MATH 5450 - Stochastic Process
Final Exam (take home)

(due: May 24, 2023)

Q1: (a) Is any Markov process a martingale? If yes, prove it. Otherwise, construct a counterexample. b) Is any martingale a Markov Process? If yes, prove it. Otherwise, construct a counterexample.

Q2: Suppose we have an urn initially containing b black and r red balls. At each draw we remove a ball at random and then replace it, together with c balls of the colour drawn. Let b_n and r_n denote, respectively, the numbers of black and red balls in the urn after the nth drawing and write $Y_n = b_n/(b_n + r_n), n \geq 0$, the proportion of black balls. Note that $b_n + r_n = b + r + cn$. Consider the following problems on this model (Polya's urn):

(i) Prove that $\{Y_n\}$ is a martingale with appropriate filtration.

(ii) Prove that there exists a random variable Y, so that $Y_n \to Y$ almost surely if $n \to \infty$.

(iii) Using certain version of martingale CLT to find the limiting distribution of $n^{1/2}(Y_n - Y)$.

Q3: Prove the following Wald equation: Let $\{X_i\}$ be i.i.d. random variables and T is a stopping time w.r.t. the natural filtration generated by X_i's. Then one has $\mathbb{E}(\sum_{i=1}^T X_i) = \mathbb{E}T\mathbb{E}X_1$, assuming $\mathbb{E}T < \infty$ and $\mathbb{E}|X_1| < \infty$.

Q4: Let $\{X_n\}$ be a martingale sequence adapted to $\{\mathcal{F}_n\}$, with $X_0 = 0$. Let $\{Y_n\}$ be the corresponding sequence of martingale difference, i.e, $Y_i = X_i - X_{i-1}$. Prove the following Burkholder's inequality for any given $1 < p < \infty$:

$$C_1\mathbb{E}\Big|\sum_{i=1}^{n} Y_i^2\Big|^{p/2} \leq \mathbb{E}\big|X_n\big|^p \leq C_2\mathbb{E}\Big|\sum_{i=1}^{n} Y_i^2\Big|^{p/2},$$

where C_1 and C_2 are some positive constants which may depend on p.

Q5: Let $\{X_n\}$ be a Markov chain containing an absorbing state s, i.e, $p(s,s) = 1$. While all other states i communicate with s, i.e., for any state i, there exists an $n = n(i)$ such that $p^n(i,s) > 0$. Show that all states other than s are transient.

Q6: The distinct pair i, j of states of a Markov chain is called symmetric if

$$\mathbb{P}_i(T_j < T_i) = \mathbb{P}_j(T_i < T_j),$$

where $T_a := \min\{n \geq 1 : X_n = a\}$. Show that if $X_0 = i$ and i, j are symmetric, the expected number of visits to j before the chain revisits i is 1.

Q7: Show that if S is finite and p is irreducible and aperiodic, then there is a uniform m so that $p^m(x,y) > 0$ for all x, y.