Midterm Examination - Suggested Solutions

MATH 3043: Honors Real Analysis

Fall 2021, HKUST

October 30th, 2021, 19:00 - 22:00

PROBLEM 1. (18 points)

Let A, B and $\{E_k\}$ be Lebesgue measurable subsets of $\mathbb{R}^N$ with finite measure.

[4] (a) Prove that if $A \subset E \subset B$ and $\mu(A) = \mu(B)$, then E is measurable.

Solution: Since $\mu(B \setminus A) = \mu(B) - \mu(A) = 0$ and $(E \setminus A) \subset (B \setminus A)$, $E \setminus A$ is measurable by completeness of Lebesgue measure. Hence $E = A \cup (E \setminus A)$ is measurable.

[4] (b) Prove that there exists a measurable subset $E \subset A$ such that $2\mu(E) = \mu(A)$.

Solution: Let $f(x) := \mu(A \cap [-x, x]^N)$. Then f is continuous: if $y > x$,

$$|f(x) - f(y)| \leq \mu([-y, y]^N \setminus [-x, x]^N) = (2y)^N - (2x)^N \to 0$$

as $y \to x$. Also $f(0) = 0$ and $\lim_{x \to +\infty} f(x) = \mu(A)$. Hence by intermediate value theorem, there exists $\xi \geq 0$ such that $f(\xi) = \frac{1}{2}\mu(A)$. Hence $E = A \cap [-\xi, \xi]^N$ is the required subset. Depending on the version of IVT used, you may need to separate the case $\mu(A) = 0$.

[4] (c) Prove that $\mu\left(\bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} E_k\right) \leq \liminf_{k \to \infty} \mu(E_k)$.

Solution: Let $F_j = \bigcap_{k=j}^{\infty} E_k$. Then $F_1 \subset F_2 \subset \cdots$, hence (by Worksheet #03 Q2)

$$\mu\left(\bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} E_k\right) = \mu\left(\bigcup_{j=1}^{\infty} F_j\right) = \lim_{j \to \infty} \mu(F_j)$$

Since $F_j \subset E_j$ for any j, $\mu(F_j) \leq \mu(E_j)$, hence by definition of liminf,

$$\lim_{j \to \infty} \mu(F_j) = \liminf_{j \to \infty} \mu(F_j) \leq \liminf_{j \to \infty} \mu(E_j).$$

[6] (d) Which of the above conclusions (a), (b), (c) hold if $(\mathbb{R}^N, \mathcal{M}_{\text{Lebesgue}}, \mu_{\text{Lebesgue}})$ is replaced by arbitrary *abstract measure space* $(X, \mathcal{M}, \mu)$? Explain your answer.

Solution: (a) does not hold in general if the measure is not complete, e.g. $(\mathbb{R}^N, \mathcal{M}_{\text{Borel}}, \mu_{\text{Borel}})$ Acceptable if you say (a) holds for *complete* abstract measure space.

(b) does not hold in general, e.g. the counting measure with A having odd number of elements.

(c) works for all measure space since we only used countable *additivity* of μ-measurable sets. Note: countable *sub*additivity is **not** enough to conclude the proof, e.g. the statement is not true in general for outer measures.

PROBLEM 2. (26 points)

Let f and $\{f_n\}$ be measurable functions defined on a measurable set $E \subset \mathbb{R}^N$. We denote by $f_n \rightsquigarrow f$ if for any $\epsilon > 0$,

$$\lim_{n\to\infty} \mu(\{x \in E \; : \; |f_n(x) - f(x)| > \epsilon\}) = 0.$$

(This is known as **convergence in measure**.)

[8] (a) Show that if $\mu(E) < \infty$ and $f_n \to f$ pointwise a.e., then $f_n \rightsquigarrow f$.

