# Chapter 2 Holomorphic functions

In this course, our main object of study is **functions of a complex variable**.

## 1. Functions of a complex variable

**Definition 2.1** A ***function of a complex variable*** (or a ***complex function***) is a function $f: U \to \mathbb{C}$ whose domain $U$ is a subset of $\mathbb{C}$.

Given a function of a complex variable, say

$$w = f(z),$$

if we write the input complex number as $z = x + iy$ and write the output complex number as $w = u + iv$, then we would get

$$f(x + iy) = u(x, y) + iv(x, y).$$

The functions $u$ and $v$ (of two real variables) are called the **real part** and the **imaginary part** of the function $f$.

**Example 2.2** Let $f: \mathbb{C} \to \mathbb{C}$ be the function $f(z) = z^2$. Then

$$\begin{aligned} f(x + iy) = (x + iy)^2 &= x^2 + 2x(iy) + (iy)^2 \\ &= (x^2 - y^2) + i(2xy), \end{aligned}$$

so its real and imaginary parts are

$$u(x, y) = x^2 - y^2 \qquad \text{and} \qquad v(x, y) = 2xy$$

respectively.

**Example 2.3** Let $f: \mathbb{C} \to \mathbb{C}$ be the function $f(z) = e^z$. Then

$$f(x + iy) = e^x(\cos y + i \sin y) = e^x \cos y + i(e^x \sin y),$$

so its real and imaginary parts are

$$u(x, y) = e^x \cos y \qquad \text{and} \qquad v(x, y) = e^x \sin y$$

respectively.

**Example 2.4** Let $f: \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be the function $f(z) = \frac{1}{z}$. Then

$$f(x + iy) = \frac{1}{x + iy} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2},$$

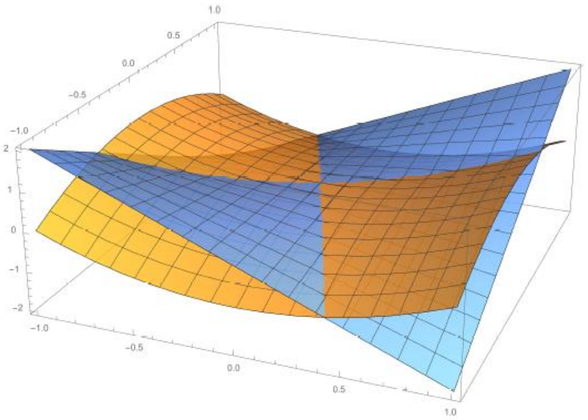
so its real and imaginary parts are

$$u(x, y) = \frac{x}{x^2 + y^2} \qquad \text{and} \qquad v(x, y) = \frac{-y}{x^2 + y^2}$$

respectively. Note that $u$ and $v$ are undefined at $(0, 0)$.

Next we study some methods to **visually present** a function of a complex variable. Note that it is not feasible to visualize a function of a complex variable using its own graph, because it would be a geometric object in four (real) dimensions. Instead, it would be easy if we treat it as **a pair of functions in two real variables**, and look at their **graphs** or **level sets** separately as in MATH2023.

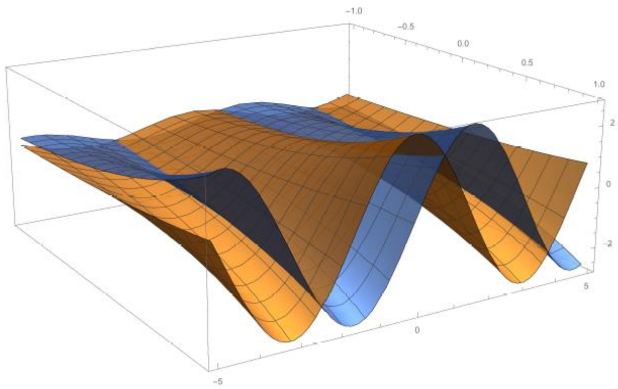
**Example 2.5** The following shows the graphs of the real and imaginary parts of the functions in Examples 2.2 – 2.4. The orange graph corresponds to the real part and the blue graph corresponds to the imaginary part. (Image credit: Prof. Frederick Fong)



$f(z) = z^2$

$u(x, y) = x^2 - y^2$

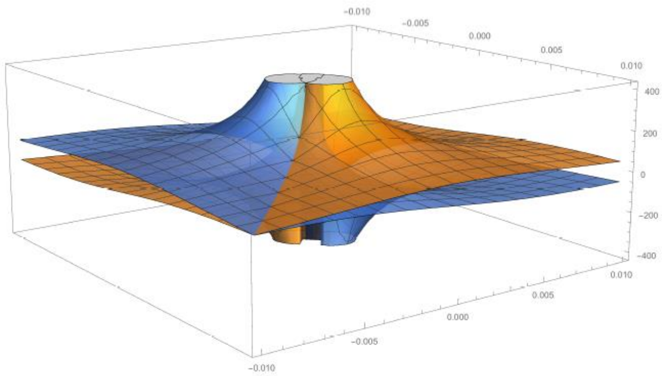
$v(x, y) = 2xy$



$f(z) = e^z$

$u(x, y) = e^x \cos y$

$v(x, y) = e^x \sin y$

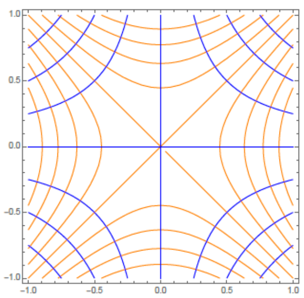


$$f(z) = \frac{1}{z}$$

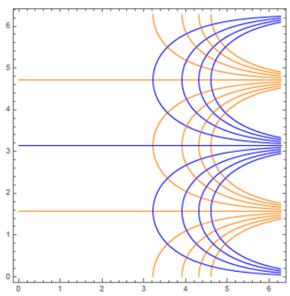
$$u(x, y) = \frac{x}{x^2 + y^2}$$

$$v(x, y) = -\frac{y}{x^2 + y^2}$$

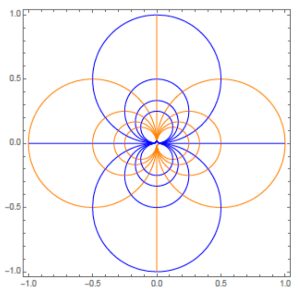
**Example 2.6** The following shows the level curves of the real and imaginary parts of the functions in Examples 2.2 – 2.4. The orange curves correspond to the real part and the blue curves correspond to the imaginary part. (Image credit: Prof. Frederick Fong)



$f(z) = z^2$



$f(z) = e^z$



$f(z) = \frac{1}{z}$

One more way to visualize a function of a complex variable is to study its **mapping properties**, i.e. how the **direct image** or **inverse image** of various simple subsets of $\mathbb{C}$ via the function look like. Recall the following set-theoretic definition from MATH2033/2043.

**Definition** Let $U$ and $V$ be sets and $f: U \to V$ be a function.

(i) For a set $A \subseteq U$, the ***direct image of $A$ via $f$*** is the set
$$f(A) := \{w \in V: w = f(z) \text{ for some } z \in A\}.$$

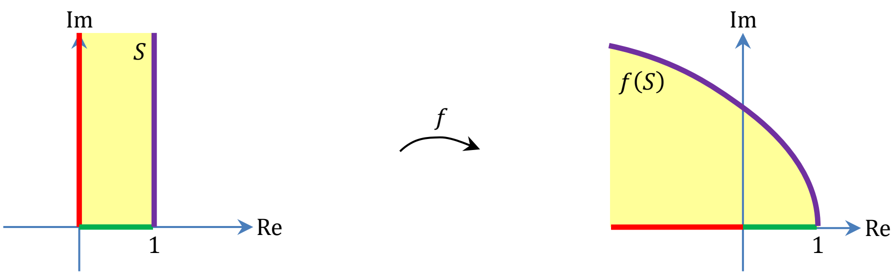
(ii) For a set $B \subseteq V$, the ***inverse image of $B$ via $f$*** is the set
$$f^{-1}(B) := \{z \in U: f(z) \in B\}.$$

**Example 2.7** Let $f: \mathbb{C} \to \mathbb{C}$ be the function $f(z) = z^2$. Find and sketch the direct image of
$$S = \{z \in \mathbb{C}: 0 \leq \operatorname{Re} z \leq 1 \text{ and } \operatorname{Im} z \geq 0\}$$
via $f$.

*Solution*: For each $z = x + iy$, if we write $w = u + iv = f(z)$, then
$$w = f(x+iy) = (x+iy)^2 = (x^2 - y^2) + i(2xy),$$
so $w$ has real and imaginary parts given by
$$u = x^2 - y^2 \qquad \text{and} \qquad v = 2xy.$$
Now we claim that
$$f(S) = \left\{ w \in \mathbb{C}: \operatorname{Re} w \leq 1 - \frac{(\operatorname{Im} w)^2}{4} \text{ and } \operatorname{Im} w \geq 0 \right\}.$$
This can be justified as follows:

- If $z \in S$, i.e. $0 \leq x \leq 1$ and $y \geq 0$, then $(1 - x^2)(1 + y^2) \geq 0$. So
$$u = x^2 - y^2 \leq 1 - x^2y^2 = 1 - v^2/4 \qquad \text{and} \qquad v = 2xy \geq 0.$$
- Conversely, if $u \leq 1 - \frac{v^2}{4}$ and $v \geq 0$, then we choose $x = \sqrt{\frac{\sqrt{u^2+v^2}+u}{2}}$ and $y = \sqrt{\frac{\sqrt{u^2+v^2}-u}{2}}$.

  Then $0 \leq x \leq 1$ and $y \geq 0$, so $z = x + iy \in S$ and $f(z) = w$.




We observe that

- The portion $\{x = 0 \text{ and } y \geq 0\}$ of the boundary $\partial S$ is mapped to $\{u \leq 0 \text{ and } v = 0\}$ via $f$.
- The portion $\{x = 1 \text{ and } y \geq 0\}$ of $\partial S$ is mapped to $\left\{u = 1 - \frac{v^2}{4} \text{ and } v \geq 0\right\}$ via $f$.
- The portion $\{0 \leq x \leq 1 \text{ and } y = 0\}$ of $\partial S$ is mapped to $\{0 \leq u \leq 1 \text{ and } v = 0\}$ via $f$.

**Example 2.8** Let $f: \mathbb{C} \to \mathbb{C}$ be the function $f(z) = e^z$. Find and sketch the inverse image of

$$S = \{w \in \mathbb{C}: |w| < 1 \text{ and } \operatorname{Re} w > 0\}$$

via $f$.

*Solution*:

For each $w = re^{i\theta} \in S$, we have $0 \le r < 1$ and $r\cos\theta > 0$. If there exists $z = x + iy$ such that $w = f(z)$, then we have

$$w = f(x + iy) = e^{x+iy} = e^x e^{iy},$$

so

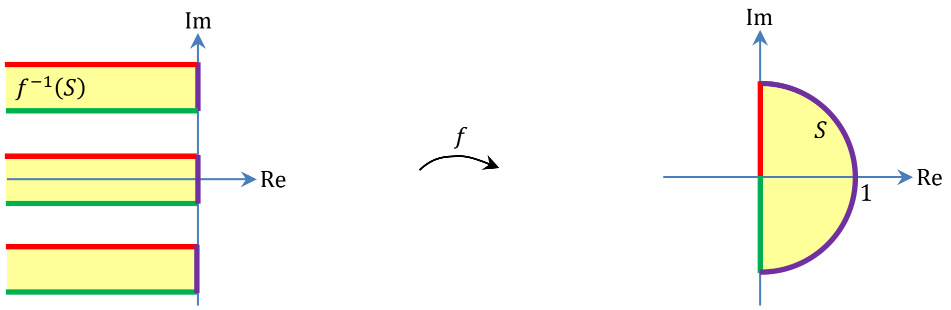
$$e^x = r < 1 \qquad \text{and} \qquad e^x \cos y = r\cos\theta > 0.$$

This happens if and only if

$$x < 0 \qquad \text{and} \qquad \cos y > 0.$$

Therefore we conclude that

$$f^{-1}(S) = \left\{z \in \mathbb{C}: \operatorname{Re} z < 0 \text{ and } \left(2n - \frac{1}{2}\right)\pi < \operatorname{Im} z < \left(2n + \frac{1}{2}\right)\pi \text{ for some } n \in \mathbb{Z}\right\}.$$




## 2. Limits and continuity

**Definition 2.9 (Limit)** Let $U \subseteq \mathbb{C}$ be an open set, let $a \in \overline{U}$, let $f: U \to \mathbb{C}$ be a function and let $L$ be a complex number. If for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(z) - L| < \varepsilon \qquad \text{whenever } z \in U \text{ and } 0 < |z - a| < \delta,$$

then we say that $L$ is a ***limit*** of $f(z)$ as $z$ tends to $a$.

