MATH4023 Complex Analysis
Final Examination
(May 23, 2022; 16:30 – 19:30)

Name: ______________________________________

Student ID Number: ______________________________________

Directions:

- This is an open-book examination. You may refer to any notes, textbooks or references offline, but notations from our lecture note should be used. You may use a calculator if needed, but all numerical answers should be given in the **exact form**.
- **DO NOT communicate with any person other than your course instructor by any means during the exam.** Physical, electronic, and internet communication with any other person are all prohibited.
- **Please write your solutions to all problems on your own pieces of paper. Write your name, student ID and signature on each page of your work.**
- You are advised to try the problems you feel more confident first, and not to spend too much time on one single problem in case you get stuck. Some problems can be tricky so be careful.
- Please submit (i) this cover page and (ii) your solutions to all problems via Canvas **NO LATER THAN 19:45**. Late submission will result in score deduction.
- Cheating is a serious violation of the HKUST Academic Code. Students caught cheating will receive a zero score for the examination, and will also be subjected to further penalties imposed by the University.

Please read the following statement and sign below it.

I have neither given nor received any unauthorized aid during this examination. The answers submitted are my own work.

I understand that sanctions will be imposed, if I am found to have violated the University's regulations governing academic integrity.

Student's Signature:

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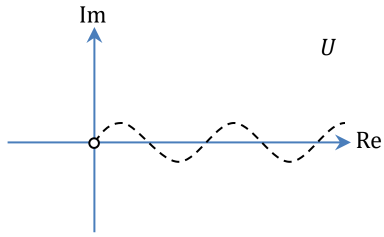
Question No.	Score	Out of
Q1 – Q6		27
Q7		6
Q8		8
Q9		8
Q10		13
Q11		12
Q12		8
Q13		8
Q14		10
Total score		**100**

Part A: Brief responses (27 points)

Write down **brief answers** to each of the following questions Q1 – Q6. **You do not need to provide any explanation or computational steps**. You are suggested to spend at most about **45 minutes** on part A in order to solve all problems in time.

1. Let $U = \mathbb{C} \setminus \{x + iy \in \mathbb{C}: x \geq 0 \text{ and } y = \sin x\}$, which is a simply connected region that does not contain 0. Let $\log: U \to \mathbb{C}$ be the holomorphic branch of complex logarithm such that



$$\log 1 = 0.$$

   (a) What is the value of $\log i$?

   (b) What is the value of 5^i?

   Write your answers either in **standard form** $a + bi$ or in **polar form** $re^{i\theta}$. (2 points)

2. Let f be the complex function

$$f(z) = \frac{1}{z^2(z-3)(z+4i)^3}.$$

   (a) Write down all the possible **annuli of convergence** for all Laurent series of f of the form $\sum_{k=-\infty}^{+\infty} a_k (z-3)^k$.

   (b) What is the **radius of convergence** of the **Taylor series of f at πi**?

   (4 points)

3. Consider the following complex functions:

$$f(z) = e^{\frac{1}{\cos z}}, \qquad g(z) = \frac{z}{\sin^2 z}, \qquad h(z) = \frac{(z-i)^2}{z^2+1}.$$

   For **each** of these functions,

   (i) write down all its **isolated singularities** in $\mathbb{C}$;

   (ii) classify each isolated singularity as a **removable singularity**, a **pole**, or an **essential singularity**; if it is a pole, also state the **order of the pole**. (6 points)

4. Let f be the function

$$f(z) = z^3 \cos\frac{1}{z}.$$

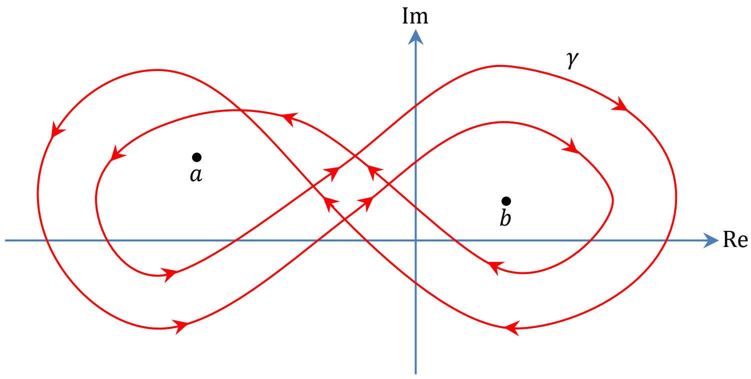
   (a) Compute the residue $\operatorname{Res}(f; 0)$.

   (b) Compute the line integral

$$\oint_{\partial D(0;10)} f(z)\,dz.$$

   (3 points)

5. The following diagram shows the image of a closed C^1 curve γ in $\mathbb{C}$, together with the positions of two complex numbers a and b.



(a) Compute the **winding numbers** $n(\gamma; a)$ and $n(\gamma; b)$.

(b) Let $f: \mathbb{C} \setminus \{b\} \to \mathbb{C}$ be a holomorphic function, whose Laurent series at b is

$$\sum_{k=-\infty}^{+\infty} a_k (z-b)^k .$$

Compute the following **line integrals** in terms of b and the Laurent series coefficients.

