Differential Geometry

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Curves

1.1 INTRODUCTION

By considering a simple example, we are trying to get some sense what differential geometry studies.

Motivation: What do we want to study?

Consider a straight line, or a round circle on the plane $\mathbb{R}^2$, or a "twisted" and "curved" curve in the space $\mathbb{R}^3$

Question:

1. describe the curve by some mathematical formulae.

2. measure the "curviness" and "twistness" of the curve.

Example 1.1. Let's study a circle of radius R. We can describe the circle by the following parametrized equation:

$$\begin{cases} x(t) = Rcos\dfrac{t}{R}, \\ y(t) = Rsin\dfrac{t}{R}. \end{cases}$$

We can also express the circle in the following way: $\alpha(t) = (x(t), y(t))$.

How to measure the curviness of the circle?

Clearly the bigger the radius, the less curved the curve. Let's compute the tangent vector $\alpha'(t)$ of the circle:

$$\alpha'(t) = (x'(t), y'(t)) = (-sin\frac{t}{R}, cos\frac{t}{R}).$$

Notice that the tangent vector $\alpha'(t)$ is a unit vector. Let's compute the second derivative of $\alpha(t)$, we get

$$\alpha''(t) = (-\frac{1}{R}cos\frac{t}{R}, -\frac{1}{R}sin\frac{t}{R}).$$

The length of $\alpha''(t)$ is $|\alpha''(t)| = 1/R$. If we take this quantity as the measurement of the curviness of the circle, we see that the smaller this quantity is, the less curved the curve is. So we can say that the length of the second derivative of a curve $\alpha(t)$ is the quantity measuring the curviness of the curve.

1.2 PARAMETRIZED CURVE

We give the definition of paramertrized curve studied in this course.

Definition 1.2. *A parametrized differentiable curve α is a differentiable map $\alpha: I \to \mathbb{R}^3$ (or $\mathbb{R}^2$ for plane curve) where I is an open interval (a, b).*

For a space curve, we write $\alpha(t) = (x(t), y(t), z(t))$. For a plane curve, we write $\alpha(t) = (x(t), y(t))$. $t \in (a, b)$ is called the parameter. The word "differentiable" in the definition means that all the functions $x(t)$, $y(t)$ and $z(t)$ are differentiable functions of t. Here are some examples of parametrized curves.

Example 1.3.

1. $\alpha(t) = (cost,\, sint,\, t),\ \ t \in \mathbb{R}^1$.
2. $\alpha(t) = (Rcos\frac{t}{R},\, Rsin\frac{t}{R})$.
3. $\alpha(t) = (Rcost,\, Rsint)$.
4. $\alpha(t) = (t^3,\, t^2)$.
5. $\alpha(t) = (t^3 - 4t,\, t^2 - 4)$. ∎

1.3 REGULAR CURVES, ARCLENGTH

We introduce the concept of regular curves.

Definition 1.4. A parametrized differentiable curve $\alpha: I \to \mathbb{R}^3$ is said to be regular if $\alpha'(t) \neq 0$ for all $t \in I$, i.e., the curve has a tangent vector everywhere.

Example 1.5. For $\alpha(t) = (t^3,\, t^2)$, $\alpha'(t) = (3t^2, 2t)$, $\alpha'(0) = (0,\, 0)$. Hence $\alpha(t)$ is not a regular curve.

$\beta(t) = (cost^2,\, sint^2)$ parametrizes the circle of radius 1. However, $\beta'(t) = (-2tsint^2,\, 2tcost^2)$. Hence $\beta'(0) = (0,\, 0)$. Hence this parametrized curve is not regular.

$\gamma(t) = (cost,\, sint)$ is another parametrization of the circle of radius 1. $\gamma'(t) = (-sint,\, cost)$ is never a zero vector. Hence $\gamma(t)$ is a regular curve. ∎

Recall the definition of the arclength of a curve.

Definition 1.6. *The arclength of a curve $\alpha(t)$ from $t = t_1$ to $t = t_2$ is defined as follows:*

$$\int_{t_1}^{t_2} |\alpha'(t)|\, dt = \begin{cases} \displaystyle\int_{t_1}^{t_2} \sqrt{x'(t)^2 + y'(t)^2}\, dt & \textit{for plane curve,} \\ \displaystyle\int_{t_1}^{t_2} \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt & \textit{for space curve.} \end{cases}$$

Pick a point $\alpha(t_0)$ on the curve α. Define $s(t) = \int_{t_0}^{t} |\alpha'(t)|\, dt$. $s(t)$ is a function of t and $s'(t) = |\alpha(t)|$. If the curve is regular, then $|\alpha'(t)| \neq 0$. By the inverse function theorem we learned in calculus, the function $s = s(t)$ will have an inverse function, denoted by $t = t(s)$. Hence the same curve can be parametrized by the arclength s in the following way: $\beta(s) = \alpha(t(s))$. So every regular curve can be parametrized by arclength.

What is good about the arclength parameter?

If t is an arclength parameter, i.e., there exists a constant C_0 such that $s = t + C_0$. Then $\frac{ds}{dt} = 1$, i.e., $|\alpha'(t)| = 1$ for all t. Conversely, if $|\alpha'(t)| = 1$ for all t, then $\frac{ds}{dt} = 1$. Therefore there exists a constant C_0 such that $s = t + C_0$, i.e., t is an arclength parameter. Thus the characterization of an arclength parameter t is that the $\alpha'(t)$ is an unit tangent vector.

Example 1.7. For the curve $\alpha(t) = (Rcost, Rsint)$, $\alpha'(t) = (-Rsint, Rcost)$, $|\alpha'(t)| = R$. Unless $R = 1$, t is not an arclength parameter.

For the curve $\alpha(t) = (Rcos\frac{t}{R}, Rsin\frac{t}{R})$, $\alpha'(t) = (-sint, cost)$, $|\alpha'(t)| = 1$. Hence t is an arclength parameter. ■

Let's talk about the orientation of a curve. For a curve $\alpha(t)$ where $t \in (a, b)$. When t moves from a to b, the point $\alpha(t)$ moves from the point $\alpha(a)$ to the point $\alpha(b)$. We get the orientation of the curve. The curve $\beta(t) = \alpha(-t)$ where $t \in (-b, -a)$ moves from $\beta(-b) = \alpha(b)$ to $\beta(-a) = \alpha(a)$ when t moves in the positive direction from $-b$ to $-a$. The trace of β is the same as that of α, but has the different orientation.

Review of Dot Product and Cross Product We do some review on dot product and cross product.

Dot Product:

For two vectors $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ (or $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$ in the case of $\mathbb{R}^2$), the dot product is defined as

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 \qquad (\text{or } \vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 \text{ in } \mathbb{R}^2).$$

Let θ be the angle between $\vec{v}$ and $\vec{w}$, then $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| cos\theta$. We also know that $\vec{v} \cdot \vec{w} = 0$ iff $\vec{v}$ is perpendicular to $\vec{w}$. If $\vec{v}(t) = (v_1(t), v_2(t), v_3(t))$ and $\vec{w}(t) = (w_1(t), w_2(t), w_3(t))$, then $\frac{d}{dt}(\vec{v}(t) \cdot \vec{w}(t)) = \vec{v}'(t) \cdot \vec{w}(t) + \vec{v}(t) \cdot \vec{w}'(t)$.

Cross Product:

For two vectors $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$, the cross product is defined as $\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$.

Let θ be the angle between $\vec{v}$ and $\vec{w}$, then $|\vec{v} \times \vec{w}| = |\vec{v}| \cdot |\vec{w}| \cdot sin\theta$. We also know that $\vec{v} \times \vec{w} = \vec{0}$ iff $\vec{v}$ and

$\vec{w}$ are parallel. If $\vec{v}(t) = (v_1(t), v_2(t), v_3(t))$ and $\vec{w}(t) = (w_1(t), w_2(t), w_3(t))$, then $\frac{d}{dt}(\vec{v}(t) \times \vec{w}(t)) = \vec{v}'(t) \times \vec{w}(t) + \vec{v}(t) \times \vec{w}'(t)$.

1.4 THE LOCAL THEORY OF CURVES IN $\mathbb{R}^3$ PARAMETRIZED BY ARCLENGTH

Let $\alpha(s)\colon I \to \mathbb{R}^3$ be a regular parametrized differentiable curve with s being the arclength parameter, i.e., $|\alpha'(s)| = 1$ for all $s \in I$.

Definition 1.8. *$\kappa(s) = |\alpha''(s)|$ is called the curvature of $\alpha(s)$ at s.*

Example 1.9. Given two fixed vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$. $\alpha(s) = \vec{u} + s\vec{v}$ parametrizes a straight line. $\alpha'(s) = \vec{v}$. Hence s is an arclength parameter iff $|\vec{v}| = 1$, i.e., $\vec{v}$ is a unit vector. Assume $\vec{v}$ is a unit vector, then the curvature of α is $|\alpha''(s)| = |\vec{0}| = 0$.

If $\alpha(s)$ is a regular curve with arclength parameter s, if the curvature $\kappa(s) = |\alpha''(s)| = 0$ for all s, then $\alpha''(s) = \vec{0}$. Hence there exists a constant unit vector $\vec{v}$ such that $\alpha'(s) = \vec{v}$. Therefore there exists a constant vector $\vec{u}$ such that $\alpha(s) = \vec{u} + s\vec{v}$ for all s, i.e., $\alpha(s)$ is a straight line.

Conclusion: curvature $\kappa(s)$ of a curve $\alpha(s)$ vanishes for all s iff the curve $\alpha(s)$ is a straight line. ∎

If we change the orientation of the curve $\alpha(s)$ where s is an arclength parameter, i.e., consider the new curve $\beta(s) = \alpha(-s)$, then $\beta'(s) = -\alpha'(-s)$ and $\beta''(s) = \alpha''(-s)$. Hence the curvature of β is $|\beta''(s)| = |\alpha''(-s)|$. Hence the curvature doesn't change under the change of the orientation.

Example 1.10. Consider a circle of radius R:

$$\alpha(t) = (Rcos\frac{t}{R}, Rsin\frac{t}{R}, 8).$$

$\alpha'(t) = (-sin\frac{t}{R}, cos\frac{t}{R}, 0)$, and $\alpha''(t) = (-\frac{1}{R}cos\frac{t}{R}, \frac{1}{R}sin\frac{t}{R}, 0)$. Since $|\alpha'(t)| = 1$ for all t, t is an arclength parameter. Hence the curvature $\kappa(t) = |\alpha''(t)| = \frac{1}{R}$.

Conclusion: the bigger the radius is, the smaller the curvature becomes. ∎

Let $\vec{t}(s) = \alpha'(s)$. Since s is an arclength parameter, $|\alpha'(s)| = 1$ for all s, i.e.,

$$1 = \alpha'(s) \cdot \alpha'(s) = \vec{t}(s) \cdot \vec{t}(s) \qquad \text{for all } s.$$

Differentiate the both sides of the equation above, we get

$$0 = \frac{d}{ds}(\vec{t}(s) \cdot \vec{t}(s)) = \vec{t}'(s) \cdot \vec{t}(s) + \vec{t}(s) \cdot \vec{t}'(s) = 2\vec{t}(s) \cdot \vec{t}'(s).$$

Hence $\vec{t}'(s)$ is perpendicular to $\vec{t}(s)$. Assume $\kappa(s) = |\alpha''(s)| > 0$ for all s. Now we define a normal vector of the curve

$$\vec{n}(s) = \frac{\vec{t}'(s)}{\kappa(s)} = \frac{\alpha''(s)}{|\alpha''(s)|}.$$

Hence $\vec{n}(s)$ is a unit vector perpendicular to the tangent vector $\vec{t}(s)$.

Define another normal vector $\vec{b}(s) = \vec{t}(s) \times \vec{n}(s)$, called the binormal vector.

Now we get three mutually perpendicular unit vectors $\vec{t}(s)$, $\vec{n}(s)$ and $\vec{b}(s)$. Unlike $\{\vec{i},\vec{j},\vec{k}\}$, $\{\vec{t}(s),\vec{n}(s),\vec{b}(s)\}$ change when s changes. How do they change?

What is $\vec{b}'(s)$?

$$\vec{b}'(s)=\vec{t}'(s)\times\vec{n}(s)+\vec{t}(s)\times\vec{n}'(s)=\vec{t}(s)\times\vec{n}'(s).$$

Hence $\vec{b}'(s)$ is perpendicular to $\vec{t}(s)$. Since $\vec{b}(s)$ is a unit vector, we must have $\vec{b}(s)\cdot\vec{b}(s)=1$. So we get

$$0=(\vec{b}(s)\cdot\vec{b}(s))'=\vec{b}'(s)\cdot\vec{b}(s)+\vec{b}(s)\cdot\vec{b}'(s)=2\vec{b}(s)\cdot\vec{b}'(s).$$

Therefore $\vec{b}'(s)$ is also perpendicular to $\vec{b}(s)$. Hence $\vec{b}'(s)$ must be parallel to $\vec{n}(s)$, i.e., there exists a function $\tau(s)$ such that $\vec{b}'(s)=\tau(s)\vec{n}(s)$.

Definition 1.11. *Let $\alpha(s)\colon I\to\mathbb{R}^3$ be a regular curve with s being an arclength parameter. Assume $\alpha''(s)\neq\vec{0}$ for all $s\in I$. Then $\tau(s)$ found above is called the torsion of the curve α.*

What is $\vec{n}'(s)$?

Since $\vec{n}(s)$ is a unit vector, $\vec{n}(s)\cdot\vec{n}(s)=1$, so we get

$$0=(\vec{n}(s)\cdot\vec{n}(s))'=2\vec{n}'(s)\cdot\vec{n}(s).$$

Hence $\vec{n}'(s)$ is perpendicular to $\vec{n}(s)$, i.e., $\vec{n}(s)$ lies in the plane spanned by $\vec{t}(s)$ and $\vec{b}(s)$. Therefore we can write

$$\vec{n}'(s)=a(s)\vec{t}(s)+c(s)\vec{b}(s)$$

where $a(s)$ and $c(s)$ are functions of s.

What are $a(s)$ and $c(s)$?

$a(s)=\vec{n}'(s)\cdot\vec{t}(s)$. Since $\vec{n}(s)$ is perpendicular to $\vec{t}(s)$, we have $0=\vec{n}(s)\cdot\vec{t}(s)$. So

$$\begin{aligned}0&=(\vec{n}(s)\cdot\vec{t}(s))'\\&=\vec{n}'(s)\cdot\vec{t}(s)+\vec{n}(s)\cdot\vec{t}'(s)\\&=a(s)+\vec{n}(s)\cdot(\kappa(s)\vec{n}(s))=a(s)+\kappa(s).\end{aligned}$$

Therefore $a(s)=-\kappa(s)$.

Similarly, $c(s)=\vec{n}'(s)\cdot\vec{b}(s)$. Since $\vec{n}(s)$ is perpendicular to $\vec{b}(s)$, we have $0=\vec{n}(s)\cdot\vec{b}(s)$. So

$$\begin{aligned}0&=(\vec{n}(s)\cdot\vec{b}(s))'\\&=\vec{n}'(s)\cdot\vec{b}(s)+\vec{n}(s)\cdot\vec{b}'(s)\\&=c(s)+\vec{n}(s)\cdot(\tau(s)\vec{n}(s))=c(s)+\tau(s).\end{aligned}$$

Therefore $c(s)=-\tau(s)$.

Now we get $\vec{n}'(s)=-\kappa(s)\vec{t}(s)-\tau(s)\vec{b}(s)$.

Conclusion: under the condition that $\alpha''(s)\neq\vec{0}$ for all s, we have the following formulae, called Frenet formulae: $\vec{t}(s)=\alpha'(s)$, and

$$\begin{cases} \vec{t}'(s) = \kappa(s)\vec{n}(s), \\ \vec{n}'(s) = -\kappa(s)\vec{t}(s) - \tau(s)\vec{b}(s), \\ \vec{b}'(s) = \tau(s)\vec{n}(s). \end{cases}$$

In summary, for a space regular curve $\alpha(s)$ satisfying the condition that $\alpha''(s) \neq 0$ where s is an arclength parameter, we get the curvature $\kappa(s)$ and the torsion $\tau(s)$ of the curve.

What does torsion $\tau(s)$ tell us about the curve?

Proposition 1.12. *Suppose $\alpha''(s) \neq \vec{0}$ for all s. Then $\tau(s) = 0$ for all s iff $\alpha(s)$ lies on a plane.*

Proof. Assume the curve $\alpha(s)$ lies on a plane. Take a point P_0 on the plane and a unit normal vector $\vec{N}$ of the plane. Then the equation of the plane is given by

$$(\overrightarrow{OP_0} - \alpha(s)) \cdot \vec{N} = 0 \qquad \text{for all } s.$$

Take the differentiation of the equation twice, we get

$$-\alpha'(s) \cdot \vec{N} = 0, \qquad -\alpha''(s) \cdot \vec{N} = 0.$$

Since $\kappa(s) = |\alpha''(s)| \neq 0$, we have the normal vector $\vec{n}(s) = \dfrac{\alpha''(s)}{\kappa(s)}$. Now we get $\vec{t}(s) \perp \vec{N}$ and $\vec{n}(s) \perp \vec{N}$. So

$$\vec{b}(s) = \vec{t}(s) \times \vec{n}(s) = \pm\vec{N}.$$

But $\vec{N}$ is a constant vector, therefore $\vec{b}'(s) = \vec{0}$, i.e., $\tau(s) = 0$ for all s.

Conversely if $\tau(s) = 0$ for all s, $\vec{b}'(s) = \tau(s)\vec{n}(s) = \vec{0}$ for all s. Hence $\vec{b}(s)$ is a constant vector, i.e., $\vec{b}(s) = \vec{b}_0$ for some constant vector $\vec{b}_0$. Since $\vec{t}(s) \perp \vec{b}(s)$, $\vec{t}(s) \cdot \vec{b}_0 = 0$ for all s, i.e., $\alpha'(s) \cdot \vec{b}_0 = 0$. So $(\alpha(s) \cdot \vec{b}_0)' = 0$ for all s, i.e., $\alpha(s) \cdot \vec{b}_0$ is a constant. Thus there exists a constant c_0 such that $\alpha(s) \cdot \vec{b}_0 = c_0$. Therefore the curve $\alpha(s)$ lies on the plane given by the equation $(x, y, z) \cdot \vec{b}_0 = c_0$. ■

In summary,

$$\textit{curviness} \Longleftrightarrow \quad \text{measured by curvature } \kappa(s),$$

$$\textit{twistness} \Longleftrightarrow \quad \text{measured by torsion } \tau(s).$$

Terminologies 1.13. The $\vec{t}\vec{b}$-plane is called the rectifying plane; $\vec{n}\vec{b}$-plane is called the normal plane; $\vec{t}\vec{n}$-plane is called the osculating plane. A line passing through the point $\alpha(s)$ and parallel to $\vec{n}(s)$ is called the principal normal; a line passing through the point $\alpha(s)$ and parallel to $\vec{b}(s)$ is called the binormal.

Example 1.14. Consider the parametrized curve

$$\alpha(s) = \frac{(\sqrt{1+s^2}, 2s, ln(s + \sqrt{1+s^2}))}{\sqrt{5}}.$$

$$\alpha'(s) = \frac{1}{\sqrt{5}}(\frac{s}{\sqrt{1+s^2}}, 2, \frac{1}{s + \sqrt{1+s^2}}).$$

We see that $|\alpha'(s)| = 1$, hence s is an arclength parameter. Since $\vec{t}(s) = \alpha'(s)$, we have

$$\vec{t}'(s) = \alpha''(s) = \frac{1}{\sqrt{5}}\left(\frac{1}{(1+s^2)\sqrt{1+s^2}}, 0, \frac{-s}{(1+s^2)\sqrt{1+s^2}}\right).$$

Hence the curvature $\kappa(s) = |\alpha''(s)| = \dfrac{1}{\sqrt{5}(1+s^2)}$. The normal vector is

$$\vec{n}(s) = \frac{\alpha''(s)}{\kappa(s)} = \left(\frac{1}{\sqrt{1+s^2}}, 0, \frac{-s}{\sqrt{1+s^2}}\right).$$

The binormal vector is

$$\begin{aligned}
\vec{b}(s) &= \vec{t}(s) \times \vec{n}(s) = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{s}{\sqrt{5+5s^2}} & \dfrac{2}{\sqrt{5}} & \dfrac{1}{\sqrt{5+5s^2}} \\ \dfrac{1}{\sqrt{1+s^2}} & 0 & \dfrac{-s}{\sqrt{1+s^2}} \end{pmatrix} \\
&= \frac{1}{\sqrt{5+5s^2}}\left(-2s, \frac{s^2}{\sqrt{1+s^2}} + \frac{1}{\sqrt{1+s^2}}, -2\right) = \frac{1}{\sqrt{5}}\left(\frac{-2s}{\sqrt{1+s^2}}, 1, \frac{-2}{\sqrt{1+s^2}}\right).
\end{aligned}$$

$$\vec{b}'(s) = \frac{-2}{\sqrt{5}(1+s^2)}\left(\frac{1}{\sqrt{1+s^2}}, 0, \frac{-s}{\sqrt{1+s^2}}\right) = \frac{-2}{\sqrt{5}(1+s^2)}\vec{n}(s).$$

Hence $\tau(s) = \dfrac{-2}{\sqrt{5}(1+s^2)}$. ■

Example 1.15. Assume that all normals of a regular parametrized curve pass through a fixed point. Prove that the trace of the curve is contained in a sphere.

Solution: Let $\alpha(s)$ be the curve with an arclength parameter s. The equation for the normal is $\beta(t) = \alpha(s) + t\vec{n}(s)$. Let P_0 be the fixed point. Then for any s, there exists a t_s (dependent on s) such that $\overrightarrow{OP_0} = \alpha(s) + t_s\vec{n}(s)$. We have

$$\overrightarrow{OP_0} \cdot \alpha'(s) = \alpha(s) \cdot \alpha'(s) + t_s\vec{n}(s) \cdot \alpha'(s).$$

Since $\vec{n}(s) \perp \alpha'(s)$, we have $\vec{n}(s) \cdot \alpha'(s) = 0$. So $\overrightarrow{OP_0} \cdot \alpha'(s) = \alpha(s) \cdot \alpha'(s)$, i.e.,

$$(\alpha(s) - \overrightarrow{OP_0}) \cdot \alpha'(s) = 0.$$

$$(\alpha(s) - \overrightarrow{OP_0}) \cdot (\alpha(s) - \overrightarrow{OP_0})' = 0.$$

$$\big((\alpha(s) - \overrightarrow{OP_0}) \cdot (\alpha(s) - \overrightarrow{OP_0})\big)' = 0.$$

Hence there exists a constant C such that

$$(\alpha(s) - \overrightarrow{OP_0}) \cdot (\alpha(s) - \overrightarrow{OP_0}) = C$$

for all s, i.e., $\alpha(s)$ lies on the sphere with center P_0 and radius $\sqrt{C}$. ■

1.5 PLANE CURVE

Let $\alpha(s)$ be a regular curve in $\mathbb{R}^2$, $\alpha(s)\colon I \to \mathbb{R}^2$ where s is an arclength parameter. $\alpha'(s)$ is the tangent vector of $\alpha(s)$. Define the normal vector $\vec{n}(s)$ by requiring $\{\vec{t}(s), \vec{n}(s)\}$ have the same orientation (right-hand rule) as that of $\{\vec{i}, \vec{j}\}$. The (signed) curvature $\kappa(s)$ of the curve $\alpha(s)$ is defined as

$$\alpha''(s) = \vec{t}'(s) = \kappa(s)\vec{n}(s).$$

The definition of the curvature for plane curves is different from that of space curves. The curvature of a space curve is always non-negative. The curvature of a plane curve may be negative.

Here is a way to find the normal vector $\vec{n}(s)$. Let $\alpha(s) = (x(s), y(s))$ where s is an arclength parameter. $\vec{t}(s) = \alpha'(s) = (x'(s), y'(s))$. The normal vector $\vec{n}(s)$ is obtained from the tangent vector $\vec{t}(s)$ by rotating $\vec{t}(s)$ 90° counterclockwise. This rotation corresponds to the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence $\vec{n}(s) = (-y'(s), x'(s))$.

We have $(x''(s), y''(s)) = \vec{t}(s)' = \kappa(s)\vec{n}(s) = \kappa(s)(-y'(s), x'(s))$. Hence we get $x''(s) = -\kappa(s)y'(s)$ and $y''(s) = \kappa(s)x'(s)$. Now $\vec{n}(s)' = (-y''(s), x''(s)) = (-\kappa(s)x'(s), -\kappa(s)y'(s)) = -\kappa(s)\vec{t}(s)$. We get a formula similar to the Frenet formula for space curve.

$$f(x) = \begin{cases} \vec{t}'(s) = \kappa(s)\vec{n}(s), \\ \vec{n}'(s) = -\kappa(s)\vec{t}(s). \end{cases}$$

Example 1.16. Consider the parametrized curve $\alpha(s) = (coss, sins)$. The trace of this curve is the unit circle and is oriented counter-clockwise. $\alpha'(s) = (-sins, coss)$. $\alpha''(s) = (-coss, -sins)$. Therefore $\vec{n}(s) = -(coss, sins)$ by the right hand rule. So $\alpha''(s) = \vec{n}(s)$, i.e., $\kappa(s) = 1$.

Consider another parametrized curve $\alpha(s) = (coss, -sins)$. The trace of this curve is also the unit circle but is oriented clockwise. $\alpha'(s) = (-sins, -coss)$. $\alpha''(s) = (-coss, sins)$. Therefore $\vec{n}(s) = (coss, -sins)$ by the right hand rule. So $\alpha''(s) = -\vec{n}(s)$, i.e., $\kappa(s) = -1$. ∎

One more remark on the right hand rule. It is equivalent to that fact that choice of the normal vector $\vec{n}$ should be such that if we move from $\vec{t}$ to $\vec{n}$, we move counter-clockwise.

1.6 CURVES NOT PARAMETRIZED BY ARCLENGTH PARAMETERS

How do we calculate curvature and torsion of a curve if the curve is parametrized by a parameter t which is not an arclength parameter?

Let $\alpha(t)$ be a regular curve, $\alpha\colon I \to \mathbb{R}^3$, $t \in I$. Suppose $t = t(s)$ where s is an arclength parameter. Recall that the arclength $s(t) = \int_{t_0}^{t} |\alpha'(u)| du$. So s is also a function of t and $\dfrac{ds(t)}{dt} = |\alpha'(t)|$. $s = s(t)$ and $t = t(s)$ are inverse functions to each other, so

$$\frac{dt(s)}{ds} = \frac{1}{\dfrac{ds(t)}{dt}} = \frac{1}{|\alpha'(t)|}.$$

Let $\beta(s) = \alpha(t(s))$. So β parametrizes the same curve but with the arclength parameter s. The curvature and the torsion can be obtained from β.

$$
\begin{aligned}
\beta'(s) &= \frac{d}{ds}\alpha(t(s)) = \alpha'(t)\frac{dt(s)}{ds}, \\
\beta''(s) &= \frac{d}{ds}(\alpha'((t)\frac{dt(s)}{ds}) \\
&= \frac{d}{ds}(\alpha'(t)|_{t=t(s)})\frac{dt(s)}{ds} + \alpha'(t)\frac{d^2t(s)}{ds^2} \\
&= \alpha''(t)(\frac{dt(s)}{ds})^2 + \alpha'(t)\frac{d^2t(s)}{ds^2}.
\end{aligned}
$$

Since $\vec{t}(s) = \beta'(s)$ and $\beta''(s) = \kappa(s)\vec{n}(s)$, we have

$$\beta'(s) \times \beta''(s) = \kappa(s)\vec{t}(s) \times \vec{n}(s) = \kappa(s)\vec{b}(s).$$

Since $|\vec{b}(s)| = 1$, we have

$$
\begin{aligned}
\kappa(s) &= |\beta'(s) \times \beta''(s)| \\
&= |\frac{d\alpha(t)}{dt}\frac{dt(s)}{ds} \times (\frac{d^2\alpha(t)}{dt^2}(\frac{dt(s)}{ds})^2 + \frac{d\alpha(t)}{dt}\frac{d^2t(s)}{ds^2})| \\
&= |\frac{d\alpha(t)}{dt} \times \frac{d^2\alpha(t)}{dt^2}||\frac{dt(s)}{ds}|^3 \\
&= \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3}.
\end{aligned}
$$

So the curvature is $\kappa(t) = \dfrac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3}$.

Note that we switch back to the parameter t in the curvature formula above. The arclength parameter s is just an auxiliary variable here. All the t's appearing in the formulas above and below should be considered as a function of s although we won't write it as such explicitly in many places for simplicity.

The formula for the torsion is a bit more complicated. We are going to use a trick to compute the torsion. Recall the definition of the torsion $\tau(s)$: $\vec{b}'(s) = \tau(s)\vec{n}(s)$. Let's compute $\vec{b}(s)$.

$$
\begin{aligned}
\vec{b}(s) &= \vec{t}(s) \times \vec{n}(s) = \beta'(s) \times \frac{\beta''(s)}{\kappa} \\
&= \frac{\alpha'(t)}{|\alpha'(t)|} \times \frac{1}{\kappa}(\frac{\alpha''(t)}{|\alpha'(t)|^2} + \alpha'(t)\frac{d^2t(s)}{ds^2}) \\
&= \frac{\alpha'(t) \times \alpha''(t)}{\kappa|\alpha'(t)|^3}.
\end{aligned}
$$

Here we used the fact that $\alpha'(t) \times \alpha'(t) = \vec{0}$.

Now we won't use the definition to compute the torsion due to the complicated computations. We shall use another method described as follows.

Since $\beta''(s) = \kappa(s)\vec{n}(s)$, we have

$$\begin{aligned}\beta'''(s) &= \kappa'(s)\vec{n}(s) + \kappa(s)\vec{n}'(s)\\ &= \kappa'(s)\vec{n}(s) + \kappa(s)(-\kappa(s)\vec{t}(s) - \tau(s)\vec{b}(s))\\ &= \kappa'(s)\vec{n}(s) - \kappa^2\vec{t}(s) - \kappa\tau\vec{b}(s).\end{aligned}$$

So take the dot product $\vec{b}(s)$ with the equation above, we get

$$\beta'''(s)\cdot\vec{b}(s) = -\kappa\tau\vec{b}(s)\cdot\vec{b}(s) = -\kappa\tau.$$

Here we used the fact that $\vec{b}\perp\vec{n}$ and $\vec{b}\perp\vec{t}$.

Hence $\tau = -\dfrac{\beta'''(s)\cdot\vec{b}(s)}{\kappa}$.

Now we need to compute $\beta'''(s)$.

$$\begin{aligned}\beta'''(s) &= \frac{d}{ds}\beta''(s) = \frac{d}{ds}\big(\alpha''(t)(\frac{dt(s)}{ds})^2 + \alpha'(t)\frac{d^2t(s)}{ds^2}\big)\\ &= \alpha'''(t)(\frac{dt(s)}{ds})^3 + \alpha''(t)\frac{d}{ds}\big((\frac{dt(s)}{ds})^2\big) + \alpha''(t)\frac{dt(s)}{ds}\frac{d^2t(s)}{ds^2} + \alpha'(t)\frac{d^3t(s)}{ds^3}.\end{aligned}$$

Hence

$$\begin{aligned}\beta'''(s)\cdot\vec{b}(s) &= \frac{\alpha'''(t)\cdot(\alpha'(t)\times\alpha''(t))}{\kappa|\alpha'(t)|^3}(\frac{d(t(s)}{ds})^3\\ &= \frac{(\alpha'(t)\times\alpha''(t))\cdot\alpha'''(t)}{\kappa|\alpha'(t)|^6}.\end{aligned}$$

Here we used that fact that $\alpha'(t)\times\alpha''(t)\perp\alpha'(t)$ and $\alpha'(t)\times\alpha''(t)\perp\alpha''(t)$. So

$$\begin{aligned}\tau &= -\frac{\beta'''(s)\cdot\vec{b}(s)}{\kappa(s)} = -\frac{(\alpha'(t)\times\alpha''(t))\cdot\alpha'''(t)}{\kappa^2(t)|\alpha'(t)|^6}\\ &= -\frac{(\alpha'(t)\times\alpha''(t))\cdot\alpha'''(t)}{|\alpha'(t)\times\alpha''(t)|^2}.\end{aligned}$$

1.7 FUNDAMENTAL THEOREM OF THE LOCAL THEORY OF CURVES

We are going to see that the curvature and the torsion of a curve basically determine the curve. The way to see this is to use some basic results in the theory of ordinary differential equations.

Question 1: Given differentiable functions $\kappa(s) > 0$ and $\tau(s)$, does there exist a regular parametrized differential curve $\alpha(s)$, $\alpha\colon I \to \mathbb{R}^3$, such that s is an arclength parameter and $\kappa(s)$ is its curvature and $\tau(s)$ is its torsion?

Question 2: Do $\kappa(s)$ and $\tau(s)$ determine the curve?

In order to answer the questions above, we need the existence and uniqueness theorem of linear ordinary differential equations which is stated as follows.

Theorem 1.17. *Given an initial condition data $(x_1^0, \ldots, x_n^0)$, then there exists a unique solution $(x_1(t), \ldots, x_n(t))$ satisfying the following system of linear ordinary differential equation (ODE for short)*

$$\begin{cases} x_1'(t) = a_{11}(t)x_1(t) + \ldots + a_{1n}(t)x_n(t) \\ \vdots \\ x_n'(t) = a_{n1}(t)x_1(t) + \ldots + a_{nn}(t)x_n(t) \end{cases}$$

where $a_{ij}(t)$ are differentiable functions.

Now we can state our fundamental theorem of the local theory of curves.

Theorem 1.18. *Given differentiable functions $\kappa(s) > 0$ and $\tau(s)$ for $s \in I$, there exists a regular parametrized curve $\alpha(s)\colon I \to \mathbb{R}^3$ such that s is an arclength parameter, $\kappa(s)$ is the curvature, $\tau(s)$ is the torsion of α. Moreover, any other curve $\tilde{\alpha}$, satisfying the same condition, differs from α by a rigid motion, i.e., there exists an orthogonal linear transformation ρ of $\mathbb{R}^3$, with positive determinant, and a vector $\vec{c}$ such that $\tilde{\alpha} = \rho \circ \alpha + \vec{c}$.*

Proof. Choose $(x_0, y_0, z_0) \in \mathbb{R}^3$, $\vec{t}_0 = (t_1^0, t_2^0, t_3^0)$ a unit vector, $\vec{n}_0 = (n_1^0, n_2^0, n_3^0)$ a unit vector perpendicular to $\vec{t}_0$. Let $\vec{b}_0 = \vec{t}_0 \times \vec{n}_0 = (b_1^0, b_2^0, b_3^0)$. Then consider the following linear system of ODE:

$$\begin{cases} \alpha'(s) = \vec{t}(s) \\ \vec{t}'(s) = \kappa(s)\vec{n}(s) \\ \vec{n}'(s) = -\kappa(s)\vec{t}(s) - \tau(s)\vec{b}(s) \\ \vec{b}'(s) = \tau(s)\vec{n}(s) \end{cases} \tag{1.1}$$

where

$$\alpha(s) = (x(s), y(s), z(s)), \qquad \vec{t}(s) = (t_1(s), t_2(s), t_3(s)),$$

$$\vec{n}(s) = (n_1(s), n_2(s), n_3(s)), \qquad \vec{b}(s) = (b_1(s), b_2(s), b_3(s)).$$

By the Theorem 1.17, the system (1.1) has a solution, i.e., there exists a parametrized curve $\alpha\colon I \to \mathbb{R}^3$ satisfying (1.1) with $\alpha(s_0) = (x_0, y_0, z_0)$ for some $s_0 \in I$.