**Lemma 2.10** *The limit of a function is unique. If $L$ is the limit of $f(z)$ as $z$ tends to $a$, then symbolically we write*

$$\lim_{z \to a} f(z) = L.$$

**Example 2.11** Let $a \in \mathbb{C}$. Show that

$$\lim_{z \to a} z^2 = a^2.$$

*Proof*:

Given any $\varepsilon > 0$, we choose $\delta = \min\left\{1, \frac{\varepsilon}{1+2|a|}\right\} > 0$. Then whenever $0 < |z-a| < \delta$, we have

$$\begin{aligned}|z^2 - a^2| &= |z+a||z-a| \\ &= |z-a+2a||z-a| \\ &\le (|z-a| + 2|a|)|z-a| \\ &< (1+2|a|)\frac{\varepsilon}{1+2|a|} = \varepsilon.\end{aligned}$$

■

The same set of **arithmetic operations** hold for limits of functions in this new context.

**Theorem 2.12** *Let $U \subseteq \mathbb{C}$ be an open set, let $a \in \overline{U}$ and let $f, g: U \to \mathbb{C}$ be functions. If $\lim_{z \to a} f(z) = L$ and $\lim_{z \to a} g(z) = M$ both exist as complex numbers, then*

- $\lim_{z \to a}\big(f(z) \pm g(z)\big) = L \pm M$,
- $\lim_{z \to a} f(z)g(z) = LM$,
- $\lim_{z \to a} \frac{f(z)}{g(z)} = \frac{L}{M}$ *if* $M \neq 0$.

**Example 2.13** Evaluate the limit

$$\lim_{z \to i} \frac{iz^3 - 1}{z^2 + 1}.$$

*Solution*:

$$\begin{aligned}\lim_{z \to i} \frac{iz^3 - 1}{z^2 + 1} &= \lim_{z \to i} \frac{(z-i)(iz^2 - z - i)}{(z-i)(z+i)} \\ &= \lim_{z \to i} \frac{iz^2 - z - i}{z + i} \\ &= \frac{i(i)^2 - (i) - i}{(i) + i} = -\frac{3}{2}.\end{aligned}$$

**Remark 2.14 (Two-path test)** Like in MATH2011/2023, if a function has **different limits** along **two different paths of approach** to the same point, then the limit **does not exist**.

**Example 2.15** Show that the limit

$$\lim_{z \to 0} \frac{z}{|z|}$$

does not exist.

*Proof*:

Write $z = x + iy$. Then $\lim_{z \to 0} \frac{z}{|z|} = \lim_{(x,y) \to (0,0)} \frac{x+iy}{\sqrt{x^2+y^2}}$. Along $y = 0$ and $x \to 0^+$, we have

$$\lim_{\substack{(x,y)\to(0,0) \\ y=0, x\to 0^+}} \frac{x+iy}{\sqrt{x^2+y^2}} = \lim_{x \to 0^+} \frac{x}{|x|} = \lim_{x \to 0^+} 1 = 1,$$

but along $y = 0$ and $x \to 0^-$, we have

$$\lim_{\substack{(x,y)\to(0,0) \\ y=0, x\to 0^-}} \frac{x+iy}{\sqrt{x^2+y^2}} = \lim_{x \to 0^-} \frac{x}{|x|} = \lim_{x \to 0^+} -1 = -1,$$

so $\lim_{z \to 0} \frac{z}{|z|}$ does not exist. ■

*Alternative proof*: [The two-path test can also be rephrased using the "**$\varepsilon$-$\delta$ language**".]

Suppose on the contrary that $\lim_{z \to 0} \frac{z}{|z|} = L$ for some $L \in \mathbb{C}$. Then there exists $\delta > 0$ such that

$$\left| \frac{z}{|z|} - L \right| < 1$$

Choose $\varepsilon = 1$ in the definition.

whenever $0 < |z| < \delta$.

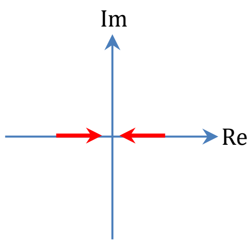
Now $\frac{\delta}{2}$ and $-\frac{\delta}{2}$ are two complex numbers satisfying $0 < \left|\frac{\delta}{2}\right| < \delta$ and $0 < \left|-\frac{\delta}{2}\right| < \delta$, so we have

$$\left| \frac{\delta/2}{|\delta/2|} - L \right| < 1 \quad \text{and} \quad \left| \frac{-\delta/2}{|-\delta/2|} - L \right| < 1,$$

i.e. $|L - 1| < 1$ and $|L + 1| < 1$. By Triangle Inequality we obtain

$$\begin{aligned} 2 &= |(L+1) - (L-1)| \\ &\leq |L+1| + |L-1| \\ &< 1 + 1 \\ &= 2, \end{aligned}$$

a contradiction. Therefore $\lim_{z \to 0} \frac{z}{|z|}$ does not exist. ■



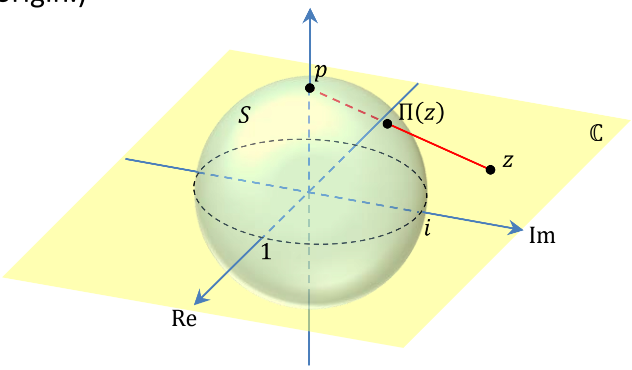

Considering these two paths of approaching $0$ is the same as considering the numbers $\delta/2$ and $-\delta/2$ in the proof.

Like in the case for functions of a real variable, one can also talk about **infinite limits** and **limits at infinity** of a function of a complex variable. The only difference is that there are "two infinities" $+\infty$ and $-\infty$ in the real case, but "only one infinity" $\infty$ in the complex case.

**Remark 2.16 (Stereographic projection)** Let $\mathbb{C}$ be identified with the $xy$-plane in $\mathbb{R}^3$. Let $S$ be the unit sphere in $\mathbb{R}^3$ centered at the origin, whose "**north pole**" $(0,0,1)$ is denoted as $p$. Now consider the function $\Pi: \mathbb{C} \to S \setminus \{p\}$ defined by

$$\Pi(z) = \Pi(x+iy) = \text{the point on } S \text{ which is collinear with both } p \text{ and } (x,y,0) \text{ in } \mathbb{R}^3.$$

It is easy to see that $\Pi$ is a continuous bijection between $\mathbb{C}$ and $S \setminus \{p\}$. (To understand the continuity of $\Pi$, one may measure the distance between two points on $S$ by the angle subtended between them from the origin.)




Now if we attach to $\mathbb{C}$ one more element called ***infinity*** $\infty$, and make this new element $\infty$ correspond to the north pole $p$, then the above function $\Pi$ should extend to be a continuous bijection between $\mathbb{C} \cup \{\infty\}$ and $S$. In this construction, $\mathbb{C} \cup \{\infty\}$ is called the ***extended complex plane***, $S$ is called the ***Riemann sphere***, and the above continuous bijection $\Pi: \mathbb{C} \cup \{\infty\} \to S$ is called the ***stereographic projection***.

Note that in the above construction, the points on $S$ that are **near** $\boldsymbol{p}$ correspond to the points on the complex plane that are in the **exterior of a large disk**. This motivates the following definitions.

**Definition 2.17 (Infinite limit)** Let $U \subseteq \mathbb{C}$ be an open set, let $a \in \overline{U}$ and let $f: U \to \mathbb{C}$ be a function. If for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(z)| > \varepsilon \qquad \text{whenever } z \in U \text{ and } 0 < |z-a| < \delta,$$

then we say that $f(z)$ has ***infinite limit*** as $z$ tends to $a$, or in symbols we write

$$\lim_{z \to a} f(z) = \infty.$$

**Theorem 2.18** *Let $U \subseteq \mathbb{C}$ be an open set, let $a \in \overline{U}$ and let $f: U \to \mathbb{C}$ be a function. Then*

$$\lim_{z \to a} f(z) = \infty \qquad \textit{if and only if} \qquad \lim_{z \to a} \frac{1}{f(z)} = 0.$$

**Definition 2.19 (Limit at infinity)** Let $U \subseteq \mathbb{C}$ be an unbounded open set, let $f: U \to \mathbb{C}$ be a function and let $L$ be a complex number. If for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(z) - L| < \varepsilon \qquad \text{whenever } z \in U \text{ and } |z| > \delta,$$

then we say that $L$ is the ***limit*** of $f(z)$ ***as*** $z$ ***tends to*** $\infty$, or in symbols we write

$$\lim_{z\to\infty} f(z) = L.$$

If for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(z)| > \varepsilon \qquad \text{whenever } z \in U \text{ and } |z| > \delta,$$

then we say that $f(z)$ has ***infinite limit as*** $z$ ***tends to*** $\infty$, or in symbols we write

$$\lim_{z\to\infty} f(z) = \infty.$$

**Theorem 2.20** *Let $U \subseteq \mathbb{C}$ be an unbounded open set, let $f: U \to \mathbb{C}$ be a function and let $L$ be a complex number. Then*

$$\lim_{z\to\infty} f(z) = L \qquad \textit{if and only if} \qquad \lim_{z\to 0} f\left(\frac{1}{z}\right) = L,$$

*and*

$$\lim_{z\to\infty} f(z) = \infty \qquad \textit{if and only if} \qquad \lim_{z\to 0} \frac{1}{f(1/z)} = 0.$$

**Example 2.21** Evaluate the limits

$$\lim_{z\to -1} \frac{iz+3}{z+1}, \qquad \lim_{z\to\infty} \frac{2z+i}{z+1} \qquad \text{and} \qquad \lim_{z\to\infty} \frac{2z^3-1}{z^2+1}.$$

*Solution*:

Since

$$\lim_{z\to -1} \frac{z+1}{iz+3} = \frac{(-1)+1}{i(-1)+3} = 0,$$

we have $\lim\limits_{z\to -1} \frac{iz+3}{z+1} = \infty$. Since

$$\lim_{z\to 0} \frac{2(1/z)+i}{(1/z)+1} = \lim_{z\to 0} \frac{2+iz}{1+z} = \lim_{z\to 0} \frac{2+i(0)}{1+0} = 2,$$

we have $\lim\limits_{z\to\infty} \frac{2z+i}{z+1} = 2$. Finally since

$$\lim_{z\to 0} \frac{(1/z)^2+1}{2(1/z)^3-1} = \lim_{z\to 0} \frac{z+z^3}{2-z^3} = \frac{0+0^3}{2-0^3} = 0,$$

we have $\lim\limits_{z\to\infty} \frac{2z^3-1}{z^2+1} = \infty$.

**Definition 2.22 (Continuity)** Let $U \subseteq \mathbb{C}$ be an open set and let $f: \overline{U} \to \mathbb{C}$ be a function.

- Given $a \in \overline{U}$, we say that $f$ is ***continuous at*** $\boldsymbol{a}$ if
$$\lim_{z \to a} f(z) = f(a),$$
i.e. for each $\varepsilon > 0$, there exists $\delta > 0$ such that
$$|f(z) - f(a)| < \varepsilon \qquad \text{whenever } z \in \overline{U} \text{ and } |z - a| < \delta.$$
- For a subset $V \subseteq \overline{U}$, we say that $f$ is ***continuous on*** $\boldsymbol{V}$ if it is continuous at every point in $V$.
- We simply say that $f$ is ***continuous*** if it is continuous on its whole domain.

**Corollary 2.23** *Let $U \subseteq \mathbb{C}$ be an open set and let $f, g: \overline{U} \to \mathbb{C}$ be continuous functions. Then the functions $f + g$, $f - g$ and $fg$ are all continuous, and the function $\frac{f}{g}$ is continuous at every number except at those $a \in \overline{U}$ where $g(a) = 0$.*

**Example 2.24** Show that the functions $\mathrm{Re}: \mathbb{C} \to \mathbb{R}$, $\mathrm{Im}: \mathbb{C} \to \mathbb{R}$, $\bar{\cdot}: \mathbb{C} \to \mathbb{C}$ and $|\cdot|: \mathbb{C} \to [0, +\infty)$ are all continuous.

