigcup_{k=1}^\infty B_k^{(1)}\right) \leq \left(1 - \frac{\omega_n}{4^n}\right)\mathcal{L}^n(U^{(1)}),$$

and hence we have

$$\mathcal{L}^n\left(U - \bigcup_{j=1}^{\infty} B_j \cup \bigcup_{k=1}^{\infty} B_k^{(1)}\right) = \mathcal{L}^n\left(U^{(1)} - \bigcup_{k=1}^{\infty} B_k^{(1)}\right) \leq \left(1 - \frac{\omega_n}{4^n}\right)^2 \mathcal{L}^n(U).$$

Repeating this process inductively, one can find a countable disjoint collection of balls $\{B'_j\}_{j=1}^{\infty}$ contained in U with $\operatorname{diam} B'_j < \delta$ for any j, such that

$$\mathcal{L}^n\left(U - \bigcup_{j=1}^{\infty} B'_j\right) \leq \left(1 - \frac{\omega_n}{4^n}\right)^k \mathcal{L}^n(U) \ \ \forall k \in \mathbb{N}.$$

Letting $k \to \infty$, we concluded that

$$\mathcal{L}^n\left(U - \bigcup_{j=1}^{\infty} B'_j\right) = 0.$$

□

Now back to the proof of the theorem. For any countable collection of open n-cubes $\{C_j\}_{j=1}^{\infty}$ such that $E \subset \bigcup_{j=1}^{\infty} C_j$. For each j, we apply the above lemma to find a disjoint collection of balls $\{B_{j,k}\}_{k=1}^{\infty}$ with $\operatorname{diam} B_{j,k} < \delta$, $\bigcup_{k=1}^{\infty} B_{j,k} \subset C_j$ and

$$\mathcal{L}^n\left(C_j - \bigcup_{k=1}^{\infty} B_{j,k}\right) = 0.$$

Because any n-cubes can be chopped into finite union of smaller n-cubes with diameter less than δ and with pairwise disjoint interiors, one can easily show that $\mathcal{L}^n(A) = 0 \implies \mathcal{H}^n_\delta(A) = 0$ for any $A \subset \mathbb{R}^n$.

Therefore, we also have

$$\mathcal{H}^n_\delta\left(C_j - \bigcup_{k=1}^{\infty} B_{j,k}\right) = 0 \implies \mathcal{H}^n_\delta(C_j) = \mathcal{H}^n_\delta\left(\bigcup_{k=1}^{\infty} B_{j,k}\right).$$

Finally, we conclude that

$$\begin{aligned}
\mathcal{H}^n_\delta(E) &\leq \mathcal{H}^n_\delta\left(\bigcup_{k=1}^{\infty} C_j\right) \leq \sum_{j=1}^{\infty} \mathcal{H}^n_\delta(C_j) \\
&= \sum_{j=1}^{\infty} \mathcal{H}^n_\delta\left(\bigcup_{k=1}^{\infty} B_{j,k}\right) \leq \sum_{j,k=1}^{\infty} \mathcal{H}^n_\delta(B_{j,k}) \\
&\leq \sum_{j,k=1}^{\infty} (\operatorname{diam} B_{j,k})^n = \sum_{j,k=1}^{\infty} \frac{2^n}{\omega_n} \mathcal{L}^n(B_{j,k}) \\
&= \sum_{j=1}^{\infty} \frac{2^n}{\omega_n} \mathcal{L}^n\left(\bigcup_{k=1}^{\infty} B_{j,k}\right) \\
&\leq \sum_{j=1}^{\infty} \frac{2^n}{\omega_n} \mathcal{L}^n(C_j) = \frac{2^n}{\omega_n} \sum_{j=1}^{\infty} |C_j|.
\end{aligned}$$

As this holds true for any countable cover $\{C_j\}_{j=1}^{\infty}$ of E open n-cubes, we conclude that $\mathcal{H}^n_\delta(E) \leq \frac{2^n}{\omega_n}(\mathcal{L}^n)^*(E)$.

To conclude, we have proved that $(\mathcal{L}^n)^*(E) = \frac{\omega_n}{2^n}\mathcal{H}^n_\delta(E)$ for any $\delta > 0$ and $E \subset \mathbb{R}^n$. As a corollary, we also have $(\mathcal{L}^n)^*(E) = \frac{\omega_n}{2^n}\mathcal{H}^n(E)$.

□

In fact it can be shown that for a C^1 k-submanifold Σ in $\mathbb{R}^n$, the $\mathcal{H}^k$-measure Σ is indeed equals to k-dimensional volume of Σ which is analytically defined using the first fundamental form. The proof requires the use of area and co-area formulae, which are important tools in geometric measure theory. Interested readers may read Evans-Gariepy's book for detail.

7.5. Hausdorff dimensions

Let's first give some motivation behind the notion of Hausdorff dimensions. Suppose $s > 0$ and E is a subset in a metric space (X, d) such that $0 < \mathcal{H}^s(E) < \infty$. Now we consider $\mathcal{H}^{s+\varepsilon}(E)$ where $\varepsilon > 0$:

For any $\delta > 0$, and any countable cover $\{C_j\}_{j=1}^\infty$ of E with $\operatorname{diam} C_j < \delta$, we have

$$\sum_{j=1}^\infty (\operatorname{diam} C_j)^{s+\varepsilon} \leq \sum_{j=1}^\infty (\operatorname{diam} C_j)^s \delta^\varepsilon.$$

Taking infimum on both sides over all such countable covers, we conclude that

$$\mathcal{H}_\delta^{s+\varepsilon}(E) \leq \mathcal{H}_\delta^s(E) \cdot \delta^\varepsilon.$$

Since $\mathcal{H}^s(E)$ is finite, we have

$$\lim_{\delta \to 0^+} \mathcal{H}_\delta^s(E) \cdot \delta^\varepsilon = \mathcal{H}^s(E) \cdot 0 = 0,$$

and hence $\mathcal{H}^{s+\varepsilon}(E) = 0$. Similarly, one could also show that $\mathcal{H}^{s-\varepsilon}(E) = \infty$ for any $\varepsilon \in (0, s)$. We leave this as an exercise for readers.

As such, the number s can be regarded as the "right" dimension to measure such an E. In this connection, we define:

Definition 7.21 (Hausdorff dimension)**.** Let (X, d) be a metric space. For any $E \subset X$, we define its **Hausdorff dimension** to be

$$\dim_{\mathcal{H}} E := \inf\{s \geq 0 : \mathcal{H}^s(E) = 0\}.$$

Example 7.22. Any rectifiable curve γ in $\mathbb{R}^n$ with non-zero arc-length has Hausdorff dimension 1.

Example 7.23. Countable sets have Hausdorff dimension 0.

Example 7.24. For any $E \subset \mathbb{R}^n$ (bounded or not) with $(\mathcal{L}^n)^*(E) > 0$, the set $E \times \{0\}$ when viewed as a subset in $\mathbb{R}^{n+m}$ has Hausdorff dimension n. To see this, we need to figure out the set $\{s \geq 0 : \mathcal{H}^s(E \times \{0\}) = 0\}$.

First we let $\mathbb{R}^n = \bigcup_{j=1}^\infty C_j$ where C_j's are unit n-cubes, then $E \times \{0\} = \bigcup_{j=1}^\infty (E \cap C_j) \times \{0\}$. Since each $(E \cap C_j) \times \{0\}$ is isometric to $E \cap C_j$ in $\mathbb{R}^n$ which is bounded, so

$$\mathcal{H}^n((E \cap C_j) \times \{0\}) = \mathcal{H}^n(E \cap C_j) = (\mathcal{L}^n)^*(E \cap C_j) < \infty.$$

This shows for any $s > n$, we have $\mathcal{H}^s((E \cap C_j) \times \{0\}) = 0$. By countable sub-additivity, we conclude that

$$\mathcal{H}^s(E \times \{0\}) = \mathcal{H}^s\left(\bigcup_{j=1}^\infty (E \cap C_j) \times \{0\}\right) = 0 \ \ \forall s > n.$$

Hence, $\{s \geq 0 : \mathcal{H}^s(E \times \{0\}) = 0\} \supset (n, \infty)$.

On the other hand, by $(\mathcal{L}^n)^*(E) > 0$ and countable sub-additivity, we must have $(\mathcal{L}^n)^*(E \cap C_j) > 0$ for at least one j. This shows

$$\mathcal{H}^n((E \cap C_j) \times \{0\}) = \mathcal{H}^n(E \cap C_j) = (\mathcal{L}^n)^*(E \cap C_j) > 0$$

for this j, and so $\mathcal{H}^s((E \cap C_j) \times \{0\}) = \infty$ for any $s < n$. Hence, $\mathcal{H}^s(E \times \{0\}) = \infty$ for any $s < n$.

It concludes that $\{s \geq 0 : \mathcal{H}^s(E \times \{0\}) = 0\}$ is either (n, ∞) or $[n, \infty)$, both of which implies $\dim_{\mathcal{H}}(E \times \{0\}) = n$.

Example 7.25. A more interesting example of calculating the Hausdorff dimension is the Cantor set constructed by successively removing the open middle middle third from the closed interval $[0,1]$. Precisely, we let

$$\begin{aligned} \mathcal{C}_0 &= [0,1] \\ \mathcal{C}_j &= \frac{1}{3}\mathcal{C}_{j-1} \cup \left(\frac{2}{3} + \frac{1}{3}\mathcal{C}_{j-1}\right) \quad \forall j \geq 1 \end{aligned}$$

The Cantor set is defined to be

$$\mathcal{C} = \bigcap_{j=0}^{\infty} \mathcal{C}_j.$$

We are going to show that $\dim_{\mathcal{H}} \mathcal{C} = \dfrac{\log 2}{\log 3} =: \gamma$. We will show that $\mathcal{H}^\gamma(\mathcal{C}) \in (0,1]$. To prove the upper bound, we observe that $\mathcal{C} = \bigcap_{j=0}^{\infty} \mathcal{C}_j$ where $\mathcal{C}_j$ consists of 2^j many disjoint closed interval $\{I_{j,k}\}_{k=1}^{2^j}$'s of length $\frac{1}{3^j}$. Each $\mathcal{C}_j$ can be regarded as cover of $\mathcal{C}$ by closed intervals with diameter $\frac{1}{3^j}$, and so

$$\mathcal{H}^\gamma_{1/3^j}(\mathcal{C}) \leq \sum_{k=1}^{2^j} (\operatorname{diam} I_{j,k})^\gamma = 2^j \cdot \left(\frac{1}{3^j}\right)^\gamma = 1.$$

Letting $j \to \infty$, we proved that $\mathcal{H}^\gamma(\mathcal{C}) \leq 1$.

To show that $\mathcal{H}^\gamma(\mathcal{C}) > 0$, we make sure of the **Cantor-Lebesgue function** (see definition in Stein p126) $F : [0,1] \to [0,1]$ which has the property that $F(\mathcal{C}) = [0,1]$ and there exists $M > 0$ such that

$$|F(x) - F(y)| \leq M\,|x-y|^\gamma \quad \forall x, y \in [0,1].$$

We leave the proof of these two facts as an exercise. Using (3) of Proposition 7.4, we have

$$1 = \mathcal{H}^1(g(\mathcal{C})) \leq M\mathcal{H}^\gamma(\mathcal{C}).$$

This shows $\mathcal{H}^\gamma(\mathcal{C}) > 0$.

To conclude, we have $\mathcal{H}^\gamma(\mathcal{C}) \in (0,\infty)$, and so $\dim_{\mathcal{H}} \mathcal{C} = \gamma$.

Exercise 7.3. Prove that the Cantor-Lebesgue function $F : [0,1] \to [0,1]$ has the property that $F(\mathcal{C}) = [0,1]$ and that it is Hölder continuous on $[0,1]$ with exponent $\gamma := \frac{\log 2}{\log 3}$.

Chapter 8

Integrations of Measurable Functions

"Love is not measured by who you love,
love is measured by how you love."

Santosh Kalwar

8.1. Measurable functions

In this chapter we introduce measurable functions on an abstract measure space and their integrals. Measurable functions are defined in a way so that we can approximate them by **simple functions** whose integrals can be naturally defined. The more precise detail will be presented later. Let's first learn about the definition of measurable functions:

Definition 8.1 (Measurable function)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space. A function $f : X \to [-\infty, \infty]$ is said to be **measurable** with respect to $(X, \mathcal{A}, \mu)$ if

$$f^{-1}\big([-\infty, a)\big) \in \mathcal{A} \ \ \forall a \in \mathbb{R}.$$

By noting that $[-\infty, a] = \bigcap_{j=1}^{\infty}[-\infty, a + \frac{1}{j})$, $[-\infty, a) = \bigcup_{j=1}^{\infty}[-\infty, a - \frac{1}{j}]$, and $(a, +\infty] = [-\infty, \infty] - [-\infty, a]$, one can show that the following are equivalent:

(1) $f^{-1}\big([-\infty, a)\big) \in \mathcal{A} \ \ \forall a \in \mathbb{R}$.

(2) $f^{-1}\big([-\infty, a]\big) \in \mathcal{A} \ \ \forall a \in \mathbb{R}$.

(3) $f^{-1}\big([a, \infty]\big) \in \mathcal{A} \ \ \forall a \in \mathbb{R}$.

(4) $f^{-1}\big((a, \infty]\big) \in \mathcal{A} \ \ \forall a \in \mathbb{R}$.

Hence, one can actually regard any of the above as the definition of measurable functions.

Like Lebesgue measurable functions, the measurable functions of an abstract measure space have the following properties. The proof is almost the same as the Lebesgue measure case.

Proposition 8.2. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Then,*

(1) If $f : X \to \mathbb{R}$ is measurable and $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous, then $\varphi \circ f$ is measurable.

(2) If $f, g : X \to \mathbb{R}$ are measurable, then so are $f + g$, $f - g$, fg.

Proof. For (1), first note that for any open intervals $(a, b) = (-\infty, b) \cap (a, \infty)$, by measurability $f^{-1}\big((a, b)\big)$ is measurable. Now given any $a \in \mathbb{R}$, as φ is finite-valued, so $\varphi^{-1}\big([-\infty, a)\big) = \varphi^{-1}\big((-\infty, a)\big)$ which is open in $\mathbb{R}$ by the continuity of φ. Open sets in $\mathbb{R}$ are countable disjoint union of open intervals, so

$$(\varphi \circ f)^{-1}\big((-\infty, a)\big) = f^{-1}\Big(\underbrace{\varphi^{-1}((-\infty, a))}_{\text{countable union of open intervals}} \Big) \in \mathcal{A}.$$

For (2): to prove $f + g$ is measurable, we note that

$$\begin{aligned} & f(x) + g(x) > a \\ \iff \quad & f(x) > a - g(x) \\ \iff \quad & \exists q \in Q \text{ such that } f(x) > q > a - g(x) \end{aligned}$$

This shows

$$\{f + g > a\} = \bigcup_{q \in \mathbb{Q}} \{f > q\} \cap \{g > a - q\},$$

which is in $\mathcal{A}$ by the measurability of f, g and that $\mathbb{Q}$ is countable. It proves $f + g$ is measurable. Similar argument shows $f - g$ is measurable too.

To prove fg is measurable, we express

$$fg = \frac{1}{4}\big((f + g)^2 - (f - g)^2\big).$$

As the functions $x \mapsto x^2$ and $x \mapsto \frac{1}{4}x$ are continuous, the above identity shows fg is measurable too. $\square$

Example 8.3. On a measurable space $(X, \mathcal{A}, \mu)$, and given $E \subset X$, we define the indicator function of E by

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

One can check easily that

$$\chi_E^{-1}\big((-\infty, a]\big) = \begin{cases} X & \text{if } a \geq 1 \\ E & \text{if } 0 \leq a < 1 \\ \emptyset & \text{if } a < 0 \end{cases}.$$

Therefore, χ_E is measurable if and only if $E \in \mathcal{A}$.

Exercise 8.1. Let $\{f_j : X \to [-\infty, \infty]\}_{j=1}^{\infty}$ be a sequence of measurable functions on a measure space $(X, \mathcal{A}, \mu)$, then

$$\sup_j f_j(x), \qquad \inf_j f_j(x), \qquad \limsup_{j \to \infty} f_j(x) \qquad \text{and} \qquad \liminf_{j \to \infty} f_j(x)$$

are measurable.

Exercise 8.2. Show that if $f : X \to [-\infty, \infty]$ is measurable, then so is $|f|$.

For Lebesgue measures, if f and g are almost everywhere equal, then one is measurable if and only if the other one is. However, one should note that such a result relies on the fact that any set of zero measure must be measurable. In order to extend this result to general measure spaces, we need to impose an extra condition about zero measure sets.

Definition 8.4 (Complete measure space)**.** A measure space $(X, \mathcal{A}, \mu)$ is said to be **complete** if whenever $E \in \mathcal{A}$ and $\mu(E) = 0$, then we have $S \in \mathcal{A}$ for any $S \subset E$.

For a measure space induced by an outer measure (such as Lebesgue and Hausdorff measures), all sets of zero measure must be measurable. However, it may not be the case in general. We will see some counter-examples later on when we discuss product measures.

Proposition 8.5. *Let $(X, \mathcal{A}, \mu)$ be a complete measure space. Given that $f : X \to [-\infty, \infty]$ is a measurable function, and $f = g$ a.e., then g is also measurable.*

Proof. Note that

$$g^{-1}\big([-\infty, a)\big) = \Big(\{f = g\} \cap f^{-1}\big([-\infty, a)\big)\Big) \cup \Big(\{f \neq g\} \cap g^{-1}\big([-\infty, a)\big)\Big).$$

As $\mu\{f \neq g\} = 0$, and by completeness we have

$$\{f \neq g\} \cap g^{-1}\big([-\infty, a)\big) \in \mathcal{A}.$$

Also, $\{f = g\} = \{f \neq g\}^c$ which is measurable too; and $f^{-1}\big([-\infty, a)\big) \in \mathcal{A}$ by the measurability of f. Combining all these, we have proved $g^{-1}\big([-\infty, a)\big) \in \mathcal{A}$ for any a, so g is measurable. $\square$

8.2. Definition of integrals

An integral of a measurable function will be defined using the integrals of **simple functions** approximating the function. By a simple function we mean:

Definition 8.6 (Simple function). Let $(X, \mathcal{A}, \mu)$ be a measure space. A function $\varphi : X \to \mathbb{R}$ is called a **simple function** if it is of the form:

$$\varphi = \sum_{j=1}^{N} c_j \chi_{E_j}$$

where c_j's are real constants, and $E_j \in \mathcal{A}$ for any j. Note that N must be finite as well.

Remark 8.7. Some books (such as Stein) require each E_j has finite measure, while some books do not. Our convention is we allow $\mu(E_j)$ to be infinite, yet c_j's need to be finite.

If $\{E_j\}_{j=1}^{\infty}$ are pairwise disjoint, it would then be natural to define the integral of a simple function by:

$$\int_X \varphi \, d\mu := \sum_{j=1}^{N} c_j \mu(E_j).$$

Any simple function can be rewritten so that $\{E_j\}$'s are pairwise disjoint and c_j's are distinct, as the domain of φ is a finite set $\{b_1, \cdots, b_M\}$ which are all distinct, one can rearrange φ by

$$\varphi = \sum_{j=1}^{M} b_j \chi_{\varphi^{-1}(b_j)}.$$

Note that $\varphi^{-1}(b_j)$'s are pairwise disjoint because b_j's are distinct.

Integrals on a measure space $(X, \mathcal{A}, \mu)$ are defined by first defining the integrals of **non-negative simple functions**, and then extending the class of functions step-by-step to all measurable functions.

As mentioned earlier, when a simple function

$$\varphi = \sum_{j=1}^{N} c_j \chi_{E_j}$$

is expressed in a way that E_j's are pairwise disjoint (note that c_j's need not be distinct here) – call it **canonical form**. For any **non-negative** simple function $\varphi : X \to \mathbb{R}$, we can define its integral by:

$$\int_X \varphi \, d\mu := \sum_{j=1}^{N} c_j \mu(E_j).$$

We take the convention that if $c_j = 0$ and $\mu(E_j) = \infty$, then $c_j \mu(E_j) = 0$. Also, we need avoid the situation of getting an indefinite-form $1 \cdot \infty + (-1) \cdot \infty$, so we restrict ourself to non-negative simple functions first so that $c_j \geq 0$ for all j.

One can also check that such as a definition is well-defined if φ can be expressed in another canonical form

$$\varphi = \sum_{j=1}^{M} b_j \chi_{F_j},$$

then we indeed have $\sum_{j=1}^{N} c_j \mu(E_j) = \sum_{j=1}^{M} b_j \mu(F_j)$ (see Stein p51 for a proof).

Integrals over a measurable subset $E \subset X$ is defined as

$$\int_E \varphi \, d\mu := \int_X \varphi \chi_E \, d\mu.$$

Note that $\varphi\chi_E$ is also a simple function because $\chi_{E_j}\chi_E = \chi_{E_j\cap E}$.

Similar to Riemann integrals, integrals of simple functions have the following properties:

Proposition 8.8. *Let $(X, \mathcal{A}, \mu)$ be a measure space, $\varphi, \psi : X \to [0, \infty)$ be two non-negative simple functions on X, and $E, F \subset X$ be any measurable sets. Then we have:*

(1) For any constants $a, b \geq 0$, we have
$$\int_E (a\varphi + b\psi)\, d\mu = a \int_E \varphi\, d\mu + b \int_E \psi\, d\mu.$$
(2) If $E \cap F = \emptyset$, then
$$\int_{E\cup F} \varphi\, d\mu = \int_E \varphi\, d\mu + \int_F \varphi\, d\mu.$$
(3) If $\varphi \leq \psi$ on E, then
$$\int_E \varphi\, d\mu \leq \int_E \psi\, d\mu.$$

Proof. The proofs are mostly straight-forwarding verifications using the definition. We first write φ and ψ in canonical forms:
$$\varphi = \sum_{i=1}^{N} c_i \chi_{E_i} \quad \text{and} \quad \psi = \sum_{j=1}^{M} b_j \chi_{F_j}.$$
Then, $\{E_i\}$'s and $\{F_j\}$'s are disjoint collections of sets. We can also assume that $\bigcup_i E_i = X$ and $\bigcup_j F_j = X$ by adding a term with c_i or b_j equal 0

Now consider the collection $\{E_i \cap F_j : 1 \leq i \leq N, 1 \leq j \leq M\}$, which is also a disjoint collection of measurable sets. By $1 = \chi_X = \sum_{i=1}^N \chi_{E_i} = \sum_{j=1}^M \chi_{F_j}$. Then φ and ψ can be written in a more "unified" way as:
$$\varphi = \sum_{\substack{1\leq i\leq N\\ 1\leq j\leq M}} c_i \chi_{E_i\cap F_j} \quad \text{and} \quad \psi = \sum_{\substack{1\leq i\leq N\\ 1\leq j\leq M}} b_j \chi_{E_i\cap F_j}.$$
Then, we have
$$\int_E \varphi\, d\mu = \int_X \varphi\chi_E\, d\mu = \int_X \sum_{\substack{1\leq i\leq N\\ 1\leq j\leq M}} c_i \chi_{E_i\cap F_j\cap E}\, d\mu = \sum_{\substack{1\leq i\leq N\\ 1\leq j\leq M}} c_i \mu(E_i \cap F_j \cap E)$$
and similarly
$$\int_E \psi\, d\mu = \sum_{\substack{1\leq i\leq N\\ 1\leq j\leq M}} b_j \mu(E_i \cap F_j \cap E),$$
$$\begin{aligned}\int_E (a\varphi + b\psi)\, d\mu &= \int_X \sum_{\substack{1\leq i\leq N\\ 1\leq j\leq M}} (ac_i + bb_j)\chi_{E_i\cap F_j\cap E}\, d\mu \\ &= \sum_{\substack{1\leq i\leq N\\ 1\leq j\leq M}} (ac_i + bb_j)\mu(E_i \cap F_j \cap E).\end{aligned}$$
From the above computations, clearly (1) holds.

(2) follows directly from the fact that $\chi_{E\cup F} = \chi_E + \chi_F$ when E, F are disjoint and the result from (1).

For (3), from the condition $\varphi \leq \psi$, we have $c_i \leq b_j$ whenever $E_i \cap F_j \neq \emptyset$. Therefore, $c_i \mu(E_i \cap F_j \cap E) \leq b_j \mu(E_i \cap F_j \cap E)$ for any i, j. This clearly shows

$$\int_E \varphi \, d\mu = \sum_{\substack{1 \leq i \leq N \\ 1 \leq j \leq M}} c_i \mu(E_i \cap F_j \cap E) \leq \sum_{\substack{1 \leq i \leq N \\ 1 \leq j \leq M}} b_j \mu(E_i \cap F_j \cap E) = \int_E \psi \, d\mu.$$

□

Definition 8.9 (Integral of a non-negative measurable function)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to [0, \infty]$ be a **non-negative** measurable function. Given any $E \in \mathcal{A}$, we define the **integral of** f **over** E by

$$\int_E f \, d\mu := \sup \left\{ \int_E \varphi \, d\mu : 0 \leq \varphi \leq f, \text{ where } \varphi \text{ is a simple function} \right\}$$

The definition is comparable to the lower Darboux integral in single-variable analysis. Writing φ as $\varphi = \sum_{i=1}^N c_i \chi_{E_i}$, then by $\varphi \leq f$ we know $c_i \leq \inf_{E_i} f$ for any i unless E_i is empty. Therefore,

$$\int_X \varphi \, d\mu = \sum_{i=1}^N c_i \mu(E_i) \leq \sum_{i=1}^N \inf_{E_i} f \cdot \mu(E_i).$$

Therefore, for each non-negative simple function $\varphi = \sum_{i=1}^N c_i \chi_{E_i} \leq f$, we can define another "better" simple function s_φ by

$$s_\varphi := \sum_{i=1}^N \inf_{E_i} f \cdot \chi_{E_i}.$$

Then, we have $\varphi \leq s_\varphi \leq f$, and s_φ is "better" in a sense that:

$$\int_X \varphi \, d\mu \leq \sum_{i=1}^N \inf_{E_i} f \cdot \mu(E_i) = \int_X s_\varphi \, d\mu \leq \int_X f \, d\mu.$$

Therefore, the definition of $\int_X f \, d\mu$ can be rewritten as

$$\int_X f \, d\mu = \sup \left\{ \sum_{i=1}^N \inf_{E_i} f \cdot \mu(E_i) : E_i \in \mathcal{A} \;\; \forall i = 1, \cdots, N \quad \text{and} \quad \bigcup_{i=1}^N E_i = X \right\}.$$

This is comparable to the lower Darboux integral in single-variable analysis. The difference is that for lower Darboux integral $E_i' s$ are restricted to sub-intervals, whereas we also allow E_i's to be any measurable sets.

Exercise 8.3. Prove from Definition 8.9 that we have

$$\int_E f \, d\mu = \int_X f \chi_E \, d\mu$$

for any measurable set E.

The integral of a measurable function satisfies the following properties:

Proposition 8.10. *Let $(X, \mathcal{A}, \mu)$ be a measure space, $E, F \in \mathcal{A}$, and $f, g : X \to [0, \infty]$ be two non-negative measurable functions. Then, the following properties hold:*

(1) If $f \leq g$ on E, then
$$\int_E f\, d\mu \leq \int_E g\, d\mu.$$
(2) If $E \subset F$, then
$$\int_E f\, d\mu \leq \int_F f\, d\mu.$$
(3) For any $c \in [0, \infty]$, we have
$$\int_E cf\, d\mu = c \int_E f\, d\mu.$$

Proof. We leave (1) and (2) as exercises for readers. They follow easily from the property of supremum. We only prove (3) here. Note that we can assume $c > 0$ otherwise it is trivial. Recall that
$$\int_E cf\, d\mu = \sup \left\{ \int_E \varphi\, d\mu : 0 \leq \varphi \leq cf \text{ where } \varphi \text{ is a simple function} \right\}.$$
For any such φ in the set, we have $0 \leq \frac{\varphi}{c} \leq f$. Note that $\frac{\varphi}{c}$ is also a simple function, so we have:
$$\int_E \frac{\varphi}{c}\, d\mu \leq \sup \left\{ \int_E \psi\, d\mu : 0 \leq \psi \leq f \text{ where } \psi \text{ is simple} \right\} =: \int_E f\, d\mu.$$
Note that by (1) of Proposition 8.8 and $\frac{\varphi}{c}$ is simple, we have
$$\frac{1}{c} \int_E \varphi\, d\mu = \int_E \frac{\varphi}{c}\, d\mu \leq \int_E f\, d\mu.$$
This shows
$$\int_E \varphi\, d\mu \leq c \int_E f\, d\mu$$
for any such φ under consideration. Taking supremum over all such φ, we proved
$$\int_E cf\, d\mu \leq c \int_E f\, d\mu \tag{8.1}$$
For the reverse inequality, we simply apply the inequality (8.1) by replacing f by cf which is also non-negative measurable. Then we get
$$\int_E f\, d\mu = \int_E \frac{1}{c} \cdot cf\, d\mu \leq \frac{1}{c} \int_E cf\, d\mu.$$
This shows
$$c \int_E f\, d\mu \leq \int_E cf\, d\mu.$$
Combining with (8.1), (3) is proved. □

Remark 8.11. It is still true that we have $\int_E (f + g)\, d\mu = \int_E f\, d\mu + \int_E g\, d\mu$, but proving it requires a deeper tool which will be discussed later.

For a non-negative *continuous* function $f : \mathbb{R} \to \mathbb{R}$ with $\displaystyle\int_a^b f(x)\, dx = 0$, we have $f(x) = 0$ on $[a, b]$. However, we do not expect this to be true without the continuity condition, as the value of an integral does not change after altering the value of f at finitely many points. For integrals on measure spaces, we expect that such an f should be *almost everywhere* zero. We will establish this fact using the following result:

Lemma 8.12 (Chebyshev's inequality). *Let $(X, \mathcal{A}, \mu)$ be a measure space, and $f : X \to [0, \infty]$ be a measurable function. Then, for any $M > 0$, we have*

$$\mu\big(f^{-1}([M, \infty])\big) \leq \frac{1}{M} \int_X f \, d\mu.$$

Proof. If $\int_X f \, d\mu = \infty$, the inequality trivially holds. From now on we assume it is finite. Note that

$$f \geq M \text{ on } f^{-1}([M, \infty]),$$

so we have

$$\begin{aligned} \int_X f \, d\mu &\geq \int_{f^{-1}([M,\infty])} f \, d\mu && \text{(by (2) of Proposition 8.10)} \\ &\geq \int_{f^{-1}([M,\infty])} M \, d\mu && \text{(by (1) of Proposition 8.10)} \\ &= M\mu\big(f^{-1}([M, \infty])\big). \end{aligned}$$

□

We say $f = g$ a.e. on X (or in full: almost everywhere on X) if the measure of the set $\{x : f(x) \neq g(x)\}$ is zero. Using the above inequality, we can prove the following results

Proposition 8.13. *Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to [0, \infty]$ be a non-negative measurable function. Then,*

(1) $f = 0$ a.e. on X if and only if $\displaystyle\int_X f \, d\mu = 0$.

(2) If $\displaystyle\int_X f \, d\mu < \infty$, then f is finite a.e.

Proof. First we prove the $\Longrightarrow$-part of (1): Suppose $f = 0$ a.e. on X, and consider any simple function φ such that $0 \leq \varphi \leq f$. Write $\varphi = \sum_{i=0}^N c_i \chi_{E_i}$, which we can assume $\{E_i\}_{i=0}^N$ is a partition of X, $c_0 = 0$, and $\{c_i\}_{i=0}^N$ are all distinct. Then by $0 \leq \varphi \leq f$ on X and $f = 0$ a.e. on X, we also have $\varphi = 0$ a.e. on X. Since $\{x : \varphi(x) \neq 0\} = \bigcup_{i=1}^N E_i$, we have

$$\mu\left(\bigcup_{i=1}^N E_i\right) = 0 \implies \sum_{i=1}^N \mu(E_i) = 0 \implies \mu(E_i) = 0 \;\; \forall i \geq 1.$$

The concludes that

$$\int_X \varphi \, d\mu = \underbrace{c_0}_{=0} \cdot \mu(E_0) + \sum_{i=1}^N c_i \underbrace{\mu(E_i)}_{=0} = 0.$$

Since φ is an arbitrary simple function such that $0 \leq \varphi \leq f$, taking supremum over all such simple functions proves $\int_X f \, d\mu = 0$.