Next we need to show that s is an arclength parameter, i.e., $|\alpha'(s)| = 1$ for all $s \in I$. Let's compute

$$\begin{cases}
(\vec{t}(s) \cdot \vec{t}(s))' = 2\vec{t}'(s) \cdot t(s) = 2\kappa(s)\vec{t}(s) \cdot \vec{n}(s) \\
(\vec{t}(s) \cdot \vec{n}(s))' = \vec{t}'(s) \cdot \vec{n}(s) + \vec{t}(s) \cdot \vec{n}'(s) \\
\qquad\qquad\qquad = \kappa(s)\vec{n}(s) \cdot \vec{n}(s) - \kappa(s)\vec{t}(s) \cdot \vec{t}(s) - \tau(s)\vec{t}(s) \cdot \vec{b}(s) \\
(\vec{n}(s) \cdot \vec{n}(s))' = 2\vec{n}(s) \cdot \vec{n}(s) = -2\kappa(s)\vec{n}(s) \cdot \vec{t}(s) - 2\tau(s)\vec{n}(s) \cdot \vec{b}(s) \\
(\vec{t}(s) \cdot \vec{b}(s))' = \vec{t}'(s) \cdot \vec{b}(s) + \vec{n}(s) \cdot \vec{b}'(s) = \kappa(s)\vec{n}(s) \cdot \vec{b}(s) + \tau(s)\vec{t}(s) \cdot \vec{n}(s) \\
(\vec{n}(s) \cdot \vec{b}(s))' = \vec{n}'(s) \cdot \vec{b}(s) + \vec{n}(s) \cdot \vec{b}'(s) \\
\qquad\qquad\qquad = -\kappa(s)\vec{t}(s) \cdot \vec{b}(s) - \tau(s)\vec{b}(s) \cdot \vec{b}(s) + \tau(s)\vec{n}(s) \cdot \vec{n}(s) \\
(\vec{b}(s) \cdot \vec{b}(s))' = 2\vec{b}(s) \cdot \vec{b}'(s) = 2\tau(s)\vec{b}(s) \cdot \vec{n}(s)
\end{cases} \tag{1.2}$$

Hence we get a system of linear ODE equations for unknown functions $\vec{t}(s) \cdot \vec{t}(s)$, $\vec{t}(s) \cdot \vec{n}(s)$, $\vec{t}(s) \cdot \vec{b}(s)$, $\vec{n}(s) \cdot \vec{n}(s)$, $\vec{n}(s) \cdot \vec{b}(s)$, $\vec{b}(s) \cdot \vec{b}(s)$ satisfying the initial condition

$$\begin{cases}
\vec{t}(s_0) \cdot \vec{t}(s_0) = \vec{t}_0 \cdot \vec{t}_0 = 1, \quad \vec{t}(s_0) \cdot \vec{n}(s_0) = \vec{t}_0 \cdot \vec{n}_0 = 0, \quad \vec{t}(s_0) \cdot \vec{b}(s_0) = 0, \\
\vec{n}(s_0) \cdot \vec{n}(s_0) = 1, \quad \vec{n}(s_0) \cdot \vec{b}(s_0) = 0, \quad \vec{b}(s_0) \cdot \vec{b}(s_0) = 1.
\end{cases} \tag{1.3}$$

Clearly

$$\begin{cases}
\vec{t}(s) \cdot \vec{t}(s) = 1, \quad \vec{t}(s) \cdot \vec{n}(s) = 0, \quad \vec{t}(s) \cdot \vec{b}(s) = 0, \\
\vec{n}(s) \cdot \vec{n}(s) = 1, \quad \vec{n}(s) \cdot \vec{b}(s) = 0, \quad \vec{b}(s) \cdot \vec{b}(s) = 0 \quad \text{for all } s \in I.
\end{cases} \tag{1.4}$$

is a solution for the system (1.2) satisfying the initial condition (1.3) above. By the uniqueness part of the Theorem 1.17, the above solution (1.4) must be the unique solution. Hence in particular $\alpha'(s) \cdot \alpha'(s) = \vec{t}(s) \cdot \vec{t}(s) = 1$ for all $s \in I$, i.e., s is an arclength parameter.

Suppose $\widetilde{\alpha}\colon I \to \mathbb{R}^3$ is another curve satisfying the same condition, i.e, $\widetilde{\kappa}(s) = \kappa(s)$, $\widetilde{\tau}(s) = \tau(s)$. Take an orthogonal linear map ρ such that $\rho\vec{t}_0 = \vec{\tilde{t}}_0$, $\rho\vec{n}_0 = \vec{\tilde{n}}_0$ and $\rho\vec{b}_0 = \vec{\tilde{b}}_0$. Let $\vec{c} = -\rho\alpha(s_0) + \widetilde{\alpha}(s_0)$.

Now consider a curve $\beta(s) = \rho(\alpha(s) - \alpha(s_0)) + \widetilde{\alpha}(s_0) = \rho \circ \alpha(s) + \vec{c}$.

Clearly the curvature of β is $\kappa(s)$ and the torsion of β is $\tau(s)$.

We still use $\vec{t}(s)$, $\vec{n}(s)$ and $\vec{b}(s)$ to represent the tangent vector, the normal vector and the binormal vector of β respectively.

Now we have $\vec{t}(s_0) = \vec{\tilde{t}}(s_0)$, $\vec{n}(s_0) = \vec{\tilde{n}}(s_0)$ and $\vec{b}(s_0) = \vec{\tilde{b}}(s_0)$.

Consider

$$\begin{aligned}
&\frac{d}{ds}\left(|\vec{t}(s) - \vec{\tilde{t}}(s)|^2 + |\vec{n}(s) - \vec{\tilde{n}}(s)|^2 + |\vec{b}(s) - \vec{\tilde{b}}(s)|^2\right) \\
= \quad & 2(\vec{t}(s) - \vec{\tilde{t}}(s)) \cdot (\vec{t}'(s) - \vec{\tilde{t}}'(s)) + 2(\vec{n}(s) - \vec{\tilde{n}}(s)) \cdot (\vec{n}'(s) - \vec{\tilde{n}}'(s)) \\
& +2(\vec{b}(s) - \vec{\tilde{b}}(s)) \cdot (\vec{b}'(s) - \vec{\tilde{b}}'(s)) \\
= \quad & 2(\vec{t} - \vec{\tilde{t}}) \cdot (\kappa\vec{n} - \tilde{\kappa}\vec{\tilde{n}}) + 2(\vec{n} - \vec{\tilde{n}}) \cdot (-\kappa\vec{t} - \tau\vec{b} + \tilde{\kappa}\vec{\tilde{t}} + \tilde{\tau}\vec{\tilde{b}}) + 2(\vec{b} - \vec{\tilde{b}}) \cdot (\tau\vec{n} - \tilde{\tau}\vec{\tilde{n}}) \\
= \quad & 2\kappa(\vec{t} - \vec{\tilde{t}}) \cdot (\vec{n} - \vec{\tilde{n}}) - 2\kappa(\vec{n} - \vec{\tilde{n}}) \cdot (\vec{t} - \vec{\tilde{t}}) - 2\tau(\vec{n} - \vec{\tilde{n}})) \cdot (\vec{b} - \vec{\tilde{b}}) \\
& +2\tau(\vec{b} - \vec{\tilde{b}}) \cdot (\vec{n} - \vec{\tilde{n}})
\end{aligned}$$

$$= \quad 0 \qquad \text{for all } s.$$

Hence the function $f(t) = |\vec{t}(s) - \vec{\tilde{t}}(s)|^2 + |\vec{n}(s) - \vec{\tilde{n}}(s)|^2 + |\vec{b}(s) - \vec{\tilde{b}}(s)|^2$ is a constant function. Use the initial conditions $\vec{t}(s_0) = \vec{\tilde{t}}(s_0)$, $\vec{n}(s_0) = \vec{\tilde{n}}(s_0)$ and $\vec{b}(s_0) = \vec{\tilde{b}}(s_0)$, we see that the function $f(t)$ must be zero for all s. Therefore we have

$$\vec{t}(s) = \vec{\tilde{t}}(s), \quad \vec{n}(s) = \vec{\tilde{n}}(s), \quad \vec{b}(s) = \vec{\tilde{b}}(s).$$

In particular, $\tilde{\alpha}'(s) = \beta'(s)$. Since $\beta(s_0) = \tilde{\alpha}(s_0)$, we must have $\beta(s) = \tilde{\alpha}(s)$ for all s.

Hence we proved the second part of the theorem. ∎

1.8 REVIEW OF THE EQUATIONS OF LINES, PLANES AND SPHERES

We will review some basic materials on lines, planes and spheres.

1. Equations of lines in $\mathbb{R}^3$*:*

Let $\vec{v} = (v_1, v_2, v_3)$ be a vector, $P_0 = (p_1, p_2, p_3)$ be a point in $\mathbb{R}^3$. The parametrized equation for the line passing through the point P_0 and parallel to the vector $\vec{v}$ is

$$\begin{cases} x = p_1 + v_1 t \\ y = p_2 + v_2 t \\ z = p_3 + v_3 t \end{cases}$$

where t is a parameter.

The vector form of the equation of the line is

$$\vec{X} = O\vec{P}_0 + t\vec{v}, \qquad \text{where } \vec{X} = (x, y, z).$$

2. Equations of planes in $\mathbb{R}^3$*:*

Let $\vec{N} = (A, B, C)$ be a vector and $P_0 = (p_1, p_2, p_3)$ be a point in $\mathbb{R}^3$.

The equation for the plane passing through the point P_0 and perpendicular to $\vec{N}$ is:

$$A(x - p_1) + B(y - p_2) + C(z - p_3) = 0, \qquad \textit{or} \quad Ax + By + Cz = D$$

where $D = Ap_1 + Bp_2 + Cp_3$.

The vector form of the equation of the plane is

$$\vec{N} \cdot (\vec{X} - O\vec{P}_0) = 0, \qquad \text{or} \quad \vec{N} \cdot \vec{X} = D$$

where $\vec{X} = (x, y, z)$ and $D = \vec{N} \cdot O\vec{P}_0$.

3. Equations of spheres in $\mathbb{R}^3$*:*

The sphere of radius r with center $P_0 = (a, b, c)$ has the following equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

The vector form of the sphere is

$$|\vec{X} - O\vec{P}_0| = r, \quad \text{or} \quad (\vec{X} - O\vec{P}_0) \cdot (\vec{X} - O\vec{P}_0) = r^2$$

where $\vec{X} = (x, y, z)$.

2

Surfaces

2.1 INTRODUCTION

Here are some examples of surfaces we have seen in calculus courses:

paraboloid $z = x^2+y^2$, sphere $x^2+y^2+z^2 = 1$, torus $z^2 = r^2-(\sqrt{x^2+y^2}-a)^2$, cone $z^2 = x^2+y^2$, cylinder $x^2+y^2 = 1$, plane $x+y+z = 0$.

Self-intersecting surfaces and the Möbius band are somewhat more complicated surfaces .

Goal: to study the curviness of surfaces in $\mathbb{R}^3$.

Following the ideas of the study of curves, we should study the rate of change of tangent planes of the surface. First of all, the surfaces we are going to study have to have tangent planes. For example, surfaces such as cones should be ruled out. To know the tangent planes of a surface, it is sufficient to know the normal vector of the surface. How to find the normal vectors?

Recall that there are three ways to describe a surface in $\mathbb{R}^3$.

1. The surface is a graph of a function of two variables, $z = f(x, y)$, as the case of paraboloid. The vector $\vec{N} = (f_x, f_y, -1)$ is a normal vector of the surface.

2. The surface is a level surface of a function of three variables, $F(x, y, z) = c$, as the case of sphere. The vector $\vec{N} = gradF = (F_x, F_y, F_z)$ is a normal vector of the surface.

3. The surface is described by a parametrization $x = x(u, v)$, $y = y(u, v)$ and $z = z(u, v)$ where u and v are parameters. The vector $\vec{N} = (x_u, y_u, z_u) \times (x_v, y_v, z_v)$ is a normal vector of the surface. In this case, we usually use $\vec{X}(u, v) = (x(u, v), y(u, v), z(u, v))$ to represent the parametrization. $\vec{X}(u, v)$ can be regarded as a map from uv-plane to $\mathbb{R}^3$. If we fix $u = u_0$ and let v change, then $\vec{X}(u_0, v)$ is a parametrized curve on the surface, called v-curve. Similarly, if we fix $v = v_0$ and let u change, then $\vec{X}(u, v_0)$ is a parametrized curve on the surface, called

u-curve. $\vec{X}_u(u_0, v_0)$ is the tangent vector of the u-curve at the point $\vec{X}(u_0, v_0)$ on the surface, and $\vec{X}_v(u_0, v_0)$ is the tangent vector of the v-curve at the point $\vec{X}(u_0, v_0)$.

A graph of a function $z = f(x, y)$ can be regarded as a level surface of the function $F(x, y, z) = f(x, y) - z$ at the level 0. A graph of a function $z = f(x, y)$ can also have the natural parametrization $\vec{X}(u, v) = (u, v, f(u, v))$.

In order to have normal vectors for a surface, we have to require that F_x, F_y and F_z cannot be all zeroes at any point of the surface in case (2) and that $\vec{X}_u$ and $\vec{X}_v$ cannot be parallel at any point of the surface in case (3), i.e., $\vec{X}_u$ and $\vec{X}_v$ are linearly independent, i.e., the matrix

$$J = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix} \tag{2.1}$$

has rank 2.

Cases (1) and (2) can describe lots of surfaces, but it turns out that the best way to describe surfaces is to use parameters as we did in the curve case. The condition that the matrix (2.1) has rank 2 guarantees that the surface has normal vectors and tangent planes just as the curve case where we require that $\alpha'(t) \neq \vec{0}$.

Definition 2.2. (first version) *A subset $S \subset \mathbb{R}^3$ is a regular surface if it is the image of the map $\vec{X}: U \subset \mathbb{R}^2 \to \mathbb{R}^3$ where U is an open subset of $\mathbb{R}^2$ and $\vec{X}(u, v) = (x(u, v), y(u, v), z(u, v))$ such that*

1. *the functions $x(u, v)$, $y(u, v)$ and $z(u, v)$ are differentiable,*

2. *$\vec{X}$ is one-to-one (not self-intersecting),*

3. *the Jacobian matrix (2.1) of the map $\vec{X}$ has rank 2 for $(u, v) \in U$.*

J is sometimes written as $d\vec{X}$ and is called the differential of the map $\vec{X}$.

Unlike the case of curves, the surface is the set, not the parametrization.

Example 2.3. For a graph $z = f(x, y)$ where $(x, y) \in U$ for an open subset U of $\mathbb{R}^2$, we use the parametrization

$$\vec{X}(u, v) = (u, v, f(u, v)).$$

Clearly if $f(x, y)$ is differentiable, then $\vec{X}$ is differentiable. $\vec{X}$ is clearly one-to-one.

$$J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ f_u & f_v \end{pmatrix}.$$

has rank 2 for all $(u, v) \in U$. So any graph of a differentiable function $f(x, y)$ is a regular surface. ∎

Example 2.4. Consider the sphere $x^2 + y^2 + z^2 = 1$. The sphere is not a graph of any function. But we can split the sphere into several pieces such that each piece is a graph of some differentiable function. For example, $z = \sqrt{1 - x^2 - y^2}$ for $x^2 + y^2 < 1$ is one piece and $z = -\sqrt{1 - x^2 - y^2}$ for $x^2 + y^2 < 1$ is another piece. Since points on the equator $\{(x, y, 0) | x^2 + y^2 = 1\}$ are not contained in the images of above two parametrizations, more parametrizations are needed. In fact, we should split the sphere into six parts and get parametrizations for each part as follows:

$$z = \pm\sqrt{1-x^2-y^2},\ \vec{z}_+(u,v) = (u, v, \sqrt{1-u^2-v^2}),\ \vec{z}_-(u,v) = (u, v, -\sqrt{1-u^2-v^2}),$$

$$x = \pm\sqrt{1-z^2-y^2},\ \vec{x}_+(u,v) = (\sqrt{1-u^2-v^2}, u, v),\ \vec{x}_-(u,v) = (-\sqrt{1-u^2-v^2}, u, v),$$

$$y = \pm\sqrt{1-x^2-z^2},\ \vec{y}_+(u,v) = (u, \sqrt{1-u^2-v^2}, v),\ \vec{y}_-(u,v) = (u, -\sqrt{1-u^2-v^2}, v).\blacksquare$$

We see that our original Definition 2.2 of regular surface doesn't include the sphere. We cut the sphere into several pieces such that each piece has a parametrization. It is impossible to give a single parametrization of the sphere. So in order to include the sphere into our category of regular surfaces, we need to modify the definition of regular surfaces.

Definition 2.5.(final version)*A subset $S \subset \mathbb{R}^3$ is a regular surface if, for each $p \in S$, there exists an open set V of $\mathbb{R}^3$ containing p and a map $\vec{X}: U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that*

1. *$\vec{X}$ is differentiable,*

2. *$\vec{X}$ is one-to-one,*

3. *$d\vec{X}$ is of rank 2.*

Question: How do we know, in case (2), which kind of functions $F(x, y, x)$ give regular surface $F(x, y, z) = c$?

Well, the surface $F(x, y, z) = c$ will have normal vectors if the gradient $gradF = (F_x, F_y, F_z)$ is not zero.

Definition 2.6. *Given a differentiable function $F: U \subset \mathbb{R}^n \to \mathbb{R}$ defined in an open set U of $\mathbb{R}^n$. We say that $p \in U$ is a critical point of F if the gradient of F at p is a zero vector, i.e., $F_x(p) = 0$, $F_y(p) = 0$, $F_z(p) = 0$. The value $F(p)$ of the function F at a critical point p is called a critical value. A value of the function F which is not a critical value is called a regular value, i.e., c is a regular value of the function F iff every point on $F^{-1}(c) = \{(x, y, z)|F(x, y, z) = c\}$ is not a critical point.*

Proposition 2.7. *Let c be a regular value of a differentiable function $F(x, y, z)$, then the level surface $F(x, y, z) = c$ is a regular surface in $\mathbb{R}^3$.*

In order to prove this Proposition, we need to recall the Implicit Function Theorem.

Theorem 2.8. (Implicit Function Theorem) *Suppose a differentiable function $G(x, y, z)$ has value zero at the point $P_0 = (x_0, y_0, z_0)$ and $G_z(x_0, y_0, z_0) \neq 0$. Then there exists an open subset U containing the point (x_0, y_0) such that there exists a unique differentiable function $z = g(x, y)$ such that $z_0 = g(x_0, y_0)$ and $G(x, y, g(x, y)) = 0$ for $(x, y) \in U$.*

Proof of Proposition 2.7: Apply Theorem 2.8 to the function $G(x, y, z) = F(x, y, z) - c$.

For any point $P = (x_0, y_0, z_0) \in F^{-1}(c)$, $gradF(P) = (F_x(P), F_y(P), F_z(P)) \neq \vec{0}$ since c is a regular value of F. Without loss of generality, we may assume that $F_z(P) \neq 0$. By the Implicit Function Theorem, there exists a differentiable function $z = f(x, y)$ such that $z_0 = (x_0, y_0)$ and $F(x, y, z(x, y)) = c$ for all $(x, y) \in U$ where U is an open set containing the point (x_0, y_0) in $\mathbb{R}^2$. Hence near the point P, the surface $F(x, y, z) = c$ can be described as the graph of the function

$z = f(x, y)$. We knew already that the graph of any differentiable function is a regular surface. Hence the level surface $F(x, y, z) = c$ is a regular surface. ∎

Example 2.9. Consider the function $F(x, y, z) = x^2 + y^2 - z^2$. Let's find out which numbers are regular values of this function. We compute $gradF = (F_x, F_y, F_z) = (2x, 2y, -2z)$. We see that only the origin $(0, 0, 0)$ is a critical point. Hence $0 = F(0, 0, 0)$ is the only critical value. All non-zero numbers are regular values. Hence the surface $x^2 + y^2 - z^2 = c$ is a regular surface if $c \neq 0$.

If $c > 0$, write $c = a^2$ for some positive number a, the surface $x^2 + y^2 - z^2 = a^2$ is the hyperboloid of one sheet.

If $c < 0$, write $c = -a^2$ for some positive number a, the surface $x^2 + y^2 - z^2 = -a^2$ is the hyperboloid of two sheets.

If $c = 0$, the surface $x^2 + y^2 - z^2 = 0$ is the cone. ∎

Example 2.10. Consider the torus. It is obtained by placing the center of a circle of radius a at the point $(r, 0, 0)$ of a circle of radius r on the xy-plane with center at the origin and then rotating the first circle along the second circle. We assume that $r > a$. Let θ and ϕ be the polar coordinate angles for the first and the second circles respectively shown in the pictures below. Let (x, y, z) be a point on the torus corresponding to angles θ and ϕ. Then we can get

$$\begin{cases} x = x(\theta, \phi) = (r + acos\theta)cos\phi, \\ y = y(\theta, \phi) = (r + acos\theta)sin\phi, \\ z = z(\theta, \phi) = asin\theta. \end{cases}$$

We can show that the torus can be described by the following equation $z^2 = a^2 - (\sqrt{x^2 + y^2} - r)^2$. If we let $F(x, y, z) = z^2 + (\sqrt{x^2 + y^2} - r)^2$, then the torus can be regarded as the level surface of the function F at the level a^2.

$$\begin{cases} F_x = 2(\sqrt{x^2 + y^2} - r)\dfrac{x}{\sqrt{x^2 + y^2}} \\ F_y = 2(\sqrt{x^2 + y^2} - r)\dfrac{y}{\sqrt{x^2 + y^2}} \\ F_z = 2z \end{cases}$$

If $(F_x, F_y, F_z) = (0, 0, 0)$, we must have $z = 0$ and either $\sqrt{x^2 + y^2} = r$ or $x = 0, y = 0$.

If $z = 0$ and $\sqrt{x^2 + y^2} = r$, we have $F(x, y, z) = 0 \neq a^2$. If $x = y = z = 0$, then $F(x, y, z) = r^2 \neq a^2$ by our assumption. Hence in any case, critical points of the function F cannot lie on the torus $F^{-1}(a^2)$. Hence a^2 is a regular value of the function F and therefore the torus is a regular surface. ∎

2.2 THE DIFFERENTIAL OF A MAP AND TANGENT PLANES

Let's review the tangent plane of a surface learned in Multi-variable Calculus. Let $\vec{X}(u, v) : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrization of a surface. Let $q = (u_0, v_0) \in U$ be a point on $\mathbb{R}^2$. Let $P = \vec{X}(q)$. A normal vector of the surface at the point P is given by $\vec{N} = \vec{X}_u(q) \times \vec{X}_v(q)$. Hence the equation of the tangent plane of the surface passing through the point P is

$$\vec{N} \cdot \big((x, y, z) - \vec{OP}\big) = 0.$$

Example 2.10. Consider the surface given by the graph of the function $z = x^2 + y^2$. At the point $P = (1, 1, 2)$, $\vec{N} = (f_x, f_y, -1) = (2x, 2y, -1)|_{(1,1,2)} = (2, 2, -1)$ is a normal vector of the surface at the point P. Hence the equation for the tangent plane of the surface passing through the point P is

$$(2, 2, -1) \cdot \big((x, y, z) - (1, 1, 2)\big) = 0.$$

If we simplify the equation, we get $2x + 2y - z = 2$. ∎

Let S be a regular surface. We define a tangent vector $\vec{v}$ to the surface S at a point $p \in S$ to be the tangent vector $\alpha'(0)$ of a differentiable parametrized curve $\alpha\colon (-\epsilon, \epsilon) \to S$ with $\alpha(0) = p$ where ϵ is a small positive number. The set of all such tangent vectors is denoted by T_pS. It is a plane passing through the origin parallel to the tangent plane of S at the point p. If the equation for the tangent plane of S at p is given by $ax + by + cz = d$, then the equation for T_pS is $ax + by + cz = 0$.

Proposition 2.11. *Let $\vec{X}\colon U \subset \mathbb{R}^2 \to S$ be a parametrization of a regular surface S and let $q \in U$ be a point. The vector subspace of dimension 2 $d\vec{X}_q(\mathbb{R}^2) \subset \mathbb{R}^3$ coincides with the set of tangent vectors to S at the point $p = \vec{X}(q)$.*

Remark: $d\vec{X}_q$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ where

$$d\vec{X}_q = \begin{pmatrix} x_u(q) & x_v(q) \\ y_u(q) & y_v(q) \\ z_u(q) & z_v(q) \end{pmatrix}.$$

The image of $d\vec{X}_q$ is the space spanned by the tangent vectors $\vec{X}_u(q)$ and $\vec{X}_v(q)$. Hence it is the plane passing through the origin parallel to the vectors $\vec{X}_u(q)$ and $\vec{X}_v(q)$. Therefore the normal vector for the plane $d\vec{X}_q(\mathbb{R}^2)$ is $\vec{X}_u(q) \times \vec{X}_v(q)$. Hence the plane $d\vec{X}_q(\mathbb{R}^2)$is parallel to the tangent plane of the surface S at the point p.

For the simplicity, we are going to use the following notation through this course: for a vector $\vec{v} = (v_1, v_2, v_3)$, we use $\vec{v}^\tau = (v_1, v_2, v_3)^\tau$ to represent the column vector

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

Proof. Let $\vec{w}$ be a tangent vector at p. So $\vec{w} = \alpha'(0)$ for some curve $\alpha\colon (-\epsilon, \epsilon) \to S$ and $\alpha(0) = p$. Since $\vec{X}$ is a one-to-one map, we can choose ϵ small enough so that the curve α is contained in the image $\vec{X}(U)$ and hence there exists a curve $\beta\colon (-\epsilon, \epsilon) \to U \subset \mathbb{R}^2$ such that $\alpha(t) = \vec{X}(\beta(t))$.

Here we regard both $\alpha(t)$ and $\beta(t)$ as column vectors, i.e., we write

$$\alpha(t) = (x(t), y(t), z(t))^\tau, \qquad \beta(t) = (u(t), v(t))^\tau.$$

Hence by the chain rule, we can compute

$$\begin{aligned} \frac{d}{dt}\vec{X}(\beta(t)) &= \begin{pmatrix} \frac{d}{dt}x(u(t), v(t)) \\ \frac{d}{dt}y(u(t), v(t)) \\ \frac{d}{dt}z(u(t), v(t)) \end{pmatrix} = \begin{pmatrix} \frac{d}{dt}x(u(t), v(t)) \\ \frac{d}{dt}y(u(t), v(t)) \\ \frac{d}{dt}z(u(t), v(t)) \end{pmatrix} \begin{pmatrix} x_u u'(t) + x_v v'(t)) \\ y_u u'(t) + y_v v'(t)) \\ z_u u'(t) + z_v v'(t)) \end{pmatrix} \\ &= \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix} \cdot \begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} \end{aligned}$$

Hence we get $\alpha'(t) = d\vec{X} \cdot \beta'(t)$. In particular, $\alpha'(0) = d\vec{X}_q(\beta'(0))$. Hence $\vec{w} = \alpha'(0)$ is in the plane $d\vec{X}_q(\mathbb{R}^2)$.

On the other hand, let $\vec{w} = d\vec{X}_q(\vec{v})$ for some vector $\vec{v} \in \mathbb{R}^2$. Take a curve $\beta(t) = t\vec{v} + q$ for $t \in (-\epsilon, \epsilon)$ in the plane $\mathbb{R}^2$. We can compute that $\beta'(t) = \vec{v}$ and $\beta'(0) = \vec{v}$. Choose $\alpha(t) = \vec{X}(\beta(t))$. Then $\alpha'(t) = d\vec{X} \cdot \beta'(t)$ and $\alpha'(0) = d\vec{X}_q \cdot \beta'(0) = d\vec{X}_q(\vec{v})$. Hence $d\vec{X}_q(\vec{v})$ is a tangent vector of S at p. ■

Next, let's talk about the differential of a (differentiable) map between two regular surfaces.

A map φ between two regular surface S_1 and S_2 is said to be differentiable if for any point $p \in S_1$ and any parametrization $\vec{X}(u,v): U \subset \mathbb{R}^2 \to S_1 \subset \mathbb{R}^3$ with $p \in \vec{X}(U)$, the map $\varphi \circ \vec{X}: U \to S_2$ is differentiable.

Example 2.12. Consider the surface S_1 given by the graph of the function $z = x^2+y^2$. $\vec{N}(x,y,z) = \dfrac{(2x, 2y, -1)}{\sqrt{4x^2+4y^2+1}}$ is a unit normal vector of the surface S_1 at the point $(x,y,z) \in S_1$. Hence $\vec{N}$ can be regarded as a map from the surface S_1 to another surface S_2 which is the unit sphere centered at the origin. If we choose $\vec{X}(u,v) = (u, v, u^2+v^2)$ as a parametrization of the surface S_1, then $\vec{N} \circ \vec{X}(u,v) = \dfrac{(2u, 2v, -1)}{\sqrt{4u^2+4v^2+1}}$. It is differentiable. So the map $\vec{N}$ from S_1 to S_2 is differentiable. ■

Given a differentiable map φ between two regular surface S_1 and S_2, let $p \in S_1$, any tangent vector $\vec{w} \in T_p(S_1)$ is $\alpha'(0)$ for some curve $\alpha(t): (-\epsilon, \epsilon) \to S_1$ with $\alpha(0) = p$. Let $\beta(t) = \varphi(\alpha(t)$. Then β is a curve on the surface S_2 passing through the point $\beta(0) = \varphi(p)$. So $\beta'(0)$ is a tangent vector of S_2 at the point $\varphi(p)$, i.e., a vector of $T_{\varphi(p)}S_2$.

Proposition 2.13. *The vector $\beta'(0)$ doesn't depend upon the choice of α, it only depends upon $\vec{w} = \alpha'(0)$. The map $d\varphi_p: T_pS_1 \to T_{\varphi(p)}S_2$ defined by $d\varphi_p(\vec{w}) = \beta'(0)$ is linear.*

Proof. With the notations as above. Choose a parametrization $\vec{X}$ of S_1 near p, i.e.,

$$\vec{X}(u,v) = (x(u,v), y(u,v), z(u,v))^\tau : U \in \mathbb{R}^2 \to S_1 \subset \mathbb{R}^3,$$

where U is an open subset of $\mathbb{R}^2$ and $p \in \vec{X}(U)$.

If we choose ϵ sufficiently small, we can have the curve α contained in $\vec{X}(U)$. Hence there exists a curve $\gamma(t) = (u(t), v(t))^\tau$ inside U such that $\alpha(t) = \vec{X}(\gamma(t))$ with $\gamma(0) = q \in U$. Write

$$\varphi \circ \vec{X}(u,v) = (\widetilde{x}(u,v), \widetilde{y}(u,v), \widetilde{z}(u,v))^\tau.$$

Since φ is differentiable, $\varphi \circ \vec{X}$ is differentiable.

$$\beta(t) = \varphi(\alpha(t)) = (\widetilde{x}(u(t),v(t)), \widetilde{y}(u(t),v(t)), \widetilde{z}(u(t),v(t)))^\tau.$$

By the Chain rule, we get

$$\begin{aligned}\beta'(0) &= (\widetilde{x}_u(q)u'(0) + \widetilde{x}_v(q)v'(0), \widetilde{y}_u(q)u'(0) + \widetilde{y}_v(q)v'(0), \widetilde{z}_u(q)u'(0) + \widetilde{z}_v(q)v'(0))^\tau \\ &= \begin{pmatrix} \widetilde{x}_u(q) & \widetilde{x}_v(q) \\ \widetilde{y}_u(q) & \widetilde{y}_v(q) \\ \widetilde{z}_u(q) & \widetilde{z}_v(q) \end{pmatrix} \cdot \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix}\end{aligned}$$

From the proof of Proposition 2.11, we know that $\vec{w} = \alpha'(0) = d\vec{X}_q(\gamma'(0))$. Since $d\vec{X}_q$ is an injection from $\mathbb{R}^2$ to $\mathbb{R}^3$, it is an isomorphism from $\mathbb{R}^2$ to T_pS_1. Let $d\vec{X}_q^{-1}$ be the inverse of $d\vec{X}_q$

which is regarded as an isomorphism from $\mathbb{R}^2$ to T_pS_1, then $d\vec{X}_q^{-1}$ is a linear transformation and

$$\gamma'(0) = d\vec{X}_q^{-1}(\vec{w}), \quad d\varphi_p = \begin{pmatrix} \widetilde{x}_u(q) & \widetilde{x}_v(q) \\ \widetilde{y}_u(q) & \widetilde{y}_v(q) \\ \widetilde{z}_u(q) & \widetilde{z}_v(q) \end{pmatrix} \circ d\vec{X}_q^{-1}$$

Hence the map $d\varphi_p$ is a linear transformation and $\beta'(0) = d\varphi_p(\vec{w})$ depends only on the map φ and the vector $\vec{w}$. ∎

Example 2.14. Let $S \subset \mathbb{R}^3$ be a regular surface and $P \subset \mathbb{R}^3$ be a plane. If all points of S are on the same side of P, prove that P is tangent to S at all points of $P \cap S$.

Proof. Let A be a point on $P \cap S$. Let $\vec{N}$ be a normal vector of the plane P. Suppose $\vec{N}$ points towards the direction of the side of P where S lies. Let $\alpha(t)\colon (-\epsilon, \epsilon) \to S \subset \mathbb{R}^3$ be an arbitrary curve on S with $\alpha(0) = A$. $\alpha'(0)$ is a tangent vector of S at A. So

$$\vec{N} \cdot (\alpha(t) - A) = |\alpha(t) - A| \cdot |\vec{N}| \cdot cos\theta$$

where θ is the angle between the vectors $(\alpha(t) - A)$ and $\vec{N}$.

Since all points of S are on the side of $\vec{N}$, θ must be between 0 and $\pi/2$. So $cos\theta \geq 0$. Therefore the function $f(t) = \vec{N} \cdot (\alpha(t) - A)$ is non-negative. But $f(0) = \vec{N} \cdot (\alpha(0) - A) = \vec{N} \cdot (A - A) = 0$. So 0 is a minimum point of the function $f(t)$. Therefore $f'(0) = 0$. But $0 = f'(0) = \vec{N} \cdot \alpha'(0)$. Hence the tangent vector $\alpha'(0)$ is perpendicular to $\vec{N}$, i.e., $\alpha'(0)$ is on the plane P. So the plane P is tangent to S at A. ∎

Example 2.15. Surface of revolution.

Consider a regular curve lying on the xz-plane:

$$x = f(v), \; y = 0, \; z = g(v), \qquad a < v < b, \; f(v) > 0, \; g'(v) \neq 0.$$

Rotating the curve along the z-axis, we get a surface of revolution S.

Let (x, y, z) be a point P on the surface S. If we let θ be the angle between the x-axis and the vector $(x, y, 0)$ which is the projection of the point P to the xy-plane, then we get

$$x = f(v)cos\theta, \quad y = f(v)sin\theta, \quad z = g(v).$$

Hence we get a parametrization of the surface given by

$$\vec{X}(\theta, v) = (f(v)cos\theta, f(v)sin\theta, g(v)), \quad 0 < \theta < 2\pi, \; a < v < b.$$

To check that $\vec{X}$ is a parametrization:

1. Since $f(v)cos\theta, f(v)sin\theta, g(v)$ are differentiable, hence $\vec{X}(\theta, v)$ is differentiable.

2. $\vec{X}$ is one-to-one. Let $(f(v_1)cos\theta_1, f(v_1)sin\theta_1, g(v_1)) = (f(v_2)cos\theta_2, f(v_2)sin\theta_2, g(v_2))$, we get $g(v_1) = g(v_2)$. Since $g'(v) \neq 0$, the function g is a monotone function of v. Therefore $v_1 = v_2$. We also have $f(v_1)cos\theta_1 = f(v_2)cos\theta_2$, $\quad f(v_1)sin\theta_1 = f(v_2)sin\theta_2$. Since $f(v_1) = f(v_2) \neq 0$, we must have $\theta_1 = \theta_2$.

