Chapter 4 Consequences of the Cauchy integral formula

1. Taylor series of a function

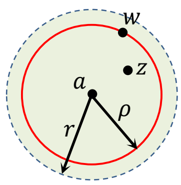
In chapter 2, we have seen that a **power series** with positive radius of convergence defines a holomorphic function on its disk of convergence. Now using Cauchy integral formula, it turns out conversely that a holomorphic function f on a disk can be "expanded" into a special power series called the **Taylor series** of f.

Theorem 4.1 (Taylor) *Let $a \in \mathbb{C}$ and $r > 0$, and let $f: D(a; r) \to \mathbb{C}$ be a holomorphic function. Then there exists a sequence of complex numbers $\{a_n\}$ such that*

$$f(z) = \sum_{k=0}^{+\infty} a_k (z-a)^k$$

for every $z \in D(a; r)$.

Proof. For each $z \in D(a; r)$, we let $\rho \in (|z-a|, r)$. Then $\left|\frac{z-a}{w-a}\right| < 1$ for every $w \in \partial D(a; \rho)$, and Cauchy integral formula gives



$$f(z) = \frac{1}{2\pi i} \oint_{\partial D(a;\rho)} \frac{f(w)}{w-z} dw = \frac{1}{2\pi i} \oint_{\partial D(a;\rho)} \frac{f(w)}{w-a} \frac{1}{1 - \frac{z-a}{w-a}} dw$$

$$= \frac{1}{2\pi i} \oint_{\partial D(a;\rho)} \frac{f(w)}{w-a} \left[\sum_{k=0}^{+\infty} \left(\frac{z-a}{w-a}\right)^k\right] dw = \frac{1}{2\pi i} \oint_{\partial D(a;\rho)} \left[\sum_{k=0}^{+\infty} \frac{f(w)}{(w-a)^{k+1}} (z-a)^k\right] dw.$$

Now for every $w \in \partial D(a; \rho)$, we have $\left|\frac{f(w)}{(w-a)^{k+1}}\right| \leq \frac{1}{\rho^{k+1}} \max\{|f(w)|: w \in \partial D(a; \rho)\}$. Since the series of real numbers $\sum_{k=0}^{+\infty} \frac{|z-a|^k}{\rho^{k+1}}$ converges, the series of functions $\sum_{k=0}^{+\infty} \frac{f(w)}{(w-a)^{k+1}} (z-a)^k$ converges uniformly on $\partial D(a; \rho)$ by Weierstrass' M-test. Therefore by Corollary 3.20,

$$f(z) = \sum_{k=0}^{+\infty} \left[\frac{1}{2\pi i} \oint_{\partial D(a;\rho)} \frac{f(w)}{(w-a)^{k+1}} dw\right] (z-a)^k.$$

Moreover, for each $n \in \mathbb{N}$, the line integral $\oint_{\partial D(a;\rho)} \frac{f(w)}{(w-a)^{n+1}} dw$ does not depend on the choice of $\rho \in (0, r)$ (according to the general version of Cauchy-Goursat, Theorem 3.45). So one may take $a_n := \frac{1}{2\pi i} \oint_{\partial D(a;\rho)} \frac{f(w)}{(w-a)^{n+1}} dw$ to finish the proof. ∎

Now we know that a holomorphic function on a disk can be expanded into a special power series. But recall (Theorem 2.89 and Corollary 2.91) that the derivative of a power series is also holomorphic on the same disk of convergence, so we deduce that **holomorphic functions are infinitely many times differentiable**.

Corollary 4.2 *Let $U \subseteq \mathbb{C}$ be an open set and $f: U \to \mathbb{C}$ be a holomorphic function. Then its derivative $f': U \to \mathbb{C}$ is also holomorphic. Consequently, a holomorphic function on U is infinitely many times differentiable at each point in U.*

This behavior of holomorphic functions is very different from differentiable functions of a real variable, as we have obvious examples of a differentiable function of a real variable which is not twice differentiable.

Combining Theorem 4.1 with Corollary 2.92, and perhaps modifying the simple closed curves using the general version of Cauchy-Goursat, we also obtain the following useful formula.

Corollary 4.3 (Generalized Cauchy integral formula) *Let $U \subseteq \mathbb{C}$ be a simply connected region, $f: U \to \mathbb{C}$ be a holomorphic function, γ be a counterclockwise oriented simple closed C^1 curve in U, a be a point in the interior of γ, and n be a non-negative integer. Then*

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-a)^{n+1}}\, dz.$$

Example 4.4 Evaluate the integrals

$$\oint_{\partial D(0;1)} \frac{\cos z}{z^3}\, dz \qquad \text{and} \qquad \oint_{\partial D(0;1)} \frac{1}{z^2(z^2-4)e^z}\, dz.$$

Solution:

- Since $\cos$ is entire, $\partial D(0;1)$ is a simple closed curve in $\mathbb{C}$ and the point 0 is in the interior of $\partial D(0;1)$, by the generalized Cauchy integral formula we have

$$\oint_{\partial D(0;1)} \frac{\cos z}{z^3}\, dz = \frac{2\pi i}{2!} \frac{d^2}{dz^2} \cos z \bigg|_{z=0} = \frac{2\pi i}{2!}(-\cos 0) = -\pi i.$$

- Let $U = D(0;2)$. Then U is a simply connected region, $1/[(z^2-4)e^z]$ is holomorphic on U, $\partial D(0;1)$ is a simple closed curve in U, and 0 is in the interior of $\partial D(0;1)$. Since

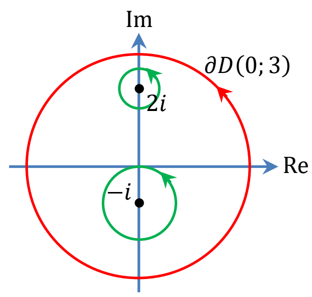
$$\frac{d}{dz} \frac{1}{(z^2-4)e^z} = \frac{4-2z-z^2}{(z^2-4)^2 e^z}$$

on U, by the generalized Cauchy integral formula we have

$$\oint_{\partial D(0;1)} \frac{1}{z^2(z^2-4)e^z}\, dz = \frac{2\pi i}{1!} \cdot \frac{4-2(0)-(0)^2}{(0^2-4)^2 e^0} = \frac{\pi i}{2}.$$

Example 4.5 Evaluate the integral

$$\oint_{\partial D(0;3)} \frac{1}{(z+i)^2(z-2i)^3}\,dz.$$



Solution:

The integrand is holomorphic away from $-i$ and $2i$. Since $\partial D(-i;1)$ and $\partial D(2i;1/2)$ are simple closed curves in the interior of $\partial D(0;3)$, whose interiors are mutually disjoint, by the general version of Cauchy-Goursat Theorem (Theorem 3.45) we have

$$\oint_{\partial D(0;3)} \frac{1}{(z+i)^2(z-2i)^3}\,dz = \oint_{\partial D(-i;1)} \frac{1}{(z+i)^2(z-2i)^3}\,dz + \oint_{\partial D\left(2i;\frac{1}{2}\right)} \frac{1}{(z+i)^2(z-2i)^3}\,dz.$$

Now applying generalized Cauchy integral formula to each integral, we have

$$\oint_{\partial D(-i;1)} \frac{1}{(z+i)^2(z-2i)^3}\,dz = \frac{2\pi i}{1!}\frac{d}{dz}\bigg|_{z=-i}\frac{1}{(z-2i)^3} = -\frac{2\pi i}{3^3},$$

$$\oint_{\partial D\left(2i;\frac{1}{2}\right)} \frac{1}{(z+i)^2(z-2i)^3}\,dz = \frac{2\pi i}{2!}\frac{d^2}{dz^2}\bigg|_{z=2i}\frac{1}{(z+i)^2} = \frac{2\pi i}{3^3}.$$

So $\oint_{\partial D(0;3)} \frac{1}{(z+i)^2(z-2i)^3}\,dz = -\frac{2\pi i}{3^3} + \frac{2\pi i}{3^3} = 0.$

Definition 4.6 Let $a \in \mathbb{C}$ and $r > 0$, and let $f: D(a;r) \to \mathbb{C}$ be a holomorphic function. The power series

$$\sum_{k=0}^{+\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k$$

is called the ***Taylor series of f at a***. The Taylor series of f at 0 is also called the ***Maclaurin series of f***.

Remark 4.7 Let f be holomorphic at a. Let b be the point(s) nearest to a such that $\lim_{z\to b} f(z) \notin \mathbb{C}$ (in particular, f is not holomorphic at b; f has a "non-removable singularity" at b).

By Theorems 4.1 and 2.89, the **radius of convergence** of the Taylor series of f at a is $|b-a|$.

Remark 4.8 ("Holomorphic" vs "Analytic") Let $U \subseteq \mathbb{C}$ be an open set and let $f: U \to \mathbb{C}$ be a function. f is said to be ***analytic at a point $a \in U$*** if there exists a power series centered at a which converges pointwise to f on some disk centered at a. (Such a power series must be the Taylor series of f at a, by the uniqueness of power series.) By Theorem 4.1 and Theorem 2.89, it turns out that **a function of one complex variable is analytic if and only if it is holomorphic**. Therefore, some authors use the words "analytic" and "holomorphic" interchangeably.

The Taylor series of **rational functions** can be obtained by the following routine:

Long division $\to$ Partial fractions $\to$ Apply $\frac{1}{1-w} = \sum_{k=0}^{+\infty} w^k$ for $|w| < 1$.

Example 4.9 For each of the following functions, find its Taylor series at 1 and the disk of convergence of the Taylor series.

(a) $f: \mathbb{C} \setminus \{-2, -3\} \to \mathbb{C}$ defined by $f(z) = \frac{z-2}{(z+3)(z+2)}$.

(b) $g: \mathbb{C} \setminus \{1+2i, 1-2i\}$ defined by $g(z) = \frac{z}{z^2-2z+5}$.

The radii of convergence can be found easily using Remark 4.7.

