Chapter 3 Line integrals

1. Line integrals

Definition 3.1 Let $[a, b]$ be a bounded closed interval in $\mathbb{R}$ and let $g: [a, b] \to \mathbb{C}$ be a bounded function (i.e. there exists $M > 0$ such that $|g(t)| < M$ for every $t \in [a, b]$). The ***integral*** of g over $[a, b]$ is defined by

$$\int_a^b g(t)dt := \int_a^b \operatorname{Re} g(t)\, dt + i \int_a^b \operatorname{Im} g(t)\, dt$$

if the Riemann integrals on the right-hand side both exist. In this case we also say that g is ***Riemann integrable*** (on $[a, b]$).

Example 3.2 Evaluate the integral

$$\int_0^{\frac{\pi}{4}} e^{2it} dt.$$

Solution:

$$\int_0^{\frac{\pi}{4}} e^{2it} dt = \int_0^{\frac{\pi}{4}} (\cos 2t + i \sin 2t) dt = \int_0^{\frac{\pi}{4}} \cos 2t\, dt + i \int_0^{\frac{\pi}{4}} \sin 2t\, dt$$
$$= \left[\frac{1}{2} \sin 2t\right]_0^{\frac{\pi}{4}} + i \left[-\frac{1}{2} \cos 2t\right]_0^{\frac{\pi}{4}} = \frac{1}{2}(1 + i).$$

Lemma 3.3 (Triangle inequality) *Let $[a, b]$ be a bounded closed interval in $\mathbb{R}$ and $g: [a, b] \to \mathbb{C}$ be a function. If g is Riemann integrable on $[a, b]$, then $|g|$ is also Riemann integrable on $[a, b]$ and we have*

$$\left|\int_a^b g(t)dt\right| \leq \int_a^b |g(t)|dt.$$

Proof. If g is Riemann integrable on $[a, b]$, then by definition $\operatorname{Re} g$ and $\operatorname{Im} g$ are also Riemann integrable on $[a, b]$, and so $|g| = \sqrt{(\operatorname{Re} g)^2 + (\operatorname{Im} g)^2}$ is also Riemann integrable on $[a, b]$. Now since $\int_a^b g(t)dt$ is a complex number, there exists $\theta \in \mathbb{R}$ (i.e. an "argument" of $\int_a^b g(t)dt$) such that $\int_a^b g(t)dt = e^{i\theta} \left|\int_a^b g(t)dt\right|$. Thus,

$$\left|\int_a^b g(t)dt\right| = e^{-i\theta} \int_a^b g(t)dt = \int_a^b \operatorname{Re}\left(e^{-i\theta} g(t)\right) dt \leq \int_a^b \left|e^{-i\theta} g(t)\right| dt = \int_a^b |g(t)|dt.$$

■

Definition 3.4 Let $[a, b]$ be an interval in $\mathbb{R}$ and let $U \subseteq \mathbb{C}$.

- A continuous function $\gamma: [a, b] \to U$ is called a ***curve*** in U.
- A curve $\gamma: [a, b] \to U$ is called a ***C^1 curve*** in U if its derivative γ' is continuous on $[a, b]$ and $\gamma'(t) \neq 0$ for any $t \in [a, b]$. (We only take $\gamma'_+(a)$ and $\gamma'_-(b)$ at the end points a and b.)

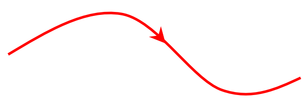


Image of a C^1 curve

Definition 3.5 (Line integral along C^1 curve) Let $U \subseteq \mathbb{C}$, let $\gamma: [a, b] \to U$ be a C^1 curve and let $f: U \to \mathbb{C}$ be a continuous function. The ***line integral*** (or ***path integral***, or ***contour integral***) of f along γ is defined by

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt.$$

$$\boxed{\begin{aligned} z &= \gamma(t) \\ dz &= \gamma'(t)dt \end{aligned}}$$

(In fact the line integral of a function can be defined not only along C^1 curves, but also along some curves that are "less nice". Interested readers may refer to Supplementary Note D for details.)

Lemma 3.6 *Let $U \subseteq \mathbb{C}$, let $f, g: U \to \mathbb{C}$ be continuous functions, let $c \in \mathbb{C}$ and let $\gamma: [a, b] \to U$ be a C^1 curve. Then*

(i) $\int_\gamma (f + g)(z)dz = \int_\gamma f(z)dz + \int_\gamma g(z)dz;$

(ii) $\int_\gamma (cf)(z)dz = c\int_\gamma f(z)dz.$

Proof. Omitted. ∎

Example 3.7 Let $\gamma: [0, 2\pi] \to \mathbb{C}$ be the curve with parametrization $\gamma(t) = e^{it}$. Evaluate

$$\int_\gamma \bar{z}\, dz.$$

Solution: We have $\gamma'(t) = ie^{it}$ for every $t \in (0, 2\pi)$, so

$$\int_\gamma \bar{z}\, dz = \int_0^{2\pi} \underbrace{\overline{e^{it}}}_{\bar{z}}\, \underbrace{ie^{it}\, dt}_{dz} = \int_0^{2\pi} e^{-it}\, ie^{it}\, dt = \int_0^{2\pi} i\, dt = 2\pi i.$$

We will see in the following two lemmas that in order to specify a line integral along a C^1 curve, it is enough to describe just the **image** (possibly with "overlapping multiplicity") and the **orientation** of the curve. In other words, **line integrals are independent of reparametrizations**.

Definition 3.8 Let $U \subseteq \mathbb{C}$, let $\gamma: [a, b] \to U$ be a C^1 curve and let $g: [g^{-1}(a), g^{-1}(b)] \to [a, b]$ be a <u>differentiable</u> and <u>strictly increasing</u> function. Then $\gamma \circ g: [g^{-1}(a), g^{-1}(b)] \to U$ is called a ***reparametrization*** of the curve γ.

Lemma 3.9 *Let $U \subseteq \mathbb{C}$, let $f: U \to \mathbb{C}$ be a continuous function and let $\gamma: [a, b] \to U$ be a C^1 curve. Then the line integral of f along γ does not depend on the parametrization of γ. In other words, if $g: [g^{-1}(a), g^{-1}(b)] \to [a, b]$ is differentiable and strictly increasing, then*

$$\int_{\gamma \circ g} f(z)dz = \int_{\gamma} f(z)dz.$$

Proof. We make the substitution $u = g(t)$. Then

$$\int_{\gamma \circ g} f(z)dz = \int_{g^{-1}(a)}^{g^{-1}(b)} f\big((\gamma \circ g)(t)\big)(\gamma \circ g)'(t)dt = \int_{g^{-1}(a)}^{g^{-1}(b)} f\Big(\gamma\big(g(t)\big)\Big)\gamma'\big(g(t)\big)g'(t)dt$$
$$= \int_a^b f\big(\gamma(u)\big)\gamma'(u)du = \int_{\gamma} f(z)dz.$$ ■

Definition 3.10 Let $U \subseteq \mathbb{C}$ and let $\gamma: [a, b] \to U$ be a curve in U. The ***reversal*** of γ is the curve $-\gamma: [-b, -a] \to U$ defined by

$$(-\gamma)(t) \coloneqq \gamma(-t)$$

for every $t \in [-b, -a]$.

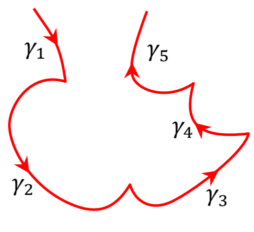
Lemma 3.11 *Let $U \subseteq \mathbb{C}$, let $f: U \to \mathbb{C}$ be a continuous function and let $\gamma: [a, b] \to U$ be a C^1 curve in U. Then*

$$\int_{-\gamma} f(z)dz = -\int_{\gamma} f(z)dz.$$

Proof. Omitted. We make the substitution $u = -t$. ■

Definition 3.12 Let $U \subseteq \mathbb{C}$, let $\gamma_1: [a, b] \to U$ and $\gamma_2: [c, d] \to U$ be curves in U such that the terminal point of γ_1 is the same as the initial point of γ_2, i.e. $\gamma_1(b) = \gamma_2(c)$ (joining end-to-end). The ***concatenation*** of γ_1 and γ_2, denoted as $\gamma_1 * \gamma_2$, is the curve in U obtained by going along γ_1 followed by γ_2, i.e. $\gamma = \gamma_1 * \gamma_2: [a, b - c + d] \to U$ is defined by

$$\gamma(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [a, b] \\ \gamma_2(t - b + c) & \text{if } t \in [b, b - c + d] \end{cases}.$$



A curve γ is called a ***piecewise C^1 curve*** if it is a concatenation of finitely many C^1 curves $\gamma_1, \gamma_2, \dots, \gamma_N$ joining end-to-end, i.e. $\gamma = \gamma_1 * \cdots * \gamma_N$.

