Solutions to Quiz Version 1

If your ID is an odd number, you should take Version 1 quiz (this one). If your ID is an even number, you should take Version 2 quiz.

Problem 1. (10 points) Determine whether the given subsets of the group $SL_2(\mathbb{R})$ is a subgroup (just answer "yes" or "no", no need to give reasons).
(1) The set of matrices in $SL_2(\mathbb{R})$ with all the entries in $\mathbb{Z}$.
(2) The set $\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in \mathbb{R}, a^2 + b^2 = 1\}$
(3) The set of matrices in $SL_2(\mathbb{R})$ with all the entries non-negative
(4) The set $\{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a > 0, b \in \mathbb{R}\}$
(5) The set $\{\begin{pmatrix} 1-t & t \\ -t & 1+t \end{pmatrix} \mid t \in \mathbb{R}\}$

Answer: (1) Yes (2) Yes (3) No (4) Yes (5) Yes

Problem 2.(10 points) Find group homomorphisms satisfying the required conditions (no need to give reasons).
(1). $\Phi : \mathbb{R}^\times \to \mathbb{Z}_2 = \{0, 1\}$, ϕ is surjective.
(2). $\Phi : GL_{10}(\mathbb{R}) \to \mathbb{R}$, , $\Phi(2I_{10}) = 3$
(3). $\Phi : \mathbb{C}^\times \to \mathbb{C}^\times$, Φ is surjective and $|Ker(\Phi)| = 2022$.
(4). $\Phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, $\Phi(1, 2, 3) = (2, 3, 1)$.
(5). $\Phi : S_3 \to S_5$, Φ is injective.

Answer: The answers for some questions are not unique.
(1) $\Phi(a) = 0$ for $a > 0$; $\Phi(a) = 1$ for $a < 0$.
(2) $\Phi(A) = \frac{3}{10 \log 2} \log |\det A|$
(3) $\Phi(z) = z^{2022}$
(4) $\Phi(x, y, z) = (y, z, x)$
(5) For $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ x & y & z \end{pmatrix}$, $\Phi(\sigma) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ x & y & z & 4 & 5 \end{pmatrix}$.

Problem 3. (15 points) Determine each of following statements true or

false (no reasons needed)
(1). Let $C[1,7]$ be the ring of continuous functions on the interval $[1,7]$, let $\alpha : [1,7] \to [1,7]$ be a continuous map, then the map $\Phi_\alpha : C[1,7] \to C[1,7]$ given by $\Phi_\alpha(f) = f \circ \alpha$ is a ring homomorphism.
(2). Let $C[1,7]$ be the ring of continuous functions on the interval $[1,7]$, the set $I = \{f \in C[1,7] \,|\, f(a) = 0 \text{ for all } a \in [2,3]\}$ is an ideal of $C[1,7]$
(3). Let R be a commutative ring, the set of units in R is a group under the multiplication.
(4). M be a module over a commutative ring R, let $End(M)$ be the set of R-module homomorphisms from M to M itself, then $End(M)$ is a ring with point-wise addition and with multiplication as the composition.
(5). Every finitely generated module over a principal ideal domain is a free module.

Answer: (1) True (2) True (3) True (4) True (5) False

Theorem 4.(15 points) Let G and G' be finite groups with $|G|$ and $|G'|$ relatively prime, if $A \subset G \times G'$ is a subgroup. Prove that there are subgroups $H \subset G$ and $H' \subset G'$ such that $A = H \times H'$.

Remark: Any proof that does not use the condition $|G|$ and $|G'|$ are relatively prime is wrong.

Proof 1. Let $\pi : A \to G$ be the map defined as $\pi(a, a') = a$, $\pi' : A \to G'$ be the map defined as $\pi'(a, a') = a'$. It is clear that π, π' are group homomorphisms. Let $H = Im\,\pi$ and $H' = Im\,\pi'$. They are subgroups of G and G' respectively. Obviously we have $A \subset H \times H'$. Because $H = \pi(A)$, by Homomorphism Theorem, $|H|$ is a divisor of $|A|$, similarly, $|H'|$ is a divisor of $|A|$. By Langrange Theorem, $|H|$ is a divisor of $|G|$ and $|H'|$ is a divisor of G'. The condition that $|G|$ and $|G'|$ are relatively prime implies that $|H|$ and $|H'|$ are relatively prime. So $|H \times H'| = |H|\,|H'|$ is a divisor of $|A|$, this together with $A \subset H \times H'$ implies that $A = H \times H'$.

Proof 2. Let $\pi : A \to G$ be the map defined as $\pi(a, a') = a$, $\pi' : A \to G'$ be the map defined as $\pi'(a, a') = a'$. It is clear that π, π' are group homomorphisms. Let $H = Im\,\pi$ and $H' = Im\,\pi'$. They are subgroups of G and G' respectively. Obviously we have $A \subset H \times H'$. We now prove the reverse inclusion $H \times H' \subset A$. For arbitrary $a \in H$, we have $(a, b) \in A$ for some $b \in G'$. Since $|G|$ and $|G'|$ are relatively prime, there are integers m, n such that $m|G| + n|G'| = 1$. Using $a^{|G|} = 1$, $b^{|G'|} = 1'$, we have

$$(a, b)^{n|G'|} = (a^{n|G'|}, b^{n|G'|}) = (a^{n|G'|}, 1') = (a^{1-m|G|}, 1') = (a, 1')$$

Since $(a, b) \in A$, so $(a, 1') \in A$. Similarly we can prove $a' \in H'$ implies $(1, a') \in A$. So for arbitrary $(a, a') \in H \times H'$, $(a, a') = (a, 1')(1, a') \in A$.