For the $\Longleftarrow$-part of (1): we assume $\int_X f \, d\mu = 0$. By the Chebyshev's inequality (Lemma 8.12), we have for any $k \in \mathbb{N}$,

$$\mu\left(f^{-1}([1/k, \infty])\right) \leq k \int_X f \, d\mu = 0.$$

Hence, $\mu\left(f^{-1}([1/k,\infty])\right) = 0$ for any $k \in \mathbb{N}$. Then, by

$$f^{-1}\big((0,\infty]\big) = \bigcup_{k=1}^{\infty} f^{-1}\left(\left(\frac{1}{k},\infty\right]\right),$$

and countable sub-additivity, we have

$$\mu\left(\{x \in X : f(x) \neq 0\}\right) = \mu\left(f^{-1}\big((0,\infty]\big)\right) \leq \sum_{k=1}^{\infty} \mu\left(f^{-1}\left((1/k,\infty]\right)\right) = 0.$$

Hence, $f = 0$ a.e. on X.

(2) can be proved by a similar fashion as the $\Longleftarrow$-part of (1). Using the Chebyshev's inequality (Lemma 8.12), we have for any $k \in \mathbb{N}$,

$$\mu\left(f^{-1}(\infty)\right) \leq \mu\left(f^{-1}([k,\infty])\right) \leq \frac{1}{k}\int_X f\,d\mu.$$

As $\int_X f\,d\mu < \infty$, letting $k \to \infty$ we get $\mu(f^{-1}(\infty)) = 0$ as desired. □

Now we define integrals of any measurable function (with mixed signs). For each function $f : X \to [-\infty,\infty]$ on a measure space $(X,\mathcal{A},\mu)$, we define the positive and negative parts of f by:

$$f_+ := \max\{0,f\} \quad \text{and} \quad f_- := -\min\{0,f\}.$$

Then, both f_+ and f_- are non-negative functions. It is easy to show (as an exercise) that f is measurable if and only if both f_+ and f_- are so. We define the integral of f by integrating f_+ and f_- separately:

Definition 8.14 (Integrals of measurable functions)**.** Let $(X,\mathcal{A},\mu)$ be a measure space, $f : X \to [-\infty,\infty]$ be a measurable function, and $E \in \mathcal{A}$. Then, we define the integral of f over E by:

$$\int_E f\,d\mu := \int_E f_+\,d\mu - \int_E f_-\,d\mu.$$

We say f is **integrable on** E if both $\int_E f_+\,d\mu$ and $\int_E f_-\,d\mu$ are finite.

8.3. Convergence theorems of integrals

In MATH 2043, we have learned that the limit and integral signs cannot always be interchanged – we need uniform convergence. However, uniform convergence is very strong condition. In this section, we will learn about several more theorems that allow us to justify the interchange of limit and integral signs.

Theorem 8.15 (Monotone Convergence Theorem (MCT)). *Let $(X, \mathcal{A}, \mu)$ be a measure space, and $f_k : X \to [0, \infty]$ be a sequence of **non-negative** measurable functions such that there exists $N \in \mathcal{A}$ with $\mu(N) = 0$, and $f_k \leq f_{k+1}$ on $X - N$ for any k. Then, for any $E \in \mathcal{A}$, we have*

$$\lim_{k\to\infty} \int_E f_k \, d\mu = \int_E \lim_{k\to\infty} f_k \, d\mu.$$

Proof. By monotonicity, $f(x) := \lim\limits_{k\to\infty} f_k(x)$ must exist or $+\infty$ for any $x \in X - N$, and f can be easily shown to be measurable. Since $f \geq f_k$ on $X - N$, we have

$$\int_E f \, d\mu \geq \int_E f_k \, d\mu \;\; \forall k \implies \int_E f \, d\mu \geq \lim_{k\to\infty} \int_E f_k \, d\mu$$

where the RHS has limit again by the monotonicity. Therefore, it suffices to prove the reverse inequality.

Take an arbitrary simple function $\varphi := \sum_{j=1}^N c_j \chi_{A_j}$ such that $0 \leq \varphi \leq f$. We need to estimate $\int_E f_k \, d\mu$ from below by the the integral $\int_E \varphi \, d\mu$. Then, taking supremum on both sides will prove the reverse inequality. Take any $\varepsilon \in (0,1)$, $k \in \mathbb{N}$, and consider the set

$$E_k^\varepsilon := \{x \in E : f_k(x) \geq (1-\varepsilon)\varphi(x)\} \in \mathcal{A}.$$

It is easy to verify that

$$E = \bigcup_{k=1}^\infty E_k^\varepsilon.$$

To see this, take any $x \in E$, and since $f(x) \geq \varphi(x) > (1-\varepsilon)\varphi(x)$ (assuming $\varphi(x) > 0$, otherwise it is trivial that $x \in E_k^\varepsilon$ for all k). As $\lim_k f_k(x) = f(x) > (1-\varepsilon)\varphi(x)$, there must be a large k (depending on x) such that $f_k(x) > (1-\varepsilon)\varphi(x)$. In fact, this is the role played by the $(1-\varepsilon)$-factor so as to allow us to use this property of limit. Then, we have

$$\begin{aligned}\int_E f_k \, d\mu \geq \int_{E_k^\varepsilon} f_k \, d\mu &\geq (1-\varepsilon) \int_{E_k^\varepsilon} \varphi \, d\mu \\ &= (1-\varepsilon) \sum_{j=1}^N c_j \mu(E_k^\varepsilon \cap A_j).\end{aligned}$$

Letting $k \to \infty$, we have

$$\begin{aligned}\lim_{k\to\infty} \int_E f_k \, d\mu &\geq (1-\varepsilon) \sum_{j=1}^\infty c_j \mu\left(\bigcup_{k=1}^\infty E_k^\varepsilon \cap A_j\right) \\ &= (1-\varepsilon) \sum_{j=1}^N c_j \mu(E \cap A_j) \\ &= (1-\varepsilon) \int_E \varphi \, d\mu.\end{aligned}$$

Letting $\varepsilon \to 0^+$, and then taking supremum of all simple functions φ's where $0 \le \varphi \le f$, we conclude that

$$\lim_{k\to\infty} \int_E f_k \, d\mu \ge \int_E f \, d\mu.$$

□

The following result is will be used to prove the Lebesgue's dominated convergence theorem (LDCT). It is sometimes useful in other context (so we call it a "theorem"):

Theorem 8.16 (Fatou's Lemma). *Let $(X, \mathcal{A}, \mu)$ be a measure space and $\{f_k\}$ be a sequence of **non-negative** measurable functions. Then, for any $E \in \mathcal{A}$, we have*

$$\int_E \liminf_{k\to\infty} f_k \, d\mu \le \liminf_{k\to\infty} \int_E f_k \, d\mu.$$

Proof. It is an easy corollary of the monotone convergence theorem (MCT). Recall that $\liminf$ can be written as

$$\liminf_{k\to\infty} f_k = \lim_{k\to\infty} \inf_{j\ge k} f_j.$$

Denote $g_k(x) := \inf_{j\ge k} f_j(x)$ which is a monotone increasing sequence of non-negative functions. Therefore, MCT can be applied on g_k. Note also that $g_k = \inf_{j\ge k} f_j \le f_k$ for any k, so we have:

$$\liminf_{k\to\infty} \int_E f_k \, d\mu \ge \liminf_{k\to\infty} \int_E g_k \, d\mu = \underbrace{\lim_{k\to\infty} \int_E g_k \, d\mu = \int_E \lim_{k\to\infty} g_k \, d\mu}_{\text{by MCT}} = \int_E \liminf_{k\to\infty} f_k \, d\mu.$$

□

Remark 8.17. Some textbook (such as Stein) proves the Fatou's lemma first, and then use it to prove the MCT.

In Stein's book, integrals of simple functions are defined in the same way as in this lecture notes. However, integrals of non-negative measurable functions are defined in a slightly different way. For any such function f, the book constructed an increasing sequence of simple functions φ_j converging to f, and define

$$\$_E f \, d\mu := \lim_{j\to\infty} \int_E \varphi_j \, d\mu.$$

Here we use \$ as the integral sign to avoid confusion with our definition of $\int$. We still use $\int$ in the RHS because the integrand is a simple function. The construction in Stein's book was written for Lebesgue measures in $\mathbb{R}^n$, yet it can be easily generalized to any measure spaces:

Proposition 8.18. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Suppose $f : X \to [0, \infty]$ is a non-negative measurable function. Then, there exists a sequence of simple functions $\varphi_k : X \to \mathbb{R}$ such that*

$$\varphi_1(x) \le \varphi_2(x) \le \varphi_3(x) \le \cdots \text{ and } \lim_{k\to\infty} \varphi_k(x) = f(x) \ \ \forall x \in X.$$

Idea of proof. For each $k \in \mathbb{N}$, we define

$$\varphi_k := \sum_{j=1}^{2^k k} \frac{j-1}{2^k} \chi_{f^{-1}[\frac{j-1}{2^k}, \frac{j}{2^k})} + k\chi_{f^{-1}[k,\infty]}.$$

We leave it as an exercise for readers to verify that this sequence fulfill the desired conditions. □

It can be shown using MCT that Stein's definition of integrals are in fact consistent with our definition. It is because we have

$$\$_E f\,d\mu = \underbrace{\lim_{j\to\infty}\int_E \varphi_j\,d\mu = \int_E \lim_{j\to\infty}\varphi_j\,d\mu}_{\text{by MCT}} = \int_E f\,d\mu.$$

MCT can also be used to prove that integrations are linear on all measurable functions (not necessarily non-negative). Precisely, we have

Proposition 8.19. *Let $(X, \mathcal{A}, \mu)$ be a measure space, $E \in \mathcal{A}$, and $f, g : X \to [-\infty, \infty]$ be two* ***integrable*** *measurable functions. Then given any $a, b \in \mathbb{R}$, we have*

$$\int_E (af + bg)\,d\mu = a\int_E f\,d\mu + b\int_E g\,d\mu.$$

Proof. We first show the result holds when $f, g \geq 0$ on X, and $a, b \geq 0$. Using Proposition 8.18 , one can find two increasing sequences of non-negative simple functions $\varphi_j, \psi_j : X \to \mathbb{R}$ such that $\varphi_j \to f$ and $\psi_j \to g$ on X. Since $a, b \geq 0$, the sequence $a\varphi_j + b\psi_j$ is also increasing, so by MCT we get:

$$\lim_{j\to\infty}\int_E (a\varphi_j + b\psi_j)\,d\mu = \int_E \lim_{j\to\infty}(a\varphi_j + b\psi_j)\,d\mu = \int_E (af + bg)\,d\mu.$$

On the other hand, by (1) of Proposition 8.8 we know that integrals on non-negative simple functions satisfy linearity, so we have:

$$\lim_{j\to\infty}\int_E (a\varphi_j + b\psi_j)\,d\mu = \lim_{j\to\infty}\left(a\int_E \varphi_j\,d\mu + b\int_E \psi_j\,d\mu\right) = a\int_E f\,d\mu + b\int_E g\,d\mu.$$

This proves the desired result when everything is non-negative. Note also that this case does not require f, g to be integrable.

Next we extend the result to any measurable functions f, g with $a = b = 1$. Observing that $f + g = f_+ - f_- + g_+ - g_-$ and $f + g = (f+g)_+ - (f+g)_-$, so by rearrangement (so that both sides consist of non-negative terms) we have:

$$(f+g)_+ + f_- + g_- = (f+g)_- + f_+ + g_+.$$

Applying the linearity on the above non-negative functions, we get:

$$\int_E (f+g)_+\,d\mu + \int_E f_-\,d\mu + \int_E g_-\,d\mu = \int_E (f+g)_-\,d\mu + \int_E f_+\,d\mu + \int_E g_+\,d\mu.$$

From the definition of integrals, we proved:

$$\begin{aligned}\int_E (f+g)\,d\mu &= \int_E (f+g)_+\,d\mu - \int_E (f+g)_-\,d\mu \\ &= \left(\int_E f_+\,d\mu + \int_E g_+\,d\mu\right) - \left(\int_E f_-\,d\mu + \int_E g_-\,d\mu\right) \\ &= \int_E f\,d\mu + \int_E g\,d\mu,\end{aligned}$$

as desired. Note that the above argument requires integrability of f and g.

To address the case with any $a, b \in \mathbb{R}$, we apply the above previous result with $g = -f$:

$$0 = \int_E (f + (-f))\,d\mu = \int_E f\,d\mu + \int_E (-f)\,d\mu.$$

This implies

$$\int_E (-f)\,d\mu = -\int_E f\,d\mu.$$

Then, for any $a \geq 0$, we have

$$\begin{aligned}
\int_E af\,d\mu &= \int_E a(f_+ - f_-)\,d\mu \\
&= \int_E af_+\,d\mu + \int_E (-af_-)\,d\mu \\
&= \int_E af_+\,d\mu - \int_E af_-\,d\mu \\
&= a\int_E f_+\,d\mu - a\int_E f_-\,d\mu \qquad \text{(by (3) of Proposition 8.10)} \\
&= a\int_E f\,d\mu.
\end{aligned}$$

If $a < 0$, we could modify the first line by

$$\int_E af\,d\mu = \int_E (-a)(f_- - f_+)\,d\mu,$$

and the rest of proof is similar, *mutatis mutandis*.

This shows $\int_E af\,d\mu = a\int_E f\,d\mu$ for any $a \in \mathbb{R}$. It completes the proof of linearity. □

Now with linearlity, the following results about integrals can be proved easily. We leave the proofs as exercises for readers.

Corollary 8.20. *Under the same assumption of the proposition, we have:*

(1) If $E, F \in \mathcal{A}$ with $E \cap F = \emptyset$, then $\displaystyle\int_{E\cup F} f\,d\mu = \int_E f\,d\mu + \int_F f\,d\mu.$

(2) If $f \leq g$ a.e. on E, then $\displaystyle\int_E f\,d\mu \leq \int_E g\,d\mu.$

(3) The triangle inequality holds:

$$\left|\int_E f\,d\mu\right| \leq \int_E |f|\,d\mu.$$

Finally, we present an very important result in measure theory. It gives a very mild condition on a sequence of measurable functions $\{f_j\}_{j=1}^{\infty}$ that allows us to interchange the limit and integral signs. Although uniform convergence of $\{f_j\}$ would also do the job, it is a very strong condition that cannot be easily met. Also, for integral over a set with infinite measure, uniform convergence in fact cannot guarantee such an interchange.

Theorem 8.21 (Lebesgue's Dominated Convergence Theorem (LDCT))**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space, $E \in \mathcal{A}$, and $\{f_j\}_{j=1}^{\infty}$ and f be measurable functions from X to $[-\infty, \infty]$ such that $f_j \to f$ a.e. on E. Suppose there exists a measurable function $g : X \to [0, \infty]$ such that*

(i) $|f_j| \leq g$ a.e. on E for any j, and

(ii) g is integrable on E, i.e. $\int_E |g|\,d\mu < \infty$,

then we have

$$\lim_{j\to\infty} \int_E f_j\,d\mu = \int_E f\,d\mu.$$

Proof. For simplicity let's just assume $|f_j| \leq g$ on the whole E for any j. Otherwise simply replace E in the following proof by $E - N$ where $N \in \mathcal{A}$ and $\mu(N) = 0$. Using the given condition (i), we have $g - f_j, g + f_j$ on non-negative functions. Applying Fatou's lemma on both functions, we have

$$\begin{aligned}\int_E g\,d\mu - \int_E f\,d\mu = \int_E (g-f)\,d\mu &= \int_E \liminf_{j\to\infty}(g-f_j)\,d\mu \\ &\leq \liminf_{j\to\infty}\int_E (g-f_j)\,d\mu \\ &= \int_E g\,d\mu - \limsup_{j\to\infty}\int_E f_j\,d\mu.\end{aligned}$$

As $\int_E g\,d\mu$ is a finite real number, the above implies

$$\int_E f\,d\mu \geq \limsup_{j\to\infty}\int_E f_j\,d\mu.$$

By applying a similar argument on $g + f_j$, we can also prove

$$\int_E g\,d\mu + \int_E f\,d\mu \leq \int_E g\,d\mu + \liminf_{j\to\infty}\int_E f_j\,d\mu \implies \int_E f\,d\mu \leq \liminf_{j\to\infty}\int_E f_j\,d\mu.$$

However, both results combined will show

$$\limsup_{j\to\infty}\int_E f_j\,d\mu \leq \int_E f\,d\mu \leq \liminf_{j\to\infty}\int_E f_j\,d\mu$$

which could only happen if they are equalities. It concludes the desired result. □

When we justify the interchange between an infinite summation and an integral sign, there is a more straight-forward condition (unlike in LDCT we need to come up with an integrable function g). This is known as:

Theorem 8.22 (Tonelli's Theorem). *Let $(X, \mathcal{A}, \mu)$ be a measure space, $E \in \mathcal{A}$, and $f_k : X \to [-\infty, \infty]$ be a sequence of measurable functions. Suppose*

$$\sum_{j=1}^{\infty}\int_E |f_j|\,d\mu < \infty,$$

then we have

$$\int_E \sum_{j=1}^{\infty} f_j\,d\mu = \sum_{j=1}^{\infty}\int_E f_j\,d\mu.$$

Proof. Denote for any $k \in \mathbb{N} \cup \{\infty\}$:

$$F_k := \sum_{j=1}^{k} f_j \quad \text{and} \quad g_k := \sum_{j=1}^{k} |f_j|\,.$$

Then by triangle inequality, we have

$$|F_k| \leq g_k \leq g_\infty \quad \text{on } X.$$

Note that g_k is an increasing sequence, so by MCT we have

$$\lim_{k\to\infty}\int_E g_k\,d\mu = \int_E g_\infty\,d\mu \implies \sum_{j=1}^{\infty}\int_E |f_j|\,d\mu = \int_E g_\infty\,d\mu.$$

Then, by the given condition, we know that g_∞ is integrable on E. Finally by LDCT, we conclude that

$$\lim_{k\to\infty}\int_E F_k\,d\mu = \int_E \lim_{k\to\infty} F_k\,d\mu,$$

which is equivalent to the desired result. □

8.3.1. Completeness of L^p. Some of the above convergence theorems can be used to show that the L^p-space is complete. On a measure space $(X, \mathcal{A}, \mu)$ and any μ-measurable function $f : X \to [-\infty, \infty]$, we define for any $p \in [1, \infty)$:

$$\|f\|_p := \left(\int_X |f|^p \, d\mu\right)^{1/p}.$$

The vector space $L^p(X)$ is the set of all measurable functions $f : X \to [-\infty, \infty]$ such that $\|f\|_p < \infty$. In order for $\| \ \|_p$ to be a norm, we usually identify two functions $f, g : X \to [-\infty, \infty]$ are "equal" if $f = g$ a.e. on X.

Theorem 8.23 (Completeness of L^p-space). *Let $(X, \mathcal{A}, \mu)$ be a measure space. Then, $\big(L^p(X), \| \ \|_p\big)$ is a Banach space for any $p \geq 1$.*

Proof. Given $\{F_n\}_{n=1}^\infty$ is a Cauchy sequence in $L^p(X)$, then for any $j \in \mathbb{N}$, one can choose an strictly increasing sequence $\{n_j\}$ in $\mathbb{N}$ such that whenever $m, n \geq n_j$, we have

$$\|F_n - F_m\|_p < \frac{1}{2^j}.$$

Then, set

$$\begin{aligned} f_1 &:= F_{n_1} \\ f_j &:= F_{n_j} - F_{n_{j-1}} \ \ \forall j \geq 2. \end{aligned}$$

It can be checked easily that

$$\sum_{j=1}^k f_j = F_{n_k}.$$

By MCT, we have

$$\begin{aligned} \int_X \left(\sum_{j=1}^\infty |f_j|\right)^p d\mu &= \int_X \lim_{N\to\infty} \left(\sum_{j=1}^N |f_j|\right)^p d\mu = \lim_{N\to\infty} \int_X \left(\sum_{j=1}^N |f_j|\right)^p d\mu \\ &\leq \lim_{N\to\infty} \left\|\sum_{j=1}^N |f_j|\right\|_p^p \leq \lim_{N\to\infty} \left(\sum_{j=1}^N \|f_j\|_p\right)^p \\ &= \left(\|F_{n_1}\|_p + \lim_{N\to\infty} \sum_{j=2}^N \left\|F_{n_j} - F_{n_{j-1}}\right\|_p\right)^p \\ &\leq \|F_{n_1}\|_p + \sum_{j=2}^\infty \frac{1}{2^{j-1}} < \infty. \end{aligned}$$

This shows $\sum_{j=1}^\infty |f_j|$ is in L^p, and so $\sum_{j=1}^\infty |f_j|$ is finite a.e. on X. By absolute convergence test, $\sum_{j=1}^\infty f_j$ converges a.e. on X. By the relation between f_j and F_{n_k}, we have F_{n_k} converges a.e. on X as $k \to \infty$.

Suppose $F_{n_k} \to F_\infty$ a.e. on X. We need to argue that in fact $\|F_{n_k} - F_\infty\|_p \to 0$ and $F_\infty \in L^p(X)$. For any $x \in X$, we have

$$|F_{n_k}(x) - F_\infty(x)|^p = \left|\sum_{j=k+1}^\infty f_j(x)\right|^p \leq \left(\sum_{j=k+1}^\infty |f_j(x)|\right)^p \leq \left(\sum_{j=1}^\infty |f_j(x)|\right)^p$$

Recall that $\left(\sum_j |f_j|\right)^p$ is integrable on X, so by LDCT we have

$$\lim_{k\to\infty}\int_X |F_{n_k} - F_\infty|^p \, d\mu = \int_X \underbrace{\lim_{k\to\infty} |F_{n_k} - F_\infty|^p}_{=0 \text{ a.e.}} \, d\mu = 0.$$

It is exactly saying that $\|F_{n_k} \to F_\infty\|_p \to 0$. The fact that $F_\infty \in L^p(X)$ follows from triangle inequality:

$$\begin{aligned}\|F_{n_k}\|_p &\leq \left\|F_{n_k} - F_{n_{k-1}}\right\|_p + \left\|F_{n_{k-1}} - F_{n_{k-2}}\right\|_p + \cdots + \|F_{n_2} - F_{n_1}\|_p + \|F_{n_1}\|_p \\ &\leq \frac{1}{2^{k-1}} + \frac{1}{2^{k-2}} + \cdots + \frac{1}{2} + \|F_{n_1}\|_p \leq 1 + \|F_{n_1}\|_p .\end{aligned}$$

Hence $\|F_\infty\|_p < 1 + \|F_{n_1}\|_p < \infty$.

Now that we have a subsequence $\{F_{n_k}\}$ converging in L^p-norm to a function $F_\infty \in L^p(X)$. Recall that the main sequence $\{F_n\}$ is Cauchy, one can easily prove that $F_n \to F_\infty$ in L^p-norm as in the real line case. □

Within the proof of the above theorem, we see that if $\{F_n\}$ is a Cauchy sequence in $L^p(X)$, then there exists a subsequence $\{F_{n_k}\}$ converging pointwise a.e. on X to a function F_∞ in $L^p(X)$. Therefore, we have the following corollary:

Corollary 8.24. *If $\{F_n\}_{n=1}^\infty$ is a sequence in $L^p(X)$ such that it converges to a function $F_\infty \in L^p(X)$ in L^p-norm, then there exists a subsequence $\{F_{n_k}\}$ converging pointwise to F_∞ a.e. on X.*

Exercise 8.4. Consider the collection of Lebesgue measurable sets $\mathcal{M}$ in $\mathbb{R}^n$. For any pair of $A, B \in \mathcal{M}$, we define

$$d(A, B) := \mathcal{L}^n\big((A - B) \cup (B - A)\big).$$

Show that $(\mathcal{M}, d)$ is a complete metric space.

Exercise 8.5. Show that the set of simple functions on a measure space $(X, \mathcal{A}, \mu)$ is dense in $L^1(X)$, meaning that for any $f \in L^1(X)$, there exists a sequence $\{\varphi_j\}$ of simple functions such that $\|\varphi_j - f\|_1 \to 0$.

Exercise 8.6. Show that the set of continuous functions $f : \mathbb{R} \to \mathbb{R}$ with compact support is dense in $L^1(X)$.

8.4. Convergence modes

Let $(X, \mathcal{A}, \mu)$ be a measure space. Given a sequence of measurable functions $f_j : X \to \mathbb{R}$, there are several different notions (or modes) of convergence of f_j. Here is a list of some common modes. Let $E \in \mathcal{A}$, then we say:

(a) $f_j \to f$ a.e. on E: there exists $N \in \mathcal{A}$ with $\mu(N) = 0$ such that $\lim_{j\to\infty} f_j(x) = f(x)$ for any $x \in E - N$.

(b) $f_j \to f$ uniformly on E:

$$\lim_{j\to\infty} \sup_E |f_j - f| = 0.$$

(c) $f_j \to f$ in L^1-norm on E:

$$\lim_{j\to\infty} \int_E |f_j - f| \, d\mu = 0.$$

(d) $f_j \to f$ in measure on E: for any $\varepsilon > 0$, we have:

$$\lim_{j\to\infty} \mu\left(\{x \in E : |f_j(x) - f(x)| \geq \varepsilon\}\right) = 0.$$

To avoid the situation of $\infty - \infty$, we assume that $f_j(x)$ is finite for any $x \in E$, yet $f(x)$ could be infinite. These convergence modes are closely related but not equivalent. There are some examples that f_j converges in one mode but not another.

Example 8.25. For each $j \in \mathbb{N}$, we express it uniquely in the form of:

$$j = 2^k + m$$

where $k \in \mathbb{N} \cup \{0\}$ and $m \in \{0, 1, \cdots, 2^k - 1\}$. Then we define $f_j : [0,1] \to \mathbb{R}$ as:

$$f_j = \chi_{\left[\frac{m}{2^k}, \frac{m+1}{2^k}\right]}.$$

For each $x \in [0,1]$, a sketch of the graphs could tell that there are infinitely many j's such that $f_j(x) = 0$ and also infinitely many j's such that $f_j(x) = 1$. Therefore, $f_j(x)$ diverges for any fixed $x \in [0,1]$. However, we have

$$\int_{[0,1]} |f_j - 0| \, d\mathcal{L}^1 = \frac{1}{2^k} \to 0 \quad \text{as } j \to \infty,$$

and also that for any $\varepsilon > 0$, we have $\{x : |f_j(x) - 0| \geq \varepsilon\} = [\frac{m}{2^k}, \frac{m+1}{2^k}]$ which has Lebesgue measure $\frac{1}{2^k} \to 0$ as $j \to \infty$. These shows $f_j \to 0$ in both L^1-norm and in measure (with respect to $\mathcal{L}^1$) on $[0,1]$. □

Example 8.26. With some modification of the above example, we can construct a sequence f_j which converges in measure but not in L^1-norm. Define $f_j : [0,1] \to \mathbb{R}$ by:

$$f_j = 2^j \chi_{[2^{-j-1}, 2^{-j}]}.$$

Then, for any $\varepsilon > 0$, we have

$$\mathcal{L}^1\left(\{x \in \mathbb{R} : |f_j(x) - 0| \geq \varepsilon\}\right) = \frac{1}{2^{j+1}} \to 0,$$

so $f_j \to 0$ in measure on $[0,1]$. However, we have

$$\int_{[0,1]} |f_j - 0| \, d\mathcal{L}^1 = \frac{1}{2},$$

so f_j does not converge to 0 in L^1-norm. Also, it is easy to see $f_j \to 0$ a.e. on $[0,1]$ (except at $x = 0$). □

Next we see how these four convergence modes are linked together.

Theorem 8.27 (Egorov's Theorem). *Let $(X, \mathcal{A}, \mu)$ be a measure space with $E \in \mathcal{A}$ and $\mu(E) < \infty$. Suppose $f, f_j : X \to \mathbb{R}$ are measurable functions, and $f_j \to f$ a.e. on E. Then, for any $\varepsilon > 0$, there exists $A_\varepsilon \in \mathcal{A}$ with $\mu(A_\varepsilon) < \varepsilon$ such that $f_j \to f$ on uniformly $E - A_\varepsilon$ as $j \to \infty$.*

Proof. We define the set:

$$C := \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{k>j} \left\{ x \in E : |f_k(x) - f(x)| < \frac{1}{2^i} \right\}.$$

Then, $x \in C$ if and only if $\forall i \in \mathbb{N}$, $\exists j \in \mathbb{N}$ such that whenever $k > j$, we have $|f_k(x) - f(x)| < \frac{1}{2^i}$, which is equivalent to $f_j(x) \to f(x)$ as $j \to \infty$. Since we are given that $f_j \to f$ a.e. on E, there exists $N \in \mathcal{A}$ with $\mu(N) = 0$ such that $f_j(x) \to f(x)$ for any $x \in E - N$. Then, we have $E - N \subset C \subset E$. By $\mu(E) < \infty$, we have $\mu(E) = \mu(C)$ and $\mu(E - C) = 0$.

Denote for each $i, j \in \mathbb{N}$,

$$C_i := \bigcup_{j=1}^{\infty} \bigcap_{k>j} \left\{ x \in E : |f_k(x) - f(x)| < \frac{1}{2^i} \right\},$$

$$C_{i,j} = \bigcap_{k>j} \left\{ x \in E : |f_k(x) - f(x)| < \frac{1}{2^i} \right\}.$$

It is easy to note that $C_i \supset C_{i+1}$ for all i, so

$$\lim_{i\to\infty} \mu(C_i) = \mu\left(\bigcap_{i=1}^{\infty} C_i \right) = \mu(C) = \mu(E).$$

On the other hand, $\mu(C_i)$ is a decreasing sequence and is bounded above by $\mu(E)$. This forces $\mu(C_i) = \mu(E)$ for any $i \in \mathbb{N}$. Moreover, for each fixed $i \in \mathbb{N}$, we have $C_{i,j} \subset C_{i,j+1}$, so

$$\lim_{j\to\infty} \mu(C_{i,j}) = \mu\left(\bigcup_{j=1}^{\infty} C_{i,j} \right) = \mu(C_i) = \mu(E).$$

Now for any $\varepsilon > 0$ and $i \in \mathbb{N}$, there exists $j(i) \in \mathbb{N}$ such that

$$\mu(E) - \frac{\varepsilon}{2^i} < \mu(C_{i,j(i)}) \le \mu(E) \implies \mu(E - C_{i,j(i)}) < \frac{\varepsilon}{2^i}.$$

Let $A := \bigcup_{i=1}^{\infty}(E - C_{i,j(i)}) = E - \bigcap_{i=1}^{\infty} C_{i,j(i)}$, then

$$\mu(A) \le \sum_{i=1}^{\infty} \mu(E - C_{i,j(i)}) < \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon.$$

We next claim that f_j converges to f uniformly on $E - A = \bigcap_{i=1}^{\infty} C_{i,j(i)}$. By the definition of $C_{i,j(i)}$, then we have

$$|f_k(x) - f(x)| \le \frac{1}{2^i} \quad \forall k > j(i).$$

Given any $\eta > 0$, we find $i \in \mathbb{N}$ sufficiently large so that $\frac{1}{2^i} < \eta$. Note that $j(i)$ depends only on i and ε but not only x. Whenever $x \in E - A \subset C_{i,j(i)}$, we have by the definition of $C_{i,j(i)}$ that for any $k > j(i)$.

$$|f_k(x) - f(x)| < \frac{1}{2^i} \implies \sup_{E-A} |f_k - f| < \frac{1}{2^i} < \eta.$$

This shows $f_k \to f$ uniformly on $E - A$, and we are done. □

It is quite clear that when $\mu(E) < \infty$, then $f_j \to f$ uniformly on E implies $f_j \to f$ in L^1-norm on E, as

$$\int_E |f_j - f| \, d\mu \leq \sup_E |f_j - f| \cdot \mu(E).$$

Convergence in L^1-norm implies convergence in measure too. It is an immediate consequence of the Chebeysev's inequality.

$$\mu\left(\{x \in E : |f_j(x) - f(x)| \geq \varepsilon\}\right) \leq \frac{1}{\varepsilon} \int_E |f_j - f| \, d\mu.$$

Letting $j \to \infty$ proves our claim.