*Proof*:
For any $a \in \mathbb{C}$ and any $\varepsilon > 0$, we choose $\delta = \varepsilon > 0$. Then whenever $|z - a| < \delta$, we have
$$|\mathrm{Re}\, z - \mathrm{Re}\, a| = |\mathrm{Re}(z - a)| \leq |z - a| < \varepsilon,$$
$$|\mathrm{Im}\, z - \mathrm{Im}\, a| = |\mathrm{Im}(z - a)| \leq |z - a| < \varepsilon,$$
$$|\overline{z} - \overline{a}| = |\overline{z - a}| = |z - a| < \varepsilon \quad \text{and}$$
$$\big||z| - |a|\big| = \big||z| - |-a|\big| \leq |z - a| < \varepsilon.$$
So the functions $\mathrm{Re}$, $\mathrm{Im}$, $\bar{\cdot}$ and $|\cdot|$ are all continuous on $\mathbb{C}$. ■

**Example 2.25** Polynomials in a complex variable are continuous. The exponential function is also continuous (cf. Q20, Problem Set 2).

**Continuous functions** have important interactions with various kinds of sets we have studied in **point-set topology**. The following is an outline of the upcoming **topological results about continuous functions**:

- The inverse image of an **open** set via a continuous function is **open**. (Lemma 2.26)
- The inverse image of a **closed** set via a continuous function is **closed**. (Corollary 2.28)
- The direct image of a **compact** set via a continuous function is **compact**. (Theorem 2.30)
- The direct image of a **connected** set via a continuous function is **connected**. (Theorem 2.35)

**Lemma 2.26** *A function $f: \mathbb{C} \to \mathbb{C}$ is continuous if and only if for every open set $U$, $f^{-1}(U)$ is open.*

*Proof.*

(⇒) Let $f: \mathbb{C} \to \mathbb{C}$ be a continuous function and $U$ be an open set. Then for each $z \in f^{-1}(U)$, we want to show that $z$ is an interior point of $f^{-1}(U)$. Since $f(z) \in U$, there exists $\varepsilon > 0$ such that $D(f(z); \varepsilon) \subseteq U$. By the continuity of $f$, there exists $\delta > 0$ such that
$$f\big(D(z;\delta)\big) \subseteq D(f(z);\varepsilon).$$
Therefore $D(z;\delta) \subseteq f^{-1}\big(D(f(z);\varepsilon)\big) \subseteq f^{-1}(U)$.

(⇐) For every $a \in \mathbb{C}$ and every $\varepsilon > 0$, since $D(f(a);\varepsilon)$ is open, $f^{-1}\big(D(f(a);\varepsilon)\big)$ is also open by assumption. Now since $a \in f^{-1}\big(D(f(a);\varepsilon)\big)$, so $a$ must be an interior point of it, i.e. there exists $\delta > 0$ such that $D(a;\delta) \subseteq f^{-1}\big(D(f(a);\varepsilon)\big)$. This is equivalent to saying that
$$|f(z) - f(a)| < \varepsilon \qquad \text{whenever } |z - a| < \delta,$$
i.e. $f$ is continuous. ■

**Corollary 2.27** *Let $f: \mathbb{C} \to \mathbb{C}$ be a continuous function and let $a \in \mathbb{C}$. If $f(a) \neq 0$, then there exists $r > 0$ such that $f(z) \neq 0$ for every $z \in D(a;r)$.*

**Corollary 2.28** *A function $f: \mathbb{C} \to \mathbb{C}$ is continuous if and only if for every closed set $U$, $f^{-1}(U)$ is closed.*

**Corollary 2.29** *The composition of continuous functions is continuous.*

**Theorem 2.30 (Extreme value theorem)** *Let $K$ be a compact subset of $\mathbb{C}$ and $f: K \to \mathbb{C}$ be a continuous function. Then $f(K)$ is compact.*

*Proof.* Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of $f(K)$. Then $\{f^{-1}(U_\alpha)\}_{\alpha \in I}$ is an open cover of $K$ by Lemma 2.26. Since $K$ is compact, this open cover must have a finite subcover $\left\{f^{-1}\big(U_{\alpha_k}\big)\right\}_{k=1}^{n}$.

It can then be verified that $\left\{U_{\alpha_k}\right\}_{k=1}^{n}$ covers $f(K)$. ■

**Corollary 2.31 (Extreme value theorem)** *Let $K$ be a compact subset of $\mathbb{C}$. If $g: K \to \mathbb{R}$ is a continuous function, then there exists $a \in K$ such that $g(z) \leq g(a)$ for all $z \in K$. In particular, if $f: K \to \mathbb{C}$ is a continuous function, then there exists $a \in K$ such that $|f(z)| \leq |f(a)|$ for all $z \in K$.*

**Example 2.32** Let $f: \mathbb{C} \to \mathbb{C}$ be a continuous function such that

$$f(z+1) = f(z+i) = f(z)$$

for every $z \in \mathbb{C}$. Show that the function $f$ is ***bounded***, i.e. there exists $M > 0$ such that

$$|f(z)| \leq M$$

for every $z \in \mathbb{C}$.

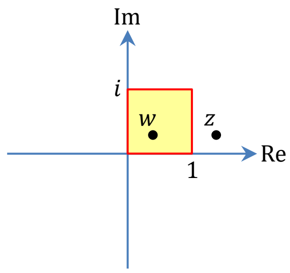
*Proof*:

By induction, the given condition easily implies that

$$f(z+m+ni) = f(z)$$

for every $z \in \mathbb{C}$ and $m, n \in \mathbb{Z}$. Let

$$K = \{z \in \mathbb{C}: 0 \leq \operatorname{Re} z \leq 1 \text{ and } 0 \leq \operatorname{Im} z \leq 1\}.$$




Then $K$ is a compact set since it is closed and bounded. Since $f$ is continuous on $K$, there exists $M > 0$ such that

$$|f(w)| \leq M$$

for every $w \in K$ by Corollary 2.31. Finally for every $z \in \mathbb{C}$, there exists $m, n \in \mathbb{Z}$ such that $w := z + m + ni \in K$, so $f(z) = f(w)$. Therefore

$$|f(z)| = |f(w)| \leq M.$$

■

Continuous functions on a compact set must be **uniformly continuous**. Intuitively this means that when a pair of points $z$ and $w$ are near each other, the values $f(z)$ and $f(w)$ are also near each other **"by the same extent" all over the set**.

**Definition 2.33** Let $U \subseteq \mathbb{C}$ and let $f: U \to \mathbb{C}$ be a function. We say that $f$ is ***uniformly continuous*** (on $U$) if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(z) - f(w)| < \varepsilon \qquad \text{whenever } z, w \in U \text{ and } |z - w| < \delta.$$

The same $\delta$ works for all $z, w \in U$.

**Theorem 2.34** *Let $K$ be a compact subset of $\mathbb{C}$ and $f: K \to \mathbb{C}$ be a continuous function. Then $f$ is uniformly continuous.*

*Proof*. Suppose on the contrary that $f$ is not uniformly continuous. Then there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$, we can find $z_n, w_n \in K$ which satisfy

$$|z_n - w_n| < \frac{1}{n} \qquad \text{and} \qquad |f(z_n) - f(w_n)| \geq \varepsilon.$$

Now $\{z_n\}$ and $\{w_n\}$ are sequences in the compact set $K$, so each of them has a subsequence which converges to a limit in $K$ (by Bolzano-Weierstrass).

Without loss of generality, we have an increasing sequence $\{n_k\}_k$ of natural numbers such that

$$\lim_{k\to+\infty} z_{n_k} = z \in K \qquad \text{and} \qquad \lim_{k\to+\infty} w_{n_k} = w \in K.$$

- Since $|z_n - w_n| < \frac{1}{n}$ for every $n \in \mathbb{N}$, we must have $|z - w| = 0$, i.e. $z = w$.
- However, since $f$ is continuous but $|f(z_n) - f(w_n)| \geq \varepsilon$ for every $n \in \mathbb{N}$, we must have
$$|f(z) - f(w)| \geq \varepsilon,$$
i.e. $f(z) \neq f(w)$.

So we have a contradiction. ■

**Theorem 2.35 (Intermediate value theorem)** *Let $U$ be a connected subset of $\mathbb{C}$ and $f: U \to \mathbb{C}$ be a continuous function. Then $f(U)$ is connected.*

*Proof.* Let $V$ and $W$ be disjoint open sets such that $f(U) \subseteq V \cup W$. Then $f^{-1}(V)$ and $f^{-1}(W)$ are disjoint open sets such that $U \subseteq f^{-1}(V) \cup f^{-1}(W)$. Since $U$ is connected, either $U \cap f^{-1}(V)$ or $U \cap f^{-1}(W)$ must be empty. Consequently, either $f(U) \cap V$ or $f(U) \cap W$ must be empty, so $f(U)$ is connected. ■

**Example 2.36** Show that there does not exist any continuous bijection from $\mathbb{C}$ to $\mathbb{R}$.

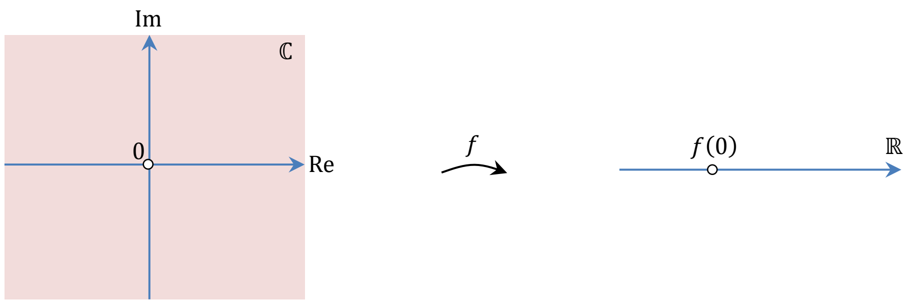
*Proof*:
Suppose that there exists a continuous bijection $f: \mathbb{C} \to \mathbb{R}$. Removing $0$ from the domain of $f$, we see that the restriction

$$f|_{\mathbb{C}\setminus\{0\}}: \mathbb{C} \setminus \{0\} \to \mathbb{R} \setminus \{f(0)\}$$

must also be a continuous bijection. Now since $\mathbb{C} \setminus \{0\}$ is connected, its direct image via the continuous function $f|_{\mathbb{C}\setminus\{0\}}$

$$f|_{\mathbb{C}\setminus\{0\}}(\mathbb{C} \setminus \{0\}) = \mathbb{R} \setminus \{f(0)\}$$

is also connected by Theorem 2.35, which is obviously a contradiction. ■



### 3. Complex differentiability, holomorphic functions

**Definition 2.37** Let $U \subseteq \mathbb{C}$ be an open set, let $a \in U$ and let $f: U \to \mathbb{C}$ be a function. We say that $f$ is (***complex***) ***differentiable at*** $\boldsymbol{a}$ if the limit

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists as a complex number. This limit is called the ***derivative of*** $\boldsymbol{f}$ ***at*** $\boldsymbol{a}$, and is denoted as $f'(a)$.

**Example 2.38** Let $f: \mathbb{C} \to \mathbb{C}$ be the function

$$f(z) = z^2.$$

Evaluate $f'(z)$ for each $z \in \mathbb{C}$.

*Solution*: [This is almost identical to what we have learnt in MATH1013/1023.]
For each $z \in \mathbb{C}$, we have

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{(z+h)^2 - z^2}{h} = \lim_{h \to 0} \frac{2hz + h^2}{h}$$
$$= \lim_{h \to 0} (2z + h) = 2z,$$

so $f'(z) = 2z$.

Although the definition of differentiability looks almost the same as the one for functions of a real variable you have seen in MATH1013/1023, it is conceptually very different because now we are dealing with a limit of a **complex function**, which becomes a limit of a **function of two real variables** if we treat the real and imaginary parts of the complex variable separately:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{(j,k) \to (0,0)} \frac{f(x + iy + j + ik) - f(x + iy)}{j + ik}.$$

**Example 2.39** Let $f: \mathbb{C} \to \mathbb{C}$ be the function

$$f(z) = |z|^2.$$

Find all the points $a \in \mathbb{C}$ at which $f$ is differentiable.

*Solution*:
At $a = 0$, we have

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|^2}{h} = \lim_{h \to 0} \overline{h} = \overline{0} = 0$$

by the continuity of the complex conjugate (Example 2.24), so $f$ is differentiable at $0$ and $f'(0) = 0$.