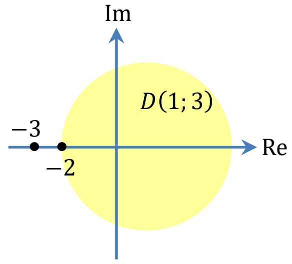
Solution:

(a) For every $z \in D(1;3)$, we have both $|z-1| < 3$ and $|z-1| < 4$, so

$$f(z) = \frac{z-2}{(z+3)(z+2)} = \frac{5}{z+3} - \frac{4}{z+2} = \frac{5}{(z-1)+4} - \frac{4}{(z-1)+3}$$

$$= \frac{5}{4} \cdot \frac{1}{1-\left(-\frac{z-1}{4}\right)} - \frac{4}{3} \cdot \frac{1}{1-\left(-\frac{z-1}{3}\right)} = \frac{5}{4}\sum_{k=0}^{+\infty}\left(-\frac{z-1}{4}\right)^k - \frac{4}{3}\sum_{k=0}^{+\infty}\left(-\frac{z-1}{3}\right)^k$$

$$= \sum_{k=0}^{+\infty}\left[\frac{5(-1)^k}{4^{k+1}} - \frac{4(-1)^k}{3^{k+1}}\right](z-1)^k.$$



The disk of convergence of the Taylor series of f at 1 is $D(1;3)$.

(b) For every $z \in D(1;2)$, we have $|z-1| < 2$, so

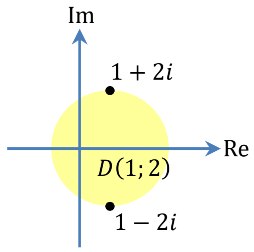
$$g(z) = \frac{z}{z^2-2z+5} = \frac{(z-1)+1}{(z-1)^2+4} = \frac{1}{4}[(z-1)+1] \cdot \frac{1}{1-\left[-\frac{(z-1)^2}{4}\right]}$$

$$= \frac{1}{4}[(z-1)+1]\sum_{k=0}^{+\infty}\frac{(-1)^k}{4^k}(z-1)^{2k}$$

$$= \sum_{k=0}^{+\infty}\frac{(-1)^k}{4^{k+1}}(z-1)^{2k+1} + \sum_{k=0}^{+\infty}\frac{(-1)^k}{4^{k+1}}(z-1)^{2k} = \sum_{k=0}^{+\infty} a_k(z-1)^k,$$

where the coefficients are given by

$$a_n = \begin{cases} (-1)^{\frac{n}{2}}/4^{\frac{n}{2}+1} & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}}/4^{\frac{n-1}{2}+1} & \text{if } n \text{ is odd} \end{cases} = \begin{cases} i^n/2^{n+2} & \text{if } n \text{ is even} \\ i^{n-1}/2^{n+1} & \text{if } n \text{ is odd} \end{cases}.$$



The disk of convergence of the Taylor series of g at 1 is $D(1;2)$.

To find the Taylor series of **combinations of exp, sin and cos**, these Maclaurin series are often useful:

$$e^z = \sum_{k=0}^{+\infty} \frac{1}{k!} z^k, \qquad \sin z = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}, \qquad \cos z = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k)!} z^{2k}$$

for every $z \in \mathbb{C}$. Verify these on your own by computing $f^{(n)}(0)$ for each $n \in \mathbb{N} \cup \{0\}$.

Example 4.10 Let $f: \mathbb{C} \to \mathbb{C}$ be the function

$$f(z) = \sin z.$$

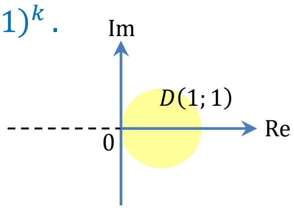
Then the Taylor series of f at $\frac{3\pi}{2}$ is given by

$$f(z) = \sin z = -\cos\left(z - \frac{3\pi}{2}\right) = -\sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k)!} \left(z - \frac{3\pi}{2}\right)^{2k} = \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{(2k)!} \left(z - \frac{3\pi}{2}\right)^{2k}.$$

The disk of convergence of this series is $\mathbb{C}$.

Example 4.11 Let $\log: \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ be a branch of logarithm. Then its n^{th} derivative at 1 is given by $(-1)^{n-1}(n-1)!$ for each $n \geq 1$. So the Taylor series of $\log$ at 1 is given by

$$\log z = \log 1 + \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}(k-1)!}{k!} (z-1)^k = \log 1 + \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k} (z-1)^k.$$



The disk of convergence of this series is $D(1;1)$.

Example 4.12 Let $f: \mathbb{C} \setminus \{3\} \to \mathbb{C}$ be the function

$$f(z) = \frac{1}{(z-3)^2}.$$

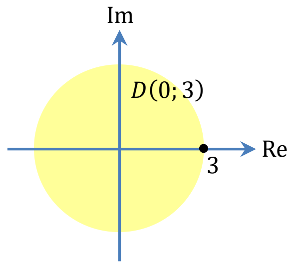
Find the Maclaurin series of f and its disk of convergence.

Solution: First consider the function $\frac{1}{3-z}$. For every $z \in D(0;3)$, we have $|z| < 3$ and so

$$\frac{1}{3-z} = \frac{1}{3} \cdot \frac{1}{1 - \frac{z}{3}} = \frac{1}{3} \sum_{k=0}^{+\infty} \left(\frac{z}{3}\right)^k = \sum_{k=0}^{+\infty} \frac{1}{3^{k+1}} z^k.$$

Now differentiating the power series term-by-term we see that

$$f(z) = \frac{1}{(z-3)^2} = \frac{d}{dz} \frac{1}{3-z} = \sum_{k=1}^{+\infty} \frac{1}{3^{k+1}} k z^{k-1} = \sum_{k=0}^{+\infty} \frac{k+1}{3^{k+2}} z^k$$



for every $z \in D(0;3)$. The disk of convergence of this series is $D(0;3)$.

A number a which satisfies $f(a) = 0$ is usually called a **zero** of f (instead of "a **root** of f") in complex analysis.

Definition 4.13 Let $a \in \mathbb{C}$ and let f be a function which is holomorphic at a. We say that f has a ***zero of order*** $\boldsymbol{m}$ (or a ***zero of multiplicity*** $\boldsymbol{m}$) at a if

$$f(a) = f'(a) = \cdots = f^{(m-1)}(a) = 0$$

but $f^{(m)}(a) \neq 0$.

Example 4.14 Let $a \in \mathbb{C}$ and let

$$f(z) = (z-a)^m.$$

Then f has a zero of order m at a.

Lemma 4.15 (Factor Theorem) *Let $a \in \mathbb{C}$ and let f be a function which is holomorphic at a. Then f has a zero of order m at a if and only if there exist $r > 0$ and a holomorphic function $g: D(a;r) \to \mathbb{C}$ such that $g(a) \neq 0$ and*

$$f(z) = (z-a)^m g(z)$$

for every $z \in D(a;r)$.

Proof.

($\Rightarrow$) Since f is holomorphic on $D(a;r)$ for some $r > 0$, there exists a sequence of complex numbers $\{a_n\}$ such that $f(z) = \sum_{k=0}^{+\infty} a_k (z-a)^k$ for every $z \in D(a;r)$. Since f has a zero of order m at a, we have $a_n = \frac{f^{(n)}(a)}{n!} = 0$ for $n \in \{0, 1, \dots, m-1\}$. Thus

$$f(z) = \sum_{k=m}^{+\infty} a_k (z-a)^k = (z-a)^m \sum_{k=0}^{+\infty} a_{k+m} (z-a)^k$$

for every $z \in D(a;r)$. Now we define $g: D(a;r) \to \mathbb{C}$ by $g(z) = \sum_{k=0}^{+\infty} a_{k+m}(z-a)^k$. Then g is holomorphic, $f(z) = (z-a)^m g(z)$ for every $z \in D(a;r)$, and

$$g(a) = a_m = \frac{f^{(m)}(a)}{m!} \neq 0.$$

($\Leftarrow$) Suppose that such a function g exists. Then by the product rule for higher derivatives,

$$f^{(n)}(z) = \frac{d^n}{d^n z}[(z-a)^m g(z)] = \sum_{k=0}^{n} \binom{n}{k} \frac{m!}{(m-k)!} (z-a)^{m-k} g^{(n-k)}(z)$$

for every $z \in D(a;r)$. Therefore

$$f(a) = f'(a) = \cdots = f^{(m-1)}(a) = 0$$

but $f^{(m)}(a) = g(a) \neq 0$. ∎

In summary, if a is a zero of f, then we can find its **order** by any of the following three methods:

- Compute $f(a), f'(a), f''(a), \ldots$, and find out which is the **first non-zero derivative**.
- Find the exponent of the **first non-zero term in the Taylor series** $f(z) = \sum_{k=0}^{+\infty} a_k (z-a)^k$ of f at a.
- Express f in the form $f(z) = (z-a)^m g(z)$, where g is holomorphic at a and $g(a) \neq 0$. By **Factor Theorem**, what we need is the number m.

Example 4.16 Find the zeros of the following functions, and determine their orders.

(a) $f(z) = \sin z$

(b) $g(z) = e^{2z} - 1$

(c) $h(z) = z^3 - 4z^2 + 5z - 2$

Solution: [Let's try using each of the three methods above.]

(a) The set of all zeros of f is $\{n\pi : n \in \mathbb{Z}\}$. For each $n \in \mathbb{Z}$, since

$$f'(n\pi) = \cos n\pi = (-1)^n \neq 0,$$

$n\pi$ is a zero of f of order 1.

(b) $g(z) = 0$ if and only if $e^{2z} = 1$, so the set of all zeros of g is $\{n\pi i : n \in \mathbb{Z}\}$. For each $n \in \mathbb{Z}$, the Taylor series of g at $n\pi i$ is given by

$$e^{2z} - 1 = e^{2(z - n\pi i)} - 1 = \sum_{k=0}^{+\infty} \frac{[2(z - n\pi i)]^k}{k!} - 1 = \sum_{k=1}^{+\infty} \frac{2^k}{k!} (z - n\pi i)^k .$$

The first non-zero term in this Taylor series has exponent 1, so $n\pi i$ is a zero of g of order 1.