The four convergence modes "almost" form a cycle when $\mu(E) < \infty$, except that (a) implies (b) only outside an arbitrarily small set (Egorov's Theorem), and (d) implies (a) only after taking subsequences:

Theorem 8.28. *Let $(X, \mathcal{A}, \mu)$ be a measure space, $E \in \mathcal{A}$, and $f_j : X \to \mathbb{R}$ be a sequence of measurable function such that $f_j \to f$ in measure on E to a measurable function f. Then, there exists a subsequence f_{j_k} converging to f pointwise a.e. on E.*

Proof. Given that f_j converges to f in measure on E, we have for any $k \in \mathbb{N}$,

$$\lim_{j \to \infty} \mu\left(\left\{x \in E : |f_j(x) - f(x)| \geq \frac{1}{2^k}\right\}\right) = 0,$$

and hence there exists $j_k \in \mathbb{N}$ such that

$$\mu\left(\left\{x \in E : |f_{j_k}(x) - f(x)| \geq \frac{1}{2^k}\right\}\right) < \frac{1}{2^k}.$$

Note that we can choose $j_1 < j_2 < j_3 < \cdots$. We claim that the subsequence f_{j_k} is what we desire, i.e. converges to f a.e. on E. Define

$$N = \bigcap_{i=1}^{\infty} \bigcup_{k>i} \left\{x \in E : |f_{j_k}(x) - f(x)| \geq \frac{1}{2^k}\right\}.$$

Clearly $N \in \mathcal{A}$ and $\mu(N) = 0$ (exercise for readers). We argue that $f_{j_k} \to f$ pointwise on $E - N$. To prove this, fix any $x \in E - N$, then from the definition of N, there exists $i \geq 1$ such that $|f_{j_k}(x) - f(x)| < \frac{1}{2^k}$ for any $k > i$. Clearly it implies $f_{j_k}(x) \to f(x)$ as $k \to \infty$. $\square$

Chapter 9

Product Measures

> "The Gross National *Product measures* everything except that which makes life worthwhile."
>
> Robert F. Kennedy

The goal of this chapter is to construct a new, naturally defined, measure on a product space $X \times Y$ where $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are two measure spaces. Then we prove the famous Fubini's Theorem which relates the integrals of functions on $X \times Y$ and the integrals over the cross-sections. Unlike other product structures such as product manifolds, product groups, product metric spaces, etc., it is not straight-forward to construct a naturally defined measure and σ-algebra on $X \times Y$ in order to make it a measure space. One reason is that $\mathcal{A} \times \mathcal{B}$ is not a σ-algebra. We are going to construct the product measure space in a number of steps.

9.1. Pre-measures

We first introduce the concept of pre-measures, which will be used to construct outer measures. In short, it is a set function which is not necessarily defined on an σ-algebra, yet it satisfies countable additivity for sets which are closed under countable union.

Definition 9.1 (Algebra on a set). Given a set X, we say $\mathcal{A}_0$ is an **algebra on** X if

(1) $\emptyset \in \mathcal{A}_0$;
(2) $E \in \mathcal{A}_0 \implies E^c := X - E \in \mathcal{A}_0$; and
(3) $E_1, E_2 \in \mathcal{A}_0 \implies E_1 \cup E_2 \in \mathcal{A}_0$.

Remark 9.2. Certainly, (1) and (2) show $X \in \mathcal{A}_0$ too, and (2) and (3) together show $\mathcal{A}_0$ is also closed under finite intersections.

Example 9.3. Let $X = \mathbb{R}^n$, and $\mathcal{A}_0$ is the set of all finite unions of disjoint rectangles. By rectangles we mean $\langle a_1, b_1\rangle \times \cdots \times \langle a_n, b_n\rangle$, where $a_i, b_i \in [-\infty, \infty]$, $\langle$ could mean (or [, and correspondingly for $\rangle$. Then $\mathcal{A}_0$ is an algebra of $\mathbb{R}^n$.

Example 9.4. Let $X = [0, 1)$, and $\mathcal{A}_0$ be the set of all finite unions of half-open intervals $[a_i, b_i)$'s. Then one can check that $\mathcal{A}_0$ is an algebra of X.

Example 9.5. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measure spaces, where $\mathcal{A}$ and $\mathcal{B}$ are σ-algebras. Then, on the product set $X \times Y$ the following is an algebra:

$$\mathcal{F}(\mathcal{A} \times \mathcal{B}) := \left\{ \bigsqcup_{j=1}^{N} (A_j \times B_j) : N \in \mathbb{N}, A_j \in \mathcal{A} \text{ and } B_j \in \mathcal{B} \ \forall j = 1, \cdots, N \right\}.$$

We usually call each $A_j \times B_j$, where $A_j \in \mathcal{A}$ and $B_j \in \mathcal{B}$ to be **measurable rectangles**. In short, $\mathcal{F}(\mathcal{A} \times \mathcal{B})$ is the collection of finite disjoint union of measurable rectangles in $X \times Y$. A sketch of some rectangles should tell us that it is an algebra on $X \times Y$. We leave it as an exercise for readers to prove it formally.

Definition 9.6 (Pre-measure, a.k.a. "measure on an algebra" in Royden's book)**.** Given a set X, and an algebra $\mathcal{A}_0$ on X. A set function $\mu_0 : \mathcal{A}_0 \to [0, \infty]$ is said to be a **pre-measure on** $(X, \mathcal{A}_0)$ if

(1) $\mu_0(\emptyset) = 0$; and

(2) whenever $E \in \mathcal{A}_0$ is a countable union of pairwise disjoint sets $\{E_j\}_{j=1}^{\infty}$ in $\mathcal{A}_0$, i.e. $E = \bigcup_{j=1}^{\infty} E_j$, then

$$\mu_0\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu_0(E_j).$$

Then, we call $(X, \mathcal{A}_0, \mu_0)$ a **pre-measure space**.

The major difference between a measure space and a pre-measure space is that the former requires that $\mathcal{A}$ is a σ-algebra whereas the later only requests that $\mathcal{A}_0$ is an algebra. If $(X, \mathcal{A}_0, \mu_0)$ a pre-measure space that $\mathcal{A}_0$ is in fact a σ-algebra, then it is indeed a measure space. Also, $\mathcal{A}_0$ is closed under finite union, so the condition stated in (2) is automatically satisfied, so a pre-measure μ_0 is necessarily finitely additive.

An example of a pre-measure space is $X = \mathbb{R}^2$, $\mathcal{A}_0$ is the finite union of disjoint rectangles (including closed, open, and half-closed), and μ_0 is defined as follows. For any $E \in \mathcal{A}_0$, we can write it $E = \bigcup_{j=1}^{\infty} R_j$ where R_j's are disjoint rectangles. We then define

$$\mu_0(E) = \sum_{j=1}^{N} |R_j|.$$

It can be shown that the above definition is independent of the choice of disjoint rectangles R_j's such that $E = \bigcup_j R_j$. Clearly μ_0 satisfies (1) in Definition 9.6. However, (2) is less trivial. We will prove (2) in a more general setting in the following proposition.

Proposition 9.7. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measure spaces, and $\mathcal{F}(\mathcal{A} \times \mathcal{B})$ be defined as in Example 9.5. We define $\mu_0 : \mathcal{F}(\mathcal{A} \times \mathcal{B}) \to [0, \infty]$ as follows. For each set $E \in \mathcal{F}(\mathcal{A} \times \mathcal{B})$, we write*

$$E = \bigsqcup_{j=1}^{N} (A_j \times B_j) \ \text{ where } A_j \in \mathcal{A} \text{ and } B_j \in \mathcal{B},$$

and define

$$\mu_0(E) := \sum_{j=1}^{N} \mu(A_j)\nu(B_j).$$

Then, $\big(X \times Y, \mathcal{F}(\mathcal{A} \times \mathcal{B}), \mu_0\big)$ is a pre-measure space.

Proof. First it is quite clear that $\mu_0(\emptyset) = 0$ as as $\mu(\emptyset) = \nu(\emptyset) = 0$. The proof of (2) is less trivial and it involves the use of MCT. Suppose $E \in \mathcal{F}(\mathcal{A} \times \mathcal{B})$ and that $E = \bigsqcup_{j=1}^{\infty} E_j$ where $E_j \in \mathcal{F}(\mathcal{A} \times \mathcal{B})$. Write

$$E := \bigsqcup_{i=1}^{N} (A_i \times B_i) \qquad \text{where } A_i \in \mathcal{A} \text{ and } B_i \in \mathcal{B}$$
$$E_j := \bigsqcup_{i=1}^{N_j} (A_{j,i} \times B_{j,i}) \qquad \text{where } A_{j,i} \in \mathcal{A} \text{ and } B_{j,i} \in \mathcal{B}$$

then we have

$$\bigsqcup_{i=1}^{N} (A_i \times B_i) = \bigsqcup_{j=1}^{\infty} \bigsqcup_{i=1}^{N_j} (A_{j,i} \times B_{j,i})$$

Taking intersection with $A_k \times B_k$ for a fixed $k \in \{1, \cdots, N\}$ on both sides, we get:

$$A_k \times B_k = \left(\bigsqcup_{j=1}^{\infty} \bigsqcup_{i=1}^{N_j} (A_{j,i} \times B_{j,i}) \right) \cap (A_k \times B_k) = \bigsqcup_{j=1}^{\infty} \bigsqcup_{i=1}^{N_j} \big((A_{j,i} \cap A_k) \times (B_{j,i} \cap B_k) \big), \tag{9.1}$$

which is also a countable union of disjoint measurable rectangles. For simplicity, we relabel the rectangles in RHS by

$$A_k \times B_k = \bigsqcup_{m=1}^{\infty} (A'_{k,m} \times B'_{k,m}), \text{ where } A'_{k,m} \in \mathcal{A} \text{ and } B'_{k,m} \in \mathcal{B}.$$

We need to prove that $\mu(A_k)\nu(B_k) = \sum_{m=1}^{\infty} \mu(A'_{k,m})\nu(B'_{k,m})$, then taking summation over all k will complete the proof of (2).

For any $x \in A_k$, we note that

$$B_k = \bigsqcup_{\{m : x \in A'_{k,m}\}} B'_{k,m},$$

and so

$$\nu(B_k) = \sum_{\{m : x \in A'_{k,m}\}} \nu(B'_{k,m}) = \sum_{m=1}^{\infty} \chi_{A'_{k,m}}(x) \nu(B'_{k,m}).$$

Note that the RHS is a series of measurable function with non-negative terms, so by MCT we have

$$\int_{A_k} \nu(B_k) \, d\mu = \sum_{m=1}^{\infty} \int_{A_k} \chi_{A'_{k,m}} \nu(B'_{k,m}) \, d\mu \implies \mu(A_k)\nu(B_k) = \sum_{m=1}^{\infty} \mu(A'_{k,m})\nu(B'_{k,m}),$$

as desired. Summing up over all k, we get

$$\mu_0(E) = \sum_{k=1}^{N} \mu(A_k)\mu(B_k) = \sum_{k=1}^{N} \sum_{m=1}^{\infty} \mu(A'_{k,m})\nu(B'_{k,m}).$$

Recall that the $A'_{k,m}$'s and $B'_{k,m}$ came from the relabelling of the union in the RHS of (9.1), so to express the collection using the previous labels, we get

$$\begin{aligned}
\mu_0(E) &= \sum_{k=1}^{N}\sum_{j=1}^{\infty}\sum_{i=1}^{N_j}\mu(A_{j,i}\cap A_k)\nu(B_{j,i}\cap B_k)\\
&= \sum_{j=1}^{\infty}\sum_{i=1}^{N_j}\sum_{k=1}^{N}\mu_0\big((A_{j,i}\times B_{j,i})\cap(A_k\times B_k)\big)\\
&= \sum_{j=1}^{\infty}\sum_{i=1}^{N_j}\mu_0\left((A_{j,i}\times B_{j,i})\cap\bigsqcup_{k=1}^{N}(A_k\times B_k)\right)\\
&= \sum_{j=1}^{\infty}\sum_{i=1}^{N_j}\mu_0\left((A_{j,i}\times B_{j,i})\cap E\right)\\
&= \sum_{j=1}^{\infty}\sum_{i=1}^{N_j}\mu_0(A_{j,i}\times B_{j,i}) = \sum_{j=1}^{\infty}\sum_{i=1}^{N_j}\mu(A_{j,i})\nu(B_{j,i}).
\end{aligned}$$

Note that we have used the finite additivity for sets in $\mathcal{F}(\mathcal{A}\times\mathcal{B})$.

On the other hand, by the definition of $\mu_0(E_j)$ we also have:

$$\sum_{j=1}^{\infty}\mu_0(E_j) = \sum_{j=1}^{\infty}\sum_{i=1}^{N_j}\mu(A_{j,i})\nu(B_{j,i}).$$

It completes the proof of (2) that $\mu_0(E) = \sum_j \mu_0(E_j)$. $\square$

For any collection $\mathcal{S}$ of subsets of a set X, we denote the smallest σ-algebra containing $\mathcal{S}$ by:

$$\sigma(\mathcal{S}) := \bigcap_{\mathcal{A}\supset\mathcal{S}}\mathcal{A}.$$

Exercise 9.1. Show that $\sigma\left(\mathcal{F}(\mathcal{A}\times\mathcal{B})\right) = \sigma(\mathcal{A}\times\mathcal{B})$.

Exercise 9.2. Let $\mathcal{A}_0$ be the algebra of all finite disjoint unions of rectangles in $\mathbb{R}^n$ (see Example 9.3). Denote by $\mathcal{B}^n$ the Borel σ-algebra of $\mathbb{R}^n$. Show that $\sigma(\mathcal{A}_0) = \mathcal{B}^n$.

Exercise 9.3. Let X be a set and $\mathcal{A}$ be a σ-algebra on X. Consider the collection $\sigma(\mathcal{A}\times\mathcal{A})$ of sets in $X\times X$. Show that if $E\in\sigma(\mathcal{A}\times\mathcal{A})$, then

$$\{x\in X : (x,x)\in E\}\in\mathcal{A}.$$

9.2. Carathéodory extension theorem

We introduced the notion of pre-measure because it can be used to induce an outer measure. In particular, the pre-measure product space $(X\times Y, \mathcal{F}(\mathcal{A}\times\mathcal{B}), \mu_0)$ in Proposition 9.7 would induce an outer measure which will be called the product measure on $X \times Y$.

Theorem 9.8 (Carathéodory extension theorem, a.k.a. Hahn-Kolmogorov theorem). *Every pre-measure $\mu_0 : \mathcal{A}_0 \to [0,\infty]$ on a set X can be extended to a countably additive measure $\mu : \sigma(\mathcal{A}_0) \to [0,\infty]$, where $\sigma(\mathcal{A}_0)$ is the smallest σ-algebra containing $\mathcal{A}_0$, and $\mu\big|_{\mathcal{A}_0} = \mu_0$.*

Proof. Such a measure can actually be constructed explicitly. For each $E \subset X$, we define

$$\mu^*(E) := \inf\left\{\sum_{j=1}^{\infty}\mu_0(C_j) : E \subset \bigcup_{j=1}^{\infty} C_j, \text{ where } C_j \in \mathcal{A}_0 \text{ for any } j\right\}.$$

We first need to verify that μ^* is an outer measure, then μ^* would be countably additive on sets which satisfy the Carethéodory's criterion (6.2).

It is clear that $\mu^*(\emptyset) = 0$ since $\emptyset \subset \emptyset$ and $\mu_0(\emptyset) = 0$. Also μ^* is clearly monotone by the property of infimum. We are left to prove the countable subadditivity. Let $\{E_j\}_{j=1}^{\infty}$ be a collection of subsets of X. Fix an $\varepsilon > 0$, and for each j, we choose a cover $\{C_{j,k}\}_{k=1}^{\infty}$ of E_j, with $C_{j,k} \in \mathcal{A}_0$, such that

$$\mu^*(E_j) \leq \sum_{k=1}^{\infty}\mu_0(C_{j,k}) < \mu^*(E_j) + \frac{\varepsilon}{2^j}.$$

Since $\bigcup_{j=1}^{\infty} E_j \subset \bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty} C_{j,k}$, we have

$$\mu^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\mu_0(C_{j,k}) < \sum_{j=1}^{\infty}\mu^*(E_j) + \varepsilon.$$

Letting $\varepsilon \to 0$, we proved $\mu^*\big(\bigcup_j E_j\big) \leq \sum_j \mu^*(E_j)$, concluding that μ^* is an outer measure.

Next we justify that μ^* is an extension of μ_0 by showing $\mu^*(E) = \mu_0(E)$ for any $E \in \mathcal{A}_0$. Since such an E is a cover of itself, we have $\mu^*(E) \leq \mu_0(E)$. To prove the reverse inequality, we pick any arbitrary cover $\bigcup_{j=1}^{\infty} C_j \supset E$ with $C_j \in \mathcal{A}_0$ for any j. Replacing $\{C_j\}$'s by disjoint sets

$$C_1' := C_1 \text{ and } C_j' := C_j - \bigcup_{k=1}^{j-1} C_k,$$

we still have $\bigcup_{j=1}^{\infty} C_j' = \bigcup_{j=1}^{\infty} C_j \supset E$ and $C_j' \in \mathcal{A}_0$ for any j. Noting that $E \in \mathcal{A}_0$, we $C_j' \cap E \in \mathcal{A}_0$ for any j and also

$$\bigcup_{j=1}^{\infty}(C_j' \cap E) = E.$$

Since μ_0 is a pre-measure on $\mathcal{A}_0$, from (2) of the definition we have

$$\mu_0(E) = \sum_{j=1}^{\infty}\mu_0(C_j' \cap E) \leq \sum_{j=1}^{\infty}\mu_0(C_j \cap E) \leq \sum_{j=1}^{\infty}\mu_0(C_j).$$

This gives a uniform lower bound $\mu_0(E)$ for $\sum_j \mu_0(C_j)$ over all coverings $\{C_j\}_{j=1}^\infty \subset \mathcal{A}_0$ of E. Taking infimum gives $\mu_0(E) \leq \mu^*(E)$. This establishes that $\mu^*(E) = \mu_0(E)$ for any $E \in \mathcal{A}_0$.

Denote the set of all μ-measurable sets by

$$\mathcal{M}_\mu := \{E \subset X : \mu^*(S) = \mu^*(S \cap E) + \mu^*(S - E) \text{ for any } S \subset X\}.$$

We finally show that $\mathcal{A}_0 \subset \mathcal{M}_\mu$, then since $\mathcal{M}_\mu$ is a σ-algebra, we have $\sigma(\mathcal{A}_0) \subset \mathcal{M}_\mu$. To prove this, we pick $E \in \mathcal{A}_0$, and S be any subset of X. It suffices to show

$$\mu^*(S) \geq \mu^*(S \cap E) + \mu^*(S - E)$$

so we may assume $\mu^*(S) < \infty$. For any $\varepsilon > 0$, we cover S by $\bigcup_{j=1}^\infty C_j$, where $C_j \in \mathcal{A}_0$, such that

$$\mu^*(S) \leq \sum_{j=1}^\infty \mu_0(C_j) < \mu^*(S) + \varepsilon.$$

Note that $C_j \cap E, C_j - E \in \mathcal{A}_0$, so we have by the finite additivity of μ_0 on sets in $\mathcal{A}_0$, we have

$$\begin{aligned}\mu^*(S) + \varepsilon > \sum_{j=1}^\infty \mu_0(C_j) &= \sum_{j=1}^\infty (\mu_0(C_j \cap E) + \mu_0(C_j - E)) \\ &= \sum_{j=1}^\infty \mu_0(C_j \cap E) + \sum_{j=1}^\infty \mu_0(C_j - E).\end{aligned}$$

Since $\bigcup_j (C_j \cap E) \supset S \cap E$ and $\bigcup_j (C_j - E) \supset S - E$, we have

$$\sum_{j=1}^\infty \mu_0(C_j \cap E) \geq \mu^*(S \cap E) \text{ and } \sum_{j=1}^\infty \mu_0(C_j - E) \geq \mu^*(S - E).$$

This proves $\mu^*(S) + \varepsilon > \mu^*(S \cap E) + \mu^*(S - E)$ for any $\varepsilon > 0$. Letting $\varepsilon \to 0^+$ completes the proof. □

To summarize, once we have a pre-measure space $(X, \mathcal{A}_0, \mu_0)$, we can explicitly construct an outer measure μ^* defined on any subset of X, and that μ^* is an extension of μ_0. Furthermore, sets in $\mathcal{A}_0$ would be μ-measurable and hence $\sigma(\mathcal{A}_0) \subset \mathcal{M}_\mu$. Combining with what we have proved about outer measures, we conclude that $(X, \mathcal{M}_\mu, \mu)$ and the smaller $(X, \sigma(\mathcal{A}_0), \mu)$ are both measure spaces.

Given that $(X \times Y, \mathcal{F}(\mathcal{A} \times \mathcal{B}), \mu_0)$ in Example 9.5 is a pre-measure space, we can construct an extension as in the proof of Carathéodory extension theorem (Theorem 9.8).

Definition 9.9 (Product measure)**.** Given two measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$, we define an outer measure $(\mu \times \nu)^*$ on the product space $X \times Y$, called the **product measure of** μ **and** ν as follows: for any $E \subset X \times Y$, we define

$$\begin{aligned}&(\mu \times \nu)^*(E) \\ &:= \inf\left\{\sum_{j=1}^\infty \mu_0(C_j) : E \subset \bigcup_{j=1}^\infty C_j \text{ where } C_j \in \mathcal{F}(\mathcal{A} \times \mathcal{B}) \text{ for any } j\right\} \\ &= \inf\left\{\sum_{j=1}^\infty \mu(A_j)\nu(B_j) : E \subset \bigcup_{j=1}^\infty (A_j \times B_j) \text{ where } A_j \in \mathcal{A} \text{ and } B_j \in \mathcal{B} \text{ for any } j\right\}.\end{aligned}$$

Here μ_0 is defined as in Proposition 9.7.

By Theorem 9.8, $\mu\times\nu$ is countably addictive on the σ-algebra generated by $\mathcal{F}(\mathcal{A}\times\mathcal{B})$, which is equal to the σ-algebra generated by measurable rectangles in $\mathcal{A}\times\mathcal{B}$. From now on we will denote this σ-algebra by $\sigma(\mathcal{A}\times\mathcal{B})$. For a set $E\in\sigma(\mathcal{A}\times\mathcal{B})$, we may simply denote $(\mu\times\nu)^*(E)$ by $(\mu\times\nu)(E)$.

In fact, if the two measure spaces $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\nu)$ are σ-finite (defined below), the product measure $\mu\times\nu$ defined above is the unique measure on $\sigma(\mathcal{A}\times\mathcal{B})$ such that $(\mu\times\nu)(A\times B)=\mu(A)\nu(B)$ for any $A\in\mathcal{A}$ and $B\in\mathcal{B}$.

Definition 9.10 (σ-finite measure)**.** A (pre-)measure μ in a (pre-)measure space $(X,\mathcal{A},\mu)$ is said to be σ**-finite** if X can be written as $X=\bigcup_{j=1}^{\infty}X_j$, where $X_j\in\mathcal{A}$ and $\mu(X_j)<\infty$ for any j.

Example 9.11. $(\mathbb{R}^n,\mathcal{M}_{\mathcal{L}^n},\mathcal{L}^n)$ is σ-finite as $\mathbb{R}^n=\bigcup_{j=1}^{\infty}[-j,j]^n$, and $\mathcal{L}^n\left([-j,j]^n\right)=(2j)^n<\infty$.

Example 9.12. Let $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\nu)$ be two σ-finite measure spaces, then $(X\times Y,\mathcal{F}(\mathcal{A}\times\mathcal{B}),\mu_0)$ in Proposition 9.7 is a σ-finite pre-measure space. To see this, let $X=\bigcup_j A_j$ and $Y=\bigcup_j B_j$ with $A_j\in\mathcal{A}$, $B_j\in\mathcal{B}$, $\mu(A_j),\nu(B_j)<\infty$ for any j. We can assume that $A_1\subset A_2\subset A_3\subset\cdots$ and $B_1\subset B_2\subset B_3\subset\cdots$. Then, we have

$$X\times Y=\bigcup_{j=1}^{\infty}(A_j\times B_j)$$

and $\mu_0(A_j\times B_j)=\mu(A_j)\nu(B_j)<\infty$.

Proposition 9.13. *Let $(X,\mathcal{A}_0,\mu_0)$ be a σ-finite pre-measure space, and $(X,\mathcal{A}:=\sigma(\mathcal{A}_0),\mu)$ be its measure space extension constructed as in the proof of Theorem 9.8. Suppose μ' is another (countably additive) measure on $(X,\mathcal{A},\mu')$ such that $\mu'(E)=\mu_0(E)$ for any $E\in\mathcal{A}_0$, then $\mu=\mu'$ on $\mathcal{A}$.*

Proof. We first claim that $\mu'(F)\le\mu(F)$ for any $F\in\mathcal{A}$. We recall that

$$\mu(F)=\inf\left\{\sum_{j=1}^{\infty}\mu_0(C_j):F\subset\bigcup_{j=1}^{\infty}C_j,\ \text{where } C_j\in\mathcal{A}_0 \text{ for any } j\right\}.$$

Let $\{C_j\}_{j=1}^{\infty}$ be any cover of F such that $C_j\in\mathcal{A}_0$ for any j. Since μ' is an extension of μ_0, $\mu'(C_j)=\mu_0(C_j)$ for any j. Hence, combining with the monotonicity and countable subadditivity of μ', we get:

$$\mu'(F)\le\mu'\left(\bigcup_{j=1}^{\infty}C_j\right)\le\sum_{j=1}^{\infty}\mu'(C_j)=\sum_{j=1}^{\infty}\mu_0(C_j).$$

Taking infimum over all such coverings, we get $\mu'(F)\le\mu(F)$. Note that we haven't used the σ-finite condition so far.

Next we prove the reverse inequality $\mu'(F)\ge\mu(F)$. We may assume that $\mu'(F)<\infty$ otherwise the reverse inequality trivially holds. We write

$$X=\bigcup_{j=1}^{\infty}X_j$$

with $X_j\in\mathcal{A}_0$ and $\mu_0(X_j)<\infty$ for any j. Consider that F is both μ- and μ'-measurable, and $X_j\in\mathcal{A}_0$, we have

$$\mu(X_j\cap F)+\mu(X_j-F)=\mu(X_j)=\mu_0(X_j)=\mu'(X_j)=\mu'(X_j\cap F)+\mu'(X_j-F)$$

Note that $\mu_0(X_j)$ is finite, so by subtraction we get:

$$\underbrace{\big(\mu'(X_j \cap F) - \mu(X_j \cap F)\big)}_{\leq 0} + \underbrace{\big(\mu'(X_j - F) - \mu(X_j - F)\big)}_{\leq 0} = 0,$$

and it implies

$$\mu'(X_j \cap F) = \mu(X_j \cap F) \text{ and } \mu'(X_j - F) = \mu(X_j - F).$$

Since it is true for all X_j, we concluded that

$$\mu'(F) = \mu'\left(\bigcup_{j=1}^{\infty}(X_j \cap F)\right) = \sum_{j=1}^{\infty}\mu'(X_j \cap F) = \sum_{j=1}^{\infty}\mu(X_j \cap F) = \mu\left(\bigcup_{j=1}^{\infty}(X_j \cap F)\right) = \mu(F).$$

This shows $\mu' = \mu$ on $\mathcal{A}$. □

Corollary 9.14. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two σ-**finite** measure spaces, then $\mu \times \nu$ defined in Definition 9.9 is the unique measure defined on $\sigma(\mathcal{A} \times \mathcal{B})$ that maps*

$$A \times B \mapsto \mu(A)\nu(B) \ \ \forall A \in \mathcal{A} \text{ and } B \in \mathcal{B}.$$

Proof. If $\lambda : \sigma(\mathcal{A} \times \mathcal{B}) \to [0, \infty]$ is another measure such that

$$\lambda(A \times B) = \mu(A)\nu(B) \ \ \forall A \in \mathcal{A} \text{ and } B \in \mathcal{B},$$

then by countable additivity we know that $\lambda = \mu \times \nu$ on the algebra $\mathcal{F}(\mathcal{A} \times \mathcal{B})$.

It remains to show that $(X \times Y, \mathcal{F}(\mathcal{A} \times \mathcal{B}), \mu \times \nu)$ is a σ-finite pre-measure space. It is so because if $X = \bigcup_j X_j$ and $Y = \bigcup_j Y_j$ where $X_j, Y_j \in \mathcal{A}_0$ and $\mu(X_j), \nu(Y_j) < \infty$ for all j, then we may assume both $\{X_j\}$ and $\{Y_j\}$ are increasing, and we have:

$$X \times Y = \bigcup_{j=1}^{\infty}(X_j \times Y_j).$$

It is clear that $(\mu \times \nu)(X_j \times Y_j) = \mu(X_j)\nu(Y_j) < \infty$. By Proposition 9.13, we conclude that $\lambda = \mu \times \nu$ on $\sigma(\mathcal{A} \times \mathcal{B})$. □

Exercise 9.4. Let $(X, \mathcal{A}, \mu)$ be a measure space which may not be complete (remainder: complete measure space means all subsets of a measure zero set is measurable). Show that there exists a **complete** measure space $(X, \mathcal{A}', \overline{\mu})$ such that $\mathcal{A} \subset \mathcal{A}'$ and $\overline{\mu}(E) = \mu(E)$ for any $E \in \mathcal{A}$.

It is nature to ask whether the two measure spaces:

$$\big(\mathbb{R}^m \times \mathbb{R}^n, \sigma(\mathcal{M}_{\mathcal{L}^m} \times \mathcal{M}_{\mathcal{L}^n}), \mathcal{L}^m \times \mathcal{L}^n\big) \text{ and } \big(\mathbb{R}^{m+n}, \mathcal{M}_{\mathcal{L}^{m+n}}, \mathcal{L}^{m+n}\big)$$

are the same. It turns out that this issue is more complicated than it seems, and the answer is negative, but by replacing *Lebesgue measurable sets* by *Borel sets*, it can be shown to be equal.

We will postpone the discussion of the above question after proving the Fubini's Theorem. Meanwhile, we can explain that why $\sigma(\mathcal{M}_{\mathcal{L}^m} \times \mathcal{M}_{\mathcal{L}^n}) \neq \mathcal{M}_{\mathcal{L}^{m+n}}$ through the following proposition. For each $E \in X \times Y$, we denote:

$$E_x := \{y \in Y : (x, y) \in E\} \text{ and } E^y := \{x \in X : (x, y) \in E\}.$$

We may call E_x and E^y the cross-sections of E.

Proposition 9.15. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measure spaces. For any $E \in \sigma(\mathcal{A} \times \mathcal{B})$, we have that cross-sections $E_x \in \mathcal{B}$ and $E^y \in \mathcal{A}$ for any $x \in X$, $y \in Y$.*

Proof. First we can easily see that for any $A, A_j \in X \times Y$, we have

$$\left(\bigcup_{j=1}^{\infty} A_j\right)_x = \bigcup_{j=1}^{\infty} (A_j)_x \text{ and } (X \times Y - A)_x = (Y - A_x)^c.$$

These show that each $x \in X$ the following collection

$$\Sigma_x := \{A \subset X \times Y : A_x \in \mathcal{B}\}.$$

is a σ-algebra on $X \times Y$. Moreover, for any $A \times B \in \mathcal{A} \times \mathcal{B}$, we have $(A \times B)_x = B \in \mathcal{B}$ and so $\mathcal{A} \times \mathcal{B} \subset \Sigma_x$. This implies

$$\sigma(\mathcal{A} \times \mathcal{B}) \subset \sigma(\Sigma_x) = \Sigma_x.$$

In other words, for any $E \in \sigma(\mathcal{A} \times \mathcal{B})$, we have $E_x \in \mathcal{B}$. Similar argument shows the $E^y \in \mathcal{A}$. □

One immediate consequence of this proposition is that for any $m, n \in \mathbb{N}$, we have

$$\sigma(\mathcal{M}_{\mathcal{L}^m} \times \mathcal{M}_{\mathcal{L}^n}) \subsetneq \mathcal{M}_{\mathcal{L}^{m+n}}.$$

The $(\subset)$-part follows from $\mathcal{M}_{\mathcal{L}^m} \times \mathcal{M}_{\mathcal{L}^n} \subset \mathcal{M}_{\mathcal{L}^{m+n}}$ (easy consequence using Stein's definition of Lebesgue measurable sets). To see why the inclusion is proper, we take any $N \subset \mathbb{R}^n$ which is not Lebesgue measurable, then define

$$E := \{0\} \times N \subset \mathbb{R}^m \times \mathbb{R}^n.$$

Clearly, $\mathcal{L}^{m+n}(E) = 0$ so $E \in \mathcal{M}_{\mathcal{L}^{m+n}}$. However, it cannot be in $\sigma(\mathcal{M}_{\mathcal{L}^m} \times \mathcal{M}_{\mathcal{L}^n})$ because the cross-section

$$E_0 = N \notin \mathcal{M}_{\mathcal{L}^n}.$$

Fortunately, one can still show that if we consider

$$\mathcal{B}^n := \text{ the Borel } \sigma\text{-algebra on } \mathbb{R}^n,$$

then we indeed have the following:

Proposition 9.16. *For any $m, n \in \mathbb{N}$, we have*

$$\big(\mathbb{R}^m \times \mathbb{R}^n, \sigma(\mathcal{B}^m \times \mathcal{B}^n), \mathcal{L}^m \times \mathcal{L}^n\big) = \big(\mathbb{R}^{m+n}, \mathcal{B}^{m+n}, \mathcal{L}^{m+n}\big).$$

Proof. We need to verify two items: (i) $\sigma(\mathcal{B}^m \times \mathcal{B}^n) = \mathcal{B}^{m+n}$, and (ii) $\mathcal{L}^m \times \mathcal{L}^n = \mathcal{L}^{m+n}$. We first show (i) and then use it to prove (ii).