Definition 3.13 (Line integral along piecewise C^1 curve) Let $U \subseteq \mathbb{C}$, let $\gamma_1, \gamma_2, \dots, \gamma_N$ be C^1 curves in U joining end-to-end, let $\gamma = \gamma_1 * \gamma_2 * \cdots * \gamma_N$, and let $f: U \to \mathbb{C}$ be a continuous function. The ***line integral*** of f along γ is defined by

$$\int_{\gamma} f(z)dz \coloneqq \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \cdots + \int_{\gamma_N} f(z)dz.$$

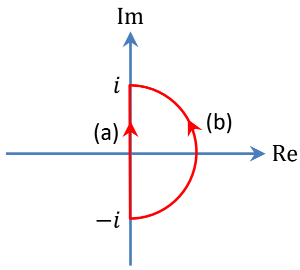
Remark 3.14 All the upcoming results concerning C^1 curves remain valid for piecewise C^1 curves, by modifying the proofs in the most trivial way. Therefore we will state and prove the results for C^1 curves only, and leave the cases for piecewise C^1 curves to the readers.

Example 3.15 For each of the following curves γ, evaluate the integral

$$\int_\gamma |z|dz.$$

(a) γ is a straight line segment joining $-i$ to i;

(b) γ is the semicircular arc of $\partial D(0;1)$ joining $-i$ to i, oriented counterclockwise.



Solution:

(a) γ has a parametrization $\gamma:[-1,1]\to\mathbb{C}$ defined by $\gamma(t)=it$. So $\gamma'(t)=i$, and

$$\int_\gamma |z|dz = \int_{-1}^{1}|it|idt = i\int_{-1}^{1}|t|dt = 2i\int_0^1 tdt = 2i\left[\frac{1}{2}t^2\right]_0^1 = i.$$

(b) γ has a parametrization $\gamma:[-\pi/2,\pi/2]\to\mathbb{C}$ defined by $\gamma(t)=e^{it}$. So $\gamma'(t)=ie^{it}$, and

$$\int_\gamma |z|dz = \int_{-\pi/2}^{\pi/2}|e^{it}|ie^{it}dt = \int_{-\pi/2}^{\pi/2} ie^{it}dt = \left[e^{it}\right]_{-\pi/2}^{\pi/2} = 2i.$$

Remark 3.16 Example 3.15 shows that the line integrals of a function along different curves joining the same pair of points are different in general. In the next section, we will identify some particular situations in which the line integral is always the same for every curve joining the same pair of points.

Definition 3.17 Let $U\subseteq\mathbb{C}$ and let $\gamma:[a,b]\to U$ be a C^1 curve. The ***arc-length*** of γ is the real number defined by

$$L := \int_a^b |\gamma'(t)|dt.$$

(In fact the arc-length can be defined not only for C^1 curves but also for some curves that are "less nice". Interested readers may refer to Supplementary Note D for details.)

Corollary 3.18 (Cauchy $\boldsymbol{ML}$-estimate) *Let $U\subseteq\mathbb{C}$, let $f:U\to\mathbb{C}$ be a continuous function and let $\gamma:[a,b]\to U$ be a C^1 curve in U. Then*

$$\left|\int_\gamma f(z)dz\right| \le ML,$$

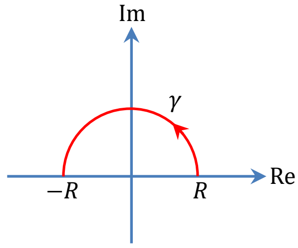
where

$$M = \max\{|f(z)|: z\in \text{image}\,\gamma\} = \max\{|f(\gamma(t))|: t\in[a,b]\}$$

(why does M exist?) and L is the arc-length of γ.

Example 3.19 For each $R > 1$, let γ be the upper half of the circle $\partial D(0; R)$, oriented counterclockwise. Show that

$$\lim_{R\to+\infty} \int_\gamma \frac{z-1}{z^3+z^2+z+1} dz = 0.$$



Proof:
For each z in the image of γ, we have $|z-1| \le |z| + |1| = R+1$, $|z+1| \ge |z| - |1| = R-1$, and $|z^2+1| \ge |z^2| - |1| = R^2 - 1$ by the triangle inequality. So

$$\left|\frac{z-1}{z^3+z^2+z+1}\right| = \frac{|z-1|}{|z+1||z^2+1|} \le \frac{R+1}{(R-1)(R^2-1)}.$$

To obtain an upper bound of a fraction, we bound its numerator from above and its denominator from below.

Now since the arc length of the semicircle γ is $L = \pi R$, the ML-estimate gives

$$\left|\int_\gamma \frac{z-1}{z^3+z^2+z+1} dz\right| \le \frac{R+1}{(R-1)(R^2-1)} \cdot \pi R = \frac{\pi R}{(R-1)^2}.$$

Since $\lim\limits_{R\to+\infty} \frac{\pi R}{(R-1)^2} = \lim\limits_{R\to+\infty} \frac{\frac{\pi}{R}}{\left(1-\frac{1}{R}\right)^2} = 0$, we have $\lim\limits_{R\to+\infty} \left|\int_\gamma \frac{z-1}{z^3+z^2+z+1} dz\right| = 0$ by squeeze theorem.

Therefore $\lim\limits_{R\to+\infty} \int_\gamma \frac{z-1}{z^3+z^2+z+1} dz = 0.$ ■

The following corollary of the ML-estimate says that **uniform convergence** also **preserves line integrals** of the terms in the limit.

Corollary 3.20 *Let $U \subseteq \mathbb{C}$, let $\gamma: [a,b] \to U$ be a C^1 curve in U, and let $\{f_n\}$ be a sequence of continuous functions $f_n: U \to \mathbb{C}$ which converges to a function $f: U \to \mathbb{C}$ uniformly on every compact subset of U. Then*

$$\lim_{n\to+\infty} \int_\gamma f_n(z) dz = \int_\gamma f(z) dz.$$

Proof. Let L be the arc-length of γ. The result is trivial if $L = 0$, so we assume that $L > 0$. Since $[a,b]$ is compact and γ is continuous, the image of γ is a compact subset of U. Therefore by the assumption, $\{f_n\}$ converges to f uniformly on the image of γ. So for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \frac{\varepsilon}{L} \qquad \text{for every } n \ge N \text{ and every } z \in \text{image}\,\gamma.$$

Now whenever $n \ge N$, we have

$$\left|\int_\gamma f_n(z) dz - \int_\gamma f(z) dz\right| = \left|\int_\gamma \big(f_n(z) - f(z)\big) dz\right| \le \frac{\varepsilon}{L} \cdot L = \varepsilon,$$

so we have the desired limit. ■

2. Cauchy-Goursat Theorem

Definition 3.21 Let $U \subseteq \mathbb{C}$ be a region and let $f: U \to \mathbb{C}$ be a function. An ***antiderivative*** of f is a holomorphic function $F: U \to \mathbb{C}$ such that $F' = f$.

Remark 3.22 Whether a function has an antiderivative or not depends not only on its defining formula but also its **domain**. We will see later that

- the function $f: \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by $f(z) = 1/z$ has no antiderivatives, but
- the function $f: \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ defined by $f(z) = 1/z$ has an antiderivative.

Remember to always specify the domain when talking about a function.

Recall that the line integrals of a function along different curves joining the same pair of points are different in general. The following theorem tells us that if a function **has an antiderivative** on a region U, then its line integral is always the same for every curve joining the same pair of points in U, i.e. its line integrals are **path-independent on $\boldsymbol{U}$**.

Theorem 3.23 (Fundamental Theorem of Calculus) *Let $U \subseteq \mathbb{C}$ be a region and $\gamma: [a, b] \to U$ be a C^1 curve. If $f: U \to \mathbb{C}$ is a continuous function which has an antiderivative $F: U \to \mathbb{C}$, then*
$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

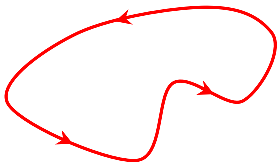
Proof. Since $F' = f$, we have
$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b F'(\gamma(t))\gamma'(t)dt$$
$$= \int_a^b (F \circ \gamma)'(t)dt = F(\gamma(b)) - F(\gamma(a)).$$
■

Example 3.24 Let γ be a piecewise C^1 curve in $\mathbb{C}$ joining 0 to $\pi + 2i$. Evaluate the integral
$$\int_\gamma \cos\frac{z}{2} dz.$$

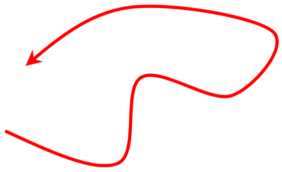
Solution:

Let $F: \mathbb{C} \to \mathbb{C}$ be the entire function $F(z) = 2\sin\frac{z}{2}$. Then $F'(z) = \cos\frac{z}{2}$ for every $z \in \mathbb{C}$. So by the Fundamental Theorem of Calculus, we have
$$\int_\gamma \cos\frac{z}{2} dz = F(\pi + 2i) - F(0) = 2\sin\left(\frac{\pi + 2i}{2}\right) - 0 = e + \frac{1}{e}.$$

Definition 3.25 A curve $\gamma:[a,b]\to\mathbb{C}$ is called a ***closed curve*** if its initial point and terminal point are the same point, i.e. $\gamma(a)=\gamma(b)$.



