MATH 5431 Final Exam – 2022

4:30 PM – 6:30 PM, December 13rd, 2022

Please show the work to the questions and late submission is not allowed.

1. Suppose $X = (X_1, \ldots, X_n)$ and $X_i \sim^{i.i.d} Possion(\theta^2)$. Please (1) calculate the probability density $P(X_1 = x)$; (2) determine the minimal sufficient statistics $T(X)$ for θ; (3) calculate the information matrix $I_X(\theta)$.

2. Suppose $X = (X_1, \ldots, X_n)$ and $X_i \sim Normal(0, i\sigma^2)$ and is independent for $i = 1, \ldots, n$. Please (1) find the uniformly minimum variance unbiased estimate $\delta(X)$ for σ^2; (2) find the maximum likelihood estimate $\hat{\sigma}^2$ for σ^2; (3) derive the limiting distribution of $\sqrt{n}(\hat{\sigma} - \sigma)$.

3. Suppose $X_i \sim^{i.i.d} F$ for $i = 1, \ldots, n$. Let ξ_p be the p-th quantile satisfying $\xi_p = \inf\{t : F(t) \geq p\}$. Please (1) show ξ_p is non-decreasing and left continuous with respect to p; (2) show $|F_n(\hat{\xi}_p) - p| = O_p(1/n)$ where F_n is the empirical cumulative distribution function and $\hat{\xi}_p$ is the sample p-th quantile satisfying $\hat{\xi}_p = \inf\{t : F_n(t) \geq p\}$; (3) if $f(\xi_{0.1}) = F'(\xi_{0.1}) = 0.1$, derive the limiting distribution of $\sqrt{n}(\hat{\xi}_{0.1} - \xi_{0.1})$.

4. Suppose $X_i \sim^{i.i.d} F$ for $i = 1, \ldots, n$. Let $\psi(X, t)$ denote the function that M-estimate is defined as the solution $\hat{t}$ satisfying $\sum_i \psi(X_i, \hat{t})/n = 0$. Please (1) figure out $\psi(X, t)$ under which $\hat{t}$ is sample median; i.e. prove $\hat{t}$ satisfies $F_n(\hat{t}-) \leq p \leq F_n(\hat{t})$ where F_n is the empirical cumulative distribution function; (2) if $\psi(t + x, t) = -\psi(t - x, t)$ for any x and t and F is symmetric in t_0, show that the M-functional is t_0; (3) describe a bootstrap procedure to estimate the variance of $\hat{t}$.

5. Suppose $X_i \sim^{i.i.d} F$ for $i = 1, \ldots, n$. Let $\theta = EX_1^2$. Please (1) specify the U-statistics $\hat{\theta}$ for θ; (2) apply Hoeffding decomposition to $\hat{\theta}$; (3) apply jackknife to $\hat{\theta}$ and specify jackknife estimate.

6. Consider a Bayesian Model where $X|\theta \sim N(\theta, 1)$ and $P(\theta = 1) = 1/3$ and $P(\theta = 0) = 2/3$. Please (1) calculate the posterior probability $P(\theta|X)$; (2) find the bayesian estimator under squared error loss; (3) find the bayesian estimator under zero one loss.