Since the L^∞- and L^2-topologies are the same on Euclidean spaces, we consider L^∞-topology so that open "balls" are open cubes. First it is easy to see that $\mathcal{B}^{m+n} \subset \sigma(\mathcal{B}^m \times \mathcal{B}^n)$, as any open cube $C \in \mathbb{R}^{m+n}$ is a product of two open cubes in $\mathbb{R}^m$ and $\mathbb{R}^n$, and so $C \in \mathcal{B}^m \times \mathcal{B}^n \subset \sigma(\mathcal{B}^m \times \mathcal{B}^n)$. This shows

$$\mathcal{T}_{\mathbb{R}^{m+n}} \subset \sigma(\mathcal{B}^m \times \mathcal{B}^n) \implies \mathcal{B}^{m+n} \subset \sigma(\mathcal{B}^m \times \mathcal{B}^n).$$

Next we prove the reverse inclusion. For each $E \in \mathcal{B}^n$ and $F \in \mathcal{B}^n m$, we define

$$\Sigma_E := \left\{A \subset \mathbb{R}^m : A \times E \in \mathcal{B}^{m+n}\right\},$$
$$\Sigma^F := \left\{B \subset \mathbb{R}^n : F \times B \in \mathcal{B}^{m+n}\right\}.$$

For any open cube $C \subset \mathbb{R}^n$, it is easy to see that Σ_C is a σ-algebra (exercise for readers to verify it) that contains all open cubes in $\mathbb{R}^m$, hence $\mathcal{T}_{\mathbb{R}^m} \subset \Sigma_C$, and so $\mathcal{B}^m \subset \Sigma_C$. This shows for any Borel set $A \subset \mathbb{R}^m$, the product $A \times C$ with any open n-cube C is a Borel set in $\mathbb{R}^{m+n}$. This also implies $A \times \mathcal{O} \in \mathcal{B}^{m+n}$ for any $A \in \mathcal{B}^m$ and $\mathcal{O} \in \mathcal{T}_{\mathbb{R}^n}$, as any open set in $\mathbb{R}^n$ is a countable union of closed n-cubes with positive volume, each of which is a countable union of open n-cubes.

Now we have shown that $\mathcal{T}_{\mathbb{R}^n} \subset \Sigma^A$ for any $A \in \mathcal{B}^m$. We again leave it as an exercise for readers to verify that Σ^A is a σ-algebra. This shows $\mathcal{B}^n \subset \Sigma^A$ for any $A \in \mathcal{B}^m$. In other words, for any $A \in \mathcal{B}^m$ and $B \in \mathcal{B}^n$, we have $A \times B \in \mathcal{B}^{m+n}$, or equivalently $\mathcal{B}^m \times \mathcal{B}^m \subset \mathcal{B}^{m+n}$, which implies

$$\sigma(\mathcal{B}^m \times \mathcal{B}^m) \subset \mathcal{B}^{m+n},$$

completing the proof of (i).

We are left to prove (ii) $\mathcal{L}^m \times \mathcal{L}^n = \mathcal{L}^{m+n}$ on $\mathcal{B}^{m+n}$. Recall that from Proposition 9.13 that the Carathéodory extension is unique when the pre-measure space is σ-finite. We consider the pre-measure space $(\mathbb{R}^m \times \mathbb{R}^n, \mathcal{A}_0, \mathcal{L}^m \times \mathcal{L}^n)$ where $\mathcal{A}_0$ is the collection of all finite unions of disjoint rectangles in $\mathbb{R}^{m+n}$ (see Example 9.3). Clearly $\mathcal{A}_0$ is an algebra. Note that $\mathcal{A}_0 \subset \sigma(\mathcal{B}^m \times \mathcal{B}^n)$ so $\sigma(\mathcal{A}_0) \subset \sigma(\mathcal{B}^m \times \mathcal{B}^n)$. On the other hand, any open set $\mathcal{O} \subset \mathbb{R}^{m+n}$ is a countable union of $(m+n)$-cubes (which are rectangles), so $\mathcal{T}_{\mathbb{R}^{m+n}} \subset \sigma(\mathcal{A}_0)$. This shows $\mathcal{B}^{m+n} \subset \sigma(\mathcal{A}_0)$. Combining these with the result from (i), we have:

$$\sigma(\mathcal{B}^m \times \mathcal{B}^n) = \mathcal{B}^{m+n} = \sigma(\mathcal{A}_0).$$

Since $\mathcal{L}^m \times \mathcal{L}^n$ is a measure on sets in $\sigma(\mathcal{B}^m \times \mathcal{B}^n)$, it is also countably additive on $\sigma(\mathcal{A}_0)$, making the following a pre-measure space:

$$(\mathbb{R}^m \times \mathbb{R}^n, \mathcal{A}_0, \mathcal{L}^m \times \mathcal{L}^n).$$

Note that for any rectangle $R \subset \mathbb{R}^{m+n}$, we can write it as $R = R' \times R''$ where R' is a rectangle in $\mathbb{R}^m$, and R'' is a rectangle in $\mathbb{R}^n$. Since

$$\mathcal{L}^{m+n}(R) = |R| = |R'|\,|R''| = \mathcal{L}^m(R')\mathcal{L}^n(R'') = (\mathcal{L}^m \times \mathcal{L}^n)(R' \times R'') = (\mathcal{L}^m \times \mathcal{L}^n)(R),$$

the measures $\mathcal{L}^{m+n}$ and $\mathcal{L}^m \times \mathcal{L}^n$ agree on $\mathcal{A}_0$. By uniqueness of Carathéodory extension (Proposition 9.13) – clearly the pre-measure space $(\mathbb{R}^m \times \mathbb{R}^n, \mathcal{A}_0, \mathcal{L}^m \times \mathcal{L}^n)$ is σ-finite, we must have $\mathcal{L}^{m+n} = \mathcal{L}^m \times \mathcal{L}^n$ on $\sigma(\mathcal{A}_0) = \mathcal{B}^{m+n}$. It completes the proof of (ii). $\square$

Exercise 9.5. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measure spaces. Suppose $f : X \times Y \to \mathbb{R}$ is a measurable function on the measure space $(X \times Y, \sigma(\mathcal{A} \times \mathcal{B}), \mu \times \nu)$. Show that $y \in Y \mapsto f(a, y)$ is a measurable function on $(Y, \mathcal{B}, \nu)$ for any $a \in X$.

9.3. Fubini's Theorem

In this section we state and prove an important result concerning integrations of functions on product measure spaces. These theorems are called Tonelli's Theorem and Fubini's Theorem. The former is restricted on non-negative functions whereas the latter is about general measurable functions. Let's first state the Tonelli's Theorem and prove it in several steps.

Theorem 9.17 (Tonelli's Theorem). *Suppose $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces. Given a* ***nonnegative*** *function $f : X \times Y \to [0, \infty]$ which is measurable with respect to the product measure space $(X \times Y, \sigma(\mathcal{A} \times \mathcal{B}), \mu \times \nu)$, then we have:*

(a) $x \in X \mapsto \displaystyle\int_Y f(x, y)\, d\nu(y)$ is measurable with respect to $(X, \mathcal{A}, \mu)$;

(b) $y \in Y \mapsto \displaystyle\int_X f(x, y)\, d\mu(x)$ is measurable with respect to $(Y, \mathcal{B}, \nu)$; and

(c) the following holds

$$\int_X \int_Y f(x, y)\, d\nu(y)\, d\mu(x) = \int_{X \times Y} f\, d(\mu \times \nu) = \int_Y \int_X f(x, y)\, d\mu(x)\, d\nu(x).$$

The proof of Tonelli's Theorem uses MCT and the monotone class theorem (to be introduced) several times. Let's first introduce a new terminology:

Definition 9.18 (Monotone class). Suppose X is a set and $\mathfrak{M} \subset \mathcal{P}_X$. We call $\mathfrak{M}$ a **monotone class on** X if the following two conditions hold:

(a) If $E_j \in \mathfrak{M}$ for any $j \in \mathbb{N}$ and $E_1 \subset E_2 \subset \cdots$, then $\displaystyle\bigcup_{j=1}^{\infty} E_j \in \mathfrak{M}$; and

(b) If $E_j \in \mathfrak{M}$ for any $j \in \mathbb{N}$ and $E_1 \supset E_2 \supset \cdots$, then $\displaystyle\bigcap_{j=1}^{\infty} E_j \in \mathfrak{M}$.

Certainly, any σ-algebra is a monotone class. One example of a monotone class but not a σ-algebra is the set of measurable rectangles in a product space. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measure spaces, we claim that $\mathcal{A} \times \mathcal{B}$ is a monotone class. To see this, we take

$$E_j = A_j \times B_j \in \mathcal{A} \times \mathcal{B} \;\; \forall j \;\text{ such that }\; E_1 \subset E_2 \subset \cdots.$$

Then, we would have $A_1 \subset A_2 \subset \cdots$ and $B_1 \subset B_2 \subset \cdots$. This implies

$$\bigcup_{j=1}^{\infty} E_j = \left(\bigcup_{j=1}^{\infty} A_j\right) \times \left(\bigcup_{j=1}^{\infty} B_j\right) \in \mathcal{A} \times \mathcal{B}.$$

This proves (a) in Definition 9.18. Similarly for (b).

Exercise 9.6. Let X be any set. Suppose $\{\mathfrak{M}_\alpha\}_{\alpha \in \mathcal{I}}$ is a family of monotone classes on X. Show that

$$\mathfrak{M} := \bigcap_{\alpha \in \mathcal{I}} \mathfrak{M}_\alpha$$

is also a monotone class on X.

The above exercise justifies that given any collection $\mathcal{A}$ of subsets in X, one can define the smallest monotone class containing $\mathcal{A}$ by:

$$\mathfrak{M}(\mathcal{A}) := \bigcap_{\mathfrak{M} \supset \mathcal{A}} \mathfrak{M}.$$

The following theorem relates several concepts (algebra, monotone class, σ-algebra) that we have learned, and will be used later in the proof of Tonelli's Theorem.

Theorem 9.19 (Monotone class theorem). *Let X be a set, and $\mathcal{A}_0$ be an algebra on X. Then we have:*

$$\sigma(\mathcal{A}_0) = \mathfrak{M}(\mathcal{A}_0).$$

Proof. Since $\sigma(\mathcal{A}_0)$ is a σ-algebra, it is also a monotone class. Therefore, we have $\sigma(\mathcal{A}_0) \supset \mathfrak{M}(\mathcal{A}_0)$.

To prove the reverse inclusion, we need to prove $\mathfrak{M}(\mathcal{A}_0)$ is indeed a σ-algebra. It is clear that $\emptyset \in \mathcal{A}_0 \subset \mathfrak{M}(\mathcal{A}_0)$. To prove the other two properties (closed under countable unions and complementations), we first construct the following collections of subsets. For each $A \in \mathcal{A}_0$, we let

$$\begin{aligned}
\mathfrak{M}_Y &:= \{E \in \mathfrak{M}(\mathcal{A}_0) : Y \cup E \in \mathfrak{M}(\mathcal{A}_0)\} \\
\mathfrak{M}' &:= \{E \in \mathfrak{M}(\mathcal{A}_0) : Y \cup E \in \mathfrak{M}(\mathcal{A}_0) \text{ for all } Y \in \mathfrak{M}(\mathcal{A}_0)\} = \bigcap_{Y \in \mathfrak{M}(\mathcal{A}_0)} \mathfrak{M}_Y \\
\mathfrak{M}'' &:= \{E \in \mathfrak{M}(\mathcal{A}_0) : X - E \in \mathfrak{M}(\mathcal{A}_0)\}
\end{aligned}$$

It is easy to see that $\mathfrak{M}_Y$ is a monotone class for any $Y \in \mathfrak{M}(\mathcal{A}_0)$, and by Exercise 9.6, $\mathfrak{M}'$ is also a monotone class. Moreover, we claim that $\mathcal{A}_0 \subset \mathfrak{M}_Y$ for any $Y \in \mathcal{A}_0$, because for any $A \in \mathcal{A}_0$, we have $Y \cup E \in \mathcal{A}_0 \subset \mathfrak{M}(\mathcal{A}_0)$. Therefore, we conclude that

$$\mathfrak{M}(\mathcal{A}_0) \subset \mathfrak{M}_Y \text{ for any } Y \in \mathcal{A}_0.$$

In other words, for any $E \in \mathfrak{M}(\mathcal{A}_0)$ and $Y \in \mathcal{A}_0$, we have $Y \cup E \in \mathfrak{M}(\mathcal{A}_0)$. Swapping the roles of E and Y in the definition of $\mathfrak{M}'$, we then have $\mathcal{A}_0 \subset \mathfrak{M}'$, and so

$$\mathfrak{M}(\mathcal{A}_0) \subset \mathfrak{M}',$$

or equivalently, for any $E, Y \in \mathfrak{M}(\mathcal{A}_0)$, we have $Y \cup E \in \mathfrak{M}(\mathcal{A}_0)$, and inductively $\mathfrak{M}(\mathcal{A}_0)$ is closed under **finite** unions.

To verify that $\mathfrak{M}(\mathcal{A}_0)$ is closed under **countable** unions, we let $\{E_j\}_{j=1}^{\infty}$ be sets in $\mathfrak{M}(\mathcal{A}_0)$, then we have

$$E_1 \subset E_1 \cup E_2 \subset E_1 \cup E_2 \cup E_3 \subset \cdots$$

and $E_1 \cup \cdots \cup E_j \in \mathfrak{M}(\mathcal{A}_0)$ for any j. Since $\mathfrak{M}(\mathcal{A}_0)$ is a monotone class, we have

$$\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} (E_1 \cup \cdots \cup E_j) \in \mathfrak{M}(\mathcal{A}_0).$$

Therefore, $\mathfrak{M}(\mathcal{A}_0)$ is closed under countable unions.

Finally, we need to check that $\mathfrak{M}(\mathcal{A}_0)$ is closed under complementation. Since $\mathcal{A}_0$ is an algebra, whenever $A \in \mathcal{A}_0$ we have $X - A \in \mathcal{A}_0 \subset \mathfrak{M}(\mathcal{A}_0)$. Therefore, $\mathcal{A}_0 \subset \mathfrak{M}''$. It is also easy to check that $\mathfrak{M}''$ is a monotone class, so we have

$$\mathfrak{M}(\mathcal{A}_0) \subset \mathfrak{M}'',$$

or in other words, $X - E \in \mathfrak{M}(\mathcal{A}_0)$ for any $E \in \mathfrak{M}(\mathcal{A}_0)$. This shows $\mathfrak{M}(\mathcal{A}_0)$ is closed under complementation.

Therefore, $\mathfrak{M}(\mathcal{A}_0)$ is an σ-algebra and it contains $\mathcal{A}_0$, so we have

$$\mathfrak{M}(\mathcal{A}_0) \supset \sigma(\mathcal{A}_0)$$

as desired. $\square$

Next we proceed to the proof of Tonelli's Theorem. The key idea is to first prove that it holds for simple functions, then we use MCT to extend it to arbitrary non-negative measurable functions.

Lemma 9.20. *Theorem 9.17 holds when $f(x,y)$ is of the form $\chi_E(x,y)$ for any $E \in \sigma(\mathcal{A} \times \mathcal{B})$.*

Proof. Assume further that $\nu(Y) < \infty$ (we will relax it later). Note that for any $x \in X$ and $y \in Y$, we have

$$\int_Y \chi_E(x,y)\, d\nu(y) = \int_Y \chi_{E_x}(y)\, d\nu(y) = \nu(E_x).$$

Consider the collection

$$\mathfrak{M} := \{E \in \sigma(\mathcal{A} \times \mathcal{B}) : x \in X \mapsto \nu(E_x) \text{ is a } \mathcal{A}\text{-measurable function}\}.$$

If one can prove that $\mathfrak{M} = \sigma(\mathcal{A} \times \mathcal{B})$, then (a) of Theorem 9.17 is proved in the case $f(x,y) = \chi_{E_x}(y)$.

First it is easy to see that $\mathcal{A} \times \mathcal{B} \subset \mathfrak{M}$: for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we have

$$\nu\big((A \times B)_x\big) = \nu(B)\chi_A(x),$$

so the map $x \mapsto \nu\big((A \times B)_x\big)$ is $\mathcal{A}$-measurable (as $A \in \mathcal{A}$). This shows $A \times B \in \mathfrak{M}$.

Next, we argue that $\mathcal{F}(\mathcal{A} \times \mathcal{B}) \subset \mathfrak{M}$: suppose $E \in \mathcal{F}(\mathcal{A} \times \mathcal{B})$ is of the form:

$$E = \bigsqcup_{j=1}^{N} (A_j \times B_j), \text{ where } A_j \in \mathcal{A} \text{ and } B_j \in \mathcal{B}.$$

Then, we have for any $x \in X$,

$$\nu(E_x) = \nu\left(\bigsqcup_{j=1}^{N} (A_j \times B_j)_x\right) = \sum_{j=1}^{N} \nu(B_j)\chi_{A_j}(x).$$

As A_j are all measurable sets, the map $x \mapsto \nu(E_x)$ is $\mathcal{A}$-measurable. This shows $E \in \mathfrak{M}$ and so $\mathcal{F}(\mathcal{A} \times \mathcal{B}) \subset \mathfrak{M}$.

To complete the proof that $\mathfrak{M} = \sigma(\mathcal{A} \times \mathcal{B})$ and noting $\mathcal{A}_0 := \mathcal{F}(\mathcal{A} \times \mathcal{B})$ is an algebra, it suffices to check that $\mathfrak{M}$ is a monotone class, then monotone class theorem would imply

$$\mathfrak{M} \supset \mathfrak{M}(\mathcal{F}(\mathcal{A} \times \mathcal{B})) = \sigma(\mathcal{F}(\mathcal{A} \times \mathcal{B})) \supset \sigma(\mathcal{A} \times \mathcal{B}) \implies \mathfrak{M} = \sigma(\mathcal{A} \times \mathcal{B}).$$

To verify that $\mathfrak{M}$ is a monotone class, we take a sequence of sets $E_j \in \mathfrak{M}$ such that:

$$E_1 \subset E_2 \subset \cdots.$$

Then, we have

$$\nu\left(\left(\bigcup_{j=1}^{\infty} E_j\right)_x\right) = \nu\left(\bigcup_{j=1}^{\infty} (E_j)_x\right) = \lim_{j\to\infty} \nu\big((E_j)_x\big)$$

by the fact that $(E_1)_x \subset (E_2)_x \subset \cdots$. Since each $x \mapsto \nu\big((E_j)_x\big)$ is $\mathcal{A}$-measurable, its limit function is also $\mathcal{A}$-measurable. This proves $\bigcup_j E_j \in \mathfrak{M}$. The intersection case is similar, as we have the condition that $\nu(Y) < \infty$.

Overall, we have shown (a) in the case $\nu(Y) < \infty$. To extend to the case $(Y, \mathcal{B}, \mu)$ is σ-finite, we let

$$Y = \bigcup_{j=1}^{\infty} Y_j$$

where $Y_j \in \mathcal{B}$ and $\nu(Y_j) < \infty$ for any j. We may assume $Y_1 \subset Y_2 \subset Y_3 \subset \cdots$, then for any $E \in \sigma(\mathcal{A} \times \mathcal{B})$, $j \in \mathbb{N}$ we consider

$$E_j := (X \times Y_j) \cap E.$$

Then $(E_j)_x = Y_j \cap E_x$. Note that $\mu(Y_j) < \infty$, we have for each j, the map

$$x \in X \mapsto \nu\big(Y_j \cap E_x\big) = \nu\big((E_j)_x\big)$$

is $\mathcal{A}$-measurable by applying the finite measure case to the set $E_j \in \sigma\left(\mathcal{A} \times \mathcal{B}|_{Y_j}\right)$. Then, noting that $E_1 \subset E_2 \subset \cdots$ and $E = \bigcup_j E_j$, we have

$$E_x = \bigcup_{j=1}^{\infty} (E_j)_x \implies \nu(E_x) = \lim_{j \to \infty} \nu\big((E_j)_x\big),$$

which is also a $\mathcal{A}$-measurable function of x. It completes the proof of (a). The proof of (b) is similar.

It remains to prove (c). First we assume μ and ν are finite measures. Let

$$\mathfrak{I} := \left\{ E \in \sigma(\mathcal{A} \times \mathcal{B}) : \int_X \int_Y \chi_E(x, y)\, d\nu(y)\, d\mu(x) = \int_{X \times Y} \chi_E\, d(\mu \times \nu) \right\}.$$

We will show $\mathfrak{I}$ is a monotone class containing $\mathcal{F}(\mathcal{A} \times \mathcal{B})$, then one can conclude $\mathfrak{I} = \sigma(\mathcal{A} \times \mathcal{B})$ by the monotone class theorem.

For any $E \in \mathcal{F}(\mathcal{A} \times \mathcal{B})$, we write

$$E = \bigsqcup_{j=1}^{\infty} (A_j \times B_j), \quad \text{where } A_j \in \mathcal{A} \text{ and } B_j \in \mathcal{B}.$$

Then, we have

$$\chi_E(x, y) = \sum_{j=1}^{N} \chi_{A_j \times B_j}(x, y) = \sum_{j=1}^{N} \chi_{A_j}(x) \chi_{B_j}(y).$$

It shows

$$\begin{aligned}
&\int_X \int_Y \chi_E(x, y)\, d\nu(y)\, d\mu(x) \\
&= \sum_{j=1}^{N} \int_X \int_Y \chi_{A_j}(x) \chi_{B_j}(y)\, d\nu(y)\, d\mu(x) \\
&= \sum_{j=1}^{N} \int_X \chi_{A_j}(x) \nu(B_j)\, d\mu(x) \\
&= \sum_{j=1}^{N} \mu(A_j) \nu(B_j).
\end{aligned}$$

On the other hand, we have

$$\int_{X \times Y} \chi_E\, d(\mu \times \nu) = \sum_{j=1}^{N} \int_{X \times Y} \chi_{A_j \times B_j}\, d(\mu \times \nu) = \sum_{j=1}^{N} \mu(A_j) \nu(B_j).$$

It implies $\mathcal{F}(\mathcal{A} \times \mathcal{B}) \subset \mathfrak{I}$.

To check that $\mathfrak{I}$ is a monotone class, we consider $E_j \in \mathfrak{I}$ such that $E_1 \subset E_2 \subset \cdots$, and denote $E := \bigcup_j E_j$. Then, we have

$$\chi_E = \lim_{j\to\infty} \chi_{E_j} \text{ and } \chi_{E_1} \le \chi_{E_2} \le \cdots.$$

Then, we have

$$\begin{aligned}
&\int_X \int_Y \chi_E(x,y)\,d\nu(y)\,d\mu(x) \\
&= \lim_{j\to\infty} \int_X \int_Y \chi_{E_j}(x,y)\,d\nu(y)\,d\mu(x) && \text{(MCT)} \\
&= \lim_{j\to\infty} \int_{X\times Y} \chi_{E_j}\,d(\mu\times\nu) && (\text{by } E_j \in \mathfrak{I}) \\
&= \int_{X\times Y} \chi_E\,d(\mu\times\nu) && \text{(MCT)}
\end{aligned}$$

This shows $E \in \mathfrak{I}$. The intersection part can be proved in a similar way, except that χ_{E_j} would be decreasing so we need $\mu(X), \nu(Y) < \infty$ to use LDCT/BCT instead.

Now given that $\mathfrak{I}$ is a monotone class containing the algebra $\mathcal{F}(\mathcal{A}\times\mathcal{B})$, the monotone class theorem shows:

$$\sigma(\mathcal{A}\times\mathcal{B}) \supset \mathfrak{I} \supset \mathfrak{M}(\mathfrak{I}) \supset \mathfrak{M}(\mathcal{F}(\mathcal{A}\times\mathcal{B})) = \sigma(\mathcal{F}(\mathcal{A}\times\mathcal{B})) = \sigma(\mathcal{A}\times\mathcal{B}).$$

Now we conclude that $\mathfrak{I} = \sigma(\mathcal{A}\times\mathcal{B})$ under the assumption $\mu(X), \nu(Y) < \infty$.

Finally we consider the general case that $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite. Write

$$X = \bigcup_{j=1}^\infty X_j \text{ and } Y = \bigcup_{k=1}^\infty Y_k$$

where $X_1 \subset X_2 \subset \cdots$ are sets in $\mathcal{A}$, and $Y_1 \subset Y_2 \subset \cdots$ are sets in $\mathcal{B}$ such that $\mu(X_j) < \infty$ and $\nu(Y_k) < \infty$ for any j, k. Given any $E \in \sigma(\mathcal{A}\times\mathcal{B})$, we let:

$$E_{j,k} := (X_j \times Y_k) \cap E.$$

The finite measure case shows for any $j, k \in \mathbb{N}$,

$$\int_{X_j} \int_{Y_k} \chi_{E_{j,k}}(x,y)\,d\nu(y)\,d\mu(x) = \int_{X_j\times Y_k} \chi_{E_{j,k}}\,d(\mu\times\nu),$$

or in other words

$$\int_X \int_Y \chi_{X_j}(x)\chi_{Y_k}(y)\chi_{E_{j,k}}(x,y)\,d\nu(y)\,d\mu(x) = \int_{X\times Y} \chi_{X_j}(x)\chi_{Y_k}(y)\chi_{E_{j,k}}\,d(\mu\times\nu).$$

By fixing k, we note that the integrands are both increasing as j increases, so by letting $j \to \infty$ the MCT shows

$$\int_X \int_Y \chi_{Y_k}(y)\chi_{(X\times Y_k)\cap E}(x,y)\,d\nu(y)\,d\mu(x) = \int_{X\times Y} \chi_{Y_k}(y)\chi_{(X\times Y_k)\cap E}\,d(\mu\times\nu).$$

Observe that the integrand also increases as k increases, by MCT we show

$$\int_X \int_Y \chi_E(x,y)\,d\nu(y)\,d\mu(x) = \int_{X\times Y} \chi_E\,d(\mu\times\nu).$$

Similar argument also proves

$$\int_{X\times Y} \chi_E\,d(\mu\times\nu) = \int_Y \int_X \chi_E(x,y)\,d\mu(x)\,d\nu(y).$$

It completes the proof of (c), and hence the proof of the lemma. $\square$

Proof of Tonelli's Theorem. Lemma 9.20 shows Tonelli's Theorem holds for χ_E for any $E \in \sigma(\mathcal{A} \times \mathcal{B})$. By linearity of integrations, it therefore holds for any simple functions in the measure space $(X \times Y, \sigma(\mathcal{A} \times \mathcal{B}), \mu \times \nu)$.

Now given any non-negative $\sigma(\mathcal{A} \times \mathcal{B})$-measurable function $f : X \times Y \to [0, \infty]$, one can find an increasing sequence φ_j of simple functions such that $\varphi_j \to f$ (see Proposition 8.18).

(a) holds for f since by MCT

$$\int_Y f(x,y)\, d\nu(y) = \lim_{j\to\infty} \int_Y \varphi_j(x,y)\, d\nu(y).$$

By Lemma 9.20 we know that

$$x \mapsto \int_Y \varphi_j(x,y)\, d\nu(y)$$

is $\mathcal{A}$-measurable for any j, hence so is its limit function. This proves (a) holds. Similarly for (b).

To prove (c), we consider taht for any j

$$\int_X \int_Y \varphi_j(x,y)\, d\nu(y)\, d\mu(x) = \int_{X\times Y} \varphi_j\, d(\mu \times \nu)$$

and the integrand increases as j increases, by MCT we have conclude that

$$\int_X \int_Y f(x,y)\, d\nu(y)\, d\mu(x) = \int_{X\times Y} f\, d(\mu \times \nu).$$

Similarly for

$$\int_{X\times Y} f\, d(\mu \times \nu) = \int_Y \int_X f(x,y)\, d\mu(x)\, d\nu(y),$$

completing the proof of (c), and hence the proof of the theorem. □

Tonelli's Theorem requires the function to be non-negative, yet it allows the integrals in (c) to be infinite. Fubini's Theorem relaxes the non-negativity assumption, but we instead require f to be integrable.

Theorem 9.21 (Fubini's Theorem)**.** *Suppose $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces. Given a function $f : X \times Y \to [-\infty, \infty]$ which is measurable with respect to the product measure space $(X \times Y, \sigma(\mathcal{A} \times \mathcal{B}), \mu \times \nu)$, and it satisfies*

$$\int_{X\times Y} |f|\, d(\mu \times \nu) < \infty. \tag{9.2}$$

Then, there exists $N \in \mathcal{A}$ and $N' \in \mathcal{B}$ with $\mu(N) = \nu(N') = 0$ such that the following holds:

(a) $x \in X \mapsto \chi_{X-N}(x) \int_Y f(x,y)\, d\nu(y)$ is measurable with respect to $(X, \mathcal{A}, \mu)$;

(b) $y \in Y \mapsto \chi_{Y-N'}(y) \int_X f(x,y)\, d\mu(x)$ is measurable with respect to $(Y, \mathcal{B}, \nu)$; and

(c) the following holds

$$\int_X \int_Y f(x,y)\, d\nu(y)\, d\mu(x) = \int_{X\times Y} f\, d(\mu \times \nu) = \int_Y \int_X f(x,y)\, d\mu(x)\, d\nu(x).$$

Proof. We first apply the Tonelli's Theorem on the non-negative function $|f|$:

$$\int_X \int_Y |f(x,y)|\, d\nu(y)\, d\mu(x) = \int_{X\times Y} |f|\, d(\mu \times \nu) = \int_Y \int_X |f(x,y)|\, d\mu(x)\, d\nu(x).$$

By (9.2), we have
$$\int_X \int_Y |f(x,y)|\, d\nu(y)\, d\mu(x) < \infty$$
and the integral $\int_Y |f(x,y)|\, d\nu(y)$ is finite μ-a.e. on X.

As $f_+ \leq |f|$ and $f_- \leq |f|$, the integrals $\int_Y f_+(x,y)\, d\nu(y)$ and $\int_Y f_-(x,y)\, d\nu(y)$ are both finite μ-a.e. on X. Let $N \in \mathcal{A}$ be the set such that
$$N := \left\{ x \in X : \int_Y f_+(x,y)\, d\nu(y) = \infty \text{ or } \int_Y f_-(x,y)\, d\nu(y) = \infty \right\},$$
then $\mu(N) = 0$. Part (a) almost follows from splitting up the integral of f into integrals of f_+ and f_-, but we need to avoid the situation of having $\infty - \infty$. To address this issue, we see that for any $x \in X$, we have
$$\chi_{X-N}(x) \int_Y f(x,y)\, d\nu(y) = \chi_{X-N}(x) \int_Y f_+(x,y)\, d\nu(y) - \chi_{X-N}(x) \int_Y f_-(x,y)\, d\nu(y),$$
where we use the convention that $0 \cdot \infty = 0$. Both
$$x \mapsto \chi_{X-N}(x) \int_Y f_+(x,y)\, d\nu(y) \quad \text{and} \quad x \mapsto \chi_{X-N}(x) \int_Y f_-(x,y)\, d\nu(y)$$
are $\mathcal{A}$-measurable functions since f_+ and f_- are non-negative measurable (hence Tonelli's Theorem can be used), and N is μ-measurable. These imply the map $x \mapsto \chi_{X-N}(x) \int_Y f(x,y)\, d\nu(y)$ is $\mathcal{A}$-measurable, proving (a) in our theorem. Part (b) can be proved in a similar way.