(c) h can be factorized as

$$h(z) = z^3 - 4z^2 + 5z - 2 = (z-1)^2 (z-2),$$

so the only zeros of h are 1 and 2. According to this factorization, since $z - 2$ is holomorphic and does not have a zero at 1, 1 is a zero of h of order 2 by Factor Theorem. Since $(z-1)^2$ is holomorphic and does not have a zero at 2, 2 is a zero of h of order 1 by Factor Theorem.

Think: What are the Taylor series of h at 1 and at 2?

(At 1: $h(z) = -(z-1)^2 + (z-1)^3$;

At 2: $h(z) = (z-2) + 2(z-2)^2 + (z-2)^3$.)

2. Morera's Theorem

Morera's Theorem is kind of a converse of Cauchy-Goursat Theorem. It says that if a continuous function has **zero line integral along any closed curve**, then it is holomorphic. The assumption in this theorem can be weakened to having zero line integral along just any simple closed curve, or even just any triangle.

Theorem 4.17 (Morera) *Let* $U \subseteq \mathbb{C}$ *be a region and* $f: U \to \mathbb{C}$ *be a continuous function. If*

$$\oint_\gamma f(z)dz = 0$$

for every triangle γ *in* U*, then* f *is holomorphic on* U.

The key to the proof of Morera's Theorem lies in the fact that in Lemma 3.41 and Lemma 3.42 about construction of antiderivatives, we just need to assume that f is **continuous**, but we need not assume that f is holomorphic or that the region is simply connected.

Proof of Theorem 4.17. By Lemma 3.41 and Lemma 3.42, f has an antiderivative $F: U \to \mathbb{C}$. Since f is the derivative of F, f is also holomorphic on U by Corollary 4.2. ∎

Example 4.18 Let $g: [0,1] \to \mathbb{C}$ be a continuous function. If $f: \mathbb{C} \to \mathbb{C}$ is a function defined by

$$f(z) = \int_0^1 g(t)e^{tz}dt,$$

show that f is entire.

Proof: [The complicated defining formula of f makes it difficult to check whether the real and imaginary parts of f satisfy the Cauchy-Riemann equations on $\mathbb{C}$. So we use Morera's Theorem.]

First, we show that f is continuous on $\mathbb{C}$. Note that the exponential function has derivative 1 at 0, so there exists $\delta > 0$ such that $\left|\frac{e^w - e^0}{w} - 1\right| < 1$ whenever $0 < |w| < \delta$. Thus

$$|e^w - 1| \leq 2|w| \qquad \text{whenever } |w| < \delta.$$

Now for each $a \in \mathbb{C}$, $M = \max\{|g(t)e^{ta}| : t \in [0,1]\}$ exists because $g(t)e^{ta}$ is continuous on the compact set $[0,1]$. For every $z \in D(a;\delta)$, we have

$$|f(z) - f(a)| = \left|\int_0^1 g(t)(e^{tz} - e^{ta})dt\right| \leq \int_0^1 |g(t)e^{ta}|\left|e^{t(z-a)} - 1\right|dt$$

$$\leq M\int_0^1 2t|z-a|dt = M|z-a|,$$

which shows that f is continuous at a.

Next, we show that $\oint_\gamma f(z)dz = 0$ for every closed C^1 curve γ in $\mathbb{C}$. Let $\gamma: [a,b] \to \mathbb{C}$ be a closed C^1 curve in $\mathbb{C}$. Then by Fubini's Theorem (MATH2023), we have

$$\begin{aligned}\oint_\gamma f(z)dz &= \oint_\gamma \left(\int_0^1 g(t)e^{tz}dt\right)dz \\ &= \int_a^b \left(\int_0^1 g(t)e^{t\gamma(s)}dt\right)\gamma'(s)ds \\ &= \int_0^1 g(t)\left(\int_a^b e^{t\gamma(s)}\gamma'(s)ds\right)dt \qquad \text{Fubini's Theorem} \\ &= \int_0^1 g(t)\underbrace{\left(\oint_\gamma e^{tz}dz\right)}_{\substack{=0 \text{ by} \\ \text{Cauchy-Goursat}}}dt = 0.\end{aligned}$$

Therefore f is entire by Morera's Theorem. ■

Example 4.19 Let $g: [0,1] \to \mathbb{C}$ be a continuous function. If $f: \mathbb{C} \setminus [0,1] \to \mathbb{C}$ is a function defined by

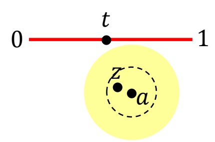
$$f(z) = \int_0^1 \frac{g(t)}{t-z}dt,$$

show that f is holomorphic on $\mathbb{C} \setminus [0,1]$.

Proof:
First, we show that f is continuous on $\mathbb{C} \setminus [0,1]$. For each $a \in \mathbb{C} \setminus [0,1]$,

- ⊙ there exists $\delta > 0$ such that $D(a;\delta) \subset \mathbb{C} \setminus [0,1]$, since $\mathbb{C} \setminus [0,1]$ is open; and
- ⊙ $M = \max\left\{\left|\frac{g(t)}{(t-a)^2}\right| : t \in [0,1]\right\}$ exists because $\frac{g(t)}{(t-a)^2}$ is continuous on the compact set $[0,1]$.

For every $z \in D\left(a;\frac{\delta}{2}\right)$ and every $t \in [0,1]$, we have $|z-a| < \frac{\delta}{2} < \frac{|t-a|}{2}$, so



$$|t-z| \geq |t-a| - |z-a| > \frac{|t-a|}{2}.$$

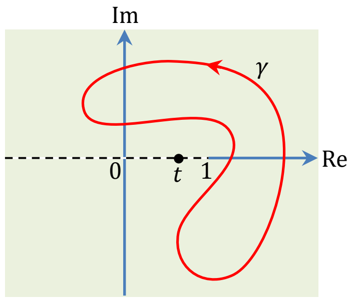
Thus

$$\begin{aligned}|f(z)-f(a)| &= \left|\int_0^1 g(t)\left(\frac{1}{t-z} - \frac{1}{t-a}\right)dt\right| \leq \int_0^1 |g(t)|\frac{|z-a|}{|t-a||t-z|}dt \\ &\leq \int_0^1 2\left|\frac{g(t)}{(t-a)^2}\right||z-a|dt \leq 2M|z-a|,\end{aligned}$$

which shows that f is continuous at a.

Next, we consider the **simply connected** regions $\mathbb{C}\setminus(-\infty,1]$ and $\mathbb{C}\setminus[0,+\infty)$ separately. Let $\gamma:[a,b]\to\mathbb{C}\setminus(-\infty,1]$ be a simple closed C^1 curve. Then by Fubini's Theorem we have

$$\begin{aligned}\oint_\gamma f(z)dz &= \oint_\gamma\left(\int_0^1 \frac{g(t)}{t-z}dt\right)dz\\ &= \int_a^b\left(\int_0^1\frac{g(t)}{t-\gamma(s)}dt\right)\gamma'(s)ds\\ &= \int_0^1 g(t)\left(\int_a^b\frac{1}{t-\gamma(s)}\gamma'(s)ds\right)dt\\ &= \int_0^1 g(t)\left(\oint_\gamma\frac{1}{t-z}dz\right)dt=0,\end{aligned}$$



where in the last step $\oint_\gamma\frac{1}{t-z}dz=0$ by Cauchy-Goursat Theorem, since every $t\in[0,1]$ is in the exterior of γ. So f is holomorphic on $\mathbb{C}\setminus(-\infty,1]$ by Morera's Theorem. By a similar argument, we can also show that f is holomorphic on $\mathbb{C}\setminus[0,+\infty)$. Therefore we conclude that f is holomorphic on $\mathbb{C}\setminus[0,1]=(\mathbb{C}\setminus(-\infty,1])\cup(\mathbb{C}\setminus[0,+\infty))$. ■

The following corollary of Morera's Theorem says that **uniform convergence** on compact sets **preserves holomorphicity** in the limit too.

Theorem 4.20 *Let $U\subseteq\mathbb{C}$ be a region and let $\{f_n\}$ be a sequence of holomorphic functions $f_n:U\to\mathbb{C}$. If $\{f_n\}$ converges to a function $f:U\to\mathbb{C}$ uniformly on every compact subset of U, then f is holomorphic. In other words, the uniform limit of a sequence of holomorphic functions is also holomorphic.*

Proof. First, f is continuous on U by Theorem 2.74 because it is the uniform limit of a sequence of continuous functions on U. Now assume further that U is an open disk and let γ be a closed C^1 curve in U. Since U is a simply connected region and each f_n is holomorphic on U, Cauchy-Goursat Theorem gives

$$\oint_\gamma f_n(z)dz=0$$

for each n. So by Corollary 3.20, we have

$$\oint_\gamma f(z)dz=\lim_{n\to+\infty}\oint_\gamma f_n(z)dz=0.$$

Therefore f is holomorphic on U by Morera's Theorem.

Finally suppose that U is an arbitrary region. For each $a\in U$, there exists a disk $D(a;r)\subseteq U$. Since $\{f_n\}$ converges uniformly to f on every compact subset of $D(a;r)$ in particular, f is holomorphic on $D(a;r)$ by the last paragraph, i.e. f is holomorphic at a. ■

Example 4.21 Let $U = \{z \in \mathbb{C}: \operatorname{Re} z > 1\}$ and $f: U \to \mathbb{C}$ be the function defined by

$$f(z) = \sum_{k=1}^{+\infty} \frac{1}{k^z}.$$

Show that f is holomorphic on U.