Next let $a = x + iy \neq 0$. Then

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{|a+h|^2-|a|^2}{h} = \lim_{(j,k)\to(0,0)}\frac{|x+iy+j+ik|^2-|x+iy|^2}{j+ik}$$
$$= \lim_{(j,k)\to(0,0)}\frac{(x+j)^2+(y+k)^2-(x^2+y^2)}{j+ik}$$
$$= \lim_{(j,k)\to(0,0)}\frac{2jx+2ky+j^2+k^2}{j+ik}.$$

Along the line $k = 0$, we have

$$\lim_{\substack{(j,k)\to(0,0)\\ k=0}}\frac{2jx+2ky+j^2+k^2}{j+ik} = \lim_{j\to 0}\frac{2jx+h^2}{j} = \lim_{j\to 0}(2x+j) = 2x;$$

but along the line $j = 0$, we have

$$\lim_{\substack{(j,k)\to(0,0)\\ j=0}}\frac{2jx+2ky+j^2+k^2}{j+ik} = \lim_{k\to 0}\frac{2ky+k^2}{ik} = \lim_{k\to 0}\frac{2y+k}{i} = -2yi.$$

Since $a \neq 0$, it follows that $2x \neq -2yi$ and so $f'(a)$ does not exist. ∎

Let's investigate on some **necessary conditions** of (complex) differentiability.

**Lemma 2.40** *Let $U \subseteq \mathbb{C}$ be an open set, let $a \in U$ and let $f: U \to \mathbb{C}$ be a function which is differentiable at $a$. Then $f$ is continuous at $a$.*

*Proof.* Identical to the real case. ∎

**Lemma 2.41** *Let $U \subseteq \mathbb{C}$ be an open set, identified as a subset of $\mathbb{R}^2$. Let $(a_1, a_2) \in U$ and let $u, v: U \to \mathbb{R}$ be functions of two real variables. If the function $f: U \to \mathbb{C}$ defined by*

$$f(x+iy) = u(x,y) + iv(x,y)$$

*is differentiable at $a = a_1 + ia_2$, then*

(i) *The partial derivatives of $u$ and $v$ both exist at $(a_1, a_2)$ and satisfy*

$$\frac{\partial u}{\partial x}(a_1,a_2) = \frac{\partial v}{\partial y}(a_1,a_2) \qquad \textit{and} \qquad \frac{\partial v}{\partial x}(a_1,a_2) = -\frac{\partial u}{\partial y}(a_1,a_2).$$

(ii) *Furthermore, $u$ and $v$ are both (real) differentiable at $(a_1, a_2)$.*

**Remark 2.42** The partial differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

are called ***Cauchy-Riemann equations***. Lemma 2.41 says that satisfying the Cauchy-Riemann equations at $(a_1, a_2)$ is a necessary condition for a function $f = u + iv$ to be differentiable at $a$.

*Proof of Lemma 2.41.* We first prove (i). Since $f$ is differentiable at $a = a_1 + ia_2$, the limit

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{(j,k)\to(0,0)}\frac{f(a_1+ia_2+j+ik)-f(a_1+ia_2)}{j+ik}$$

exists. In particular, the limit along any direction must be the same. Along $k=0$, we have

$$\begin{aligned} f'(a) &= \lim_{j\to 0}\frac{f(a_1+ia_2+j)-f(a_1+ia_2)}{j} \\ &= \lim_{j\to 0}\frac{[u(a_1+j,a_2)+iv(a_1+j,a_2)]-[u(a_1,a_2)+iv(a_1,a_2)]}{j} \\ &= \lim_{j\to 0}\left[\frac{u(a_1+j,a_2)-u(a_1,a_2)}{j}+i\frac{v(a_1+j,a_2)-v(a_1,a_2)}{j}\right] = u_x(a_1,a_2)+iv_x(a_1,a_2); \end{aligned}$$

and by a similar computation, we see that along $j=0$ we have

$$f'(a) = \lim_{k\to 0}\frac{f(a_1+ia_2+ik)-f(a_1+ia_2)}{ik} = v_y(a_1,a_2)-iu_y(a_1,a_2).$$

Therefore

$$u_x(a_1,a_2)+iv_x(a_1,a_2) = v_y(a_1,a_2)-iu_y(a_1,a_2),$$

which implies that the Cauchy-Riemann equations are satisfied at $(a_1,a_2)$.

It now remains to prove (ii), i.e. that $u$ and $v$ are (real) differentiable at $(a_1,a_2)$. Note that in the last paragraph we have shown that $f'(a) = u_x(a_1,a_2)+iv_x(a_1,a_2)$. This together with the Cauchy-Riemann equations imply that for every $z = x+iy \in U$,

$$\begin{aligned} &f(z)-f(a)-f'(a)(z-a) \\ &= \big(u(x,y)+iv(x,y)\big)-\big(u(a_1,a_2)+iv(a_1,a_2)\big) \\ &\qquad -\big(u_x(a_1,a_2)+iv_x(a_1,a_2)\big)\big((x-a_1)+i(y-a_2)\big) \\ &= [u(x,y)-u(a_1,a_2)-u_x(a_1,a_2)(x-a_1)+v_x(a_1,a_2)(y-a_2)] \\ &\qquad + i[v(x,y)-v(a_1,a_2)-v_x(a_1,a_2)(x-a_1)-u_x(a_1,a_2)(y-a_2)] \\ &= \big[u(x,y)-u(a_1,a_2)-u_x(a_1,a_2)(x-a_1)-u_y(a_1,a_2)(y-a_2)\big] \\ &\qquad + i\big[v(x,y)-v(a_1,a_2)-v_x(a_1,a_2)(x-a_1)-v_y(a_1,a_2)(y-a_2)\big] \end{aligned}$$

Since $f$ is differentiable at $a$, we have

$$\lim_{z\to a}\frac{|f(z)-f(a)-f'(a)(z-a)|}{|z-a|} = 0,$$

(why?), so

$$\lim_{(x,y)\to(a_1,a_2)}\frac{\big|u(x,y)-u(a_1,a_2)-u_x(a_1,a_2)(x-a_1)-u_y(a_1,a_2)(y-a_2)\big|}{\sqrt{(x-a_1)^2+(y-a_2)^2}} = 0 \quad \text{and}$$

$$\lim_{(x,y)\to(a_1,a_2)}\frac{\big|v(x,y)-v(a_1,a_2)-v_x(a_1,a_2)(x-a_1)-v_y(a_1,a_2)(y-a_2)\big|}{\sqrt{(x-a_1)^2+(y-a_2)^2}} = 0,$$

which show that $u$ and $v$ are both differentiable at $(a_1,a_2)$. ∎

**Example 2.39** Let $f: \mathbb{C} \to \mathbb{C}$ be the function

$$f(z) = |z|^2.$$

Find all the points $a \in \mathbb{C}$ at which $f$ is differentiable.

*Solution*: [Let's try this again with the new tool of Cauchy-Riemann equations.]

Note that $f(x+iy) = |x+iy|^2 = (x^2+y^2) + i(0)$. Let $u, v: \mathbb{R}^2 \to \mathbb{R}$ be the functions

$$u(x,y) = \operatorname{Re} f(x+iy) = x^2+y^2 \qquad \text{and} \qquad v(x,y) = \operatorname{Im} f(x+iy) = 0.$$

Then

$$u_x(x,y) = 2x, \qquad u_y(x,y) = 2y,$$
$$v_x(x,y) = 0, \qquad v_y(x,y) = 0.$$

The Cauchy-Riemann equations are satisfied at $(0,0)$ only, so $f$ is not differentiable at any point in $\mathbb{C} \setminus \{0\}$. On the other hand, we can check that

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|^2}{h} = \lim_{h\to 0} \overline{h} = \overline{0} = 0,$$

so $f$ is differentiable only at $0$, and $f'(0) = 0$.

**Example 2.43** Let $f: \mathbb{C} \to \mathbb{C}$ be the function

$$f(z) = \overline{z}.$$

Find all the points $a \in \mathbb{C}$ at which $f$ is differentiable.

*Solution*:

Note that $f(x+iy) = x - iy$. Let $u, v: \mathbb{R}^2 \to \mathbb{R}$ be the functions

$$u(x,y) = \operatorname{Re} f(x+iy) = x \qquad \text{and} \qquad v(x,y) = \operatorname{Im} f(x+iy) = -y.$$

Then

$$u_x(x,y) = 1, \qquad u_y(x,y) = 0,$$
$$v_x(x,y) = 0, \qquad v_y(x,y) = -1.$$

The Cauchy-Riemann equations are never satisfied, so $f$ is nowhere differentiable on $\mathbb{C}$.

**Remark 2.44** Note that just satisfying the Cauchy-Riemann equations at $(a_1, a_2)$ (without $u$, $v$ being (real) differentiable at $a$) is not sufficient for concluding that $f$ is differentiable at $a$. Consider the function $f: \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \begin{cases} (\operatorname{Re} z \operatorname{Im} z) \,/\, \overline{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}.$$

One can easily check that $u_x(0,0) = u_y(0,0) = v_x(0,0) = v_y(0,0) = 0$, so the Cauchy-Riemann equations are satisfied at $(0,0)$. However, the limit

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{(x,y)\to(0,0)} \frac{f(x+iy)-f(0)}{x+iy} = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

does not exist as we have seen in MATH2023. Therefore $f$ is not differentiable at $0$.

It turns out that (i) and (ii) in Lemma 2.41 together are also **sufficient** for differentiability.

**Lemma 2.45** *Let $U \subseteq \mathbb{C}$ be an open set, identified as a subset of $\mathbb{R}^2$. Let $(a_1, a_2) \in U$ and let $u, v: U \to \mathbb{R}$ be functions of two real variables. If*

(i) *$u$ and $v$ satisfy the Cauchy-Riemann equations at $(a_1, a_2)$, and*

(ii) *$u$ and $v$ are both (real) differentiable at $(a_1, a_2)$,*

*then the function $f: U \to \mathbb{C}$ defined by*

$$f(x+iy) = u(x,y) + iv(x,y)$$

*is differentiable at $a = a_1 + ia_2$.*

*Proof*. For simplicity, let $E_1, E_2: U \to \mathbb{R}$ be the functions defined by

$$E_1(x,y) = u(x,y) - u(a_1,a_2) - u_x(a_1,a_2)(x-a_1) - u_y(a_1,a_2)(y-a_2) \quad \text{and}$$

$$E_2(x,y) = v(x,y) - v(a_1,a_2) - v_x(a_1,a_2)(x-a_1) - v_y(a_1,a_2)(y-a_2).$$

Since $u$ and $v$ are differentiable at $(a_1, a_2)$, it follows that

$$\lim_{(x,y)\to(a_1,a_2)} \frac{|E_1(x,y)|}{\sqrt{(x-a_1)^2 + (y-a_2)^2}} = \lim_{(x,y)\to(a_1,a_2)} \frac{|E_2(x,y)|}{\sqrt{(x-a_1)^2 + (y-a_2)^2}} = 0.$$

Now the issue is to show that the limit $\lim_{z\to a} \frac{f(z)-f(a)}{z-a}$ exists. First consider the numerator. For each $z = x + iy \in U$, since $u$ and $v$ satisfy the Cauchy-Riemann equations at $(a_1, a_2)$, we have

$$\begin{aligned}
& f(z) - f(a) \\
&= [u(x,y) + iv(x,y)] - [u(a_1,a_2) + iv(a_1,a_2)] \\
&= \big[E_1(x,y) + u_x(a_1,a_2)(x-a_1) + u_y(a_1,a_2)(y-a_2)\big] \\
&\qquad + i\big[E_2(x,y) + v_x(a_1,a_2)(x-a_1) + v_y(a_1,a_2)(y-a_2)\big] \\
&= [E_1(x,y) + u_x(a_1,a_2)(x-a_1) - v_x(a_1,a_2)(y-a_2)] \\
&\qquad + i[E_2(x,y) + v_x(a_1,a_2)(x-a_1) + u_x(a_1,a_2)(y-a_2)] \\
&= E_1(x,y) + iE_2(x,y) + u_x(a_1,a_2)(z-a) + iv_x(a_1,a_2)(z-a).
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{z\to a} \frac{f(z)-f(a)}{z-a} &= \underbrace{\lim_{z\to a} \frac{E_1(x,y)}{z-a}}_{=0} + i \underbrace{\lim_{z\to a} \frac{E_2(x,y)}{z-a}}_{=0} + u_x(a_1,a_2) + iv_x(a_1,a_2) \\
&= u_x(a_1,a_2) + iv_x(a_1,a_2)
\end{aligned}$$

exists, i.e. $f$ is differentiable at $a$. ∎

**Corollary 2.46** *Let $U \subseteq \mathbb{C}$ be an open set, identified as a subset of $\mathbb{R}^2$. Let $(a_1, a_2) \in U$ and let $u, v: U \to \mathbb{R}$ be functions of two real variables. If the function $f: U \to \mathbb{C}$ defined by*

$$f(x+iy) = u(x,y) + iv(x,y)$$

*is differentiable at $a = a_1 + ia_2$, then its derivative at $a$ is given by*

$$f'(a) = u_x(a_1,a_2) + iv_x(a_1,a_2) = u_x(a_1,a_2) - iu_y(a_1,a_2).$$

It is usually difficult to check the (real) differentiability conditions for the two-variable functions $u$ and $v$ at each point, but thanks to the following theorem from MATH2023/3033, if $u$ and $v$ are of class $C^1$ (i.e. their **partial derivatives are continuous**) **on an open set**, then $u$ and $v$ are differentiable at each point within the open set automatically.