For (c), we compute that:
$$\begin{aligned}
&\int_X \int_Y f(x,y)\, d\nu(y)\, d\mu(x) \\
&= \int_X \chi_{X-N}(x) \int_Y \big(f_+(x,y) - f_-(x,y)\big)\, d\nu(y)\, d\mu(x) \\
&= \int_X \int_Y \chi_{X-N}(x) f_+(x,y)\, d\nu(y)\, d\mu(x) - \int_X \int_Y \chi_{X-N}(x) f_-(x,y)\, d\nu(y)\, d\mu(x) \\
&= \underbrace{\int_{X\times Y} \chi_{X-N}(x) f_+(x,y)\, d(\mu\times\nu)}_{<\infty} - \underbrace{\int_{X\times Y} \chi_{X-N}(x) f_-(x,y)\, d(\mu\times\nu)}_{<\infty} \\
&= \int_{X\times Y} \chi_{X-N}(x) f(x,y)\, d(\mu\times\nu).
\end{aligned}$$
Note that $(x,y) \in (X-N)\times Y$ if and only if $x \in X - N$, so
$$\chi_{(X-N)\times Y}(x,y) = \chi_{X-N}(x) \ \ \forall (x,y) \in X \times Y.$$
Note that $(\mu \times \nu)(N \times Y) = 0$ as $\mu(N) = 0$ and $(Y, \mathcal{B}, \nu)$ is σ-finite. Therefore $\chi_{(X-N)\times Y} f = f$ a.e. on $X \times Y$, so
$$\int_{X\times Y} \chi_{X-N}(x) f(x,y)\, d(\mu\times\nu) = \int_{X\times Y} f(x,y)\, d(\mu\times\nu).$$
This shows
$$\int_X \int_Y f(x,y)\, d\nu(y)\, d\mu(x) = \int_{X\times Y} f(x,y)\, d(\mu\times\nu).$$
The other side
$$\int_{X\times Y} f(x,y)\, d(\mu\times\nu) = \int_Y \int_X f(x,y)\, d\mu(x)\, d\nu(y)$$

can be proved in a similar way. This completes the proof of (c) and also the theorem. □

Recall that

$$\big(\mathbb{R}^m \times \mathbb{R}^n, \sigma(\mathcal{B}^m \times \mathcal{B}^n), \mathcal{L}^m \times \mathcal{L}^n\big) = \big(\mathbb{R}^{m+n}, \mathcal{B}^{m+n}, \mathcal{L}^{m+n}\big).$$

Fubini's Theorem implies the following corollary:

Corollary 9.22. *Given any $f : \mathbb{R}^{m+n} \to [-\infty, \infty]$ which is Borel measurable, i.e. $f^{-1}\big([a,\infty]\big) \in \mathcal{B}^{m+n}$ for any $a \in \mathbb{R}$, such that*

$$\int_{\mathbb{R}^{m+n}} |f| \, d\mathcal{L}^{m+n} < \infty,$$

then we have

$$\begin{aligned}\int_{x\in\mathbb{R}^m} \int_{y\in\mathbb{R}^n} f(x,y) \, d\mathcal{L}^n(y) \, d\mathcal{L}^m(x) &= \int_{\mathbb{R}^{m+n}} f \, d\mathcal{L}^{m+n} \\ &= \int_{y\in\mathbb{R}^n} \int_{x\in\mathbb{R}^m} f(x,y) \, d\mathcal{L}^m(x) \, d\mathcal{L}^n(y).\end{aligned}$$

This justifies the switching of the x- and y-integrals in many examples we have seen in MATH 2023. The condition that f needs to be Borel measurable is very mild, and includes continuous functions. The integrability condition also holds if the support of the function is compact, and the function is bounded.

Chapter 10

Lipschitz Continuous Functions

> "Nowadays, there are only three really great English mathematicians: Hardy, Littlewood, and Hardy-Littlewood."
>
> Harold Bohr

The goal of this chapter is to study the differentiability of Lipschitz continuous functions. We will prove that every Lipschitz continuous function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ from an open set U must be differentiable $\mathcal{L}^n$-a.e. on U. This result is commonly called the Rademacher's Theorem. The proof relies on the special case when $n = 1$. It is in fact the most technical case which we will prove it separately first. Furthermore, we will study the area and coarea formulae, and their applications in analysis and geometry. The proofs of the area and coarea formulae will not be presented in this lecture notes. Interested readers could read Evans-Gariepy's book for detail.

10.1. Functions of Bounded Variation on $\mathbb{R}$

10.1.1. Definitions. We first introduce two new concepts about functions on $\mathbb{R}$.

Definition 10.1 (Absolute continuous functions). A function $f : [a, b] \to \mathbb{R}$ is said to be **absolutely continuous on** $[a, b]$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $\{(c_j, d_j)\}_{j=1}^n$ is a finite disjoint collection of open intervals in $[a, b]$ such that

$$\sum_{j=1}^n (d_j - c_j) < \delta,$$

then we have

$$\sum_{j=1}^n |f(d_j) - f(c_j)| < \varepsilon.$$

Definition 10.2 (Bounded variation)**.** Consider a function $f : [a,b] \to \mathbb{R}$. Given any partition $P : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ of $[a,b]$, we denote

$$V_a^b(f,P) := \sum_{i=1}^n |f(x_i) - f(x_{i-1})| \,.$$

Then, f is said to be of **bounded variation on** $[a,b]$ if

$$V_a^b(f) := \sup\{V_a^b(f,P) : P \text{ is a partition of } [a,b]\} < \infty.$$

The reason why we introduce these two new notions is because they are related to the main theme of this chapter in the following way:

$$\begin{aligned}&\{\text{Lipschitz continuous}\} \subset \{\text{Absolutely continuous}\}\\ &\subset \{\text{continuous and of bounded variation}\} \subset \{\text{differentiable a.e.}\}\end{aligned}$$

It is not hard to prove the first two inclusions, whereas the last one is the hardest to prove. If $f : [a,b] \to \mathbb{R}$ is Lipschitz continuous, then there exists $L > 0$ such that $|f(x) - f(y)| \leq L\,|x-y|$ for any $x, y \in [a,b]$. To show that such an f is absolutely continuous on $[a,b]$, we take any $\varepsilon > 0$, and choose $\delta = \frac{\varepsilon}{L}$, then for any finite disjoint collection of intervals $\{(c_i,d_i)\}_{i=1}^n$ in $[a,b]$ with $\sum_{i=1}^n (d_i - c_i) < \delta = \frac{\varepsilon}{L}$, we have

$$\sum_{i=1}^n |f(d_i) - f(c_i)| \leq \sum_{i=1}^n L\,|d_i - c_i| < L\delta = \varepsilon.$$

This proves $\{\text{Lipschitz continuous}\} \subset \{\text{Absolutely continuous}\}$.

Exercise 10.1. Prove that any absolutely continuous function $f : [a,b] \to \mathbb{R}$ is continuous and of bounded variation on $[a,b]$.

Exercise 10.2. Prove directly (without considering absolute continuity) that any Lipschitz continuous function $f : [a,b] \to \mathbb{R}$ must be of bounded variation on $[a,b]$.

Exercise 10.3. Show that any monotone bounded function $f : [a,b] \to \mathbb{R}$ is of bounded variation.

Exercise 10.4. Show that the function

$$f(x) = \begin{cases} x\cos(\pi/2x) & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$$

is not of bounded variation on $[0,1]$.

Exercise 10.5. Suppose $f : [a,b] \to \mathbb{R}$ is $\mathcal{L}^1$-integrable on $[a,b]$. Show that the function

$$F(x) := \int_a^x f\, d\mathcal{L}^1$$

is absolutely continuous on $[a,b]$.

You may wonder why the notion absolute continuity is introduced here, as Lipschitz continuity directly implies the bounded variation property. It is because the Newton-Leibniz formula:

$$\int_a^b f'(x)\, dx = f(b) - f(a)$$

holds for absolutely continuous functions but generally not true for functions of bounded variation. The Newton-Leibniz formula will be needed to derive the higher dimensional case of the Rademacher's Therorem.

10.1.2. Jordan's Theorem for BV functions. Our first goal is to prove that functions of bounded variation on $[a, b]$ can be expressed as the difference between two monotone functions – known as Jordan's Theorem. After that, we then show any monotone function is differentiable almost everywhere. These two results combine to prove that any function of bounded variation must be differentiable almost everywhere.

Theorem 10.3 (Jordan). *A function $f : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$ if and only if it is the difference of two increasing functions on $[a, b]$.*

Proof. ($\Longrightarrow$)-part: For any $x \in [a, b]$, we consider the interval $[a, x]$ and the total variation

$$V_a^x(f) := \sup\{V_a^x(f, P) : P \text{ is a partition of } [a, x]\}. \tag{10.1}$$

We first argue that for any $x, y \in [a, b]$ with $x < y$, we then have

$$V_a^x(f) + V_x^y(f) = V_a^y(f).$$

For any partition P_1 of $[a, x]$, and P_2 of $[x, y]$, then union $P_3 := P_1 \cup P_2$ is a partition of $[a, y]$, and clearly we have

$$V_a^x(f, P_1) + V_x^y(f, P_2) = V_a^y(f, P_3) \leq V_a^y(f).$$

The last inequality follows from the definition of $V_a^y(f)$. Taking supremum over all partitions P_1 of $[a, x]$ and P_2 of $[x, y]$, we proved

$$V_a^x(f) + V_x^y(f) \leq V_a^y(f).$$

For the reverse inequality, we take an arbitrary partition P_4 of $[a, y]$, then $P_5 := P_4 \cup \{x\}$ is a refinement of P_4, and so

$$V_a^y(f, P_5) \geq V_a^y(f, P_4).$$

Clearly, $V_a^y(f, P_5) = V_a^x(f, P_5 \cap [a, x]) + V_x^y(f, P_5 \cap [x, y])$, therefore we have

$$V_a^x(f) + V_x^y(f) \geq V_a^x(f, P_5 \cap [a, x]) + V_x^y(f, P_5 \cap [x, y]) = V_a^y(f, P_5) \geq V_a^y(f, P_4).$$

Taking supremum over all P_4 of $[a, y]$, we proved

$$V_a^x(f) + V_x^y(f) \geq V_a^y(f),$$

as desired.

An immediate consequence of (10.1) is that $x \mapsto V_a^x(f)$ is increasing. Given $x, y \in [a, b]$ with $x < y$, we have

$$V_a^y(f) = V_a^x(f) + V_x^y(f) \geq V_a^x(f),$$

proving the monotonicity of the map $x \mapsto V_a^x(f)$.

Another immediate consequence of (10.1) is that the map $x \mapsto f(x) + V_a^x(f)$ is also increasing. Given any $x, y \in [a, b]$ with $x < y$, we consider the partition $\{x, y\}$ of $[x, y]$.

$$f(x) - f(y) \leq |f(x) - f(y)| = V_x^y(f, \{x, y\}) \leq V_x^y(f) = V_a^y(f) - V_a^x(f).$$

Note that the last step requires the finiteness of $V_a^x(f)$, and that's where we used the fact that f is of bounded variation. By rearrangement, we conclude that $f(x) + V_a^x(f) \leq f(y) + V_a^y(f)$ as desired.

Since $f(x) = \big(f(x) + V_a^x(f)\big) - V_a^x(f)$, the two monotonicity results show f can be written as the difference between two increasing functions.

The ($\Longleftarrow$)-part follows easily from Exercise 10.3. $\square$

Exercise 10.6. Show that if f is absolutely continuous on $[a, b]$, then $V_a^x(f)$ is a continuous function on $[a, b]$. As a corollary, f can be written as the difference of two **continuous** increasing functions on $[a, b]$.

10.1.3. Vitali's Covering.

Our next goal is to prove that all monotone functions $f : [a, b] \to \mathbb{R}$ are differentiable almost everywhere. Then, combining with Jordan's theorem, it would imply all functions of bounded variation on $[a, b]$, including Lipschitz and absolutely continuous functions, are differentiable almost everywhere.

While monotone functions are *continuous* almost everywhere can be easily proved by density of rational numbers, the proof that they are *differentiable* almost everywhere is much more technical. Royden's and Stein's books used different approach. In this lecture notes, we follow the former's approach. Readers are recommended to learn also the latter's approach. We begin with the notion of Vitali's covering, which is a very "refined" cover of a set in $\mathbb{R}^n$

Definition 10.4 (Vitali's covering)**.** Let $E \subset \mathbb{R}^n$ be a set. We say a collection $\mathcal{V} = \{V_\alpha\}_{\alpha \in \mathcal{A}}$ sets in $\mathbb{R}^n$ to be a **Vitali's covering of** E if for each point $x \in E$ and any $\varepsilon > 0$, $V_\alpha \in \mathcal{V}$ such that

$$x \in V_\alpha \quad \text{and } 0 < \text{diam} V_\alpha < \varepsilon.$$

Example 10.5. One clear example of a Vitali's covering of a set $E \subset \mathbb{R}^n$ is the collection of all open balls in $\mathbb{R}^n$ intersecting E.

Let's learn a less trivial example. Consider a non-empty open set $U \subset \mathbb{R}^n$, and the collection:

$$\mathcal{V} := \{B_{1/k}(y) \subset \mathbb{R}^n : y \in \mathbb{Q}^n \cap U, k \in \mathbb{N}\}.$$

We claim that $\mathcal{V}$ is a Vitali's covering of U. Given any $x \in U$, and $\varepsilon > 0$, then by the openness of U, there exists a large $k \in \mathbb{N}$ such that $\frac{1}{k} < \frac{\varepsilon}{2}$ and $B_{1/k}(x) \subset U$. By density of rational numbers, one can find $y \in B_{1/k}(x) \cap \mathbb{Q}^n$. Then, we have $B_{1/k}(y) \in \mathcal{V}$, $x \in B_{1/k}(y)$ and $\text{diam} B_{1/k}(y) = \frac{2}{k} < \varepsilon$, as desired.

The theorem below will be used to prove the almost everywhere differentiability of monotone functions:

Theorem 10.6 (Vitali's Covering Theorem)**.** *Let $E \subset \mathbb{R}^n$ be a set with $\mathcal{L}^*(E) < \infty$, and $\mathcal{V}$ be a family of closed balls with positive radii in $\mathbb{R}^n$ such that it is a Vitali's covering of E. Then, there exists a finite or countably infinite sequence $\{B_i\}$ of disjoint balls in $\mathcal{V}$ such that*

$$\mathcal{L}^*\left(E - \bigcup_i B_i\right) = 0.$$

In order to prove the theorem, we need to introduce another result also due to Vitali. We call them Vitali's Covering Lemmas. From now on, when B denotes an open (resp. closed) ball in $\mathbb{R}^n$, we then let $3B$ be the open (resp. closed) ball in $\mathbb{R}^n$ with the same center, and 3 times the radius. Precisely, if $B = B_r(x_0)$, then $3B = B_{3r}(x_0)$. Similarly, we edenote $5B = B_{5r}(x_0)$. Using these notations, we introduce the following beautiful result:

Lemma 10.7 (Vitali's 3-Covering Lemma)**.** *Let $B_1, \cdots, B_m$ be a finite collection of balls in $\mathbb{R}^n$, then there exists a subcollection $\{B_{i_1}, \cdots, B_{i_k}\}$ of these balls which are disjoint, and satisfy*

$$\bigcup_{i=1}^{m} B_i \subset \bigcup_{j=1}^{k} 3B_{i_j}.$$

Proof. The key observation is that given two balls that intersect each other, if one enlarge the bigger ball by 3 times its radius, then it will cover the smaller ball. Precisely, given $B_r(x)$ and $B_R(y)$ with $R > r$ and $B_r(x) \cap B_R(y) \neq \emptyset$, then $B_r(x) \subset 3B_R(y)$. This observation simply follows from triangle inequality.

Using the above observation, then one can first find the largest ball, call it B_{i_1}, among $\{B_1, \cdots, B_m\}$, which must exist in a finite collection. We also denote $B_{i_1,1}, B_{i_1,2}, \cdots, B_{i_1,k}$ to be the collection of balls with non-empty intersection with B_{i_1}. Then, by the above observation, we have

$$B_{i_1,1} \cup \cdots \cup B_{i_1,k} \subset B_{i_1}.$$

Repeat the above process in the subcollection $\{B_1, \cdots, B_m\} - \{B_{i_1}, B_{i_1,1}, \cdots, B_{i_1,k}\}$, i.e. find the largest ball, called it B_{i_2} which must then be disjoint from B_{i_1}, in this smaller collection, then $3B_{i_2}$ will cover all other balls intersecting it. The whole process must stop since the collection is finite. □

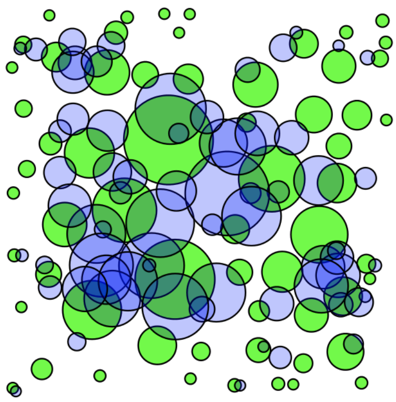


Figure 10.1. the green balls are $B_{i_1}, \cdots B_{i_k}$ in Lemma 10.7

The proof of Lemma 10.7 requires the collection of balls to be finite, because we need to make sure the largest ball exists in the collection. If the collection is infinite, it may not have the largest ball even if the radii are uniformly bounded in the collection.

Luckily, the lemma can be extended (with a much more difficult proof) to infinite collection of balls, with 5-times of radii instead of 3-times:

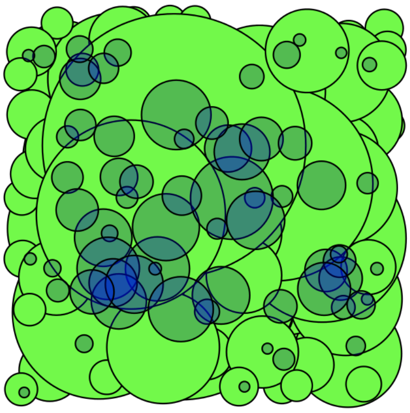


Figure 10.2. after enlarging the green balls by 3 times, they cover all other balls

Lemma 10.8 (Vitali's 5-Covering Lemma)**.** *Let $\mathcal{B}$ be any collection of balls in $\mathbb{R}^n$ such that*

$$\sup\{\text{diam}(B) : B \in \mathcal{B}\} < \infty.$$

Then, there exists a countable subcollection of disjoint balls $\{B_j\}_{j=1}^{\infty} \subset \mathcal{B}$ such that for any $B \in \mathcal{B}$, there exists B_j in that countable subcollection so that $B \cap B_j \neq \emptyset$ and $B \subset 5B_j$. In particular, it implies

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_{j=1}^{\infty} 5B_j.$$

The proof of Lemma 10.8 is much more difficult than that of Lemma 10.7. One needs to use the Zorn's Lemma (which is proved by the Axiom of Choice). We omit the proof of all these lemmas here. We will use this theorem to give a proof of Theorem 10.6.

Proof of Theorem 10.6. Given a set $E \subset \mathbb{R}^n$ with $\mathcal{L}^*(E) < \infty$, and $\mathcal{V}$ is a Vitali's covering of E by closed balls with positive radii. We consider a subcollection

$$\mathcal{V}' := \{B \in \mathcal{V} : \text{diam}(B) < 1\}.$$

Since balls in $\mathcal{V}'$ have uniformly bounded diameter, Lemma 10.8 asserts that there exists a countable subcollection of disjoint balls $\{B_j\}_{j=1}^{\infty} \subset \mathcal{V}'$ such that for any $B \in \mathcal{V}'$, there exists B_j so that $B \cap B_j \neq \emptyset$ and $B \subset 5B_j$.

Next we argue that for any $k \in \mathbb{N}$, we have

$$E - \bigcup_{j=1}^{k} B_j \subset \bigcup_{j=k+1}^{\infty} 5B_j. \tag{10.2}$$

To prove this, pick $x \in E$ and $x \notin \bigcup_{j=1}^{k} B_j$, then we want to show $x \in \bigcup_{j=k+1}^{\infty} 5B_j$. Recall that B_j's are closed balls, so $\bigcup_{j=1}^{k} B_j$ is closed. Hence, there exists $r \in (0,1)$ such that $B_r(x) \subset \mathbb{R}^n - \bigcup_{j=1}^{k} B_j$. As $\mathcal{V}$ is a Vitali's covering of E, there exists a ball $B \in \mathcal{V}$ such that $x \in B$ and $0 < \text{diam}B < r$. This ball B is also in $\mathcal{V}'$ (as $\text{diam}B < r < 1$), hence there exists B_j such that $B \cap B_j \neq \emptyset$ and $B \subset 5B_j$. This j cannot be in $\{1, \cdots, k\}$ since $B \subset B_r(x)$ is disjoint from $\bigcup_{j=1}^{k} B_j$. This shows $x \in B \subset 5B_j$ for some $j > k$, completing the proof of (10.2).

Using (10.2), we can conclude that for any $k \in \mathbb{N}$,

$$\mathcal{L}^*\left(E - \bigcup_{j=1}^{k} B_j\right) \leq \sum_{j=k+1}^{\infty} \mathcal{L}^*(5B_j) = \sum_{j=k+1}^{\infty} 5^n \mathcal{L}^*(B_j) = 5^n \mathcal{L}^*\left(\coprod_{j=k+1}^{\infty} B_j\right).$$

The last equality follows from the fact that B_j's are disjoint. Recall that $\mathcal{L}^*(E) < \infty$, so by letting $k \to \infty$, we get

$$\mathcal{L}^*\left(\bigcap_{k=1}^{\infty}\left(E - \bigcup_{j=1}^{k} B_j\right)\right) \leq 5^n \mathcal{L}^*\left(\bigcap_{k=1}^{\infty} \coprod_{j=k+1}^{\infty} B_j\right),$$

which implies

$$\mathcal{L}^*\left(E - \bigcup_{j=1}^{\infty} B_j\right) \leq 5^n \mathcal{L}^*(\emptyset) = 0.$$

Here we use the fact that $x \in \bigcap_{k=1}^{\infty} \bigcup_{j=k+1}^{\infty} B_j$ if and only if x is in infinitely many B_j's. However, it is not possible as B_j's are disjoint.

It completes the proof of the theorem. □

10.1.4. Monotone functions are differentiable a.e. After a necessary detour to those covering lemmas and theorem, we are now ready to prove the main result that monotone functions are almost everywhere differentiable.

Theorem 10.9 (Lebesgue). *Any increasing function $f : [a, b] \to \mathbb{R}$ is differentiable $\mathcal{L}^1$-a.e. on $[a, b]$.*

Corollary 10.10. *Any function $f : [a, b] \to \mathbb{R}$ of bounded variation is differentiable $\mathcal{L}^1$-a.e. on $[a, b]$.*

In order to prove this theorem, we need to introduce two new notions, called Dini's derivatives. Consider a function $f : [a, b] \to \mathbb{R}$, and let $x \in (a, b)$, we denote:

$$\overline{D}f(x) := \lim_{\delta \to 0^+} \sup_{0<|t|\leq\delta} \frac{f(x+t) - f(x)}{t} = \inf_{\delta>0} \sup_{0<|t|\leq\delta} \frac{f(x+t) - f(x)}{t},$$

$$\underline{D}f(x) := \lim_{\delta \to 0^+} \inf_{0<|t|\leq\delta} \frac{f(x+t) - f(x)}{t} = \sup_{\delta>0} \inf_{0<|t|\leq\delta} \frac{f(x+t) - f(x)}{t}$$

These derivatives always exist or equal to $+\infty$ or $-\infty$, as both $\sup_{0<|t|\leq\delta}$ and $\inf_{0<|t|\leq\delta}$ are monotone as δ goes to 0. Also, we always have $\overline{D}f(x) \geq \underline{D}f(x)$ for any $x \in (a, b)$, and equality holds if and only if $f'(x)$ exists or equal to $\pm\infty$.

Exercise 10.7. Let $f : [-1, 1] \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} -1 & \text{if } x \in [-1, 0) \\ 1 & \text{if } x \in [0, 1] \end{cases}$$

Find $\overline{D}f(0)$ and $\underline{D}f(0)$.

Exercise 10.8. Let $f(x) = |x|$. Find $\overline{D}f(0)$ and $\underline{D}f(0)$.

The proof of this theorem uses the following useful lemma, which is an analogue of the mean-value theorem in single-variable calculus:

Lemma 10.11. *Let $f : [a,b] \to \mathbb{R}$ be an increasing function. Then, for any $\alpha > 0$, we have*

$$\alpha\mathcal{L}^*\{x \in (a,b) : \overline{D}f(x) \geq \alpha\} \leq f(b) - f(a).$$

Proof. The proof of the lemma uses Vitali's covering theorem. Denote

$$E_\alpha := \{x \in (a,b) : \overline{D}f(x) \geq \alpha\}.$$

Consider an arbitrary $\beta \in (0,\alpha)$, and the collection

$$\mathcal{V}_\beta := \{[c,d] \subset (a,b) : \beta(d-c) \leq f(d) - f(c)\}.$$

We first show that $\mathcal{V}_\beta$ is a Vitali's covering of E_α for any $\beta \in (0,\alpha)$. For any $x \in E_\alpha$ and $\varepsilon > 0$, we have

$$\lim_{\delta\to 0^+} \sup_{0<|t|\leq\delta} \frac{f(x+t)-f(x)}{t} = \overline{D}f(x) \geq \alpha > \beta.$$

Hence, there exists $\delta_0 \in (0,\varepsilon)$ such that

$$\sup_{0<|t|\leq\delta_0} \frac{f(x+t)-f(x)}{t} > \beta.$$

We assume also assume δ_0 is sufficiently small so that $[x-\delta_0, x+\delta_0] \subset (a,b)$. The above inequality shows there exists $t_0 \in [-\delta_0,\delta_0]\backslash\{0\}$ such that

$$\sup_{0<|t|\leq\delta_0} \frac{f(x+t)-f(x)}{t} \geq \frac{f(x+t_0)-f(x)}{t_0} > \beta.$$

Then, the closed interval $[x, x+t_0]$ if $t_0 > 0$ (or $[x+t_0, x]$ if $t_0 < 0$) is in $\mathcal{V}_\beta$, contains x and has diameter $|t_0| < \delta_0 < \varepsilon$. This verifies that $\mathcal{V}_\beta$ is a Vitali's covering of E_α.

By Vitali's covering theorem, there exists a countable disjoint collection of closed intervals $\{[c_j,d_j]\}_{j=1}^\infty$ in $\mathcal{V}_\beta$ such that

$$\mathcal{L}^*\left(E_\alpha - \bigcup_{j=1}^\infty [c_j,d_j]\right) = 0.$$

Then by subadditivity of $\mathcal{L}^*$, we have

$$\begin{aligned}
\mathcal{L}^*(E_\alpha) &\leq \mathcal{L}^*\left(\left(E_\alpha - \bigcup_{j=1}^\infty [c_j,d_j]\right) \cup \bigcup_{j=1}^\infty [c_j,d_j]\right) \\
&= \mathcal{L}^*\left(\bigcup_{j=1}^\infty [c_j,d_j]\right) = \sum_{j=1}^\infty (d_j - c_j).
\end{aligned}$$

The last step follows from the fact that $\{[c_j,d_j]\}_{j=1}^\infty$ are disjoint. Then by the definition of $\mathcal{V}_\beta$, we have for any j,

$$\beta(d_j - c_j) \leq f(d_j) - f(c_j),$$

and hence

$$\beta\mathcal{L}^*(E_\alpha) \leq \sum_{j=1}^\infty \beta(d_j - c_j) \leq \sum_{j=1}^\infty \big(f(d_j) - f(c_j)\big) \leq f(b) - f(a).$$

The last step follows from the fact that f is increasing on $[a,b]$.

As β is an arbitrary number in $(0,\alpha)$, letting $\beta \to \alpha^-$ in the above inequality will complete the proof of this lemma. $\square$

Remark 10.12. We call the above lemma to be an analogue of mean-value theorem because if f is continuous on $[a,b]$ and differentiable on (a,b) with $f'(x) \geq \alpha$ on $[c,d] \subset (a,b)$, then the classical mean-value asserts that

$$f(d) - f(c) = f'(\xi)(d-c) \geq \alpha(d-c) = \alpha\mathcal{L}^*([c,d]).$$

The lemma allows f' to be $\overline{D}f$, and the interval $[c,d]$ can be a more general set.

Exercise 10.9. Let $f : [a,b] \to \mathbb{R}$ be an increasing function. Suppose $E \subset U \subset (a,b)$ where E is a set such that $\underline{D}f(x) \leq \alpha$ for any $x \in E$, and U is an open set, then the following is a Vitali's covering of E:

$$\mathcal{V} := \{[c,d] \subset U : f(d) - f(c) \leq \alpha(d-c)\}.$$

Exercise 10.10. Given the same assumption as in the above lemma, show that

$$\mathcal{L}^*\{x \in (a,b) : \overline{D}f(x) = +\infty\} = 0.$$

Proof of Theorem 10.9. Let $f : [a,b] \to \mathbb{R}$ be an increasing function. We only need to show f is differentiable $\mathcal{L}^1$-a.e. on (a,b), since $\{a,b\}$ has zero measure.

The function f is not differentiable at x_0 if $\overline{D}f(x_0) > \underline{D}f(x_0)$ or $\overline{D}f(x_0) = +\infty$ (note that $\underline{D}f(x_0) = -\infty$ is not possible for an increasing function f). From Exercise 10.10, the points at which the later occurs form a set of measure zero, so we only need to show the following set has zero Lebesgue measure:

$$E := \{x \in (a,b) : \underline{D}f(x) < \overline{D}f(x)\}.$$

One can express E as:

$$E := \bigcup_{\substack{\alpha,\beta \in \mathbb{Q} \\ 0<\alpha<\beta}} \underbrace{\{x \in (a,b) : \underline{D}f(x) \leq \alpha < \beta \leq \overline{D}f(x)\}}_{=:E_{\alpha,\beta}}.$$

By countable subadditivity, it suffices to show $E_{\alpha,\beta}$ has zero Lebesgue measure for any fixed rational numbers α and β with $0 < \alpha < \beta$.