> **Solution:** By Egorov's Theorem, for any $\delta > 0$, there exists closed $A \subset E$ such that $\mu(E \backslash A) \leq \delta$ and $f_n \to f$ uniformly on A.
>
> That means there exists N such that $n > N$ implies $|f_n(x) - f(x)| \leq \epsilon$ for any $x \in A$.
>
> Hence for any $\delta > 0$, there exists N such that $n > N$ implies
>
> $$\mu(\{x \in E \; : \; |f_n(x) - f(x)| > \epsilon\}) \leq \mu(E \setminus A) \leq \delta$$
>
> which exactly means the limit is zero.

[5] (b) Give a counterexample to (a) if $\mu(E) = \infty$.

> **Solution:** Take $N = 1$ and the same example where Egorov's Theorem failed. Let $E = \mathbb{R}$ and $f_n(x) = \mathbf{1}_{[n,n+1)}(x)$. Then $f_n \to 0$ pointwise, but
>
> $$\mu(\{x \in \mathbb{R} \; : \; |f_n(x)| > \epsilon\} = 1$$
>
> for any n if $0 < \epsilon < 1$.

[5] (c) The converse to (a) is not true. Give an example where $f_n \rightsquigarrow f$ but $f_n \not\to f$ *everywhere*.

> **Solution:** Take $E = [0, 1] \subset \mathbb{R}$ and take the sequence f_n to be the enumeration of length $\frac{1}{2^k}$ subintervals of $[0, 1]$, i.e.
>
> $$\mathbf{1}_{[0,1]}, \mathbf{1}_{[0,\frac{1}{2}]}, \mathbf{1}_{[\frac{1}{2},1]}, \mathbf{1}_{[0,\frac{1}{4}]}, \mathbf{1}_{[\frac{1}{4},\frac{1}{2}]}, \mathbf{1}_{[\frac{1}{2},\frac{3}{4}]} \cdots$$
>
> Then $f_n \rightsquigarrow 0$ but $f_n(x)$ does not converge to 0 for any point $x \in [0, 1]$.
>
> Some students took $E = \mathbb{Q}$, $f_n = 0$ and $f = \mathbf{1}_{\mathbb{Q}}$. This is technically correct for the question, but not really a counterexample to the converse of (a) since it is still converging pointwise a.e....

[8] (d) However, show that if $f_n \rightsquigarrow f$, there exists a subsequence such that $f_{n_k} \to f$ pointwise a.e..

> **Solution:** Let $\epsilon = \frac{1}{2^k}$. By assumption, there exists a subsequence n_k such that
>
> $$\mu\{x \in E \; : \; |f_{n_k}(x) - f(x)| \geq \frac{1}{2^k}\} < \frac{1}{2^k}$$
>
> Let us call $E_k := \{x \in E \; : \; |f_{n_k}(x) - f(x)| \geq \frac{1}{2^k}\}$.
>
> Then $f_{n_k}(x) \to f(x)$ if x lies in finitely many E_k.
>
> By Borel–Cantelli Lemma, since $\sum_{k=1}^{\infty} \mu(E_k) < 1$, the set $\{x \text{ lies in infinitely many } E_k\}$ has measure zero, hence $f_{n_k} \to f$ pointwise a.e. as required.

PROBLEM 3. (28 points)

Let $B := B(0,1) \subset \mathbb{R}^N$ be the unit open ball (in the standard Euclidean norm) and let $\overline{B}$ be its closure. Let $F : \mathbb{R}^N \to \mathbb{R}^N$ be a function such that

- F is continuous on $\overline{B}$ and $\mathcal{C}^1$ on B,
- $F(x) = x$ if $x \in \partial B := \overline{B} \setminus B$,
- The Jacobian $DF(x)$ is invertible for any $x \in B$.

[8] (a) Show that if $U \subset B$ is an open subset, then $F(U)$ is open.

Solution: For any $x \in B$, since $DF(x)$ is invertible, by Inverse Function Theorem, there exists open set $x \in U_x$ such that F has a continuous inverse $G = F^{-1}$ locally. In particular if $V \subset U_x$ is open, $F(V) = G^{-1}(V)$ is open.