**Theorem ($C^1 \Rightarrow$ differentiable)** *Let $u$ be a function of two real variables and let $D$ be an open disk in $\mathbb{R}^2$. If the **partial derivatives** $u_x$ and $u_y$ are **continuous** on $D$, then $u$ is differentiable at every point in $D$.*

The above theorem motivates us to focus on complex functions that are (complex) **differentiable on an open set**. Because such kind of functions are so important in complex analysis, instead of calling them to be "differentiable", we invent a fancy wording and call them to be "**holomorphic**".

**Definition 2.47 (Holomorphic function)** Let $U \subseteq \mathbb{C}$ be an open set and let $f: U \to \mathbb{C}$ be a function. We say that $f$ is ***holomorphic at a point $a \in U$*** if $f$ is differentiable on some open disk centered at $a$. We say that $f$ is ***holomorphic on $U$*** if $f$ is differentiable at every point in $U$. A function $f: \mathbb{C} \to \mathbb{C}$ which is holomorphic on $\mathbb{C}$ is also called an ***entire*** function.

**Corollary 2.48 (Cauchy-Riemann)** *Let $U \subseteq \mathbb{C}$ be an open set, identified as a subset of $\mathbb{R}^2$. If $u, v: U \to \mathbb{R}$ are functions having continuous partial derivatives on $U$ (i.e. they are $C^1$ on $U$) and they satisfy the Cauchy-Riemann equations on $U$, then the function $f: U \to \mathbb{C}$ defined by*

$$f(x+iy) = u(x,y) + iv(x,y)$$

*is holomorphic on $U$.*

**Remark 2.49** Corollary 2.48 does <u>**not**</u> say that if $f$ is holomorphic on an open set $U$, then $u$ and $v$ have continuous partial derivatives on $U$, although this turns out to be true as we will see later.

**Example 2.50** Show that the exponential function $f: \mathbb{C} \to \mathbb{C}$ defined by $f(z) = e^z$ is entire.

*Proof*: Note that $f(x+iy) = e^{x+iy} = e^x(\cos y + i \sin y)$. Let $u, v: \mathbb{R}^2 \to \mathbb{R}$ be the functions

$$u(x,y) = \operatorname{Re} f(x+iy) = e^x \cos y \qquad \text{and} \qquad v(x,y) = \operatorname{Im} f(x+iy) = e^x \sin y.$$

Their partial derivatives are given by

$$u_x(x,y) = e^x \cos y, \qquad u_y(x,y) = -e^x \sin y,$$
$$v_x(x,y) = e^x \sin y, \qquad v_y(x,y) = e^x \cos y.$$

These partial derivatives are continuous on $\mathbb{R}^2$ and satisfy the Cauchy-Riemann equations on $\mathbb{R}^2$. Therefore by Corollary 2.48, the exponential function is entire. And by the way, we also have

$$f'(x+iy) = u_x(x,y) + iv_x(x,y) = e^x(\cos y + i \sin y) = e^{x+iy},$$

i.e. $f'(z) = e^z$, the same as seen in the real case. ∎

**Example 2.51** Let $f:\mathbb{C}\to\mathbb{C}$ be an entire function. Show that the function $g:\mathbb{C}\to\mathbb{C}$ defined by

$$g(z)=\overline{f(\bar{z})}$$

is also entire.

*Proof*: Let $u(x,y)=\operatorname{Re} f(x+iy)$ and $v(x,y)=\operatorname{Im} f(x+iy)$. Since $f$ is entire, by Lemma 2.41, $u$ and $v$ are (real) differentiable on $\mathbb{R}^2$ and satisfy Cauchy-Riemann equations

$$u_x(x,y)=v_y(x,y) \qquad \text{and} \qquad v_x(x,y)=-u_y(x,y)$$

for every $(x,y)\in\mathbb{R}^2$. Now $g(x+iy)=\overline{f(x-iy)}=u(x,-y)-iv(x,-y)$, so the real and imaginary parts of $g$ are given by

$$U(x,y)\coloneqq \operatorname{Re} g(x+iy)=u(x,-y) \qquad \text{and} \qquad V(x,y)\coloneqq \operatorname{Im} g(x+iy)=-v(x,-y)$$

respectively. $U$ and $V$ are also (real) differentiable on $\mathbb{R}^2$ since $u$ and $v$ are. Moreover,

$$\begin{aligned}
U_x(x,y)&=1\cdot u_x(x,-y)+0\cdot u_y(x,-y)=u_x(x,-y),\\
U_y(x,y)&=0\cdot u_x(x,-y)+(-1)\cdot u_y(x,-y)=-u_y(x,-y),\\
V_x(x,y)&=1\cdot[-v_x(x,-y)]+0\cdot[-v_y(x,-y)]=-v_x(x,-y),\\
V_y(x,y)&=0\cdot[-v_x(x,-y)]+(-1)\cdot[-v_y(x,-y)]=v_y(x,-y),
\end{aligned}$$

Chain rule

so $U_x(x,y)=V_y(x,y)$ and $V_x(x,y)=-U_y(x,y)$ for every $(x,y)\in\mathbb{R}^2$. Therefore $g$ is entire by Lemma 2.45. ∎

**Example 2.52** Find an entire function $f:\mathbb{C}\to\mathbb{C}$ such that

$$\operatorname{Re} f(x+iy)=2x-x^3+3xy^2$$

for every $x,y\in\mathbb{R}$.

*Solution*: Let $u(x,y)=2x-x^3+3xy^2$ for every $(x,y)\in\mathbb{R}^2$. Then

$$u_x(x,y)=2-3x^2+3y^2 \qquad \text{and} \qquad u_y(x,y)=6xy,$$

so $u_x$ and $u_y$ are both continuous on $\mathbb{R}^2$. To find an entire function $f:\mathbb{C}\to\mathbb{C}$ whose real part is $\operatorname{Re} f(x+iy)=u(x,y)$, we need to solve the Cauchy-Riemann equations

$$u_x(x,y)=v_y(x,y) \qquad \text{and} \qquad v_x(x,y)=-u_y(x,y)$$

for the unknown function $v$, which will be the imaginary part of $f$. From the second equation we have $v_x(x,y)=-u_y(x,y)=-6xy$, so

$$v(x,y)=-3x^2y+C(y),$$

where $C(y)$ is a real differentiable function independent of $x$. Now $v_y(x,y)=-3x^2+C'(y)$, so together with the first equation $u_x(x,y)=v_y(x,y)$ we have

$$2-3x^2+3y^2=-3x^2+C'(y).$$

Thus $C'(y)=2+3y^2$, which gives $C(y)=2y+y^3+C$, where $C$ is an arbitrary real constant. We may just choose $C=0$ and get a solution $v(x,y)=2y-3x^2y+y^3$. Finally, an entire function $f$ with $\operatorname{Re} f(x+iy)=u(x,y)$ is then given by

$$\begin{aligned}
f(z)=f(x+iy)&=u(x,y)+iv(x,y)=(2x-x^3+3xy^2)+i(2y-3x^2y+y^3)\\
&=2(x+iy)-(x+iy)^3=2z-z^3.
\end{aligned}$$

**Remark 2.53** Let $U$ be a region in $\mathbb{C}$, identified as an open subset of $\mathbb{R}^2$. A function $u: U \to \mathbb{R}$ (of two real variables) is said to be ***harmonic*** on $U$ if it satisfies the Laplace equation

$$u_{xx} + u_{yy} = 0.$$

If a function $u: U \to \mathbb{R}$ has continuous second partial derivatives and is the real part of a holomorphic function on $U$, then according to Cauchy-Riemann equations, $u$ must be harmonic. (Check that the real part given in Example 2.52 is indeed harmonic.) But conversely, a harmonic function on $U$ is not always the real part of some holomorphic function on $U$. As a counter-example, the harmonic function $u: \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ defined by

$$u(x,y) = \ln\sqrt{x^2 + y^2}$$

is not the real part of any holomorphic function on the region $\mathbb{C} \setminus \{0\}$.

In MATH1013/1023, we used the mean value theorem to establish the fact that the **only functions having zero derivative are the constant functions** (i.e. "$\int 0\,dx = C$"). In the context of complex functions, we can still obtain the same result by applying the mean value theorem to the real and imaginary parts separately. Note that the **connectedness** assumption on the domain is essential.

**Theorem 2.54** *Let $U$ be a region in $\mathbb{C}$ (i.e. open and connected) and let $f: U \to \mathbb{C}$ be a function. If $f$ is holomorphic on $U$ and $f' = 0$ on $U$, then $f$ is a constant function.*

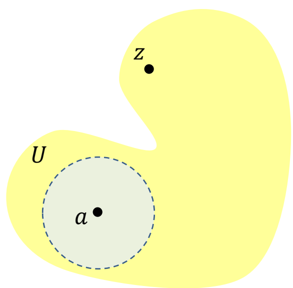
*Proof*. Let $a \in U$. The issue is to show that $f(z) = f(a)$ for every $z \in U$. Since $U$ is open, there exists $\varepsilon > 0$ such that $D(a;\varepsilon) \subseteq U$. We first show that $f$ is constant on this $D(a;\varepsilon)$. For each $z \in D(a;\varepsilon)$, $a$ and $z$ can be joined by a line segment completely contained in $D(a;\varepsilon)$. So we consider the function $g: [0,1] \to \mathbb{R}$ defined by

$$g(t) = \operatorname{Re} f\big((1-t)a + tz\big).$$

Since $g$ is continuous on $[0,1]$ and differentiable on $(0,1)$, mean value theorem implies that

$$\operatorname{Re} f(z) - \operatorname{Re} f(a) = g(1) - g(0) = g'(c)$$

for some $c \in (0,1)$. But $g'$ is just a certain directional derivative of $u \coloneqq \operatorname{Re} f$, so it is a linear combination of $u_x$ and $u_y$ (we have $g' = (\operatorname{Re}(z-a))u_x + (\operatorname{Im}(z-a))u_y$, to be precise). Since $u_x = u_y = 0$ by the hypothesis, we must have $g' = 0$ too. Therefore $\operatorname{Re} f(z) = \operatorname{Re} f(a)$. Similarly, we can also show that $\operatorname{Im} f(z) = \operatorname{Im} f(a)$ and so $f(z) = f(a)$.




Finally since $U$ is open and connected, it is polygonally path-connected. So for each $z \in U$, $a$ and $z$ can be joined by a path which consists of finitely many line segments completely contained in $U$. Applying the same argument on each line segment as in the previous paragraph, we have $f(z) = f(a)$ again. Therefore $f$ is a constant function. ∎

**Example 2.55** Let $U$ be a region in $\mathbb{C}$, identified as a region in $\mathbb{R}^2$. If $u, v: U \to \mathbb{R}$ are functions of two real variables such that the functions $f, g: U \to \mathbb{C}$ defined by

$$f(x+iy) = u(x,y) + iv(x,y) \qquad \text{and} \qquad g(x,y) = v(x,y) + iu(x,y)$$

are both holomorphic, show that $f$ and $g$ are both constant functions.

*Proof*: Since $f$ and $g$ are holomorphic on $U$, the pairs of Cauchy-Riemann equations

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \qquad \text{and} \qquad \begin{cases} v_x = u_y \\ v_y = -u_x \end{cases}$$

hold simultaneously on $U$. These four equations together imply that

$$u_x \equiv u_y \equiv v_x \equiv v_y \equiv 0$$

on $U$, so $f' = u_x + iv_x = 0$ and $g' = v_x + iu_x = 0$ on $U$. Since $U$ is a region, $f$ and $g$ are both constant by Theorem 2.54. ∎

**Example 2.56** Let $U$ be a region in $\mathbb{C}$ and $f: U \to \mathbb{C}$ be a holomorphic function. If $f(z) \in \mathbb{R}$ for every $z \in U$, show that $f$ is constant.

*Proof*: Let $u, v: U \to \mathbb{R}$ be the functions $u(x,y) = \operatorname{Re} f(x+iy)$ and $v(x,y) = \operatorname{Im} f(x+iy)$. Since $f$ is holomorphic on $U$, it follows that $u$ and $v$ satisfy the Cauchy-Riemann equations

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x$$

on $U$. Now since $f(z) \in \mathbb{R}$ for every $z \in U$, we have $v(x,y) = 0$ for every $(x,y) \in U$. This implies that $v_x = 0 = v_y$ on $U$, and as a result $u_x = v_y = 0$ on $U$ also. So $f' = u_x + iv_x = 0$ on $U$. Since $U$ is a region, $f$ is constant by Theorem 2.54. ∎

**Example 2.57** Let $U$ be a region in $\mathbb{C}$ and $f: U \to \mathbb{C}$ be a holomorphic function. If $|f(z)| = 1$ for every $z \in U$, show that $f$ is constant.