3. $\vec{X}_\theta = (-f(v)sin\theta, f(v)cos\theta, 0)$ and $\vec{X}_v = (f'(v)cos\theta, f'(v)sin\theta, g'(v))$.

$$\vec{X}_\theta \times \vec{X}_v = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ -f(v)sin\theta & f(v)cos\theta & 0 \\ f'(v)cos\theta & f'(v)sin\theta & g'(v) \end{pmatrix} = (g'(v)f(v)cos\theta, f(v)g'(v)sin\theta, -f(v)f'(v)).$$

Since both $f(v)$ and $g'(v)$ are non-vanishing, $g'(v)f(v)cos\theta$ and $f(v)g'(v)sin\theta$ cannot be zero at the same time. Hence $\vec{X}_\theta \times \vec{X}_v$ is not zero, i.e., $\vec{X}_\theta$ and $\vec{X}_v$ are linearly independent. Therefore $d\vec{X}$ is of rank two.

The tangent plane of the surface at the point $p = \vec{X}(\theta, v)$ is given by the equation

$$(g'(v)f(v)cos\theta, f(v)g'(v)sin\theta, -f(v)f'(v)) \cdot (x - f(v)cos\theta, y - f(v)sin\theta, z - g(v)) = 0.$$

The equation of T_pS is given by

$$(g'(v)f(v)cos\theta, f(v)g'(v)sin\theta, -f(v)f'(v)) \cdot (x, y, z) = 0. \blacksquare$$

2.3 THE FIRST FUNDAMENTAL FORM AND AREA

Question: How do we calculate the area of the surface and the length of a curve on a surface?

For any two vectors $\vec{v}$ and $\vec{w}$, we use $< \vec{v}, \vec{w} >$ to represent the inner product (the dot product) of the two vectors, i.e., $< \vec{v}, \vec{w} >= \vec{v} \cdot \vec{w}$.

1. Let $\alpha(t)\colon (a, b) \to S \subset \mathbb{R}^3$ be a curve on the surface, the arclength of the curve is

$$\int_a^b |\alpha'(t)| dt = \int_a^b \sqrt{< \alpha'(t), \alpha'(t) >} dt.$$

2. Suppose a piece of the surface S has a parametrization

$$\vec{X}(u, v) = (x(u, v), y(u, v), z(u, v)), \qquad (u, v) \in U.$$

Then the area of this piece of the surface equals

$$\int\int_U |\vec{X}_u \times \vec{X}_v| dudv.$$

Let θ be the angle between the two vectors $\vec{X}_u$ and $\vec{X}_v$, then

$$\begin{aligned} & |\vec{X}_u \times \vec{X}_v|^2 = |\vec{X}_u|^2 |\vec{X}_v|^2 sin^2\theta \\ = \; & |\vec{X}_u|^2 |\vec{X}_v|^2 (1 - cos^2\theta) \\ = \; & |\vec{X}_u|^2 |\vec{X}_v|^2 - |\vec{X}_u|^2 |\vec{X}_v|^2 cos^2\theta \\ = \; & < \vec{X}_u, \vec{X}_u >< \vec{X}_v, \vec{X}_v > - < \vec{X}_u, \vec{X}_v >^2 . \end{aligned}$$

Hence the area of the surface is

$$\int\int_U \sqrt{<\vec{X}_u, \vec{X}_u><\vec{X}_v, \vec{X}_v> - <\vec{X}_u, \vec{X}_v>^2}dudv.$$

Now we see that the formulae of the length of a curve on a surface S and the area of the surface S depend on the inner product of tangent vectors $\alpha'(t), \vec{X}_u$, and $\vec{X}_v$. Hence the inner product $<,>$ is crucial here. This leads to the definition of the first fundamental form.

Definition 2.16. *Let S be a regular surface, p be a point on S and T_pS be the two-dimensional vector subspace consisting of tangent vectors of S at p. Let $<,>_p$ be the restriction of the inner product $<>$ on $\mathbb{R}^3$ to T_pS. The first fundamental form I_p is a function on T_pS such that $I_p(\vec{v}) =<\vec{v}, \vec{v}>_p$ for $\vec{v} \in T_pS$.*

Let $\vec{X}(u,v) = (x(u,v), y(u,v), z(u,v))$ be a parametrization of S, $\vec{X}: U \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$, $q \in U$ and $p = \vec{X}(q) \in S$. Let $\vec{v} \in T_pS$ be a tangent vector of the surface S at p. Hence there exists a curve $\gamma(t) = (u(t), v(t))$ for $-\epsilon < t < \epsilon$ inside U with $\gamma(0) = q$ such that $\vec{v} = \alpha'(0)$ where $\alpha(t) = \vec{X}(u(t), v(t))$. From the chain rule, we get

$$\alpha'(t) = \vec{X}_u u'(t) + \vec{X}_v v'(t), \quad \alpha'(0) = \vec{X}_u u'(0) + \vec{X}_v v'(0).$$

Hence the first fundamental form $I_p(\vec{v})$ is

$$\begin{aligned} & I_p(\vec{v}) =<\vec{v}, \vec{v}>_p=<\alpha'(0), \alpha'(0)>_p \\ = & <\vec{X}_u u'(0) + \vec{X}_v v'(0), \vec{X}_u u'(0) + \vec{X}_v v'(0)>_p \\ = & <\vec{X}_u(q), \vec{X}_u(q)> u'(0)^2 + 2<\vec{X}_u(q), \vec{X}_v(q)> u'(0)v'(0) + <\vec{X}_v(q), \vec{X}_v(q)> v'(0)^2. \end{aligned}$$

Define

$$\begin{cases} E(u,v) =<\vec{X}_u(u,v), \vec{X}_u(u,v)>, \\ F(u,v) =<\vec{X}_u(u,v), \vec{X}_v(u,v)>, \\ G(u,v) =<\vec{X}_v(u,v), \vec{X}_v(u,v)>. \end{cases}$$

Then we have

$$I_p(a\vec{X}_u(q) + b\vec{X}_v(q)) = E(q)a^2 + 2F(q)ab + G(q)b^2.$$

Hence the first fundamental form is determined by E, F and G with respect to the parametrization $\vec{X}$.

Example 2.17. Let L be a plane passing through a point P_0 and parallel to vectors $\vec{w}_1$ and $\vec{w}_2$ where $\vec{w}_1$ and $\vec{w}_2$ are unit vectors and perpendicular to each other.

The plane L has the parametrization

$$\vec{X}(u,v) = \vec{OP_0} + u\vec{w}_1 + v\vec{w}_2, \qquad (u,v) \in \mathbb{R}^2.$$

$$\vec{X}_u = \vec{w}_1, \quad \vec{X}_v = \vec{w}_2.$$

$$\begin{cases} E =<\vec{X}_u, \vec{X}_u>=<\vec{w}_1, \vec{w}_1>= 1, \\ F =<\vec{X}_u, \vec{X}_v>=<\vec{w}_1, \vec{w}_2>= 0, \\ E =<\vec{X}_v, \vec{X}_v>=<\vec{w}_2, \vec{w}_2>= 1. \end{cases}$$

Hence the first fundamental form with respect to this parametrization is

$$I_p(a\vec{X}_u + b\vec{X}_v) = a^2 + b^2.\blacksquare$$

Example 2.18. Consider the right cylinder over the circle given by $x^2 + y^2 = 1$ in $\mathbb{R}^3$. Choose a parametrization $\vec{X}: \mathbb{R}^2 \to \mathbb{R}^3$, $\vec{X}(u, v) = (cosu, sinu, v)$. $\vec{X}_u = (-sinu, cosu, 0)$ and $\vec{X}_v = (0, 0, 1)$. Hence

$$E =< \vec{X}_u, \vec{X}_u >= 1, \quad F =< \vec{X}_u, \vec{X}_v >== 0, \quad E =< \vec{X}_v, \vec{X}_v >= 1.$$

Hence the first fundamental form with respect to this parametrization is

$$I_p(a\vec{X}_u + b\vec{X}_v) = a^2 + b^2.$$

Notice that the first fundamental form of the cylinder under the parametrization above is the same as that of the plane in the previous Example 2.17. ■

We saw that the knowledge of the first fundamental form, equivalently E, F and G for a parametrization of a surface S, can be used to find the area of the surface, the arclength of a curve on the surface and the angles between two tangent vectors.

For example, let $\alpha(t) = \vec{X}(u(t), v(t))$ be a curve on the surface where $a < t < b$, then the arclength of the curve is given by

$$\int_a^b \sqrt{E(u(t), v(t))u'(t)^2 + 2F(u(t), v(t))u'(t)v'(t) + G(u(t), v(t))v'(t)^2}dt.$$

Given two tangent vectors $\vec{v}_1$ and $\vec{v}_2$, the angle θ between them is given by

$$cos\theta = \frac{< \vec{v}_1, \vec{v}_2 >}{|\vec{v}_1||\vec{v}_2|}.$$

In particular, the angle θ between the tangent vectors $\vec{X}_u$ and $\vec{X}_v$ is given by

$$cos\theta = \frac{< \vec{X}_u, \vec{X}_v >}{|\vec{X}_u||\vec{X}_v|} = \frac{F}{\sqrt{EG}}.$$

Hence if $F = 0$, the coordinate curves of the parametrization are orthogonal.

Example 2.19. Use the spherical coordinates to parametrize the sphere of radius r:

$$\vec{X}(\theta, \varphi) = (rcos\theta sin\varphi, rsin\theta sin\varphi, rcos\varphi), \quad 0 < \theta < 2\pi,\ 0 < \varphi < \pi.$$

$$\vec{X}_\theta = (-rsin\theta sin\varphi, rcos\theta sin\varphi, 0),$$

$$\vec{X}_\varphi = (rcos\theta cos\varphi, rsin\theta cos\varphi, -rsin\varphi).$$

$$E =< \vec{X}_\theta, \vec{X}_\theta >= r^2 sin^2\theta sin^2\varphi + r^2 cos^2\theta sin^2\varphi = r^2 sin^2\varphi,$$

$$F =< \vec{X}_\theta, \vec{X}_\varphi >= -r^2 sin\theta sin\varphi cos\theta cos\varphi + r^2 sin\theta sin\varphi cos\theta cos\varphi = 0,$$

$$G =< \vec{X}_\varphi, \vec{X}_\varphi >= r^2 cos^2\theta cos^2\varphi + r^2 sin^2\theta cos^2\varphi + r^2 sin^2\varphi = r^2.$$

Hence the first fundamental form with respect to this parametrization is

$$I(a\vec{X}_\theta + b\vec{X}_\varphi) = Ea^2 + 2Fab + Gb^2 = r^2 sin^2\varphi a^2 + r^2 b^2.$$

Since $F = 0$, the coordinate curves of this parametrization of the sphere are orthogonal.
The area of the sphere of radius r is:

$$\begin{aligned}
&\int\int_{\substack{0<\theta<2\pi \\ 0<\varphi<\pi}} \sqrt{EG - F^2}\, d\theta d\varphi \\
= &\int_0^{2\pi}\int_0^{\pi} \sqrt{r^2 sin^2\varphi \cdot r^2 - 0}\, d\varphi d\theta = r^2 \int_0^{2\pi}\int_0^{\pi} sin\varphi d\theta \\
= &\ 4\pi r^2. \blacksquare
\end{aligned}$$

Example 2.20. The coordinate curves of a parametrization $\vec{X}(u, v)$ constitute a Tchebyshef net if the lengths of the opposite sides of any quadrilateral formed by them are equal. Show that a necessary and sufficient condition for this is

$$E_v = G_u = 0.$$

Solution: Let $u = u_1$, $u = u_2$, $v = v_1$ and $v = v_2$ be lines in the uv-plane. Then we have four corresponding coordinate curves on the surface

$$\vec{X}(u_1, v), \quad \vec{X}(u_2, v), \quad \vec{X}(u, v_1), \quad \vec{X}(u, v_2).$$

Let A, B, C, C be four vertices of the quadrilateral formed by these four coordinate curves. The condition that "the lengths of the opposite sides of any quadrilateral formed by them are equal" means that the lengths of the sides AB and CD are equal and the lengths of the sides AC and BD are equal for any u_1, u_2, v_1, v_2.

The length of the side AB of the quadrilateral is

$$\int_{v_1}^{v_2} \sqrt{< \vec{X}_v(u_1, v), \vec{X}_v(u_1, v) >}\, dv = \int_{v_1}^{v_2} \sqrt{G(u_1, v)}\, dv.$$

Similarly the length of the side CD is $\int_{v_1}^{v_2} \sqrt{G(u_2, v)}dv$. Hence the lengths of the sides AB and CD are equal iff

$$\int_{v_1}^{v_2} \sqrt{G(u_1, v)}\, dv = \int_{v_1}^{v_2} \sqrt{G(u_2, v)}\, dv \qquad \text{for any } u_1, u_2, v_1, v_2$$

iff

$$\frac{d}{dv_2}\int_{v_1}^{v_2} \sqrt{G(u_1, v)}\, dv = \frac{d}{dv_2}\int_{v_1}^{v_2} \sqrt{G(u_2, v)}\, dv \qquad \text{for any } u_1, u_2, v_1, v_2$$

iff $\sqrt{G(u_1, v_2)} = \sqrt{G(u_2, v_2)}$ for any u_1, u_2, v_2 iff $G(u, v_2)$ is a constant function of u when v_2 is an arbitrary fixed number iff $G_u(u, v) = 0$ for all u, v.

By the same method, we can also show that the lengths of the sides AC and BD are equal iff $E_v(u, v) = 0$ for all u, v. $\blacksquare$

Suppose we have two surfaces S_1 and S_2 intersecting at a point P. Let $\vec{N}_1$ and $\vec{N}_2$ be normal vectors of S_1 at P and S_2 at P respectively. Let θ be the angle between $\vec{N}_1$ and $\vec{N}_2$. Then θ is called the angle of intersecting surfaces S_1 and S_2 at the point P.

3

The Gauss Map and Gaussian Curvature

3.1 THE ORIENTATION OF A REGULAR SURFACE

A differentiable field of unit normal vectors on a regular surface S is a differentiable map $\vec{N}: S \to \mathbb{R}^3$ which associates to each point $q \in S$ a unit normal vector $\vec{N}(q) \in \mathbb{R}^3$ to S.

Definition 3.1. *A regular surface $S \subset \mathbb{R}^3$ is orientable if there exists a differentiable field of unit normal vectors $\vec{N}: \to \mathbb{R}^3$ on S.*

If a regular surface S is the level surface $S = \{(x, y, z) \in \mathbb{R}^3 | F(x, y, z) = a\}$ where $F: U \subset \mathbb{R}^3 \to \mathbb{R}^1$ is a function whose second derivatives are all exist and a is a regular value of f, then $gradF = (F_x, F_y, F_z)$. Since a is a regular value of F, $gradF \neq 0$ on S. So

$$\vec{N}(p) = \frac{(F_x(p), F_y(p), F_z(p))}{\sqrt{F_x^2(p) + F_y^2(p) + F_z^2(p)}} \qquad \text{for } p \in S$$

is a diffferentiable field of unit normal vectors on S. Hence S is orientable. For example, the sphere $x^2 + y^2 + z^2 = r^2$ is orientable. The graph of $z = f(x, y)$ is orientable if $f(x, y)$ is differentiable. In fact, let $F(x, y, z) = f(x, y) - z$, then $gradF = (f_z, f_y, -1) \neq \vec{0}$. Hence the surface $z = f(x, y)$ is the level surface $F(x, y, z) = 0$ with 0 being a regular value of F. So the surface $z = f(x, y)$ is orientable.

3.2 THE GAUSS MAP

Let S be an oriented regular surface, i.e., there exists a differentiable field $\vec{N}$ of unit normal vectors on S. The field $\vec{N}$ can be regarded as a differentiable map from S to the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$.

Definition 3.2. *The map* $\vec{N}: S \to S^2$ *is called the Gauss map.*

Recall that the curviness of a surface is caused by the change of the unit normal vector fields of S. Hence the "differential of the field of unit normal vectors" (the rate of change of the unit normal vectors) should measure the curviness of the surface.

Now $\vec{N}$ is a differentiable map between two surfaces S and S^2. The differential $d\vec{N}_p$ of the map $\vec{N}$ at a point p on S is a linear transformation

$$d\vec{N}_p: T_pS \to T_{\vec{N}(p)}S^2.$$

Recall the definition of $d\vec{N}_p$. Take a tangent vector $\vec{v} \in T_pS$. Let $\alpha(t): (-\epsilon, \epsilon) \to S \subset \mathbb{R}^3$ be a curve on S with $\alpha(0) = p$ and $\alpha'(0) = \vec{v}$. Then

$$d\vec{N}_p(\vec{v}) = \frac{d}{dt}\vec{N}(\alpha(t))|_{t=0}.$$

Since $\vec{N}(p) \perp T_pS$, $\vec{N}(p) \perp T_{\vec{N}(p)}S^2$ and both of the two planes pass through the origin, we have $T_pS = T_{\vec{N}(p)}S^2$. So we can regard $d\vec{N}_p$ as a linear transformation from T_pS to itself.

Example 3.3. (i) For a plane L given by $ax + by + cz = d$, we can take $\vec{N} = (a, b, c)/\sqrt{a^2 + b^2 + c^2}$ as the field of unit normal vectors of S. Let $\alpha(t)$ be a curve on L and p be a point on L with $p = \alpha(0)$, then

$$d\vec{N}_p(\alpha'(0)) = \frac{d}{dt}\vec{N}(\alpha(t))|_{t=0} = \frac{d}{dt}(a, b, c)/\sqrt{a^2 + b^2 + c^2} = \vec{0}.$$

Hence $d\vec{N}_p = 0$ is the zero map for any p on L.

(ii) Consider the sphere S of radius r, $S = \{(x, y, z)|x^2 + y^2 + z^2 = r^2\}$.

$$\vec{N}(x, y, x) = \frac{grad(x^2 + y^2 + z^2)}{|grad(x^2 + y^2 + z^2)|} = (x, y, z)/r$$

is a differentiable field of unit normal vectors of S. Let $\alpha(t) = (x(t), y(t), z(t))$ be a curve on S, then

$$\frac{d}{dt}\vec{N}(\alpha(t))|_{t=0} = \frac{d}{dt}\big((x(t), y(t), z(t))/r\big)|_{t=0} = \frac{1}{r}(x'(0), y'(0), z'(0)) = \frac{1}{r}\alpha'(0).$$

Hence $d\vec{N}_p(\alpha'(0)) = \frac{1}{r}\alpha'(0)$, i.e.,

$$d\vec{N}_p(\vec{v}) = \frac{1}{r}\vec{v} \qquad \text{for } \vec{v} \in T_pS.$$

(iii) Consider the cylinder S: $x^2 + y^2 = 1$ in $\mathbb{R}^3$.

$$\vec{N} = \frac{grad(x^2 + y^2)}{|grad(x^2 + y^2)|} = (x, y, 0)$$

is a differential field of unit normal vectors of S. Let $\alpha(t) = (x(t), y(t), z(t))$ be a curve on S with $\alpha(0) = p$ and $\alpha'(0) = \vec{v} = (a, b, c)$. Then the differential of $\vec{N}$ is

$$d\vec{N}_p(\alpha'(0)) = \frac{d}{dt}(x(t), y(t), 0)|_{t=0} = (x'(0), y'(0), 0).$$

Hence $d\vec{N}_p\big((a,b,c)\big) = (a,b,0)$. ■

Before we discuss some property of $d\vec{N}_p$, let's recall some concepts from linear algebra.

Definition 3.4. *Let V be a vector space with an inner product $<,>$. A linear transformation $L\colon V \to V$ is said to be self-adjoint if $< L\vec{v}, \vec{w} > = < \vec{v}, L\vec{w} >$ for any vectors $\vec{v}$ and $\vec{w}$ in V.*

A self-adjoint linear transformation is related to a symmetric matrix in the following way. Choose an orthonormal basis $\{\vec{e}_1, \dots, \vec{e}_n\}$ of V, i.e.,

$$< \vec{e}_i, \vec{e}_j > = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

Write $L\vec{e}_i = \sum\limits_{i=1}^{n} a_{ij}\vec{e}_j$. We get a matrix

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & a_{ij} & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}.$$

Hence we get

$$< L\vec{e}_i, \vec{e}_j > = < \sum_{k=1}^{n} a_{ik}\vec{e}_k, \vec{e}_j > = \sum_{k=1}^{n} a_{ik} < \vec{e}_k, \vec{e}_j > = \sum_{k=1}^{n} a_{ik}\delta_{kj} = a_{ij},$$

$$< \vec{e}_i, L\vec{e}_j > = < \vec{e}_i, \sum_{k=1}^{n} a_{jk}\vec{e}_k > = \sum_{k=1}^{n} a_{jk} < \vec{e}_i, \vec{e}_k > = \sum_{k=1}^{n} a_{jk}\delta_{ik} = a_{ji}.$$

Since $< L\vec{e}_i, \vec{e}_j > = < \vec{e}_i, L\vec{e}_j >$, we have $a_{ij} = a_{ji}$. Hence the matrix A is a symmetric matrix. A nice property of symmetric matrices is that they can be diagonalizable.

Proposition 3.5. *The differential $d\vec{N}_p\colon T_pS \to T_pS$ of the Gauss map $\vec{N}$ is a self-adjoint linear transformation.*

Proof. Choose a parametrization $\vec{X}(u,v)$ of S near a point $p \in S$. We know that $\vec{X}_u$ and $\vec{X}_v$ form a basis of T_pS. Let $\alpha(t) = \vec{X}(u(t), v(t))$ be a parametrized curve on S such that $\alpha(0) = p$ and $t \in (-\epsilon, \epsilon)$. $\alpha'(t) = \vec{X}_u u'(t) + \vec{X}_v v'(t)$. Choose the Gauss map $\vec{N}(u,v) = \dfrac{\vec{X}_u(u,v) \times \vec{X}_v(u,v)}{|\vec{X}_u(u,v) \times \vec{X}_v(u,v)|}$. Then

$$d\vec{N}_p(\vec{X}_u u'(0) + \vec{X}_v v'(0)) = d\vec{N}_p(\alpha'(0)) = \frac{d}{dt}\big(\vec{N}(u(t), v(t))\big)|_{t=0} = \vec{N}_u u'(0) + \vec{N}_v v'(0).$$

In particular, $d\vec{N}_p(\vec{X}_u) = \vec{N}_u$ and $d\vec{N}_p(\vec{X}_v) = \vec{N}_v$. Since $< \vec{N}, \vec{X}_u > = 0$ for all u and v, we get

$$0 = \frac{\partial}{\partial v} < \vec{N}, \vec{X}_u > = < \vec{N}_v, \vec{X}_u > + < \vec{N}, \vec{X}_{uv} > .$$

Hence $< \vec{N}_v, \vec{X}_u > = - < \vec{N}, \vec{X}_{uv} >$. Similarly, since $< \vec{N}, \vec{X}_v > = 0$ for all u and v, we get

$$0 = \frac{\partial}{\partial u} < \vec{N}, \vec{X}_v > = < \vec{N}_u, \vec{X}_v > + < \vec{N}, \vec{X}_{vu} > .$$

Hence $<\vec{N}_u, \vec{X}_v>= -<\vec{N}, \vec{X}_{vu}>$. Therefore we get

$$<\vec{N}_u, \vec{X}_v>=<\vec{N}_v, \vec{X}_u>.$$

Let $\vec{v} = a\vec{X}_u + b\vec{X}_v$ and $\vec{w} = c\vec{X}_u + d\vec{X}_v$ be two vectors in T_pS. Now since $d\vec{N}_p(a\vec{X}_u + b\vec{X}_v) = a\vec{N}_u + b\vec{N}_v$, we get

$$\begin{aligned}
& <d\vec{N}_p(\vec{v}), \vec{w}>=<d\vec{N}_p(a\vec{X}_u + b\vec{X}_v), c\vec{X}_u + d\vec{X}_v> \\
= \; & ac<\vec{N}_u, \vec{X}_u> + bc<\vec{N}_v, \vec{X}_u> + ad<\vec{N}_u, \vec{X}_v> + bd<\vec{N}_v, \vec{X}_v> \\
= \; & ac<\vec{N}_u, \vec{X}_u> + bc<\vec{N}_u, \vec{X}_v> + ad<\vec{N}_v, \vec{X}_u> + bd<\vec{N}_v, \vec{X}_v> \\
= \; & ac<\vec{X}_u, \vec{N}_u> + bc<\vec{X}_v, \vec{N}_u> + ad<\vec{X}_u, \vec{N}_v> + bd<\vec{X}_v, \vec{N}_v> \\
= \; & <a\vec{X}_u + b\vec{X}_v, d\vec{N}_p(c\vec{X}_u + d\vec{X}_v)>=<\vec{v}, d\vec{N}_p(\vec{w})>.
\end{aligned}$$

Hence the linear transformation $d\vec{N}_p$ is self-adjoint. ■

Definition 3.6. *The quadratic form II_p, defined over T_pS by $II_p(\vec{v}) = -<d\vec{N}_p(\vec{v}), \vec{v}>$, is called the second fundamental form of the surface S at p.*

Choose an orthonormal basis $\{\vec{\epsilon}_1, \vec{\epsilon}_2\}$ of T_pS. If we write $d\vec{N}_p(\vec{\epsilon}_1) = a\vec{\epsilon}_1 + b\vec{\epsilon}_2$ and $d\vec{N}_p(\vec{\epsilon}_2) = c\vec{\epsilon}_1 + d\vec{\epsilon}_2$, since the linear transformation $d\vec{N}_p$ is self-adjoint, b must equal c, i.e., the following matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is symmetric. From Linear Algebra, we know that the matrix A can be diagonalized under some orthonormal basis, i.e., there exist eigenvectors $\vec{s} = (s_1, s_2)$ and $\vec{t} = (t_1, t_2)$ of A with $s_1^2 + s_2^2 = 1$, $t_1^2 + t_2^2 = 1$ and $s_1t_1 + s_2t_2 = 0$ such that

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} \cdot \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} a & b \\ b & d \end{pmatrix} \cdot \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

where λ_1 and λ_2 are the corresponding eigenvalues of A. Hence

$$as_1 + bs_2 = \lambda_1 s_1, \quad bs_1 + ds_2 = \lambda_1 s_2, \quad at_1 + bt_2 = \lambda_1 t_1, \quad bt_1 + dt_2 = \lambda_2 t_2.$$

Take $\vec{e}_1 = s_1\vec{\epsilon}_1 + s_2\vec{\epsilon}_2$ and $\vec{e}_2 = t_1\vec{\epsilon}_1 + t_2\vec{\epsilon}_2$. One can check that

$$\begin{aligned}
& <\vec{e}_1, \vec{e}_1>=<s_1\vec{\epsilon}_1 + s_2\vec{\epsilon}_2, s_1\vec{\epsilon}_1 + s_2\vec{\epsilon}_2> \\
= \; & s_1^2<\vec{\epsilon}_1, \vec{\epsilon}_1> + 2s_1s_2<\vec{\epsilon}_1, \vec{\epsilon}_2> + s_2^2<\vec{\epsilon}_2, \vec{\epsilon}_2>= s_1^2 + s_2^2 = 1.
\end{aligned}$$

Similarly we get $<\vec{e}_1, \vec{e}_2>= 0$ and $<\vec{e}_2, \vec{e}_2>= 1$. Hence $\vec{e}_1$ and $\vec{e}_2$ form an orthonormal basis of T_pS.

$$d\vec{N}_p(\vec{e}_1)$$

$$
\begin{aligned}
&= s_1 d\vec{N}_p(\vec{\epsilon}_1) + s_2 d\vec{N}_p(\vec{\epsilon}_2) = s_1(a\vec{\epsilon}_1 + b\vec{\epsilon}_2) + s_2(b\vec{\epsilon}_1 + d\vec{\epsilon}_2)\\
&= (s_1 a + s_2 b)\vec{\epsilon}_1 + (bs_1 + ds_2)\vec{\epsilon}_2 = \lambda_1 s_1 \vec{\epsilon}_1 + \lambda_1 s_2 \vec{\epsilon}_2 = \lambda_1(s_1\vec{\epsilon}_1 + s_2\vec{\epsilon}_2)\\
&= \lambda_1 \vec{e}_1.
\end{aligned}
$$

$$
\begin{aligned}
& d\vec{N}_p(\vec{e}_2)\\
&= t_1 d\vec{N}_p(\vec{\epsilon}_1) + t_2 d\vec{N}_p(\vec{\epsilon}_2) = t_1(a\vec{\epsilon}_1 + b\vec{\epsilon}_2) + t_2(b\vec{\epsilon}_1 + d\vec{\epsilon}_2)\\
&= (t_1 a + t_2 b)\vec{\epsilon}_1 + (bt_1 + dt_2)\vec{\epsilon}_2 = \lambda_2 t_1 \vec{\epsilon}_1 + \lambda_2 t_2 \vec{\epsilon}_2 = \lambda_2(t_1\vec{\epsilon}_1 + t_2\vec{\epsilon}_2)\\
&= \lambda_2 \vec{e}_2.
\end{aligned}
$$

If we take $k_1 = \max\{-\lambda_1, -\lambda_2\}$ and $k_2 = \min\{-\lambda_1, -\lambda_2\}$, the computations above say that there exist an orthonormal basis $\{\vec{e}_1, \vec{e}_2\}$ of T_pS such that

$$d\vec{N}_p(\vec{e}_1) = -k_1\vec{e}_1, \qquad d\vec{N}_p(\vec{e}_2) = -k_2\vec{e}_2.$$

So the matrix representation of $d\vec{N}_p$ under the basis $\{\vec{e}_1, \vec{e}_2\}$ is the diagonal matrix

$$A = \begin{pmatrix} -k_1 & 0 \\ 0 & -k_2 \end{pmatrix}.$$

Definition 3.7. *With the notations as above, $k_1 \cdot k_2$ is called the Gauss curvature K of the surface S at p and $H = (k_1 + k_2)/2$ is called the mean curvature of S at p.*

For any tangent vector $\vec{v} \in T_pS$, if we use $\{\vec{e}_1, \vec{e}_2\}$ as the basis of T_pS, we can write $\vec{v} = a\vec{e}_1 + b\vec{e}_2$. Then

$$
\begin{aligned}
& I\!I_p(\vec{v}) = - < dN_p(\vec{v}), \vec{v} >\\
&= - < adN_p(\vec{e}_1) + bdN_p(\vec{e}_2), a\vec{e}_1 + b\vec{e}_2 >= - < -ak_1\vec{e}_1 - bk_2\vec{e}_2, a\vec{e}_1 + b\vec{e}_2 >\\
&= k_1 a^2 + k_2 b^2.
\end{aligned}
$$

Example 3.8. Consider the three surfaces in the Example 3.3.

(i) For the plane, since $d\vec{N}_p = 0$ is the zero map, $k_1 = k_2 = 0$. Hence the Gauss curvature $K = k_1 k_2 = 0$.

(ii) For the sphere of radius r, since $d\vec{N}_p(\vec{v}) = \frac{1}{r}\vec{v}$ for $\vec{v} \in T_pS$, the eigenvalues of $d\vec{N}_p$ is $1/r$. Hence $k_1 = k_2 = -1/r$ and the Gauss curvature $K = k_1 k_2 = 1/r^2$.

(iii) For the cylinder, let $\alpha(t) = (x(t), y(t), z(t))$ be a curve on S with $\alpha(0) = p$ and $\alpha'(0) = \vec{v} = (a, b, c)$. Then $x(t)^2 + y(t)^2 = 1$. Differentiating the above equation, we get $x'(0)x(0) + y'(0)y(0) = 0$. If we take $p = (p_1, p_2, p_3) = \alpha(0) = (x(0), y(0), z(0))$, then the set T_pS of tangent vectors consisits of vectors $\vec{v} = (a, b, c)$ such that $ap_1 + bp_2 = 0$. Since the vector $\vec{v}_0 = (0, 0, 1)$ satisfies the equation $0 \cdot p_1 + 0 \cdot p_2 = 0$, it is a tangent vector in T_pS. Recall that $d\vec{N}_p\big((a, b, c)\big) = (a, b, 0)$. Hence

$d\vec{N}_p\big((0,0,1)\big) = (0,0,0) = 0\cdot(0,0,1)$, i.e., 0 is an eigenvalue of $d\vec{N}_p$. Therefore $0 = k_1$ or $0 = k_2$. In any case, the Gauss curvature $K = k_1k_2 = 0$. ■

Example 3.9. Consider the surface $S\colon z = x^2 + ky^2$. $(2x, 2ky, -1)$ is a normal vector. We can take

$$\vec{N} = \frac{(2x, 2ky, -1)}{\sqrt{4x^2 + 4k^2y^2 + 1}}$$

as the field of unit normal vectors of S. At the point $P = (0,0,0)$, let $\alpha(t)$ be a curve on S with $\alpha(0) = P$. Write $\alpha(t) = (x(t), y(t), z(t))$. We have $z(t) = x(t)^2 + 2ky(t)^2$. Differentiate the both sides of the equation, we get $z'(t) = 2x(t)x'(t) + 2ky(t)y'(t)$. Hence $z'(0) = 2x(0)x'(0) + 2ky(0)y'(0) = 0$. So any tangent vector $\vec{v}$ of S at P is of the form $(a, b, 0)$. Thus $T_PS = \{(a, b, 0)\}$. Let's compute the differential of the Gauss map:

$$\begin{aligned}
& d\vec{N}_P(\alpha'(0)) \\
= \;& \frac{d}{dt}\vec{N}(\alpha(t))|_{t=0} = \frac{d}{dt}\frac{(2x(t), 2ky(t), -1)}{\sqrt{4x(t)^2 + 4k^2y(t)^2 + 1}}|_{t=0} \\
= \;& \frac{d}{dt}\frac{1}{\sqrt{4x(t)^2 + 4k^2y(t)^2 + 1}}|_{t=0}(0,0,1) + (2x'(0), 2ky'(0), 0) \\
= \;& (2x'(0), 2ky'(0), 0)
\end{aligned}$$

Hence we get

$$d\vec{N}_P\big((x'(0), y'(0), 0)\big) = d\vec{N}_P(\alpha'(0)) = (2x'(0), 2ky'(0), 0).$$

Hence we get $d\vec{N}_P\big((a, b, 0)\big) = (2a, 2kb, 0)$. Vectors $\vec{v}_1 = (1,0,0)$ and $\vec{v}_2 = (0,1,0)$ form an orthonormal basis of T_PS and we can check that $d\vec{N}_P(\vec{v}_1) = 2\vec{v}_1$ and $d\vec{N}_P(\vec{v}_2) = 2k\vec{v}_2$. Hence we get $k_1 = -2$, $k_2 = -2k$ or $k_1 = -2k$, $k_2 = -2$. In any case, the Gauss curvature $K = k_1 \cdot k_2 = -2\cdot(-2k) = 4k$. ■

From the discussions above, we see that for the plane, the Gauss curvature K of the plane at the point P is zero and the differential of the Gauss map $d\vec{N}_P$ is zero. For the cylinder $x^2 + y^2 = 1$, the Gauss curvature K at the point P is zero and the differential of the Gauss map $d\vec{N}_P$ isn't zero. For the paraboloid $z = x^2 + ky^2$ with $k > 0$, the Gauss curvature K at the point $P = (0,0,0)$ is $4k > 0$. For the hyperbolic paraboloid $z = x^2 + ky^2$ with $k < 0$, the Gauss curvature K at the point $P = (0,0,0)$ is $4k < 0$. These four cases are the representatives of all the points on a regular surface.

Definition 3.10. *A point p of a surface S is called*

1. *elliptic if the Gauss curvature K at p is positive.*

2. *hyperbolic if the Gauss curvature K at p is negativve.*

3. *parabolic if the Gauss curvature K at p is zero but the differential $d\vec{N}_p$ of the Gauss map at p is not zero.*

4. *planar if the Gauss curvature K at p is zero and the differential $d\vec{N}_p$ of the Gauss map at p is zero.*

Next, we shall study the interplay between the curvature of a curve on a regular surface and the Gauss curvature of the surface.

