Solution for 2.

"$\Rightarrow$" If $\alpha(s)$ lies on a plane, its torsion $\tau(s)=0$. Thus

$$\vec{b}'(s) = \tau(s)\vec{n}(s) = 0.$$

Hence $\vec{b}(s)$ is a constant, say $\vec{b}(s) = \vec{b}_0$.

$\vec{b}(s)\cdot\alpha'(s) = \vec{b}(s)\cdot\vec{t}(s) = 0$, $\quad \vec{b}_0\cdot\alpha'(s)=0$. $(\vec{b}_0\cdot\alpha(s))' = 0$,

hence $\vec{b}_0\cdot\alpha(s)$ is a constant. We can write this constant as $\vec{b}_0\cdot\vec{OP}$ for some point P.

Then $\vec{b}_0\cdot\alpha(s) = \vec{b}_0\cdot\vec{OP}$, $\quad \vec{b}_0\cdot(\alpha(s)-\vec{OP}) = 0$.

Hence ~~the~~ every osculating plane given by the equation $\vec{b}_0\cdot((x,y,z)-\alpha(s)) = 0$ contains the point P.

"$\Leftarrow$" If ~~the~~ the osculating plane contains a point P,

then $\vec{b}(s)\cdot(\alpha(s)-\vec{OP}) = 0$. ①

$$\vec{b}'(s)\cdot(\alpha(s)-\vec{OP}) + \underbrace{\vec{b}(s)\cdot\alpha'(s)}_{=0} = 0, \Rightarrow \vec{b}'(s)\cdot(\alpha(s)-\vec{OP}) = 0$$

i.e. $\tau(s)\cdot\vec{n}(s)\cdot(\alpha(s)-\vec{OP}) = 0$.

If $\tau(s)$ ~~is nowhere zero~~ then $\vec{n}(s)\cdot(\alpha(s)-\vec{OP}) = 0$. ②

[is not zero, then it is nowhere zero in an open interval]

$$\vec{n}'(s)\cdot(\alpha(s)-\vec{OP}) + \underbrace{\vec{n}(s)\cdot\alpha'(s)}_{=0} = 0 \Rightarrow \vec{n}'(s)\cdot(\alpha(s)-\vec{OP}) = 0.$$

$$\Rightarrow \left[-k\vec{t}(s) - \tau(s)\vec{b}(s)\right]\cdot(\alpha(s)-\vec{OP}) = 0 \Rightarrow -k(s)\vec{t}(s)\cdot(\alpha(s)-\vec{OP}) = 0$$

Since $k(s)$ is nowhere zero, $\vec{t}(s)\cdot(\alpha(s)-\vec{OP}) = 0$ ③

Since $\{\vec{t}, \vec{n}, \vec{b}\}$ is a basis of $\mathbb{R}^3$, from ①, ② and ③

$\alpha(s)-\vec{OP} = 0$ for all s, i.e. the curve is a point P, a contradiction. Hence $\tau(s)=0$ for all s, i.e. $\alpha(s)$ is a plane curve.

Solution for 3. i) $f(x,y,z) = x^2 - xz + z^2$

$\text{grad} f = (2x - z,\ 0,\ -x + 2z) = (0, 0, 0)$

$$\begin{cases} 2x - z = 0 \\ -x + 2z = 0 \end{cases} \Rightarrow (x, y, z) = (0, y, 0).$$

The critical values are $f(0, y, 0) = 0$.

1 is not a critical value, hence it is a regular value.

Thus $x^2 - xz + z^2 = 1$ is a regular surface.

(ii). Consider the vector $\vec{v} = (0, 1, 0)$

grad f is a normal vector of the surface,

$\text{grad} f \cdot \vec{v} = (2x - z,\ 0,\ -x + 2z) \cdot (0, 1, 0) = 0$

Hence $\text{grad} f \perp \vec{v}$, i.e., $\vec{v}$ is a tangent vector of the surface at any point of the surface

$\Rightarrow$ every tangent plane is parallel to the vector $\vec{v}$.