A closed curve



Not a closed curve

Remark 3.26 The line integral of a continuous function f along a closed curve γ is sometimes also denoted as

$$\oint_\gamma f(z)dz.$$

Corollary 3.27 *Let $U\subseteq\mathbb{C}$ be a region, $f:U\to\mathbb{C}$ be a continuous function having an antiderivative on U and $\gamma:[a,b]\to U$ be a closed C^1 curve. Then*

$$\oint_\gamma f(z)dz=0.$$

Example 3.28 (Key example) Let $f:\mathbb{C}\setminus\{0\}\to\mathbb{C}$ be the function

$$f(z)=\frac{1}{z}.$$

Show that f does not have an antiderivative on the region $\mathbb{C}\setminus\{0\}$.

Proof:

Let $\gamma:[0,2\pi]\to\mathbb{C}\setminus\{0\}$ be the closed C^1 curve defined by $\gamma(t)=e^{it}$ (i.e. the unit circle centered at 0 oriented counterclockwise). Then

$$\oint_\gamma f(z)dz=\int_0^{2\pi}\frac{1}{e^{it}}ie^{it}dt=\int_0^{2\pi}i\,dt=2\pi i\neq 0,$$

so f does not have an antiderivative on $\mathbb{C}\setminus\{0\}$ according to Corollary 3.27. ■

Remark 3.29 In Example 3.28, one may think that "the complex logarithm"

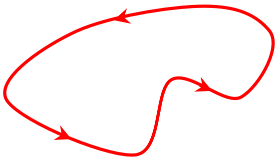
$$F(z)=\ln z$$

is an antiderivative of f on $\mathbb{C}\setminus\{0\}$. This is in fact not true, and later in this chapter we will look into what "complex logarithm" actually is.

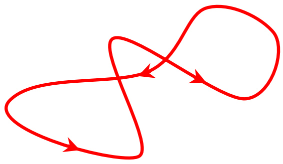
Definition 3.30 A closed curve $\gamma:[a,b]\to\mathbb{C}$ is called a ***simple closed curve*** if it does not have any "self-intersection", i.e. if

$$\gamma(t_1)\neq\gamma(t_2)$$

for any distinct $t_1,t_2\in[a,b]$, except when $t_1=a$ and $t_2=b$.



A simple closed curve in $\mathbb{C}$



A closed curve in $\mathbb{C}$ which is not simple

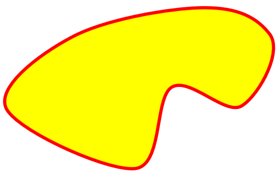
Theorem 3.31 (Jordan curve theorem) *The image of a simple closed curve in $\mathbb{C}$ divides the whole plane $\mathbb{C}$ into two regions. One of these regions is bounded, and the other one is unbounded.*

Proof. Very difficult. Interested students may refer to Supplementary Note E for details. ∎

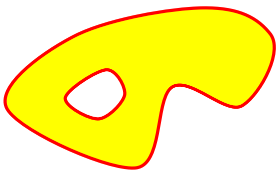
Definition 3.32 Let γ be a simple closed curve in $\mathbb{C}$. The bounded region as described in the above Jordan curve theorem is called the ***interior*** of γ. The other unbounded region is called the ***exterior*** of γ.

Definition 3.33 Let $U\subseteq\mathbb{C}$ be a region. We say that U is a ***simply connected region*** if for every simple closed curve γ in U, the interior of γ also lies completely in U.

Remark 3.34 Intuitively, a simply connected region in $\mathbb{C}$ can be regarded as a region that has no "holes".



A simply connected region in $\mathbb{C}$



A region in $\mathbb{C}$ which is not simply connected

Remark 3.35 Let $U\subset\mathbb{C}$ be a bounded simply connected region, whose boundary ∂U is a simple closed piecewise C^1 curve. When we write the line integral

$$\oint_{\partial U} f(z)dz,$$

the **counterclockwise orientation** of ∂U is always assumed.

The following is the **most important theorem in complex analysis**. Many beautiful results in complex analysis about holomorphic functions are all consequences of this fundamental theorem.

Theorem 3.36 (Cauchy-Goursat) *Let $U \subseteq \mathbb{C}$ be a simply connected region, $f: U \to \mathbb{C}$ be a holomorphic function and γ be a closed C^1 curve in U. Then*

$$\oint_\gamma f(z)dz = 0.$$

Example 3.37 Thanks to Cauchy-Goursat Theorem, one easily tells that if γ is a closed C^1 curve in $\mathbb{C}$, then

$$\oint_\gamma e^z dz = 0, \qquad \oint_\gamma \sin z\, dz = 0 \qquad \text{and} \qquad \oint_\gamma z^2 dz = 0,$$

etc. without actually doing any calculation, since these integrands are all entire functions.

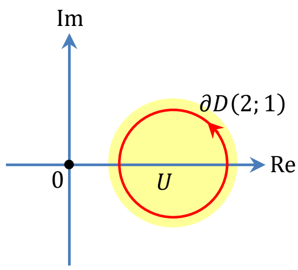
Example 3.38 Evaluate the integral

$$\oint_{\partial D(2;1)} \frac{1}{z} dz.$$

Solution: [Although the integrand $1/z$ is not entire, its only "singularity" 0 (only point at which it is not holomorphic) is in the **exterior** of $\partial D(2;1)$. So we can still construct a simply connected region U on which $1/z$ is holomorphic.]

Let $U = D(2;1.1)$. Then U is a simply connected region, the function $f: U \to \mathbb{C}$ defined by $f(z) = 1/z$ is holomorphic and $\partial D(2;1)$ is a closed C^1 curve in U. So by Cauchy-Goursat Theorem we have

$$\oint_{\partial D(2;1)} \frac{1}{z} dz = 0.$$



Remark 3.39 In Cauchy-Goursat Theorem, the conditions that U is **simply connected** and that f is **holomorphic** are both essential. Refer to Example 3.28 and Example 3.7.

- Let $U = \mathbb{C} \setminus \{0\}$, which is a **non-simply connected** region. Then the function $f: U \to \mathbb{C}$ defined by $f(z) = 1/z$ is holomorphic and $\partial D(0;1)$ is a closed C^1 curve in U, but

$$\oint_{\partial D(0;1)} \frac{1}{z} dz = 2\pi i \neq 0.$$

- Let $U = \mathbb{C}$ which is a simply connected region and let $f: U \to \mathbb{C}$ be the function $f(z) = \overline{z}$, which is **not holomorphic**. Then $\partial D(0;1)$ is a closed C^1 curve in U, but

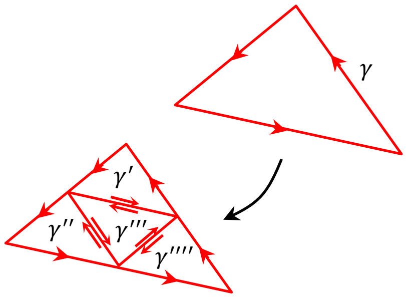
$$\oint_{\partial D(0;1)} \overline{z}\, dz = 2\pi i \neq 0.$$

The proof of this powerful theorem consists of a series of lemmas.

Lemma 3.40 *Let* $U \subseteq \mathbb{C}$ *be a simply connected region,* $f: U \to \mathbb{C}$ *be a holomorphic function and* γ *be a triangle in* U *(i.e. a closed curve in* U *which consists of three line segments). Then*

$$\oint_\gamma f(z)dz = 0.$$

Proof. Let $I := \left|\oint_\gamma f(z)dz\right|$, so the issue is to show that $I = 0$.



Let L be the arc-length of γ, i.e. the perimeter of the triangle. Let $\gamma', \gamma'', \gamma''', \gamma''''$ be the triangles obtained by subdividing each edge of γ at its mid-point (refer to the diagram). Then the perimeter of each of these smaller triangles is $\frac{L}{2}$, and

$$I = \left|\oint_\gamma f(z)dz\right| \leq \left|\oint_{\gamma'} f(z)dz\right| + \left|\oint_{\gamma''} f(z)dz\right| + \left|\oint_{\gamma'''} f(z)dz\right| + \left|\oint_{\gamma''''} f(z)dz\right|.$$

In particular, one of these four terms is $\geq \frac{I}{4}$, and let's denote such a term by $\left|\oint_{\gamma_1} f(z)dz\right| \geq \frac{I}{4}$.

Next we subdivide γ_1 into four smaller triangles in the same way. Then there is a certain triangle γ_2 with perimeter $\frac{L}{2^2}$ such that $\left|\oint_{\gamma_2} f(z)dz\right| \geq \frac{I}{4^2}$. Repeating this procedure, we see that for each $n \in \mathbb{N}$, there exists a triangle γ_n with perimeter $\frac{L}{2^n}$ such that $\left|\oint_{\gamma_n} f(z)dz\right| \geq \frac{I}{4^n}$.