Consider a curve $\alpha(s)$ on a regular surface S with s being the arclength parameter, $s \in (-\epsilon, \epsilon)$ and $p = \alpha(0)$. Let $\vec{N}$ be the field of unit normal vectors of S and $\vec{N}(s) = \vec{N}(\alpha(s))$ be the restriction of $\vec{N}$ to the curve $\alpha(s)$. Since $\alpha'(s)$ is a tangent vector of S at the point $\alpha(s)$, we get $\alpha'(s) \perp \vec{N}(s)$, i.e., $< \alpha'(s), \vec{N}(s) >= 0$ for all s. Differentiate the equation above, we get

$$0 = \frac{d}{ds} < \alpha'(s), \vec{N}(s) >=< \alpha'(s), \frac{d}{ds}\vec{N}(s) > + < \alpha''(s), \vec{N}(s) > .$$

$$< \alpha''(s), \vec{N}(s) >= - < \alpha'(s), \frac{d}{ds}\vec{N}(s) > .$$

Let $s = 0$, we get

$$< \alpha''(0), \vec{N}(0) >= - < \alpha'(0), d\vec{N}_p(\alpha'(0)) >= II_p(\alpha'(0)).$$

Let $\vec{n}$ be the unit normal vector of the curve α at p and κ be the curvature of the curve α at p, then we get

$$II_p(\alpha'(0)) = \kappa < \vec{N}_p, \vec{n} > .$$

This leads to the following definition.

Definition 3.11. *With assumptions as above, $\kappa_n = \kappa < \vec{N}_p, \vec{n} >$ is called the normal curvature of the curve $\alpha(s)$ at p.*

If we use θ to denote the angle between $\vec{n}$ and $\vec{N}_p$, then the normal curvature $\kappa_n = \kappa cos\theta$. Hence the value of the second fundamental form II_p at a unit tangent vector $\vec{v} \in T_pS$ equals the normal curvature of a regular parametrized curve α passing through p and tangent to $\vec{v}$, i.e.,

$$II_p(\vec{v}) = II(\alpha'(0)) = \kappa_n$$

where $\alpha'(0) = \vec{v}$ and we use the arclength parameter to parametrize the curve α. We see that the normal curvature of the curve α only depends on the unit tangent vector $\alpha'(0)$. Therefore if there are two curves on S passing through the same point p and having the same unit tangent vectors (equivalently having the same tangent lines), they must have the same normal curvatures at p. Hence we proved the following proposition.

Proposition 3.12. *All curves lying on a regular surface S and having the same tangent lines at a given point $p \in S$ must have the same normal curvatures at this point.*

This leads to the following definition.

Definition 3.13. *Given a unit tangent vector $\vec{v}$ of S at p, the normal curvature along the direction $\vec{v}$ is defined to be $II_p(\vec{v})$.*

Let $\vec{v}$ be a tangent vector of S at p. Choose a plane L passing through the point p and parallel to the tangent vector v and the normal vector $\vec{N}_p$ of the surface S at p. This plane intersects with the surface S along a curve C on S whose tangent vector is $\vec{v}$ and normal vector $\pm\vec{N}_p$. The curve C is called the normal section.

The curvature κ of the curve C at the point p is $II_p(\vec{v})$. If we use $\{\vec{e}_1, \vec{e}_2\}$ to denote the orthonormal basis of T_pS such that $dN_p(\vec{e}_1) = -k_1\vec{e}_1$ and $dN_p(\vec{e}_2) = -k_2\vec{e}_2$, then $\vec{v}$ can be written as $\vec{v} =$

$cos\theta\vec{e}_1 + sin\theta\vec{e}_2$ where θ is the angle between $\vec{e}_1$ and $\vec{v}$. Now we get the curvature of the normal section C at p is given by

$$\kappa = |II_p(\vec{v})| = |k_1 cos^2\theta + k_2 sin^2\theta|.$$

This formula is called Euler formula.

Let $\{\vec{e}_1, \vec{e}_2\}$ be the orthonormal basis of T_pS such that

$$d\vec{N}_p(\vec{e}_1) = -k_1\vec{e}_1, \qquad d\vec{N}_p(\vec{e}_2) = -k_2\vec{e}_2, \quad \text{where } k_1 \geq k_2.$$

For any unit tangent vector $\vec{v} \in T_pS$, we can write $\vec{v} = cos\theta\vec{e}_1 + sin\theta\vec{e}_2$ where θ is the angle between $\vec{v}$ and $\vec{e}_1$. Hence the normal curvature along the direction $\vec{v}$ is

$$\begin{aligned} & II_p(\vec{v}) = -< d\vec{N}_p(cos\theta\vec{e}_1 + sin\theta\vec{e}_2), cos\theta\vec{e}_1 + sin\theta\vec{e}_2 > \\ = \; & -< cos\theta d\vec{N}_p(\vec{e}_1) + sin\theta d\vec{N}_p(\vec{e}_2), cos\theta\vec{e}_1 + sin\theta\vec{e}_2 > \\ = \; & -< cos\theta(-k_1)\vec{e}_1 + sin\theta(-k_2)\vec{e}_2, cos\theta\vec{e}_1 + sin\theta\vec{e}_2 > \\ = \; & k_1 cos^2\theta + k_2 sin^2\theta. \end{aligned}$$

$$k_2 = k_2 cos^2\theta + k_2 sin^2\theta \leq k_1 cos^2\theta + k_2 sin^2\theta \leq k_1 cos^2\theta + k_1 sin^2\theta = k_1.$$

Hence k_1 is the maximum normal curvature and k_2 is the minimum normal curvature.

Definition 3.14. *The maximum and minimum normal curvatures k_1 and k_2 are called the principal curvatures. The corresponding directions (eigenvectors) $\vec{e}_1$ and $\vec{e}_2$ are called principal directions.*

Example 3.15. Consider a plane. From the computations in Example 3.3, we know that $d\vec{N}_p = 0$ for any point p on the plane. Hence the second fundamental form $II_p(\vec{v}) = -< d\vec{N}_p(\vec{v}), \vec{v} >= 0$ for all $\vec{v}$. Therefore the maximum and minimum normal curvatures k_1 and k_2 are equal to zero. Hence every direction is a principal direction.

Consider the sphere S of radius r. Let $\vec{v}$ be a unit tangent vector at p. Cut the sphere by the plane L passing through the point p and parallel to the vectors $\vec{v}$ and the normal vector $\vec{N}$ of S at p. $II_p(\vec{v})$ is the normal curvature κ_n of the curve C which is the intersection of S and L. Since the curve C is on the plane L, the normal vector $\vec{n}$ of C at p is $\vec{N}$. Hence the angle θ between $\vec{n}$ and $\vec{N}$ is 0 and $II_p(\vec{v}) = \kappa_n = \kappa cos\theta = \kappa$ where κ is the curvature of the curve C. But the curve C, which is a circle of radius r, has curvature equal to $1/r$. So $II_p(\vec{v}) = 1/r$ for all unit tangent vectors $\vec{v}$ at p. Therefore all normal curvatures are equal and every direction is a principal direction. ∎

Definition 3.16. *Given a point p on a regular surface S. If the principal curvatures at p are all equal, equivalently $k_1 = k_2$, the point p is called an umbilical point of S.*

Every point on a plane is an umbilical point. The same is true for any sphere.

Proposition 3.17. *If all points of a connected regular surface S are umbilical points, then S is either contained in a sphere or in a plane.*

Proof. Choose a parametrization $\vec{X}(u,v)\colon U \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$ such that U is a connected open subset of $\mathbb{R}^2$. Let $V = \vec{X}(U)$. For each point $q \in V$, q is an umbilical point, i.e., $k_1(q) = k_2(q) = -\lambda(q)$. For any tangent vector $\vec{v}$ of S at q, since $\vec{X}_u$ and $\vec{X}_v$ form a basis of T_qS, we can write $\vec{v} = a\vec{X}_u + b\vec{X}_v$. Let $\vec{e}_1$ and $\vec{e}_2$ be two perpendicular principal directions at q and write $\vec{v} = c\vec{e}_1 + d\vec{e}_2$. Then

$$d\vec{N}_q(\vec{v}) = d\vec{N}_q(c\vec{e}_1 + d\vec{e}_2) = cd\vec{N}_q(\vec{e}_1) + dd\vec{N}_q(\vec{e}_2) = -ck_1\vec{e}_1 - dk_2\vec{e}_2 = \lambda(q)(c\vec{e}_1 + d\vec{e}_2) = \lambda(q)\vec{v}.$$

Therefore we have $d\vec{N}_q(a\vec{X}_u + b\vec{X}_v) = \lambda(q)(a\vec{X}_u + b\vec{X}_v)$. In particular we have

$$\vec{N}_u = d\vec{N}_q(\vec{X}_u) = \lambda(q)\vec{X}_u, \quad \text{and} \quad \vec{N}_v = d\vec{N}_q(\vec{X}_v) = \lambda(q)\vec{X}_v.$$

Take differentiations of the equations above, we get

$$\frac{\partial}{\partial v}\vec{N}_u = \frac{\partial}{\partial v}(\lambda\vec{X}_u), \quad \frac{\partial}{\partial u}\vec{N}_v = \frac{\partial}{\partial u}(\lambda\vec{X}_v).$$

Hence we get

$$\vec{N}_{uv} = \lambda_v\vec{X}_u + \lambda\vec{X}_{uv}, \quad \vec{N}_{vu} = \lambda_u\vec{X}_v + \lambda\vec{X}_{vu}.$$

Since $\vec{N}_{uv} = \vec{N}_{vu}$ and $\vec{X}_{uv} = \vec{X}_{vu}$, we get $\lambda_v\vec{X}_u = \lambda_u\vec{X}_v$. Since $\vec{X}_u$ and $\vec{X}_v$ are linearly independent, we must have $\lambda_u = \lambda_v = 0$ for all $(u, v) \in U$. Since U is connected, λ must be a constant function of $(u, v) \in U$.

Two cases can occur:

1. If $\lambda = 0$, then $\vec{N}_u = \vec{N}_v = 0$ for all $(u, v) \in U \subset \mathbb{R}^2$. Hence $\vec{N}$ must be a constant vector for all $(u, v) \in U \subset \mathbb{R}^2$. Thus V is a piece of a plane.

2. If $\lambda = c \neq 0$ for some constant c, then $\vec{N}_u = c\vec{X}_u$. Therefore $\frac{\partial}{\partial u}(\frac{\vec{N}}{c} - \vec{X}) = \vec{0}$. $\vec{N}_v = c\vec{X}_v$ implies $\frac{\partial}{\partial v}(\frac{\vec{N}}{c} - \vec{X}) = \vec{0}$. Hence $\frac{\vec{N}}{c} - \vec{X}$ is a constant vector, denoted by $-\vec{C}$. So $\frac{\vec{N}}{c} - \vec{X} = -\vec{C}$, i.e., $\vec{X} - \vec{C} = \frac{\vec{N}}{c}$. Remember that $\vec{N}$ is a unit vector, so $|\vec{X} - \vec{C}|^2 = 1/c^2$. Hence V is on the sphere of radius $1/c$ with center at $\vec{C}$.

Now we know that the surface S is covered by either a piece of a sphere or a plane. By the regularity and connectedness of the surface, it is either a piece of a sphere or a piece of a plane. ∎

Definition 3.18. *If a regular connected curve C on a regular surface S is such that the unit tangent vector of C at $p \in C$ is a principal direction for all $p \in C$, then the curve C is called a line of curvature of S.*

Definition 3.19. *Let p be a point on a regular surface S. An asymptotic direction of S at p is a unit tangent vector in T_pS along which the normal curvature is zero. An asymptotic curve of S is a regular connected curve $C \subset S$ such that for each $p \in C$ the tangent line of C at p is parallel to an asymptotic direction.*

Definition 3.20. *Let p be a point on a surface S. Two nonzero vectors $\vec{w}_1, \vec{w}_2 \in T_pS$ are conjugate if $< d\vec{N}_p(\vec{w}_1), \vec{w}_2 >= 0 =< \vec{w}_1, d\vec{N}_p(\vec{w}_2) >$.*

Let's discuss a bit more about asymptotic directions. Let $\vec{e}_1$ and $\vec{e}_2$ be the principal directions of T_pS such that $d\vec{N}_p(\vec{e}_1) = -k_1\vec{e}_1$ and $d\vec{N}_p(\vec{e}_2) = -k_1\vec{e}_2$ where $k_1 \geq k_2$. Let $\vec{v} \in T_pS$ be a direction. Write $\vec{v} = a\vec{e}_1 + b\vec{e}_2$, then the normal curvature along $\vec{v}$ is $II_p(\vec{v}) = k_1a^2 + k_2b^2$.

If p is an elliptic point, i.e., $K = k_1k_2 > 0$, then k_1 and k_2 have the same sign. Hence $II_p(\vec{v}) = k_1a^2 + k_2b^2 \neq 0$ since $a^2 + b^2 = 1$. Therefore there are no asymptotic directions at p.

If p is a planar point, then $k_1 = k_2 = 0$. Hence $II_p(\vec{v}) = k_1a^2 + k_2b^2 = 0$ for any tangent direction $\vec{v}$. So any direction is an asymptotic direction.

If p is a hyperbolic point, then $K = k_1k_2 < 0$. Hence k_1 and k_2 have different signs. Since $k_1 \geq k_2$, we must have $k_1 > 0$ and $k_2 < 0$. In this case $II_p(\vec{v}) = k_1a^2 + k_2b^2 = k_1a^2 - (-k_2)b^2 = (\sqrt{k_1}a + \sqrt{-k_2}b)(\sqrt{k_1}a - \sqrt{-k_2}b)$. Hence there are two asymptotic directions

$$\vec{v} = a\vec{e}_1 + b\vec{e}_2 \quad \text{with } \sqrt{k_1}a + \sqrt{-k_2}b = 0 \quad \text{or } \sqrt{k_1}a - \sqrt{-k_2}b = 0.$$

$$\vec{v}_1 = \pm\frac{1}{\sqrt{k_1 - k_2}}(-\sqrt{-k_2}\vec{e}_1 + \sqrt{k_1}\vec{e}_2), \qquad \vec{v}_2 = \pm\frac{1}{\sqrt{k_1 - k_2}}(\sqrt{-k_2}\vec{e}_1 + \sqrt{k_1}\vec{e}_2).$$

If p is a parabolic point, then $d\vec{N}_p \neq 0$ but $K = k_1k_2 = 0$. Hence one of k_1 and k_2 is zero and the other isn't. Suppose $k_1 = 0$ and $k_2 \neq 0$, then $II_p(\vec{v}) = k_1a^2 + k_2b^2 = k_2b^2$. Hence the asymptotic directions are $\pm\vec{e}_1$. If $k_1 \neq 0$ and $k_2 = 0$, then the asymptotic directions are $\pm\vec{e}_2$.

Next, let's look at the relations among principal directions, asymptotic directions and conjugate directions.

Fact 3.21. Two orthogonal principal directions are conjugate.
Proof. Let $\vec{e}_1$ and $\vec{e}_2$ be the two orthogonal principal directions such that $d\vec{N}_p(\vec{e}_1) = -k_1\vec{e}_1$ and $d\vec{N}_p(\vec{e}_2) = -k_2\vec{e}_2$. Hence we have

$$< d\vec{N}_p(\vec{e}_1), \vec{e}_2 >=< -k_1\vec{e}_1, \vec{e}_2 >= 0, \qquad < \vec{e}_1, d\vec{N}_p(\vec{e}_2) >=< \vec{e}_1, -k_2\vec{e}_2 >= 0.$$

Hence $\vec{e}_1$ and $\vec{e}_2$ are conjugate. ∎

Fact 3.22. An asymptotic direction is conjugate to itself.
Proof. Let $\vec{v}$ be an asymptotic direction, i.e., $II_p(\vec{v}) = 0$. Then

$$< d\vec{N}_p(\vec{v}), \vec{v} >=< \vec{v}, d\vec{N}_p(\vec{v}) >= -II_p(\vec{v}) = 0.$$

Hence by the defination $\vec{v}$ is conjugate to itself. ∎

Fact 3.23. At a nonplanar umbilical point, every orthogonal pair of directions is a pair of conjugate directions.
Proof. Nonplanar means that one of k_1 and k_2 is not zero. Umbilical point means that $k_1 = k_2 = \lambda$. Let $\vec{v} = a\vec{e}_1 + b\vec{e}_2$. Then

$$d\vec{N}_p(\vec{v}) = ad\vec{N}_p(\vec{e}_1) + bd\vec{N}_p(\vec{e}_2) = a(-k_1)\vec{e}_1 + b(-k_2)\vec{e}_2 = -\lambda\vec{v}.$$

Let $\vec{v}_1, \vec{v}_2$ be two orthogonal pair of directions, i.e., $< \vec{v}_1, \vec{v}_2 >= 0$, then we have

$$< d\vec{N}_p(\vec{v}_1), \vec{v}_2 >=< -\lambda\vec{v}_1, \vec{v}_2 >= 0, \qquad < \vec{v}_1, d\vec{N}_p(\vec{v}_2) >=< \vec{v}_1, -\lambda\vec{v}_2 >= 0.$$

Hence $\vec{v}_1$ and $\vec{v}_2$ are conjugate to each other. ∎

Fact 3.24. At a planar point p, any two pairs of directions are conjugate.
Proof. Since p is a planar point, $d\vec{N}_p(\vec{v}) = 0$ for any $\vec{v} \in T_pS$. Let $\vec{w}$ be another arbitrary direction at p, then

$$< d\vec{N}_p(\vec{v}), \vec{w} >=< \vec{0}, \vec{w} >= 0, \qquad < \vec{v}, d\vec{N}_p(\vec{w}) >=< \vec{v}, \vec{0} >= 0.$$

Hence $\vec{v}$ is conjugate to $\vec{w}$. ∎

Example 3.25. Show that if the mean curvature is zero at a nonplanar point, then this point has two orthogonal asymptotic directions.

Proof. Mean curvature is $H = (k_1 + k_2)/2$. Hence $H = 0$ implies $k_1 = -k_2$. Since the point p is not planar, k_1 and k_2 cannot be zero at the same time. Hence k_1 must be positive and k_2 must be negative. Let $k_1 = k > 0$ and then $k_2 = -k$. For any direction $\vec{v} = a\vec{e}_1 + b\vec{e}_2 \in T_pS$, the normal curvature along $\vec{v}$ is $II_p(\vec{v}) = k_1a^2 + k_2b^2 = ka^2 - kb^2 = k(a^2 - b^2)$. Hence the aymptotic directions are those $\vec{v} = a\vec{e}_1 + b\vec{e}_2$ such that $b^2 - a^2 = 0$. So $b = \pm a$. Since $a^2 + b^2 = 1$, we must have $a = b = \pm\sqrt{2}/2$ or $a = -b = \pm\sqrt{2}/2$. Hence $\vec{v}_1 = \pm\sqrt{2}/2(\vec{e}_1 + \vec{e}_2)$ and $\vec{v}_2 = \pm\sqrt{2}/2(\vec{e}_1 - \vec{e}_2)$ are asymptotic directions. Now

$$< \vec{v}_1, \vec{v}_2 > = < \pm\sqrt{2}/2(\vec{e}_1 + \vec{e}_2), \pm\sqrt{2}/2(\vec{e}_1 - \vec{e}_2) > = 0.$$

Hence $\vec{v}_1$ and $\vec{v}_2$ are orthogonal. ■

Let $\vec{r}$ and $\vec{r'}$ be conjugate directions at p. Let θ be the angle between $\vec{e}_1$ and $\vec{r}$ and ϕ be the angle between $\vec{e}_1$ and $\vec{r'}$. We can write

$$\vec{r} = cos\theta\vec{e}_1 + sin\theta\vec{e}_2, \qquad \vec{r'} = cos\phi\vec{e}_1 + sin\phi\vec{e}_2.$$

Now we have

$$\begin{aligned} & < d\vec{N}_p(\vec{r}), \vec{r'} > = < \vec{r}, d\vec{N}_p(\vec{r'}) > \\ = \; & < cos\theta d\vec{N}_p(\vec{e}_1) + sin\theta d\vec{N}_p(\vec{e}_2), cos\phi\vec{e}_1 + sin\phi\vec{e}_2 > \\ = \; & < -k_1cos\theta\vec{e}_1 - k_2sin\theta\vec{e}_2, cos\phi\vec{e}_1 + sin\phi\vec{e}_2 > \\ = \; & -k_1cos\theta cos\phi - k_2sin\theta sin\phi. \end{aligned}$$

Hence we see that $\vec{r}$ and $\vec{r'}$ are conjuage iff $k_1cos\theta cos\phi + k_2sin\theta sin\phi = 0$.

Let's use this condition to solve the following problem.

Example 3.26. Let p be a non-umbilical elliptic point of a surface S and let $\vec{r}$ and $\vec{r'}$ be conjugate directions at p. Let $\vec{r}$ vary in T_pS and show that the minimum of the angle of $\vec{r}$ with $\vec{r'}$ is reached at a unique pair of directions in T_pS that are symmetric with respect to the principal directions.

Proof. With the notation,. $|\phi - \theta|$ is the angle between $\vec{r}$ and $\vec{r'}$. We have $|\phi - \theta| < 2\pi$. $\phi - \theta = 0$ or $\pm\pi$ are not possible since otherwise from (3.1), we get $k_1cos^2\theta + k_2sin^2\theta = 0$. Since p is elliptic, k_1 and k_2 have the same sign. Hence we must have $cos^2\theta = sin^2\theta = 0$, a contradiction.

We need to show if $|\phi - \theta|$ gets minimum, then $\phi = \pi - \theta$, $\phi = 2\pi - \theta$, or $\phi = 3\pi - \theta$.

If the angle $|\phi - \theta|$ gets minimum, then $cos(\phi - \theta)$ has a minimum or maximum. Since $\vec{r}$ and $\vec{r'}$ are conjugate, we must have

$$k_1cos\theta cos\phi + k_2sin\theta sin\phi = 0. \tag{3.1}$$

Hence the question is reduced to find the extrema of $cos(\phi - \theta)$, which is a function of two variables ϕ and θ, under the constraint (3.1). The method to solve this type of the problem is the Lagrange method.

$$\begin{cases} gradcos(\phi - \theta) = \lambda grad(k_1cos\theta cos\phi + k_2sin\theta sin\phi), \\ k_1cos\theta cos\phi + k_2sin\theta sin\phi = 0. \end{cases}$$

$$gradcos(\phi-\theta)=(\frac{\partial}{\partial\phi}cos(\phi-\theta),\frac{\partial}{\partial\theta}cos(\phi-\theta))=(-sin(\phi-\theta),sin(\phi-\theta)),$$

$$grad(k_1cos\theta cos\phi+k_2sin\theta sin\phi)=(-k_1cos\theta sin\phi+k_2sin\theta cos\phi,-k_1sin\theta cos\phi+k_2cos\theta sin\phi).$$

Hence we get

$$-sin(\phi-\theta)=\lambda(-k_1cos\theta sin\phi+k_2sin\theta cos\phi),$$

$$sin(\phi-\theta)=\lambda(-k_1sin\theta cos\phi+k_2cos\theta sin\phi).$$

If $\lambda=0$, then $sin(\phi-\theta)=0$. Hence $\phi-\theta=k\pi$ for some integer k. From (3.1), we get $k_1cos^2\theta+k_2sin^2\theta=0$. Since p is elliptic, k_1 and k_2 have the same sign. Hence we must have $cos^2\theta=sin^2\theta=0$, a contradiction. Therefore λ cannot be zero. Thus we have

$$k_1cos\theta sin\phi-k_2sin\theta cos\phi=-k_1sin\theta cos\phi+k_2cos\theta sin\phi,$$

$$k_1(cos\theta sin\phi+sin\theta cos\phi)-k_2(sin\theta cos\phi+cos\theta sin\phi)=0,$$

$$k_1sin(\theta+\phi)-k_2sin(\theta+\phi)=0.$$

Hence we get $(k_1-k_2)sin(\theta+\phi)=0$. Since p is not an umbilical point, $k_1\neq k_2$. Therefore $sin(\theta+\phi)=0$, i.e., $\theta=k\pi-\phi$ where $k=1,2,3$. Therefore θ and ϕ is symmetric with respect to $\vec{e}_1$, or $-\vec{e}_1$, or $\vec{e}_2$, or $-\vec{e}_2$. ∎

3.3 THE GAUSS MAP IN LOCAL COORDINATES

Let $\vec{X}:U\subset\mathbb{R}^2\to S\subset\mathbb{R}^3$ be a parametrization such that $\vec{N}(u,v)=\dfrac{\vec{X}_u\times\vec{X}_v}{|\vec{X}_u\times\vec{X}_v|}$ is the orientation (field of unit normal vectors) of S. We want to represent the second fundamental form in terms of u,v as we did for the first fundamental form and to study all the concepts in the previous section again in this new setup.

Recall that, at a point $p\in S$, the tangent plane T_pS has $\{\vec{X}_u,\vec{X}_v\}$ as a basis. Since $\vec{N}(u,v)$ is a unit normal vector, we have $<\vec{N}(u,v),\vec{N}(u,v)>=1$ for all $(u,v)\in U$. Hence we get

$$0=\frac{\partial}{\partial u}<\vec{N}(u,v),\vec{N}(u,v)>=2<\vec{N}_u(u,v),\vec{N}(u,v)>,$$

$$0=\frac{\partial}{\partial v}<\vec{N}(u,v),\vec{N}(u,v)>=2<\vec{N}_v(u,v),\vec{N}(u,v)>.$$

Hence we get $\vec{N}_u(u,v)\perp\vec{N}(u,v)$ and $\vec{N}_v(u,v)\perp\vec{N}(u,v)$. Therefore $\vec{N}_u(u,v)$ and $\vec{N}_v(u,v)$ are tangent vectors of S at p. So $\vec{N}_u=\vec{N}_u(u,v)$ and $\vec{N}_v=\vec{N}_v(u,v)$ can be written as a linear combination of $\vec{X}_u$ and $\vec{X}_v$.

$$\vec{N}_u=a_{11}\vec{X}_u+a_{21}\vec{X}_v,\qquad\vec{N}_v=a_{12}\vec{X}_u+a_{22}\vec{X}_v.$$

Recall that $d\vec{N}_p(\vec{X}_u) = \vec{N}_u$ and $d\vec{N}_p(\vec{X}_v) = \vec{N}_v$. So for any tangent vector $\vec{v} = a\vec{X}_u + b\vec{X}_v$, we get

$$\begin{aligned} & d\vec{N}_p(\vec{v}) = d\vec{N}_p(a\vec{X}_u + b\vec{X}_v) \\ = \; & ad\vec{N}_p(\vec{X}_u) + bd\vec{N}_p(\vec{X}_v) = a\vec{N}_u + b\vec{N}_v \\ = \; & a(a_{11}\vec{X}_u + a_{21}\vec{X}_v) + b(a_{12}\vec{X}_u + a_{22}\vec{X}_v) \\ = \; & (aa_{11} + ba_{12})\vec{X}_u + (aa_{21} + ba_{22})\vec{X}_v \end{aligned}$$

Hence the matrix representation of the linear transformation $d\vec{N}_p$ from T_pS to T_pS under the basis $\{\vec{X}_u, \vec{X}_v\}$ is

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \text{ i.e., } d\vec{N}_p(a\vec{X}_u + b\vec{X}_v) = c_1\vec{X}_u + c_2\vec{X}_v \text{ where } \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = A \cdot \begin{pmatrix} a \\ b \end{pmatrix}.$$

Hence the Gauss curvature $K = k_1k_2 = detA = a_{11}a_{22} - a_{21}a_{12}$.

Now the second fundamental form takes the form

$$\begin{aligned} & II(a\vec{X}_u + b\vec{X}_v) = - < d\vec{N}(a\vec{X}_u + b\vec{X}_v), a\vec{X}_u + b\vec{X}_v > \\ = \; & - < ad\vec{N}(\vec{X}_u) + bd\vec{N}(\vec{X}_v), a\vec{X}_u + b\vec{X}_v >= - < a\vec{N}_u + b\vec{N}_v, a\vec{X}_u + b\vec{X}_v > \\ = \; & - < \vec{N}_u, \vec{X}_u > a^2 - (< \vec{N}_u, \vec{X}_v > + < \vec{N}_v, \vec{X}_u >)ab - < \vec{N}_v, \vec{X}_v > b^2. \end{aligned}$$

Let $e = - < \vec{N}_u, \vec{X}_u >$, $f = -(< \vec{N}_u, \vec{X}_v > + < \vec{N}_v, \vec{X}_u >)/2$ and $g = - < \vec{N}_v, \vec{X}_v >$. Since $< \vec{N}, \vec{X}_u >= 0$, we get

$$0 = \frac{\partial}{\partial u} < \vec{N}, \vec{X}_u >=< \vec{N}_u, \vec{X}_u > + < \vec{N}, \vec{X}_{uu} > .$$

Hence $e = - < \vec{N}_u, \vec{X}_u >=< \vec{N}, \vec{X}_{uu} >$.

Similarly from $< \vec{N}, \vec{X}_v >= 0$, we get

$$0 = \frac{\partial}{\partial v} < \vec{N}, \vec{X}_v >=< \vec{N}_v, \vec{X}_v > + < \vec{N}, \vec{X}_{vv} > .$$

Hence $g = - < \vec{N}_v, \vec{X}_v >=< \vec{N}, \vec{X}_{vv} >$.

From

$$\begin{aligned} & 0 = \frac{\partial}{\partial v} < \vec{N}, \vec{X}_u >=< \vec{N}_v, \vec{X}_u > + < \vec{N}, \vec{X}_{uv} >, \\ and \qquad & 0 = \frac{\partial}{\partial u} < \vec{N}, \vec{X}_v >=< \vec{N}_u, \vec{X}_v > + < \vec{N}, \vec{X}_{vu} >, \end{aligned}$$

we get $f = - < \vec{N}_u, \vec{X}_v >= - < \vec{N}_v, \vec{X}_u >=< \vec{N}, \vec{X}_{vu} >$.

Hence the second fundamental form takes the following form

$$II(a\vec{X}_u + b\vec{X}_v) = ea^2 + 2fab + gb^2.$$

Recall that we wrote $\vec{N}_u = a_{11}\vec{X}_u + a_{21}\vec{X}_v$ and $\vec{N}_v = a_{12}\vec{X}_u + a_{22}\vec{X}_v$. Take the inner product of the first equation with $\vec{X}_v$, we get

$$-f =< \vec{N}_u, \vec{X}_v >=< a_{11}\vec{X}_u + a_{21}\vec{X}_v, \vec{X}_v >= a_{11}F + a_{21}G.$$

Similarly we get

$$-f =< \vec{N}_v, \vec{X}_u >=< a_{12}\vec{X}_u + a_{22}\vec{X}_v, \vec{X}_u >= a_{12}E + a_{22}F.$$

$$-e =< \vec{N}_u, \vec{X}_u >=< a_{11}\vec{X}_u + a_{21}\vec{X}_v, \vec{X}_u >= a_{11}E + a_{21}F.$$

$$-g =< \vec{N}_v, \vec{X}_v >=< a_{21}\vec{X}_u + a_{22}\vec{X}_v, \vec{X}_v >= a_{21}F + a_{22}G.$$

Write the above three equations in the matrix form:

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

Hence

$$A^{\tau} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix} \cdot \begin{pmatrix} E & F \\ F & E \end{pmatrix}^{-1}.$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \frac{1}{EG - F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}.$$

So we get

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix} \cdot \frac{1}{EG - F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$
$$= \quad -\frac{1}{EG - F^2}\begin{pmatrix} eG - fF & fE - eF \\ fG - gF & -fF + gE \end{pmatrix}.$$

Therefore we obtain

$$a_{11} = \frac{fF - eG}{EG - F^2},\ a_{21} = \frac{eF - fE}{EG - F^2},\ a_{12} = \frac{gF - fG}{EG - F^2},\ a_{22} = \frac{fF - gE}{EG - F^2}.$$

The Gauss curvature is $K = a_{11}a_{22} - a_{21}a_{12} = detA = detA^{\tau}$. Hence

$$K = det\begin{pmatrix} e & f \\ f & g \end{pmatrix} \cdot det\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \frac{eg - f^2}{EG - F^2}.$$

Example 3.27. Compute the Gauss curvature of the points of the torus covered by the parametrization found in Example 2.10 in the Chapter 2.

$$\vec{X}(\theta, \phi) = ((r + acos\theta)cos\phi, (r + acos\theta)sin\phi, asin\theta).$$

$$\vec{X}_\theta = (-asin\theta cos\phi, -asin\theta sin\phi, acos\theta),$$
$$\vec{X}_\phi = (-(r+acos\theta)sin\phi, (r+acos\theta)cos\phi, 0),$$
$$\vec{X}_{\theta\theta} = (-acos\theta cos\phi, -acos\theta sin\phi, -asin\theta),$$
$$\vec{X}_{\theta\phi} = (asin\theta sin\phi, -asin\theta cos\phi, 0),$$
$$\vec{X}_{\phi\phi} = (-(r+acos\theta)cos\phi, -(r+acos\theta)sin\phi, 0).$$

$$E = <\vec{X}_\theta, \vec{X}_\theta> = a^2sin^2\theta cos^2\phi + a^2sin^2\theta sin^2\phi + a^2cos^2\theta = a^2,$$
$$F = <\vec{X}_\theta, \vec{X}_\phi> = asin\theta cos\phi(r+acos\theta)sin\phi - asin\theta sin\phi(r+acos\theta)cos\phi = 0,$$
$$G = <\vec{X}_\phi, \vec{X}_\phi> = (r+acos\theta)^2sin^2\phi + (r+acos\theta)^2cos^2\phi = (r+acos\theta)^2.$$

$$\begin{aligned}
e &= <\vec{N}, \vec{X}_{\theta\theta}> = <\frac{\vec{X}_\theta \times \vec{X}_\phi}{|\vec{X}_\theta \times \vec{X}_\phi|}, \vec{X}_{\theta\theta}> = \frac{(\vec{X}_\theta \times \vec{X}_\phi)\cdot\vec{X}_{\theta\theta}}{|\vec{X}_\theta \times \vec{X}_\phi|} \\
&= \frac{1}{\sqrt{EG-F^2}} \cdot \begin{vmatrix} -asin\theta cos\phi & -asin\theta sin\phi & acos\theta \\ -(r+acos\theta)sin\phi & (r+acos\theta)cos\phi\phi & 0 \\ -acos\theta cos\phi & -acos\theta sin\phi & -asin\theta \end{vmatrix} \\
&= \frac{1}{a(r+acos\theta)}\big(a^2sin^2\theta cos^2\phi(r+acos\theta) + a^2cos^2\theta sin^2\phi(r+acos\theta) \\
&\quad + a^2cos^2\theta cos^2\phi(r+acos\theta) + a^2sin^2\theta sin^2\phi(r+acos\theta)\big) \\
&= \frac{a^2(r+acos\theta)}{a(r+acos\theta)} = a.
\end{aligned}$$

Similarly we get

$$f = <\vec{N}, \vec{X}_{\theta\phi}> = <\frac{\vec{X}_\theta \times \vec{X}_\phi}{|\vec{X}_\theta \times \vec{X}_\phi|}, \vec{X}_{\theta\phi}> = \frac{(\vec{X}_\theta \times \vec{X}_\phi)\cdot\vec{X}_{\theta\phi}}{|\vec{X}_\theta \times \vec{X}_\phi|} = 0.$$

$$g = <\vec{N}, \vec{X}_{\phi\phi}> = <\frac{\vec{X}_\theta \times \vec{X}_\phi}{|\vec{X}_\theta \times \vec{X}_\phi|}, \vec{X}_{\phi\phi}> = \frac{(\vec{X}_\theta \times \vec{X}_\phi)\cdot\vec{X}_{\phi\phi}}{|\vec{X}_\theta \times \vec{X}_\phi|} = (r+acos\theta)cos\theta.$$

Hence the Gauss curvature is

$$K = \frac{eg-f^2}{EG-F^2} = \frac{a(r+acos\theta)cos\theta}{a^2(r+acos\theta)^2} = \frac{cos\theta}{a(r+acos\theta)}.$$

Since $r > a$, $r + acos\theta > 0$. Hence the sign of K is determined by the sign of $cos\theta$.