Now for any $x \in U$, there exists open neighborhood $B(x,\epsilon) \subset U$. Then $V_x = B(x,\epsilon) \cap U_x \subset U$ is open in U_x, so $F(V_x)$ is open by above, and

$$F(U) = F\left(\bigcup_{x \in U} V_x\right) = \bigcup_{x \in U} F(V_x)$$

is open.

[12] (b) Show that the image $F(B) = B$.

Solution: Consider the restriction of F to $\overline{B}$.

Since $\overline{B}$ is compact, the continuous function $\|F\|$ has a maximum by Extreme Value Theorem. Since B is open, by (a) the image $F(B)$ is open. Hence $\|F\|$ cannot attain a maximum on B. Therefore $\|F\|$ must attain maximum on the boundary $\partial B = \overline{B} \setminus B$, where F acts as identity. Hence $\|F(x)\| < 1$ for any $x \in B$, which means $F(B) \subset B$.

On the other hand, if $B \setminus F(B)$ is nonempty, let $x \in B \setminus F(B)$.
Since it does not lie on the boundary, $x \in B \setminus F(\overline{B})$ which is open since $\overline{B}$ is compact. We also have $B \setminus F(\overline{B}) \subset B \setminus F(B)$. Therefore $B \setminus F(B) = B \setminus F(\overline{B})$ is open.
Hence

$$B = (B \setminus F(B)) \coprod F(B)$$

is a disjoint union of two open sets, a contradiction since B is connected.
(If $B = U \coprod V$, then the line segment (excluding the endpoints) joining any $x \in U$ and $y \in V$ will be split into 2 or more disjoint open intervals, which is impossible.)

[8] (c) Show that for each $y_0 \in F(B)$, $F^{-1}(y_0) \cap B$ is a finite set.

Solution: Assume there exists $y_0 \in F(B) = B$ such that $F^{-1}(y_0) \cap B \subset \overline{B}$ is infinite. Since $\overline{B}$ is compact, it has an accumulation point $\alpha \in \overline{B}$.

If $\alpha \in B$, then for any $\epsilon > 0$ the ball $B(\alpha,\epsilon)$ contains infinitely many points $\{x \in B : F(x) = y_0\}$, a contradiction since F is locally invertible by (a).

On the other hand, if $\alpha \in \partial B$, then $F(\alpha) = \alpha$. However by continuity $F(\alpha) = y_0$ also, again a contradiction since $y_0 \notin \partial B$.

PROBLEM 4. (28 points)

Let V be a Banach space and $f : V \to \mathbb{R}$ be a function such that $f^{-1}(-\infty, a)$ is open for any $a \in \mathbb{R}$. (This is known as **upper semi-continuous function**.)

Let $m : V \to \mathbb{R} \cup \{-\infty\}$ be defined by

$$m(x) := \liminf_{y \to x} f(y), \qquad x \in V$$
$$:= \lim_{\delta \to 0^+} \left(\inf_{0 < \|y - x\| < \delta} f(y) \right).$$

Finally let

$$S_n := \{x \in V : f(x) - m(x) \geq \frac{1}{n}\}, \qquad n \in \mathbb{Z}_{>0}.$$

[10] (a) Show that S_n is closed in V.

Solution: If $x_0 \in S_n^c$, then there exists $\epsilon > 0$ such that

$$f(x_0) - m(x_0) < \frac{1}{n} - \epsilon < \frac{1}{n}.$$

By definition, $x_0 \in f^{-1}(-\infty, f(x_0) + \frac{\epsilon}{2})$ is open, hence there exists $\delta_1 > 0$ such that $|x - x_0| < \delta_1$ implies