Now for each $n \in \mathbb{N}$, let Δ_n denote the closure of the interior of γ_n (i.e. a closed triangular region). Then $\{\Delta_n\}$ is a sequence of nested non-empty compact sets ($\Delta_{n+1} \subseteq \Delta_n$ for all $n \in \mathbb{N}$), so by **Cantor intersection theorem** (Theorem 1.74), there exists a such that $a \in \Delta_n$ for every n. Since U is simply connected, we must have $a \in U$, so f is holomorphic at a. For each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left|\frac{f(z) - f(a)}{z - a} - f'(a)\right| < \varepsilon \qquad \text{whenever } 0 < |z - a| < \delta,$$

and there exists a large enough $N \in \mathbb{N}$ so that $\Delta_n \subset D(a;\delta)$ for every $n \geq N$, so

$$\frac{I}{4^n} \leq \left|\oint_{\gamma_n} f(z)dz\right| = \left|\oint_{\gamma_n} f(z)dz - \underbrace{\oint_{\gamma_n} [f(a) + f'(a)(z-a)]dz}_{=0 \because \text{ a polynomial has antiderivatives on } \mathbb{C}}\right|$$

$$= \left|\oint_{\gamma_n} [f(z) - f(a) - f'(a)(z-a)]dz\right| \leq \frac{\varepsilon L}{2^n} \cdot \frac{L}{2^n} = \frac{\varepsilon L^2}{4^n}$$

by ML-estimate. This shows that $I \leq \varepsilon L^2$ for every $\varepsilon > 0$, and so $I = 0$. ■

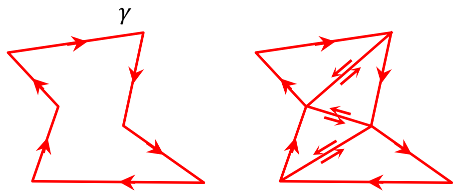
Lemma 3.41 *Let $U \subseteq \mathbb{C}$ be a region and $f: U \to \mathbb{C}$ be a continuous function. If*

$$\oint_\gamma f(z)dz = 0$$

for every triangle γ in U, then

$$\oint_\gamma f(z)dz = 0$$

for every polygonal closed curve γ in U.



A decomposition of a polygonal closed curve into triangles. (*Think*: What if the region is not simply connected?)

Proof. A polygonal closed curve can be decomposed into finitely many triangles. ∎

Lemma 3.42 *Let $U \subseteq \mathbb{C}$ be a region and $f: U \to \mathbb{C}$ be a continuous function. If*

$$\oint_\gamma f(z)dz = 0$$

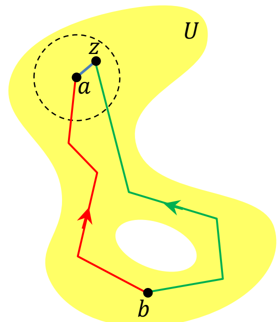
for every polygonal closed curve γ in U, then f has an antiderivative on U.

In Lemma 3.41 and 3.42 we need not assume that f is holomorphic. **Continuity** is enough.

Proof. Fix $b \in U$ and define a function $F: U \to \mathbb{C}$ by

$$F(z) = \int_{\text{a polygonal curve in } U \text{ joining } b \text{ to } z} f(w)dw.$$

Then F is well-defined (independent of the choice of the polygonal curve γ) because of the assumption that $\oint_\gamma f(w)dw = 0$ for every polygonal closed curve γ in U.



Now we want show that F is an antiderivative of f, i.e. $F'(a) = f(a)$ for every $a \in U$. Since U is open, there exists $r > 0$ such that $D(a; r) \subseteq U$. Now for each $z \in D(a; r) \setminus \{a\}$, let L denote the line segment joining a to z. Then $F(z) - F(a) = \int_L f(w)dw$, so together with ML-estimate we have

$$\left|\frac{F(z) - F(a)}{z - a} - f(a)\right| = \frac{1}{|z - a|}\left|\int_L (f(w) - f(a))dw\right| \leq \max\{|f(w) - f(a)| : w \in L\},$$

which tends to 0 as z tends to a, by the continuity of f at a. Hence $F'(a) = f(a)$. ∎

Note that Lemma 3.42 is a converse to Corollary 3.27. Combining Lemmas 3.40, 3.41 and 3.42, we obtain the following conclusion.

Theorem 3.43 *Let $U \subseteq \mathbb{C}$ be a simply connected region and $f: U \to \mathbb{C}$ be a holomorphic function. Then f has an antiderivative on U.*

Now combining Theorem 3.43 and Corollary 3.27, we finish the proof of Cauchy-Goursat Theorem.

Example 3.44 Let n be an integer.

(a) Let $a \in \mathbb{C}$ and $r > 0$. Compute

$$\oint_{\partial D(a,r)} (z-a)^n dz.$$

(b) Let U be the rectangular region $\{z \in \mathbb{C}: 0 < \operatorname{Re} z < 3 \text{ and } 0 < \operatorname{Im} z < 2\}$. Compute

$$\oint_{\partial U} (z-2-i)^n dz$$

using the result from (a).

Solution:

(a) $\partial D(a;r)$ has a parametrization $\gamma: [0, 2\pi] \to \mathbb{C}$ defined by $\gamma(t) = a + re^{it}$. So

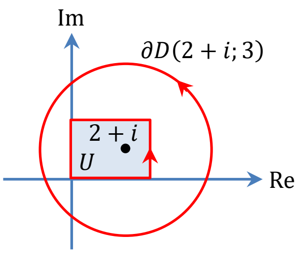
$$\oint_{\partial D(a;r)} (z-a)^n dz = \int_0^{2\pi} \left(re^{it}\right)^n ire^{it} dt = r^{n+1} \int_0^{2\pi} ie^{i(n+1)t} dt$$

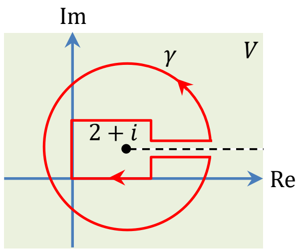
$$= \begin{cases} [it]_0^{2\pi} & \text{if } n = -1 \\ r^{n+1}\left[\dfrac{1}{n+1} e^{i(n+1)t}\right]_0^{2\pi} & \text{if } n \neq -1 \end{cases} = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{if } n \neq -1 \end{cases}.$$

(b) Since n is an integer, the function $(z-2-i)^n$ is holomorphic on $\mathbb{C} \setminus \{2+i\}$. In particular, it is holomorphic on the **simply connected region**

$$V = \mathbb{C} \setminus \{z \in \mathbb{C}: \operatorname{Re} z \geq 2 \text{ and } \operatorname{Im} z = 1\}.$$

Now let γ be the curve in V as shown in the following diagram on the right.





Since γ is a **closed** piecewise C^1 **curve** in V, by Cauchy-Goursat Theorem we have

$$0 = \oint_\gamma (z-2-i)^n dz = \oint_{\partial D(2+i;3)} (z-2-i)^n dz - \oint_{\partial U} (z-2-i)^n dz.$$

Therefore the result from (a) gives

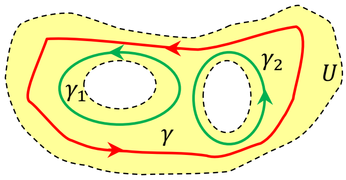
$$\oint_{\partial U} (z-2-i)^n dz = \oint_{\partial D(2+i;3)} (z-2-i)^n dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{if } n \neq -1 \end{cases}.$$

Example 3.44 (b) shows that the only importance of the **simply connectedness** of the region U in Cauchy-Goursat Theorem is to ensure that <u>f is always holomorphic in the interior of γ</u>. So with a similar technique, one can extend Cauchy-Goursat Theorem to ***multiply connected*** regions, i.e. regions having "holes".

Theorem 3.45 (Cauchy-Goursat, general version) *Let $U \subseteq \mathbb{C}$ be a region and $f: U \to \mathbb{C}$ be a holomorphic function. If $\gamma, \gamma_1, \dots, \gamma_n$ are simple closed curves in U oriented counterclockwise such that*

- image γ_1, …, image γ_n *are in the interior of* γ,
- *the interiors of* $\gamma_1, \dots, \gamma_n$ *are **mutually disjoint**, and*
- U *contains the intersection of the interior of* γ *and all the exteriors of* $\gamma_1, \dots, \gamma_n$,

then

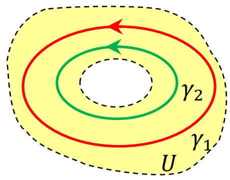


$$\oint_{\gamma} f(z)dz = \sum_{k=1}^{n} \oint_{\gamma_k} f(z)dz.$$

In particular, if γ_1 *and* γ_2 *are simple closed curves in* U *oriented counterclockwise such that*

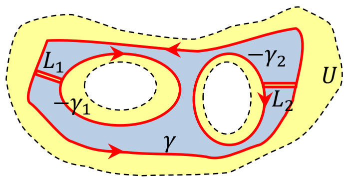
- image γ_2 *is in the interior of* γ_1 *and*
- U *contains the intersection of the interior of* γ_1 *and the exterior of* γ_2,

then



$$\oint_{\gamma_1} f(z)dz = \oint_{\gamma_2} f(z)dz.$$

Sloppy proof. [We follow the same technique as in Example 3.44 (b).] For each $k \in \{1, 2, \dots, n\}$, let L_k be a line segment joining a point on γ to a point on γ_k, such that the line segments $L_1, \dots, L_n$ are all disjoint. Then the curve defined by the concatenation



$$\gamma * L_1 * -\gamma_1 * -L_1 * L_2 * -\gamma_2 * -L_2 * \cdots * L_n * -\gamma_n * -L_n$$

is a piecewise C^1 curve which encloses a **simply connected** region on which f is holomorphic, so

$$\oint_{\gamma} f(z)dz + \int_{L_1} f(z)dz - \oint_{\gamma_1} f(z)dz - \int_{L_1} f(z)dz + \cdots + \int_{L_n} f(z)dz - \oint_{\gamma_n} f(z)dz - \int_{L_n} f(z)dz = 0$$

by Cauchy-Goursat Theorem. ∎

It turns out that Cauchy-Goursat Theorem can also be applied in computing **real integrals**.