Proof: For each $n \in \mathbb{N}$, the function $f_n: U \to \mathbb{C}$ defined by

$$f_n(z) = 1/n^z = e^{-z \ln n}$$

is holomorphic. Now let $K \subset U$ be a compact set. Then $m := \min\{\operatorname{Re} z : z \in K\} > 1$ exists since the function $\operatorname{Re}: K \to \mathbb{C}$ is continuous on the compact set K. Now for every $z \in K$ and every $n \in \mathbb{N}$, we have

$$|f_n(z)| = \left|e^{-z \ln n}\right| = e^{-(\operatorname{Re} z) \ln n} = \frac{1}{n^{\operatorname{Re} z}} \leq \frac{1}{n^m}.$$

Since $m > 1$, $\sum_{k=1}^{+\infty} \frac{1}{k^m}$ converges by p-test; and so $\sum_{k=1}^{+\infty} \frac{1}{k^z}$ converges uniformly on K by Weierstrass' M-test. Therefore by Theorem 4.20, its uniform limit f is holomorphic on U. ∎

Remark 4.22 The function f in the above example can actually be "extended" to a holomorphic function $\zeta: \mathbb{C} \setminus \{1\} \to \mathbb{C}$, which is commonly known as the ***Riemann zeta function***.

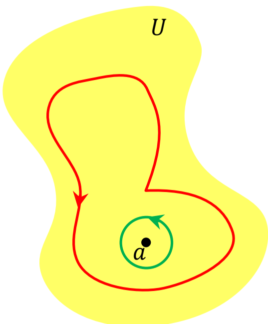
The following is another important corollary of Morera's Theorem, which basically says that **"almost holomorphic" functions are holomorphic**.

Theorem 4.23 *Let $U \subseteq \mathbb{C}$ be a region, $a \in U$, and $f: U \to \mathbb{C}$ be a continuous function which is holomorphic on $U \setminus \{a\}$. Then f is holomorphic on U.*

Proof. Let γ be any closed C^1 curve in U. The issue is to show that $\oint_\gamma f(z)dz = 0$. We consider the following three cases.

(i) If a is in the exterior of γ, then $\oint_\gamma f(z)dz = 0$ by Cauchy-Goursat.

(ii) If a is in the interior of γ, then we mimic the proof of Cauchy integral formula as follows. For every sufficiently small $\varepsilon > 0$, $\partial D(a;\varepsilon)$ also lies in the interior of γ, thus



$$\begin{aligned}\oint_\gamma f(z)dz &= \oint_{\partial D(a;\varepsilon)} f(z)dz \\ &= \oint_{\partial D(a;\varepsilon)} [f(z) - f(a)]dz + \oint_{\partial D(a;\varepsilon)} f(a)dz.\end{aligned}$$

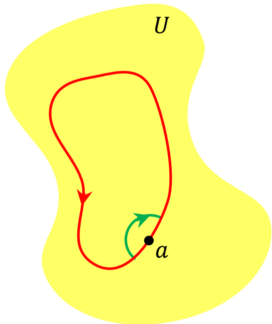
The second integral is zero by Cauchy-Goursat Theorem. For the first integral, since f is continuous at a, there exists $\delta > 0$ such that $|f(z) - f(a)| < 1$ whenever $|z - a| < \delta$. So for $\varepsilon \in (0, \delta)$, ML-estimate gives

$$\left|\oint_{\partial D(a;\varepsilon)} [f(z) - f(a)]dz\right| \leq 1 \cdot 2\pi\varepsilon,$$

which tends to 0 as $\varepsilon \to 0^+$. So $\lim\limits_{\varepsilon\to 0^+} \left|\oint_{\partial D(a;\varepsilon)}[f(z) - f(a)]dz\right| = 0$ by squeeze theorem, and so $\lim\limits_{\varepsilon\to 0^+} \oint_{\partial D(a;\varepsilon)}[f(z) - f(a)]dz = 0$. Therefore

$$\oint_\gamma f(z)dz = \lim_{\varepsilon\to 0^+}\left(\oint_{\partial D(a;\varepsilon)} [f(z) - f(a)]dz + \oint_{\partial D(a;\varepsilon)} f(a)dz\right) = 0.$$

(iii) If a is in the image of γ, then we slightly modify γ so that it goes around a along a semicircular arc of small radius $\varepsilon > 0$ instead, and a is in the exterior of this modified curve γ_ε. Then $\oint_{\gamma_\varepsilon} f(z)dz = 0$ by Cauchy-Goursat. Since f is continuous at a, the line integral of f along the small circular arc and along the original small portion of γ both tend to zero as $\varepsilon \to 0^+$ by ML-estimate. Therefore



$$\oint_\gamma f(z)dz = \lim_{\varepsilon\to 0^+} \oint_{\gamma_\varepsilon} f(z)dz = 0.$$

Combining all three cases, f is holomorphic on U by Morera's Theorem. ∎

Example 4.24 Let $U \subseteq \mathbb{C}$ be a region, $a \in U$ and $f: U \to \mathbb{C}$ be a holomorphic function. Then the continuous function $g: U \to \mathbb{C}$ defined by

$$g(z) = \begin{cases} \dfrac{f(z) - f(a)}{z - a} & \text{if } z \in U \setminus \{a\} \\ f'(a) & \text{if } z = a \end{cases}$$

is also holomorphic on U. In particular, consider the entire function $f(z) = \sin z$. In Example 4.16 (a), we saw that f has a zero of order 1 at 0. So by Factor Theorem, there exist $r > 0$ and a holomorphic function $g: D(0; r) \to \mathbb{C}$ such that

$$\sin z = zg(z)$$

for every $z \in D(0; r)$. It turns out that we can take $r = +\infty$ and take $g: \mathbb{C} \to \mathbb{C}$ to be the continuous function

$$g(z) = \begin{cases} \dfrac{\sin z}{z} & \text{if } z \in \mathbb{C} \setminus \{0\} \\ 1 & \text{if } z = 0 \end{cases}.$$

Then g is entire by Theorem 4.23.

Theorem 4.23 has the following strengthened version.

Theorem 4.25 *Let* $U \subseteq \mathbb{C}$ *be a region,* $\gamma: \mathbb{R} \to U$ *be a piecewise* C^1 *curve (whose image may be unbounded), and* $f: U \to \mathbb{C}$ *be a continuous function which is holomorphic on* $U \setminus \operatorname{image}\gamma$*. Then* f *is holomorphic on* U*.*

Sketch of proof. Similar to Theorem 4.23, we use Morera's Theorem. When handing the case that a portion of the arbitrary closed curve Γ in the integral $\oint_\Gamma f(z)dz$ coincides with the given piecewise C^1 curve γ, one may need to construct a uniformly convergent sequence and apply the integration theorem for uniform convergence. Details are omitted. ∎

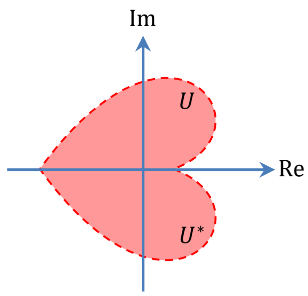
As a strong corollary of Morera's Theorem, Theorem 4.25 leads to the following two reflection principles, which help **extending the domains of holomorphic functions by reflection**.

Theorem 4.26 (Schwarz reflection) *Let* $H = \{z \in \mathbb{C}: \operatorname{Im} z > 0\}$ *be the open upper half-plane. Let* $U \subseteq H$ *be a region, and denote the reflection of* U *across the real axis by*

$$U^* \coloneqq \{z \in \mathbb{C}: \bar{z} \in U\}.$$

Let $f: \overline{U} \to \mathbb{C}$ *be a continuous function which is holomorphic on* U*. If* f *sends real numbers to real numbers (i.e.* $f(z) \in \mathbb{R}$ *for every* $z \in \overline{U} \cap \mathbb{R}$*), then* f *can be extended to a holomorphic function on* $U \cup U^*$*, i.e. there exists a continuous function* $F: \overline{U \cup U^*} \to \mathbb{C}$ *which is holomorphic on the interior of its domain and* $F(z) = f(z)$ *for every* $z \in \overline{U}$*.*

⊙ *In particular, if* $f: \overline{H} \to \mathbb{C}$ *is continuous on* $\overline{H}$*, holomorphic on* H*, and* $f(z) \in \mathbb{R}$ *for every* $z \in \mathbb{R}$*, then* f *can be extended to an entire function* $F: \mathbb{C} \to \mathbb{C}$*.*



Sketch of proof. Define the function $F: \overline{U \cup U^*} \to \mathbb{C}$ by

$$F(z) = \begin{cases} f(z) & \text{if } z \in \overline{U} \\ \overline{f(\bar{z})} & \text{if } z \in \overline{U^*} \end{cases}.$$

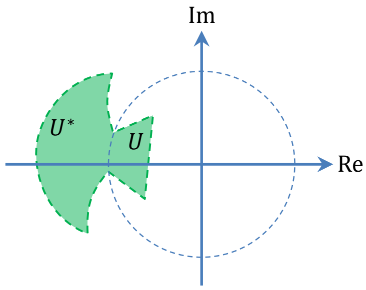
Then F is well-defined and is continuous (because $f(z) = \overline{f(\bar{z})}$ when $z \in \overline{U} \cap \mathbb{R}$ by the assumption). F is clearly holomorphic on U, and is also holomorphic on U^* by the Cauchy-Riemann equations (see Example 2.51). Now F is continuous on $\overline{U \cup U^*}$ and holomorphic everywhere on the interior of $\overline{U \cup U^*}$ except possibly on the real line. Therefore F is holomorphic on the whole interior of $\overline{U \cup U^*}$ by Theorem 4.25. ∎

Theorem 4.27 (Schwarz reflection) *Let* $U \subseteq D(0;1)$ *be a region, and denote the reflection of* U *across* $\partial D(0;1)$ *by*

$$U^* \coloneqq \{z \in \mathbb{C} : 1/\bar{z} \in U\}.$$

Let $f : \overline{U} \to \mathbb{C}$ *be a continuous function which is holomorphic on* U. *If* f *sends* $\partial D(0;1)$ *to real numbers (i.e.* $f(z) \in \mathbb{R}$ *for every* $z \in \overline{U} \cap \partial D(0;1)$*), then* f *can be extended to a holomorphic function on* $U \cup U^*$*, i.e. there exists a continuous function* $F : \overline{U \cup U^*} \to \mathbb{C}$ *which is holomorphic on the interior of its domain and* $F(z) = f(z)$ *for every* $z \in \overline{U}$.