$$f(x) - f(x_0) < \frac{\epsilon}{2}$$

On the other hand, there exists $\delta_2 > 0$ such that

$$m(x_0) - \inf_{0 < \|x - x_0\| < \delta_2} f(x) < \frac{\epsilon}{2}$$

hence rearranging, $\|x - x_0\| < \delta_2$ implies

$$m(x_0) < f(x) + \frac{\epsilon}{2}$$

Also

$$m(x) = \liminf_{y \to x} f(y) \geq m(x_0) - \frac{\epsilon}{2}$$

Hence for $\delta = \min(\delta_1, \delta_2)$ and $x \in B(x_0, \delta)$,

$$f(x) - m(x) < f(x_0) + \frac{\epsilon}{2} - m(x_0) + \frac{\epsilon}{2} < \frac{1}{n} - \epsilon + \epsilon = \frac{1}{n}$$

so that $B(x_0, \delta) \subset S_n^c$, and S_n^c is open.

[6] (b) Show that if U is an open set contained in S_n, then $m(x) = -\infty$ for any $x \in U$.

Solution: Suppose there is $x_0 \in U$ with $m(x_0) > -\infty$.

Then as in (a) there exists $\delta > 0$ such that $m(x) > m(x_0) - \frac{1}{2n}$ for $x \in B(x_0, \delta) \subset U$.

By definition of $m(x)$, if $m(x_0) < \infty$, there exists $\delta_1 < \delta$ such that

$$\inf_{0 < \|y - x_0\| < \delta_1} f(y) - m(x_0) < \frac{1}{2n},$$

hence there must exists $\overline{x} \in B(x_0, \delta_1) \subset B(x_0, \delta) \subset U$ such that $f(\overline{x}) < m(x_0) + \frac{1}{2n}$. Therefore

$$f(\overline{x}) - m(\overline{x}) < m(x_0) + \frac{1}{2n} - m(x_0) + \frac{1}{2n} < \frac{1}{n},$$

i.e. $\overline{x} \notin S_n$, a contradiction.

[12] (c) Show that the set $\mathcal{S} := \{x \in V \ : \ f(x) \leq m(x)\}$ is dense in V.

Solution: We show that S_n is (closed) nowhere dense. Then since V is a complete metric space,

$$\mathcal{S} = \{x \in V \ : \ f(x) - m(x) \leq 0\} = \left(\bigcup_{n=1}^{\infty} S_n \right)^c$$

is dense by Baire's Category Theorem.

Suppose S_n is not nowhere dense. Since S_n is closed, it has nonempty interior, e.g. an open ball $U \subset S_n$.

By (b), $m(x) = -\infty$ for $x \in U$, hence the set

$$A_1 = \{x \in U \ : \ f(x) < -1\} = U \cap f^{-1}(-\infty, -1)$$

is *non-empty* and open. Take open ball $U_1 = B(x_1, 1)$ such that $\overline{U_1} \subset A_1$.

In the same way, the set $A_2 = \{x \in U_1 \ : \ f(x) < -2\} \subset U_1$ is non-empty and open. Take open ball $U_2 := B(x_2, \frac{1}{2})$ such that $\overline{U_2} \subset A_2 \subset U_1$. Hence $f(x) < -2$ for $x \in \overline{U_2}$.

By induction, we obtain nested closed balls

$$\overline{U_1} \supset \overline{U_2} \supset \overline{U_3} \supset \cdots$$

where $U_n = B(x_n, \frac{1}{n})$ for some $x_n \in V$ and $f(x) < -n$ for any $x \in \overline{U_n}$.

But then $\displaystyle\bigcap_{n=1}^{\infty} \overline{U_n}$ is nonempty since V is complete and diam $U_n \to 0$.

(Lecture #03: Take any $x_k \in U_k$, then $x_i, x_j \in U_k$ for $i, j > k$, hence $\|x_i - x_j\| < \frac{1}{2k}$ and we have a Cauchy sequence, which converges to $x_0 \in \bigcap_{n=1}^{\infty} \overline{U_n}$ since V is complete.)

Therefore $f(x_0) = -\infty$ for $x_0 \in \displaystyle\bigcap_{n=1}^{\infty} \overline{U_n}$, a contradiction since f is finite-valued.

–End of Paper–