Example 3.46 Evaluate the improper integral

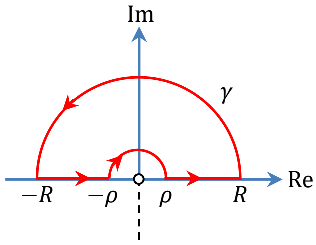
$$\int_0^{+\infty} \frac{1 - \cos x}{x^2} dx.$$

Solution:

Let $U = \mathbb{C} \setminus \{iy \in \mathbb{C}: y \leq 0\}$ which is a simply connected region in $\mathbb{C}$, and let $f: U \to \mathbb{C}$ be the holomorphic function

$$f(z) = \frac{1 - e^{iz}}{z^2}.$$

Let $R > 1$ and $\rho \in (0,1)$, and define



- ⦿ $\gamma_1 : [0, \pi] \to \mathbb{C}$ by $\gamma_1(t) = Re^{it}$,
- ⦿ $\gamma_2 : [-R, -\rho] \to \mathbb{C}$ by $\gamma_2(x) = x$,
- ⦿ $\gamma_3 : [0, \pi] \to \mathbb{C}$ by $\gamma_3(t) = \rho e^{i(\pi - t)}$, and
- ⦿ $\gamma_4 : [\rho, R] \to \mathbb{C}$ by $\gamma_4(x) = x$.

Let $\gamma = \gamma_1 * \gamma_2 * \gamma_3 * \gamma_4$ be the closed piecewise C^1 curve in U as shown in the diagram. Then by Cauchy-Goursat Theorem, we have

$$\int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz + \int_{\gamma_4} f(z)dz = \oint_{\gamma} f(z)dz = 0.$$

On the other hand, we have

$$\int_{\gamma_2} f(z)dz = \int_{-R}^{-\rho} \frac{1 - e^{ix}}{x^2} dx = \int_{\rho}^{R} \frac{1 - e^{-ix}}{x^2} dx \qquad \text{and} \qquad \int_{\gamma_4} f(z)dz = \int_{\rho}^{R} \frac{1 - e^{ix}}{x^2} dx\,;$$

and we next aim to let $\rho \to 0^+$ and $R \to +\infty$.

- ⦿ For every $z \in \operatorname{image} \gamma_1$, we have $\operatorname{Im} z \geq 0$ and so

$$\left|1 - e^{iz}\right| \leq 1 + \left|e^{iz}\right| = 1 + e^{-\operatorname{Im} z} \leq 2.$$

  Thus ML-estimate gives

$$\left|\int_{\gamma_1} f(z)dz\right| = \left|\int_{\gamma_1} \frac{1 - e^{iz}}{z^2} dz\right| \leq \frac{2}{R^2} \cdot \pi R = \frac{2\pi}{R},$$

  which tends to 0 as $R \to +\infty$, so $\lim\limits_{R \to +\infty} \int_{\gamma_1} f(z)dz = 0$.

- ⦿ For every $z \in \operatorname{image} \gamma_3$, we have $|z| = \rho < 1$ and so

$$\left|\frac{1 - e^{iz}}{z^2} + \frac{i}{z}\right| = \left|\sum_{k=2}^{+\infty} \frac{(iz)^k}{k!\, z^2}\right| \leq \sum_{k=2}^{+\infty} \frac{|z|^{k-2}}{k!} \leq \sum_{k=2}^{+\infty} \frac{1}{k!} \leq e.$$

  Thus ML-estimate gives

$$\left|\int_{\gamma_3} \left(f(z) + \frac{i}{z}\right) dz\right| = \left|\int_{\gamma_3} \left(\frac{1 - e^{iz}}{z^2} + \frac{i}{z}\right) dz\right| \leq e \cdot \pi\rho,$$

  which tends to 0 as $\rho \to 0^+$, so

$$\lim_{\rho \to 0^+} \int_{\gamma_3} f(z)dz = \lim_{\rho \to 0^+} \int_{\gamma_3} \frac{-i}{z} dz = \lim_{\rho \to 0^+} \int_0^{\pi} \frac{-i}{\rho e^{i(\pi - t)}} \left(-i\rho e^{i(\pi - t)}\right) dt = -\int_0^{\pi} dt = -\pi.$$

Thus letting $\rho \to 0^+$ and $R \to +\infty$ in

$$\int_{\gamma_1} f(z)dz + \int_{\rho}^{R} \frac{1 - e^{-ix}}{x^2} dx + \int_{\gamma_3} f(z)dz + \int_{\rho}^{R} \frac{1 - e^{ix}}{x^2} dx = 0,$$

we obtain $0 - \pi + \int_0^{+\infty} \frac{2 - e^{ix} - e^{-ix}}{x^2} dx = 0$. Therefore

$$\int_0^{+\infty} \frac{1 - \cos x}{x^2} dx = \frac{\pi}{2}.$$

$\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

3. Cauchy integral formula

Cauchy-Goursat Theorem is useful when dealing with an integrand which is **holomorphic on a simply connected region** containing the closed curve. When handling an integrand that is not holomorphic at some points (has "**singularities**") in the interior of a simple closed curve, the following **Cauchy integral formula** will be useful instead.

Theorem 3.47 (Cauchy integral formula) *Let* $U \subseteq \mathbb{C}$ *be a simply connected region,* $f: U \to \mathbb{C}$ *be a holomorphic function,* γ *be a counterclockwise oriented simple closed* C^1 *curve in* U*, and* a *be a point in the interior of* γ*. Then*

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-a}\,dz = f(a).$$

Remark 3.48 In Theorem 3.47, if the point $a \in U$ is in the exterior of γ instead, then the integrand $\frac{f(z)}{z-a}$ is still holomorphic on a simply connected region (smaller than U) containing image γ, so we still have $\oint_\gamma \frac{f(z)}{z-a}dz = 0$ by Cauchy-Goursat (cf. Example 3.38). In summary,

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-a}\,dz = \begin{cases} f(a) & \text{if } a \text{ is in the interior of } \gamma \\ 0 & \text{if } a \text{ is in the exterior of } \gamma \end{cases}.$$

Example 3.49 Evaluate the line integral

$$\oint_\gamma \frac{e^z}{z^2+4}\,dz$$

(a) if $\gamma = \partial D(0;1)$,
(b) if $\gamma = \partial D(2i;1)$, and
(c) if $\gamma = \partial D(0;4)$.

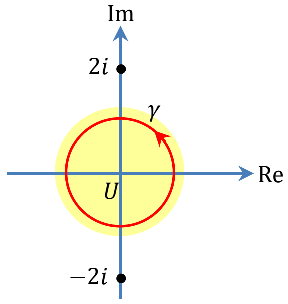
Solution: Note that the integrand

$$\frac{e^z}{z^2+4} = \frac{e^z}{(z-2i)(z+2i)}$$

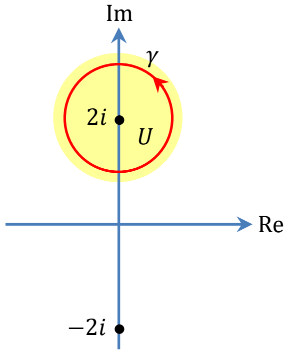
is holomorphic everywhere on $\mathbb{C}$ except at the two points $2i$ and $-2i$.