(i) $\oint_\gamma f(z)dz$

(ii) $\oint_\gamma z f(z)dz$

(c) Let φ be a counterclockwise oriented simple closed C^1 curve in $\mathbb{C}$, and let g be a meromorphic function on $\mathbb{C}$ such that g has no zeros or poles on image φ. Suppose that g has **8 poles in the interior of $\boldsymbol{\varphi}$**, counting multiplicities. If $g \circ \varphi$ is the closed C^1 curve γ whose image is shown above,

(i) **how many zeros in the interior of $\boldsymbol{\varphi}$** does g have, counting multiplicities?

(ii) **how many zeros in the interior of $\boldsymbol{\varphi}$** does $g - a$ have, counting multiplicities?

(8 points)

6. Give an explicit example of each of the following:

(a) **Two power series** f and g in the complex variable z, both centered at 0, such that

⦿ f and g both have radius of convergence 1, but

⦿ the power series $f + g$ has radius of convergence strictly greater than 1.

(b) **A function** $f: \mathbb{C} \setminus \{i, -i\} \to \mathbb{C}$ which

⦿ has simple zeros at 0 and at 1,

⦿ has double poles at i and at $-i$,

⦿ is holomorphic on $\mathbb{C} \setminus \{i, -i\}$.

(4 points)

Part B: Short problems (73 points)

Show all your work in detail for each of the following problems Q7 – Q14:

- ⊙ Make clear definition to any mathematical symbol you use, whenever it is not given in the context of the problem and not defined in our lecture note.
- ⊙ Diagrams may help explaining your work, but will not be counted as any proof on its own.
- ⊙ You may apply without proof **any result from Chapter 1 – Chapter 5** of our lecture note.

7. Let $f: \overline{D(0;1)} \to \mathbb{C}$ be a continuous function which is holomorphic on $D(0;1)$, and let M, N be non-negative real numbers. If

$$\left|f\left(e^{i\theta}\right)\right| \leq M \qquad \text{for every } \theta \in [0, \pi] \qquad \text{and}$$
$$\left|f\left(e^{i\theta}\right)\right| \leq N \qquad \text{for every } \theta \in [\pi, 2\pi],$$

show that

$$|f(0)| \leq \sqrt{MN}.$$

(6 points)

8. Let $f: \mathbb{C} \to \mathbb{C}$ be an entire function such that

$$\operatorname{Re} f(z) \neq \operatorname{Im} f(z)$$

for any $z \in \mathbb{C}$. Show that f is a constant function.

(8 points)

9. Let $f: \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be a holomorphic function such that

$$f(z) = f\left(\frac{1}{z}\right)$$

for every $z \in \mathbb{C} \setminus \{0\}$. If $f(z) \in \mathbb{R}$ for every $z \in \partial D(0;1)$, show that $f(z) \in \mathbb{R}$ for every $z \in \mathbb{R} \setminus \{0\}$.

Hint: Schwarz reflection principle may be useful.

(8 points)

10. Let $f: \mathbb{C} \setminus \{0, 2, 3\} \to \mathbb{C}$ be the function

$$f(z) = \frac{1}{z} + \frac{1}{(z-2)^2} + \frac{1}{z-3}.$$

(a) Compute the Taylor series of f at 1. What is its disk of convergence?

(7 points)

(b) Compute the Laurent series of f **centered at 3** which **converges at 1**. What is its annulus of convergence?

(6 points)

11. Let $a \in (0, \pi)$. Evaluate the improper integral

$$\int_{-\infty}^{+\infty} \frac{\cos x}{x^2 - 2x\cos a + 1}\,dx$$

in terms of a, **by considering a line integral of a relevant complex holomorphic function**.

Note: If you obtain the numerical answer just by transforming some other known real integrals, or by any other method that does not involve complex analysis, you will receive credits for the numerical answer only.

(12 points)

12. Let $\varphi\colon [0,1] \to \mathbb{C}$ be a closed C^1 curve, let $a \in \mathbb{C} \setminus (\operatorname{image} \varphi)$, and let $\gamma\colon [0,1] \to \mathbb{C}$ be a closed C^1 curve such that

$$|\gamma(t) - \varphi(t)| < |\varphi(t) - a|$$

for every $t \in [0,1]$. Show that

$$n(\gamma; a) = n(\varphi; a).$$

Hint: It may be useful to consider the function $\psi\colon [0,1] \to \mathbb{C}$ defined by $\psi(t) = \frac{\gamma(t)-a}{\varphi(t)-a}$.

Pictorial proof will not be accepted.

(8 points)

13. Let $f\colon \mathbb{C} \to \mathbb{C}$ be the polynomial

$$f(z) = z^5 - 3z^4 + 2z - 10i.$$

How many zeros of f are there in the annulus $A(0; 1, 2)$, counting multiplicities?

(8 points)

14. Let n be a positive integer and let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function such that

$$|f(z+w)|^{\frac{1}{n}} \le |f(z)|^{\frac{1}{n}} + |f(w)|^{\frac{1}{n}}$$

for every $z, w \in \mathbb{C}$.

(a) Using mathematical induction, show that

$$|f(kz)|^{\frac{1}{n}} \le k|f(z)|^{\frac{1}{n}}$$

for every positive integer k and every $z \in \mathbb{C}$.

(2 points)

(b) Using (a) or otherwise, show that f is a polynomial of degree at most n.

(8 points)

END OF PAPER