$K = 0$ iff $cos\theta = 0$ iff $\theta = \pi/2$ or $\theta = 3\pi/2$. These are either parabolic or planar point. In fact, they are parabolic point.

$K > 0$ iff $cos\theta > 0$ iff $0 < \theta < \pi/2$ or $3\pi/2 < \theta < 2\pi$. These are elliptic points.
$K < 0$ iff $cos\theta < 0$ iff $\pi/2 < \theta < 3\pi/2$. These are hyperbolic points. ∎

Next let's study principal directions and asymptotic directions under local coordinates.

Let $\vec{X}(u, v)$ be a parametrization of S. Recall that a connected regular curve $\alpha(t)$ on S is an asymptotic curve iff the tangent vectors $\alpha'(t)$ of α are parallel to asymptotic directions at the point $\alpha(t)$ iff $II_{\alpha(t)}(\alpha'(t)) = 0$.

If we write $\alpha(t) = \vec{X}(u(t), v(t))$ where $\beta(t) = (u(t), v(t))$ is a regular curve in $\mathbb{R}^2$, we have $\alpha'(t) = u'(t)\vec{X}_u + v'(t)\vec{X}_v$. Hence $\alpha(t)$ is an asymptotic curve iff

$$II_{\alpha(t)}(\alpha'(t)) = eu'(t)^2 + 2fu'(t)v'(t) + gv'(t)^2 = 0.$$

Fact 3.28. A necessary and sufficient condition for a parametriation in a neighbourhood of a hyperbolic point ($eg - f^2 < 0$) to be such that the coordinate curves of the parametrization are asymptotic curves is that $e = 0 = g$.

Proof. The coordinates curves are $\vec{X}(u_0, v)$ and $\vec{X}(u, v_0)$. These two curves correspond to the curves $u = u_0, v(t) = t$ and $u(t) = t, v = v_0$ in $\mathbb{R}^2$ respectively. If coordinates curves are asymptotic curves, then the curves $\beta(t) = (u_0, t)$ and $\gamma(t) = (t, v_0)$ should satisfy the equation

$$eu'(t)^2 + 2fu'(t)v'(t) + gv'(t)^2 = 0,$$

i.e., $g = 0$ for the curve $\beta(t) = (u_0, t)$ and $e = 0$ for the curve $\gamma(t) = (t, v_0)$.

Conversely, if $g = 0$ and $e = 0$, the equation

$$eu'(t)^2 + 2fu'(t)v'(t) + gv'(t)^2 = 0$$

becomes $fu'(t)v'(t) = 0$. Clearly the curves $\beta(t) = (u_0, t)$ and $\gamma(t) = (t, v_0)$ satisfy the equation. Hence the coordinate curves $\vec{X}(u_0, v)$ and $\vec{X}(u, v_0)$ are asymptotic curves. ∎

Example 3.29. Consider the surface $z = xy$. Let's use the following parametrization $\vec{X}(u, v) = (u, v, uv)$. We compute

$$\vec{X}_u = (1, 0, v), \qquad \vec{X}_v = (0, 1, u).$$

$$\vec{X}_{uu} = (0, 0, 0), \qquad \vec{X}_{vv} = (0, 0, 0).$$

Hence $e =< \vec{N}, \vec{X}_{uu} >= 0$ and $g =< \vec{N}, \vec{X}_{vv} >= 0$. By the Fact 3.28, we see that coordinate curve $z = x_0 y, x = x_0$ is an asymptotic curve, and the same is true for the other coordinate curve $z = xy_0, y = y_0$. ∎

Recall that a connected regular curve $\alpha(t) = \vec{X}(u(t), v(t))$ is a line of curvature iff the tangent vector $\alpha'(t)$ is parallel to one of the principal directions iff the tangent vector $\alpha'(t)$ is an eigenvector of $d\vec{N}_{\alpha(t)}$, i.e., $d\vec{N}_{\alpha(t)}(\alpha'(t)) = \lambda(t)\alpha'(t)$ for some function $\lambda(t)$. Also recall that, since $\alpha'(t) = u'(t)\vec{X}_u + v'(t)\vec{X}_v$, if we write

$$d\vec{N}_{\alpha(t)}(u'(t)\vec{X}_u + v'(t)\vec{X}_v) = c\vec{X}_u + d\vec{X}_v,$$

we get

$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = \begin{pmatrix} a_{11}u'(t) + a_{12}v'(t) \\ a_{21}u'(t) + a_{22}v'(t) \end{pmatrix}.$$

Hence we get

$$a_{11}u'(t) + a_{12}v'(t) = c = \lambda(t)u'(t),$$
$$a_{21}u'(t) + v'(t)a_{22} = d = \lambda(t)v'(t).$$

Multiply the first equation with $v'(t)$, minus the second the equation multiplied by $u'(t)$, we get

$$a_{12}v'(t)^2 + (a_{11} - a_{22})v'(t)u'(t) - a_{21}u'(t)^2 = 0. \tag{3.2}$$

Recall that

$$a_{11} = \frac{fF - eG}{EG - F^2},\ a_{21} = \frac{eF - fE}{EG - F^2},\ a_{12} = \frac{gF - fG}{EG - F^2},\ a_{22} = \frac{fF - gE}{EG - F^2}.$$

So $a_{11} - a_{22} = \dfrac{gE - eG}{EG - F^2}$. So the equation (3.2) becomes

$$(fE - eF)u'(t)^2 + (gE - eG)u'(t)v'(t) + (gF - fG)v'(t)^2 = 0,$$

which can be written as

$$\begin{vmatrix} v'(t)^2 & -u'(t)v'(t) & u'(t)^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0. \tag{3.3}$$

Recall that coordinate curves are perpendicular iff $F = 0$ and if p is not an umbilical point, then the two principal directions at p must be perpendicular.

Fact 3.30. Suppose the surface S contains no umbilical points. Let $\vec{X}(u, v)$ be a parametrization. Then coordinate curves are lines of curvature iff $F = f = 0$.

Proof. If coordinate curves are line of curvatures, coordinate curves must be perpendicular, hence $F = 0$. And $\beta(t) = (u_0, t)$ and $\gamma(t) = (t, v_0)$ should satisfy the equation (3.3) respectively, i.e., $-fG = 0$ for $\beta(t) = (u_0, t)$. Since G vanishes nowhere, hence $f = 0$.

Conversely, if $F = 0 = f$, the equation (3.3) becomes $(gE - eG)v'(t)u'(t) = 0$. Clearly $\beta(t) = (u_0, t)$ and $\gamma(t) = (t, v_0)$ satisty this equation. Hence corrdinate curves are line of curvatures. ∎

Example 3.31. Surface of revolution.

Consider a regular curve lying on the xz-plane:

$$x = f(v),\ y = 0,\ z = g(v), \qquad a < v < b,\ f(v) > 0.$$

Assume v is the arclength parameter of the curve, i.e., $f'(v)^2 + g'(v)^2 = 1$. Rotating the curve along the z-axis, we get a surface of revolution S.

Let (x, y, z) be a point P on the surface S. If we let θ be the angle between the x-axis and the vector $(x, y, 0)$ which is the projection of the point P to the xy-plane, then we get

$$x = f(v)cos\theta, \quad y = f(v)sin\theta, \quad z = g(v).$$

Hence we get a parametrization of the surface given by

$$\vec{X}(\theta, v) = (f(v)cos\theta, f(v)sin\theta, g(v)), \quad 0 < \theta < 2\pi,\ a < v < b.$$

We checked in Example 2.15 of Chapter 2 that $\vec{X}$ is a parametrization.

$$\begin{aligned}
\vec{X}_\theta &= (-f(v)sin\theta, f(v)cos\theta, 0),\\
\vec{X}_v &= (f'(v)cos\theta, f'(v)sin\theta, g'(v)),\\
\vec{X}_{\theta\theta} &= (-f(v)cos\theta, -f(v)sin\theta, 0),\\
\vec{X}_{v\theta} &= (-f'(v)sin\theta, f'(v)cos\theta, 0),\\
\vec{X}_{vv} &= (f''(v)cos\theta, f''(v)sin\theta, g''(v)).
\end{aligned}$$

$$\begin{aligned}
E &=< \vec{X}_\theta, \vec{X}_\theta >= f(v)^2 sin^2\theta + f(v)^2 cos^2\theta = f(v)^2,\\
F &=< \vec{X}_\theta, \vec{X}_v >= -f(v)f'(v)sin\theta cos\theta + f(v)f'(v)cos\theta sin\theta = 0,\\
G &=< \vec{X}_v, \vec{X}_v >= f'(v)^2 cos^2\theta + f'(v)^2 sin^2\theta + g'(v)^2 = f'(v)^2 + g'(v)^2 = 1.
\end{aligned}$$

$$\begin{aligned}
e &=< \vec{N}, \vec{X}_{\theta\theta} >=< \frac{\vec{X}_\theta \times \vec{X}_v}{\sqrt{EG-F^2}}, \vec{X}_{\theta\theta} >= \frac{\vec{X}_{\theta\theta}\cdot(\vec{X}_\theta \times \vec{X}_v)}{\sqrt{EG-F^2}}\\
&= \frac{1}{f(v)}\begin{vmatrix} -f(v)cos\theta & -f(v)sin\theta & 0\\ -f(v)sin\theta & f(v)cos\theta & 0\\ f'(v)cos\theta & f'(v)sin\theta & g'(v)\end{vmatrix} = -f(v)g'(v),\\
f &=< \vec{N}, \vec{X}_{v\theta} >=< \frac{\vec{X}_\theta \times \vec{X}_v}{\sqrt{EG-F^2}}, \vec{X}_{v\theta} >= \frac{\vec{X}_{v\theta}\cdot(\vec{X}_\theta \times \vec{X}_v)}{\sqrt{EG-F^2}}\\
&= \frac{1}{f(v)}\begin{vmatrix} -f'(v)sin\theta & f'(v)cos\theta & 0\\ -f(v)sin\theta & f(v)cos\theta & 0\\ f'(v)cos\theta & f'(v)sin\theta & g'(v)\end{vmatrix} = 0,\\
g &=< \vec{N}, \vec{X}_{vv} >=< \frac{\vec{X}_\theta \times \vec{X}_v}{\sqrt{EG-F^2}}, \vec{X}_{vv} >= \frac{\vec{X}_{\theta\theta}\cdot(\vec{X}_\theta \times \vec{X}_v)}{\sqrt{EG-F^2}}\\
&= \frac{1}{f(v)}\begin{vmatrix} f''(v)cos\theta & f''sin\theta & g''(v)\\ -f(v)sin\theta & f(v)cos\theta & 0\\ f'(v)cos\theta & f'(v)sin\theta & g'(v)\end{vmatrix} = f''(v)g'(v) - g''(v)f'(v).
\end{aligned}$$

Coordinate curves of this parametrization correspond to θ-curves and v-curves. Since $F = f = 0$, by Fact 3.30, the coordinate curves are lines of curvatures. The Gauss curvature is

$$\begin{aligned}
K &= \frac{eg - f^2}{EG - F^2} = \frac{-f(v)g'(v)(f''(v)g'(v) - g''(v)f'(v))}{f(v)^2}\\
&= \frac{-g'(v)f''(v)g'(v) + g'(v)g''(v)f'(v)}{f(v)}.
\end{aligned}$$

Since $f'(v)^2 + g'(v)^2 = 1$, differentiate the equation, we get $2f''(v)f'(v) + 2g''(v)g'(v) = 0$. Hence $f''(v)f'(v) = -g''(v)g'(v)$. Therefore Gauss curvature is

$$\begin{aligned} K &= \frac{-g'(v)f''(v)g'(v) - f''(v)f'(v)f'(v)}{f(v)} \\ &= -\frac{(g'(v)g'(v) + f'(v)f'(v))f''(v)}{f(v)} = -\frac{f''(v)}{f(v)}. \end{aligned}$$

From the above formula for the Gauss curvature K, we see that if $f''(v) < 0$, we get $K > 0$ and we have elliptic points. If $f''(v) > 0$, we get $K < 0$ and we have hyperbolic points.

If $K = 0$, then we must have $f''(v) = 0$ and $g'(v)(f''(v)g'(v) - g''(v)f'(v)) = 0$.

If both $g'(v)$ and $f''(v)g'(v) - g''(v)f'(v)$ are zeros, then $e = 0$ and $g = 0$. Adding that $f = 0$, we see that in this case the second fundamental form

$$II(a\vec{X}_u + b\vec{X}_v) = ea^2 + 2fab + gb^2$$

will be zero and hence the differential $d\vec{N}$ will be zero. Therefore we have a planar point. In this situation, because $f'(v)^2 + g'(v)^2 = 1$, we have $f'(v) \neq 0$. So we get $g'(v) = 0$ and $g''(v) = 0$.

If $g'(v) = 0$ and $f''(v)g'(v) - g''(v)f'(v) \neq 0$, we get $e = 0$, $f = 0$ and $g \neq 0$. Hence the second fundamental form $II(a\vec{X}_u + b\vec{X}_v) = gb^2$ is not zero. Therefore the differential $d\vec{N}$ isn't zero. Thus we get a parabolic point. In this siutation, we have equivalently $g'(v) = 0$ and $g''(v) \neq 0$.

If $g'(v) \neq 0$ and $f''(v)g'(v) - g''(v)f'(v) = 0$, we get $e \neq 0$, $f = 0$ and $g = 0$. Hence the second fundamental form $II(a\vec{X}_u + b\vec{X}_v) = ea^2$ is not zero. Therefore the differential $d\vec{N}$ isn't zero. Thus again we get a parabolic point. ∎

Proposition 3.32. *Let p be a point of a surface S such that the Gauss curvature $K(p) \neq 0$ and let V be a connected neighbourhood of p where K does not change sign. Then $|K(p)| = \lim_{A\to 0} \frac{A'}{A}$ where A is the area of a region $B \subset V$ containing p, A' is the area of the image of B by the Gauss map $\vec{N}: S \to S^2$, and the limit is taken through a sequence of regions B_n that converges to the point p in the sense that any ball around p contains all B_n for n sufficiently large.*

Proof. Let $\vec{X}(u, v)$ be a parametrization near p whose image contains V, R be the region in uv-plane corresponding to B, $\vec{N}^*(u, v) = \dfrac{\vec{X}_u \times \vec{X}_v}{|\vec{X}_u \times \vec{X}_v|}$ be the field of unit normal vector vectors. Assume that $\vec{X}(\vec{0}) = p$. Then since $\vec{N}^*_u = a_{11}\vec{X}_u + a_{21}\vec{X}_v$, $\vec{N}^*_v = a_{12}\vec{X}_u + a_{22}\vec{X}_v$, we have

$$\begin{aligned} &\vec{N}^*_u \times \vec{N}^*_u = (a_{11}\vec{X}_u + a_{21}\vec{X}_v) \times (a_{12}\vec{X}_u + a_{22}\vec{X}_v) \\ = \ & a_{11}a_{22}\vec{X}_u \times \vec{X}_v + a_{21}a_{12}\vec{X}_u \times \vec{X}_v = (a_{11}a_{22} - a_{21}a_{12})\vec{X}_u \times \vec{X}_v \\ = \ & K\vec{X}_u \times \vec{X}_v. \end{aligned}$$

Since $K \neq 0$, one can show that if we choose the region R sufficiently small, then $\vec{N}^*: R \to S^2$ is a parametrization. Hence

$$A = \int\int_R |\vec{X}_u \times \vec{X}_v| dudv,$$

$$A' = \int\int_R |\vec{N}_u^* \times \vec{N}_v^*| dudv = \int\int_R |K| \cdot |\vec{X}_u \times \vec{X}_v| dudv.$$

Let r be the area of the region R. Then we have

$$\begin{aligned} \lim_{A\to 0} \frac{A'}{A} &= \lim_{A\to 0} \frac{A'/r}{A/r} = \lim_{r\to 0} \frac{(1/r)\int\int_R |K||\vec{X}_u \times \vec{X}_v| dudv}{(1/r)\int\int_R |\vec{X}_u \times \vec{X}_v| dudv} \\ &= \frac{|K(p)||\vec{X}_u(\vec{0}) \times \vec{X}_v(\vec{0})|}{|\vec{X}_u(\vec{0}) \times \vec{X}_v(\vec{0})|} = |K(p)|. \end{aligned}$$

Here we used a result from analysis that $\lim_{r\to 0}(1/r)\int\int_R f(u,v)dudv = f(0,0)$. ∎

Proposition 3.33. *Let $p \in S$ be an elliptic point of a surface S. Then there exists a neighbourhood V of p in S such that all points in V belong to the same side of the tangent plane of V at p. If $p \in S$ is a hyperbolic point, then in each neighbourhood of p there exists points of S in both sides of the tangent plane of S at p.*

Proof. Let $\vec{X}(u,v)$ be a parametrization $\vec{X}: U \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$ such that $(0,0) \in U$ and $\vec{X}(0,0) = p$. Consider the following function

$$h(u,v) =< \vec{X}(u,v) - \vec{X}(0,0), \vec{N}(p) >= |\vec{X}(u,v) - \vec{X}(0,0)| \cdot |\vec{N}(p)| \cdot cos\theta$$

where θ is the angle between $\vec{X}(u,v) - \vec{X}(0,0)$ and $\vec{N}(p)$.

If the points $\vec{X}(u,v)$ are on one side of the tangent plane of S at p, then the angle θ must be either all $< \pi/2$ or all $> \pi/2$.

By the Taylor's formula, we get

$$\begin{aligned} \vec{X}(u,v) &= \vec{X}(0,0) + \vec{X}_u(0,0)u + \vec{X}_v(0,0)v \\ &+ \tfrac{1}{2}\big(\vec{X}_{uu}(0,0)u^2 + 2\vec{X}_{uv}(0,0)uv + \vec{X}_{vv}(0,0)v^2\big) + \vec{R}(u,v) \end{aligned}$$

$\vec{R}$ is the remainder satisfying the property $\lim_{(u,v)\to(0,0)} \frac{\vec{R}}{u^2+v^2} = (0,0,0)$. Since $\vec{X}_u(0,0) \perp \vec{N}(p)$ and $\vec{X}_v(0,0) \perp \vec{N}(p)$, we have

$$\begin{aligned} h(u,v) &=< \frac{1}{2}\big(\vec{X}_{uu}(0,0)u^2 + 2\vec{X}_{uv}(0,0)uv + \vec{X}_{vv}(0,0)v^2\big) + \vec{R}(u,v), \vec{N}(p) > \\ &= \frac{1}{2}\big(< \vec{X}_{uu}, \vec{N} > u^2 + 2 < \vec{X}_{uv}, \vec{N} > uv + < \vec{X}_{vv}, \vec{N} > v^2\big) + < \vec{R}(u,v), \vec{N} > \\ &= \frac{1}{2}\big(eu^2 + 2fuv + gv^2\big) + < \vec{R}(u,v), \vec{N} > \\ &= \frac{1}{2} II_p(\vec{w}) + R(u,v). \end{aligned}$$

where $\vec{w} = u\vec{X}_u + v\vec{X}_v$, $e =< \vec{X}_{uu}(0,0), \vec{N}(p) >$, $f =< \vec{X}_{uv}(0,0), \vec{N}(p) >$, $g =< \vec{X}_{vv}(0,0), \vec{N}(p) >$ and $R(u,v) =< \vec{R}(u,v), \vec{N}(p) >$.

Let p be an elliptic point of S. Since $EG - F^2 = |\vec{X}_u \times \vec{X}_v|^2 > 0$, we must have $eg - f^2 > 0$. By completing squares, we get

$$eu^2 + 2fuv + gv^2 = e(u + \frac{f}{e}v)^2 + \frac{eg - f^2}{e}v^2 = g(v + \frac{f}{g}u)^2 + \frac{eg - f^2}{g}u^2.$$

Since $eg - f^2 > 0$, we have $eg > f^2 \geq 0$. Hence either $e > 0$ and $g > 0$ or $e < 0$ and $g < 0$. Let's assume that $e > 0$ and $g > 0$. In this case, we have

$$\begin{aligned} eu^2 + 2fuv + gv^2 &= \tfrac{1}{2}\big(e(u + \tfrac{f}{e}v)^2 + \tfrac{eg-f^2}{e}v^2 + g(v + \tfrac{f}{g}u)^2 + \tfrac{eg-f^2}{g}u^2\big) \\ &\geq \tfrac{1}{2}\big(\tfrac{eg-f^2}{e}v^2 + \tfrac{eg-f^2}{g}u^2\big). \end{aligned}$$

Take $m = min\{\frac{eg - f^2}{g}, \frac{eg - f^2}{e}\} > 0$. Since $|R| =< \vec{R}, \vec{N}(p) >\leq |\vec{R}| \cdot |\vec{N}(p)| = |\vec{R}|$ and $\lim_{(u,v)\to(0,0)} \frac{\vec{R}}{u^2 + v^2} = 0$, we have $\lim_{(u,v)\to(0,0)} \frac{|\vec{R}|}{u^2 + v^2} = 0$. Take $\epsilon = m/8$. There exists $\delta > 0$ such that whenever $0 < u^2 + v^2 < \delta$, we have $\frac{|R|}{u^2 + v^2} < \epsilon$. Hence if (u, v) satisfies $0 < u^2 + v^2 < \delta$, we have

$$h(u, v) \geq \frac{m}{4}(u^2 + v^2) - \frac{m}{8}(u^2 + v^2) = \frac{m}{8}(u^2 + v^2) > 0.$$

Hence for the angle θ, $cos\theta$ is always positive, i.e., $0 < \theta < \pi/2$. Therefore all such points $\vec{X}(u, v)$ lie on the same side of the tangent plane of S at p as the normal vector $\vec{N}(p)$ if $0 < u^2 + v^2 < \delta$.

Apply the similar arguments to the case of $e < 0$ and $g < 0$, we get all points $\vec{X}(u, v)$ lie on the same side of the tangent plane of S at p as the normal vector $-\vec{N}(p)$ if $0 < u^2 + v^2 < \delta$.

For a hyperbolic point p, $eg - f^2 < 0$. We have the following cases: $e \neq 0$, or $g \neq 0$, or $e = g = 0$ and $f \neq 0$.

Let's assume that $e > 0$. Consider points $(u, 0)$. Then $h(u, 0) = \frac{e}{2}u^2 + R(u, 0)$. There exists $\delta_1 > 0$ such that whenever $|u| < \delta_1$ we have $\frac{|R(u, 0)|}{u^2} < \frac{e}{4}$. Hence

$$h(u, 0) \geq \frac{e}{2}u^2 - \frac{e}{4}u^2 = \frac{e}{4}u^2 > 0$$

whenever $0 < |u| < \delta_1$. Therefore the points $\vec{X}(u, 0)$ is on the same side of the tangent plane of S at p as the normal vector $\vec{N}(p)$ when $|u| < \delta_1$.

Consider points (u, v) such that $u + \frac{f}{e}v = 0$. $h(u, v) = \frac{eg - f^2}{2e}v^2 + R(u, v)$. There exists $\delta_2 > 0$ such that whenever $u^2 + v^2 < \delta_2$ we have $\frac{|R(u, v)|}{u^2 + v^2} < -\frac{eg - f^2}{4e(\frac{f^2}{e^2} + 1)}$. Hence

$$h(u, v) \leq \frac{eg - f^2}{2e}v^2 - \frac{eg - f^2}{4e}v^2 = \frac{eg - f^2}{4e}v^2 < 0$$

whenever $0 < u^2 + v^2 < \delta_2$ and $u + \frac{f}{e}v = 0$. Therefore the points $\vec{X}(u, v)$ is on the same side of tangent plane of S at p as the normal vector $-\vec{N}(p)$ when $0 < u^2 + v^2 < \delta_2$ and $u + \frac{f}{e}v = 0$. Therefore for $0 < u^2 + v^2 < \delta < min\{\delta_1, \delta_2\}$, there are always points $\vec{X}(u, v)$ on different sides of the tangent plane of S at p.

The similar arguments also work for the cases when $e < 0$, $g > 0$ and $g < 0$.

If $e = g = 0$, then $f \neq 0$. We can make the change of variables $u = \widetilde{u} + \widetilde{v}$ and $v = \widetilde{u} - \widetilde{v}$. Then $h(\widetilde{u}, \widetilde{v}) = 2f(\widetilde{u}^2 - \widetilde{v}^2) + \widetilde{R}(\widetilde{u}, \widetilde{v})$. Here $\frac{\widetilde{R}(\widetilde{u}, \widetilde{v})}{\widetilde{u}^2 + \widetilde{v}^2} \to 0$ as $(\widetilde{u}, \widetilde{v}) \to (0, 0)$. Then similar argument as above can be applied to get the conclusion stated in the Proposition. ∎

How about parabolic points and planar points? Do we have similar results? The answer is no by the following examples.

Example 3.34: Consider the surface $\vec{X}(u, v) = (u, v, u^3 - 3v^2u)$.

$$\vec{X}_u = (1, 0, 3u^2 - 3v^2), \quad \vec{X}_v = (0, 1, -6uv).$$

$$\vec{X}_u(0,0) = (1, 0, 0), \quad \vec{X}_v(0, 0) = (0, 1, 0), \quad \vec{N}(0, 0) = \vec{X}_u(0, 0) \times \vec{X}_v(0, 0) = (0, 0, 1).$$

$$\vec{X}_{uu} = (0, 0, 6u), \quad \vec{X}_{uv} = (0, 0, -6v), \quad \vec{X}_{vv} = (0, 0, -6u).$$

$$e =< \vec{X}_{uu}(0, 0), \vec{N}(0, 0) >= 0,\ f =< \vec{X}_{uv}(0, 0), \vec{N}(0, 0) >= 0,\ g =< \vec{X}_{vv}(0, 0), \vec{N}(0, 0) >= 0.$$

Hence the point $(0, 0, 0)$ is a planar point. Consider points $\vec{X}(u, 0) = (u, 0, u^3)$. The tangent plane of the surface at the origin is the xy-plane. For $u > 0$, it is on the same side of the tangent plane as $\vec{N}(0, 0)$. For $u < 0$, it is on the other side of the tangent plane. ∎

Example 3.35: Consider the surface of revolution obtained by rotating the curve $z = y^4$ about the z-axis. The curve can be parametrized by $\alpha(t) = (0, t, t^4)$.

$$\alpha'(t) = (0, 1, 4t^3), \qquad \alpha''(t) = (0, 0, 12t^2).$$

$$\alpha'(0) = (0, 1, 0), \qquad \alpha''(0) = (0, 0, 0).$$

Using the formula 12(b) on page 25 in the text book, we get the curvature of the curve at the origin $O = (0, 0, 0)$ is

$$\kappa = \frac{|\alpha'(0) \times \alpha''(0)|}{|\alpha'(0)|^3} = 0.$$

Hence the normal curvature along the direction $(0, 1, 0)$ is zero. Since the surface is a rotation of the curve along the z-axis, the normal curvature along any direction at the origin is zero. Hence the second fundamental form is a zero form. So $d\vec{N}_O$ is the zero map. This implies the origin is a planar point of the surface. Clearly, every point except the origin is on the same side of the tangent plane of the surface at the origin which is the xy-plane. ∎

4

Gauss Theorem and Geodesics

4.1 ISOMTRIES

Recall that the Gauss curvatures of the plane and the cylinder $x^2+y^2=1$ in $\mathbb{R}^3$ are zero eventhough the cylinder is curved. We shall explain why this is so.

Definition 4.1. *Let S and $\overline{S}$ be two regular surfaces. A diffeomorphism $\varphi\colon S \to \overline{S}$ is an isometry if for all $p \in S$ and all pairs $\vec{w}_1, \vec{w}_2 \in T_pS$ we have*

$$< \vec{w}_1, \vec{w}_2 >_p = < d\varphi_p(\vec{w}_1), d\varphi_p(\vec{w}_2) >_{\varphi(p)} .$$

The surfaces S and $\overline{S}$ are said to be isometric.

A map $\varphi\colon V \to \overline{S}$ of a neighbourhood V of $p \in S$ is a local isometry at p if there exists a neighbourhood $\overline{V}$ of $\varphi(p) \in \overline{S}$ such that $\varphi\colon V \to \overline{V}$ is an isometry.

Example 4.2. Take two orthonormal vectors $\vec{w}_1$ and $\vec{w}_2$ and a point P in $\mathbb{R}^3$.

Consider the plane S passisng through the point P and parallel to $\vec{w}_1$ and $\vec{w}_2$. Choose $\vec{X}(u,v) = \vec{OP} + u\vec{w}_1 + v\vec{w}_2$ as the parametrization.

Consider another surface $\overline{S}$ which is the cylinder $x^2+y^2=1$. Choose a parametrization $\vec{Y}(u,v) = (cosu, sinu, v)$ for the cylinder where $0 < u < 2\pi$ and $-\infty < v < \infty$.

Consider the map $\varphi\colon S \to \overline{S}$, $\varphi = \vec{Y} \circ \vec{X}^{-1}$. φ is an isometry. In fact, take two tangent vectors $\vec{w}_1$ and $\vec{w}_2$ of S at a point $p \in S$. There exist two curves $(u_1(t), v_1(t))$ and $(u_2(t), v_2(t))$ such that $\vec{X}(u_1(0), v_1(0)) = p$, $\vec{X}(u_2(0), v_2(0)) = p$ and

$$\vec{w}_1 = \frac{d}{dt}\vec{X}(u_1(t), v_1(t))|_{t=0} = u_1'(0)\vec{X}_u + v_1'(0)\vec{X}_v,$$

$$\vec{w}_2 = \frac{d}{dt}\vec{X}(u_2(t), v_2(t))|_{t=0} = u_2'(0)\vec{X}_u + v_2'(0)\vec{X}_v.$$

$$\begin{aligned}
& < \vec{w}_1, \vec{w}_2 >_p = < u_1'(0)\vec{X}_u + v_1'(0)\vec{X}_v, u_2'(0)\vec{X}_u + v_2'(0)\vec{X}_v > \\
= \; & < \vec{X}_u, \vec{X}_u > u_1'(0)u_2'(0) + < \vec{X}_u, \vec{X}_v > \big(u_1'(0)v_2'(0) + u_2'(0)v_1'(0)\big) + < \vec{X}_v, \vec{X}_v > v_1'(0)v_2'(0) \\
= \; & E u_1'(0)u_2'(0) + F(u_1'(0)v_2'(0) + u_2'(0)v_1'(0)) + G v_1'(0)v_2'(0).
\end{aligned}$$

Since $\varphi(\vec{X}) = \vec{Y}$, we have $\varphi\big(\vec{X}(u_1(t), v_1(t))\big) = \vec{Y}(u_1(t), v_1(t))$. Hence we get

$$\begin{aligned}
& d\varphi(\vec{w}_1) = \frac{d}{dt}\varphi\big(\vec{X}(u_1(t), v_1(t))\big)|_{t=0} \\
= \; & \frac{d}{dt}\vec{Y}(u_1(t), v_1(t))|_{t=0} = u_1'(0)\vec{Y}_u + v_1'(0)\vec{Y}_v.
\end{aligned}$$

Similarly we get $d\varphi(\vec{w}_2) = u_2'(0)\vec{Y}_u + v_2'(0)\vec{Y}_v$. Hence we get

$$\begin{aligned}
& < d\varphi(\vec{w}_1), d\varphi(\vec{w}_2) > = < u_1'(0)\vec{Y}_u + v_1'(0)\vec{Y}_v, u_2'(0)\vec{Y}_u + v_2'(0)\vec{Y}_v > \\
= \; & < \vec{Y}_u, \vec{Y}_u > u_1'(0)u_2'(0) + < \vec{Y}_u, \vec{Y}_v > \big(u_1'(0)v_2'(0) + u_2'(0)v_1'(0)\big) + < \vec{Y}_v, \vec{Y}_v > v_1'(0)v_2'(0) \\
= \; & \overline{E} u_1'(0)u_2'(0) + \overline{F}(u_1'(0)v_2'(0) + u_2'(0)v_1'(0)) + \overline{G} v_1'(0)v_2'(0).
\end{aligned}$$

We computed in Example 2.17 and Example 2.18 of Chapter 2 that $E = \overline{E}$, $F = \overline{F}$ and $G = \overline{G}$. Hence

$$< d\varphi(\vec{w}_1), d\varphi(\vec{w}_2) > = < \vec{w}_1, \vec{w}_2 > .$$

Therefore φ is an isometry, and the plane and the cylinder are locally isometric. ■

If φ is an isometry between S and $\overline{S}$, then for a tangent vector $\vec{w}$ of S at p, we have

$$I_p(\vec{w}) = < \vec{w}, \vec{w} > = < d\varphi(\vec{w}), d\varphi(\vec{w}) > = I_{\varphi(p)}(d\varphi(\vec{w})).$$

Hence $d\varphi$ keeps the first fundamental form unchanged. Conversely, if a diffeomorphism φ preserves the first fundamental form, then for any two tangent vectors $\vec{w}_1 \in T_pS$ and $\vec{w}_2 \in T_pS$, we have

$$\begin{aligned}
& 2 < \vec{w}_1, \vec{w}_2 > = I_p(\vec{w}_1 + \vec{w}_2) - I_p(\vec{w}_1) - I_p(\vec{w}_2) \\
= \; & I_p(d\varphi(\vec{w}_1 + \vec{w}_2)) - I_p(d\varphi(\vec{w}_1)) - I_p(d\varphi(\vec{w}_2)) \\
= \; & 2 < d\varphi(\vec{w}_1), d\varphi(\vec{w}_2) > .
\end{aligned}$$

Therefore φ is an isometry.

Pproposition 4.3. *Assume that there exist parametrizations $\vec{X}: U \to S$ and $\vec{Y}: U \to \overline{S}$ such that $E = \overline{E}$, $F = \overline{F}$ and $G = \overline{G}$ in U where $E = < \vec{X}_u, \vec{X}_u >$, $F = < \vec{X}_u, \vec{X}_v >$, $G = < \vec{X}_v, \vec{X}_v >$, $\overline{E} = < \vec{Y}_u, \vec{Y}_u >$, $\overline{F} = < \vec{Y}_u, \vec{Y}_v >$ and $\overline{G} = < \vec{Y}_v, \vec{Y}_v >$. Then the map $\varphi = \vec{Y} \circ \vec{X}^{-1}: \vec{X}(U) \to \overline{S}$ is a local isometry.*

Proof. Take a tangent vector $\vec{w}$ of S at a point $p \in \vec{X}(U)$. There exists a curve $(u(t), v(t))$ such that $p = \vec{X}(u(0), v(0))$ and

$$\vec{w} = \frac{d}{dt}\vec{X}(u(t), v(t))|_{t=0} = u'(0)\vec{X}_u + v'(0)\vec{X}_v.$$

$$
\begin{aligned}
& I_p(\vec{w}) = I_p(u'(0)\vec{X}_u + v'(0)\vec{X}_v) \\
= \quad & Eu'(0)^2 + 2Fu'(0)v'(0) + Gv'(0)^2.
\end{aligned}
$$

Since $\varphi(\vec{X}) = \vec{Y}$, we have $\varphi\big(\vec{X}(u(t), v(t))\big) = \vec{Y}(u(t), v(t))$. Hence we get

$$
\begin{aligned}
& d\varphi_p(\vec{w}) = \frac{d}{dt}\varphi\big(\vec{X}(u(t), v(t))\big)|_{t=0} \\
= \quad & \frac{d}{dt}\vec{Y}(u(t), v(t))|_{t=0} = u'(0)\vec{Y}_u + v'(0)\vec{Y}_v.
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
& I_{\varphi(p)}(d\varphi_p(\vec{w})) = I_{\varphi(p)}(u'(0)\vec{Y}_u + v'(0)\vec{Y}_v) \\
= \quad & \overline{E}u'(0)^2 + 2\overline{F}(u'(0)v'(0) + \overline{G}v'(0)^2.
\end{aligned}
$$

Since $E = \overline{E}$, $F = \overline{F}$ and $G = \overline{G}$, we must have $I_{\varphi(p)}(d\varphi_p(\vec{w})) = I_p(\vec{w})$. Therefore φ is a local isometry. ∎

4.2 GAUSS THEOREM

Let S be a regular surface with $\vec{X}(u, v)$ as a parametrization. We know that $\vec{X}_u$, $\vec{X}_v$ and $\vec{N}$ form a basis of $\mathbb{R}^3$. Hence we can write $\vec{X}_{uu}$ as a linear combination of $\vec{X}_u$, $\vec{X}_v$ and $\vec{N}$. The same is true for $\vec{X}_{uv}$, $\vec{X}_{vu}$ and $\vec{X}_{vv}$. Hence we can write

$$
\vec{X}_{uu} = \Gamma^1_{11}\vec{X}_u + \Gamma^2_{11}\vec{X}_v + L_1\vec{N}, \quad \vec{X}_{uv} = \Gamma^1_{12}\vec{X}_u + \Gamma^2_{12}\vec{X}_v + L_2\vec{N},
$$

$$
\vec{X}_{vu} = \Gamma^1_{21}\vec{X}_u + \Gamma^2_{21}\vec{X}_v + \overline{L}_2\vec{N}, \quad \vec{X}_{vv} = \Gamma^1_{22}\vec{X}_u + \Gamma^2_{22}\vec{X}_v + L_3\vec{N}.
$$

One can see that $L_1 = \langle \vec{X}_{uu}, \vec{N}\rangle = e$ and $L_2 = \langle \vec{X}_{uv}, \vec{N}\rangle = f$

The coefficients Γ^k_{ij} are called Christoffel symbols. We can check easily that $\Gamma^1_{12} = \Gamma^1_{21}$ and $\Gamma^2_{12} = \Gamma^2_{21}$.