⦿ *In particular, if* $f : \overline{D(0;1)} \to \mathbb{C}$ *is continuous on* $\overline{D(0;1)}$, *holomorphic on* $D(0;1)$, *and* $f(z) \in \mathbb{R}$ *for every* $z \in \partial D(0;1)$*, then* f *can be extended to an entire function* $F : \mathbb{C} \to \mathbb{C}$.



Sketch of proof. Define the function $F : \mathbb{C} \to \mathbb{C}$ by

$$F(z) = \begin{cases} f(z) & \text{if } z \in \overline{U} \\ \overline{f(1/\bar{z})} & \text{if } z \in \overline{U^*} \end{cases}.$$

Then the rest is similar to the proof of Theorem 4.26. ■

3. Isolated zeros theorem, identity theorem

The following **isolated zeros theorem** has a similar spirit as the uniqueness theorem of power series. It says that the only holomorphic function that has "convergent" zeros is the constant function 0.

Theorem 4.28 (Isolated zeros) *Let* $U \subseteq \mathbb{C}$ *be a region and* $f : U \to \mathbb{C}$ *be a holomorphic function. If there exists* $a \in U$ *and a sequence* $\{z_n\}$ *of points in* U *converging to* a *such that* $z_n \neq a$ *for all* $n \in \mathbb{N}$ *and*

$$f(z_n) = 0$$

for all $n \in \mathbb{N}$*, then* f *must be the constant function zero, i.e.* $f(z) = 0$ *for all* $z \in U$.

Proof. Since f is holomorphic at a, there exist $r > 0$ such that

$$f(z) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k$$

for every $z \in D(a;r)$. But by the hypothesis and the uniqueness of power series, this Taylor series of f at a must in fact be the zero power series, i.e. $f^{(n)}(a) = 0$ for every $n \in \mathbb{N} \cup \{0\}$.

Now consider the set

$$S = \left\{z \in U : f^{(n)}(z) = 0 \text{ for every } n \in \mathbb{N} \cup \{0\}\right\} = \bigcap_{n=0}^{+\infty} \left(f^{(n)}\right)^{-1}(\{0\}).$$

Then

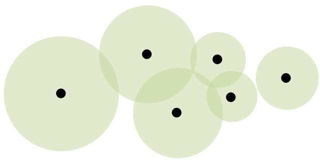
- S is **non-empty** since $a \in S$.
- S is **closed** since the singleton $\{0\}$ is closed, $f^{(n)}$ are all continuous, and the intersection of closed sets is closed.
- S is **open** since for each $z \in S$, the Taylor series of f at z is the zero power series, so $f \equiv 0$ on some disk centered at z, and so every point in this disk also belongs to S.

Since U is connected, we are left with the only possibility that $S = U$. Consequently $f(z) = 0$ for every $z \in U$. ■

Definition 4.29 A set $S \subset \mathbb{C}$ is ***isolated*** if for each $a \in S$, there exists $r > 0$ such that

$$(D(a;r) \setminus \{a\}) \cap S = \emptyset,$$

i.e. $D(a;r)$ does not contain any other element of S apart from a.



The black dots form an ***isolated*** set: Each dot is the center of some disk which contains no other dots.

Corollary 4.30 (Isolated zeros) *Let $U \subseteq \mathbb{C}$ be a region and $f: U \to \mathbb{C}$ be a holomorphic function such that $f \not\equiv 0$. Then the zeros of f are* ***isolated****, i.e. $f^{-1}(\{0\})$ is an isolated set.*

Corollary 4.31 *Let $U \subseteq \mathbb{C}$ be a region and $f: U \to \mathbb{C}$ be a holomorphic function such that $f \not\equiv 0$. Then for each compact set $K \subset U$, f has only finitely many zeros in K.*

Proof. If f has infinitely many zeros in K, then there exists an infinite sequence of zeros of f in K. This sequence must have a convergent subsequence since K is compact, so $f \equiv 0$ by the isolated zeros theorem. ■

Example 4.32 Let $f: \mathbb{C} \to \mathbb{C}$ be an entire function such that $f \not\equiv 0$. Show that f has at most countably many zeros, i.e. the set $f^{-1}(\{0\}) = \{z \in \mathbb{C} : f(z) = 0\}$ is countable.

Proof:

For each $n \in \mathbb{N}$, $f^{-1}(\{0\}) \cap \overline{D(0;n)}$ is finite by Corollary 4.31 since $\overline{D(0;n)}$ is compact. So $f^{-1}(\{0\}) = \bigcup_{n=1}^{+\infty} \left(f^{-1}(\{0\}) \cap \overline{D(0;n)}\right)$ is countable. ■

Corollary 4.33 (Identity theorem) *Let $U \subseteq \mathbb{C}$ be a region and $f, g: U \to \mathbb{C}$ be holomorphic functions. If there exists $a \in U$ and a sequence $\{z_n\}$ of points in U converging to a such that $z_n \neq a$ for all $n \in \mathbb{N}$ and*

$$f(z_n) = g(z_n)$$

for all $n \in \mathbb{N}$, then f and g must be the same function, i.e. $f(z) = g(z)$ for all $z \in U$.

Example 4.34 Let $f, g: D(0;2) \to \mathbb{C}$ be holomorphic functions which have no zeros. If

$$\frac{f'(1/n)}{f(1/n)} = \frac{g'(1/n)}{g(1/n)}$$

for every $n \in \mathbb{N}$, show that $f = cg$ for some $c \in \mathbb{C} \setminus \{0\}$.

Proof: Since f and g are holomorphic and have no zeros, f'/f and g'/g are holomorphic on $D(0;2)$ as well. Now we have $\frac{f'}{f}\left(\frac{1}{n}\right) = \frac{g'}{g}\left(\frac{1}{n}\right)$ for every $n \in \mathbb{N}$ and $\lim_{n\to+\infty} \frac{1}{n} = 0 \in D(0;2)$, so by identity theorem, we have

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)}$$

for all $z \in D(0;2)$. This gives $f'(z)g(z) = g'(z)f(z)$, i.e. $(f/g)'(z) = 0$ for all $z \in D(0;2)$. Since f/g has zero derivative on the region $D(0;2)$, f/g must be a constant function (this constant must be non-zero since f has no zeros). In other words, there exists $c \in \mathbb{C} \setminus \{0\}$ such that $f(z)/g(z) = c$ for every $z \in D(0;2)$. Therefore $f = cg$. ■

Remark 4.35 By identity theorem, we can conclude that the holomorphic function $F: \overline{U \cup U^*} \to \mathbb{C}$ constructed in the proofs of Schwarz reflection principle (Theorem 4.26 and 4.27) is unique. If another holomorphic function $G: \overline{U \cup U^*} \to \mathbb{C}$ also extends the given function f, then since

$$F(z) = f(z) = G(z)$$

for every $z \in U$ and U is an open set, the identity theorem will force $F = G$. We call F the ***analytic continuation*** of f to the larger domain $\overline{U \cup U^*}$.

Example 4.36 Let $f: D(0;1) \to \mathbb{C}$ be the holomorphic function defined by the power series $f(z) = \sum_{k=0}^{+\infty} z^k$, and let $F: \mathbb{C} \setminus \{1\} \to \mathbb{C}$ be the holomorphic function defined by $F(z) = \frac{1}{1-z}$.

Since $\sum_{k=0}^{+\infty} z^k = \frac{1}{1-z}$ for every $z \in D(0;1)$, F is the analytic continuation of f to $\mathbb{C} \setminus \{1\}$.

The idea of analytic continuation also helps us understand more about the complex logarithm. To be precise, we can construct the **complete complex logarithm** by performing analytic continuation on its branches. Interested students may refer to Supplementary Note F for details.

4. Maximum modulus principle

Cauchy integral formula leads to the following **mean value property** of holomorphic functions, which says that the mean value of a holomorphic function on a circle equals to its value at the center of the circle.

Theorem 4.37 (Mean value property) *Let $U \subseteq \mathbb{C}$ be an open set, let $f: U \to \mathbb{C}$ be a holomorphic function, and let $a \in U$ and $r > 0$ such that $\overline{D(a;r)} \subset U$. Then*
$$\frac{1}{2\pi}\int_0^{2\pi} f\left(a + re^{it}\right)dt = f(a).$$

Proof. $\partial D(a;r)$ has a parametrization $\gamma: [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = a + re^{it}$. Then
$$2\pi i f(a) = \oint_{\partial D(a;r)} \frac{f(z)}{z - a}dz = \int_0^{2\pi} \frac{f\left(a + re^{it}\right)}{re^{it}} ire^{it}dt = i\int_0^{2\pi} f\left(a + re^{it}\right)dt$$
by Cauchy integral formula. ∎

Theorem 4.38 (Maximum modulus principle) *Let $U \subseteq \mathbb{C}$ be a region and let $f: U \to \mathbb{C}$ be a non-constant holomorphic function. Then $|f|$ does not attain relative maximum in U.*

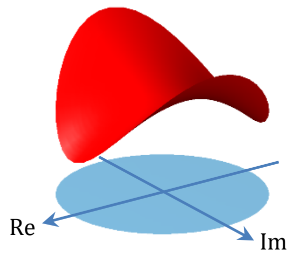
Proof. Suppose that $|f|$ attains relative maximum at $a \in U$, i.e. there exists $r > 0$ such that $|f(z)| \le |f(a)|$ for every $z \in D(0;r)$. Then for each $\rho \in [0, r)$, mean value property gives
$$\begin{aligned} |f(a)| &= \frac{1}{2\pi}\left|\int_0^{2\pi} f\left(a + \rho e^{it}\right)dt\right| \le \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(a + \rho e^{it}\right)\right|dt \\ &\le \frac{1}{2\pi}\int_0^{2\pi} |f(a)|dt = |f(a)|, \end{aligned}$$
so all the above are equalities. This implies that
$$|f(a)| = \left|f\left(a + \rho e^{it}\right)\right|$$
for every $t \in [0, 2\pi]$ and for every $\rho \in [0, r)$, i.e. f is a constant function on $D(0;r)$. So f is a constant function on U by identity theorem. ∎