Given any $\varepsilon > 0$, one can pick an open set $U \supset E_{\alpha,\beta}$ such that

$$\mathcal{L}^*(E_{\alpha,\beta}) \leq \mathcal{L}^*(U) < \mathcal{L}^*(E_{\alpha,\beta}) + \varepsilon.$$

As $E_{\alpha,\beta} \subset (a,b)$, we may assume – by replacing U by $U \cap (a,b)$ – that $U \subset (a,b)$. Then, consider the collection of closed intervals:

$$\mathcal{V}_\alpha := \{[c,d] \subset U : f(d) - f(c) \leq \alpha(d-c)\},$$

which is a Vitali's covering of $E_{\alpha,\beta}$ by Exercise 10.9. Therefore, Vitali's covering theorem shows there exists a countable subcollection of disjoint intervals $\{[c_j,d_j]\}_{j=1}^\infty$ such that

$$\mathcal{L}^*\left(E_{\alpha,\beta} - \bigcup_{j=1}^\infty [c_j,d_j]\right) = 0.$$

Recall that from the definition of $\mathcal{V}_\alpha$, we have

$$f(d_j) - f(c_j) \leq \alpha(d_j - c_j) \ \ \forall j.$$

Summing up, we get

$$\begin{aligned}\sum_{j=1}^\infty \big(f(d_j) - f(c_j)\big) \leq \alpha \sum_{j=1}^\infty (d_j - c_j) &= \alpha\mathcal{L}^*\left(\bigsqcup_{j=1}^\infty [c_j,d_j]\right) \\ &\leq \alpha\mathcal{L}^*(U) < \alpha\left(\mathcal{L}^*(E_{\alpha,\beta}) + \varepsilon\right).\end{aligned}$$

On the other hand, since $E_{\alpha,\beta} \cap (c_j, d_j) \subset \{x \in (c_j, d_j) : \overline{D}f(x) \geq \beta\}$ for each j, by Lemma 10.11 we have

$$\beta\mathcal{L}^*(E_{\alpha,\beta} \cap (c_j, d_j)) \leq \beta\mathcal{L}^*\{x \in (c_j, d_j) : \overline{D}f(x) \geq \beta\} \leq f(d_j) - f(c_j).$$

Summing up all j, we get

$$\begin{aligned}\beta\mathcal{L}^*\left(\bigcup_{j=1}^{\infty} E_{\alpha,\beta} \cap (c_j, d_j)\right) &\leq \beta\sum_{j=1}^{\infty} \mathcal{L}^*(E_{\alpha,\beta} \cap (c_j, d_j)) \\ &\leq \beta\sum_{j=1}^{\infty} \big(f(d_j) - f(c_j)\big) \\ &\leq \alpha\big(\mathcal{L}^*(E_{\alpha,\beta}) + \varepsilon\big).\end{aligned}$$

Recall that $E_{\alpha,\beta} - \bigcup_{j=1}^{\infty}[c_j, d_j]$ has zero Lebesgue measure, so

$$\mathcal{L}^*(E_{\alpha,\beta}) = \mathcal{L}^*\left(\bigcup_{j=1}^{\infty} E_{\alpha,\beta} \cap (c_j, d_j)\right) \leq \frac{\alpha}{\beta}\left(\mathcal{L}^*(E_{\alpha,\beta}) + \varepsilon\right).$$

Since the above inequality holds true for any arbitrary $\varepsilon > 0$, letting $\varepsilon \to 0^+$, we conclude that

$$\mathcal{L}^*(E_{\alpha,\beta}) \leq \frac{\alpha}{\beta}\mathcal{L}^*(E_{\alpha,\beta})$$

for any fixed rational numbers α, β such that $0 < \alpha < \beta$. Since $\frac{\alpha}{\beta} < 1$, it could only happen when $\mathcal{L}^*(E_{\alpha,\beta}) = 0$. It completes the proof that $\mathcal{L}^*(E) = 0$. $\square$

As any Lipschitz continuous function $f : [a, b] \to \mathbb{R}$ is of bounded variation, they are differentiable $\mathcal{L}^1$-a.e. on $[a, b]$. Before we extend this result to higher dimensions, we need to take a detour. In the next section, we focus on the notion of absolutely continuous functions, which include Lipschitz continuous functions. We will prove that absolutely continuous functions $f : [a, b] \to \mathbb{R}$ satisfies the Newton-Leibniz's formula:

$$\int_a^b f' \, d\mathcal{L}^1 = f(b) - f(a).$$

Meanwhile, let's learn a famous example of a function which is of bounded variation but the Newton-Leibniz's formula does not hold:

10.1.5. Cantor-Lebesgue function: devil's staircase. Recall that the Cantor set $\mathcal{C}$ was constructed by successively removing the middle one-third open intervals:

$$\mathcal{C} = [0, 1] - \big(I_{1/3} \cup 2I_{1/3^2} \cup 2^2 I_{1/3^3} \cup \cdots\big)$$

where $I_{1/3} = (1/3, 2/3)$, $2I_{1/3^2} = (1/9, 2/9) \cup (7/9, 8/9)$, etc.

Based on this construction, we construct a sequence of function $\{f_k\}_{k=1}^{\infty}$ as follows: for $f_1 : [0, 1] \to \mathbb{R}$, we let

$$f_1(x) = \begin{cases} \frac{3x}{2} & \text{if } x \in [0, \frac{1}{3}] \\ \frac{1}{2} & \text{if } x \in (\frac{1}{3}, \frac{2}{3}) \\ \frac{3x}{2} - \frac{1}{2} & \text{if } x \in [\frac{2}{3}, 1] \end{cases}.$$

In other words, f_1 is a constant $\frac{1}{2}$ on $I_{1/3}$, and being (affine) linear outside of $I_{1/3}$.

Similarly, f_2 is constructed so that it equals $\frac{1}{4}$ on $(1/9, 2/9)$, equals $\frac{1}{2} = \frac{2}{4}$ on $(1/3, 2/3)$, and equals $\frac{3}{4}$ on $(7/9, 8/9)$. Then interpolation with linear graphs in the gap in-between.

Exercise 10.11. Write down the general expression of f_k. Show also that
$$|f_k(x) - f_{k-1}(x)| \leq \frac{1}{2^k}$$
for any $x \in [0, 1]$ and $k \geq 2$.

From the above exercise, one can easily show that $\{f_k\}$ converges uniformly on $[0, 1]$. By our construction, if $x \in I_{1/3^j}$ for some j, then $f_k(x)$ is independent of x when $k > j$. This shows the limit $f_\infty := \lim\limits_{k\to\infty} f_k$ is constant on the open interval $I_{1/3^j}$ for any j. Since these open intervals is almost equal to $\mathcal{C}$, we have that $f'_\infty = 0$ a.e. on $[0, 1]$.

However, f_∞ is an increasing function since every f_k is so. It is a function of bounded variation on $[0, 1]$. However, it is easy to see that
$$\int_0^1 f' \, d\mathcal{L}^1 = 0 \quad \text{but } f(1) - f(0) = 1 - 0 = 1.$$

This example shows two things:

(1) Newton-Leibniz's formula may not hold even for increasing functions.

(2) Even if a continuous function f satisfies $f' = 0$ a.e. on $[0, 1]$, f may not be a constant.

The next section introduces a stronger type of function so that (1) and (2) both hold.

10.2. Absolutely Continuous Functions on $\mathbb{R}$

The definition of absolute continuity on $\mathbb{R}$ was defined in the previous section. It is essentially the multi-interval generalization of uniform continuity. Absolutely continuous functions are special because we are about to show that they satisfies the Newton-Leibniz's formula:

$$\int_a^b f' d\mathcal{L}^1 = f(b) - f(a).$$

Although this result can be easily proven when f is C^1, its a generalization to absolutely continuous functions is highly non-trivial. The proof even involves the use of Vitali's covering lemma again!

Recall that in single-variable calculus, the Newton-Leibniz's formula was a corollary of the result that

$$\frac{d}{dx}\int_a^x f'(t)\,dt = f'(x) \tag{10.3}$$

where f is a C^1 function. Then, f and $\displaystyle\int_a^x f'(t)\,dt$ are both anti-derivatives of f', so they differ by a constant:

$$\int_a^x f'(t)\,dt - f(x) = C \;\; \forall x \in [a,b].$$

Put $x = a$, we then get that $C = -f(a)$. This shows

$$\int_a^x f'(t)\,dt = f(x) - f(a) \;\; \forall x \in [a,b].$$

Now that f is not a C^1 function but just a Lipschitz/absolutely continuous function. In order to prove the Newton-Leibniz's formula, we first need to establish a more generalized result than (10.3). This generalized result is called Lebesgue's differentiation theorem, but before we introduce and proof it, we first need to prove an intermediate result due to Hardy and Littlewood.

10.2.1. Hardy-Littlewood maximal inequality. Now $f : \mathbb{R}^n \to \mathbb{R}$ is a $\mathcal{L}^n$-integrable function on $\mathbb{R}^n$, we define $Mf : \mathbb{R}^n \to [0,\infty]$, called the **Hardy-Littlewood maximal function**, as follows:

$$Mf(x) := \sup_{x\in B} \frac{1}{\mathcal{L}^n(B)}\int_B |f| \; d\mathcal{L}^n,$$

where the supremum is taken over all balls B containing x. In particular, on $\mathbb{R}$ it can be written as:

$$Mf(x) = \sup_{\varepsilon,\delta>0} \frac{1}{\varepsilon+\delta}\int_{x-\varepsilon}^{x+\delta} |f| \; d\mathcal{L}^1.$$

It can be easily verified that Mf is a $\mathcal{L}^n$-measurable function. In fact, the set

$$E_\alpha := \{x \in \mathbb{R}^n : Mf(x) > \alpha\}$$

is open for any $\alpha > 0$. To see this, pick $x \in E_\alpha$, then

$$\sup_{x\in B} \frac{1}{\mathcal{L}^n(B)}\int_B |f| \; d\mathcal{L}^n > \alpha.$$

There exists a ball B_x containing x such that

$$\frac{1}{\mathcal{L}^n(B_x)}\int_{B_x} |f| \; d\mathcal{L}^n > \alpha.$$

It shows that for any $y \in (B_x)^\circ$, we have

$$Mf(y) = \sup_{y \in B} \frac{1}{\mathcal{L}^n(B)} \int_B |f| \, d\mathcal{L}^n \geq \frac{1}{\mathcal{L}^n(B_x)} \int_{B_x} |f| \, d\mathcal{L}^n > \alpha.$$

In other words, $(B_x)^\circ \subset E_\alpha$. This shows for any $\alpha > 0$, the set E_α is open, and hence measurable. From the definition of Mf, it is easy to see that when $\alpha = 0$, $E_\alpha = \mathbb{R}^n$; and when $\alpha < 0$, $E_\alpha = \emptyset$. It concludes that Mf is a $\mathcal{L}^n$-measurable function.

Next we will prove an important result concerning this Mf function.

Proposition 10.13 (Hardy-Littlewood maximal inequality)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be an $\mathcal{L}^n$-integrable function on $\mathbb{R}^n$, then the following inequality holds for any $\alpha > 0$.*

(10.4)
$$\mathcal{L}^n\{x \in \mathbb{R}^n : Mf(x) > \alpha\} \leq \frac{5^n}{\alpha} \|f\|_{L^1(\mathbb{R}^n)}$$

Proof. The inequality is magically proved by the Vitali's 5-covering lemma. For any $\alpha > 0$, we denote

$$E_\alpha := \{x \in \mathbb{R}^n : Mf(x) > \alpha\}.$$

For any $x \in E_\alpha$, one can find an open ball B_x containing x such that

$$\frac{1}{\mathcal{L}^n(B_x)} \int_{B_x} |f| \, d\mathcal{L}^n > \alpha.$$

Then, this collection of balls $\{B_x\}_{x \in E_\alpha}$ forms an open cover of E_α. Also, since

$$\mathcal{L}^n(B_x) < \frac{1}{\alpha} \int_{B_x} |f| \, d\mathcal{L}^n \leq \frac{1}{\alpha} \|f\|_{L^1(\mathbb{R}^n)} < \infty,$$

the collection $\{B_x\}_{x \in E_\alpha}$ has bounded Lebesgue measure, and hence bounded radii. By Vitali's 5-covering lemma, there exists a countable subcollection of disjoint balls $\{B_{x_j}\}_{j=1}^\infty$, where $x_j \in E_\alpha$, such that

$$E_\alpha \subset \bigcup_{x \in E_\alpha} B_x \subset \bigcup_{j=1}^\infty 5B_{x_j}.$$

It implies

(10.5)
$$\mathcal{L}^n(E_\alpha) \leq \mathcal{L}^n\left(\bigcup_{j=1}^\infty 5B_{x_j}\right) = \sum_{j=1}^\infty \mathcal{L}^n(5B_{x_j}) = 5^n \sum_{j=1}^\infty \mathcal{L}^n(B_{x_j}).$$

Recall that by our choice of B_x, we have for any j,

$$\mathcal{L}^n(B_{x_j}) < \frac{1}{\alpha} \int_{B_{x_j}} |f| \, d\mathcal{L}^n.$$

Note that $\{B_{x_j}\}_{j=1}^\infty$ is a disjoint collection, so we have

(10.6)
$$\sum_{j=1}^\infty \mathcal{L}^n(B_{x_j}) < \sum_{j=1}^\infty \frac{1}{\alpha} \int_{B_{x_j}} |f| \, d\mathcal{L}^n = \frac{1}{\alpha} \int_{\sqcup_j B_{x_j}} |f| \, d\mathcal{L}^n \leq \frac{1}{\alpha} \|f\|_{L^1(\mathbb{R}^n)}.$$

Combining (10.5) and (10.6), we complete the proof of (10.4). □

Exercise 10.12. Using the Hardy-Littlewood maximal inequality, prove that for any $\mathcal{L}^n$-integrable function f on $\mathbb{R}^n$, the function Mf is finite $\mathcal{L}^n$-a.e. on $\mathbb{R}^n$.

Exercise 10.13. It is possible to modify the proof of the Hardy-Littlewood maximal inequality so that only Vitali's 3-covering theorem is needed. It is easy if E_α is compact. Generally, one can take an increasing sequence of compact sets $K_1 \subset K_2 \subset \cdots$ such that $E_\alpha = \bigcup_j K_j$. Try to write down the whole proof.

Question: How is the result better than (10.4) if we prove it using 3-covering?

10.2.2. Lebesgue's differentiation theorem. Next we give a generalization of the Fundamental Theorem of Calculus (which originally require f to be continuous)

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

to any integrable function f, and also to higher dimension. The above result can be restated as

$$\lim_{h\to 0}\frac{1}{h}\int_{[x,x+h]} f(t)\,dt = f(x).$$

Note that h is the measure of $[x, x+h]$. This result is generalized to allow measurable functions in the following form:

Theorem 10.14 (Lebesgue Differentiation Theorem). *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{L}^n$-integrable on $\mathbb{R}^n$, then*

$$\lim_{\delta\to 0^+}\sup_{\substack{\mathcal{L}^n(B)<\delta\\ x\in B}}\frac{1}{\mathcal{L}^n(B)}\int_B f\,d\mathcal{L}^n = f(x) \qquad \text{for } \mathcal{L}^n\text{-a.e. } x\in\mathbb{R}^n. \tag{10.7}$$

Here the supremum is taken over all balls B containing x with $\mathcal{L}^n(B) < \delta$.

Proof. To begin, we first prove that (10.7) holds for all continuous function $g : \mathbb{R}^n \to \mathbb{R}$ with compact support. Then, we use an approximation argument to extend the result to any $\mathcal{L}^n$-integrable function.

Let $g : \mathbb{R}^n \to \mathbb{R}$ be a continuous function with compact support. Such a function is also uniformly continuous on $\mathbb{R}^n$, so for any $\varepsilon > 0$, there exists $\delta' > 0$ such that if $|x-y| < \delta'$, then $|g(x) - g(y)| < \varepsilon$. Let $\delta = C_n\delta'$ be a constant such that $\mathcal{L}^n(B) = \delta$ if and only if $\operatorname{diam} B = \delta'$, then for any B containing x with $\mathcal{L}^n(B) < \delta$, $y \in B$ would imply $|x-y| < \operatorname{diam} B = \delta'$. From the above, we can conclude that for any such B, we have

$$\begin{aligned}\left|\frac{1}{\mathcal{L}^n(B)}\int_B g(y)\,d\mathcal{L}^n(y) - g(x)\right| &= \left|\frac{1}{\mathcal{L}^n(B)}\int_B \big(g(y)-g(x)\big)\,d\mathcal{L}^n(y)\right|\\ &\le \frac{1}{\mathcal{L}^n(B)}\int_B |g(y)-g(x)|\,d\mathcal{L}^n(y)\\ &\le \frac{1}{\mathcal{L}^n(B)}\int_B \varepsilon\,d\mathcal{L}^n(y) = \varepsilon.\end{aligned}$$

Taking supremum over all balls B containing x with $\mathcal{L}^n(B) < \delta$, and then letting $\delta \to 0^+$, one can show (10.7) holds (even for all $x \in \mathbb{R}^n$) when such a function g.

Next we extend the result to any integrable function f on $\mathbb{R}^n$. We use a density argument that for any $\mathcal{L}^n$-integrable function f on $\mathbb{R}^n$, and any $\varepsilon > 0$, there exists a continuous function g with compact support such that

$$\|f-g\|_{L^1(\mathbb{R}^d)} < \varepsilon.$$

To prove (10.7) holds for f, we first express

$$\begin{aligned}&\frac{1}{\mathcal{L}^n(B)}\int_B f(y)\,d\mathcal{L}^n(y)-f(x)\\&=\frac{1}{\mathcal{L}^n(B)}\int_B \big(f(y)-g(y)\big)\,d\mathcal{L}^n(y)-\big(f(x)-g(x)\big)+\frac{1}{\mathcal{L}^n(B)}\int_B g(y)\,dy-g(x).\end{aligned}$$

Taking supremum over all balls B containing x with $\mathcal{L}^n(B)$, we get

$$\begin{aligned}&\sup_{\substack{\mathcal{L}^n(B)<\delta\\ x\in B}}\left|\frac{1}{\mathcal{L}^n(B)}\int_B f(y)\,d\mathcal{L}^n(y)-f(x)\right|\\&\leq M(f-g)(x)+|f(x)-g(x)|+\sup_{\substack{\mathcal{L}^n(B)<\delta\\ x\in B}}\left|\frac{1}{\mathcal{L}^n(B)}\int_B g(y)\,dy-g(x)\right|.\end{aligned}$$

Letting $\delta\to 0^+$, by the fact that (10.7) holds for g, we get

$$\lim_{\delta\to 0^+}\sup_{\substack{\mathcal{L}^n(B)<\delta\\ x\in B}}\left|\frac{1}{\mathcal{L}^n(B)}\int_B f(y)\,d\mathcal{L}^n(y)-f(x)\right|\leq M(f-g)(x)+|f(x)-g(x)|\,.$$

For any $\alpha>0$, we denote

$$E_\alpha:=\left\{x\in\mathbb{R}^n:\lim_{\delta\to 0^+}\sup_{\substack{\mathcal{L}^n(B)<\delta\\ x\in B}}\left|\frac{1}{\mathcal{L}^n(B)}\int_B f(y)\,d\mathcal{L}^n(y)-f(x)\right|>\alpha\right\}.$$

We will show that $\mathcal{L}^n(E_\alpha)=0$ for any $\alpha>0$. From the above inequality, we have

$$E_\alpha\subset\{x\in\mathbb{R}^n:M(f-g)(x)>\alpha/2\}\cup\{x\in\mathbb{R}^n:|f(x)-g(x)|>\alpha/2\}.$$

By Hardy-Littlewood maximal inequality, we have

$$\mathcal{L}^n\{x\in\mathbb{R}^n:M(f-g)(x)>\alpha/2\}\leq\frac{2\cdot 5^n}{\alpha}\,\|f-g\|_{L^1(\mathbb{R}^n)}<\frac{2\cdot 5^n}{\alpha}\varepsilon.$$

Also, Chebychev's inequality shows

$$\begin{aligned}&\mathcal{L}^n\{x\in\mathbb{R}^n:|f(x)-g(x)|>\alpha/2\}\\&\leq\frac{2}{\alpha}\int_{|f-g|>\alpha/2}|f-g|\,d\mathcal{L}^n\\&\leq\frac{2}{\alpha}\,\|f-g\|_{L^1(\mathbb{R}^n)}<\frac{2}{\alpha}\varepsilon.\end{aligned}$$

These estimates show that

$$\mathcal{L}^n(E_\alpha)\leq\frac{2\cdot 5^n}{\alpha}\varepsilon+\frac{2}{\alpha}\varepsilon.$$

Letting $\varepsilon\to 0^+$ (noting that E_α is independent of ε), we get $\mathcal{L}^n(E_\alpha)=0$ for any $\alpha>0$. Since

$$E:=\left\{x\in\mathbb{R}^n:\lim_{\delta\to 0^+}\sup_{\substack{\mathcal{L}^n(B)<\delta\\ x\in B}}\left|\frac{1}{\mathcal{L}^n(B)}\int_B f(y)\,d\mathcal{L}^n(y)-f(x)\right|>0\right\}=\bigcup_{k=1}^{\infty}E_{1/k},$$

we conclude that $\mathcal{L}^n(E)=0$, and hence proving (10.7) holds for f. □

Exercise 10.14. Show, as a corollary to the Lebesgue's differentiation theorem, that if $f:[a,b]\to\mathbb{R}$ is an $\mathcal{L}^1$-integrable function on $\mathbb{R}$, then

$$\frac{d}{dx}\int_a^x f'\,d\mathcal{L}^1=f(x)\qquad\forall\text{ a.e.-}x\in\mathbb{R}.$$

10.2.3. Newton-Leibniz's formula. Now that the Lebesgue's differential theorem is proved. In order to mimic the proof the Newton-Leibniz's formula in single-variable calculus, we need one more result – that if two functions have the same derivative a.e., then they differ by a constant. We have seen that it is not always true in the Cantor-Lebesgue's function example, but luckily it still holds for absolutely continuous functions.

Proposition 10.15. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function such that $f' = 0$ a.e. on $[a,b]$. Then f is a constant function on $[a,b]$.*

Proof. The proof uses Vitali's covering theorem (again). It suffices to show $f(a) = f(b)$. For other points $c \in (a,b)$, one can show $f(a) = f(c)$ with the same argument by replacing $[a,b]$ by $[a,c]$.

Let E be the set of $x \in (a,b)$ where $f'(x) = 0$. From our assumption, we have $\mathcal{L}^1(E) = b - a$. For any $\varepsilon > 0$, we consider the following collection of closed intervals:

$$\mathcal{V}_\varepsilon := \{[c,d] \subset (a,b) : |f(d) - f(c)| \leq \varepsilon\,|d - c|\}.$$

We claim that $\mathcal{V}_\varepsilon$ is a Vitali's covering of E: for any $x \in E$, we have

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = 0,$$

hence there exists $\delta > 0$ such that whenever $0 < |y - x| < \delta$, we have

$$\left|\frac{f(y) - f(x)}{y - x}\right| < \varepsilon.$$

In particular, whenever $\eta \in (0,\delta)$, the closed interval $[x, x+\eta]$ contains x and is belong to $\mathcal{V}_\varepsilon$. As η can be arbitrarily small, this shows $\mathcal{V}_\varepsilon$ is a Vitali's covering of E.

By Vitali's covering theorem, there exists a countable subcollection of disjoint intervals $\{[c_i, d_i]\}_{i=1}^\infty$ of $\mathcal{V}_\varepsilon$ such that

$$\mathcal{L}^1\left(E - \bigcup_{i=1}^\infty [c_i, d_i]\right) = 0 \implies \mathcal{L}^1\left([a,b] - \bigcup_{i=1}^\infty (c_i, d_i)\right) = 0.$$

By the definition of $\mathcal{V}_\varepsilon$, we have

$$|f(d_i) - f(c_i)| \leq \varepsilon(d_i - c_i) \;\; \forall i.$$

Recall that f is absolutely continuous on $[a,b]$, there exists $\delta > 0$ such that whenever a finite disjoint collection of open intervals $\{(\alpha_i, \beta_i)\}_{i=1}^N$ have total length $< \delta$, we have

$$\sum_{i=1}^N |f(\beta_i) - f(\alpha_i)| < \varepsilon.$$

Using this $\delta > 0$, we can find a large integer k such that

$$\mathcal{L}^1\left([a,b] - \bigcup_{i=1}^k (c_i, d_i)\right) < \delta.$$

Note that $[a,b] - \cup_{i=1}^k (c_i, d_i)$ is a finite disjoint union of closed intervals. WLOG assume $c_1 < d_1 < c_2 < d_2 < \cdots < c_k < d_k$, then

$$[a,b] - \bigcup_{i=1}^k (c_i, d_i) = [a, c_1] \sqcup [d_1, c_2] \sqcup [d_2, c_3] \sqcup \cdots \sqcup [d_{k-1}, c_k] \sqcup [d_k, b].$$

The total length of these intervals are less than δ, so we have

$$|f(a) - f(c_1)| + |f(d_1) - f(c_2)| + \cdots + |f(d_{k-1}) - f(c_k)| + |f(d_k) - f(b)| < \varepsilon.$$

Then, by triangle inequality and using the above estimates, we have

$$\begin{aligned}
&|f(a) - f(b)| \\
&\leq |f(a) - f(c_1)| + |f(c_1) - f(d_1)| + |f(d_1) - f(c_2)| \\
&\qquad + \cdots + |f(d_{k-1}) - f(c_k)| + |f(c_k) - f(d_k)| + |f(d_k) - f(b)| \\
&< \varepsilon + \sum_{i=1}^{k} |f(c_k) - f(d_k)| \\
&\leq \varepsilon + \sum_{i=1}^{k} \varepsilon(d_k - c_k) \\
&\leq \varepsilon + \varepsilon(b-a).
\end{aligned}$$

Letting $\varepsilon \to 0^+$, we get $f(a) = f(b)$ as desired. □

Now we are almost ready to prove the Newton-Leibniz formula of absolutely continuous function, by mimicking the single-variable calculus argument. Lebesgue's differentiation theorem shows

$$g(x) = \int_a^x f' \, d\mathcal{L}^1$$

has derivative equal to $f'(x)$ a.e., so both f and g are anti-derivative of f' almost everywhere. If one can further show that $f - g$ is absolutely continuous, then it must be a constant by the previous proposition. That will give the proof of Newton-Leibniz's formula as in single-variable calculus. Therefore, we need one more result:

Finally, we can now state and prove our main result:

Proposition 10.16. *Let $f : [a,b] \to \mathbb{R}$ be an increasing continuous function, then*

$$\int_a^b f' \, d\mathcal{L}^1 \leq f(b) - f(a).$$

Hence, f' is $\mathcal{L}^1$-integrable on $[a,b]$.

Proof. First extend f so that $f(x) = f(a)$ for any $x < a$, and $f(x) = f(b)$ for any $x > b$. The extension is still an increasing, continuous function.

Consider the sequence of functions:

$$g_n(x) := \frac{f(x + \frac{1}{n}) - f(x)}{\frac{1}{n}}.$$

Since f' exists a.e., $g_n(x) \to f'(x)$ for a.e. $x \in [a,b]$. Also, as f is increasing, $g_n \geq 0$. By Fatou's Lemma, we get

$$\int_a^b f' \, d\mathcal{L}^1 = \int_a^b \liminf_{n\to\infty} g_n \, d\mathcal{L}^1 \leq \liminf_{n\to\infty} \int_a^b g_n \, d\mathcal{L}^1. \tag{10.8}$$

Then we consider:

$$\begin{aligned}
&\int_a^b g_n \, d\mathcal{L}^1 = \int_a^b \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}} \, d\mathcal{L}^1(x) = \int_{a+\frac{1}{n}}^{b+\frac{1}{n}} nf(y) \, d\mathcal{L}^1(y) - \int_a^b nf(x) \, d\mathcal{L}^1(x) \\
&= \int_b^{b+\frac{1}{n}} nf(x) \, d\mathcal{L}^1(x) - \int_a^{a+\frac{1}{n}} nf(x) \, d\mathcal{L}^1(x) \\
&= f(b) - \int_a^{a+\frac{1}{n}} nf(x) \, d\mathcal{L}^1(x).
\end{aligned}$$

Since f is continuous at a, we have

$$\lim_{h\to 0}\frac{1}{h}\int_a^{a+h} f(x)\,d\mathcal{L}^1(x) = f(a) \implies \lim_{n\to\infty}\int_a^{a+\frac{1}{n}} nf(x)\,d\mathcal{L}^1(x) = f(a).$$

Combining this result with (10.8), we finish the proof of the proposition. □

Corollary 10.17. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function, then*

$$g(x) := \int_a^x f'\,d\mathcal{L}^1$$

is also absolutely continuous on $[a,b]$.

Proof. By the Jordan's theorem and Exercise 10.6, f can be expressed as $f = f_1 - f_2$ where f_1, f_2 are two increasing continuous functions on $[a,b]$.

From the above proposition, both f_1' and f_2' are $\mathcal{L}^1$-integrable on $[a,b]$. Hence, $f' = f_1' - f_2'$ is also integrable on $[a,b]$. From Exercise 10.5, we conclude that g is absolutely continuous on $[a,b]$. □

Theorem 10.18. *Suppose $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$, then we have*

$$\int_a^b f'\,d\mathcal{L}^1 = f(b) - f(a).$$

Proof. By Corollary 10.17, the function

$$g(x) := \int_a^x f'\,d\mathcal{L}^1$$

is absolutely continuous on $[a,b]$, and by Lebesgue's differentiation theorem its derivative $g' = f'$ a.e. on $[a,b]$. Hence, $g - f$ is an absolutely continuous function on $[a,b]$ such that its derivative is zero almost everywhere on $[a,b]$. By Proposition 10.15, $g(x) - f(x) = C$ on $[a,b]$.

Therefore, we have

$$\int_a^x f'\,d\mathcal{L}^1 = f(x) + C \qquad \forall x \in [a,b].$$

Putting $x = a$, we get $C = -f(a)$. Hence, we conclude that

$$\int_a^x f'\,d\mathcal{L}^1 = f(x) - f(a) \qquad \forall x \in [a,b].$$

The theorem is proved by putting $x = b$. □

10.3. Rademacher's Theorem

From now on we assume that the function $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous on its open domain U, which means that there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \leq C\,|x - y| \quad \forall x, y \in U.$$

The smallest such C is denoted by $\text{Lip}_U(f)$, or simply $\text{Lip}\,(f)$.

A function $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is **differentiable** at $x_0 \in U$ means that there exists a linear map, denoted by $Df(x_0) : \mathbb{R}^n \to \mathbb{R}^m$ (alternative notations: Df_{x_0}, df_{x_0}, f_{*x_0}, Tf_{x_0}, etc.) such that

$$f(x) = f(x_0) + Df(x_0)(x - x_0) + o\big(|x - x_0|\big) \quad \text{as } x \to x_0.$$

In view of the fact that f is differentiable at $x_0 \in U$ if and only if $\pi_i \circ f$ is differentiable at $x_0 \in U$ for any $i = 1, \cdots, m$, where $\pi_i : \mathbb{R}^n \to \mathbb{R}$ is the projection of the i-th coordinate, we may restrict to the case $f : U \subset \mathbb{R}^n \to \mathbb{R}$. Furthermore, for any $f : U \subset \mathbb{R}^n \to \mathbb{R}$, one can extend the function to $\overline{f} : \mathbb{R}^n \to \mathbb{R}$ by defining

$$\overline{f}(x) := \inf_{a \in U} \big(f(a) + \text{Lip}\,(f)\,|x - a|\big) \ \ \forall x \in \mathbb{R}^n,$$

then one can check (exercise) that $\overline{f}$ is Lipschitz continuous on $\mathbb{R}^n$, and $\overline{f} \equiv f$ on U. Therefore, we may focus on Lipschitz continuous functions of type $f : \mathbb{R}^n \to \mathbb{R}$.

Theorem 10.19 (Rademacher's Theorem)**.** *Any Lipschitz continuous function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable $\mathcal{L}^n$-a.e.*

The proof of Rademacher's Theorem relies on the result of the special case when $n = 1$ discussed in the previous sections.

Proof of Rademacher's Theorem. Now assuming the special case $f : \mathbb{R} \to \mathbb{R}$ is proved. We proceed the whole proof in a number of steps.

Step 1: Prove that for each unit vector $v \in \mathbb{R}^n$, $D_v f(x)$ exists $\mathcal{L}^n$-a.e. on $\mathbb{R}^n$.

For a fixed unit vector $v \in \mathbb{R}^n$, we denote the set

$$N_v := \{x \in \mathbb{R}^n : D_v f(x) \text{ does not exist}\}.$$

Recall that

$$D_v f(x) := \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}.$$

Denote $L_{x,v} := \{x + tv : t \in \mathbb{R}\}$ the unique straight line passing through x and parallel to v. Since $t \mapsto f(x + tv)$ is Lipschitz continuous on $\mathbb{R}$, it is differentiable $\mathcal{L}^1$-a.e. on the line $L_{x,v}$ by the special case of the theorem when $n = 1$. In other words, we have $\mathcal{H}^1(N_v \cap L_{x,v}) = 0$ for any $x \in \mathbb{R}^n$ and unit vector $v \in \mathbb{R}^n$.

Since f is continuous, so is a $x \mapsto \frac{f(x+tv)-f(x)}{t}$ is a Borel measurable function for any fixed $t \neq 0$, the set

$$N_v = \mathbb{R}^n - \left\{x \in \mathbb{R}^n : \limsup_{t \to 0} \frac{f(x + tv) - f(x)}{t} = \liminf_{t \to 0} \frac{f(x + tv) - f(x)}{t} < \infty\right\}$$

is a Borel set. We can then use the Fubini Theorem (after rotation) to conclude that:

$$\mathcal{L}^n(N_v) = \int_{\{\langle x,v \rangle = 0\}} \mathcal{H}^1(N_v \cap L_{x,v})\, d\mathcal{L}^1(x) = 0.$$

As a corollary, $\nabla f(x)$ exists $\mathcal{L}^n$-a.e. on $\mathbb{R}^n$ too.