Take the inner product of the four equations above with $\vec{X}_u$ and $\vec{X}_v$ we get

$$
\left\{ \begin{array}{l}
\Gamma^1_{11}E + \Gamma^2_{11}F =< \vec{X}_{uu}, \vec{X}_u >= \frac{1}{2}E_u, \\
\Gamma^1_{11}F + \Gamma^2_{11}G =< \vec{X}_{uu}, \vec{X}_v >= F_u - \frac{1}{2}E_v.
\end{array} \right.
$$

$$
\left\{ \begin{array}{l}
\Gamma^1_{12}E + \Gamma^2_{12}F =< \vec{X}_{uv}, \vec{X}_u >= \frac{1}{2}E_v, \\
\Gamma^1_{12}F + \Gamma^2_{12}G =< \vec{X}_{uv}, \vec{X}_v >= \frac{1}{2}G_u.
\end{array} \right.
$$

$$
\left\{ \begin{array}{l}
\Gamma^1_{22}E + \Gamma^2_{22}F =< \vec{X}_{vv}, \vec{X}_u >= F_v - \frac{1}{2}G_u, \\
\Gamma^1_{22}F + \Gamma^2_{22}G =< \vec{X}_{vv}, \vec{X}_v >= \frac{1}{2}G_v.
\end{array} \right.
$$

Since $EG - F^2 \neq 0$, by the results in linear algebra, we can solve each of the three systems of linear equations in variables Γ_{ij}^k above. The conclusion is that the Christoffel symbols depend on E, F and G (and their derivatives) only.

Start with the identity $(\vec{X}_{uu})_v = (\vec{X}_{uv})_u$, we get

$$(\Gamma_{11}^1 \vec{X}_u + \Gamma_{11}^2 \vec{X}_v + e\vec{N})_v = (\Gamma_{12}^1 \vec{X}_u + \Gamma_{12}^2 \vec{X}_v + f\vec{N})_u,$$

$$\begin{aligned} & \Gamma_{11}^1 \vec{X}_{uv} + (\Gamma_{11}^1)_v \vec{X}_u + \Gamma_{11}^2 \vec{X}_{vv} + (\Gamma_{11}^2)_v \vec{X}_v + e\vec{N}_v + e_v\vec{N} \\ = \; & \Gamma_{12}^1 \vec{X}_{uu} + (\Gamma_{12}^1)_u \vec{X}_u + \Gamma_{12}^2 \vec{X}_{vu} + (\Gamma_{12}^2)_u \vec{X}_v + f\vec{N}_u + f_u\vec{N}. \end{aligned}$$

Now we write $\vec{X}_{uv}$, $\vec{X}_{vv}$, $\vec{X}_{uu}$, $\vec{X}_{vu}$, $\vec{N}_v$ and $\vec{N}_u$ as linear combinations of $\vec{X}_u$, $\vec{X}_v$ and $\vec{N}$ and pick out the coefficients of the vector $\vec{X}_v$, we get

$$\Gamma_{11}^1 \Gamma_{12}^2 + \Gamma_{11}^2 \Gamma_{22}^2 + e a_{22} + (\Gamma_{11}^2)_v = \Gamma_{12}^1 \Gamma_{11}^2 + \Gamma_{12}^2 \Gamma_{21}^2 + f a_{21} + (\Gamma_{12}^2)_u.$$

Recall that $(EG - F^2)a_{22} = fF - gE$ and $(EG - F^2)a_{21} = eF - fE$. Simplify the equation above and we get the following formula

$$-EK = (\Gamma_{12}^2)_u - (\Gamma_{11}^2)_v + \Gamma_{12}^1 \Gamma_{11}^2 + \Gamma_{12}^2 \Gamma_{12}^2 - \Gamma_{11}^2 \Gamma_{22}^2 - \Gamma_{11}^1 \Gamma_{12}^2$$

where K is the Gauss curvature.

This is a very complicated formula for the Gauss curvature K. The other formula $K = \dfrac{eg - f^2}{EG - F^2}$ is much simpler. However the complicated formula above does say something extremely significant, that is, that the Gauss curvature K depends only on E, F and G since Christoffel symbols depend only on E, F and G. Therefore we can assert that K is determined by the first fundamental form. Hence if two surfaces are locally isometric, the Gauss curvatures of these two surfaces will be the same. This explains why the plane and cylinder in the Example 4.2 have Gauss curvatures equal.

From the mostly used formula $K = \dfrac{eg - f^2}{EG - F^2}$, a priori one tends to draw the conclusion that the Gauss curvature depends on the first and the second fundamental forms. The most remarkable application of the complicated formula above is that it says that K is independent of the second fundamental form.

Gauss's Thoerema Egregium *The Gauss curvature K of a surface is invariant by local isometris.*

From the identity

$$(\vec{X}_{uu})v = (\vec{X}_{uv})u,$$

we can also consider coefficients of $\vec{X}_u$ and $\vec{N}$. Write out all the terms:

$$\begin{aligned} & \Gamma_{11v}^1 \vec{X}_u + \Gamma_{11}^1 \vec{X}_{uv} + \Gamma_{11v}^2 \vec{X}_v + \Gamma_{11}^2 \vec{X}_{vv} + L_{1v}\vec{N} + L_1\vec{N}_v \\ = \; & \Gamma_{12u}^1 \vec{X}_u + \Gamma_{12}^1 \vec{X}_{uu} + \Gamma_{12u}^2 \vec{X}_v + \Gamma_{12}^2 \vec{X}_{vu} + L_{2u}\vec{N} + L_2\vec{N}_u. \end{aligned}$$

$$\begin{aligned} & \Gamma_{11v}^1 \vec{X}_u + \Gamma_{11}^1(\Gamma_{12}^1 \vec{X}_u + \Gamma_{12}^2 \vec{X}_v + L_2\vec{N}) + \Gamma_{11v}^2 \vec{X}_v + \Gamma_{11}^2(\Gamma_{22}^1 \vec{X}_u + \Gamma_{22}^2 \vec{X}_v + L_3\vec{N}) + L_{1v}\vec{N} + L_1(a_{12}\vec{X}_u + a_{22}\vec{X}_v) \\ = \; & \Gamma_{12u}^1 \vec{X}_u + \Gamma_{12}^1(\Gamma_{11}^1 \vec{X}_u + \Gamma_{11}^2 \vec{X}_v + L_1\vec{N}) + \Gamma_{12u}^2 \vec{X}_v + \Gamma_{12}^2(\Gamma_{21}^1 \vec{X}_u + \Gamma_{21}^2 \vec{X}_v + L_2\vec{N}) + L_{2u}\vec{N} + L_2(a_{11}\vec{X}_u + a_{21}\vec{X}_v). \end{aligned}$$

For the coefficients of $\vec{X}_u$, we obtain

$$\Gamma^1_{11v} + \Gamma^1_{11}\Gamma^1_{12} + \Gamma^2_{11}\Gamma^1_{22} + L_1 a_{12} = \Gamma^1_{12u} + \Gamma^1_{12}\Gamma^1_{11} + \Gamma^2_{12}\Gamma^1_{21} + L_2 a_{11}.$$

Recall that $L_1 = e, L_2 = f$, $a_{11} = \dfrac{fF - eG}{EG - F^2}$, $a_{12} = \dfrac{gF - fG}{EG - F^2}$. We obtain

$$\Gamma^1_{11v} + \Gamma^2_{11}\Gamma^1_{22} - \Gamma^1_{12u} - \Gamma^2_{12}\Gamma^1_{21} = f a_{11} - e a_{12} = -KF,$$

which is another formula expressing the Gaussian curvature K in terms of Γ^k_{ij} and their derivatives.

The coefficients of $\vec{N}$:

$$\Gamma^1_{11} f + \Gamma^2_{11} g + e_v = \Gamma^1_{12} e + \Gamma^2_{12} f + f_u.$$

This is a new identity, one of the Mainardi-Codazzi equations.

The identity $(\vec{X}_{vv})_u = (\vec{X}_{vu})_v$ also gives three identities. Write out all the terms, we obtain

$$(\Gamma^1_{22}\vec{X}_u + \Gamma^2_{22}\vec{X}_v + L_3\vec{N})_u = (\Gamma^1_{21}\vec{X}_u + \Gamma^2_{21}\vec{X}_v + Ł_2\vec{N})_v.$$

$$\begin{aligned} &\Gamma^1_{22u}\vec{X}_u + \Gamma^1_{22}\vec{X}_{uu} + \Gamma^2_{22u}\vec{X}_v + \Gamma^2_{22}\vec{X}_{vu} + L_{3u}\vec{N} + L_3\vec{N}_u \\ &= \Gamma^1_{21v}\vec{X}_u + \Gamma^1_{21}\vec{X}_{uv} + \Gamma^2_{21v}\vec{X}_v + \Gamma^2_{21}\vec{X}_{vv} + L_{2v}\vec{N} + L_2\vec{N}_v. \end{aligned}$$

$$\begin{aligned} & \Gamma^1_{22u}\vec{X}_u + \Gamma^1_{22}(\Gamma^1_{11}\vec{X}_u + \Gamma^2_{11}\vec{X}_v + L_1\vec{N}) + \Gamma^2_{22u}\vec{X}_v + \Gamma^2_{22}(\Gamma^1_{21}\vec{X}_u + \Gamma^2_{21}\vec{X}_v + L_2\vec{N}) \\ & \qquad + L_{3u}\vec{N} + L_3(a_{11}\vec{X}_u + a_{21}\vec{X}_v) \\ = \; & \Gamma^1_{21v}\vec{X}_u + \Gamma^1_{21}(\Gamma^1_{21}\vec{X}_u + \Gamma^2_{21}\vec{X}_v + L_2\vec{N}) + \Gamma^2_{21v}\vec{X}_v + \Gamma^2_{21}(\Gamma^1_{22}\vec{X}_u + \Gamma^2_{22}\vec{X}_v + L_3\vec{N}) \\ & \qquad + L_{2v}\vec{N} + L_2(a_{12}\vec{X}_u + a_{22}\vec{X}_v). \end{aligned}$$

The coefficients of $\vec{X}_u$ is:

$$\Gamma^1_{22}\Gamma^1_{11} + \Gamma^1_{22u} + \Gamma^2_{22}\Gamma^1_{21} + L_3 a_{11} = \Gamma^1_{21}\Gamma^1_{21} + \Gamma^1_{21v} + \Gamma^2_{21}\Gamma^1_{22} + L_2 a_{12}.$$

$$\Gamma^1_{22}\Gamma^1_{11} + \Gamma^1_{22u} + \Gamma^2_{22}\Gamma_{21} - \Gamma^1_{21}\Gamma^1_{21} - \Gamma^2_{21}\Gamma_{22} - \Gamma^1_{21v} = f a_{11} - g a_{12} = f\frac{gF - fG}{EG - F^2} - g\frac{fF - eG}{EG - F^2} = -GK,$$

which is another formula for Gaussian curvature.

The coefficients of $\vec{X}_v$ is:

$$\Gamma^1_{22}\Gamma^2_{11} + \Gamma^2_{22u} + \Gamma^2_{22}\Gamma^2_{21} + L_3 a_{21} = \Gamma^1_{21}\Gamma^2_{21} + \Gamma^2_{21v} + \Gamma^2_{21}\Gamma^2_{22} + L_2 a_{22}.$$

$$\Gamma_{22}^1\Gamma_{11}^2 + \Gamma_{22u}^2 - \Gamma_{21}^1\Gamma_{21}^2 - \Gamma_{21v}^2 = L_2a_{22} - L_3a_{21} = f\frac{fF-gE}{EG-F^2} - g\frac{eF-fE}{EG-F^2} = -FK,$$

which is another formula for Gaussian curvature.

The coefficients of $\vec{N}$ is

$$g_u + \Gamma_{22}^1 e + \Gamma_{22}^2 f = \Gamma_{21}^1 f + \Gamma_{21}^2 g + f_v,$$

which is another equation of Mainardi-Codazzi equations.

One can consider the identity $\vec{N}_{uv} = \vec{N}_{vu}$. We obtain

$$(a_{11}\vec{X}_u + a_{21}\vec{X}_v)_v = (a_{12}\vec{X}_u + a_{22}\vec{X}_v)_u.$$

$$\begin{aligned} & a_{11v}\vec{X}_u + a_{11}(\Gamma_{12}^1\vec{X}_u + \Gamma_{12}^2\vec{X}_v + L_2\vec{N}) + a_{21v}\vec{X}_v + a_{21}(\Gamma_{22}^1\vec{X}_u + \Gamma_{22}^2\vec{X}_v + L_3\vec{N}) \\ = \; & a_{12u}\vec{X}_u + a_{12}(\Gamma_{11}^1\vec{X}_u + \Gamma_{11}^2\vec{X}_v + L_1\vec{N}) + a_{22u}\vec{X}_v + a_{22}(\Gamma_{21}^1\vec{X}_u + \Gamma_{21}^2\vec{X}_v + L_2\vec{N}). \end{aligned}$$

The coefficients of $\vec{N}$ gives:

$$a_{11}f + a_{21}g - a_{12}e - a_{22}f = \frac{fF-eG}{EG-F^2}f + \frac{eF-fE}{EG-F^2}g - \frac{gF-fG}{EG-F^2}e - \frac{fF-gE}{EG-F^2}f = 0,$$

which is a trivial identity.

The coefficients of $\vec{X}_u$ and $\vec{X}_v$ will give two identities which can be derived from Mainardi-Codazzi equations.

Gauss identity and Mainardi-Codazzi equations are all the compatibility conditions by the following theorem due to a French mathematician Bonnet. Recall that the Christoffel symbols can be expressed in terms of E, F, G and their derivatives. Thus the Gauss formula and Mainardi-Codazzi equations involves E, F, G, e, f, g and their derivatives only.

Theorem (Bonnet) Let E, F, G, e, f, g be differentiable functions, defined in an open set $V \subset \mathbb{R}^2$, with $E > 0$ and $G > 0$. Assume that the given functions satisfy formally the Gauss and Mainardi-Codazzi equations and that $EG - F^2 > 0$. Then, for every $q \in V$ there exists a neighbourhood $U \subset V$ of q and a diffeomorphism $\vec{X} \colon \vec{X}(U) \subset \mathbb{R}^3$ such that the regular surface $\vec{X}(U)$ has E, F, G and e, f, g as coefficients of the first and the second fundamental forms respectively under the parametrization $\vec{X}$.

Furthermore, if U is connected and if $\vec{Y} \colon U \to \vec{Y}(U) \subset \mathbb{R}^3$ is another diffeomorphism satisfying the same conditions, then there exist a translation T and a proper linear orthogonal transformation ρ on $\mathbb{R}^3$ such that $\vec{Y} = T \circ \rho \circ \vec{X}$.

4.3 VECTOR FIELDS

Definition 4.4. *A vector field $\vec{w}$ on an open subset $U \subset \mathbb{R}^2$ is a map from U to $\mathbb{R}^2$,*

$$\vec{w}(x, y) = \big(f(x, y), g(x, y)\big), \quad (x, y) \in \mathbb{R}^2.$$

The vector field $\vec{w}$ is said to be differentiable if both $f(x, y)$ and $g(x, y)$ are differentiable functions on U.

Definition 4.5. *A vector field $\vec{w}$ on an open set U of a regular surface S is a map from U to $\mathbb{R}^3$ such that $\vec{w}(p)$ is a tangent vector of S at p, i.e., $\vec{w}(p) \in T_p(S)$.*

Given a parameterization $\vec{X}(u, v)$ of S near p, we can write

$$\vec{w} \circ \vec{X}(u, v) = f(u, v)\vec{X}_u(u, v) + g(u, v)\vec{X}_v(u, v).$$

If $f(u, v)$ and $g(u, v)$ are differentiable, we say $\vec{w}$ is *differentiable.*

Example 4.6. Consider the unit sphere. Choose spherical coordinates to parametrize the sphere

$$\vec{X}(\theta, \varphi) = (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi), \quad 0 < \varphi < \pi.$$

$\vec{X}_\varphi$ is a vector field on the open set of the sphere with both north and south poles deleted. It cannot be extended to a vector field on the whole sphere. However the vector field $\vec{w} = \sin\varphi \vec{X}_\varphi$ can be extended to the whole sphere by requiring $\vec{w}$ to be zero at north and south poles.

To see that $\vec{w}$ is differentiable, we choose another parameterization $\vec{Y}(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$ near the north point $(0, 0, 1)$. Since $u = \cos\theta \sin\varphi$ and $v = \sin\theta \sin\varphi$, we have

$$\sin\varphi = \sqrt{u^2 + v^2}, \qquad \cos\theta = \frac{u}{\sqrt{u^2 + v^2}}.$$

Therefore the vector field $\vec{w}$, under the parameterization $\vec{Y}$, takes the form

$$\vec{w} = (u\sqrt{1 - u^2 - v^2}, v\sqrt{1 - u^2 - v^2}, -(u^2 + v^2)).$$

$$\vec{Y}_u = \big(1, 0, -\frac{u}{\sqrt{1 - u^2 - v^2}}\big), \quad \vec{Y}_v = \big(0, 1, \frac{-v}{\sqrt{1 - u^2 - v^2}}\big).$$

$$\vec{w} = u\sqrt{1 - u^2 - v^2}\vec{Y}_u + v\sqrt{1 - u^2 - v^2}\vec{Y}_v.$$

The functions $u\sqrt{1 - u^2 - v^2}$ and $v\sqrt{1 - u^2 - v^2}$ are differentiable in a neighbourhood of $(0, 0)$. Thus $\vec{w}$ is differentiable near the north pole. Using the same method, we can also get the same conclusion for the south pole. Therefore. $\vec{w}$ is a differentiable vector field over the whole sphere. ■

A vector field $\vec{w}(x, y) = (f(x, y), g(x, y))$ defines a system of linear differential equations

$$\frac{dx}{dt} = f(x, y), \qquad \frac{dy}{dt} = g(x, y).$$

The solution $\alpha(t) = (x(t), y(t))$ of the system is also called a *trajectory* of $\vec{w}$.

For example, if the vector field $\vec{w}(x, y) = (x, y)$, then a trajectory of $\vec{w}$ is $\alpha(t) = (ae^t, be^t)$.

From the existence and uniqueness theorem and continuity theorem in the theory of ordinary differential equations, we have the following theorems.

Theorem 4.7. *Given a differentiable vector field $\vec{w}$ in an open set U in $\mathbb{R}^2$ and a point $p \in U$, there exists a trajectory α: $0 \in (a, b) \to U$ with $\alpha(0) = p$. The trajectory is unique in the sense that for any other trajectory β: $0 \in (c, d) \to U$ with $\beta(0) = p$, we have $\alpha = \beta$ over the interval $(a, b) \cap (c, d)$.*

Theorem 4.8. *With the assumptions as above. There exists a neighbourhood $V \subset U$ of p, an interval I, and a map F: $V \times I \to U$ such that F is differentiable and for any fixed point $q \in V$, the curve $\alpha(t) = F(q, t)$ for $t \in I$ is the trajectory of $\vec{w}$ passing through q, i.e.,*

$$F(q, 0) = q, \quad \frac{\partial F}{\partial t}(q, t) = \vec{w}(F(q, t)).$$

The map F above is called the (local) *flow* of $\vec{w}$ at p.

Lemma 4.9. *Let $\vec{w}$ be a differentiable vector field on an open set $U \subset \mathbb{R}^2$ with $\vec{w}(p) \neq \vec{0}$ where $p \in U$. Then there exists a neighbourhood $W \subset U$ of p and a differentiable function g over W with $dg_q \neq \vec{0}$ for all $q \in W$ such that g is constant on each trajectory of $\vec{w}$.*

Such function g is called the (local) first integral of $\vec{w}$ in a neighbourhood of p.

Proof. Choose a coordinate system of $\mathbb{R}^2$ such that $p = (0,0)$ and $\vec{w}(p)$ is parallel to the x-axis, i.e., $\vec{w}(p) = (a, 0)$ for some nonzero number a. Let $F: V \times I \to U$ be the local flow from the Theorem 4.8. Let $Z = \{(0, y, t) \in V \times I\}$. Consider the restriction f of F to the subset Z. f is a map from the two dimensional space Z to the two dimensional space U. Let's calculate the differential df of f at the point $r = (0,0,0)$. Firstly,

$$\begin{aligned} df_r((0,0,1)) &= df_r(\frac{d}{dt}(0,0,t)|_{t=0}) = \frac{d}{dt}f((0,0,t))|_{t=0} \\ &= \frac{d}{dt}F((0,0,t))|_{t=0} = \vec{w}(F(0,0,t))|_{t=0} = \vec{w}(F(0,0,0)) = \vec{w}(p). \end{aligned}$$

$$df_r((0,1,0)) = df_r(\frac{d}{dy}(0,y,0)) = \frac{d}{dy}f(0,y,0)|_{y=0} = \frac{d}{dy}F(0,y,0)|_{y=0} = \frac{d}{dy}(0,y) = (0,1).$$

Since $\vec{w}(p)$ and $(0,1)$ are linearly independent, the linear transformation df_r is invertible. By the inverse function theorem, the map f has an inverse in an open subset $W \subset U$ containing p. The inverse f^{-1} is also differentiable and maps W to Z. Write $f^{-1}(x,y) = (0, g(x,y), h(x,y))$. The function $g(x,y)$ is differentiable. For a trajectory $\alpha(t)$ with $\alpha(0) = (0,c)$, we have $F((0,c),t) = \alpha(t)$. Therefore $f(0,c,t) = \alpha(t)$. Equivalently $f^{-1}(\alpha(t)) = (0,c,t)$. Thus $g(\alpha(t)) = c$. In summary, the function g is a constant on the trajectory $\alpha(t)$. Since df_r is invertible, df_r^{-1} is also invertible. Thus $dg_p \neq \vec{0}$. We can choose W sufficiently small so that $dg_q \neq \vec{0}$ for any $q \in W$. ∎

For example, if the vector field $\vec{w} = (-y, x)$ on $\mathbb{R}^2$, then a first integral is $f(x,y) = x^2 + y^2$. In fact, $x(t) = a\cos t$ and $y(t) = a \sin t$ is a trajectory of the vector field, and

$$f(x(t), y(t)) = (a\cos t)^2 + (a \sin t)^2 = a^2.$$

Theorem 4.10. *Given a regular surface S and a point $p \in S$. There exists a parametrization $\vec{X}(u,v)$ in a neighbourhood V of p such that the coordinate curves $u = constant$ and $v = constant$ intersect orthogonally. Such an $\vec{X}$ is called an orthogonal parametrization.*

Proof. Take an arbitrary parametrization $\vec{Y}(r,s): U \to S$ at p. We can assume $\vec{Y}(0,0) = p$. Let $E =< \vec{Y}_r, \vec{Y}_r >$ and $F =< \vec{Y}_r, \vec{Y}_s >$. Consider the vector fields

$$\vec{w}_1 = \vec{Y}_r, \qquad \vec{w}_2 = -\frac{F}{E}\vec{Y}_r + \vec{Y}_s.$$

Since $< \vec{w}_1, \vec{w}_2 >= 0$, w_1 and w_2 are orthogonal. Recall that $d\vec{Y}$, considered as a map from $\mathbb{R}^2$ to T_qS for $q \in \vec{Y}(U)$, is invertible. Define $\vec{v}_1 = d\vec{Y}^{-1}(\vec{w}_1)$ and $\vec{v}_2 = d\vec{Y}^{-1}(\vec{w}_2)$. $\vec{v}_1$ and $\vec{v}_2$ are vector fields on U and linearly independent at every point of U. From Lemma 4.9, there exists a neighbourhood

W of the origin and the first integrals $f_1(s,t)$ and $f_2(s,t)$, defined on W, of $\vec{v}_1$ and $\vec{v}_2$ respectively with $df_{1q} \neq 0$ and $df_{2q} \neq 0$ for $q \in W$. If $\alpha(t)$ is a trajectory of $\vec{v}_1$, then $f_1(\alpha(t)) = const$. Therefore,

$$df_1(\vec{v}_1(\alpha(t))) = df_1(\alpha'(t)) = \frac{d}{dt} f_1(\alpha(t)) = 0.$$

Therefore $df_1(\vec{v}_1(q)) = 0$ for all $q \in W$. Similarly $df_2(\vec{v}_2) = 0$ for all $q \in W$. Since $df_{1q} \neq 0$, $df_1(\vec{v}_2) \neq 0$. Similarly $df_2(\vec{v}_2) \neq 0$.

Define a map $\varphi(s,t) = (f_1(s,t), f_2(s,t))$. From the discussions above, we can write $d\varphi(\vec{v}_1) = (df_1(\vec{v}_1), df_2(\vec{v}_1)) = (0,a)$ with $a \neq 0$ and $d\varphi(\vec{v}_2) = (df_1(\vec{v}_2), df_2(\vec{v}_2)) = (b,0)$ with $b \neq 0$. Now we see that the linear transformation $d\varphi$ has rank two. Thus $d\varphi$ is invertible. By the inverse function theorem, φ is invertible.

Consider $\vec{X}(u,v) = \vec{Y} \circ \varphi^{-1}(u,v)$. Let $V = \vec{X}(\varphi(W))$. $d\vec{X} = d\vec{Y} \circ d\varphi^{-1}$ has rank two. Therefore $\vec{X}$ is a parametrization. We knew that $d\varphi(\vec{v}_1)$ is parallel to $(0,1)$. Thus $d\varphi^{-1}(0,1) = d\varphi^{-1}(\frac{\partial}{\partial v}(u,v))$ is parallel to $\vec{v}_1$, i.e. $d\varphi^{-1}(0,1) = c\vec{v}_1$. Therefore

$$\begin{aligned} \vec{X}_v &= \frac{\partial}{\partial v}(\vec{Y} \circ \varphi^{-1}(u,v)) = d(\vec{Y} \circ \varphi^{-1})(\frac{\partial}{\partial v}(u,v)) \\ &= d\vec{Y} \circ d\varphi^{-1}(0,1) = d\vec{Y}(c\vec{v}_1) = c\vec{w}_1. \end{aligned}$$

Similarly $\vec{X}_u = h\vec{w}_2$. Therefore $< \vec{X}_u, \vec{X}_v >= ch < \vec{w}_1, \vec{w}_2 >= 0$. Thus $\vec{X}$ is an orthogonal parametrization. ∎

The proof of the theorem in fact proves a more general theorem as follows.

Theorem 4.11. *Given a regular surface $S \subset \mathbb{R}^3$. Let $\vec{w}_1$ and $\vec{w}_2$ be two vector fields in an open subset $U \subset S$ which are linearly independent at some point $p \in U$. Then it is possible to parametrize a neighbourhood $V \subset U$ of p such that for each $q \in V$ the coordinate curves of this parametrization passing through q are tangent to $\vec{w}_1(q)$ and $\vec{w}_2(q)$ respectively.*

Corollary 4.12. *Let $p \in S$ be a hyperbolic point of a regular surface S. Then it is possible to parametrize a neighbourhood of p such that the coordinate curves of this parametrization are the asymptotic curves of S.*

Proof. Choose a parametrization $\vec{X}(u,v){:}\, U \to S$ of the surface near p such that the Gaussian curvature is negative in U. Let $\alpha(t) = (u(t), v(t))$ be an asymptotic curve. Then $\alpha(t)$ satisfies the equation

$$e(u')^2 + 2fu'v' + g(v')^2 = 0.$$

If both e and g are identically zero, then the coordinate curves are asymptotic curves. So our parametrization satisfies the requirement.

If one of e and g nonzero, without loss of generality, assume $e \neq 0$. Also assume $e(p) > 0$, and the proof for $e(p) < 0$ is similar. Then choose an open neighbourhood $V \subset U$ such that $e > 0$ in V. Then the equation of the asymptotic curves can be factorized as

$$\left(\sqrt{e}u' + \left(\frac{f}{\sqrt{e}} + \sqrt{\frac{-eg+f^2}{e}}\right)v'\right)\left(\sqrt{e}u' + \left(\frac{f}{\sqrt{e}} - \sqrt{\frac{-eg+f^2}{e}}\right)v'\right) = 0.$$

For simplicity, let $A=\sqrt{e}$, $B=\left(\frac{f}{\sqrt{e}}+\sqrt{\frac{-eg+f^2}{e}}\right)$, and $D=\left(\frac{f}{\sqrt{e}}-\sqrt{\frac{-eg+f^2}{e}}\right)$. Consider two vector fields

$$\vec{w}_1=(B,-A),\quad \vec{w}_2=(D,-A).$$

Let $f_1(u,v)$ and $f_2(u,v)$ be the first integrals of $\vec{w}_1$ and $\vec{w}_2$ respectively. Therefore, if $\alpha_i(t)$ is a trajectory of $\vec{w}_i$, then

$$\alpha_i'(t)=\vec{w}_i(t),\quad \text{and } f_i(\alpha_i(t))=\text{constant}.$$

Clearly, $\alpha_i(t)$ is an asymptotic curve since it satisfies the differential equation of the asymptotic curves. Taking derivatives, we obtain

$$f_{iu}u'+f_{iv}v'=0.$$

Hence (f_{1u},f_{1v}) is parallel to (A,B), and (f_{2u},f_{2v}) is parallel to (A,D). Since $B\neq D$, we have (f_{1u},f_{1v}) and (f_{2u},f_{2v}) are linearly independent. Therefore, the transformation

$$\tilde{u}=f_1(u,v),\quad \tilde{v}=f_2(u,v)$$

is a change of variables. Taking inverse, we obtain a change of variables

$$u=g_1(\tilde{u},\tilde{v}),\quad v=g_2(\tilde{u},\tilde{v}).$$

Let $\vec{Y}(\tilde{u},\tilde{v})=\vec{X}(g_1(\tilde{u},\tilde{v}),g_2(\tilde{u},\tilde{v}))$. Then the asymptotic curve $\alpha_1(t)=(u(t),v(t))$ satisfies the equation

$$\tilde{u}'=f_{1u}u'+f_{1v}v'=0,\quad \text{hence } \tilde{u}=\text{constant}.$$

Same holds for $\alpha_2(t)$. Hence the coordinate curves of $\vec{Y}$ are asymptotic curves.

■

Example 4.13. Consider the hyperbolic paraboloid $z=x^2-y^2$. We choose the natural parametrization

$$\vec{X}(u,v)=(u,v,u^2-v^2).$$

We can calculate to obtain

$$e=\frac{2}{\sqrt{1+4u^2+4v^2}},\quad f=0,\quad g=-\frac{2}{\sqrt{1+4u^2+4v^2}}.$$

Therefore, the equation of asymptotic curves is

$$\frac{2}{\sqrt{1+4u^2+4v^2}}\big((u')^2-(v')^2\big)=0.$$

Factorize, we obtain

$$u'+v'=0,\quad u'-v'=0.$$

The two vectors fields $\vec{w}_1(u,v)=(1,-1)$ and $\vec{w}_2(u,v)=(-1,-1)$. The corresponding first integrals are $f_1(u,v)=u+v$ and $f_2(u,v)=u-v$ respectively. Choose the change of variables:

$$\tilde{u}=u+v,\quad \tilde{v}=u-v,\quad \text{equivalently } u=\frac{\tilde{u}+\tilde{v}}{2},\quad v=\frac{\tilde{u}-\tilde{v}}{2}.$$

The equation for asymptotic curves are

$$\tilde{u}' \cdot \tilde{v}' = 0.$$

Hence the new parametrization is

$$\vec{Y}(\tilde{u}, \tilde{v}) = \left(\frac{\tilde{u} + \tilde{v}}{2}, \frac{\tilde{u} - \tilde{v}}{2}, \tilde{u}\tilde{v} \right).$$

The coordinate curves for this parametrization are asymptotic curves.

Using the same idea of proof, we obtain the following corollary.

Corollary 4.14. *Let $p \in S$ be a non-umbilical point of a regular surface S. Then it is possible to parametrize a neighbourhood of p such that the coordinate curves of this parametrization are the lines of curvatures of S.*

Remark: Given a function $f(x, y)$ on an open set U of $\mathbb{R}^2$, the differential $df = f_x dx + f_y dy$ as defined in Calculus. If we regard f as a map from U to $\mathbb{R}$, df has a better interpretation in that it is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}$ as follows. Given a vector $\vec{v} \in \mathbb{R}^2$. Regard $\vec{v}$ as a tangent vector of a curve $\alpha(t) = (x(t), y(t))$, i.e., $\vec{v} = \alpha'(0)$. Assume $\alpha(0) = p$. Then $df_p(\vec{v})$ is defined to be $\frac{d}{dt}(f(\alpha(t))|_{t=0}$. Therefore $df_p(\alpha'(0)) = f_x(p)x'(0) + f_y(p)y'(0)$. $f(x, y) = x$ is a function on $\mathbb{R}^2$. We can show that $dx(\alpha'(t)) = x'(t)$. Similarly $dy(\alpha'(t)) = y'(t)$. Therefore $df(\vec{v}) = (f_x dx + f_y dy)(\vec{v})$. Thus as linear functionals over the vector space $T_p\mathbb{R}^2$, we have $df = f_x dx + f_y dy$. Now the symbols dx and dy are elements of the dual space $(T_p\mathbb{R}^2)^*$.

Definition 4.15. *A parametrized curve $\alpha \colon [0, \ell] \to S$ on a surface S is the restriction to $[0, \ell]$ of a differentiable mapping of $(-\epsilon, \ell + \epsilon)$ where $\epsilon > 0$ to S. If $\alpha(0) = p$ and $\alpha(\ell) = q$, we say that α joins p to q. We say α is regular if $\alpha'(t) \neq \vec{0}$ for $t \in [0, \ell]$.*

Definition 4.16. *Let $\alpha \colon I \to S$ be a curve on a surface S. A vector field $\vec{w}$ along α is a correspondence that assigns to each $t \in I$ a tangent vector $\vec{w}(t)$ of S at $\alpha(t)$. The vector field $\vec{w}$ is differentiable at $t_0 \in I$ if for some parametrization $\vec{X}(u, v)$ at $\alpha(t_0)$, the functions $a(t)$ and $b(t)$ in $\vec{w}(t) = a(t)\vec{X}_u(\alpha(t)) + b(t)\vec{X}_v(\alpha(t))$ are differentiable functions of t at t_0. $\vec{w}$ is differentiable in I if it is differentiable for every $t \in I$.*

4.4 COVARIANT DERIVATIVE, GEODESICS

Let $\vec{w}$ be a vector field in an open set $U \subset S$. Let $\alpha \colon (-\epsilon, \epsilon) \to U$ be a curve on S. Let $p = \alpha(0)$ and $\vec{v} = \alpha'(0)$. The vector field $\vec{w}(\alpha(t))$ on the curve α can be considered as a map from $(-\epsilon, \epsilon)$ to $\mathbb{R}^3$. The derivative $\frac{d}{dt}(\vec{w}(\alpha(t))|_{t=0})$ is vector in $\mathbb{R}^3$. However it may not be a tangent vector of S at p in general. Consider the tangent plane T_pS and the unit normal vector $\vec{N}_p$ of the surface S at p. Take the projection of $\frac{d}{dt}(\vec{w}(\alpha(t))|_{t=0})$ to the tangent plane T_pS with respect to the normal vector $\vec{N}_p$. Let's use $\frac{D\vec{w}}{dt}(0)$ to denote the projection. It is a tangent vector of S at p.