Corollary 4.39 (Minimum modulus principle) *Let $U \subseteq \mathbb{C}$ be a region and let $f: U \to \mathbb{C}$ be a non-constant holomorphic function such that $f(z) \neq 0$ for any $z \in U$. Then $|f|$ does not attain relative minimum in U.*

Proof. Apply maximum modulus principle to $\frac{1}{f}$. ∎

Corollary 4.40 (Maximum modulus principle) *Let $U \subseteq \mathbb{C}$ be a bounded region and let $f: \overline{U} \to \mathbb{C}$ be a continuous function which is holomorphic on U. Then the absolute maximum of $|f|$ in $\overline{U}$ is attained on ∂U, i.e.*

$$\max\{|f(z)|: z \in \overline{U}\} = \max\{|f(z)|: z \in \partial U\}.$$



The red surface is a plot of $|\cos z|$ for $z \in \overline{D(0;1)}$ (in blue). The highest point on the red surface must occur somewhere along its edge according to Corollary 4.40.

Example 4.41 Let a and b be complex numbers. Calculate $\max\{|az+b|: z \in \overline{D(0;1)}\}$.

Solution:

Let $f: \overline{D(0;1)} \to \mathbb{C}$ be the function $f(z) = az + b$. Since f is continuous on $\overline{D(0;1)}$ and holomorphic on $D(0;1)$, the absolute maximum of $|f|$ is attained on $\partial D(0;1)$ according to maximum modulus principle. Now for every $z \in \partial D(0;1)$,

$$|f(z)| = |az+b| \le |az| + |b| = |a| + |b|.$$

On the other hand, if we write $a = |a|e^{i\alpha}$ and $b = |b|e^{i\beta}$, then $e^{i(\beta-\alpha)} \in \partial D(0;1)$ and

$$\left|f\left(e^{i(\beta-\alpha)}\right)\right| = \left|ae^{i(\beta-\alpha)} + b\right| = \left|ae^{-i\alpha} + be^{-i\beta}\right| = |a| + |b|,$$

i.e. the absolute maximum of $|f|$ on $\partial D(0;1)$ is attained at $e^{i(\beta-\alpha)}$. Therefore

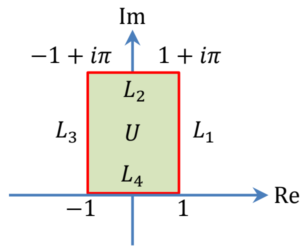
$$\max\{|az+b|: z \in \overline{D(0;1)}\} = \max\{|az+b|: z \in \partial D(0;1)\} = |a| + |b|.$$

Example 4.42 Let U be the rectangular region

$$U = \{z \in \mathbb{C}: -1 < \operatorname{Re} z < 1 \text{ and } 0 < \operatorname{Im} z < \pi\}$$

and let $f: \overline{U} \to \mathbb{C}$ be the function $f(z) = z^2 e^{-z}$. Find the maximum modulus of $f(z)$ on $\overline{U}$.

Solution:



Since f is continuous on $\overline{U}$ and holomorphic on U, the maximum modulus principle asserts that

$$\max\{|f(z)|: z \in \overline{U}\} = \max\{|f(z)|: z \in \partial U\}.$$

Let L_1, L_2, L_3, L_4 be the four portions of ∂U as shown in the diagram.

By calculus of a real variable, we find that

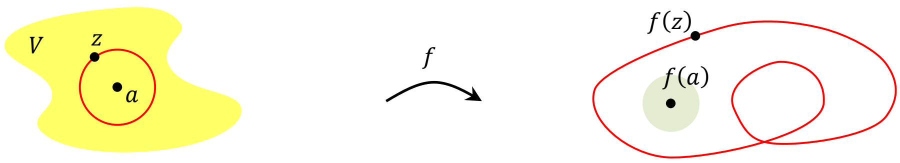
$$\begin{aligned}
\max\{|f(z)|: z \in L_1\} &= \max\{(1+y^2)e^{-1}: y \in [0,\pi]\} = (1+\pi^2)e^{-1}, \\
\max\{|f(z)|: z \in L_2\} &= \max\{(x^2+\pi^2)e^{-x}: x \in [-1,1]\} = (1+\pi^2)e, \\
\max\{|f(z)|: z \in L_3\} &= \max\{(1+y^2)e: y \in [0,\pi]\} = (1+\pi^2)e, \\
\max\{|f(z)|: z \in L_4\} &= \max\{x^2 e^{-x}: x \in [-1,1]\} = e.
\end{aligned}$$

So $\max\{|f(z)|: z \in \overline{U}\} = \max\{|f(z)|: z \in \partial U\} = (1+\pi^2)e.$

The following **open mapping theorem** gives another good property of holomorphic functions that differentiable functions of a real variable do not have in general. It states that a non-constant holomorphic function **maps open sets onto open sets**. The open mapping theorem can be regarded as a strengthened version of the maximum modulus principle.

Theorem 4.43 (Open mapping) *The direct image of an open set via a non-constant holomorphic function is open.*

Proof. Let $U \subseteq \mathbb{C}$ be a region, let $f: U \to \mathbb{C}$ be a non-constant holomorphic function, and let $V \subseteq U$ be an open set. The issue is to show that $f(V)$ is open.



For each $a \in V$, we aim to find a disk centered at $f(a)$ which is contained completely in $f(V)$. By the isolated zeros theorem, there exists $r > 0$ such that $\overline{D(a;r)} \subseteq V$ and $f(z) - f(a)$ has no zeros on $\partial D(a;r)$. Let $2\delta = \min\{|f(z) - f(a)| : z \in \partial D(a;r)\} > 0$, which exists because $\partial D(a;r)$ is compact. Then for each $w \in D(f(a);\delta)$, we have

$$|f(z) - w| \geq |f(z) - f(a)| - |f(a) - w| \geq \delta$$

for every $z \in \partial D(a;r)$, while $|f(a) - w| < \delta$. So $|f - w|$ attains its minimum somewhere in $D(a;r)$. Since $f - w$ is non-constant, it must have a zero in $D(a;r)$ by minimum modulus principle, i.e. $w \in f\big(D(a;r)\big)$. Thus $D(f(a);\delta) \subseteq f\big(D(a;r)\big) \subseteq f(V)$, and so $f(V)$ is open. ■

Theorem 4.44 (Biholomorphic regions) *Let $U, V \subseteq \mathbb{C}$ be regions. If $f: U \to V$ is a holomorphic function which has an inverse $g: V \to U$, then g is also holomorphic.*

Proof. First, g is continuous because for each open set $W \subseteq U$, $g^{-1}(W) = f(W)$ is open by the open mapping theorem. Next, for each $a \in U$ such that $f'(a) \neq 0$, we have

$$\lim_{z \to f(a)} \frac{g(z) - g\big(f(a)\big)}{z - f(a)} = \lim_{z \to f(a)} \frac{1}{\dfrac{f\big(g(z)\big) - f(a)}{g(z) - a}} = \lim_{w \to a} \frac{1}{\dfrac{f(w) - f(a)}{w - a}} = \frac{1}{f'(a)}$$

by the continuity of g, so g is differentiable at $f(a)$ whenever $f'(a) \neq 0$. Finally, since the zeros of f' are isolated, we have proved that g is continuous on V and is holomorphic except perhaps at some isolated points in V. Therefore g is holomorphic on V by Theorem 4.23. ■

Remark 4.45 In fact, if $f: U \to V$ is a holomorphic function which has an inverse $g: V \to U$, then f' has no zeros in U. To see this, we note that g is proved to be holomorphic in Theorem 4.44, so chain rule gives $g'\big(f(z)\big)f'(z) = 1$ for every $z \in U$, and so $f'(z) \neq 0$ for any $z \in U$.

An application of the maximum modulus principle yields the following **Schwarz lemma**, which characterizes all the **holomorphic functions from $\boldsymbol{D(0;1)}$ to $\boldsymbol{D(0;1)}$**.

Theorem 4.46 (Schwarz lemma) *Let $f: D(0;1) \to D(0;1)$ be a holomorphic function such that $f(0) = 0$. Then,*

(i) $|f(z)| \leq |z|$ *for every* $z \in D(0;1)$*, and* $|f'(0)| \leq 1$*;*

(ii) *Either one of the equalities in (i) holds for some $w \in D(0;1)$ if and only if there exists $\theta \in \mathbb{R}$ such that*
$$f(z) = e^{i\theta} z \qquad \text{for every } z \in D(0;1).$$

Proof. Let $F: D(0;1) \to \mathbb{C}$ be the function defined by
$$F(z) = \begin{cases} f(z)/z & \text{if } z \in D(0;1) \setminus \{0\} \\ f'(0) & \text{if } z = 0 \end{cases}.$$
Then F is holomorphic by Theorem 4.23.

Now to prove (i), we note that for each $r \in (0,1)$, F is continuous on $\overline{D(0;r)}$ and holomorphic on $D(0;r)$, so by maximum modulus principle,
$$\max\left\{|F(z)|: z \in \overline{D(0;r)}\right\} = \max\{|F(z)|: z \in \partial D(0;r)\}.$$
Thus for every $z \in D(0;r)$, we have
$$|F(z)| \leq \max\left\{\left|\frac{f(z)}{z}\right|: z \in \partial D(0;r)\right\} = \frac{\max\{|f(z)|: z \in \partial D(0;r)\}}{r} \leq \frac{1}{r},$$
i.e. $\left|\frac{f(z)}{z}\right| \leq \frac{1}{r}$ for every $z \in D(0;r) \setminus \{0\}$ and $|f'(0)| \leq \frac{1}{r}$. These inequalities hold for every $r \in (0,1)$, so $|f(z)| \leq |z|$ for every $z \in D(0;1)$, and $|f'(0)| \leq 1$.

