Final Examination - Suggested Solutions

MATH 3043: Honors Real Analysis

Fall 2021, HKUST

December 17th, 2021, 16:30 - 19:30

In this Exam, unless otherwise specified,

- **every function is real-valued (including L^p-spaces),**
- **"measurable" means Lebesgue measurable,**
- **μ denotes the Lebesgue measure on $\mathbb{R}^N$.**
- **We assume the Axiom of Choice.**

PROBLEM 1. (30 points)

In this question, $f \in L^p$ means $|f|^p$ is integrable only, ignoring the equivalent classes. In other words, we distinguish two functions even if they are the same a.e..

Let $1 \le p < \infty$, $M > 0$, and $\{f_n\}$ be a sequence of functions on $(0, 1)$.

Prove or give a counterexample to the following statements.

[5] (a) If $f_n \in L^p[0,1]$ and $f_n \to f \in L^p[0,1]$ pointwise a.e., then $f_n \to f$ in L^p-norm.

Solution: False. Take $f_n = \begin{cases} n^{1/p} & 0 \le x \le \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$. Then $f_n \to 0$ a.e., but $\|f_n - 0\|_{L^p} = 1$ for any n.

[5] (b) If $f_n \in L^p[0,1]$ and $f_n \to f \in L^p[0,1]$ in L^p-norm, then $f_n \to f$ pointwise a.e..

Solution: False. Use the same example from Midterm Q2(c):

Take the sequence f_n to be the enumeration of length $\frac{1}{2^k}$ subintervals of $[0, 1]$, i.e.

$$\mathbf{1}_{[0,1]}, \mathbf{1}_{[0,\frac{1}{2}]}, \mathbf{1}_{[\frac{1}{2},1]}, \mathbf{1}_{[0,\frac{1}{4}]}, \mathbf{1}_{[\frac{1}{4},\frac{1}{2}]}, \mathbf{1}_{[\frac{1}{2},\frac{3}{4}]} \cdots$$

Then $\|f_n - 0\|_{L^p} = \frac{1}{2^k} \to 0$ but f_n does not converge to 0 for any point $x \in [0, 1]$.

[5] (c) If $f_n \in L^p[0,1]$ and $f_n \to f \in L^p[0,1]$ uniformly, then $\lim\limits_{n\to\infty} \|f_n\|_{L^p} = \|f\|_{L^p}$.

Solution: True. For any $\epsilon > 0$, there exists N such that $n > N$ implies $|f_n - f| < \epsilon$. Then

$$\|f_n - f\|_{L^p}^p = \int |f_n - f|^p < \epsilon^p$$

Hence $\|f_n - f\|_{L^p} \to 0$ and $|\|f_n\|_{L^p} - \|f\|_{L^p}| \leq \|f_n - f\|_{L^p} \to 0$ as well.

[5] (d) If $f_n \in L^1[0,1]$, $f_n \to f \in L^1[0,1]$ uniformly, and f_n is differentiable a.e., then f is differentiable a.e..

Solution: False.

Example 1. The Weierstrass function $f(x)$ is continuous nowhere differentiable. In particular it is Riemann integrable and hence in $L^1[0,1]$.

By Stone–Weierstrass Theorem, we can approximate $f(x)$ by a sequence of polynomials uniformly.

Example 2. Let q_n be the enumeration of $\mathbb{Q} \cap [0,1]$ with increasing denominator. If $q_n = \frac{a}{b}$ in the lowest form, define $d(q_n) := \frac{1}{b}$.

Then $f_n := \sum_{k=1}^n d(q_k)$ is constant zero except finitely many points, hence differentiable a.e., but f_n converges to the Thomae's function uniformly, which is not differentiable everywhere.

[5] (e) If $f_n \in AC[0,1]$, $f_n \to f$ uniformly, and $|f_n(x)| \leq M$ a.e., then $f \in AC[0,1]$, where $AC[0,1]$ denotes absolutely continuous functions on $[0,1]$.

Solution: False. Let

$$f_n(x) := \begin{cases} \frac{\sin n}{n} & 0 \leq x \leq \frac{1}{n} \\ x \sin \frac{1}{x} & x > \frac{1}{n} \end{cases} .$$

Then f_n is AC and bounded by 1, but converges uniformly to $x \sin \frac{1}{x}$ which is not AC.

[5] (f) If $f_n \in BV[0,1]$, $f_n \to f$ uniformly, and $T_{f_n} \leq M$, then $f \in BV[0,1]$, where $BV[0,1]$ denotes bounded variation functions on $[0,1]$ and T_f its total variation.

Solution: True. Let $P = (x_i)$ be any partition. Then

$$\sum |f_n(x_i) - f_n(x_{i-1})| \leq M$$

Uniform convergence is also pointwise convergence, hence taking limit, we have

$$\sum |f(x_i) - f(x_{i-1})| \leq M$$

for any partition, so taking supremum over all partitions, we have $T_f \leq M$ and f is BV.

PROBLEM 2. (20 points)

In this question, $\int_{\mathbb{R}_{\geq 0}}$ denotes Lebesgue integration, and $\int_0^\infty$ denotes (improper) Riemann integration.

[5] (a) For $\alpha > 0$, prove that $e^{-\alpha x}$ is Lebesgue integrable over $\mathbb{R}_{\geq 0}$ and evaluate

$$\int_{\mathbb{R}_{\geq 0}} e^{-\alpha x} dx.$$

Justify your steps carefully.