Definition 4.13. $\frac{D\vec{w}}{dt}(0)$ *is called the covariant derivative of the vector field $\vec{w}$ relative to the vector $\vec{v}$.*

The covariant derivative depends only upon the vector $\vec{v}$ and doesn't depend upon the curve α. To see this, choose a parametrization $\vec{X}(u,v)$ near p. Let $\alpha(t) = \vec{X}(u(t),v(t))$ and

$$\vec{w}(t) = \vec{w}(\alpha(t)) = a(u(t),v(t))\vec{X}_u + b(u(t),v(t))\vec{X}_v = a(t)\vec{X}_u(u(t),v(t)) + b(t)\vec{X}_v(u(t),v(t)).$$

$$\begin{aligned}
& \frac{dw}{dt} = a(t)(\vec{X}_{uu}u'(t) + \vec{X}_{uv}v'(t)) + b(t)(\vec{X}_{vu}u'(t) + \vec{X}_{vv}v'(t)) + a'(t)\vec{X}_u + b'(t)\vec{X}_v \\
= \ & a((\Gamma^1_{11}\vec{X}_u + \Gamma^2_{11}\vec{X}_v + e\vec{N})u' + (\Gamma^1_{12}\vec{X}_u + \Gamma^2_{12}\vec{X}_v + f\vec{N})v') \\
& +b((\Gamma^1_{21}\vec{X}_u + \Gamma^2_{21}\vec{X}_v + f\vec{N})u' + (\Gamma^1_{22}\vec{X}_u + \Gamma^2_{22}\vec{X}_v + g\vec{N})v') + a'\vec{X}_u + b'\vec{X}_v \\
= \ & (a\Gamma^1_{11}u' + a\Gamma^1_{12}v' + b\Gamma^1_{21}u' + b\Gamma^1_{22}v' + a')\vec{X}_u + (a\Gamma^2_{11}u' + a\Gamma^2_{12}v' + b\Gamma^2_{21}u' + b\Gamma^2_{22}v' + b')\vec{X}_v \\
& +(aeu' + afv' + bfu' + bgv')\vec{N}
\end{aligned}$$

Therefore the projection to the tangent plane is

$$\begin{aligned}
\frac{D(\vec{w})}{dt} & = (a\Gamma^1_{11}u' + a\Gamma^1_{12}v' + b\Gamma^1_{21}u' + b\Gamma^1_{22}v' + a')\vec{X}_u \\
+ & \quad (a\Gamma^2_{11}u' + a\Gamma^2_{12}v' + b\Gamma^2_{21}u' + b\Gamma^2_{22}v' + b')\vec{X}_v.
\end{aligned}$$

We see that the covariant derivative depends upon $\vec{v} = (u', v')$ only, not on the curve α.

From the expression above, we can also conclude that the covariant derivative depends upon the Christoffel symbols only. Therefore, it depends upon the first fundamental form only. In another word, the covariant derivative is a concept of the intrinsic geometry.

Definition 4.14. *Let $\vec{w}(t)$ be a differential vector field along a curve $\alpha(t)\colon I \to S$. The expression $\frac{D\vec{w}}{dt}$ is well defined and is called the covariant derivative of $\vec{w}$ at t.*

Let $\vec{w}$ be a differential field of unit vectors along a parametrized curve $\alpha\colon I \to S$ on an oriented surface S. Since $\vec{w}(t)$, $t \in I$, is a unit vector field, $\frac{d\vec{w}}{dt}(t)$ is perpendicular to $\vec{w}(t)$. We get $\frac{D\vec{w}}{dt}$ is also perpendicular to $\vec{w}$. This is because

$$\frac{D\vec{w}}{dt} = \frac{d\vec{w}}{dt} - <\frac{d\vec{w}}{dt}, \vec{N}> \vec{N} \quad \text{and} \quad \frac{D\vec{w}}{dt}\cdot\vec{w} = \frac{d\vec{w}}{dt}\cdot\vec{w} - <\frac{d\vec{w}}{dt}, \vec{N}> \vec{N}\cdot\vec{w} = 0.$$

Therefore $\frac{D\vec{w}}{dt}$ is parallel to $\vec{N} \times \vec{w}(t)$.

Definition 4.15. *With assumptions as above. We have*

$$\frac{D\vec{w}}{dt} = \lambda(\vec{N} \times \vec{w}(t)).$$

The real number $\lambda = \lambda(t)$, denoted by $[\frac{D\vec{w}}{dt}]$, is called the algebraic value of the covariant derivative of $\vec{w}$ at t.

Definition 4.16. *Let C be a curve on an oriented surface S and $\alpha(s)$ be a parameterization of C by the arc length s. The algebraic value of the covariant derivative $[\frac{D\alpha'(s)}{ds}]$ is called the geodesic curvature of C at the point $p = \alpha(s)$, denoted by κ_g.*

With the assumption as in the Definition 4.16, we have $\frac{d\alpha'(s)}{ds} = \alpha''(s)$. Using the Gram-Schmidt method, we see that

$$\frac{D\vec{w}}{dt} = \frac{d\vec{w}}{dt} - < \frac{d\vec{w}}{dt}, \vec{N} > \vec{N}.$$

Therefore we can write

$$\begin{aligned} & \alpha''(s) = < \alpha''(s), \vec{N} > \vec{N} + \frac{D\alpha'}{ds} \\ = \; & \kappa_n \vec{N} + \kappa_g \vec{N} \times \alpha'(s). \end{aligned}$$

Since $\vec{N}$ is perpendicular to $\vec{N} \times \alpha'(s)$, we have

$$\kappa^2 = |\alpha''(s)|^2 = \kappa_g^2 + \kappa_n^2.$$

Given two differential unit vector fields $\vec{v}$ and $\vec{w}$ along a curve $\alpha(t)$ on the surface S. Choose another vector field $\vec{u}$ along the curve α such that $\vec{v}$ is perpendicular to $\vec{u}$ and that $\vec{v} \times \vec{u} = \vec{N}$. We write

$$\vec{w} = a(t)\vec{v}(t) + b(t)\vec{u}(t),$$

where $a(t)$ and $b(t)$ are differentiable functions and $a^2 + b^2 = 1$.

Lemma 4.17. *Let φ_0 be such that $a(t_0) = \cos\varphi_0$ and $b(t_0) = \sin\varphi_0$. Then the differentiable function*

$$\varphi(t) = \varphi_0 + \int_{t_0}^{t} (a(t)b'(t) - b(t)a'(t))dt$$

is such that $\cos\varphi(t) = a(t)$, $\sin\varphi(t) = b(t)$, $\varphi(t_0) = \varphi_0$.

Proof. It suffices to prove that the function

$$(a(t) - \cos\varphi(t))^2 + (b(t) - \sin\varphi(t))^2 = 2 - 2(a(t)\cos\varphi(t) + b(t)\sin\varphi(t))$$

is identically zero.

Take the derivative of the expression $a(t)^2 + b(t)^2 = 1$, we get $a(t)a'(t) + b(t)b'(t) = 0$. Thus $a(t)a'(t) = -b(t)b'(t)$, and using $\varphi'(t) = a(t)b'(t) - a'(t)b(t)$, we have

$$\begin{aligned} & (a(t)\cos\varphi(t) + b(t)\sin\varphi(t))' \\ = \; & -a(t)\sin\varphi(t)\varphi'(t) + a'(t)\cos\varphi(t) + b(t)\cos\varphi(t)\varphi'(t) + b'(t)\sin\varphi(t) \\ = \; & a'(t)\cos\varphi(t) + b'(t)\sin\varphi(t) - b'(t)\sin\varphi(t)(a(t)^2 + b(t)^2) - a'(t)\cos\varphi(t)(a(t)^2 + b(t)^2) \\ = \; & 0 \end{aligned}$$

Therefore $f(t) = 2 - 2(a(t)\cos\varphi(t) + b(t)\sin\varphi(t))$ is a constant. Since $f(t_0) = 0$, we have $f(t) = 0$ for all t. ∎

Lemma 4.18. *Let $\vec{v}$ and $\vec{w}$ be two differential unit vector fields along the curve α on S. Let φ be the differentiable function in Lemma 4.17. Then*

$$[\frac{D\vec{w}}{dt}] - [\frac{D\vec{v}}{dt}] = \frac{d\varphi}{dt}.$$

Proof. Take $\vec{u} = \vec{N} \times \vec{v}$. Then by Lemma 4.17,

$$\vec{w} = \cos\varphi\vec{v} + \sin\varphi\vec{u},$$

$$\vec{N} \times \vec{w} = \cos\varphi\vec{N} \times \vec{v} + \sin\varphi\vec{N} \times \vec{u} = \cos\varphi\vec{u} - \sin\varphi\vec{v}.$$

$$\vec{w}' = -\sin\varphi\varphi'\vec{v} + \cos\varphi\vec{v}' + \cos\varphi\varphi'\vec{u} + \sin\varphi\vec{u}'.$$

Take the inner product with $\vec{N} \times \vec{w}$, using the fact that $< \vec{u}, \vec{v} >= 0$, $< \vec{v}, \vec{v}' >= 0$, and $< \vec{u}, \vec{u}' >= 0$, we get

$$\begin{aligned} & < \vec{w}', \vec{N} \times \vec{w} > \\ = \; & \sin^2\varphi\varphi' + \cos^2\varphi < \vec{v}', \vec{u} > + \cos^2\varphi\varphi' - \sin^2\varphi < \vec{u}', \vec{v} > . \end{aligned}$$

Take the derivative of $< \vec{v}, \vec{u} >= 0$, we get $< \vec{v}', \vec{u} >= - < \vec{v}, \vec{u}' >$. Therefore

$$< \vec{w}', \vec{N} \times \vec{w} >= \varphi' + (\cos^2\varphi + \sin^2\varphi) < \vec{v}', \vec{u} >= \varphi' + < \vec{v}', \vec{u} > .$$

Since

$$\begin{aligned} & < \vec{w}', \vec{N} \times \vec{w} > \\ = \; & < \frac{D\vec{w}}{dt} + < \vec{N}, \vec{w}' > \vec{N}, \vec{N} \times \vec{w} >=< \frac{D\vec{w}}{dt}, \vec{N} \times \vec{w} >=< [\frac{D\vec{w}}{dt}]\vec{N} \times \vec{w}, \vec{N} \times \vec{w} > \\ = \; & [\frac{D\vec{w}}{dt}], \end{aligned}$$

and similarly $< \vec{v}', \vec{u} >= [\frac{D\vec{v}}{dt}]$, we have

$$[\frac{D\vec{w}}{dt}] =< \vec{w}', \vec{N} \times \vec{w} >= \varphi' + < \vec{v}', \vec{u} >= \frac{d\varphi}{dt} + [\frac{D\vec{v}}{dt}].$$

∎

Proposition 4.19. *Let $\vec{X}(u, v)$ be an orthogonal parametrization of a neighbourhood of an oriented surface S, and $\vec{w}(t)$ be a differentiable field of unit vectors along the curve $\vec{X}(u(t), v(t))$. Then*

$$[\frac{D\vec{w}}{dt}] = \frac{1}{2\sqrt{EG}}\big(G_u\frac{dv}{dt} - E_v\frac{du}{dt}\big) + \frac{d\varphi}{dt},$$

where $\varphi(t)$ is the angle from $\vec{X}_u$ to $\vec{w}(t)$ in the given orientation.

Proof. Let $\vec{U} = \vec{X}_u/\sqrt{E}$ and $\vec{V} = \vec{X}_v/\sqrt{G}$ be the unit tangent vectors of the coordinate curves. We have $\vec{N} = \vec{U} \times \vec{V}$. Write $\vec{U}(t) = \vec{U}(u(t), v(t))$. From Lemma 4.18, we get

$$[\frac{D\vec{w}(t)}{dt}] = [\frac{D\vec{U}(t)}{dt}] + \frac{d\varphi}{dt}.$$

$$[\frac{D\vec{U}(t)}{dt}] =< \frac{d\vec{U}(t)}{dt}, \vec{N} \times \vec{U}(t) >=< \frac{d\vec{U}(t)}{dt}, \vec{V} >=< \vec{U}_u \frac{du}{dt} + \vec{U}_v \frac{dv}{dt}, \vec{V} > .$$

Since $< \vec{X}_u, \vec{X}_v >= F = 0$, we have $< \vec{X}_{uu}, \vec{X}_v >= -E_v/2$, and

$$\begin{aligned} & < \vec{U}_u, \vec{V} > \\ = \quad & < (\frac{\vec{X}_u}{\sqrt{E}})_u, \frac{\vec{X}_v}{\sqrt{G}} >=< \frac{\vec{X}_{uu}}{\sqrt{E}} + \frac{-E_u}{2\sqrt{E}^3}\vec{X}_u, \frac{\vec{X}_v}{\sqrt{G}} >=< \vec{X}_{uu}, \vec{X}_v > /\sqrt{EG} = -\frac{E_v}{2\sqrt{EG}}. \end{aligned}$$

Similarly we get

$$< \vec{U}_v, \vec{V} >= \frac{G_u}{2\sqrt{EG}}.$$

Therefore we get the formula stated in the Proposition. ∎

Proposition 4.20. *Let $\alpha(s)$ be a parametrization by arc length of a regular oriented curve C on an oriented surface S. Let p be a point on the curve α. Let $\vec{X}(u, v)$ be an orthogonal parametrization of S in p and $\varphi(s)$ be the angle that $\vec{X}_u$ makes with $\alpha'(s)$ in the given orientation. Then*

$$\kappa_g = (\kappa_g)_1 \cos\varphi + (\kappa_g)_2 \sin\varphi + \frac{d\varphi}{ds},$$

where $(\kappa_g)_1$ and $(\kappa_g)_2$ are the geodesic curvatures of the coordinate curves $v =$ constant and $u =$ constant respectively.

Proof. Consider the coordinate curve $\vec{X}(u, v_0)$. Let t be its arc length such that $u = u(t)$ is an increasing function of t. Then we get $\frac{d}{dt}\vec{X}(u(t), v_0) = \vec{X}_u \frac{du(t)}{dt}$. We have

$$1 = |\frac{d}{dt}\vec{X}(u(t), v_0)| = |\vec{X}_u||\frac{du(t)}{dt}| = \sqrt{E}\frac{du(t)}{dt}.$$

Therefore $\frac{du(t)}{dt} = 1/\sqrt{E}$. Apply Proposition 4.19 to the vector field $\frac{d}{dt}\vec{X}(u(t), v_0)$, we get

$$(\kappa_g)_1 = [\frac{D\vec{X}(u(t), v_0)}{dt}] = \frac{-E_v}{2\sqrt{EG}} \frac{du}{dt} = \frac{-E_v}{2E\sqrt{G}}.$$

Similarly we get

$$(\kappa_g)_2 = \frac{G_u}{2G\sqrt{E}}.$$

Now write $\alpha(s) = \vec{X}(u(s), v(s))$. We get $\alpha'(s) = \vec{X}_u u'(s) + \vec{X}_v v'(s)$. Since $< \vec{X}_u, \vec{X}_v >= F = 0$, we get

$$\sqrt{E}\cos\varphi = |\vec{X}_u| \cos\varphi =< \vec{X}_u, \alpha'(s) >= Eu'(s),$$

$$\sqrt{G}\sin\varphi = |\vec{X}_v|\cos\varphi =< \vec{X}_v, \alpha'(s) >= Gv'(s).$$

Apply Proposition 4.19 to the vector field $\alpha'(s)$, we get

$$\begin{aligned}
\kappa_g &= \frac{1}{2\sqrt{EG}}\big(G_u v'(s) - E_v u'(s)\big) + \frac{d\varphi}{ds} \\
&= \frac{1}{2\sqrt{EG}}\big(2G\sqrt{E}(\kappa_g)_2\frac{\sin\varphi}{\sqrt{G}} + 2E\sqrt{G}(\kappa_g)_1\frac{\cos\varphi}{\sqrt{E}}\big) + \frac{d\varphi}{ds} \\
&= (\kappa_g)_1\cos\varphi + (\kappa_g)_2\sin\varphi + \frac{d\varphi}{ds}.
\end{aligned}$$

■

Definition 4.21. *A vector field $\vec{w}$ along a parametrized curve $\alpha\colon I \to S$ is said to be parallel if $\frac{D\vec{w}}{dt} = 0$ for every $t \in I$.*

Proposition 4.22. *Let $\vec{w}$ and $\vec{v}$ be parallel vector fields along $\alpha\colon I \to S$. Then $< \vec{w}(t), \vec{v}(t) >$ is constant. In particular, $|\vec{w}(t)|$ and $|\vec{v}(t)|$ are constant, and the angle between $\vec{w}(t)$ and $\vec{v}(t)$ is constant.*

Proof. Since $\vec{v}$ is parallel, $\vec{v}'(t)$ is perpendicular to the tangent plane of S at $\alpha(t)$. In particular $\vec{v}'(t)$ is perpendicular to $\vec{w}(t)$. Thus $< \vec{v}'(t), \vec{w}(t) >= 0$. Similarly we have $< \vec{v}(t), \vec{w}'(t) >= 0$. Then

$$< \vec{v}(t), \vec{w}(t) >' =< \vec{v}'(t), \vec{w}(t) > + < \vec{v}(t), \vec{w}'(t) >= 0.$$

Therefore $< \vec{v}(t), \vec{w}(t) >$ is a constant. ■

Here is a property of covariant derivative we need to use later. Let $f(t)$ be a function over the interval I.

$$\begin{aligned}
\frac{D(f(t)\vec{w}(t))}{dt} &= \frac{d}{dt}(f(t)\vec{w}(t)) - < \frac{d}{dt}(f(t)\vec{w}(t)), \vec{N} > \vec{N} \\
&= f'(t)\vec{w}(t) + f(t)\vec{w}'(t) - < f'(t)\vec{w}(t) + f(t)\vec{w}'(t), \vec{N} > \vec{N} \\
&= f'(t)\vec{w}(t) + f(t)\big((\vec{w}'(t) - < \vec{w}'(t), \vec{N} > \vec{N}\big) \\
&= f'(t)\vec{w}(t) + f(t)\frac{D\vec{w}(t)}{dt}.
\end{aligned}$$

Proposition 4.23. *Let $\alpha\colon I \to S$ be a parametrized curve in S and let $\vec{w}_0 \in T_{\alpha(t_0)}S$, $t_0 \in I$. Then there exists a unique parallel vector field $\vec{w}(t)$ along $\alpha(t)$, with $\vec{w}(t_0) = \vec{w}_0$.*

Proof. Firstly, we assume that the curve α is contained in the image of an orthogonal parametrization $\vec{X}(u, v)$. We write $\alpha(t) = \vec{X}(u(t), v(t))$. Let's prove the uniqueness first. Suppose $\vec{w}$ is parallel. From Proposition 4.22, $|\vec{w}(t)|$ is a constant, say, equal to $|\vec{w}_0| = 1/c$. Therefore $\frac{D(c\vec{w})}{dt} = c\frac{D\vec{w}}{dt} = 0$. Apply Proposition 4.19 to the unit vector field $c\vec{w}$, we get

$$0 = \frac{D(c\vec{w})}{dt} = \frac{1}{2\sqrt{EG}}\big(G_u\frac{dv}{dt} - E_v\frac{du}{dt}\big) + \frac{d\varphi}{dt},$$

where φ is the angle from $\vec{X}_u$ to $\vec{w}$. We get

$$\frac{d\varphi}{dt} = -\frac{1}{2\sqrt{EG}}(G_u\frac{dv}{dt} - E_v\frac{du}{dt}).$$

Denote the right hand term by $B(t)$ and let φ_0 be the angle from $\vec{X}_u(u(t_0), v(t_0))$ to $\vec{w}_0$. We obtain

$$\varphi = \varphi_0 + \int_{t_0}^{t} B(t)dt.$$

Thus $\vec{w}(t) = \cos\varphi\vec{X}_u/|c\vec{X}_u| + \sin\varphi\vec{X}_v/c|\vec{X}_v|$. Therefore $\vec{w}$ is unique.

For the existence, take

$$\varphi = \varphi_0 + \int_{t_0}^{t} B(t)dt.$$

Then $\vec{w}(t) = |\vec{w}_0|(\frac{\cos\varphi\vec{X}_u}{\sqrt{E}} + \frac{\sin\varphi\vec{X}_v}{\sqrt{G}})$ is parallel along α by going backwards in the argument above.

For the general case, we can use the compactness of $\alpha(I)$ to cover $\alpha(I)$ by finitely many parametrizations. Then apply the conclusion above to each parameterization. ∎

Definition 4.24. *Let $\alpha\colon I \to S$ be a parametrized curve and $\vec{w}_0 \in T_{\alpha(t_0)}S$, $t_0 \in I$. Let $\vec{w}$ be the parallel vector field along α with $\vec{w}(t_0) = \vec{w}_0$. The vector $\vec{w}(t_1)$ for $t_1 \in I$ is called the parallel transport of $\vec{w}_0$ along α at the point $\alpha(t_1)$.*

If α is regular, then the parallel transport doesn't depend upon the parametrization of $\alpha(I)$. In fact, let $\beta(r)\colon J \to S$ be another regular parametrization of $\alpha(I)$. Recall that if we write $\vec{w}(t) = a(t)\vec{X}_u + b(t)\vec{X}_v$ and $\alpha(t) = \vec{X}(u(t), v(t))$, we have the formula

$$\begin{aligned}\frac{D(\vec{w})}{dt} &= (a\Gamma^1_{11}u' + a\Gamma^1_{12}v' + b\Gamma^1_{21}u' + b\Gamma^1_{22}v' + a')\vec{X}_u \\ &+(a\Gamma^2_{11}u' + a\Gamma^2_{12}v' + b\Gamma^2_{21}u' + b\Gamma^2_{22}v' + b')\vec{X}_v.\end{aligned}$$

Therefore we have $\frac{D\vec{w}}{dr} = \frac{D\vec{w}}{dt}\frac{dt}{dr}$. Since $\frac{dt}{dr} \neq 0$, $\vec{w}(t)$ is parallel if and only if $\vec{w}(r)$ is parallel.

Fix two points p and q on S. Take a curve α on S with $\alpha(0) = p$ and $\alpha(1) = q$. Denote by $P_\alpha\colon T_pS \to T_qS$ the map that assigns to each $\vec{v} \in T_pS$ its parallel transport along α at q. One can check that P_α is a linear transformation. Proposition 4.22 implies P_α is an isometry.

Given two surfaces S_1 and S_2 which are tangent along a parametrized curve α and a tangent vector $\vec{w}_0$ at $\alpha(t_0)$. $\vec{w}(t)$ is the parallel transport of $\vec{w}_0$ relative to the surface S_1 if and only if $\vec{w}(t)$ is the parallel transport of $\vec{w}_0$ relative to S_2. In fact, that $\vec{w}$, considered as tangent vectors of S_1, is parallel along α is equivalent to that $\frac{d\vec{w}}{dt}$ is perpendicular to the tangent plane $T_{\alpha(t)}S_1 = T_{\alpha(t)}S_2$, which is equivalent to that $\vec{w}$, considered as tangent vectors of S_2, is parallel along α. By the uniqueness of the parallel transport, the assertion holds.

Example 4.25. Consider a circle C on the unit sphere $x^2 + y^2 + z^2 = 1$ given by $z = \cos\varphi$, where $0 < \varphi < \pi/2$. φ is fixed. Let p be a point on the circle and $\vec{w}_0$ be a unit tangent vector to the circle at p. Let $\alpha(s)$ be the parametrization of the circle by the arc length parameter s so that $p = \alpha(0)$. Let's determine the parallel transport of $\vec{w}_0$ along the circle.

Consider the cone which is tangent to the sphere along C. The angle ϕ at the vertex of the cone is equal to $\phi = \pi/2 - \varphi$. The parallel transport of $\vec{w}_0$ along C on the unit sphere is the parallel transport of $\vec{w}_0$ along C on the cone. The cone minus one ruling line is isometric to the open set in $\mathbb{R}^2$ given, in polar coordinate, by

$$0 < r < +\infty, \qquad 0 < \theta < 2\pi \sin\phi.$$

The part of cone from the vertex to the curve C is a fan U. This can be seen as follows. The arc length of the circle C is $2\pi\sin\varphi = 2\pi\cos\phi$. The fan U has radius $\cot\phi$. Thus the angle θ of the fan is $2\pi\cos\phi/\cot\phi = 2\pi\sin\phi$.

Since the covariant derivative depends only on the first fundamental form, parallel vector fields remain parallel under isometry. The covariant derivative on the plane is the ordinary directional derivative. Thus the parallel vector field $\vec{w}(s)$ along C on the fan with $\vec{w}(0) = \vec{w}_0$ is the constant $\vec{w}_0$. We parametrize the curve C on the fan by the arc length s

$$\beta(s) = (\cot\phi\cos(s\tan\phi), \cot\phi\sin(s\tan\phi)).$$

The point p corresponds to the point $(\cot\phi, 0)$ on the x-axis and $\vec{w}_0$ corresponds to the vector $(0,1)$. The parallel of $\vec{w}_0$ at the point $\beta(s)$ on the fan is still $\vec{w}_0$. If the angle between the tangent vector $\beta'(s)$ of β and $\vec{w}_0$ is δ, then $\cos\delta =< \vec{w}_0, \beta'(s) >= \cos(s\tan\phi)$. Thus $\delta = s\tan\phi$. Now we come back to the sphere. The transport of $\vec{w}_0$ along C at the arc length s on the sphere is the tangent vector at $\alpha(s)$ whose angle with $\alpha'(s)$ is $s\tan\phi$. ∎

Definition 4.26. *A non-constant parametrized curve $\alpha\colon I \to S$ is said to be geodesic at $t \in I$ if the field of its tangent vectors $\alpha'(t)$ is parallel along α at t, i.e.,*

$$\frac{D\alpha'(t)}{dt} = 0.$$

α is a parametrized geodesic if it is geodesic for all $t \in I$.

A regular connected curve C on S is said to be a geodesic if, for every point $p \in C$, the parametrization $\alpha(s)$ of C near p by arc length parameter s is parametrized geodesic.

From the definition above, we see that a regular connected curve C is geodesic if the parametrization $\alpha(s)$ of C by arc length parameter s is parallel, i.e., α'' is perpendicular to the tangent plane of S at $\alpha(s)$. This is equivalent to that α'' is parallel to the normal vector of the surface S at $\alpha(s)$, i.e., the normal of the curve at $\alpha(s)$ is parallel to the normal of the surface at the same point.

Let's use the observation above to study geodesics on some special surfaces.

Example 4.27. Let C be a line on a surface S. The unit tangent vectors of C are all equal. Therefore the derivative of the unit tangent vector field $\vec{t}$ of C is zero. Hence the covariant derivative of $\vec{t}$ is also zero. Thus C is a geodesic. In particular, lines on planes are geodesics. ∎

Example 4.28. The great circles of a sphere $S\colon x^2 + y^2 + y^2 = r^2$ are geodesics. The great circle is obtained by cutting the sphere by a plane passing through the origin. The normal $\vec{n}$ of the circle at the point p is parallel to the line connecting the origin with p. The normal $\vec{N}$ of the sphere at the point p is also parallel to the line passing through the origin and p. Therefore $\vec{n}$ is parallel to N. Thus great circles are geodesics. ∎

Example 4.29. Let S be the cylinder $x^2 + y^2 = 1$. Lines on the cylinder are geodesics. Circles on S are geodesics by the similar argument as in the previous example. Are there any other geodesics on S?

Consider the parametrization of the cylinder

$$\vec{X}(u, v) = (\cos u, \sin u, v).$$

Let's assume that $\vec{X}(0, 0) = p$. We knew that $\vec{X}(u, v)$ is a local isometry between the plane and the cylinder. Geodesics remain geodesic under local isometry. Therefore if $\vec{X}(u(s), v(s))$ is a geodesic on S passing through p, $(u(s), v(s))$ must be a line on the plane passing through the origin. Thus $u(s) = as$ and $v(s) = bs$ with $a^2 + b^2 = 1$ so that s is the arc length parameter of $\vec{X}(u(s), v(s))$. Thus the geodesic is $\vec{X}(u(s), v(s)) = (\cos(as), \sin(as), bs)$. It is a helix. ∎

Let C be a geodesic on S. Then the geodesic curvature κ_g of C is zero. This can be seen as follows. Take a parametrization $\alpha(s)$ of C by arc length s. C is a geodesic if the covariant derivative of α' is zero. But

$$\frac{D\alpha'}{ds} = \kappa_g \vec{N} \times \alpha'.$$

Thus $\kappa_g = 0$.

Let's study the equations satisfied by geodesics in local coordinates.

Let $\vec{X}(u, v)$ be a parametrization of a regular surface and $\alpha(t) = \vec{X}(u(t), v(t))$ be a curve on S. Then

$$\alpha'(t) = u'(t)\vec{X}_u + v'(t)\vec{X}_v.$$

The covariant derivative of the vector field $\alpha'(t)$ is

$$\begin{aligned} & \frac{D(\vec{\alpha}'(t))}{\partial t} \\ = \; & (u'\Gamma^1_{11}u' + u'\Gamma^1_{12}v' + v'\Gamma^1_{21}u' + v'\Gamma^1_{22}v' + u'')\vec{X}_u \\ & +(u'\Gamma^2_{11}u' + u'\Gamma^2_{12}v' + v'\Gamma^2_{21}u' + v'\Gamma^2_{22}v' + v'')\vec{X}_v. \end{aligned}$$

Thus the equations of geodesic is

$$\Gamma^1_{11}(u')^2 + 2\Gamma^1_{12}u'v' + \Gamma^1_{22}(v')^2 + u'' = 0,$$

$$\Gamma^2_{11}(u')^2 + 2\Gamma^2_{12}u'v' + \Gamma^2_{22}(v')^2 + v'' = 0.$$

Notice that $\Gamma^1_{12} = \Gamma^1_{21}$ and $\Gamma^2_{12} = \Gamma^2_{21}$.

Corollary 4.30. *Given a point $p \in S$ and a non-zero vector $\vec{w} \in T_pS$, there exist an $\epsilon > 0$ and a unique parametrization geodesic $\alpha\colon (-\epsilon, \epsilon) \to S$ such that $\alpha(0) = p$ and $\alpha'(0) = \vec{w}$.*

Proof. Apply the uniqueness and existence theorem of ordinary differential equations to the geodesic equations. ∎

Example 4.31. Consider the surface of revolution given by the parametrization

$$\vec{X}(u, v) = (f(v) \cos u, f(v) \sin u, g(v))$$

where $f(v) > 0$.

$$E =< \vec{X}_u, \vec{X}_u >= f(v)^2, \quad F =< \vec{X}_u, \vec{X}_v >= 0, \quad G =< \vec{X}_v, \vec{X}_v >= f'(v)^2 + g'(v)^2.$$

To get the Christoffel symbols, consider

$$\begin{aligned}
&\Gamma^1_{11}E + \Gamma^2_{11}F = \tfrac{1}{2}E_u = 0,\\
&\Gamma^1_{11}F + \Gamma^2_{11}G = F_u - \tfrac{1}{2}E_v = -f'(v)f''(v).\\
&\Gamma^1_{12}E + \Gamma^2_{12}F = \tfrac{1}{2}E_v = f'(v)f''(v),\\
&\Gamma^1_{12}F + \Gamma^2_{12}G = \tfrac{1}{2}G_u = 0.\\
&\Gamma^1_{22}E + \Gamma^2_{22}F = F_v - \tfrac{1}{2}G_u = 0,\\
&\Gamma^1_{22}F + \Gamma^2_{22}G = \tfrac{1}{2}G_v = f'(v)f''(v) + g'(v)g''(v).
\end{aligned}$$

Therefore we get

$$\Gamma^1_{11} = \Gamma^1_{22} = \Gamma^2_{12} = 0, \quad \Gamma^1_{12} = \frac{ff'}{f^2}, \quad \Gamma^2_{11} = -\frac{ff'}{(f')^2 + (g')^2}, \quad \Gamma^2_{22} = \frac{f'f'' + g'g''}{(f')^2 + (g')^2}.$$

Thus the geodesic equations are

$$\begin{aligned}
u'' + \frac{2ff'}{f^2}u'v' &= 0,\\
v'' - \frac{ff'}{(f')^2 + (g')^2}(u')^2 + \frac{f'f'' + g'g''}{(f')^2 + (g')^2}(v')^2 &= 0.
\end{aligned}$$

Let's consider the meridians $u = c$ and $v = v(s)$ where c is a constant and s is an arc length parameter of the curve $\vec{X}(c, v(s))$. The first equation of the geodesic equations is satisfied since $u' = 0$. Since $|\frac{d\vec{X}(c, v(s))}{ds}| = 1$, we get

$$(f'(v)^2 + g'(v)^2)(v')^2 = |\vec{X}_v(c, v(s))v'|^2 = 1.$$

Differentiate the equation above with respect to s, we get

$$2(f'f'' + g'g'')v'(v')^2 + 2((f')^2 + (g')^2)v'v'' = 0.$$

Since $v' \neq 0$, we get

$$v'' = -\frac{f'f'' + g'g''}{(f')^2 + (g')^2}(v')^2.$$

This is the second equation of geodesic equations. Therefore meridians are geodesics.

Consider coordinate curves $u = u(s)$ and $v = c$ where c is a constant and s is the arc length parameter of the curve $\vec{X}(u(s), c)$. Such coordinate curve is called a parallel. It is a circle. Since $v' = 0$, the first equation gives $u'' = 0$, i.e., u' is a constant. The second equation becomes

$$-\frac{ff'}{(f')^2 + (g')^2}(u')^2 = 0.$$

Since $u' \neq 0$, we must have $ff' = 0$. Since $f(v) > 0$, we have $f' = 0$. Therefore $\vec{X}(u(s), c)$ is a geodesic if and only if $f'(c) = 0$.

Let's examine geodesics carefully. Since

$$\frac{d}{ds}((f^2(v(s))\frac{du(s)}{ds})) = 2f(v(s))\frac{df(v)}{dv}\frac{dv(s)}{ds}\frac{du(s)}{ds} + f^2(v(s))\frac{d^2u(s)}{ds^2} = 0,$$

we get $f^2(v(s))u'(s) = c$, where c is a constant. Let θ, $0 \leq \theta \leq \pi/2$, be the angle between a geodesic $\vec{X}(u(s), v(s))$ and a parallel at an intersection point of the two curves. Here s is the arc length parameter for the geodesic. Then

$$\cos\theta = \frac{|<\vec{X}_u, \vec{X}_u u' + \vec{X}_v v'>|}{|\vec{X}_u|} = |fu'|.$$

Since $r = f(v)$ is the radius of the parallel at the intersection point, we obtain *Clairaut's relation*:

$$r\cos\theta = const. = |c|.$$

Are there any other geodesics?

Let $\vec{X}(u(s), v(s))$ be a geodesic parametrized by arc length s. Assume the geodesic is not a meridian nor a parallel of the surface. From the first equation of geodesic equations, we have that $f^2(v(s))u'(s) = c$, where c is a constant. $c \neq 0$ since otherwise we will have $u'(s) = 0$ and thus the curve is a meridian.