*Proof*: Let $u, v: U \to \mathbb{R}$ be the functions $u(x,y) = \operatorname{Re} f(x+iy)$ and $v(x,y) = \operatorname{Im} f(x+iy)$. Since $f$ is holomorphic on $U$, it follows that $u$ and $v$ satisfy the Cauchy-Riemann equations

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x$$

on $U$. Now since $|f(z)| = 1$ for every $z \in U$, we have $u(x,y)^2 + v(x,y)^2 = 1$ for every $(x,y) \in U$. Partial differentiation gives

$$2uu_x + 2vv_x = 0 \qquad \text{and} \qquad 2uu_y + 2vv_y = 0$$

on $U$, so

$$uu_x + vv_x = 0 \qquad \text{and} \qquad -uv_x + vu_x = 0$$

on $U$. These two equations imply that

$$(u^2 + v^2)u_x = (u^2 + v^2)v_x = 0,$$

$u^2 + v^2$ is the constant function 1.

so $u_x = v_x = 0$ on $U$. Therefore $f' = u_x + iv_x = 0$ on $U$. Since $U$ is a region, $f$ is constant by Theorem 2.54. ∎

As in the case for differentiable functions of a real variable, we have the following results about how holomorphic functions interact with various **arithmetic operations**. The proofs are all exactly the same as in the case for functions of a real variable.

**Lemma 2.58** *Let $U \subseteq \mathbb{C}$ be an open set, and let $f, g: U \to \mathbb{C}$ be holomorphic functions. Then*

(i) *$f \pm g$ and $fg$ are holomorphic on $U$, with $(f \pm g)' = f' \pm g'$ and $(fg)' = f'g + fg'$. In other words, the set of all holomorphic functions on $U$ forms a ring.*

(ii) *$f/g$ is holomorphic on the open set $U \setminus g^{-1}(\{0\}) = U \setminus \{z \in U: g(z) = 0\}$, with*

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

**Theorem 2.59 (Chain rule)** *Let $U, V \subseteq \mathbb{C}$ be open sets. If $f: V \to \mathbb{C}$ and $g: U \to V$ are holomorphic functions, then $f \circ g: U \to \mathbb{C}$ is also a holomorphic function. Moreover, we have*

$$(f \circ g)'(z) = f'\big(g(z)\big)g'(z)$$

*for every $z \in U$.*

**Example 2.60** Similar to Example 2.38, for each non-negative integer $n$, if $f(z) = z^n$ then we have $f'(z) = nz^{n-1}$. So all polynomials are entire by Lemma 2.58. All rational functions are holomorphic everywhere except at the roots of the denominators.

**Definition 2.61** The (***complex***) ***cosine function*** and ***sine function*** $\cos, \sin: \mathbb{C} \to \mathbb{C}$ are defined by

$$\cos z \coloneqq \frac{e^{iz} + e^{-iz}}{2} \qquad \text{and} \qquad \sin z \coloneqq \frac{e^{iz} - e^{-iz}}{2i}$$

respectively. The other four ***trigonometric functions*** are defined as usual:

$$\tan z \coloneqq \frac{\sin z}{\cos z}, \qquad \cot z \coloneqq \frac{\cos z}{\sin z}, \qquad \sec z \coloneqq \frac{1}{\cos z}, \qquad \csc z \coloneqq \frac{1}{\sin z}.$$

**Example 2.62** $\cos$ and $\sin$ are entire functions. Chain rule gives

$$\frac{d}{dz}\cos z = \frac{d}{dz}\frac{e^{iz} + e^{-iz}}{2} = \frac{ie^{iz} - ie^{-iz}}{2} = i^2 \cdot \frac{e^{iz} - e^{-iz}}{2i} = -\sin z,$$

$$\frac{d}{dz}\sin z = \frac{d}{dz}\frac{e^{iz} - e^{-iz}}{2i} = \frac{ie^{iz} + ie^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos z,$$

which are the same as in the real case. Thus by the same proofs as in MATH1013 we obtain

$$\frac{d}{dz}\tan z = \sec^2 z, \qquad \frac{d}{dz}\cot z = -\csc^2 z,$$

$$\frac{d}{dz}\sec z = \sec z \tan z, \qquad \frac{d}{dz}\csc z = -\csc z \cot z$$

again.

**Example 2.63** Find the inverse image of the singleton $\{2\}$ via the complex cosine function; or in other words, find all the (complex number) solutions to the equation

$$\cos z = 2.$$

*Solution*:

If $\cos z = 2$, then $\frac{e^{iz}+e^{-iz}}{2} = 2$ and so $e^{2iz} - 4e^{iz} + 1 = 0$. By the quadratic formula we have

$$e^{iz} = 2 \pm \sqrt{3}.$$

If we write $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$, then this gives $e^{-y}e^{ix} = 2 \pm \sqrt{3}$. Thus

- ⦿ We have $e^{-y} = 2 \pm \sqrt{3} > 0$, so $y = -\ln(2 \pm \sqrt{3}) = \pm \ln(2 + \sqrt{3})$.
- ⦿ De Moivre's Theorem gives $x = 2k\pi$ for some $k \in \mathbb{Z}$.

$\ln(2-\sqrt{3}) = -\ln(2+\sqrt{3})$

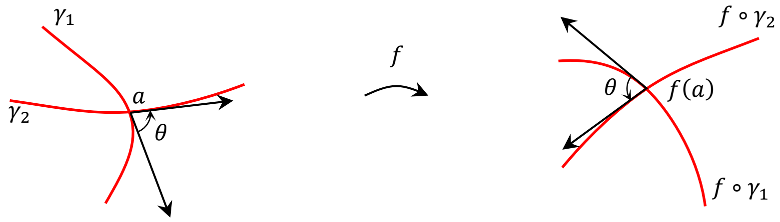
Therefore, $z = x + iy = 2k\pi \pm i\ln(2 + \sqrt{3})$ for some $k \in \mathbb{Z}$.
Conversely, it is easy to verify that $\cos(2k\pi \pm i\ln(2 + \sqrt{3})) = 2$ for every $k \in \mathbb{Z}$, so these are all the solutions to the equation $\cos z = 2$ in $\mathbb{C}$.

The following are some **geometric interpretations** of the Cauchy-Riemann equations.

**Remark 2.64** Recall the following facts:

- ⦿ In MATH2121/2131, we learnt that a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ with a matrix of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ is a composition of a scaling (by a factor $\sqrt{a^2 + b^2}$) and a rotation.
- ⦿ In MATH2023/3033/3043, we learnt that if $(u, v)$ is a pair of functions of two real variables, then the matrix of the linear approximation $T: \mathbb{R}^2 \to \mathbb{R}^2$ of $(u, v)$ at a point (relative to the standard basis) is the Jacobian $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$.

So geometrically speaking, the pair of functions $(u, v)$ satisfies the **Cauchy-Riemann equations** if and only if its **Jacobian is a composition of a scaling and a rotation**. In this situation, the Jacobian determinant of $(u, v)$ must be non-negative, and it is zero at a point $(x, y)$ if and only if $u_x(x, y) = v_x(x, y) = 0$. This geometric interpretation explains the "rigidity" of complex holomorphic functions when compared with real differentiable functions. In particular, a holomorphic function is ***conformal***, i.e. it "preserves angles", wherever its derivative is non-zero.




$f$ "preserves angles" because all tangent vectors are scaled and rotated by the same matrix.

**Remark 2.65** Recall from MATH2023 that the gradient vectors of a function of two real variables are always perpendicular to its level curves. Now if the pair of functions $(u, v)$ satisfies the **Cauchy-Riemann equations**, then at each point in their domain we have

$$\nabla v = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{pmatrix} \nabla u,$$

i.e. $\nabla v$ can be obtained from $\nabla u$ via a **counterclockwise rotation by a right angle**. In particular, this implies that $\nabla u$ and $\nabla v$ are perpendicular to each other at each point, and explains why in Example 2.6 the orange and blue level curves (i.e. the level curves of $u$ and $v$) are always perpendicular to each other.

## 4. Power series

Our next goal is to study **power series**, which is a generalization of polynomials to infinitely many terms, as in the real case. As we will see later (in Theorem 2.89), power series provide a large class of **examples of holomorphic functions**. We need some preparation before we study power series.

**Definition 2.66** Let $U \subseteq \mathbb{C}$, let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions $f_n: U \to \mathbb{C}$, and let $f: U \to \mathbb{C}$ be a function. We say that the sequence of functions $\{f_n\}$ ***converges pointwise on U*** (to the function $f$) if for each $\varepsilon > 0$ and for each $z \in U$, there exists $N \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \varepsilon \qquad \text{whenever } n \geq N.$$

$N$ may depend on different $z \in U$.

In other words,

$$\lim_{n\to+\infty} f_n(z) = f(z) \qquad \text{for each } z \in U.$$

**Definition 2.67** Let $U \subseteq \mathbb{C}$, let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions $f_n: U \to \mathbb{C}$, and let $f: U \to \mathbb{C}$ be a function. We say that the sequence of functions $\{f_n\}$ ***converges uniformly on U*** (to the function $f$) if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $z \in U$ and all $n \geq N$,

$$|f_n(z) - f(z)| < \varepsilon.$$

The same $N$ works for **all** $z \in U$.

In other words,

$$\lim_{n\to+\infty} \sup\{|f_n(z) - f(z)| : z \in U\} = 0.$$

**Remark 2.68** It is essential to **always specify the set** on which a sequence of functions converges uniformly. It is meaningless to just say that "$\{f_n\}$ converges uniformly" instead of saying that "$\{f_n\}$ converges uniformly on $U$".

**Remark 2.69** If $\{f_n\}$ converges pointwise (resp. uniformly) on $U$, then it converges pointwise (resp. uniformly) on every subset of $U$.

**Corollary 2.70 (Uniform convergence $\Rightarrow$ Pointwise convergence)** *Let $U \subseteq \mathbb{C}$ and $\{f_n\}$ be a sequence of functions $f_n: U \to \mathbb{C}$. If $\{f_n\}$ converges uniformly on $U$, then $\{f_n\}$ converges pointwise on $U$.*

**Corollary 2.71 (Uniform limit $\equiv$ Pointwise limit)** *Let $U \subseteq \mathbb{C}$ and $\{f_n\}$ be a sequence of functions $f_n: U \to \mathbb{C}$. If $\{f_n\}$ converges uniformly on $U$ and converges pointwise on $U$ to a function $f: U \to \mathbb{C}$, then $\{f_n\}$ converges uniformly on $U$ to the same limit function $f$.*

**Example 2.72** For each $n \in \mathbb{N}$, let $f_n: \mathbb{C} \to \mathbb{C}$ be the function defined by

$$f_n(z) = \frac{z^n}{n}.$$

Show that $\{f_n\}$ converges uniformly on $\overline{D(0;1)}$.

*Proof*:

For every $z \in \overline{D(0;1)}$ we have $\lim_{n\to+\infty} f_n(z) = \lim_{n\to+\infty} \frac{z^n}{n} = 0$, so the pointwise limit on $\overline{D(0;1)}$ is the constant function $f(z) = 0$. Now for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. Then for every $z \in \overline{D(0;1)}$ and every $n \geq N$, we have

$$|f_n(z) - f(z)| = \left|\frac{z^n}{n} - 0\right| = \frac{|z|^n}{n} \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon,$$

so $\{f_n\}$ converges uniformly on $\overline{D(0;1)}$ to the constant function $0$.

**Example 2.73** For each $n \in \mathbb{N}$, let $f_n: \mathbb{C} \to \mathbb{C}$ be the function defined by

$$f_n(z) = z^n.$$

Show that $\{f_n\}$ converges pointwise but not uniformly on $D(0;1)$.

*Proof*:

For every $z \in D(0;1)$ we have $\lim_{n\to+\infty} f_n(z) = \lim_{n\to+\infty} z^n = 0$, so the pointwise limit on $D(0;1)$ is the constant function $f(z) = 0$. However, for every $N \in \mathbb{N}$, we have

$$\left|f_{N+1}\left(2^{-\frac{1}{N+2}}\right) - f\left(2^{-\frac{1}{N+2}}\right)\right| = 2^{-\frac{N+1}{N+2}} - 0 > \frac{1}{2}$$

even though $2^{-\frac{1}{N+2}} \in D(0;1)$ and $N+1 > N$. Therefore $\{f_n\}$ does not converge uniformly on $D(0;1)$ to the constant function $0$, and so by Corollary 2.71, $\{f_n\}$ does not converge uniformly on $D(0;1)$. ∎

A **uniformly convergent** sequence of functions preserves in its limit many good properties that each of its terms possesses. The following theorem says that it **preserves continuity** in the limit, and we will see many other such properties later.