(a) Let $U = D(0;1.1)$, which is a simply connected region. Then the integrand is holomorphic on U and $\gamma = \partial D(0;1)$ is a closed curve in U, so by Cauchy-Goursat Theorem we have

$$\oint_{\partial D(0;1)} \frac{e^z}{z^2+4}\,dz = 0.$$



(b) Let $U = D(2i; 1.1)$, which is a simply connected region. Then $\frac{e^z}{z+2i}$ is holomorphic on U, $\gamma = \partial D(2i; 1)$ is a simple closed curve in U, and $2i$ is in the interior of γ. So by Cauchy integral formula,

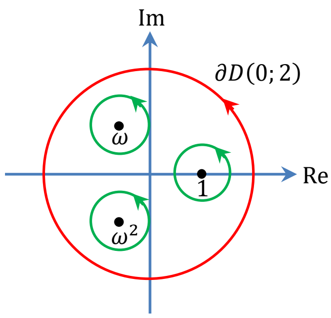


$$\oint_{\partial D(2i;1)} \frac{e^z}{z^2+4} dz = \oint_{\partial D(2i;1)} \frac{e^z/(z+2i)}{z-2i} dz$$
$$= 2\pi i \cdot \frac{e^{(2i)}}{(2i)+2i} = \frac{\pi}{2}(\cos 2 + i \sin 2).$$

(c) This time both points $2i$ and $-2i$ are in the interior of $\gamma = \partial D(0; 4)$. We handle each of them separately by a **partial fraction** decomposition of the integrand. We have
$$\oint_{\partial D(0;4)} \frac{e^z}{z^2+4} dz = \oint_{\partial D(0;4)} \frac{e^z}{4i}\left(\frac{1}{z-2i} - \frac{1}{z+2i}\right) dz$$
$$= \frac{1}{4i}\left(\oint_{\partial D(0;4)} \frac{e^z}{z-2i} dz - \oint_{\partial D(0;4)} \frac{e^z}{z+2i} dz\right).$$
Now we may apply Cauchy integral formula on each of the two new integrals to get
$$\oint_{\partial D(0;4)} \frac{e^z}{z-2i} dz = 2\pi i \cdot e^{2i} \qquad \text{and} \qquad \oint_{\partial D(0;4)} \frac{e^z}{z+2i} dz = 2\pi i \cdot e^{-2i},$$
so
$$\oint_{\partial D(0;4)} \frac{e^z}{z^2+4} dz = \frac{1}{4i}\left(2\pi i \cdot e^{2i} - 2\pi i \cdot e^{-2i}\right) = \pi i \sin 2.$$

It is sometimes too tedious to find the partial fraction decomposition of the integrand if its denominator consists of a polynomial of high degree. In this situation, we may instead try to modify the given curve using the **general version of Cauchy-Goursat Theorem** (Theorem 3.45).

Example 3.50 Evaluate the integral
$$\oint_{\partial D(0;2)} \frac{1}{z^3-1} dz.$$



Solution: Let $\omega \coloneqq e^{2\pi i/3}$ denote a cube root of 1. The integrand
$$\frac{1}{z^3-1} = \frac{1}{(z-1)(z-\omega)(z-\omega^2)}$$
is holomorphic away from 1, ω and ω^2. Since $\partial D\left(1; \frac{1}{2}\right)$, $\partial D\left(\omega; \frac{1}{2}\right)$, $\partial D\left(\omega^2; \frac{1}{2}\right)$ are simple closed curves in the interior of $\partial D(0; 2)$, whose interiors are mutually disjoint, Theorem 3.45 gives
$$\oint_{\partial D(0;2)} \frac{1}{z^3-1} dz = \oint_{\partial D\left(1;\frac{1}{2}\right)} \frac{1}{z^3-1} dz + \oint_{\partial D\left(\omega;\frac{1}{2}\right)} \frac{1}{z^3-1} dz + \oint_{\partial D\left(\omega^2;\frac{1}{2}\right)} \frac{1}{z^3-1} dz.$$

$$\boxed{(z-\omega)(z-\omega^2)=z^2+z+1}$$

Now applying Cauchy integral formula to each integral, we have

$$\oint_{\partial D\left(1;\frac{1}{2}\right)} \frac{1}{z^3-1}\,dz = \oint_{\partial D\left(1;\frac{1}{2}\right)} \frac{\frac{1}{z^2+z+1}}{z-1}\,dz = 2\pi i\cdot\frac{1}{1^2+1+1} = \frac{2\pi i}{3},$$

$$\oint_{\partial D\left(\omega;\frac{1}{2}\right)} \frac{1}{z^3-1}\,dz = \oint_{\partial D\left(\omega;\frac{1}{2}\right)} \frac{\frac{1}{(z-1)(z-\omega^2)}}{z-\omega}\,dz = 2\pi i\cdot\frac{1}{(\omega-1)(\omega-\omega^2)} = \frac{2\pi i}{3}\omega,$$

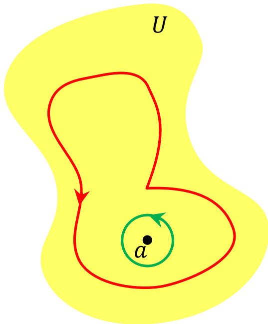
$$\oint_{\partial D\left(\omega^2;\frac{1}{2}\right)} \frac{1}{z^3-1}\,dz = \oint_{\partial D\left(\omega^2;\frac{1}{2}\right)} \frac{\frac{1}{(z-1)(z-\omega)}}{z-\omega^2}\,dz = 2\pi i\cdot\frac{1}{(\omega^2-1)(\omega^2-\omega)} = \frac{2\pi i}{3}\omega^2.$$

So

$$\oint_{\partial D(0;2)} \frac{1}{z^3-1}\,dz = \frac{2\pi i}{3}(1+\omega+\omega^2) = 0.$$

Now let's come back to the **proof of Cauchy integral formula**.

Proof of Theorem 3.47. Note that a is in the interior of γ. So for every sufficiently small $\varepsilon>0$, $\partial D(a;\varepsilon)$ also lies in the interior of γ, thus by the general version of Cauchy-Goursat Theorem (Theorem 3.45), we have



$$\oint_\gamma \frac{f(z)}{z-a}\,dz = \oint_{\partial D(a;\varepsilon)} \frac{f(z)}{z-a}\,dz$$
$$= \oint_{\partial D(a;\varepsilon)} \frac{f(z)-f(a)}{z-a}\,dz + f(a)\oint_{\partial D(a;\varepsilon)} \frac{1}{z-a}\,dz.$$

Now for the second integral, we always have

$$\oint_{\partial D(a;\varepsilon)} \frac{1}{z-a}\,dz = \int_0^{2\pi} \frac{1}{\varepsilon e^{it}}\, i\varepsilon e^{it}\,dt = 2\pi i.$$

For the first integral, since f is differentiable at a, there exists $\delta>0$ such that

$$\left|\frac{f(z)-f(a)}{z-a} - f'(a)\right| < 1,$$

i.e. $\left|\frac{f(z)-f(a)}{z-a}\right| < 1+|f'(a)|$, whenever $0<|z-a|<\delta$. So for $\varepsilon\in(0,\delta)$, ML-estimate gives

$$\left|\oint_{\partial D(a;\varepsilon)} \frac{f(z)-f(a)}{z-a}\,dz\right| \le (1+|f'(a)|)\cdot 2\pi\varepsilon.$$

Since $\lim\limits_{\varepsilon\to 0^+}(1+|f'(a)|)\cdot 2\pi\varepsilon = 0$, we have $\lim\limits_{\varepsilon\to 0^+}\left|\oint_{\partial D(a;\varepsilon)} \frac{f(z)-f(a)}{z-a}\,dz\right| = 0$ by squeeze theorem, and so

$$\lim_{\varepsilon\to 0^+}\oint_{\partial D(a;\varepsilon)} \frac{f(z)-f(a)}{z-a}\,dz = 0.$$

Therefore recalling that $\oint_\gamma \frac{f(z)}{z-a}dz = \oint_{\partial D(a;\varepsilon)} \frac{f(z)}{z-a}dz$ for **every** sufficiently small $\varepsilon > 0$, we have

$$\oint_\gamma \frac{f(z)}{z-a}dz = \lim_{\varepsilon\to 0^+}\left(\oint_{\partial D(a;\varepsilon)} \frac{f(z)-f(a)}{z-a}dz + f(a)\oint_{\partial D(a;\varepsilon)} \frac{1}{z-a}dz\right)$$
$$= 0 + f(a)\cdot 2\pi i = 2\pi i f(a).$$

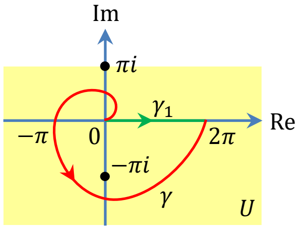
■

Let's look at some more trickier examples of applying the Cauchy integral formula.

Example 3.51 Let $\gamma: [0, 2\pi] \to \mathbb{C}$ be the curve defined by $\gamma(t) = te^{it}$. Evaluate the integral

$$\int_\gamma \frac{1}{z^2+\pi^2}dz.$$

Solution: [Although we have an explicit parametrization of γ, the line integral is too tedious (if not impossible) to evaluate by definition. We can still make use of Cauchy integral formula if we "complete γ to become a closed curve".]



Let $U = \{z \in \mathbb{C}: \operatorname{Im} z < \pi\}$. Then U is a simply connected region, and $1/(z-\pi i)$ is holomorphic on U. Also let $\gamma_1: [0, 2\pi] \to \mathbb{C}$ be the line segment defined by $\gamma_1(t) = t$. Then the curve $\gamma * -\gamma_1$ becomes a counterclockwise oriented simple closed curve in U, and the point $-\pi i$ is in the interior of $\gamma * -\gamma_1$. So

$$\int_\gamma \frac{1}{z^2+\pi^2}dz - \int_{\gamma_1} \frac{1}{z^2+\pi^2}dz = \oint_{\gamma*-\gamma_1} \frac{1/(z-\pi i)}{z+\pi i}dz = 2\pi i \cdot \frac{1}{(-\pi i)-\pi i} = -1$$

by Cauchy integral formula. Therefore,

$$\int_\gamma \frac{1}{z^2+\pi^2}dz = \int_{\gamma_1} \frac{1}{z^2+\pi^2}dz - 1 = \int_0^{2\pi} \frac{1}{t^2+\pi^2}dt - 1 = \frac{\arctan 2}{\pi} - 1.$$

Example 3.52 Evaluate the integral

$$\oint_{\partial D(0;1)} \frac{1}{\overline{z}+2}dz.$$

Solution: [Since the integrand $1/(\overline{z}+2)$ is **holomorphic nowhere**, we can apply neither Cauchy-Goursat Theorem nor Cauchy integral formula directly. So let's go back to basics and start with a parametrization of $\partial D(0;1)$.]

