MATH4023 Complex Analysis
Mid-term Test
(March 30, 2022; 19:30 – 21:30)

Name: ____

Student ID Number: ____

Directions:

- This is an open-book examination. You may refer to any notes, textbooks or references offline, but notations from our lecture note should be used. You may use a calculator if needed, but all numerical answers should be given in the **exact form**.
- **DO NOT communicate with any person other than your course instructor by any means during the exam.** Physical, electronic, and internet communication with any other person are all prohibited.
- **Please write your solutions to all problems on your own pieces of paper. Write your name, student ID and signature on each page of your work.**
- You are advised to try the problems you feel more confident first, and not to spend too much time on one single problem in case you get stuck. Some problems can be tricky so be careful.
- Please submit (i) this cover page and (ii) your solutions to all problems via Canvas **NO LATER THAN 21:45**. Late submission will result in score deduction.
- Cheating is a serious violation of the HKUST Academic Code. Students caught cheating will receive a zero score for the examination, and will also be subjected to further penalties imposed by the University.

Please read the following statement and sign below it.

I have neither given nor received any unauthorized aid during this examination. The answers submitted are my own work.

I understand that sanctions will be imposed, if I am found to have violated the University's regulations governing academic integrity.

Student's Signature:

____

Question No.	**Score**	**Out of**
Q1 – 5		13
Q6		4
Q7		4
Q8		4
Q9		6
Q10		7
Q11		7
Q12		7
Q13		8
Total score		**60**

Part A: Brief responses (13 points)

Write down **brief answers** to each of the following questions Q1 – Q5. No explanation or computational steps are required, except for Q2.

1. Consider the following subsets of $\mathbb{C}$:

$$\begin{aligned} A &= \text{the set of all roots of the polynomial } z^{4023} + z^{2022} + 1, \\ B &= \text{the set of all solutions to the equation } \sin z = 2, \\ C &= \{z \in \mathbb{C} : |z| > 10\}, \\ D &= \{z \in \mathbb{C} : |z| \geq 10 \text{ and } \operatorname{Im} z > 0\}, \\ E &= \{z \in \mathbb{C} : |z| = 10\}. \end{aligned}$$

   (a) Which of them is/are **open** subset(s) of $\mathbb{C}$?
   (b) Which of them is/are **closed** subset(s) of $\mathbb{C}$?
   (c) Which of them is/are **compact** subset(s) of $\mathbb{C}$?
   (d) Which of them is/are **connected** subset(s) of $\mathbb{C}$?

(4 points)

2. Consider the following regions in $\mathbb{C}$:

$$\begin{aligned} U &= D(0; 2), \\ V &= U \setminus \overline{D(0; 1)}, \\ W &= V \cap \{z \in \mathbb{C} : \operatorname{Re} z < 0\}. \end{aligned}$$

   On which of these regions U, V and W does there exist a holomorphic branch of complex logarithm? Also explain briefly
   (i) **why there is** such a branch on each of your choice(s) and
   (ii) **why there is no** such branch on each of the other(s).

(3 points)

3. Let $f: \mathbb{C} \to \mathbb{C}$ be the function

$$f(z) = e^z.$$

   What is the **direct image** of the set

$$S = \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1 \text{ and } 2 \leq \operatorname{Im} z \leq 3\}$$

   via f? Write down your answer in an appropriate set notation as in the following box:

$$\boxed{f(S) = \{w \in \mathbb{C} : \qquad\qquad\qquad\qquad \}}$$

   Make sure that your set notation is precise and unambiguous.

(2 points)

4. Let $\sum_{k=0}^{+\infty} a_k z^k$ be a power series with radius of convergence $R > 0$. What is the **radius of convergence** of the power series

$$\sum_{k=0}^{+\infty} (|a_{3k}| + |a_{3k+1}|) z^{4k}$$

in terms of R?

(2 points)

5. Evaluate the line integral

$$\oint_{\partial D(2;1)} \frac{\cos z}{z(z-2)} dz\,.$$

(2 points)

Part B: Short problems (47 points)

In Q6 – Q13, **show all your work in detail** and make clear definition to any symbol you use whenever it is not given in the context of the problem and not defined in class. You may apply without proof **any result from Chapter 1 – Chapter 3** of our lecture note.

6. Let $z \in \mathbb{C} \setminus \{i, -i\}$. Show that $\frac{z}{1+z^2}$ is a real number **if and only if** either

$$\operatorname{Im} z = 0 \qquad \text{or} \qquad |z| = 1.$$

(4 points)

7. Let z be a root of the polynomial $p(z) = (z+1)^n + z^n$. Show that

$$\operatorname{Re} z = -\frac{1}{2}.$$

(4 points)

8. Let $b \in \mathbb{C}$ and let $f: \mathbb{C} \to \mathbb{C}$ be defined as

$$f(z) = \begin{cases} \dfrac{\operatorname{Im} z}{z} & \text{if } z \neq 0 \\ b & \text{if } z = 0 \end{cases}.$$

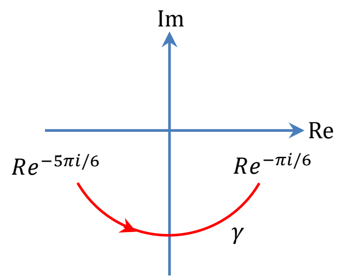
Determine all possible value(s) of b such that f is continuous on $\mathbb{C}$.

(4 points)

9. For each $R > 1$, let γ be the portion of $\partial D(0;R)$ joining $Re^{-\frac{5\pi i}{6}}$ to $Re^{-\frac{\pi i}{6}}$, oriented counterclockwise about 0 (see the diagram). Show that

$$\lim_{R\to+\infty} \int_\gamma \frac{1}{1+e^{iz}}\,dz = 0.$$

(6 points)



10. Let $U \subseteq \mathbb{C}$ be a region and $f: U \to \mathbb{C}$ be a holomorphic function such that for **each** $z \in U$, either $\operatorname{Re} f(z) = 1$ or $\operatorname{Im} f(z) = 1$. Show that f is a constant function.

(7 points)

11. Let $v: \mathbb{R}^2 \to \mathbb{R}$ be the function

$$v(x,y) = y \sin x \cosh y + x \cos x \sinh y.$$

Remark: Just in case you forgot, the hyperbolic functions are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{and} \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

(a) Show that v is harmonic on $\mathbb{R}^2$.

(b) Find an entire function $f: \mathbb{C} \to \mathbb{C}$ such that

$$\operatorname{Im} f(x+iy) = v(x,y)$$

for every $(x,y) \in \mathbb{R}^2$. Express $f(z)$ in terms of the complex variable z only.

(7 points)

12. Evaluate the line integral

$$\oint_{\partial D(0;2)} \frac{e^{z^2+|z|^2}}{z^2 - z}\,dz\,.$$

Show all your work with explanations in full detail.

(7 points)

13. Let $f(z) = \sum_{k=0}^{+\infty} a_k z^k$ be a power series such that

$$f\left(e^{\frac{2\pi i}{4023}} \frac{1}{n}\right) = e^{\frac{2\pi i}{4023}} f\left(\frac{1}{n}\right)$$

for every $n \in \mathbb{N}$. Show that $a_k = 0$ whenever $k-1$ is not divisible by 4023.

(8 points)

END OF PAPER