**Theorem 2.74** *Let $U \subseteq \mathbb{C}$ and $f_n: U \to \mathbb{C}$ be continuous functions. If $\{f_n\}$ converges uniformly on $U$ to a function $f: U \to \mathbb{C}$, then $f$ is also continuous. In other words, the uniform limit of a sequence of continuous functions is also continuous.*

*Proof*. [The "$\varepsilon/3$ trick".] Let $a \in U$ and $\varepsilon > 0$ be arbitrary.

- ⊙ Since $\{f_n\}$ converges uniformly on $U$ to $f$, there exists $N \in \mathbb{N}$ such that
$$|f_n(z) - f(z)| < \frac{\varepsilon}{3} \qquad \text{for all } n \geq N \text{ and all } z \in U.$$
- ⊙ Since $f_{N+1}$ is continuous at $a$, there exists $\delta > 0$ such that
$$|f_{N+1}(z) - f_{N+1}(a)| < \frac{\varepsilon}{3} \qquad \text{whenever } z \in U \text{ and } |z - a| < \delta.$$

Therefore whenever $z \in U$ and $|z - a| < \delta$, we have
$$\begin{aligned} |f(z) - f(a)| &\leq |f(z) - f_{N+1}(z)| + |f_{N+1}(z) - f_{N+1}(a)| + |f_{N+1}(a) - f(a)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$
which shows that $f$ is continuous at $a$. ■

**Remark 2.75** The **pointwise limit** of a sequence of continuous functions may not be continuous. As a quick counter-example, let $f_n(z) = z^n$. Then each $f_n$ is continuous on $U = D(0;1) \cup \{1\}$ and $\{f_n\}$ converges pointwise on $U$, but its pointwise limit function is discontinuous at $1$.

**Theorem 2.76 (Cauchy criterion)** *Let $U \subseteq \mathbb{C}$ and let $\{f_n\}$ be a sequence of functions $f_n: U \to \mathbb{C}$. $\{f_n\}$ converges uniformly on $U$ if and only if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $z \in U$ and all $m, n \geq N$,*
$$|f_n(z) - f_m(z)| < \varepsilon.$$

*Proof*. ($\Rightarrow$) Let $\varepsilon > 0$ be arbitrary. Since $\{f_n\}$ converges uniformly on $U$ to $f$, there exists $N \in \mathbb{N}$ such that
$$|f_n(z) - f(z)| < \frac{\varepsilon}{2} \qquad \text{for all } n \geq N \text{ and all } z \in U.$$
Thus, for all $z \in U$ and all $m, n \geq N$, we have
$$|f_n(z) - f_m(z)| \leq |f_n(z) - f(z)| + |f(z) - f_m(z)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

($\Leftarrow$) Let $\varepsilon > 0$ be arbitrary.

⊙ By the assumption, there exists $N \in \mathbb{N}$ such that

$$|f_n(z) - f_m(z)| < \frac{\varepsilon}{2} \qquad \text{for all } m, n \geq N \text{ and all } z \in U.$$

⊙ For each $z \in U$, $\{f_n(z)\}$ is a Cauchy sequence (of complex numbers) by the assumption, so it converges to a limit by completeness of $\mathbb{C}$. Let's call such a limit $f(z)$. Then there exists $k \in \mathbb{N}$ ($k$ may depend on $z$) such that

$$|f_n(z) - f(z)| < \frac{\varepsilon}{2} \qquad \text{whenever } n \geq k.$$

Now for all $z \in U$ and all $n \geq N$, we have

$$|f_n(z) - f(z)| \leq |f_n(z) - f_{n+k}(z)| + |f_{n+k}(z) - f(z)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

so $\{f_n\}$ converges uniformly on $U$ to the function $f$. ■

**Definition 2.77** Let $U \subseteq \mathbb{C}$ and let $\{f_n\}$ be a sequence of functions $f_n: U \to \mathbb{C}$. A ***series*** of functions, denoted by

$$\sum_{k=0}^{+\infty} f_k,$$

is defined as the sequence of its partial sums, i.e. $\{\sum_{k=0}^{n} f_k\}_n$. We say that a series ***converges pointwise*** (resp. ***uniformly***) ***on U*** if it converges pointwise (resp. uniformly) on $U$ as a sequence. We often abuse the notation and denote the limit of a series also by the same notation $\sum_{k=0}^{+\infty} f_k$, if such a limit exists.

For series, there is a notion of **absolute convergence**. If we replace the word "convergent" in Definition 2.66 – 2.67 by "absolutely-convergent", we obtain the following notions.

**Definition 2.78** Let $U \subseteq \mathbb{C}$ and let $\{f_n\}$ be a sequence of functions $f_n: U \to \mathbb{C}$. We say that the series of functions $\sum_{k=0}^{+\infty} f_k$ is

(i) ***pointwise absolutely-convergent on U*** if $\sum_{k=0}^{+\infty} |f_k|$ converges pointwise on $U$, i.e. the series of complex numbers $\sum_{k=0}^{+\infty} |f_k(z)|$ converges for each $z \in U$;

(ii) ***uniformly absolutely-convergent on U*** if $\sum_{k=0}^{+\infty} |f_k|$ converges uniformly on $U$.

**Lemma 2.79** A series of functions converges pointwise (resp. uniformly) on a set if it is pointwise (resp. uniformly) absolutely-convergent on the set.

*Proof*. Apply Cauchy criterion to $|\sum_{k=n+1}^{m} f_k| \leq \sum_{k=n+1}^{m} |f_k|$. ■

**Weierstrass' $\boldsymbol{M}$-test** is a convenient method of proving the **uniform convergence** of a **series of functions**.

**Theorem 2.80 (Weierstrass' $\boldsymbol{M}$-test)** *Let $U \subseteq \mathbb{C}$, let $\{M_n\}$ be a sequence of non-negative real numbers and let $\{f_n\}$ be a sequence of functions $f_n: U \to \mathbb{C}$ such that $f_n$ is bounded above by $M_n$ for each $n$, i.e.*

$$|f_n(z)| \leq M_n$$

*for every $z \in U$. If the series of real numbers $\sum_{k=0}^{+\infty} M_k$ converges, then the series of functions $\sum_{k=0}^{+\infty} f_k$ is uniformly absolutely-convergent on $U$.*

*Proof*. Since $\sum_{k=0}^{+\infty} M_k$ converges, for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{k=m+1}^{n} M_k < \varepsilon \qquad \text{whenever } n > m \geq N.$$

Now whenever $n > m \geq N$, we have

$$\sum_{k=m+1}^{n} |f_k(z)| \leq \sum_{k=m+1}^{n} M_k < \varepsilon$$

for every $z \in U$, so $\sum_{k=0}^{+\infty} |f_k|$ converges uniformly on $U$ by Cauchy criterion. ∎

With all the previous preparation, we are now ready to study **power series**.

**Definition 2.81** Let $a \in \mathbb{C}$ and let $\{a_n\}$ be a sequence of complex numbers. A ***power series*** is a series of functions of the form

$$f(z) = \sum_{k=0}^{+\infty} a_k (z-a)^k,$$

i.e. a series of monomials in $(z-a)$. The number $a$ is called the ***center*** of the power series $f$, and the $a_n$'s are called the ***coefficients*** of $f$.

**Remark 2.82** Note that the domain of a power series $f$ is not specified in Definition 2.81, although it is obvious that this domain must at least contain the center $a$. A main goal of this section is to study the **natural domain** of a power series.

**Definition 2.83** Let $a \in \mathbb{C}$ and let $\{a_n\}$ be a sequence of complex numbers. The ***radius of convergence*** of the power series $\sum_{k=0}^{+\infty} a_k(z-a)^k$ is the number $R \in [0, +\infty]$ defined by

$$R := \frac{1}{\limsup_n |a_n|^{\frac{1}{n}}}.$$

Note that $\limsup|a_n|^{1/n}$ always belongs to $[0, +\infty]$. If $\limsup|a_n|^{1/n} = 0$, then $R$ is understood as $+\infty$; if $\limsup|a_n|^{1/n} = +\infty$, then $R$ is understood as $0$.

The ***disk of convergence*** of the power series $\sum_{k=0}^{+\infty} a_k(z-a)^k$ is

- $D(a; R)$ if $0 < R < +\infty$;
- $\mathbb{C}$ if $R = +\infty$;
- the singleton $\{a\}$ (but <u>not</u> the empty set $\emptyset$) if $R = 0$.

Please refer to Supplementary Note C if you have not taken MATH2043/3033 or is not familiar with the **limit superior** of a sequence of real numbers, which was used in the above definition.

**Example 2.84** Find the disk of convergence of the power series

$$\frac{1}{3} + \frac{3-z}{3^2} + \frac{(3-z)^2}{3^3} + \cdots.$$

*Solution*: The power series can be rewritten as

$$\frac{1}{3} + \frac{3-z}{3^2} + \frac{(3-z)^2}{3^3} + \cdots = \sum_{k=0}^{+\infty} \frac{(3-z)^k}{3^{k+1}} = \sum_{k=0}^{+\infty} \frac{(-1)^k}{3^{k+1}} (z-3)^k,$$

so $R = \dfrac{1}{\limsup_n \left|\frac{(-1)^n}{3^{n+1}}\right|^{1/n}} = \dfrac{1}{\lim_{n\to+\infty} 3^{-\frac{n+1}{n}}} = 3$. The disk of convergence of this power series is $D(3; 3)$.

**Lemma 2.85** *Let $L \in [0, +\infty]$ and let $\{x_n\}$ be a sequence of positive real numbers such that $\lim_{n\to+\infty} \frac{x_{n+1}}{x_n} = L$. Then $\lim_{n\to+\infty} \sqrt[n]{x_n} = L$.*

*Proof.* Omitted. A good exercise in mathematical analysis (MATH1023/2033). ■

**Corollary 2.86** *Let $a \in \mathbb{C}$ and let $\{a_n\}$ be a sequence of complex numbers. If $\lim_{n\to+\infty} \left|\frac{a_n}{a_{n+1}}\right|$ exists (either as a non-negative real number or $+\infty$), then the radius of convergence of the power series $\sum_{k=0}^{+\infty} a_k(z-a)^k$ is also given by*

$$R = \lim_{n\to+\infty} \left|\frac{a_n}{a_{n+1}}\right|.$$

**Example 2.87** Find the disk of convergence of each of the following power series.

(a) $\sum_{k=0}^{+\infty} \frac{z^k}{k!}$ (b) $\sum_{k=1}^{+\infty} \frac{(z+3)^k}{k^3 2^{k+1}}$ (c) $\sum_{k=0}^{+\infty} k!\, z^k$

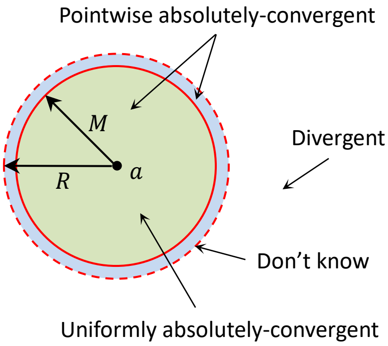
*Solution*:

(a) The radius of convergence is $R = \lim_{n\to+\infty} \left|\frac{1/n!}{1/(n+1)!}\right| = \lim_{n\to+\infty}(n+1) = +\infty$. So the disk of convergence is $\mathbb{C}$.

(b) The radius of convergence is $R = \lim_{n\to+\infty} \left|\frac{1/n^3 2^{n+1}}{1/(n+1)^3 2^{n+2}}\right| = \lim_{n\to+\infty} 2\left(1+\frac{1}{n}\right)^3 = 2$. So the disk of convergence is $D(-3;2)$.

(c) The radius of convergence is $R = \lim_{n\to+\infty} \left|\frac{n!}{(n+1)!}\right| = \lim_{n\to+\infty} \frac{1}{n+1} = 0$. So the disk of convergence is a singleton $\{0\}$ which consists of just the center.