The curve $\partial D(0;1)$ has a parametrization $\gamma: [-\pi, \pi] \to \mathbb{C}$ defined by $\gamma(t) = e^{it}$, so

$$\oint_{\partial D(0;1)} \frac{1}{\overline{z}+2}dz = \int_{-\pi}^{\pi} \frac{1}{e^{-it}+2} ie^{it}dt.$$

However, one also notes that

$$\int_{-\pi}^{\pi}\frac{1}{e^{-it}+2}ie^{it}dt=\oint_{\partial D(0;1)}\frac{1}{(1/z)+2}dz=\oint_{\partial D(0;1)}\frac{z/2}{z+1/2}dz.$$

$\overline{z}=\frac{1}{z}$ for every $z\in\partial D(0;1)$.

Now in the last line integral, the function $z/2$ is entire, $\partial D(0;1)$ is a simple closed C^1 curve in $\mathbb{C}$, and the point $-1/2$ is in the interior of $\partial D(0;1)$. So we have

$$\oint_{\partial D(0;1)}\frac{z/2}{z+1/2}dz=2\pi i\cdot\frac{(-1/2)}{2}=-\frac{\pi i}{2}$$

by Cauchy integral formula. Therefore

$$\oint_{\partial D(0;1)}\frac{1}{\overline{z}+2}dz=-\frac{\pi i}{2}.$$

4. <u>Complex logarithm</u>

Unlike the real exponential function which is one-to-one on $\mathbb{R}$, we have seen that the complex exponential function has period $2\pi i$, so in particular it is not one-to-one on $\mathbb{C}$. Therefore, it takes some more tricks for one to define a **complex logarithm** appropriately. Recall the following result we obtained when proving Cauchy-Goursat Theorem, which is important on its own.

Theorem 3.43 *Let $U\subseteq\mathbb{C}$ be a <u>simply connected</u> region and $f:U\to\mathbb{C}$ be a holomorphic function. Then f has an antiderivative on U.*

Let $U\subseteq\mathbb{C}$ be a region, identified as an open subset of $\mathbb{R}^2$. In Remark 2.53, we have seen that a harmonic function on U is not always the real part of a holomorphic function on U. However, it turns out that if the region U is **simply connected**, then a harmonic function on U must indeed be the real part of a holomorphic function on U.

Corollary 3.53 *Let $U\subseteq\mathbb{C}$ be a simply connected region, identified as an open subset of $\mathbb{R}^2$. If $u:U\to\mathbb{R}$ is a harmonic function, then there exists a holomorphic function $f:U\to\mathbb{C}$ such that*

$$\operatorname{Re}f=u.$$

Proof. Let $g:U\to\mathbb{C}$ be the function defined by

$$g(x+iy)=u_x(x,y)-iu_y(x,y)$$

Since u is harmonic, u_x and $-u_y$ satisfy the Cauchy-Riemann equations on U. So g is holomorphic. Since U is a simply connected region, Theorem 3.43 implies that g has an antiderivative on U, i.e. there exists a holomorphic function $h:U\to\mathbb{C}$ such that $h'=g$. By Corollary 2.46 we have $(\operatorname{Re}h)_x=u_x$ and $(\operatorname{Re}h)_y=u_y$, so $u-\operatorname{Re}h$ is a constant function on U, say $u(x,y)-\operatorname{Re}h(x+iy)=c$ for every $(x,y)\in U$. Consequently the requirement is satisfied by the function $f:U\to\mathbb{C}$ defined by $f(z)=h(z)+c$. ∎

More importantly, Theorem 3.43 also enables the construction of the **complex logarithm**.

Theorem 3.54 *Let $U \subset \mathbb{C}$ be a simply connected region such that $0 \notin U$. Then there exists a holomorphic function $f: U \to \mathbb{C}$ such that $e^{f(z)} = z$ for every $z \in U$.*

Proof. Since $0 \notin U$, the function $g: U \to \mathbb{C}$ defined by $g(z) = \frac{1}{z}$ is holomorphic. Since U is a simply connected region, Theorem 3.43 implies that g has an antiderivative on U, i.e. there exists a holomorphic function $h: U \to \mathbb{C}$ such that $h' = g$. We verify that

$$\frac{d}{dz}\left(ze^{-h(z)}\right) = e^{-h(z)} - zh'(z)e^{-h(z)} = e^{-h(z)} - z \cdot \frac{1}{z}e^{-h(z)} = 0$$

on U, so $ze^{-h(z)}$ is a constant function, say $ze^{-h(z)} = c$ for every $z \in U$. Since $c \neq 0$ (why?), there exists $k \in \mathbb{C}$ such that $c = e^k$. Consequently the requirement is satisfied by the function $f: U \to \mathbb{C}$ defined by $f(z) = h(z) + k$. ■

Definition 3.55 Let $U \subset \mathbb{C}$ be a simply connected region such that $0 \notin U$. A ***branch of (complex) logarithm*** is a choice of a holomorphic function $f: U \to \mathbb{C}$ such that $e^{f(z)} = z$ for every $z \in U$. (The existence of such f is guaranteed by Theorem 3.54)

Corollary 3.56 *Let $U \subset \mathbb{C}$ be a simply connected region such that $0 \notin U$. For every branch $\log: U \to \mathbb{C}$ of logarithm on U, we always have*

$$\frac{d}{dz}\log z = \frac{1}{z}$$

on U.

Example 3.57 Let $U = \mathbb{C} \setminus (-\infty, 0]$. Then U is a simply connected region which does not contain 0.

⊙ For each $z \in U$, there exist a unique $r > 0$ and a unique $\theta \in (-\pi, \pi)$ such that $z = re^{i\theta}$. Then the holomorphic function $\mathrm{Log}: U \to \mathbb{C}$ defined by

$$\mathrm{Log}\, z \coloneqq \ln r + i\theta,$$

or in other words,

$$\mathrm{Log}\, z \coloneqq \ln|z| + i\,\mathrm{Arg}\, z\,,$$

is a branch of logarithm called the ***principal branch of logarithm***. The symbol Log (with capital letter L) is reserved for this particular principal branch of complex logarithm.

⊙ For each $z \in U$, there also exist unique $r > 0$ and $\theta \in (5\pi, 7\pi)$ such that $z = re^{i\theta}$. Then the holomorphic function $\log: U \to \mathbb{C}$ defined by

$$\log z \coloneqq \ln r + i\theta$$

is another branch of logarithm defined on U.

Example 3.58 Let $U = \mathbb{C} \setminus [0, +\infty)$. Then U is also a simply connected region which does not contain 0. For each $z \in U$, there exists a unique $r > 0$ and a unique $\theta \in (0, 2\pi)$ such that $z = re^{i\theta}$. Then the holomorphic function $\log: U \to \mathbb{C}$ defined by

$$\log z \coloneqq \ln r + i\theta$$

is a branch of logarithm defined on $U = \mathbb{C} \setminus [0, +\infty)$.

Remark 3.59 Let $U \subset \mathbb{C}$ be a simply connected region such that $0 \notin U$. A branch of logarithm on U can be defined explicitly when we revisit the proof of Theorem 3.43. Fixing any $a \in U$ and choosing $k \in \mathbb{C}$ such that $e^k = a$, a branch of logarithm $\log: U \to \mathbb{C}$ can be defined by

$$\log z \coloneqq k + \int_{\text{a curve in } U \text{ joining } a \text{ to } z} \frac{1}{w} dw,$$

in which the line integral is path-independent in U. To be explicit, here choosing k means choosing a value for the expression "$\log a$". Every two different choices of k differ by an integer multiple of $2\pi i$, and result in a different branch of logarithm on U.

- The principal branch $\operatorname{Log}: \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ in Example 3.57 corresponds to the choice $\operatorname{Log} 1 = 0$. The other branch $\log: \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ mentioned in Example 3.57 corresponds to the choice $\log 1 = 6\pi i$.
- The branch $\log: \mathbb{C} \setminus [0, +\infty) \to \mathbb{C}$ in Example 3.58 corresponds to the choice $\log(-1) = \pi i$.

Example 3.60 Let $U \subset \mathbb{C}$ be the simply connected region defined by

$$U = \mathbb{C} \setminus \left\{ re^{i\theta} \in \mathbb{C} : r \geq 0 \text{ and } \theta = \frac{\pi}{4} \right\}.$$

Let $\log: U \to \mathbb{C}$ be a branch of logarithm on U such that $\log 1 = 2\pi i$. Find the value of $\log 2i$.