Step 2: Prove that for each unit vector $v \in \mathbb{R}^n$, we have $D_v f(x) = \nabla f(x) \cdot v$ for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$.

This result appeared in any standard multivariable calculus course, in which we assume f is C^1 and the proof uses the chain rule. We cannot do so here because f is just assumed to be Lipschitz. To prove this, we verify that

$$\int_{\mathbb{R}^n} \varphi(x)\ D_v f(x)\, d\mathcal{L}^n(x) = \int_{\mathbb{R}^n} \varphi(x)\ \nabla f(x) \cdot v\, d\mathcal{L}^n(x)$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n)$.

To do so we note that by translation $x \mapsto x - tv$, we have for any $t \neq 0$:

$$\int_{\mathbb{R}^n} \frac{f(x+tv) - f(x)}{t}\ \varphi(x)\, d\mathcal{L}^n(x) = \int_{\mathbb{R}^n} \frac{\varphi(x-tv) - \varphi(x)}{t}\ f(x)\, d\mathcal{L}^n(x).$$

As f is Lipschitz, the integrand on the LHS is bounded, and since φ has compact support, we can use LDCT/BCT to show

$$\begin{aligned}
&\lim_{t\to\infty} \int_{\mathbb{R}^n} \frac{f(x+tv) - f(x)}{t}\ \varphi(x)\, d\mathcal{L}^n(x) \\
&= \int_{\mathbb{R}^n} \lim_{t\to\infty} \frac{f(x+tv) - f(x)}{t}\ \varphi(x)\, d\mathcal{L}^n(x) \\
&= \int_{\mathbb{R}^n} D_v f(x)\ \varphi(x)\, d\mathcal{L}^n.
\end{aligned}$$

For the integrand in the RHS, we have φ being C^1 on a compact set, by MVT the integrand is bounded too, and for $0 < |t| < 1$, the integrand has compact support. By LDCT/BCT, one can also show that

$$\begin{aligned}
&\lim_{t\to\infty} \int_{\mathbb{R}^n} \frac{\varphi(x-tv) - \varphi(x)}{t}\ f(x)\, d\mathcal{L}^n(x) \\
&= \int_{\mathbb{R}^n} \lim_{t\to\infty} \frac{\varphi(x-tv) - \varphi(x)}{t}\ f(x)\, d\mathcal{L}^n(x) \\
&= -\int_{\mathbb{R}^n} D_v \varphi(x)\ f(x)\, d\mathcal{L}^n.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} D_v f(x)\ \varphi(x)\, d\mathcal{L}^n \\
&= -\int_{\mathbb{R}^n} D_v \varphi(x)\ f(x)\, d\mathcal{L}^n \\
&= -\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sum_{j=1}^n f(x) v_j \frac{\partial \varphi}{\partial x_j}(x)\, d\mathcal{L}^1(x_1) \cdots d\mathcal{L}^1(x_n).
\end{aligned}$$

Here we have used the Fubini's Theorem applied on Borel measurable functions. Next we apply integration by parts (which holds as $f\varphi$ is Lipschitz). For the $j = 1$ term,

$$\begin{aligned}
&-\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \int_{\mathbb{R}} f(x) v_1 \frac{\partial \varphi}{\partial x_1}(x)\, d\mathcal{L}^1(x_1) \cdots d\mathcal{L}^1(x_n) \\
&= -\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\frac{\partial}{\partial x_1}(v_1 f \varphi) - v_1 \varphi \frac{\partial f}{\partial x_1} \right) d\mathcal{L}^1(x_1) \cdots d\mathcal{L}^1(x_n) \\
&= -\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \underbrace{[v_1 f \varphi]_{x_1=-\infty}^{x_1=+\infty}}_{=0} d\mathcal{L}^1(x_2) \cdots d\mathcal{L}^1(x_n) \\
&\quad + \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \int_{\mathbb{R}} v_1 \varphi \frac{\partial f}{\partial x_1}\, d\mathcal{L}^1(x_1) \cdots d\mathcal{L}^1(x_n) \\
&= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \int_{\mathbb{R}} v_1 \varphi \frac{\partial f}{\partial x_1}\, d\mathcal{L}^1(x_1) \cdots d\mathcal{L}^1(x_n).
\end{aligned}$$

Similarly for $j \geq 2$. Summing up we can conclude that

$$\int_{\mathbb{R}^n} (D_v f)\, \varphi\, d\mathcal{L}^n = \sum_{j=1}^{n} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} v_j \varphi \frac{\partial f}{\partial x_j}\, d\mathcal{L}^1(x_1) \cdots d\mathcal{L}^1(x_n) = \int_{\mathbb{R}^n} (\nabla f \cdot v)\, \varphi\, d\mathcal{L}^n.$$

Since the above holds for any $\varphi \in C_0^\infty(\mathbb{R}^n)$, we conclude that $D_v f = \nabla f \cdot v$ almost everywhere on $\mathbb{R}^n$.

Step 3: Construct a Lebesgue measurable set $E \subset \mathbb{R}^n$ with $\mathcal{L}^n(\mathbb{R}^n - E) = 0$ and f is differentiable on E.

Let $\{v_j\}_{j=1}^\infty$ be a countable **dense** subset of the unit sphere $\mathbb{S}^{n-1}$, and consider:

$$E_j := \{x \in \mathbb{R}^n : D_{v_j} f(x) \text{ and } \nabla f(x) \text{ exist, and } D_{v_j} f(x) = \nabla f(x) \cdot v_j\}.$$

From the previous two steps, we know that $\mathcal{L}^n(\mathbb{R}^n - E_j) = 0$ for any $j \in \mathbb{N}$. Therefore, by letting

$$E = \bigcap_{j=1}^\infty E_j,$$

we have $\mathcal{L}^n(\mathbb{R}^n - E) = \mathcal{L}^n\left(\bigcup_{j=1}^\infty (\mathbb{R}^n - E_j)\right) = 0$ too. We claim that f is differentiable on E.

To prove this claim, we need to show that for any $x_0 \in E$, we have

$$\lim_{x \to x_0} \underbrace{\frac{|f(x) - f(x_0) - \nabla f(x_0) \cdot (x - x_0)|}{|x - x_0|}}_{=:Q(x, x_0)} = 0.$$

One key observation is that when $x_0 \in E$ and x is moving away from x_0 along the direction of one of the v_j's, then the above limit holds. The other directions v will be settled by a density argument.

For any $x \neq x_0$, we consider $v = \dfrac{x - x_0}{|x - x_0|}$ and write $x = x_0 + tv$ where $t = |x - x_0|$. We can estimate $Q(x, x_0)$ by

$$|Q(x, x_0)| = |Q(x_0 + tv, x_0)| \leq |Q(x_0 + tv_j, x_0)| + |Q(x_0 + tv_j, x_0) - Q(x_0 + tv, x_0)|.$$

For the first term on the RHS, we have

$$Q(x_0 + tv_j, x_0) = \left| \frac{f(x_0 + tv_j) - f(x_0)}{t} - \nabla f(x_0) \cdot v_j \right|,$$

which can be arbitrarily small for each fixed j provided that t is sufficiently small (by the definition of E_j). For the second term, we have

$$
\begin{aligned}
&|Q(x_0 + tv_j, x_0) - Q(x_0 + tv, x_0)| \\
&\leq \left| \left(\frac{f(x_0 + tv_j) - f(x_0)}{t} - \nabla f(x_0) \cdot v_j \right) - \left(\frac{f(x_0 + tv) - f(x_0)}{t} - \nabla f(x_0) \cdot v \right) \right| \\
&\leq \Big(\mathrm{Lip}\,(f) + |\nabla f(x_0)| \Big) \, |v_j - v| \leq \Big(1 + \sqrt{n}\Big) \mathrm{Lip}\,(f) \;\; |v_j - v| \, .
\end{aligned}
$$

To wrap up the proof, given any $\varepsilon > 0$, we first take v_j such that $|v_j - v| < \dfrac{\varepsilon}{2(1+\sqrt{n})\mathrm{Lip}\,(f)}$. For this v_j, as $x_0 \in E$, we know that

$$
\lim_{t \to 0} Q(x_0 + tv_j, x_0) = \big| D_{v_j} f(x_0) - \nabla f(x_0) \cdot v_j \big| = 0,
$$

and so there exists $\delta > 0$ such that when $0 < |t| = |x - x_0| < \delta$, we have

$$
|Q(x_0 + tv_j, x_0)| < \frac{\varepsilon}{2}.
$$

In other words, whenever $0 < |x - x_0| < \delta$, we have

$$
|Q(x, x_0)| \leq |Q(x_0 + tv_j, x_0)| + |Q(x_0 + tv_j, x_0) - Q(x_0 + tv, x_0)| < \varepsilon.
$$

This shows $\lim\limits_{x \to x_0} Q(x, x_0) = 0$ when $x_0 \in E$, or equivalently f is differentiable at $x_0 \in E$. $\square$

10.4. Area and Coarea Formulae

The area and coarea formulae are beautiful and significant results about Lipschitz continuous functions on Euclidean spaces. The area formula gives the geometric meaning of Hausdorff measures, proving that $\mathcal{H}^1$ of a parametrized curve is its length (as result we proved in Proposition 7.16 using definitions), $\mathcal{H}^2$ of a regular surface $\Sigma^2 \subset \mathbb{R}^3$ is the surface area, and more generally $\mathcal{H}^k$ of a k-dimensional submanifold Σ^k of $\mathbb{R}^n$ is the k-dimensional volume defined using the first fundamental form. The co-area formula is a "curved" version of the Fubini-Tonelli's Theorem. It has some far-reaching applications on differential geometry especially the analysis of minimal surfaces.

10.4.1. k-th Jacobian determinant. Given a Lipschitz function $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ from an open set U, we have proved that F is differentiable $\mathcal{L}^n$-a.e. on U. Therefore, the Jacobian matrix $DF(a) = \left[\frac{\partial F^j}{\partial x_i}(a)\right]$ is defined $\mathcal{L}^n$-a.e. on U. The area and coarea formulae involve one quantity called the k-th Jacobian determinant, which is defined as:

$$J_k F(a) := \sqrt{\text{sum of squares of all } k \times k \text{ sub-determinants of } DF(a)}.$$

Clearly, it is necessary that $k \leq m$ and $k \leq n$ for the above definition to make sense. As an example, "sum of squares of all 2×2 sub-determinants" of the 2×3 matrix

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

means

$$\begin{vmatrix} a & b \\ d & e \end{vmatrix}^2 + \begin{vmatrix} a & c \\ d & f \end{vmatrix}^2 + \begin{vmatrix} b & c \\ e & f \end{vmatrix}^2.$$

Exercise 10.15. Consider a Lipschitz continuous function $F : \mathbb{R}^n \to \mathbb{R}^m$. Show that:

(a) when $n \leq m$, then $J_n F(a) = \sqrt{\det\left(DF(a)^T DF(a)\right)}$, while

(b) when $n \geq m$, then $J_m F(a) = \sqrt{\det\left(DF(a) DF(a)^T\right)}$.

10.4.2. Area formula. The area formula concerns about Lipschitz function from a lower dimensional space to a higher (or equal) one:

Theorem 10.20 (Area formula). *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz continuous where $n \leq m$, and E be a Lebesgue measurable set in $\mathbb{R}^n$. Then, we have:*

$$\int_E J_n F(x)\, d\mathcal{H}^n(x) = \int_{\mathbb{R}^n} \#\big(E \cap F^{-1}(y)\big)\, d\mathcal{H}^n(y) \tag{10.9}$$

where $\# : \mathcal{P}_{\mathbb{R}^n} \to [0, \infty]$ is the counting measure. Furthermore, when $g : \mathbb{R}^n \to [-\infty, \infty]$ is a $\mathcal{L}^n$-measurable function which is either non-negative or integrable, we then have:

$$\int_{\mathbb{R}^n} g(x) J_n F(x)\, d\mathcal{H}^n(x) = \int_{\mathbb{R}^m} \sum_{x \in F^{-1}(y)} g(x)\, d\mathcal{H}^n(y). \tag{10.10}$$

The proof of (10.10) when g is non-negative makes use of (10.9), and the approximation of g by an increasing sequence of simple functions. Applying MCT yields 10.10. Note that another way of writing $\#\big(E \cap F^{-1}(y)\big)$ is $\displaystyle\sum_{x \in F^{-1}(y)} \chi_E(x)$. For integrable g but not necessarily non-negative, g_+ and g_- are simultaneously $+\infty$ only at a set of Lebesgue measure zero. Therefore, (10.10) can be proved by splitting g into $g_+ - g_-$. The proof of (10.9) can be found in Evans-Gariepy's book (p114-122, revised edition).

Exercise 10.16. Write down the detail of the proof of (10.10) using (10.9).

Let's discuss the significance of the area formula in differential geometry. The formula shows that the Hausdorff measure of an appropriate dimension coincides with the old concept of length, area, volume, etc. of submanifolds.

Example 10.21 (Length of a parametrized curve)**.** Let $\gamma(t) : [a,b] \to \mathbb{R}^n$ be a C^1 curve in $\mathbb{R}^n$, and in particular, it is Lipschitz continuous. Write $\gamma(t) = \big(\gamma_1(t), \cdots, \gamma_n(t)\big)$. Then, we have

$$D\gamma(t) = \begin{bmatrix} \gamma_1'(t) & \cdots & \gamma_n'(t) \end{bmatrix}^T \implies J_1\gamma(t) = \sqrt{\gamma_1'(t)^2 + \cdots + \gamma_n'(t)^2} = |\gamma'(t)|$$

Therefore, the LHS of (10.9) is given by:

$$\int_{[a,b]} J_1\gamma(t)\, d\mathcal{H}^1(t) = \int_{[a,b]} |\gamma'(t)|\; d\mathcal{H}^1(t) = \int_a^b |\gamma'(t)|\; dt,$$

which is the arc-length of the curve. For the RHS of (10.9), the term $\#\big([a,b] \cap \gamma^{-1}(y)\big)$ counts how many $t \in [a,b]$ such that $\gamma(t) = y$. If we suppose γ is **injective** on $[a,b]$, then $\#\big([a,b] \cap \gamma^{-1}(y)\big)$ equals 1 if $y \in \gamma[a,b]$ (i.e. on the curve), and equals 0 otherwise. In other words, we have

$$\#\big([a,b] \cap \gamma^{-1}(y)\big) \equiv \chi_{\gamma[a,b]}(y).$$

Hence the RHS of (10.9) is

$$\int_{\mathbb{R}^n} \#\big([a,b] \cap \gamma^{-1}(y)\big)\, d\mathcal{H}^1 = \mathcal{H}^1\big(\gamma[a,b]\big).$$

Therefore, it shows that the $\mathcal{H}^1$ of a C^1 parametrized curve is the arc-length.

Example 10.22 (Surface area of the graph $z = f(x,y)$)**.** Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function on an open set U in $\mathbb{R}^2$. The graph $z = f(x,y)$ is a C^1 surface in $\mathbb{R}^3$. Define $F : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ by

$$F(x,y) = \big(x, y, f(x,y)\big),$$

then we have $DF(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$ and hence

$$JF(x,y) = \sqrt{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}^2 + \begin{vmatrix} 1 & 0 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{vmatrix}^2 + \begin{vmatrix} 0 & 1 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{vmatrix}^2} = \sqrt{1 + |\nabla f|^2}.$$

Also, for any $q \in \mathbb{R}^3$, we have $\#\big(U \cap F^{-1}(q)\big) = \chi_{F(U)}(q)$ since F is injective.

Then, (10.9) shows

$$\underbrace{\int_U \sqrt{1 + |\nabla f(x,y)|^2}\, dx\, dy}_{\int_U JF(p)\, d\mathcal{H}^2(p)} = \underbrace{\int_{\mathbb{R}^3} \chi_{F(U)}\, d\mathcal{H}^2}_{\int_{\mathbb{R}^3} \#\left(U \cap F^{-1}(q)\right)\, d\mathcal{H}^2(q)} = \mathcal{H}^2\big(F(U)\big).$$

This shows $\mathcal{H}^2$ of the graph $\{z = f(x,y)\}$ coincides with the surface area.

Example 10.23 (Area of a C^1 submanifold)**.** Let $\Sigma^n \subset \mathbb{R}^{m>n}$ be a C^1 submanifold, meaning that there is an atlas of local coordinate charts $\{F_\alpha(u_\alpha^1, \cdots, u_\alpha^n) : U_\alpha \subset \mathbb{R}^n \to \mathbb{R}^{n+1}\}_{\alpha=1}^N$ such that $\Sigma^n = \bigcup_{\alpha=1}^N F_\alpha(U_\alpha)$, each F_α is C^1 on U and is a homeomorphism onto its image, and DF_α has rank n on U_α for any α. Here N could be countably infinite. In differential geometry, the surface area of Σ^n is defined as

$$\sum_{\alpha=1}^N \int_{U_\alpha} (\rho_\alpha \circ F_\alpha)\sqrt{\det[g_\alpha]}\, du_\alpha^1 \cdots du_\alpha^n.$$

Here $\{\rho_\alpha : \Sigma^n \to [0,1]\}_{\alpha=1}^N$ are the partition of unity subordinate to the above atlas, and g_α is the first fundamental form whose (i,j)-th entry is $\left\langle \frac{\partial F_\alpha}{\partial u_\alpha^i}, \frac{\partial F_\alpha}{\partial u_\alpha^j} \right\rangle$.

We will show that the surface area of Σ^n is in fact equal to its $\mathcal{H}^n$ using (10.10). For each α we write

$$F_\alpha = (f_\alpha^1, \cdots, f_\alpha^m)$$

where each f_α^j is a C^1 scalar function on U_α. Then we have:

$$(DF_\alpha)^T(DF_\alpha) = \begin{bmatrix} \frac{\partial f^1}{\partial u_1} & \cdots & \frac{\partial f^m}{\partial u_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^1}{\partial u_n} & \cdots & \frac{\partial f^m}{\partial u_n} \end{bmatrix} \begin{bmatrix} \frac{\partial f^1}{\partial u_1} & \cdots & \frac{\partial f^1}{\partial u_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial u_1} & \cdots & \frac{\partial f^m}{\partial u_n} \end{bmatrix}$$

Here for simplicity we write u_α^i as u^i, and f_α^j as f^j. The (i,j)-th entry of $(DF_\alpha)^T(DF_\alpha)$ is then given by

$$\sum_{k=1}^m \frac{\partial f^k}{\partial u_i}\frac{\partial f^k}{\partial u_j} = \left\langle \frac{\partial F_\alpha}{\partial u_\alpha^i}, \frac{\partial F_\alpha}{\partial u_\alpha^j} \right\rangle = (g_\alpha)_{ij}.$$

This shows $J_n F_\alpha = \sqrt{\det[g_\alpha]}$. For each α, we apply (10.10) with the function $\rho_\alpha \circ F_\alpha$, the map F_α over the set U_α:

$$\begin{aligned} \text{LHS} &= \int_{U_\alpha} (\rho_\alpha \circ F_\alpha) JF_\alpha \, d\mathcal{H}^n = \int_{U_\alpha} (\rho_\alpha \circ F_\alpha)\sqrt{\det[g_\alpha]}\, d\mathcal{H}^n, \\ \text{RHS} &= \int_{\mathbb{R}^m} \sum_{u_\alpha \in F_\alpha^{-1}(y)} (\rho_\alpha \circ F_\alpha)(u_\alpha)\, d\mathcal{H}^n(y) = \int_{\mathbb{R}^m} \rho_\alpha(y)\chi_{F_\alpha(U_\alpha)}(y)\, d\mathcal{H}^n(y) \\ &= \int_{F_\alpha(U_\alpha)} \rho_\alpha(y)\, d\mathcal{H}^n(y) = \int_{\Sigma^n} \rho_\alpha(y)\, d\mathcal{H}^n(y). \end{aligned}$$

The last equality follows from the fact that ρ_α has support inside $F_\alpha(U_\alpha)$.

Therefore, by summing up over α, we get

$$\sum_{\alpha=1}^N \int_{U_\alpha} (\rho_\alpha \circ F_\alpha)\sqrt{\det[g_\alpha]}\, d\mathcal{H}^n = \sum_{\alpha=1}^N \int_{\Sigma^n} \rho_\alpha(y)\, d\mathcal{H}^n(y) = \int_{\Sigma^n} 1\, d\mathcal{H}^n = \mathcal{H}^n(\Sigma^n)$$

as $\sum_\alpha \rho_\alpha \equiv 1$ on Σ^n. This again shows the $\mathcal{H}^n$ of the C^1 surface Σ^n coincides with the surface area defined analytically using the first fundamental form.

Example 10.24 (Change of variables formula)**.** When $n = m$ and $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is injective, the area formula 10.10 is simply the change-of-variables formula. In this case,

$$JF_n = \sqrt{\det(DF)^T \cdot \det DF} = |\det DF|\,,$$

and by replacing g by $(g \circ F)\chi_U$ in (10.10), we get:

$$\int_U g \circ F(x)\ |\det DF(x)|\ d\mathcal{H}^n(x) = \int_{\mathbb{R}^m} g(y)\chi_U(F^{-1}(y))\, d\mathcal{H}^n(y),$$

or equivalently,

$$\int_U g \circ F(x)\ |\det DF(x)|\ d\mathcal{L}^n(x) = \int_{F(U)} g(y)\, d\mathcal{L}^n(y),$$

which is exactly the change-of-variable formula in multivariable calculus.

10.4.3. Coarea formula.

The coarea formula is about Lipschitz function from a higher dimensional space to a lower (or equal) one.

Theorem 10.25 (Coarea formula)**.** *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz continuous where $n \geq m$, and E be a Lebesgue measurable set in $\mathbb{R}^n$. Then, we have*

$$\omega_n \int_E J_m F \, d\mathcal{H}^n = \omega_{n-m}\omega_m \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\big(E \cap F^{-1}(y)\big) \, d\mathcal{H}^m(y). \tag{10.11}$$

Furthermore, when $g : \mathbb{R}^n \to [-\infty, \infty]$ is a $\mathcal{L}^n$-measurable function which is either non-negative or integrable, we then have

$$\omega_n \int_{\mathbb{R}^n} g(x) \, J_m F(x) \, d\mathcal{H}^n(x) = \omega_{n-m}\omega_m \int_{\mathbb{R}^m} \int_{F^{-1}(y)} g(z) \, d\mathcal{H}^{n-m}(z) \, d\mathcal{H}^m(y). \tag{10.12}$$

Exercise 10.17. Prove (10.12) using (10.11).

Remark 10.26. When $n = m$, $\mathcal{H}^{n-m} = \mathcal{H}^0$ is the counting measure. Therefore, the area and coarea formulae coincide in this case.

The proof of the coarea formula can be found in Evans-Gariepy's book (p126-138, revised edition). Let's discuss its significance in geometry and analysis in this lecture note.

Example 10.27 (Fubini-Tonelli's Theorem)**.** The coarea formula is in fact a generalization of the Fubini-Tonelli's Theorem on $\mathbb{R}^n$. Let $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^m$ be the projection map

$$(x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+n}) \mapsto (x_1, \cdots, x_m).$$

Then, the Jacobian $D\pi$ equals to $[I_m | \, 0_{m,n} \,]$, and so

$$J_m \pi = \sqrt{\det\big([\, I \mid 0 \,][\, I \mid 0 \,]^T\big)} = 1.$$

For simplicity we denote $(x_1, \cdots, x_{m+n})$ by $(\mathbf{x}, \mathbf{y})$ where $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$. Then, (10.10) becomes:

$$\omega_{m+n} \int_{\mathbb{R}^{m+n}} g \, d\mathcal{H}^{m+n} = \omega_n \omega_m \int_{\mathbf{x} \in \mathbb{R}^m} \int_{(\mathbf{u},\mathbf{v}) \in \Pi(\mathbf{x})} g(\mathbf{u}, \mathbf{v}) \, d\mathcal{H}^n(\mathbf{u}, \mathbf{v}) \, d\mathcal{H}^m(\mathbf{x}),$$

where for each $\mathbf{x} \in \mathbb{R}^m$, $\Pi(\mathbf{x}) := \{(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in \mathbb{R}^n\}$. We next simplify the inner integral of the RHS. Denote that for each fixed $\mathbf{x} \in \mathbb{R}^m$, consider the map $\Phi : \mathbb{R}^n \to \Pi(\mathbf{x}) \subset \mathbb{R}^{m+n}$ defined by $\Phi(\mathbf{y}) := (\mathbf{x}, \mathbf{y})$. Applying area formula (10.10) on the integrand $g(\mathbf{x}, \cdot)$ and the map Φ, we get:

$$\int_{\mathbf{y} \in \mathbb{R}^n} g(\mathbf{x}, \mathbf{y}) \, d\mathcal{H}^n(\mathbf{y}) = \int_{\mathbb{R}^{m+n}} \sum_{\mathbf{w} \in \Phi^{-1}(\mathbf{u},\mathbf{v})} g(\mathbf{x}, \mathbf{w}) \, d\mathcal{H}^n(\mathbf{u}, \mathbf{v}).$$

Since $\mathbf{w} \in \Phi^{-1}(\mathbf{u}, \mathbf{v})$ if and only if $\mathbf{u} = \mathbf{x}$ and $\mathbf{w} = \mathbf{v}$, or equivalently, $(\mathbf{u}, \mathbf{v}) \in \Pi(\mathbf{x})$ and $\mathbf{w} = \mathbf{v}$, we can further get that:

$$\int_{\mathbb{R}^{m+n}} \sum_{\mathbf{w} \in \Phi^{-1}(\mathbf{u},\mathbf{v})} g(\mathbf{x}, \mathbf{w}) \, d\mathcal{H}^n(\mathbf{u}, \mathbf{v}) = \int_{\mathbb{R}^{m+n}} g(\mathbf{x}, \mathbf{v}) \chi_{\Pi(\mathbf{x})}(\mathbf{u}, \mathbf{v}) \, d\mathcal{H}^n(\mathbf{u}, \mathbf{v}).$$

This shows

$$\int_{(\mathbf{u},\mathbf{v}) \in \Pi(\mathbf{x})} g(\mathbf{u}, \mathbf{v}) \, d\mathcal{H}^n(\mathbf{u}, \mathbf{v}) = \int_{\mathbf{y} \in \mathbb{R}^n} g(\mathbf{x}, \mathbf{y}) \, d\mathcal{H}^n(\mathbf{y}).$$

Combining with the above results, we demonstrated that

$$\omega_{m+n} \int_{\mathbb{R}^{m+n}} g \, d\mathcal{H}^{m+n} = \omega_n \omega_m \int_{\mathbf{x} \in \mathbb{R}^m} \int_{\mathbf{y} \in \mathbb{R}^n} g(\mathbf{x}, \mathbf{y}) \, d\mathcal{H}^n(\mathbf{y}) \, d\mathcal{H}^m(\mathbf{x}),$$

which is exactly the Fubini-Tonelli's Theorem on Euclidean spaces (note that $\mathcal{H}^n = \frac{\omega_n}{2^n}\mathcal{L}^n$ on $\mathbb{R}^n$).

Remark 10.28. Although we proved the Fubini-Tonelli's Theorem for Borel functions on Euclidean spaces only, one can further extend it to Lebesgue measurable functions by approximation techniques.

Example 10.29 (Integration on level sets)**.** Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous, then $Df = (\nabla f)^T$, and so

$$J_1 f = \sqrt{\det\left((Df)(Df)^T\right)} = |\nabla f|\,.$$

For any Lebesgue measurable set $E \subset \mathbb{R}^n$, (10.11) shows

$$\omega_n \int_E |\nabla f| \; d\mathcal{H}^n = \omega_{n-1}\omega_1 \int_{\mathbb{R}} \mathcal{H}^{n-1}\big(\{x \in E : f(x) = t\}\big)\, d\mathcal{H}^1(t).$$

Exercise 10.18. Let $K \subset \mathbb{R}^n$ be a non-empty compact set. For each $x \in \mathbb{R}^n$, we let $d(x,K) = \inf\{d(x,y) : y \in K\}$. Show that for any interval $[a,b]$ with $0 < a < b$, we have:

$$\int_{[a,b]} \mathcal{H}^{n-1}\big(\{x \in \mathbb{R}^n : d(x,K) = t\}\big)\, dt = \mathcal{L}^n\big(\{x \in \mathbb{R}^n : a \leq d(x,K) \leq b\}\big).$$

Chapter 11

Radon-Nikodym's Theorem

Given a measure space $(X, \mathcal{A}, \mu)$ and a non-negative measurable funtion $f : X \to [0, \infty]$, one can naturally define a new measure ν on $(X, \mathcal{A})$ by defining $\nu(E)$, $E \in \mathcal{A}$, as

$$\nu(E) := \int_E f \, d\mu.$$

It is not hard to check that ν is a measure, since f is non-negative and so the countable additivity of ν follows easily from MCT. One natural question concerning measures of this form is: "under what condition on an arbitrarily given measure ν on $(X, \mathcal{A})$ that there exists $f : X \to [0, \infty]$ such that ν is of the form $E \mapsto \displaystyle\int_E f \, d\mu$? Clearly, one necessary condition is that whenever $\mu(E) = 0$, we must have $\nu(E) = 0$. For simplicity, we call this condition as:

Definition 11.1 (Absolute continuity of measures). Let μ, ν be two measures on $(X, \mathcal{A})$. We say ν is **absolutely continuous with respect to** μ, denoted by $\nu \ll \mu$, if whenever $\mu(E) = 0$, $E \in \mathcal{A}$, we must have $\nu(E) = 0$.

It turns out that if $(X, \mathcal{A}, \mu)$ is σ-finite, then one has $\nu \ll \mu$ if and only if ν is of the form $E \in \mathcal{A} \mapsto \displaystyle\int_E f \, d\mu$. This is known as Radon-Nikodym's Theorem. We also hope to extend the theorem by relaxing the non-negativity assumption of f and ν. Therefore, we first introduce the concept of signed measures.

11.1. Signed Measures

A signed measure is a "measure" that could take both positive and negative values, and also countable additivity.

Definition 11.2 (Signed measure)**.** Let X be a set, and $\mathcal{A}$ be an σ-algebra on X. A set function $\mu : \mathcal{A} \to (-\infty, +\infty]$ is said to be a **signed measure** on $(X, \mathcal{A})$ if

(i) $\mu(\emptyset) = 0$, and

(ii) μ is countably additive, i.e. for any disjoint countable collection $\{E_j\}_{j=1}^{\infty}$ in $\mathcal{A}$, we have

$$\mu\left(\bigsqcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j).$$

Remark 11.3. We exclude the possibility of having $\mu(E) = -\infty$ for some $E \in \mathcal{A}$, so as to avoid the issue of $\mu(A \sqcup B) = \mu(A) + \mu(B) = \infty + (-\infty)$.

Remark 11.4. The union $\bigsqcup_j E_j$ on the LHS of (ii) is invariant under any permutation of j's, but the summation in the RHS could change after permutation. Therefore, we require that whenever there is a disjoint countable collection $\{E_j\}_{j=1}^{\infty}$ in $\mathcal{A}$ such that $\mu(\cup_j E_j)$ is finite, the infinite sum needs to $\sum_{j=1}^{\infty} \mu(E_j)$ converges absolutely.

Exercise 11.1. Consider a signed measure μ on $(X, \mathcal{A})$, and sets $E, F \in \mathcal{A}$ such that $E \subset F$.

(a) Show that if $\mu(E)$ is finite, then $\mu(F - E) = \mu(F) - \mu(E)$.

(b) Show that if $\mu(F)$ is finite, then $\mu(E)$ is finite too.

Example 11.5. For each any integrable function $f : X \to [-\infty, \infty]$ on a measure space $(X, \mathcal{A}, \mu)$, the following set function ν is a signed measure:

$$\nu(E) := \int_E f \, d\mu.$$

It is clear that $\nu(\emptyset) = 0$. Since we are given that $\int_X |f| \, d\mu < \infty$, we also have $\int_X f_+ \, d\mu < \infty$ and $\int_X f_- \, d\mu < \infty$. This shows

$$\nu(E) = \int_E f_+ \, d\mu - \int_E f_- \, d\mu \in (-\infty, \infty) \;\; \forall E \in \mathcal{A}.$$

To prove that ν is countably additive, we need to show that for any countable disjoint collection $\{E_j\}_{j=1}^{\infty}$ of sets in $\mathcal{A}$, we have

$$\int_{\sqcup_j E_j} f \, d\mu = \sum_j \int_{E_j} f \, d\mu,$$

which is equivalent to

$$\int_X \sum_j f \chi_{E_j} \, d\mu = \sum_j \int_X f \chi_{E_j} \, d\mu.$$

We can use the LDCT by verifying that

$$\sum_j \int_X \left| f \chi_{E_j} \right| \, d\mu = \int_X \sum_j |f| \, \chi_{E_j} \, d\mu = \int_X |f| \, \chi_{\sqcup_j E_j} \, d\mu \leq \int_X |f| \, d\mu < \infty.$$

Note that we have used MCT in the first step.