Since s is arc length, we have

$$1 = f^2(\frac{du(s)}{ds})^2 + ((\frac{df(v)}{dv})^2 + (\frac{dg(v)}{dv})^2)(\frac{dv(s)}{ds})^2.$$

Substitute $f^2u' = c$ into the equation above, we get

$$((\frac{df(v)}{dv})^2 + (\frac{dg(v)}{dv})^2)(\frac{dv(s)}{ds})^2 = -\frac{c^2}{f^2} + 1.$$

Differentiate the expression above, we get

$$\begin{aligned} & 2\frac{dv}{ds}\frac{d^2v}{ds^2}((\frac{df(v)}{dv})^2 + (\frac{dg(v)}{dv})^2) + (2\frac{df(v)}{dv}\frac{d^2f(v)}{dv^2} + 2\frac{dg(v)}{dv}\frac{d^2g(v)}{dv^2})(\frac{dv(s)}{ds})^3 \\ = \; & 2f(v)f'(v)(\frac{du}{ds})^2\frac{dv(s)}{ds}. \end{aligned}$$

$v'(s)$ cannot be zero in an open interval, for otherwise the geodesic is a parallel over this open interval. Consider this part of the geodesic. Therefore, by dividing v', we get the second equation of the geodesic equations. Thus we can replace the second equation of the geodesic equations by the equation

$$1 = f^2(\frac{du(s)}{ds})^2 + ((\frac{df(v)}{dv})^2 + (\frac{dg(v)}{dv})^2)(\frac{dv(s)}{ds})^2.$$

Since $u'(s) \neq 0$, by the inverse function theorem, s is also a function of u. Let's write $s = s(u)$. Then $v = v(s) = v(s(u))$. v is a function of u. Multiplying $(\frac{ds}{du})^2$ to the equation above, we get

$$
\begin{aligned}
(\frac{ds}{du})^2 &= (\frac{ds}{du})^2 \big(f^2(\frac{du(s)}{ds})^2 + ((\frac{df(v)}{dv})^2 + (\frac{dg(v)}{dv})^2)(\frac{dv(s)}{ds})^2\big) \\
&= f^2 + ((\frac{df(v)}{dv})^2 + (\frac{dg(v)}{dv})^2)(\frac{dv(s(u))}{du})^2.
\end{aligned}
$$

Here we used $\frac{dv}{ds}\frac{ds}{du} = \frac{dv}{du}$. Using the fact that $(\frac{ds}{du})^2 = (\frac{du}{ds})^{-2} = f^4/c^2$, we have

$$f^2 = c^2 + c^2\frac{f'(v)^2 + g'(v)^2}{f^2}(\frac{dv}{du})^2,$$

$$\frac{dv}{du} = \pm\frac{f}{c}\sqrt{\frac{f^2 - c^2}{f'(v)^2 + g'(v)^2}}.$$

Therefore we get the equation of a segment of a geodesic in turns of an equation in u and v

$$u = \pm c\int \frac{1}{f(v)}\sqrt{\frac{f'(v)^2 + g'(v)^2}{f(v)^2 - c^2}}dv.$$

■

Example 4.32. Let's consider geodesics of a paraboloid $z = x^2 + y^2$, which are not meridians.

We can parametrize the surface by

$$\vec{X}(u, v) = (v\cos u, v\sin u, v^2), \quad 0 < v < \infty, \quad 0 < u < 2\pi.$$

Let p_0 be a point of the paraboloid and C_0 be the parallel of radius r_0 passing through p_0. Let $\alpha(s) = \vec{X}(u(s), v(s))$ be a parametrized geodesic passing through p_0 and making an angle θ_0 with C_0. Recall the Clairaut's relation

$$r\cos\theta = const. = |c|, \quad 0 \leq \theta \leq \pi/2.$$

Here $r = v(s)$. When r increases, $\cos\theta$ decreases and therefore θ increases. If $r = v(s)$ doesn't change over an open interval, then $\alpha(s)$ over this interval is a part of parallel. But parallels on the paraboloid cannot be geodesics. Therefore r is not a constant over any open interval. When r changes, θ changes.

Claim: When r increases, the geodesic α meets all meridians. Firstly, the geodesic cannot be tangent to a meridian since otherwise, by Corollary 4.30, the geodesic will coincide with the meridian.

Suppose the claim is false. Since θ increases when r increases, α must approach a meridian M, say $u = u_0$, asymptotically. Therefore $u \to u_0$ as $r = v \to \infty$. Recall that we have

$$u = \pm c\int \frac{1}{f(v)}\sqrt{\frac{f'(v)^2 + g'(v)^2}{f(v)^2 - c^2}}dv.$$

We can make a choice of the orientation of the curve α so that we have $c > 0$ and

$$u = c\int_{v_0}^{v} \frac{1}{f(v)}\sqrt{\frac{f'(v)^2 + g'(v)^2}{f(v)^2 - c^2}}dv + const. > c\int_{v_0}^{v} \frac{dv}{v} + const. = c\ln\frac{v}{v_0} + const.$$

Here we used the inequality $\dfrac{1+4v^2}{v^2-c^2} > 1$. As $r = v \to \infty$, u increases beyond any value, which contradicts to $u \to u_0$. The claim is proved.

When $r = v$ decreases, the angle θ decreases and approaches the value 0, which corresponds to a parallel of radius $|c|$. It can be shown that no geodesic of the paraboloid can be asymptotic to a parallel, which we won't do here. Therefore α must be tangent to the parallel of radius $|c|$ at a point p_1, ∎

4.5 REVISIT OF GEODESICS

Everybody knows that on the plane, the shortest distance between two points is achieved by the straight line connecting these two points. We can ask, given two points on a regular surface, which curve on the surface achieves the shortest distance between these two points.

Choose a parameterization $\vec{X}(u,v)\colon U \to S$ of the surface S. Let $\alpha\colon [a,b] \to S$ be a regular curve on S contained in $\vec{X}(U)$. Choose the arclength s as the parameter of the curve α. Therefore we have a curve $(u(s), v(s))$ in U such that $\vec{X}(u(s), v(s)) = \alpha(s)$. We want to get the condition on the curve α such that the shortest distance between $\alpha(a)$ and $\alpha(b)$ is achieved by the curve α. So we suppose that α is the curve whose length is the smallest among all curves on S connecting $\alpha(a)$ and $\alpha(b)$.

Let $p(s)$ and $q(s)$ be functions defined over $[a,b]$ such that $p(a) = q(a) = p(b) = q(b) = 0$. Let ϵ be a number small enough such that $(u(s) + \epsilon p(s), v(s) + \epsilon q(s))$ is contained in U. Let $\alpha(s,\epsilon) = \vec{X}(u(s) + \epsilon p(s), v(s) + \epsilon q(s))$. For fixed ϵ, $\alpha(s,\epsilon)$ is a curve on S connecting $\alpha(a)$ and $\alpha(b)$. The length of the curve $\alpha(s,\epsilon)$ is given by

$$L(\epsilon) = \int_a^b \sqrt{\langle \frac{d}{ds}\alpha(s,\epsilon), \frac{d}{ds}\alpha(s,\epsilon)\rangle}ds.$$

Since $L(\epsilon)$ has minimum at $\epsilon = 0$, we must have $\dfrac{d}{d\epsilon}L(\epsilon)|_{\epsilon=0} = 0$.

$$\frac{d}{d\epsilon}L(\epsilon) = \int_a^b \frac{d}{d\epsilon}\sqrt{\langle \frac{d}{ds}\alpha(s,\epsilon), \frac{d}{ds}\alpha(s,\epsilon)\rangle}ds = \int_a^b \frac{\frac{d}{d\epsilon}\langle \frac{d}{ds}\alpha(s,\epsilon), \frac{d}{ds}\alpha(s,\epsilon)\rangle}{2\sqrt{\langle \frac{d}{ds}\alpha(s,\epsilon), \frac{d}{ds}\alpha(s,\epsilon)\rangle}}ds.$$

$$\begin{aligned}
& \frac{d}{d\epsilon}\langle \frac{d}{ds}\alpha(s,\epsilon), \frac{d}{ds}\alpha(s,\epsilon)\rangle \\
= \; & \frac{d}{d\epsilon}\langle \vec{X}_u(u(s)+\epsilon p(s), v(s)+\epsilon q(s))(u'+\epsilon p') + \vec{X}_v(u(s)+\epsilon p(s), v(s)+\epsilon q(s))(v'+\epsilon q'), \\
& \vec{X}_u(u(s)+\epsilon p(s), v(s)+\epsilon q(s))(u'+\epsilon p') + \vec{X}_v(u(s)+\epsilon p(s), v(s)+\epsilon q(s))(v'+\epsilon q')\rangle
\end{aligned}$$

$$\begin{aligned}
= \quad & 2\langle \vec{X}_{uu}(u+\epsilon p, v+\epsilon q)(u'+\epsilon p')p + \vec{X}_{uv}(u+\epsilon p, v+\epsilon q)(u'+\epsilon p')q \\
& + \vec{X}_{vu}(u+\epsilon p, v+\epsilon q)(v'+\epsilon q')p + +\vec{X}_{vv}(u+\epsilon p, v+\epsilon q)(v'+\epsilon q')q, \\
& \vec{X}_u(u+\epsilon p, v+\epsilon q)(u'+\epsilon p') + \vec{X}_v(u+\epsilon p, v+\epsilon q)(v'+\epsilon q')\rangle \\
& +2\langle \vec{X}_u(u+\epsilon p, v+\epsilon q)p' + \vec{X}_v(u+\epsilon p, v+\epsilon q)q', \\
& \vec{X}_u(u+\epsilon p, v+\epsilon q)(u'+\epsilon p') + \vec{X}_v(u+\epsilon p, v+\epsilon q)(v'+\epsilon q')\rangle.
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2}\frac{d}{d\epsilon}\langle \frac{d}{ds}\alpha(s,\epsilon), \frac{d}{ds}\alpha(s,\epsilon)\rangle|_{\epsilon=0} \\
= \quad & \langle \vec{X}_{uu}(u,v)u'p + \vec{X}_{uv}(u,v)u'q + \vec{X}_{vu}(u,v)v'p + +\vec{X}_{vv}(u,v)v'q, \vec{X}_u(u,v)u' + \vec{X}_v(u,v)v'\rangle \\
& +\langle \vec{X}_u(u,v)p' + \vec{X}_v(u,v)q', \vec{X}_u(u,v)u' + \vec{X}_v(u,v)v'\rangle \\
= \quad & \langle \vec{X}_{uu}, \vec{X}_u\rangle(u')^2p + \langle \vec{X}_{uv}, \vec{X}_u\rangle(u')^2q + \langle \vec{X}_{vu}, \vec{X}_u\rangle u'v'p + \langle \vec{X}_{vv}, \vec{X}_u\rangle u'v'q \\
& + \langle \vec{X}_{uu}, \vec{X}_v\rangle u'v'p + \langle \vec{X}_{uv}, \vec{X}_v\rangle u'v'q + \langle \vec{X}_{vu}, \vec{X}_v\rangle (v')^2p + \langle \vec{X}_{vv}, \vec{X}_v\rangle (v')^2q \\
& + Eu'p' + Fu'q' + Fv'p' + Gv'q' \\
= \quad & \langle \vec{X}_{uu}, \vec{X}_u\rangle(u')^2p + \langle \vec{X}_{vu}, \vec{X}_u\rangle u'v'p + \langle \vec{X}_{uu}, \vec{X}_v\rangle u'v'p + \langle \vec{X}_{vu}, \vec{X}_v\rangle (v')^2p \\
& + \langle \vec{X}_{uv}, \vec{X}_u\rangle(u')^2q + \langle \vec{X}_{vv}, \vec{X}_u\rangle u'v'q + \langle \vec{X}_{uv}, \vec{X}_v\rangle u'v'q + \langle \vec{X}_{vv}, \vec{X}_v\rangle (v')^2q \\
& + (Eu'p)' + (Fu'q)' + (Fv'p)' + (Gv'q)' \\
& - (E_u(u')^2 + E_vu'v' + Eu'')p - (F_u(u')^2 + F_vu'v' + Fu'')q \\
& - (F_uu'v' + F_v(v')^2 + Fv'')p - (G_uu'v' + G_v(v')^2 + Gv'')q
\end{aligned}$$

Recall that $E_u = 2\langle \vec{X}_{uu}, \vec{X}_u\rangle$, $2\langle \vec{X}_{uv}, \vec{X}_u\rangle = E_v$, $G_u = 2\langle \vec{X}_{vu}, \vec{X}_v\rangle$, $2\langle \vec{X}_{vv}, \vec{X}_v\rangle = G_v$, $F_u = \langle \vec{X}_{uu}, \vec{X}_v\rangle + \langle \vec{X}_u, \vec{X}_{vu}\rangle$ and $F_v = \langle \vec{X}_{uv}, \vec{X}_v\rangle + \langle \vec{X}_u, \vec{X}_{vv}\rangle$. We can have the simplications as follows

$$\begin{aligned}
& \langle \vec{X}_{uu}, \vec{X}_u\rangle(u')^2 + \langle \vec{X}_{vu}, \vec{X}_u\rangle u'v' + \langle \vec{X}_{uu}, \vec{X}_v\rangle u'v' + \langle \vec{X}_{vu}, \vec{X}_v\rangle (v')^2 \\
& - (E_u(u')^2 + E_vu'v' + Eu'') - (F_uu'v' + F_v(v')^2 + Fv'') \\
= \quad & -\langle \vec{X}_{uu}, \vec{X}_u\rangle(u')^2 - \langle \vec{X}_{vu}, \vec{X}_u\rangle u'v' - \langle \vec{X}_u, \vec{X}_{vu}\rangle u'v' - \langle \vec{X}_u, \vec{X}_{vv}\rangle (v')^2 - Eu'' - Fv'' \\
= \quad & -(E\Gamma^1_{11} + F\Gamma^2_{11})(u')^2 - 2(E\Gamma^1_{12} + F\Gamma^2_{12})u'v' - (E\Gamma^1_{22} + F\Gamma^2_{22})(v')^2 \\
& - Eu'' - Fv'',
\end{aligned}$$

$$\begin{aligned}
& \langle \vec{X}_{uv}, \vec{X}_u\rangle(u')^2 + \langle \vec{X}_{vv}, \vec{X}_u\rangle u'v' + \langle \vec{X}_{uv}, \vec{X}_v\rangle u'v' + \langle \vec{X}_{vv}, \vec{X}_v\rangle (v')^2 \\
& - (F_u(u')^2 + F_vu'v' + Fu'') - (G_uu'v' + G_v(v')^2 + Gv'') \\
= \quad & -\langle \vec{X}_{uu}, \vec{X}_v\rangle(u')^2 - \langle \vec{X}_{vu}, \vec{X}_v\rangle u'v' - \langle \vec{X}_v, \vec{X}_{vu}\rangle u'v' - \langle \vec{X}_v, \vec{X}_{vv}\rangle (v')^2 - Fu'' - Gv'' \\
= \quad & -(F\Gamma^1_{11} + G\Gamma^2_{11})(u')^2 - 2(F\Gamma^1_{12} + G\Gamma^2_{12})u'v' - (F\Gamma^1_{22} + G\Gamma^2_{22})(v')^2 \ - Fu'' - Gv''.
\end{aligned}$$

$$
\begin{aligned}
& \frac{d}{d\epsilon} L(\epsilon)|_{\epsilon=0} \\
= & \int_a^b \frac{\frac{d}{d\epsilon}\langle \frac{d}{ds}\alpha(s,\epsilon), \frac{d}{ds}\alpha(s,\epsilon)\rangle|_{\epsilon=0}}{2\sqrt{\langle \frac{d}{ds}\alpha(s,0), \frac{d}{ds}\alpha(s,0)\rangle}} ds \\
= & \frac{1}{2}\int_a^b \frac{d}{d\epsilon}\langle \frac{d}{ds}\alpha(s,\epsilon), \frac{d}{ds}\alpha(s,\epsilon)\rangle|_{\epsilon=0} ds
\end{aligned}
$$

since s is the arclength parameter and $\langle \frac{d}{ds}\alpha(s,0), \frac{d}{ds}\alpha(s,0)\rangle = \langle \alpha'(s), \alpha'(s)\rangle = 1$. Now we get

$$
\begin{aligned}
& \frac{d}{d\epsilon} L(\epsilon)|_{\epsilon=0} \\
= & \int_a^b \big((Eu'p)' + (Fu'q)' + (Fv'p)' + (Gv'q)'\big) ds \\
& - \int_a^b \big((E\Gamma_{11}^1 + F\Gamma_{11}^2)(u')^2 + 2(E\Gamma_{12}^1 + F\Gamma_{12}^2)u'v' + (E\Gamma_{22}^1 + F\Gamma_{22}^2)(v')^2 + Eu'' + Fv''\big) p ds \\
& - \int_a^b \big((F\Gamma_{11}^1 + G\Gamma_{11}^2)(u')^2 + 2(F\Gamma_{12}^1 + G\Gamma_{12}^2)u'v' + (F\Gamma_{22}^1 + G\Gamma_{22}^2)(v')^2 + Fu'' + Gv''\big) q ds \\
= & \big((Eu'p) + (Fu'q) + (Fv'p) + (Gv'q)\big)|_a^b \\
& - \int_a^b \big((E\Gamma_{11}^1 + F\Gamma_{11}^2)(u')^2 + 2(E\Gamma_{12}^1 + F\Gamma_{12}^2)u'v' + (E\Gamma_{22}^1 + F\Gamma_{22}^2)(v')^2 + Eu'' + Fv''\big) p ds \\
& - \int_a^b \big((F\Gamma_{11}^1 + G\Gamma_{11}^2)(u')^2 + 2(F\Gamma_{12}^1 + G\Gamma_{12}^2)u'v' + (F\Gamma_{22}^1 + G\Gamma_{22}^2)(v')^2 + Fu'' + Gv''\big) q ds \\
= & - \int_a^b \big((E\Gamma_{11}^1 + F\Gamma_{11}^2)(u')^2 + 2(E\Gamma_{12}^1 + F\Gamma_{12}^2)u'v' + (E\Gamma_{22}^1 + F\Gamma_{22}^2)(v')^2 + Eu'' + Fv''\big) p ds \\
& - \int_a^b \big((F\Gamma_{11}^1 + G\Gamma_{11}^2)(u')^2 + 2(F\Gamma_{12}^1 + G\Gamma_{12}^2)u'v' + (F\Gamma_{22}^1 + G\Gamma_{22}^2)(v')^2 + Fu'' + Gv''\big) q ds
\end{aligned}
$$

where we used $p(a) = p(b) = q(a) = q(b) = 0$. Therefore $\frac{d}{d\epsilon} L(\epsilon)|_{\epsilon=0} = 0$ implies that

$$
\begin{aligned}
& \int_a^b (E\Gamma_{11}^1 + F\Gamma_{11}^2)(u')^2 + 2(E\Gamma_{12}^1 + F\Gamma_{12}^2)u'v' + (E\Gamma_{22}^1 + F\Gamma_{22}^2)(v')^2 + Eu'' + Fv''\big) p ds \\
& + \int_a^b \big((F\Gamma_{11}^1 + G\Gamma_{11}^2)(u')^2 + 2(F\Gamma_{12}^1 + G\Gamma_{12}^2)u'v' + (F\Gamma_{22}^1 + G\Gamma_{22}^2)(v')^2 + Fu'' + Gv''\big) q ds \\
= & \ 0
\end{aligned}
$$

for all functions p and q. Therefore we must have

$$
(E\Gamma_{11}^1 + F\Gamma_{11}^2)(u')^2 + 2(E\Gamma_{12}^1 + F\Gamma_{12}^2)u'v' + (E\Gamma_{22}^1 + F\Gamma_{22}^2)(v')^2 + Eu'' + Fv'' = 0,
$$

$$(F\Gamma^1_{11} + G\Gamma^2_{11})(u')^2 + 2(F\Gamma^1_{12} + G\Gamma^2_{12})u'v' + (F\Gamma^1_{22} + G\Gamma^2_{22})(v')^2 + Fu'' + Gv'' = 0.$$

We can write the two equations as an equation of matrix

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\left(\begin{pmatrix} u'' \\ v'' \end{pmatrix} + \begin{pmatrix} \Gamma^1_{11} \\ \Gamma^2_{11} \end{pmatrix}(u')^2 + 2\begin{pmatrix} \Gamma^1_{12} \\ \Gamma^2_{12} \end{pmatrix}u'v' + \begin{pmatrix} \Gamma^1_{22} \\ \Gamma^2_{22} \end{pmatrix}(v')^2\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence we get the equations for the geodesic

$$u'' + \Gamma^1_{11}(u')^2 + 2\Gamma^1_{12}u'v' + \Gamma^1_{22}(v')^2 = 0,$$
$$v'' + \Gamma^2_{11}(u')^2 + 2\Gamma^2_{12}u'v' + \Gamma^2_{22}(v')^2 = 0.$$

Let's consider an example. Take a curve $(x,z) = \big(f(v), g(v)\big)$, $a < v < b$ and $f(v) > 0$ in the xz-plane and rotate the curve about the z-axis. We get a surface of revolution S. We can choose $\vec{X}(u,v) = \big(f(v)cosu, f(v)sinu, g(v)\big)$ as the parameterization of S.

$$\vec{X}_u = \big(-f(v)sinu, f(v)cosu, 0\big), \quad \vec{X}_v = \big(f'(v)cosu, f'(v)sinu, g'(v)\big).$$
$$\vec{X}_{uu} = \big(-f(v)cosu, -f(v)sinu, 0\big), \quad \vec{X}_{uv} = \big(-f'(v)sinu, f'(v)cosu, 0\big),$$
$$\vec{X}_{vv} = \big(f''(v)cosu, f''(v)sinu, g''(v)\big).$$
$$E = f^2(v), \quad F = 0, \quad G = f'(v)^2 + g'(v)^2.$$
$$\langle \vec{X}_{uu}, \vec{X}_u\rangle = 0, \quad \langle \vec{X}_{uu}, \vec{X}_v\rangle = -ff', \quad \langle \vec{X}_{vu}, \vec{X}_u\rangle = ff',$$
$$\langle \vec{X}_{vu}, \vec{X}_v\rangle = 0, \quad \langle \vec{X}_{vv}, \vec{X}_u\rangle = 0, \quad \langle \vec{X}_{vv}, \vec{X}_v\rangle = f'f'' + g'g''.$$
$$\Gamma^1_{11} = 0, \quad \Gamma^2_{11} = \frac{-ff'}{(f')^2 + (g')^2}, \quad \Gamma^2_{12} = 0,$$
$$\Gamma^1_{12} = \frac{ff'}{f^2} = \frac{f'}{f}, \quad \Gamma^1_{22} = 0, \quad \Gamma^2_{22} = \frac{f'f'' + g'g''}{(f')^2 + (g')^2}.$$

Hence the geodesic equations are

$$u'' + 2f'u'v' = 0, \quad v'' - \frac{-ff'}{(f')^2 + (g')^2}(u')^2 + \frac{f'f'' + g'g''}{(f')^2 + (g')^2}(v')^2 = 0.$$

Here $u' = \dfrac{du(s)}{ds}$, $v' = \dfrac{dv(s)}{ds}$, $f' = \dfrac{df(v)}{dv}$ and $g' = \dfrac{dg(v)}{dv}$.

For the cylinder $x^2 + y^2 = 1$, it is a surface of revolution by rotating the curve $(x,z) = (1,v)$ around the z-axis. Hence the geodesic equations are

$$u'' = 0, \quad v'' = 0.$$

The solution for the system of the differential equations is $u(s) = u_1 + u_2 s$ and $v(s) = v_1 + v_2 s$, i.e., $(u,v) = (u_1, v_1) + s(u_2, v_2)$, a straight line. For the unit sphere $x^2 + y^2 + z^2 = 1$, it is a surface of revolution by rotating the curve $(x,z) = (sin\varphi, cos\varphi)$ around the z=axis. If we use the spherical coordinates to parameterize the sphere, i.e., $\vec{X}(\theta,\varphi) = (cos\theta sin\varphi, sin\theta sin\varphi, cos\varphi)$, the geodesic equations are

$$\theta'' + 2\frac{cos\varphi}{sin\varphi}\varphi'\theta' = 0, \quad \varphi'' - sin\varphi cos\varphi(\theta')^2 = 0.$$

Clearly the curve $(\theta,\varphi) = (\theta_0, s)$ satisfies the differential equations where θ_0 is a constant angle. The curve $\vec{X}(\theta_0, s)$ is the big circle starting from the north pole $(0,0,1)$ to the south pole $(0,0,-1)$.

4.6 MINIMAL SURFACES

Let's consider the surface analogue of geodesics, i.e., under what condition a surface has minimal area among certain variations of the surface.

Let $\vec{X}: U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular parameterized surface. Choose a bounded domain $D \subset U$ such that the closure $\overline{D}$ of D is contained in U. Let $h: \overline{D} \to \mathbb{R}^1$ be a differentiable function. The normal variation of $\vec{X}(\overline{D})$, determined by h, is the map given by

$$\varphi: \overline{D} \times (-\epsilon, \epsilon) \to \mathbb{R}^3,$$

$$\varphi(u, v, t) = \vec{X}(u, v) + t \cdot h(u, v)\vec{N}(u, v)$$

where $\vec{N}(u, v)$ is the unit normal vector of the surface $\vec{X}(\overline{D})$. For fixed t, $\varphi(u, v, t)(D)$ is a surface.

$$\varphi(u, v, t)_u = \vec{X}_u + t \cdot h_u \vec{N} + t \cdot h\vec{N}_u, \quad \varphi(u, v, t)_v = \vec{X}_v + t \cdot h_v \vec{N} + t \cdot h\vec{N}_v.$$

$$\begin{aligned}
E^t &= \langle \varphi(u, v, t)_u, \varphi(u, v, t)_u \rangle \\
&= E + th(\langle \vec{X}_u, \vec{N}_u \rangle + \langle \vec{X}_u, \vec{N}_u \rangle) + t^2 h^2 \langle \vec{N}_u, \vec{N}_u \rangle + t^2 h_u h_u \\
&= E - 2the + t^2 R_1, \\
F^t &= \langle \varphi(u, v, t)_u, \varphi(u, v, t)_v \rangle \\
&= F + th(\langle \vec{X}_u, \vec{N}_v \rangle + \langle \vec{X}_v, \vec{N}_u \rangle) + t^2 h^2 \langle \vec{N}_u, \vec{N}_v \rangle + t^2 h_u h_v \\
&= F - 2thf + t^2 R_2, \\
G^t &= \langle \varphi(u, v, t)_v, \varphi(u, v, t)_v \rangle \\
&= E + th(\langle \vec{X}_v, \vec{N}_v \rangle + \langle \vec{X}_v, \vec{N}_v \rangle) + t^2 h^2 \langle \vec{N}_v, \vec{N}_v \rangle + t^2 h_v h_v \\
&= E - 2thg + t^2 R_3.
\end{aligned}$$

$$E^t G^t - (F^t)^2 = EG - F^2 - 2t \cdot h(Eg - 2Ff + Ge) + t^2 R = (EG - F^2)(1 - 4thH) + t^2 R$$

where R is a function defined in $\overline{D}$ and $H = \dfrac{Eg - 2fF + Ge}{2(EG - F^2)}$ is the mean curvature of the surface $\vec{X}(U)$.

$E^t G^t - (F^t)^2$ will be positive in $\overline{D}$ when t is very small. Hence, for fixed t small enough, $\varphi(u, v, t)$ is a parameterization of a regular surface $\varphi(u, v, t)(D)$. The area $A(t)$ of the surface $\varphi(u, v, t)(D)$ is given by

$$A(t) = \int_{\overline{D}} \sqrt{E^t G^t - (F^t)^2} dudv = \int_{\overline{D}} \sqrt{EG - F^2} \sqrt{1 - 4thH + t^2 \frac{R}{EG - F^2}} dudv.$$

Hence $A'(0) = -\int_{\overline{D}} 2hH\sqrt{EG - F^2} dudv$. If the surface $\vec{X}(D)$ has the minimal area among all the normal variations, i.e., $h(u, v)$ is arbitrary, then $A'(0)$ must be zero for all $h(u, v)$. If we take $h = H$, we get $\int_{\overline{D}} 2H^2\sqrt{EG - F^2} dudv = 0$. Since $EG - F^2 > 0$, we must have $H(u, v) = 0$ for all $(u, v) \in D$. A regular surface whose mean curvature is zero everywhere is called a *minimal surface.*

The examples of minimal surfaces include the Helicoid given by the the parameterization $\vec{X}(u, v) = (asinhvcosu, asinhvsinu, au)$ and the Enneper's surface given by the parameterization$\vec{X}(u, v) = \left(u - \dfrac{u^3}{3} + uv^2, v - \dfrac{v^3}{3} + vu^2, u^2 - v^2\right)$.

4.7 GAUSS-BONNET THEOREM

Let S be a regular surface. A geodesic triangle T on S is a region on S bounded by three regular curves which are geodesics. Let α_1, α_2 and α_3 be three sides of the triangle and φ_1 be the interior angle between the edges α_2 and α_3, φ_2 be the interior angle between the edges α_3 and α_1, and φ_3 be the interior angle between the edges α_1 and α_2 respectively. Then we have the Gauss Bonnet formula (local version)

$$\varphi_1 + \varphi_2 + \varphi_3 = \pi + \int\int_T K d\sigma$$

where K is the Gaussian curvature and $\int\int_T K d\sigma$ is the surface integral of the function K over the region T on the surface S.

From the Gauss-Bonett formula, we can conclude that the sum of interior angles of a geodesic triangle T is

1. equal to π if $K = 0$ in T.

2. greater than π if $K > 0$ in T.

3. smaller than π if $K < 0$ in T.

If the surface S is a plane or cylinder given by $x^2 + y^2 = 1$, then the Gaussian curvature is zero in both cases. Hence, for any geodesic triangle on such S, the sum of interior angles is π.

If the surface S is the unit sphere $x^2 + y^2 + z^2 = 1$, the Gaussian curvature is 1 everywhere. Hence, for any geodesic triangle on such S, the sum of interior angles is greater than π.

If the surface S is the pseudosphere (see exercise 6 on page 168 of the textbook), the Gaussian curvature is -1 everywhere. Hence, for any geodesic triangle on such S, the sum of interior angles is smaller than π.

The latter two cases provide models for non-Euclidean geometry.

We will study and prove the local version of Gauss-Bonnet formula.

Definition 4.33. Let $\alpha\colon [0, \ell] \to S$ be a continuous map to a regular surface S. We say α is a *simple, closed, and piecewise regular paremetrized curve* if

1. $\alpha(0) = \alpha(\ell)$;

2. For any $t_1 \neq t_2$, $\alpha(t_1) \neq \alpha(t_2)$;

3. There exists a subdivision $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = \ell$ such that α is differentiable and regular in each $[t_i, t_{i+1}]$ for $i = 0, \ldots, n$.

The points $\alpha(t_i$ for $i = 0, \ldots, n$ are called the vertices of α, and the traces $\alpha([t_i, t_{i+1}]$ are called the regular arcs of α.

For each vertex $\alpha(t_i)$, the following one-sided limits exist by the regularity of α:

$$\alpha'(t_i - 0) := \lim_{t \to t_i^-} \alpha'(t) \neq 0, \quad \alpha'(t_i + 0) := \lim_{t \to t_i^+} \alpha'(t) \neq 0.$$

Assume that the surface is an oriented surface with $\vec{N}$ as its unit normal vector field giving the orientation. Let θ_i be the angle between vectors $\alpha'(t_i - 0)$ and $\alpha'(t_i + 0)$ such that $0 < |\theta_i| \leq \pi$. If

$|\theta_i| = \pi$, the vertex is a called a cusp. If θ_i is not a cusp, the sign of θ_i is determined by right hand rule of three linearly independent vectors $\alpha'(t_i - 0)$, $\alpha'(t_i + 0)$, $\vec{N}(\alpha(t_i))$, i.e. the sign is positive if the cross product $\alpha'(t_i - 0) \times \alpha'(t_i + 0)$ is a positive multiple of $\vec{N}(\alpha(t_i))$, and negative otherwise. The signed angle $-\pi < \theta_i < \pi$ is called the external angle of the vertex $\alpha(t_i)$. For simplicity, let's assume that the curve α has no cusps.

Let $\vec{X}: U \subset \mathbb{R}^2 \to S$ be a parametrization compatible with the orientation of S. Assume that U is homeomorphic to an open disk. Let $\varphi_i: [t_i, t_{i+1}] \to \mathbb{R}$ be differentiable functions which the positive angel from $\vec{X}_u$ to $\alpha'(t)$ at the point $\alpha(t)$.

Theorem 4.34.(Turning Tangents)

$$\sum_{i=0}^{n} \big(\varphi_i(t_{i+1}) - \varphi_i(t_i)\big) + \sum_{i=0}^{n} \theta_i = \pm 2\pi.$$

Here the sign depends on the orientation of α.

A region R on the oriented surface S is called a *simple region* if R is homeomorphic to a disk and the boundary ∂R of R is the trace of a simple, closed, piecewise regular parametrized curve $\alpha: I \to S$. α is *positively oriented* if one walks on the curve in the positive direction and with one's head pointing in the direction of $\vec{N}$, then the region is on the left.

Gauss-Bonnet Theorem (Local Version) Let $\vec{X}: U \to S$ be an orthogonal parametrization of an oriented surface S, where $U \subset \mathbb{R}^2$ is homeomorphic to an opne disk and $\vec{X}_u \times \vec{X}_v$ is the orientation of S. Let $R \subset \vec{X}(U)$ be a simple region of S, and let $\alpha: I \to S$ be a curve such that $\partial R = \alpha(I)$. Assume that α is positively oriented, parametrized by arc length s, let $\alpha(s_0), \ldots, \alpha(s_n)$ be the vertices, and let $\theta_0, \ldots, \theta_n$ be the external angles. Then

$$\sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} \kappa_g(s) ds + \int_R K dS + \sum_{i=0}^{n} \theta_i = 2\pi.$$

Here $\kappa_g(s)$ is the geodesic curvature of the regular arcs of α, K is the Gaussian curvature of S, and $\int_R K dS$ is the surface integral.

Proof: Recall a basic fact on surface integrals. Given a function on an oriented surface S paremetrized by $\vec{X}(u, v): U \to S$ with the normal vector $\vec{X}_u \times \vec{X}_v$ being the orientation. Let A be a region in U and $R = \vec{X}(A)$ be the image in S. Let f be a function on S, the surface integral

$$\int_R f dS = \int \int_A f(\vec{X}(u, v)) \sqrt{EG - F^2} du dv.$$

Let $\alpha(s) = \vec{X}(u(s), v(s))$. Then we knew

$$\kappa_g(s) = \frac{1}{2\sqrt{EG}} \left(G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right) + \frac{d\varphi_i}{ds}.$$

Thus we have the integration

$$\sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} \kappa_g(s) ds = \sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} \frac{1}{2\sqrt{EG}} \left(G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right) ds + \sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} \frac{d\varphi_i}{ds} ds.$$

Note that α is a closed simple curve bounding the region R, let $R = \vec{X}(A)$ for a region A in U. Recall the Green theorem,

$$\sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} \left(P\frac{du}{ds} + Q\frac{dv}{ds} \right) ds = \int\int_A \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v} \right) dudv.$$

Applying the Green theorem to the first term on the right hand side, we obtain

$$\begin{aligned} & \sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} \frac{1}{2\sqrt{EG}} \left(G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right) ds = \int\int_A \left(\left(\frac{E_v}{2\sqrt{EG}} \right)_v + \left(\frac{G_u}{2\sqrt{EG}} \right)_u \right) dudv \\ = \quad & -\int\int_A K\sqrt{EG} dudv = -\int_R K dS, \end{aligned}$$

where the first equality follows from the formula in an exercise in the textbook.

The second term on the right hand side is

$$\sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} \frac{d\varphi_i}{ds} ds = \sum_{i=0}^{n} \big(\varphi_i(s_{i+1}) - \varphi_i(s_i)\big) = 2\pi - \sum_{i=0}^{n} \theta_i,$$

where the last equality follows from the theorem of turning tangents. The sign in front of 2π is positive since α is positively oriented.

∎