MATH 5431 Midterm Exam – 2023

9:00 AM – 11:50 AM, October 18rd, 2023

Please show the work to the questions and late submission is not allowed.

1. Suppose X follows a k_1-parameter exponential family distribution and Y follows a k_2-parameter exponential family distribution. Please (1) write down the density and the natural parameter space of X; (2) show that the natural parameter space of Y is convex and write down the conditions when Y is of full rank; (3) show that if X is independent of Y, (X, Y) follows a $(k_1 + k_2)$-parameter exponential family distribution, and if X and Y are i.i.d. ($k_1 = k_2$), then (X, Y) follows a k_1-parameter exponential family distribution.

   Solution: (1) by definition; (2) $\int e^{\sum_{i=1}^{i=k_1}(\alpha\eta_{1,i}+(1-\alpha)\eta_{2,i})(\theta)T_i(x)}h(x)dx \le \alpha \int e^{\sum_{i=1}^{i=k_1}\eta_{1,i}(\theta)T_i(x)}h(x)dx + (1-\alpha)\int e^{\sum_{i=1}^{i=k_1}\eta_{2,i}(\theta)T_i(x)}h(x)dx < \infty$; (3) by definition.

2. Suppose $X = (X_1, \ldots, X_n)$ where $n > 1$ and $X_i \sim^{i.i.d} F_\theta$ for $i = 1, \ldots, n$. Please (1) show that $(X_1, \ldots, X_n)$ is sufficient statistics and $(X_1, \ldots, X_{n-1})$ is not sufficient statistics; (2) find minimal sufficient statistics for θ when F_θ denotes the uniform distribution $U(-\theta, \theta)$ and $\theta > 0$; (3) show that $X_{(n)} - X_{(1)}$ is independent of $\bar{X} = \sum_{i=1}^{i=n} X_i/n$ when F_θ denotes the normal distribution $N(\theta, 1)$.

   Solution: (1) by definition; (2) $T = \max(-X_{(1)}, X_{(n)}, 0)$; (3) Basu theorem.

3. Suppose $X = (X_1, \ldots, X_n)$ where $n > 10$ and $X_i \sim^{i.i.d} F_\theta$ for $i = 1, \ldots, n$. Please (1) find the UMVUE of θ and $P_\theta(\sum_{i=1}^{i=n-1} X_i > X_n)$ when F_θ denotes the bernoulli distribution $Bin(\theta)$; (2) find the UMVUE of μ and $\mu + \sigma^2$ when F_θ denotes the normal distribution $N(\mu, \sigma^2)$ and $\theta = (\mu, \sigma^2)$; (3) provide an example that UMVUE does not exist and show UMVUE is unique if it exists.

   Solution: (1) $\bar{X}$ and by Lehmann-Shceffe Theorem, $1 - \bar{X}$ when $\bar{X} \le 2/n$ and 1 when $\bar{X} > 2/n$; (2) $\bar{X}$ and by characterization theorem, $\bar{X} + S^2$; (3) $X \sim Bin(n, p)$ and $\theta = \sqrt{p}$.

4. Suppose $X = (X_1, \ldots, X_n)$ where $n > 1$ and $X_i \sim^{i.i.d} F_\theta$ for $i = 1, \ldots, n$. Please (1) write down the likelihood of X and calculate the MLE for θ

when F_θ denotes the uniform distribution $U(\theta, \theta+1)$; (2) calculate the MLE for θ^2 and show its asymptotic distribution when F_θ denotes the normal distribution $N(1, \theta^2)$; (3) calculate the MLE for θ and θ^2 when F_θ denotes the bernoulli distribution $Bin(\theta)$ and $\theta \geq 0.5$.

Solution: (1) $\hat{\theta} \in (X_{(n)} - 1, X_{(1)})$; (2) $\hat{\theta}^2 = 1/n \sum_{i=1}^{i=n} (X_i - 1)^2$ and $\sqrt{n}(\hat{\theta}^2 - \theta^2) \longrightarrow N(0, 2\theta^4)$; (3) $\max(0.5, \bar{X})$ and $\max(0.5, \bar{X})^2$.

5. Suppose $X = (X_1, \ldots, X_n)$ where $n > 1$ and $X_i \sim^{i.i.d} F_\theta$ for $i = 1, \ldots, n$. Please (1) write down the definition of information matrix $I_X(\theta)$ and show that $I_X(\theta) = nI_{X_1}(\theta)$; (2) show that if $\theta = h(\eta)$ and h is one to one map, $I_X(\theta) = I_X(\eta)$; (3) show that asymptotic efficient estimate is not unique if it exists and provide an example that C-R lower bound is not achievable.

   Solution: (1) $I_X(\theta) = E_\theta(l_X^{'}(\theta))^T(l_X^{'}(\theta))$; (3) if $\hat{\theta}$ is asymptotic efficient, $\hat{\theta} + 1/n$ is asymptotic efficient. $X \sim Bin(n, p)$ and $\theta = p^2$.