Example 11.6. Let μ_i, $i = 1, 2$, be two (positive) measures on $(X, \mathcal{A})$. Suppose $\mu_2(X) < \infty$, then

$$\mu := \mu_1 - \mu_2$$

is a signed measure.

Given a signed measure μ on $(X, \mathcal{A})$, one can associate it with a measure $|\mu|$, which is defined as

$$|\mu|(E) := \sup \left\{ \sum_{j=1}^{\infty} |\mu(E_j)| : E = \bigsqcup_{j=1}^{\infty} E_j \text{ where } E_j \in \mathcal{A} \text{ for any } j \right\}.$$

The supremum is taken over all possible measurable countable partitions of E. It is easy to see that $|\mu|$ is monotone, i.e. when $E \subset F$, we have $|\mu|(E) \leq |\mu|(F)$. It is left as an exercise for readers.

The measure $|\mu|$ is called the **total variation** of μ. The name is inspired from the concept of total variations for functions on the real line.

Remark 11.7. $|\mu|(E)$ is **different** from $|\mu(E)|$.

Proposition 11.8. *For any signed measure μ on the space $(X, \mathcal{A})$, the total variation $|\mu|$ is a (positive) measure on $(X, \mathcal{A})$.*

Proof. To verify that $|\mu|$ is a measure, we mainly need to verify countable additivity, as it is clear that $|\mu|(\emptyset) = 0$. Take an arbitrary disjoint countable sets $\{F_k\}_{k=1}^{\infty}$ in $\mathcal{A}$, we need to show

$$|\mu| \left(\bigsqcup_{k=1}^{\infty} F_k \right) = \sum_{k=1}^{\infty} |\mu|(F_k).$$

For the $(\leq)$-part, we take an arbitrary measurable partition $\sqcup_j E_j$ of $F := \sqcup_k F_k$. Then, we consider the refinement of the two partitions

$$\begin{aligned}
\sum_{j=1}^{\infty} |\mu(E_j)| &= \sum_{j=1}^{\infty} \left| \mu \left(E_j \cap \bigcup_{k=1}^{\infty} F_k \right) \right| = \sum_{j=1}^{\infty} \left| \sum_{k=1}^{\infty} \mu(E_j \cap F_k) \right| \\
&\leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |\mu(E_j \cap F_k)| = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\mu(E_j \cap F_k)| \\
&\leq \sum_{k=1}^{\infty} |\mu|(F_k).
\end{aligned}$$

The last step follows from the fact that for each fixed k, the collection $\{E_j \cap F_k\}_{j=1}^{\infty}$ is a measurable partition of F_k. Taking supremum over all partitions $\sqcup_j E_j$ of F, we get

$$|\mu|(F) \leq \sum_{k=1}^{\infty} |\mu|(F_k).$$

For the $(\geq)$-part, we may assume $|\mu|(F_k) < \infty$ for any k, since $|\mu|(F_k) = \infty$ implies $|\mu|(F) = \infty$ by monotonicity. Now for any $\varepsilon > 0$ and any $k \in \mathbb{N}$, we take a measurable partition $F_k = \sqcup_j F_{k,j}$ such that

$$|\mu|(F_k) - \frac{\varepsilon}{2^k} < \sum_{j=1}^{\infty} |\mu(F_{k,j})| \leq |\mu|(F_k).$$

Summing over k, we get:

$$\sum_{k=1}^{\infty} |\mu|(F_k) - \varepsilon \leq \sum_{k,j=1}^{\infty} |\mu(F_{k,j})| \leq |\mu|(F).$$

The right-most inequality follows from the fact that $\{F_{k,j}\}_{k,j=1}^{\infty}$ is a countable measurable partition of F. Letting $\varepsilon \to 0^+$, we proved

$$\sum_{k=1}^{\infty} |\mu|(F_k) \leq |\mu|(F),$$

as desired. □

It is easy to see that for any $E \in \mathcal{A}$, we have

$$|\mu(E)| \leq |\mu|(E)$$

since $\{E\}$ itself is a measurable partition of E. When $|\mu|$ is a finite measure, μ is then a finite signed measure. The converse is also true, although it is not trivial.

Proposition 11.9. *Suppose μ is a finite signed measure on $(X, \mathcal{A})$, then $|\mu|$ is a finite measure.*

Proof. We first argue that if $|\mu|(E) = \infty$ for some $E \in \mathcal{A}$, then there are disjoint measurable sets $A, B \subset E$ such that $E = A \cup B$ and $|\mu(A)|, |\mu(B)| \geq 1$.

To prove this, we use the definition of $|\mu|(E)$ to get a measurable partition $E = \sqcup_j E_j$ such that

$$\sum_{j=1}^{\infty} |\mu(E_j)| > 2|\mu(E)| + 2.$$

Note that μ is a finite signed measure so $|\mu(E)| < \infty$. Then, there exists $N \in \mathbb{N}$ such that

$$\sum_{j=1}^{N} |\mu(E_j)| > 2|\mu(E)| + 2.$$

By splitting the sum according to whether $\mu(E_j) \geq 0$, we get:

$$\sum_{\{j:1\leq j\leq N, \mu(E_j)<0\}} \big(-\mu(E_j)\big) + \sum_{\{j:1\leq j\leq N, \mu(E_j)\geq 0\}} \mu(E_j) > 2|\mu(E)| + 2.$$

At least one of the summations on the LHS must be greater than $|\mu(E)| + 1$. If it is the first summation, then we can let

$$A := \bigcup_{\{j:1\leq j\leq N, \mu(E_j)<0\}} E_j \quad \text{and} \quad B := E - A.$$

Then we have

$$|\mu(A)| = -\sum_{\{j:1\leq j\leq N, \mu(E_j)<0\}} \mu(E_j) \implies |\mu(A)| > |\mu(E)| + 1 > 1,$$

and also that

$$\begin{aligned} |\mu(B)| &= |\mu(E) - \mu(A)| && \text{(from Exercise 11.1)} \\ &\geq |\mu(A)| - |\mu(E)| \\ &> |\mu(E)| + 1 - |\mu(E)| = 1. \end{aligned}$$

If it is the second summation being greater than $|\mu(E)| + 1$, then proof is similar.

Now back to the proof of the proposition. We prove by contradiction. Assume on the contrary that $|\mu|(X) = \infty$. According to the above result, there exist disjoint measurable sets $A_1, B_1 \subset X$ such that $X = A_1 \cup B_1$ and $|\mu(A_1)|, |\mu(B_1)| \geq 1$. Since $\mu(X) = \mu(A_1) + \mu(B_1)$, at least one of the $|\mu(A_1)|$ and $|\mu(B_1)|$ must be infinite. Assume (after relabelling if necessary) that $\mu(A_1) = \infty$.

Then, apply the above lemma again on A_1, then we get disjoint measurable sets $A_2, B_2 \subset A_1$ such that $A_1 = A_2 \cup B_2$, with $|\mu(A_1)| = \infty$ and $|\mu(B_2)| \geq 1$.

Inductively, one can get two sequences of measurable sets $\{A_j\}_{j=1}^{\infty}$ and $\{B_j\}_{j=1}^{\infty}$ such that $A_j = A_{j+1} \sqcup B_{j+1}$ with $|\mu(A_j)| = \infty$ and $|\mu(B_j)| \geq 1$ for all $j \geq 1$. Note that by such a construction $\{B_j\}_{j=1}^{\infty}$ must be disjoint. Let

$$B := \bigsqcup_{j=1}^{\infty} B_j,$$

then we have

$$\mu(B) = \sum_{j=1}^{\infty} \mu(B_j).$$

Since μ is a finite signed measure, we have $\mu(B) \in \mathbb{R}$ and so the above infinite series converges. However, it is not possible with $|\mu(B_j)| \geq 1$ for all $j \geq 1$. $\square$

Exercise 11.2. Let $(X, \mathcal{A}, \mu)$ be a measure space and f be an integrable function. Consider the signed measure defined by

$$\nu(E) := \int_E f \, d\mu.$$

Show that

$$|\nu|(E) = \int_E |f| \, d\mu.$$

For any signed measure μ on $(X, \mathcal{A})$, we define:

$$\mu_+ := \frac{1}{2}(|\mu| + \mu)$$
$$\mu_- := \frac{1}{2}(|\mu| - \mu)$$

which map any $E \in \mathcal{A}$ to a non-negative extended real number. It is easy to check that both μ_+ and μ_- are measures on $(X, \mathcal{A})$. Clearly, we have

$$\mu = \mu_+ - \mu_- \quad \text{and} \quad |\mu| = \mu_+ + \mu_-.$$

It is called the **Jordan decomposition** of μ. Another approach of decomposing μ into μ_+ and μ_- is using the notion of positive and negative sets – known as **Hahn decomposition**. Interested readers may read Royden's book Chapter 17 for detail. We will prove that two approaches would give the same μ_+ and μ_- using the Radon-Nikodym's Theorem.

Denote by $M(X, \mathcal{A})$ the collection of all **finite** signed measures on $(X, \mathcal{A})$. Since the finiteness assumption rules out the possibility of having $\infty - \infty$, the collection $M(X, \mathcal{A})$ is a vector space. Furthermore, by Proposition 11.9 we know that $|\mu|(X) < \infty$ for any $\mu \in M(X, \mathcal{A})$. We leave it as an exercise for readers to prove that $\|\cdot\| : M(X, \mathcal{A}) \to [0, \infty)$ defined by:

$$\|\mu\| := |\mu|(X),$$

is a complete norm, making $\big(M(X, \mathcal{A}), \|\cdot\|\big)$ a Banach space.

11.2. Lebesgue-Radon-Nikodym's Theorem

In this section we are ready to state and prove the Radon-Nikodym's Theorem. We will state and prove an even stronger result, the Lebesgue-Radon-Nikodym's Theorem, which gives a structural decomposition between any pair of an σ-finite measure μ and an σ-finite signed measure ν on the same space $(X, \mathcal{A})$. It will imply the Radon-Nikodym's Theorem as a special case.

To begin, let's define some new terminologies:

- a measure (or a signed measure) ν is said to be **supported on a set** A if $\nu(E) = \nu(E \cap A)$ for all $E \in \mathcal{A}$.
- two signed/positive measures μ and ν on $(X, \mathcal{A})$ are said to be **mutually singular** if there exist disjoint sets $A, B \in \mathcal{A}$ such that μ is supported on A, and ν is supported on B. We use $\mu \perp \nu$ to denote such a situation.

Example 11.10. Let $a, b \in X$ with $a \neq b$. Then it is easy to see that the Dirac delta measures δ_a and δ_b are mutually singular.

Theorem 11.11 (Lebesgue-Radon-Nikodym's Theorem). *Let μ be an σ-finite (positive) measure and ν be a σ-finite signed measure, both defined on $(X, \mathcal{A})$. Then, there exist unique signed measures ν_a and ν_s defined on $\mathcal{A}$ such that*

$$\nu = \nu_a + \nu_s, \quad \nu_a \ll \mu \quad \textit{and} \quad \nu_s \perp \mu.$$

Furthermore, there exists a measurable function $f : X \to [-\infty, \infty]$ such that

$$\nu_a(E) = \int_E f \, d\mu, \ \ \forall E \in \mathcal{A}.$$

Corollary 11.12 (Radon-Nikodym's Theorem). *With the same assumption given in Theorem 11.11, and assume further that $\nu \ll \mu$. Then, there exists a measurable function $f : X \to [-\infty, \infty]$ such that*

$$\nu(E) = \int_E f \, d\mu, \ \ \forall E \in \mathcal{A}.$$

Proof of Corollary. Using Theorem 11.11 (to be proved later), we write $\nu = \nu_a + \nu_s$, and we will show $\nu_s = 0$. To argue this, we let $A, B \in \mathcal{A}$ be disjoint sets such that $\nu_s(E) = \nu_s(E \cap A)$ and $\mu(E) = \mu(E \cap B)$ for any $E \in \mathcal{A}$. We argue that $\nu_s(S) = 0$ for any $S \subset A$ and $S \in \mathcal{A}$. For such S, we have $S \cap B = \emptyset$ and so $\mu(S) = \mu(S \cap B) = 0$. Since we have both $\nu \ll \mu$ and $\nu_a \ll \mu$, this shows $\nu(S) = 0$ and $\nu_a(S) = 0$. Recall that $\nu = \nu_a + \nu_s$. We conclude that $\nu_s(S) = 0$ too.

Now for any $E \in \mathcal{A}$, the set $E \cap A \subset A$ and $E \cap A \in \mathcal{A}$. Therefore, $\nu_s(E) = \nu_s(E \cap A) = 0$. Hence, $\nu = \nu_a$, and the result follows. □

To prove Theorem 11.11, we need to invoke the Riesz's Representation Theorem on Hilbert spaces.

Theorem 11.13 (Riesz's Representation Theorem). *Let l be a bounded linear functional on a Hilbert space H, i.e.*

$$\|l\| := \sup_{x \neq 0} \frac{|l(x)|}{\|x\|} < \infty.$$

Then, there exists $y \in H$ such that

$$l(x) = \langle x, y \rangle \ \ \forall x \in H.$$

The proof of Theorem 11.13 can be found in some standard functional analysis textbooks. Meanwhile, we will apply it on L^2 spaces to give a proof of Theorem 11.11. This elegant proof is due to von Neumann.

Proof of Theorem 11.11. We first assume that μ and ν are positive and finite measures. Define

$$\rho := \mu + \nu,$$

then ρ is also a finite (positive) measure on $(X, \mathcal{A})$. We consider the linear functional $l \in L^2(X, \rho)^*$ defined as:

$$l(\psi) := \int_X \psi \, d\nu \;\; \forall \psi \in L^2(X, \rho).$$

Note that $\nu \leq \rho$, so l is well-defined. It is easy to check that l is bounded:

$$\begin{aligned} |l(\psi)| &\leq \int_X |\psi| \; d\nu \leq \int_X |\psi| \; d\rho \\ &\leq \left(\int_X 1 \, d\rho \right)^{1/2} \left(\int_X |\psi|^2 \; d\rho \right)^{1/2} \\ &= \rho(X)^{1/2} \, \|\psi\|_{L^2(X,\rho)} \, . \end{aligned}$$

This shows $\|l\| \leq \rho(X)^{1/2} < \infty$. By Theorem 11.13, there exists $g \in L^2(X, \rho)$ such that

$$l(\psi) = \int_X \psi g \, d\rho \;\; \forall \psi \in L^2(X, \rho).$$

In other words, we have

$$\int_X \psi \, d\nu = \int_X \psi g \, d\rho \;\; \forall \psi \in L^2(X, \rho). \tag{11.1}$$

Next we argue that $0 \leq g \leq 1$ a.e. on $(X, \mathcal{A}, \rho)$. Let $E \in \mathcal{A}$, and put $\psi = \chi_E$ into (11.1), we have

$$\nu(E) = \int_E g \, d\rho.$$

Therefore, $\int_E g \, d\rho \geq 0$ for any $E \in \mathcal{A}$. We must have $g \geq 0$ a.e. on $(X, \mathcal{A}, \rho)$. Recall also that $\rho \geq \nu$, we know

$$\rho(E) \geq \nu(E) = \int_E g \, d\rho \implies \int_E (1 - g) \, d\rho \geq 0 \;\; \forall E \in \mathcal{A}.$$

This shows $g \leq 1$ a.e. on $(X, \mathcal{A}, \rho)$. To summarize, we have $0 \leq g \leq 1$ on $X - N$ where $N \in \mathcal{A}$ and $\rho(N) = 0$. Note that we have $\mu(N) = 0$ and $\nu(N) = 0$ too.

Recall that $\rho = \mu + \nu$. We rewrite (11.1) as

$$\int_X \psi(1 - g) \, d\nu = \int_X \psi g \, d\mu \;\; \forall \psi \in L^2(X, \rho). \tag{11.2}$$

Now we let

$$\begin{aligned} \nu_a(E) &:= \nu\big(g^{-1}[0,1) \cap E\big) \\ \nu_s(E) &:= \nu\big(g^{-1}(1) \cap E\big) \end{aligned}$$

and argue that they are the ν_a and ν_s as desired.

To show $\nu_s \perp \mu$, we first apply $\psi = \chi_{g^{-1}(1)}$ on (11.2), and get:

$$\int_{\{g=1\}} (1 - g) \, d\nu = \int_{\{g=1\}} g \, d\mu \implies 0 = \mu\big(g^{-1}(1)\big).$$

Let $A = X - g^{-1}(1)$ and $B = g^{-1}(1)$, then clearly they are disjoint sets in $\mathcal{A}$, and $X = A \sqcup B$. Hence, we have

$$\mu(A \cap E) = \mu(X \cap E) - \mu(B \cap E) = \mu(X \cap E) - 0 = \mu(E),$$

and that

$$\nu_s(B \cap E) = \nu\big(g^{-1}(1) \cap B \cap E\big) = \nu\big(g^{-1}(1) \cap E\big) = \nu_s(E).$$

Therefore, $\nu_s \perp \mu$.

To prove that ν_a takes the form of $E \mapsto \int_E f\,d\mu$ for some f, we apply $\psi = \chi_E(1 + g + \cdots + g^n)$ on (11.2):

$$\int_E (1 + g + \cdots + g^n)(1 - g)\,d\nu = \int_E g(1 + g + \cdots + g^n)\,d\mu.$$

Summing up the integrand in LHS, we have

$$\int_E (1 - g^{n+1})\,d\nu = \int_E g(1 + g + \cdots + g^n)\,d\mu.$$

Recall that $0 \leq g \leq 1$ a.e. on $(X, \mathcal{A}, \rho)$, and hence also hold a.e. on $(X, \mathcal{A}, \mu)$ and $(X, \mathcal{A}, \nu)$. Therefore, we may decompose both sides of the above integrals into $\int_{E \cap g^{-1}[0,1)} + \int_{E \cap g^{-1}(1)}$. This gives:

$$\int_{E \cap g^{-1}[0,1)} (1 - g^{n+1})\,d\nu = \int_{E \cap g^{-1}[0,1)} g(1 + g + \cdots + g^n)\,d\mu$$

where we have used the fact that $\mu(g^{-1}(1)) = 0$ for the RHS. The integrands on both sides are monotone increasing as $n \to \infty$, by MCT we get:

$$\int_{E \cap g^{-1}[0,1)} 1\,d\nu = \int_{E \cap g^{-1}[0,1)} \frac{g}{1 - g}\,d\mu.$$

Recall again that $\mu(g^{-1}(1)) = 0$, we finally proved:

$$\nu_a(E) = \nu\big(E \cap g^{-1}[0,1)\big) = \int_E \frac{g}{1 - g}\,d\mu.$$

Therefore, we can take $f := \frac{g}{1-g}$, completing the proof of the case μ and ν are positive finite measures.

To extend the result to the case μ and ν are σ-finite (and still positive), we may write $X = \bigsqcup_{j=1}^{\infty} X_j$ such that $X_j \in \mathcal{A}$, $\mu(X_j) < \infty$ and $\nu(X_j) < \infty$ for any j. Then for each j, we define the measures

$$\mu_j(E) := \mu(E \cap X_j) \qquad\qquad \nu_j(E) := \nu(E \cap X_j)$$

for any $E \in \mathcal{A}$. Clearly μ_j and ν_j are finite measures on $(X, \mathcal{A})$, so by our proven result each ν_j can be decomposed into

$$\nu_j = \nu_{j,a} + \nu_{j,s}$$

such that $\nu_{j,s} \perp \mu_j$ and $\nu_{j,a}(E) = \int_E f_j\,d\mu_j$ for a measurable function $f_j : X \to [0, \infty]$. Then, one can define

$$\nu_a := \sum_j \nu_{j,a}, \qquad \nu_s := \sum_j \nu_{j,s} \quad \text{and} \quad f := \sum_j f_j.$$

Then, clearly $\nu = \nu_a + \nu_s$. One can check $\nu_a(E) = \int_E f\,d\mu$ by MCT (note f_j are non-negative), and $\nu_s \perp \mu$ by countable additivity (left as an exercise for readers). This completes the proof for the case of σ-finite measures (where ν is still positive).

When ν is signed, we could apply the above argument separately on ν_+ and ν_-, which are positive measures. The final result then follows from linearity.

The uniqueness part follows easy from the fact that if a signed measure λ satisfies both $\lambda \ll \mu$ and $\lambda \perp \mu$, then $\lambda = 0$. $\square$

11.3. Applications of Radon-Nikodym's Theorem

The Radon-Nikodym's Theorem has many applications in probability theory. In this course, we will discuss mostly its applications in analysis. Two major applications will be discussed. One is to give an alternative definition of μ_+ and μ_- of a signed measure μ. We defined μ_+ and μ_- using the total variations $|\mu|$, instead of using positive and negative sets as in Royden's book. Another application is to prove that the dual space of $L^p(X, \mu)$, denoted by $L^p(X, \mu)^*$ which is the set of all bounded linear functional on $L^p(X, \mu)$, is isometrically isomorphic to $L^q(X, \mu)$ where $\frac{1}{p} + \frac{1}{q} = 1$.

Proposition 11.14. *Let μ be a finite signed measure on $(X, \mathcal{A})$. Then, there exists a measurable function $h : X \to \mathbb{R}$ such that $|h| = 1$ a.e. on $(X, \mathcal{A}, |\mu|)$, and*

$$\mu(E) = \int_E h \, d|\mu| \quad \forall E \in \mathcal{A}.$$

Furthermore, we have

$$\mu_+(E) = \mu\big(E \cap h^{-1}(1)\big)$$
$$\mu_-(E) = -\mu\big(E \cap h^{-1}(-1)\big)$$

for any $E \in \mathcal{A}$. Consequently, $\mu_+ \perp \mu_-$.

Proof. Note that $\mu \ll |\mu|$, and both μ and $|\mu|$ are finite. Radon-Nikodym's Theorem shows there exists a measurable function h with respect to $(X, \mathcal{A})$ such that

$$\mu(E) = \int_E h \, d|\mu| \quad \forall E \in \mathcal{A}.$$

Since $|\mu|(E) - |\mu(E)| \geq 0$, we have

$$\int_E (1 - h) \, d|\mu| = |\mu|(E) - \mu(E) \geq 0 \ \ \forall E \in \mathcal{A}.$$

This shows $h \leq 1$ a.e. on $(X, \mathcal{A}, |\mu|)$. Similarly, using the fact that $|\mu|(E) + |\mu(E)| \geq 0$, one can show $-1 \leq h$ a.e. on $(X, \mathcal{A}, |\mu|)$.

Now we have proved that $|h| \leq 1$ a.e. on $(X, \mathcal{A}, |\mu|)$. We claim further that $|h| = 1$ a.e. Consider the level region $E_j := |h|^{-1}\left([0, 1 - \frac{1}{j})\right)$ where $j \in \mathbb{N}$. For any measurable countable partition $E_j = \sqcup_k E_{j,k}$, we have

$$\begin{aligned}
\sum_k |\mu(E_{j,k})| &= \sum_k \left| \int_{E_{j,k}} h \, d|\mu| \right| \leq \sum_k \int_{E_{j,k}} |h| \, d|\mu| \\
&\leq \sum_k \int_{E_{j,k}} \left(1 - \frac{1}{j}\right) d|\mu| = \left(1 - \frac{1}{j}\right) \sum_k |\mu|(E_{j,k}) \\
&= \left(1 - \frac{1}{j}\right) |\mu|(E_j)
\end{aligned}$$

Taking supremum over all measurable countable partitions of E_j, we conclude that

$$|\mu|(E_j) \leq \left(1 - \frac{1}{j}\right) |\mu|(E_j).$$

We must have $|\mu|(E_j) = 0$. Since it is true for any j, we conclude that:

$$|\mu|\left(|h|^{-1}[0,1)\right) = |\mu|\left(\bigcup_j E_j\right) = 0.$$

Combining with the fact that $|h| \leq 1$ a.e. with respect to $|\mu|$, we conclude that $|h| = 1$ a.e. with respect to $|\mu|$.

To prove the results about μ_+ and μ_-, we consider that

$$\begin{aligned}
\mu_+(E) &= \frac{1}{2}\big(\mu(E) + |\mu|(E)\big) \\
&= \frac{1}{2}\int_E (h+1)\, d\,|\mu| \\
&= \frac{1}{2}\left(\int_{E\cap h^{-1}(1)} (h+1)\, d\,|\mu| + \int_{E\cap h^{-1}(-1)} (h+1)\, d\,|\mu|\right) \\
&= \frac{1}{2}\left(2\,|\mu|\,(E\cap h^{-1}(1)) + 0\right) \\
&= |\mu|\,(E\cap h^{-1}(1)).
\end{aligned}$$

The proof about $\mu_-(E)$ is similar. □

Next we use the Radon-Nikodym's Theorem to argue that the dual space of L^p is isometrically isometric to L^q, where $\frac{1}{p} + \frac{1}{q} = 1$. The isomorphism is given by the "pairing" map. Precisely, for any $g \in L^q(X,\mu)$, we define $I_g : L^p(X,\mu) \to \mathbb{R}$ by

$$I_g(f) := \int_X fg\, d\mu \quad \forall f \in L^p(X,\mu).$$

By Cauchy-Schwarz's inequality, $I_g(f)$ is finite since

$$|I_g(f)| \leq \|fg\|_{L^1(X,\mu)} \leq \|f\|_{L^p(X,\mu)} \|g\|_{L^q(X,\mu)}.$$

Clearly I_g is linear. The above also shows that I_g is a bounded linear functional, hence $I_g \in L^p(X,\mu)^*$.

Define the linear map $\Phi : L^q(X,\mu) \to L^p(X,\mu)^*$ given by $\Phi(g) = I_g$, then Φ is injective, and we leave it as an exercise for readers to check that Φ is norm-perserving. In order to prove that L^q is isometrically isomorphic to $(L^p)^*$, we need to show Φ is surjective.

Proposition 11.15. *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. Let $1 \leq p < \infty$ and suppose that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for any bounded linear functional l on $L^p(X,\mu)$, there exists $g \in L^q(X,\mu)$ such that $l = I_g$.*

Proof. We prove only the case when $\mu(X) < \infty$, and leave the σ-finite case for readers to fill in. First we define a set function (which will be shown to be a measure) $\nu : \mathcal{A} \to [0,\infty)$ by

$$\nu(E) := l(\chi_E) \quad \forall E \in \mathcal{A}.$$

Clearly, $\nu(\emptyset) = 0$. To prove ν satisfies countable additivity, we recall that l is bounded, hence is continuous.

Given any disjoint collection of sets $\{E_j\}_{j=1}^{\infty}$ in $\mathcal{A}$, we denote $A_\infty := \bigcup_{j=1}^{\infty} E_j$ and $A_n := \bigcup_{j=1}^{n} E_j$. We first note that $\chi_{A_n} \to \chi_{A_\infty}$ in $L^p(X,\mu)$-norm. To see this, we consider

$$\|\chi_{A_n} - \chi_{A_\infty}\|^p_{L^p(X,\mu)} = \int_X |\chi_{A_n} - \chi_{A_\infty}|^p \, d\mu = \int_X \left(\sum_{j=n+1}^{\infty} \chi_{A_j} \right)^p d\mu.$$

Note that $\left(\sum_{j=n+1}^{\infty} \chi_{A_j}\right)^p = \left(\chi_{\sqcup_{j=n+1}^{\infty} A_j}\right)^p \leq 1$ and $\mu(X) < \infty$, by LDCT/BCT, we conclude that

$$\lim_{n\to\infty} \int_X \left(\sum_{j=n+1}^{\infty} \chi_{A_j} \right)^p d\mu = \int_X \lim_{n\to\infty} \left(\sum_{j=n+1}^{\infty} \chi_{A_j} \right)^p d\mu = 0,$$

as the infinite series $\sum_j \chi_{A_j}$ converges.

Now that $\chi_{A_n} \to \chi_{A_\infty}$ in L^p-norm, and l is continuous, we have

$$\nu(A_\infty) = l(\chi_{A_\infty}) = \lim_{n\to\infty} l(\chi_{A_n}) = \lim_{n\to\infty} \sum_{j=1}^{n} l(\chi_{E_j}) = \sum_{j=1}^{\infty} \nu(E_j).$$

This proves ν is countably additive, so it is a measure on $(X, \mathcal{A})$.

It is easy to observe that $\nu \ll \mu$, since when $\mu(E) = 0$, we have $\nu(E) = l(0) = 0$. Using the Radon-Nikodym's Theorem, there exists a measurable function $g : X \to [0, \infty]$ such that

$$\nu(E) = \int_E g \, d\mu \;\; \forall E \in \mathcal{A}.$$

In other words, we have

$$l(\chi_E) = \int_X \chi_E g \, d\mu = I_g(\chi_E) \;\; \forall E \in \mathcal{A}.$$

We will later show that g is in $L^q(X, \mu)$. Let's assume so in the next paragraph.

By linearity, it then implies $l(\varphi) = I_g(\varphi)$ for any simple functions φ. Next we extend this result to any L^p functions. Take $f \in L^p(X, \mu)$ and decompose it into $f = f_+ - f_-$. Then both f_+ and f_- are non-negative L^p functions. One can use an increasing sequence $\{\varphi_j\}$ of simple functions to approximate each of them. Say $\varphi_j \to f_+$, then by LDCT we have

$$\|f_+ - \varphi_j\|_{L^p(X,\mu)} \to 0.$$

By the continuity of l, we know

$$\lim_{j\to\infty} l(\varphi_j) = l(f_+).$$

On the other hand, we have

$$\lim_{j\to\infty} l(\varphi_j) = \lim_{j\to\infty} I_g(\varphi_j) = \lim_{j\to\infty} \int_X g\varphi_j \, d\mu.$$

Note that φ_j increases, it is bounded above by f_+. Hence $|g\varphi_j| \leq |gf_+|$ a.e. on X, and $|gf_+|$ is integrable by Hölder inequality. Thus, LDCT implies

$$\lim_{j\to\infty} \int_X g\varphi_j \, d\mu = \int_X gf_+ \, d\mu = I_g(f_+).$$

Combining the above results, we conclude that

$$l(f_+) = I_g(f_+),$$

and by a similar argument on f_-, we have $l(f_-) = I_g(f_-)$. By linearity, we conclude that $l = I_g$ on $L^p(X, \mu)$.

Next we need to check that $g \in L^q(X, \mu)$. When $p > 1$, we construct a sequence $\{g_j\}$ defined truncations of g:

$$g_j(x) := \begin{cases} |g(x)|^{q-1} \operatorname{sgn} g(x) & \text{if } |g(x)|^{q-1} \leq j \\ j \operatorname{sgn} g(x) & \text{if } |g(x)|^{q-1} > j \end{cases}.$$

Then, $g_j(x)$ and $g(x)$ always have the same sign, so

$$g_j g = |g_j g| \geq |g_j| \, |g_j|^{\frac{1}{q-1}} = |g_j|^p .$$

Hence,

$$\int_X |g_j|^p \, d\mu = \int_X g_j g \, d\mu = I_g(g_j) = l(g_j) \leq \|l\| \, \|g_j\|_{L^p(X,\mu)} ,$$

which implies:

$$\|g_j\|_{L^p(X,\mu)}^{p-1} \leq \|l\| .$$

Since $|g_j| \to |g|^{q-1}$ pointwise, one can use LDCT to argue that

$$\|l\| \geq \lim_{j\to\infty} \|g_j\|_{L^p} = \left(\int_X \lim_{j\to\infty} |g_j|^p \, d\mu \right)^{1/p} = \left(\int_X |g|^{p/(q-1)} \, d\mu \right)^{1/p} = \|g\|_{L^q(X,\mu)}^{q/p} .$$

This shows g is L^q, completing the proof of the case $\mu(X) < \infty$ and $p > 1$.

We leave it as an exercise to readers to prove the remaining cases, i.e. $p = 1$ and $q = \infty$, and when $\mu(X) = \infty$ but X is σ-finite. $\square$