Theorem 5.(10 points) Let $\phi : \mathbb{C} \to \mathbb{C}$ be a ring homomorphism such that $\phi(r) = r$ for all $r \in \mathbb{R}$. Prove that ϕ is either the identity map or the complex conjugate map, that is, $\phi(z) = \bar{z}$.

Proof. First we note that $\phi(i)\phi(i) = \phi(i^2) = \phi(-1) = -1$. So $\phi(i) = i$ or $\phi(i) = -i$. If $\phi(i) = i$, $\phi(a + bi) = \phi(a) + \phi(b)\phi(i) = a + bi$, this proves ϕ is the identity map. If $\phi(i) = -i$, $\phi(a + bi) = \phi(a) + \phi(b)\phi(i) = a - bi$, this proves ϕ is the complex conjugate.

Theorem 6.(15 points) Let X_k be the set of all k-dimensional subspaces in $\mathbb{R}^n$ $(1 \le k < n)$ Let $GL_n(\mathbb{R})$ act on X_k as follows, for $g \in GL_n(\mathbb{R})$, $V \in X_k$, since g is a linear isomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$, g transforms the subspace V to a subspace which we denote by gV. (1) How many orbits does X_k have? (2) Let $GL_n(\mathbb{R})$ act on $X_k \times X_k$ by $g(V_1, V_2) = (gV_1, gV_2)$, how many orbits does $X_k \times X_k$ have?

Proof. (1) there is only one orbit. For any $V \in X_k$, so V is a k-dimensional subspace in $\mathbb{R}^n$. Take a basis of $v_1, \dots, v_k$ of V, we extend the basis to a basis $v_1, \dots, v_k, v_{k+1}, \dots, v_n$ of $\mathbb{R}^n$, let g be the linear map sends e_i to v_i, then $g\mathrm{Span}(e_1, \dots, e_k) = V$. This proves every $V \in X_k$ is in the orbit of $\mathrm{Span}(e_1, \ldots, e_k)$.

(2). We note the following fact: for $(V_1, V_2), (W_1, W_2) \in X_k \times X_k$, (V_1, V_2) and (W_1, W_2) are in the same orbit if and only if $\dim(V_1 \cap V_2) = \dim(W_1 \cap W_2)$.

Proof of the fact: If (V_1, V_2) and (W_1, W_2) are in the same orbit, so there is $g \in GL_n(\mathbb{R})$ such that $W_1 = gV_1, W_2 = gV_2$, $W_1 \cap W_2 = (gV_1) \cap (gV_2) = g(V_1 \cap V_2)$, so it has the same dimension as $V_1 \cap V_2$. Conversely, suppose $\dim(V_1 \cap V_2) = \dim(W_1 \cap W_2) = j \leq k$, take a basis $v_1, \ldots, v_j$ of $V_1 \cap V_2$, we extend the basis to a basis of $\mathbb{R}^n$

$$v_1, \ldots, v_j, a_{j+1}, \ldots, a_k, b_{j+1}, \ldots, b_k, c_{2j-k+1}, \ldots, c_n$$

such that it satisfies the following conditions:
(a) $v_1, \ldots, v_j, a_{j+1}, \ldots, a_k$ is a basis for V_1,
(b) $v_1, \ldots, v_j, b_{j+1}, \ldots, b_k$ is a basis for V_2. Similarly we take a basis $w_1, \ldots, w_j$ of $W_1 \cap W_2$, we extend the basis to a basis of $\mathbb{R}^n$

$$v_1, \ldots, v_j, a'_{j+1}, \ldots, a'_k, b'_{j+1}, \ldots, b'_k, c'_{2j-k+1}, \ldots, c'_n$$

satisfying the similar conditions as (a) (b). Let g be the linear isomorphism on $\mathbb{R}^n$ that maps v_i to w_i, a_i to a'_i, b_i to b'_i, c_i to c'_i. Then $g(V_1, V_2) = (W_1, W_2)$.

Final answer: If $2k \leq n$, $\dim(V_1 \cap V_2)$ can be $0, 1, \ldots, k$, so there are $k+1$ orbits. If $2k > n$, $\dim(V_1 \cap V_2)$ can be $2k-n, 2k-n+1, \ldots, k$, there are $n-k+1$ orbits.

Theorem 7.(15 points) Let R be an integral domain. Let $I \subset R$ be an ideal, suppose I is NOT a principal ideal, prove that I is NOT a free R-module.

Proof. Assume I is a free module, so it has a basis $\{v_i\}_{i \in I}$. Since I is NOT principal, $|I| \geq 2$. let $i, j \in I$, $i \neq j$, then $v_j v_i + (-v_i) v_j = 0$, it contradicts to that v_i, v_j are R-linearly independent.

Theorem 8.(10 points) Let F be a field, $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in F[x]$ be a monic polynomial of degree $n \geq 1$. We define a map $\Phi : F[x]/(f) \to F(x)/(f)$ by

$$\Phi(h(x) + (f)) = xh(x) + (f)$$

It is clear that Φ is a linear map over F. Prove that the characteristic polynomial of Φ, defined as $\det(\lambda I - \Phi)$ (where I is the identity map on $F[x]/(f)$), is

$$f(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

Proof. We take the following basis $1 + (f), x + (f), \ldots, x^{n-1} + (f)$ of $F[x]/(f)$. With respect to this basis, the linear operator Φ has the following matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & 0 & \ldots & 0 & -a_1 \\ 0 & 1 & 0 & \ldots & 0 & -a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -a_{n-1} \end{pmatrix}$$

One compute (use induction on n) $\det(\lambda I_n - A)$ to get $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$.