To prove (ii), we note that if either one of these equalities hold for some $w \in D(0;1)$, then $|F|$ attains relative maximum at w, so F is constant by maximum modulus principle. This constant must have absolute value 1 because $|F(w)| = 1$. The converse is trivial. ∎

The following is an important family of holomorphic functions from $D(0;1)$ to $D(0;1)$.

Lemma 4.47 *For each $a \in D(0;1)$, let $\varphi_a: D(0;1) \to D(0;1)$ be the function*
$$\varphi_a(z) = \frac{z - a}{1 - \bar{a}z}.$$
Then

⊙ φ_a *is holomorphic and invertible (i.e. an* ***automorphism*** *of* $D(0;1)$*) with* ${\varphi_a}^{-1} = \varphi_{-a}$*, and*

⊙ ${\varphi_a}'(z) = \frac{1-|a|^2}{(1-\bar{a}z)^2}$ *for every* $z \in D(0;1)$*.*

Example 4.48 Let $f: D(0;1) \to D(0;1)$ be a holomorphic function. Show that

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}$$

for every $z \in D(0;1)$, and that this bound for $|f'|$ is sharp, i.e. for each $z \in D(0;1)$ there exists a holomorphic function f such that the equality is attained at z.

Proof:

For each $a \in D(0;1)$, let $\varphi_a: D(0;1) \to D(0;1)$ be the function $\varphi_a(w) = \frac{w-a}{1-\bar{a}w}$ in Lemma 4.47.

Now if $f: D(0;1) \to D(0;1)$ is a holomorphic function, then for each $z \in D(0;1)$, the composite function $g = \varphi_{f(z)} \circ f \circ \varphi_{-z}: D(0;1) \to D(0;1)$ is also holomorphic and satisfies $g(0) = 0$. By Schwarz lemma (i) we have $|g'(0)| \leq 1$, i.e.

$$1 \geq \left|\varphi_{f(z)}'\big(f(z)\big) \cdot f'(z) \cdot \varphi_{-z}'(0)\right| = \frac{1}{1 - |f(z)|^2} \cdot |f'(z)| \cdot (1 - |z|^2),$$

so $|f'(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2}$ as desired. This bound for $|f'|$ is sharp because for each $z \in D(0;1)$,

$$|\varphi_z'(z)| = \frac{1}{1 - |z|^2} = \frac{1 - |\varphi_z(z)|^2}{1 - |z|^2}.$$

∎

Example 4.49 Does there exist a holomorphic function $f: D(0;1) \to D(0;1)$ such that

$$f(0) = \frac{1}{2} \qquad \text{and} \qquad f'(0) = \frac{3}{4}?$$

If yes, is it unique?

Solution:

Suppose that $f: D(0;1) \to D(0;1)$ is such a holomorphic function. For each $a \in D(0;1)$, let $\varphi_a: D(0;1) \to D(0;1)$ be the function $\varphi_a(w) = \frac{w-a}{1-\bar{a}w}$. Then $g = \varphi_{\frac{1}{2}} \circ f: D(0;1) \to D(0;1)$ is a holomorphic function. It also satisfies that $g(0) = 0$ and

$$g'(0) = \varphi_{\frac{1}{2}}'\big(f(0)\big)f'(0) = \frac{1 - (1/2)^2}{1 - (1/2)f(0)} f'(0) = 1,$$

so by Schwarz lemma (ii), there exists $\theta \in \mathbb{R}$ such that $g(z) = e^{i\theta} z$ for every $z \in D(0;1)$. Since $g'(0) = e^{i\theta}$, it follows that $e^{i\theta} = 1$ and so in fact $g(z) = z$ for every $z \in D(0;1)$. Therefore the function f is uniquely given by

$$f(z) = \varphi_{-\frac{1}{2}}\big(g(z)\big) = \varphi_{-\frac{1}{2}}(z) = \frac{2z + 1}{2 - z}$$

for every $z \in D(0;1)$. It can be easily checked that $f(0) = \frac{1}{2}$ and $f'(0) = \frac{3}{4}$ indeed.

5. Liouville's Theorem

When we apply the ML-estimate to the (generalized) Cauchy integral formula on circles, we often get useful **estimates of the modulus** of a holomorphic function and its derivatives.

Lemma 4.50 (Cauchy estimate) *Let $U \subseteq \mathbb{C}$ be an open set, let $f: U \to \mathbb{C}$ be a holomorphic function, and let $a \in U$ and $r > 0$ such that $\overline{D(a;r)} \subset U$. Then for each $n \in \mathbb{N} \cup \{0\}$,*

$$\left|f^{(n)}(a)\right| \leq \frac{n!\, M_r}{r^n},$$

where $M_r \coloneqq \max\{|f(z)|: z \in \partial D(a;r)\}$.

Note that the $n = 0$ case of this lemma is compatible with the maximum modulus principle (Corollary 4.40).

Proof. By generalized Cauchy integral formula, we have

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{\partial D(a;r)} \frac{f(z)}{(z-a)^{n+1}}\, dz,$$

so ML-estimate gives

$$\left|f^{(n)}(a)\right| = \frac{n!}{2\pi} \left| \oint_{\partial D(a;r)} \frac{f(z)}{(z-a)^{n+1}}\, dz \right| \leq \frac{n!}{2\pi} \cdot \frac{M_r}{r^{n+1}} \cdot 2\pi r = \frac{n!\, M_r}{r^n}.$$

■

Liouville's Theorem states that a **bounded entire function must be constant**. This property of holomorphic functions is again very different from differentiable functions of a real variable, as there exist functions of a real variable which are bounded, infinitely differentiable on $\mathbb{R}$, but non-constant.

Theorem 4.51 (Liouville) *Let $f: \mathbb{C} \to \mathbb{C}$ be a bounded entire function. Then f is a constant function.*

Proof. Suppose that $|f(z)| \leq M$ for every $z \in \mathbb{C}$. Then for each $a \in \mathbb{C}$ and for every $r > 0$, Cauchy estimate gives

$$|f'(a)| = \left| \frac{1}{2\pi i} \oint_{\partial D(a;r)} \frac{f(z)}{(z-a)^2}\, dz \right| \leq \frac{1}{2\pi} \cdot \frac{M}{r^2} \cdot 2\pi r = \frac{M}{r}.$$

Since this inequality is true for every $r > 0$, it follows that $|f'(a)| = 0$, i.e. $f'(a) = 0$. Now we have shown that $f'(a) = 0$ for every $a \in \mathbb{C}$, so f is a constant function. ■

One useful application of Liouville's Theorem is in proving the **Fundamental Theorem of Algebra**. Although the statement of this theorem is completely algebraic, a purely algebraic proof is yet to be found. Apart from the two proofs below, we will give two more analytic proofs to this theorem later in chapter 5.

Theorem 4.52 (Fundamental Theorem of Algebra) *A non-constant polynomial with complex coefficients must have a zero in $\mathbb{C}$. In other words, the field of complex numbers $\mathbb{C}$ is algebraically closed.*

Proof. Let p be a non-constant polynomial, and suppose on the contrary that p has no zeros in $\mathbb{C}$. Then $\frac{1}{p}$ is entire. Since $\lim_{z\to\infty}\frac{1}{p(z)} = 0$ (why?), $\frac{1}{p}$ is also bounded. Therefore $\frac{1}{p}$ is a constant function by Liouville's Theorem, i.e. p is a constant function, which is a contradiction. ■

Alternative proof. Let p be a non-constant polynomial, suppose on the contrary that p has no zeros in $\mathbb{C}$. Then $\frac{1}{p}$ is entire. Since $\lim_{z\to\infty}\frac{1}{p(z)} = 0$, there exists $r > 0$ such that $\left|\frac{1}{p(z)}\right| < \left|\frac{1}{p(0)}\right|$ for every $z \in \mathbb{C} \setminus D(0;r)$. This implies that $\frac{1}{p}$ attains absolute maximum, and in particular relative maximum, at some point in $D(0;r)$. Therefore $\frac{1}{p}$ is a constant function by maximum modulus principle (Theorem 4.38), i.e. p is a constant function, which is a contradiction. ■

Liouville's Theorem is a powerful tool to show that **non-constant entire functions cannot "grow too slowly"**. It can be easily applied when there is some bound to the modulus of a function.

Example 4.53 Let $f: \mathbb{C} \to \mathbb{C}$ be an entire function and there exists $M > 0$ such that
$$|f(z)| \leq M|e^z|$$
for every $z \in \mathbb{C}$. Show that there exists $c \in \mathbb{C}$ such that $f(z) = ce^z$ for every $z \in \mathbb{C}$.

Proof:
Let $g: \mathbb{C} \to \mathbb{C}$ be the function $g(z) = f(z)e^{-z}$. Since f is entire, g is also entire. On the other hand, we also have
$$|g(z)| = \frac{|f(z)|}{|e^z|} \leq M$$
for every $z \in \mathbb{C}$, so g is bounded. Therefore by Liouville's Theorem, g is a constant function, i.e. there exists $c \in \mathbb{C}$ such that $g(z) = c$ for every $z \in \mathbb{C}$. This implies that $f(z) = ce^z$ for every $z \in \mathbb{C}$. ■

Example 4.54 Let $f: \mathbb{C} \to \mathbb{C}$ be an entire function such that

$$f(z+1) = f(z+i) = f(z)$$

for every $z \in \mathbb{C}$. Show that f is a constant function.