**Theorem 2.88 (Cauchy-Hadamard)** *Let $a \in \mathbb{C}$ and let $\{a_n\}$ be a sequence of complex numbers. Let $R$ be the radius of convergence of the power series $\sum_{k=0}^{+\infty} a_k (z-a)^k$. Then*
(i) *this power series is uniformly absolutely-convergent on every compact subset of $D(a;R)$;*
(ii) *this power series diverges at every point in $\mathbb{C} \setminus \overline{D(a;R)}$.*




*Proof*. To prove (i), we assume that $R > 0$ as the case $R = 0$ is trivial. Let $K$ be a compact subset of $D(a;R)$, then the continuous function $d(z) \coloneqq |z-a|$ attains maximum on $K$, so we may set

$$M \coloneqq \max\{|z-a| : z \in K\} < R.$$

Now choose $\rho \in (M,R)$. Then we have

$$\frac{1}{\rho} > \limsup_n |a_n|^{\frac{1}{n}},$$

so there exists $N \in \mathbb{N}$ such that

$$|a_n|^{\frac{1}{n}} < \frac{1}{\rho} \qquad \text{whenever } n \geq N.$$

Now for each $n$ we have

$$|a_n(z-a)^n| = |a_n||z-a|^n < |a_n|M^n$$

for every $z \in K$. Since $|a_n|M^n < \left(\frac{M}{\rho}\right)^n$ for every $n \geq N$ and the geometric series $\sum_{k=0}^{+\infty}\left(\frac{M}{\rho}\right)^k$ converges, the series of real numbers $\sum_{k=0}^{+\infty}|a_k|M^k$ also converges by comparison test. Therefore the power series is uniformly absolutely-convergent on $K$ by Weierstrass' $M$-test.

Next to prove (ii), we assume that $R < +\infty$ as the case $R = +\infty$ is trivial. Now for each point $z \in \mathbb{C} \setminus \overline{D(a;R)}$, we have $|z-a| > R$. Since

$$\frac{1}{|z-a|} < \limsup_n |a_n|^{\frac{1}{n}},$$

there exists infinitely many $n \in \mathbb{N}$ such that

$$|a_n|^{\frac{1}{n}} > \frac{1}{|z-a|}.$$

So for these infinitely many $n$ we have

$$|a_n(z-a)^n| = |a_n||z-a|^n > 1.$$

Now there are infinitely many terms in the sequence of complex numbers $\{a_n(z-a)^n\}$ whose absolute value is $1$, so $\{a_n(z-a)^n\}$ does not converge to $0$. Therefore the series of complex numbers $\sum_{k=0}^{+\infty} a_k(z-a)^k$ diverges by term test (Lemma 1.35). ∎

The following main theorem of this section shows that **power series** provide a large class of **examples of holomorphic functions on open disks**.

**Theorem 2.89** *Let $a \in \mathbb{C}$ and let $\{a_n\}$ be a sequence of complex numbers. If the power series $\sum_{k=0}^{+\infty} a_k(z-a)^k$ has a positive radius of convergence $R > 0$, then the function $f: D(a;R) \to \mathbb{C}$ defined by*

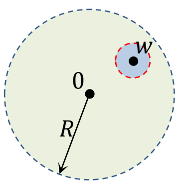
$$f(z) = \sum_{k=0}^{+\infty} a_k(z-a)^k$$

*is holomorphic, and its derivative is given by another power series*

$$f'(z) = \sum_{k=1}^{+\infty} k a_k (z-a)^{k-1} = \sum_{k=0}^{+\infty} (k+1) a_{k+1} (z-a)^k.$$

i.e. a power series can be differentiated "**term-by-term**".

*Proof*. Without loss of generality we consider $a = 0$ only. Let $w \in D(0;R)$. The issue is to show that $f$ is differentiable at $w$.




Now for every $z \in D\left(w; \frac{R-|w|}{2}\right)$, we have

$$|z| \le |z-w| + |w| < \frac{R-|w|}{2} + |w| = \frac{R+|w|}{2},$$

so the power series $f$ converges absolutely both at $z$ and at $w$.

So we may rearrange the terms of the power series $f$ and obtain

$$\left|\frac{f(z)-f(w)}{z-w}-\sum_{k=1}^{+\infty} k a_k w^{k-1}\right| = \left|\sum_{k=1}^{+\infty} a_k \frac{z^k-w^k}{z-w}-\sum_{k=1}^{+\infty} k a_k w^{k-1}\right|$$

$$=\left|\sum_{k=1}^{+\infty} a_k(z^{k-1}+z^{k-2}w+\cdots+zw^{k-2}+w^{k-1})-\sum_{k=1}^{+\infty} k a_k w^{k-1}\right|$$

$$=\left|\sum_{k=2}^{+\infty} a_k(z^{k-1}+z^{k-2}w+\cdots+zw^{k-2}-(k-1)w^{k-1})\right|$$

$$=\left|(z-w)\sum_{k=2}^{+\infty} a_k(z^{k-2}+2z^{k-3}w+\cdots+(k-2)zw^{k-3}+(k-1)w^{k-2})\right|$$

$$\le |z-w|\sum_{k=2}^{+\infty}|a_k|\frac{k(k-1)}{2}\left(\frac{R+|w|}{2}\right)^{k-2},$$

which tends to zero as $z$ tends to $w$, because the final series of real numbers converges to some non-negative real number $M$ by root test (how?). ■

**Example 2.90** We have seen in Example 2.87 (a) that the power series

$$\sum_{k=0}^{+\infty}\frac{z^k}{k!}$$

has radius of convergence $R=+\infty$, so by Theorem 2.89 this power series defines a holomorphic function on $\mathbb{C}$, i.e. an entire function. So apart from Example 2.50, this gives an alternative proof of the fact that the exponential function is entire.

Theorem 2.89 can be applied again to the derivative $f'$, because $f'$ is also a power series with the same disk of convergence. So by induction we have the following.

**Corollary 2.91** *A power series is infinitely many times differentiable on its disk of convergence.*

**Corollary 2.92** *If a power series* $f(z)=\sum_{k=0}^{+\infty} a_k(z-a)^k$ *has a positive radius of convergence* $R>0$*, then for each* $n\in\mathbb{N}$*, we have*

$$f^{(n)}(a)=n!\,a_n.$$

**Example 2.93** Let $r > 0$ and suppose that the power series $\sum_{k=0}^{+\infty} a_k z^k$ converges pointwise on $D(0;r)$ to a function $f: D(0;r) \to \mathbb{C}$. Express the power series $\sum_{k=1}^{+\infty} k^2 a_k z^k$ in terms of $f$ and its derivatives.

*Solution*:
Since the given power series converges pointwise on $D(0;r)$, its radius of convergence is at least $r$. So by Theorem 2.89, the limit function $f$ is holomorphic function on $D(0;r)$, and term-by-term differentiation gives

$$f'(z) = \sum_{k=1}^{+\infty} k a_k z^{k-1} \qquad \text{and} \qquad f''(z) = \sum_{k=2}^{+\infty} k(k-1) a_k z^{k-2}$$

for every $z \in D(0;r)$, and so

$$zf'(z) = \sum_{k=1}^{+\infty} k a_k z^{k} \qquad \text{and} \qquad z^2 f''(z) = \sum_{k=2}^{+\infty} k(k-1) a_k z^{k}$$

for every $z \in D(0;r)$. Therefore

$$\sum_{k=1}^{+\infty} k^2 a_k z^k = \sum_{k=1}^{+\infty} [k(k-1)+k] a_k z^k = \sum_{k=2}^{+\infty} k(k-1) a_k z^k + \sum_{k=1}^{+\infty} k a_k z^k = z^2 f''(z) + zf'(z)$$

for every $z \in D(0;r)$.

The following theorem says that the only power series that has zeros (roots) converging to its center is the zero power series.

**Theorem 2.94 (Uniqueness of power series)** *Let $f(z) = \sum_{k=0}^{+\infty} a_k z^k$ be a power series. If there exists a sequence of complex numbers $\{z_n\}$ converging to $0$ such that $z_n \neq 0$ for all $n \in \mathbb{N}$ and*

$$f(z_n) = 0$$

*for all $n \in \mathbb{N}$, then $f$ must be the zero power series, i.e. $a_n = 0$ for all $n$.*

*Proof*. First of all, we have

$$a_0 = f(0) = \lim_{n \to +\infty} f(z_n) = \lim_{n \to +\infty} 0 = 0$$

by the continuity of $f$. Now we assume that the first $N$ coefficients of $f$ are all $0$, i.e.

$$a_0 = a_1 = \cdots = a_{N-1} = 0$$

for some $N \geq 1$, so that $f(z) = \sum_{k=N}^{+\infty} a_k z^k$. If we let $g$ be the power series

$$g(z) = \sum_{k=0}^{+\infty} a_{k+N} z^k,$$

then we have $g(z) = \frac{f(z)}{z^N}$ for every $z \in D(0, |z_1|) \setminus \{0\}$. This implies that

$$a_N = g(0) = \lim_{z \to 0} g(z) = \lim_{z \to 0} \frac{f(z)}{z^N} = \lim_{n \to +\infty} \frac{f(z_n)}{{z_n}^N} = \lim_{n \to +\infty} 0 = 0.$$

Consequently, we have $a_n = 0$ for all $n$ by mathematical induction. ■

**Corollary 2.95 (Uniqueness of power series)** *Let $a \in \mathbb{C}$ and let $f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k$ and $g(z) = \sum_{k=0}^{\infty} b_k (z-a)^k$ be power series both centered at $a$. If there exists a sequence of complex numbers $\{z_n\}$ converging to $a$ such that $z_n \neq a$ for all $n \in \mathbb{N}$ and*

$$f(z_n) = g(z_n)$$

*for all $n \in \mathbb{N}$, then $f$ and $g$ must be the same power series, i.e. $a_n = b_n$ for all $n$.*

**Example 2.96** Does there exist a power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ such that

$$f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n^2}$$

for every $n \in \mathbb{N}$?

*Solution*:

Suppose that there exists a power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ centered at $0$ such that

$$f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n^2}$$

for every $n \in \mathbb{N}$. Since $f$ converges at $1$, the radius of convergence of $f$ is at least $1$ by Theorem 2.88 (ii). Now consider the two power series

$$g(z) = z^2 \qquad \text{and} \qquad h(z) = -z^2$$

centered at $0$ (in particular, they both have only one non-zero coefficient). We have

$$f\left(\frac{1}{2m}\right) = \frac{1}{(2m)^2} = g\left(\frac{1}{2m}\right) \qquad \text{and} \qquad f\left(\frac{1}{2m+1}\right) = -\frac{1}{(2m+1)^2} = h\left(\frac{1}{2m+1}\right)$$

for every $m \in \mathbb{N}$. Since $\left\{\frac{1}{2m}\right\}$ and $\left\{\frac{1}{2m+1}\right\}$ are both sequences with non-zero terms that converge to $0$, by the uniqueness of power series we have

$$f = g \qquad \text{and} \qquad f = h$$

on $D(0; 1)$. This implies that $g = h$ on $D(0; 1)$, which is obviously a contradiction. Therefore such a power series $f$ does not exist.

## Summary of Chapter 2

The following are what you need to know in this chapter in order to get a pass (a distinction) in this course:

- ✓ **Functions of a complex variable**
  - ⊙ **Domain**, **codomain** and **range**
  - ⊙ Visual presentations
  - ⊙ **Direct image** and **inverse image**

- ✓ **Limit** and **continuity** of functions of a complex variable
  - ⊙ $\varepsilon$-$\delta$ **definition of limit** of a function of a complex variable
  - ⊙ **Arithmetic operations** of limits, **two-path test** for **non-existence** of limits
  - ⊙ **Infinite limits** and **limits at infinity**, **stereographic projection**, **complex infinity** $\infty$
  - ⊙ $\varepsilon$-$\delta$ **definition of continuity**
  - ⊙ **Topological** characterizations or properties of continuous functions: $f^{-1}(\text{open})$ is open; $f^{-1}(\text{closed})$ is closed; $f(\text{compact})$ is compact; $f(\text{connected})$ is connected
  - ⊙ **Uniform continuity**; continuous functions on compact set are uniformly continuous

- ✓ **Holomorphic functions**
  - ⊙ **Complex differentiability** at a point
  - ⊙ **Cauchy-Riemann equations**, geometric interpretations of Cauchy-Riemann equations
  - ⊙ **Necessary** and **sufficient conditions** of complex differentiability at a point
  - ⊙ **Holomorphic**: differentiable <u>near</u> a point
  - ⊙ **Holomorphic functions** on **open sets**, **entire functions**
  - ⊙ **Examples** of holomorphic functions: Polynomials, rational functions, **exponential** function, **trigonometric** functions, **power series** (with positive radius of convergence)
  - ⊙ How is a holomorphic function different from a $C^1$ vector field in two real dimensions?

- ✓ **Sequences** and **series** of functions
  - ⊙ **Pointwise** and **uniform convergence** of sequences and series of functions
  - ⊙ **Cauchy criterion**
  - ⊙ **Continuity theorem**: Uniform limit of a sequence of continuous functions is continuous
  - ⊙ **Pointwise** and **uniform absolute-convergence** of series of functions
  - ⊙ **Weierstrass' $M$-test** for series of functions
  - ⊙ **Power series**, **radius of convergence** and **disk of convergence**
  - ⊙ **Term-by-term differentiation** of power series
  - ⊙ **Uniqueness** of power series