Solution:

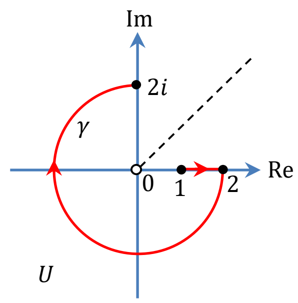
The given branch of logarithm on U is defined by

$$\log z \coloneqq 2\pi i + \int_{\text{a curve in } U \text{ joining } 1 \text{ to } z} \frac{1}{w} dw.$$

In particular, for $z = 2i$, we may let γ_1 be the line segment joining 1 to 2, let γ_2 be the portion of $\partial D(0; 2)$ oriented **clockwise** joining 2 to $2i$, and let $\gamma = \gamma_1 * \gamma_2$. Then

$$\begin{aligned}
\log 2i &= 2\pi i + \int_\gamma \frac{1}{w} dw \\
&= 2\pi i + \left(\int_1^2 \frac{1}{t} dt - \int_{\frac{\pi}{2}}^{2\pi} \frac{1}{2e^{it}} 2ie^{it} dt \right) \\
&= \ln 2 + \frac{\pi}{2} i.
\end{aligned}$$

$\int_\gamma \frac{1}{w} dw$ is easy to compute by parametrization if γ consists of line segments through 0 and/or circular arcs centered at 0 only.



Remark 3.61 In Remark 2.53, we have seen that the harmonic function $u\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ defined by

$$u(x,y) = \ln\sqrt{x^2+y^2}$$

is not the real part of any holomorphic function on $\mathbb{C}\setminus\{0\}$, because $\mathbb{C}\setminus\{0\}$ is not simply connected. In fact, this counter-example came from the idea of complex logarithms. If $U \subset \mathbb{C}$ is a simply connected region not containing 0, then for every branch of logarithm $f\colon U \to \mathbb{C}$, we have

$$\operatorname{Re} f(x+iy) = \ln\sqrt{x^2+y^2}.$$

Now with any choice of a branch of logarithm on a simply connected region not containing 0, we can give meaning to **exponential functions with a (non-zero) complex base**.

Definition 3.62 Let $U \subset \mathbb{C}$ be a simply connected region <u>such that $0 \notin U$</u>. For a fixed choice of a branch of logarithm $\log\colon U \to \mathbb{C}$ and for each $z \in U$, the **exponential function with base z** is defined by

$$z^w := e^{w\log z}$$

for every $w \in \mathbb{C}$.

Remark 3.63 Let $U \subset \mathbb{C}$ be a simply connected region not containing 0. For a fixed branch of complex logarithm $\log\colon U \to \mathbb{C}$, the following identities **<u>no longer hold</u>** in general:

- $\log(zw) \neq \log z + \log w$ in general;
- $\log\frac{z}{w} \neq \log z - \log w$ in general;
- $\log z^w \neq w\log z$ in general;
- $\log e^z \neq z$ in general; but $e^{\log z} = z$ still holds.

As an example, when we consider the principal branch of logarithm $\operatorname{Log}\colon \mathbb{C}\setminus(-\infty,0] \to \mathbb{C}$, we have

$$2\operatorname{Log} e^{\frac{2\pi i}{3}} = \operatorname{Log} e^{\frac{2\pi i}{3}} + \operatorname{Log} e^{\frac{2\pi i}{3}} = \frac{2\pi i}{3} + \frac{2\pi i}{3} = \frac{4\pi i}{3} \text{ but } \operatorname{Log}\left(e^{\frac{2\pi i}{3}}\right)^2 = \operatorname{Log}\left(e^{\frac{2\pi i}{3}}e^{\frac{2\pi i}{3}}\right) = -\frac{2\pi i}{3}.$$

Example 3.64 Using the principal branch of logarithm $\operatorname{Log}\colon \mathbb{C}\setminus(-\infty,0] \to \mathbb{C}$, express the complex numbers $(-i)^i$ and $(1+i)^i$ in polar form.

Solution:

On $\mathbb{C}\setminus(-\infty,0]$, we have $-i = 1e^{i(-\pi/2)}$ and $(1+i) = \sqrt{2}e^{i\pi/4}$. Thus we have

$$(-i)^i = e^{i\operatorname{Log}(-i)} = e^{i\left(\ln 1 + i\frac{-\pi}{2}\right)} = e^{\frac{\pi}{2}},$$

and

$$(1+i)^i = e^{i\operatorname{Log}(1+i)} = e^{i\left(\ln\sqrt{2}+i\frac{\pi}{4}\right)} = e^{-\frac{\pi}{4}}\left(\cos\frac{\ln 2}{2} + i\sin\frac{\ln 2}{2}\right).$$

Example 3.65 Let $U = \mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Re} z = 0 \text{ and } \operatorname{Im} z \le 0\}$, let $\log : U \to \mathbb{C}$ be the branch of logarithm on U defined by

$$\log\left(re^{i\theta}\right) = \ln r + i\theta \qquad \text{for every } r > 0 \text{ and } \theta \in \left(\frac{3\pi}{2}, \frac{7\pi}{2}\right),$$

and let γ be the semicircle centered at 0 joining 1 to -1 oriented counterclockwise. Evaluate

$$\int_\gamma z^i dz\,.$$

Solution:

The curve γ has a parametrization $\gamma : [2\pi, 3\pi] \to U$ given by $\gamma(t) = e^{it}$. Thus,

$$\int_\gamma z^i dz = \int_\gamma e^{i\log z} dz = \int_{2\pi}^{3\pi} e^{i(\ln 1 + it)} ie^{it} dt = i\int_{2\pi}^{3\pi} e^{(i-1)t} dt$$
$$= -\frac{e^{-3\pi} + e^{-2\pi}}{2} + i\frac{e^{-3\pi} + e^{-2\pi}}{2}.$$

Alternative solution:

With the chosen branch of logarithm, $\frac{1}{i+1} z^{i+1}$ is an antiderivative of z^i on U (why?). Thus by Fundamental Theorem of Calculus (Theorem 3.23), we have

$$\int_\gamma z^i dz = \frac{1}{i+1} e^{(i+1)\log(-1)} - \frac{1}{i+1} e^{(i+1)\log 1} = \frac{1}{i+1} e^{(i+1)(3\pi i)} - \frac{1}{i+1} e^{(i+1)(2\pi i)}$$
$$= -\frac{e^{-3\pi} + e^{-2\pi}}{2} + i\frac{e^{-3\pi} + e^{-2\pi}}{2}.$$

The construction of branches of logarithm can be extended to a more general setting. The following results are about **branches of "$\log g$" and "$g^{1/n}$"**, given a non-vanishing holomorphic function g.

Theorem 3.66 *Let $U \subset \mathbb{C}$ be a simply connected region and let $g : U \to \mathbb{C}$ be a holomorphic function such that $g(z) \neq 0$ for any $z \in U$. Then there exists a holomorphic function $f : U \to \mathbb{C}$ such that $e^{f(z)} = g(z)$ for every $z \in U$.*

Theorem 3.67 *Let $U \subset \mathbb{C}$ be a simply connected region and let $g : U \to \mathbb{C}$ be a holomorphic function such that $g(z) \neq 0$ for any $z \in U$. Then for each $n \in \mathbb{N}$, there exists a holomorphic function $f : U \to \mathbb{C}$ such that $\left(f(z)\right)^n = g(z)$ for every $z \in U$.*

Proof of Theorem 3.66 and Theorem 3.67. Mimic the proof of Theorem 3.54; to be left as an exercise (Q22, Problem Set 4). ∎

Summary of Chapter 3

The following are what you need to know in this chapter in order to get a pass (a distinction) in this course:

✓ **Line integrals**

- **Riemann integral** of a complex valued function on $[a, b]$
- **Line integrals** of continuous functions along **piecewise** C^1 **curves**
- **Line integrals** of Riemann integrable complex functions along **rectifiable curves**
- Line integral is specified by **image** and **orientation** of the curve
- **Arc-length** of a piecewise C^1 curve (rectifiable curve)
- ML**-estimate** of line integrals, **uniform convergence** and line integration

✓ **Cauchy-Goursat Theorem** and **Cauchy integral formula**

- Existence of antiderivatives $\Rightarrow$ **Fundamental Theorem of Calculus**
- **Interior** and **exterior** of a simple closed curve; **simply connected** regions
- **Cauchy-Goursat Theorem**: Holomorphic functions on simply connected regions have zero line integrals along closed curves.
- **Proof of Cauchy-Goursat Theorem**: Triangle case $\Rightarrow$ Polygonal closed curves $\Rightarrow$ Construction of an antiderivative
- **General version** of Cauchy-Goursat Theorem: Regions with "holes"
- Proof of the general version: Introduce **line segments** to "cut open" the region and make it simply connected
- **Cauchy integral formula**
- To **compute line integrals** using (a combination of) Cauchy-Goursat Theorem, its general version, as well as Cauchy integral formula

✓ **Complex logarithm**

- Branches of logarithm can be defined on a **simply connected region not containing** $\mathbf{0}$
- **Principal branch of logarithm** $\mathrm{Log}: \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$
- **Other branches of logarithm** defined via line integrals
- Exponential functions with a (non-zero) **complex base**
- Branches of "$\log g$" given a non-vanishing holomorphic function g