Proof: Let $U = \{z \in \mathbb{C}: 0 \leq \operatorname{Re} z \leq 1 \text{ and } 0 \leq \operatorname{Im} z \leq 1\}$. Since U is a compact set and f is continuous, the direct image $f(U)$ is bounded. By the periodicity of f, $f(\mathbb{C}) = f(U)$ is also bounded (cf. Example 2.32). Now f is a bounded entire function, so by Liouville's Theorem, f is a constant function. ∎

Example 4.55 Let $f: \mathbb{C} \to \mathbb{C}$ be an entire function whose real part is always non-negative. Show that f is a constant function.

Proof: Let $g: \mathbb{C} \to \mathbb{C}$ be the function $g(z) = e^{-f(z)}$. Since f is entire, g is also entire. Now since $\operatorname{Re} f(z) \geq 0$ for every $z \in \mathbb{C}$, we have

$$|g(z)| = \left|e^{-f(z)}\right| = e^{-\operatorname{Re} f(z)} \leq 1$$

for every $z \in \mathbb{C}$, so g is bounded. Therefore by Liouville's Theorem, g is a constant function, i.e. there exists $c \in \mathbb{C} \setminus \{0\}$ such that $g(z) = c$ for every $z \in \mathbb{C}$. Since $c \neq 0$, there exists $a \in \mathbb{C}$ such that $e^a = c$, i.e.

$$e^{-f(z)} = e^a$$

for every $z \in \mathbb{C}$. Now since f is continuous, the image of f must be a non-empty **connected** subset of $\{-a + 2\pi k i: k \in \mathbb{Z}\}$, which must be a singleton. In other words, f is constant. ∎

Example 4.56 Let $n \in \mathbb{N}$ and let $f: \mathbb{C} \to \mathbb{C}$ be an entire function such that

$$|f(z)| \leq |z|^n$$

for every $z \in \mathbb{C}$. Show that f is a polynomial of degree at most n.

Proof: [We mimic the proof of Liouville's Theorem.]

For each $a \in \mathbb{C}$ and for every $r > 0$, Cauchy estimate gives

$$\begin{aligned}\left|f^{(n+1)}(a)\right| &= \left|\frac{(n+1)!}{2\pi i} \oint_{\partial D(a;r)} \frac{f(w)}{(w-a)^{n+2}} dw\right| \\ &\leq \frac{(n+1)!}{2\pi} \cdot \frac{(|a|+r)^n}{r^{n+2}} \cdot 2\pi r = \frac{(n+1)!\,(|a|+r)^n}{r^{n+1}}.\end{aligned}$$

Since this inequality is true for every $r > 0$ and since

$$\lim_{r \to +\infty} \frac{(n+1)!\,(|a|+r)^n}{r^{n+1}} = 0,$$

it follows that $\left|f^{(n+1)}(a)\right| = 0$, i.e. $f^{(n+1)}(a) = 0$. Now we have shown that $f^{(n+1)}(a) = 0$ for every $a \in \mathbb{C}$, so f is a polynomial of degree at most n. ∎

At the first look, the following result seems to be much stronger than Liouville's Theorem. But it turns out to be an easy consequence of Liouville's Theorem.

Example 4.57 Let $f: \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Show that

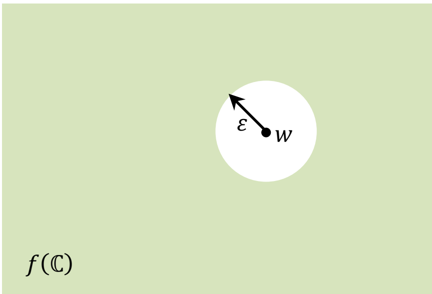
$$\overline{f(\mathbb{C})} = \mathbb{C},$$

i.e. for each $w \in \mathbb{C}$ and each $\varepsilon > 0$, there exists $z \in \mathbb{C}$ such that $|f(z) - w| < \varepsilon$.

Proof:
Suppose on the contrary that there exist $w \in \mathbb{C}$ and $\varepsilon > 0$ such that

$$|f(z) - w| \geq \varepsilon$$

for every $z \in \mathbb{C}$.



Let $g: \mathbb{C} \to \mathbb{C}$ be the function defined by

$$g(z) = \frac{1}{f(z) - w}.$$

Then g is entire since the denominator is holomorphic and never zero. On the other hand,

$$|g(z)| = \left|\frac{1}{f(z) - w}\right| \leq \frac{1}{\varepsilon}$$

for every $z \in \mathbb{C}$, so g is bounded. Therefore by Liouville's Theorem, g is a constant function, i.e. there exists $c \in \mathbb{C} \setminus \{0\}$ such that $g(z) = c$ for every $z \in \mathbb{C}$. This implies that $f(z) = \frac{1}{c} + w$ for every $z \in \mathbb{C}$, so f is a constant function, which gives a contradiction. ■

Remark 4.58 Let $f: \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Liouville's Theorem in its original form asserts that the image of f is an **unbounded** subset of $\mathbb{C}$. Example 4.57 is a stronger result which asserts that the image of f is not only unbounded, but also **dense** in $\mathbb{C}$ (i.e. the image does not miss out any disk in $\mathbb{C}$). In fact, there is an even stronger result which asserts that the image of f is actually **the whole $\mathbb{C}$ minus at most one point**. In other words, an entire function whose image misses two distinct points in $\mathbb{C}$ must be a constant function. This deep result is known as ***Picard's Little Theorem***. It is the best one can achieve, because the non-constant entire function $\exp$ has image $\mathbb{C} \setminus \{0\}$.

Remark 4.59 The following is a summary on properties of **holomorphic functions in a complex variable**. Let $U \subseteq \mathbb{C}$ be a region. A function $f: U \to \mathbb{C}$ is said to be ***holomorphic*** if it is differentiable on U. By this moment, we have seen many properties of holomorphic functions which are not shared by real differentiable functions.

(i) **Cauchy-Riemann equations**:
A function $f(x+iy) = u(x,y) + iv(x,y): U \to \mathbb{C}$ is holomorphic if and only if the partial derivatives u_x, u_y, v_x, v_y are all continuous on U and satisfy the Cauchy-Riemann Equations
$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}.$$

(ii) **Path-independent integrals**: (Cauchy-Goursat Theorem / Morera's Theorem)
If U is a **simply connected** region and $f: U \to \mathbb{C}$ is continuous, then f is holomorphic if and only if along any **closed** piecewise C^1 curve γ lying completely inside U, we have
$$\oint_\gamma f(z)dz = 0.$$

(iii) **Taylor series**:
Given any $a \in \mathbb{C}$ and $r > 0$, a function $f: D(a;r) \to \mathbb{C}$ is holomorphic if and only if it is analytic, i.e. it has a convergent Taylor series
$$f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k$$
on $D(a;r)$. This series is uniformly absolutely-convergent on compact subsets of $D(a;r)$.

(iv) **Infinite differentiability**:
A function $f: U \to \mathbb{C}$ is holomorphic if and only if it is infinitely many times differentiable (i.e. smooth / of class C^∞) on U.

(v) **Mean value property, maximum modulus principle**
- ⊙ Mean value of a holomorphic function f on a circle equals to its value at the center, i.e.
$$\frac{1}{2\pi}\int_0^{2\pi} f\left(a + re^{it}\right)dt = f(a).$$
- ⊙ A non-constant holomorphic function $f: U \to \mathbb{C}$ cannot attain relative maximum in U.
- ⊙ If U is a bounded region, then a continuous function $f: \overline{U} \to \mathbb{C}$ holomorphic on U must attain absolute maximum on ∂U, i.e. $\max\left\{|f(z)|: z \in \overline{U}\right\} = \max\{|f(z)|: z \in \partial U\}$.

(vi) **Open mapping**:
A non-constant holomorphic function $f: U \to \mathbb{C}$ is an open mapping, i.e. it maps open sets onto open sets.

(vii) **Unboundedness on $\mathbb{C}$**: (Liouville's Theorem)
An entire function $f: \mathbb{C} \to \mathbb{C}$ is bounded if and only if it is constant.

All these striking properties of holomorphic functions provide a foundation to the area of study of **theory of holomorphic functions in a complex variable**.

Summary of Chapter 4

The following are what you need to know in this chapter in order to get a pass (a distinction) in this course:

✓ **Taylor series** of a holomorphic function
- ⊙ To compute the **Taylor series** of a holomorphic function
- ⊙ Holomorphic functions are **infinitely differentiable**
- ⊙ **Order of a zero** of a holomorphic function

✓ **Morera's Theorem** and its consequences
- ⊙ **Morera's Theorem**: f is continuous and $\oint_\gamma f(z)dz = 0$ for every closed C^1 curve γ $\Rightarrow$ f is holomorphic
- ⊙ **Uniform limit** of a sequence of holomorphic functions is holomorphic
- ⊙ Continuous and "**almost holomorphic**" functions must be holomorphic
- ⊙ **Schwarz reflection principle**

✓ **Isolated zeros theorem** and **identity theorem**
- ⊙ **Isolated zeros theorem**: Zeros of non-constant holomorphic functions are isolated
- ⊙ **Identity Theorem**: If the values of holomorphic functions f and g agree on a convergent sequence, then $f = g$
- ⊙ **Analytic continuation** of a holomorphic function

✓ **Maximum modulus principle** and its consequences
- ⊙ **Mean value property**
- ⊙ **Maximum modulus principle**: f non-constant holomorphic $\Rightarrow$ Relative maximum of $|f|$ cannot occur in the interior of the domain
- ⊙ **Maximum modulus principle**: f holomorphic $\Rightarrow$ Absolute maximum of $|f|$ occurs in the boundary of the domain
- ⊙ **Open mapping theorem**
- ⊙ **Schwarz lemma,** holomorphic functions from $D(0;1)$ to $D(0;1)$

✓ **Liouville's Theorem** and its consequences
- ⊙ **Cauchy estimates**: Cauchy integral formula + ML-estimate
- ⊙ **Liouville's Theorem**: Bounded entire functions are constants
- ⊙ **Fundamental Theorem of Algebra**
- ⊙ **Picard's Little Theorem**