Solution:

$$\begin{aligned}
\int_{\mathbb{R}_{\geq 0}} e^{-\alpha x} dx &= \int \mathbf{1}_{[0,\infty)} e^{-\alpha x} dx \\
&= \int \lim_{M\to\infty} \mathbf{1}_{[0,M]} e^{-\alpha x} dx \\
&= \lim_{M\to\infty} \int \mathbf{1}_{[0,M]} e^{-\alpha x} dx && \text{(MCT)} \\
&= \lim_{M\to\infty} \int_{[0,M]} e^{-\alpha x} dx \\
&= \lim_{M\to\infty} \int_0^M e^{-\alpha x} dx && \text{(Riemann = Lebesgue)} \\
&= \lim_{M\to\infty} (\frac{1}{\alpha} - e^{-\alpha M}\alpha) \\
&= \frac{1}{\alpha}
\end{aligned}$$

In particular $e^{-\alpha x}$ is Lebesgue integrable over $\mathbb{R}_{\geq 0}$.

[5] (b) For $\beta > 0$, compute

$$\int_0^M e^{-\beta x} \sin x dx.$$

(This is MATH1024 Level.)

Solution: By integration by parts twice,

$$\begin{aligned}\int_0^M e^{-\beta x}\sin x dx &= [-e^{-\beta x}\cos x]_0^M - \beta\int_0^M e^{-\beta x}\cos x dx\\ &= 1 - e^{-\beta M}\cos M - \beta[e^{-\beta x}\sin x]_0^M - \beta^2\int_0^M e^{-\beta x}\sin x dx\\ &= 1 - e^{-\beta M}\cos M - \beta e^{-\beta M}\sin M - \beta^2\int_0^M e^{-\beta x}\sin x dx\end{aligned}$$

Hence

$$\int_0^M e^{-\beta x}\sin x dx = \frac{1 - e^{-\beta M}\cos M - \beta e^{-\beta M}\sin M}{1+\beta^2}$$

[10] (c) For $\gamma > 0$, using (a) and (b), evaluate

$$\int_0^\infty \frac{\sin x}{x} e^{-\gamma x} dx.$$

Justisfy your steps carefully.

Solution:

$$\begin{aligned}&\int_0^\infty \frac{\sin x}{x} e^{-\gamma x} dx\\ &= \lim_{M\to\infty}\int_0^M \frac{\sin x}{x} e^{-\gamma x} dx\\ &= \lim_{M\to\infty}\int_0^M \int_{\mathbb{R}_{\geq 0}} e^{-xt}\sin x e^{-\gamma x} dt dx \qquad \text{(by (a))}\\ &= \lim_{M\to\infty}\int_0^M \int_{\mathbb{R}_{\geq 0}} e^{-(\gamma+t)x}\sin x dt dx\end{aligned}$$

Since $|\sin x| \leq |x|$,

$$\int_0^M \int_{\mathbb{R}_{\geq 0}} |e^{-(\gamma+t)x}\sin x| dt dx = \int_0^M |\frac{\sin x}{x} e^{-\gamma x}| dx \leq \int_{\mathbb{R}_{\geq 0}} e^{-\gamma x} dx = \frac{1}{\gamma}$$

is integrable by (a), hence by Tonelli-Fubini, we can interchange the limit of integration and obtain

$$\begin{aligned}&= \lim_{M\to\infty}\int_{\mathbb{R}_{\geq 0}}\int_0^M e^{-(\gamma+t)x}\sin x dx dt\\ &= \lim_{M\to\infty}\int_{\mathbb{R}_{\geq 0}} \frac{1 - e^{-(\gamma+t)M}\cos M - (\gamma+t)e^{-(\gamma+t)M}\sin M}{1+(\gamma+t)^2} dt\end{aligned}$$

We calculate each part:

$$\lim_{M\to\infty}\int_{\mathbb{R}_{\geq 0}} \frac{1}{1+(\gamma+t)^2}dt = [\arctan(\gamma+t)]_0^\infty = \frac{\pi}{2} - \arctan\gamma$$

$$\begin{aligned}
&\lim_{M\to\infty}\int_{\mathbb{R}_{\geq 0}} \left| \frac{-e^{-(\gamma+t)M}\cos M - (\gamma+t)e^{-(\gamma+t)M}\sin M}{1+(\gamma+t)^2} \right| dt \\
&\leq \lim_{M\to\infty}\int_{\mathbb{R}_{\geq 0}} e^{-(\gamma+t)M}\frac{1+\gamma+t}{(1+(\gamma+t)^2)}dt \\
&\leq \lim_{M\to\infty}\int_{\mathbb{R}_{\geq 0}} e^{-(\gamma+t)M}A dt && \text{Since } \tfrac{1+a}{1+a^2} \text{ is bounded} \\
&= \lim_{M\to\infty}\frac{Ae^{-\gamma M}}{M} = 0 && \text{by (a)}
\end{aligned}$$

We conclude that

$$\int_0^\infty \frac{\sin x}{x}e^{-\gamma x}dx = \frac{\pi}{2} - \arctan\gamma = \arctan\frac{1}{\gamma}$$

Note. Differentiation under the integral sign is not useful for this problem: We only allow doing that for closed interval, you need to integrate the result over $\mathbb{R}_{\geq 0}$, and you end up calculating $\int_0^\infty \frac{\sin x}{x}dx$ which is not much different from this problem.

The grading for this problem puts heavy emphasis on justifying the integrations.

PROBLEM 3. (25 points)

[10] (a) Prove that there exists a complete orthonormal basis $\{\phi_k\}_{k=1}^\infty$ of $L^2[0,1]$ such that each $\phi_k(x)$ is a polynomial in x^2.

Solution: Take $\{1, x^2, x^4, ...\}$ which is linearly independent. Applying Gram-Schmidt process, we obtain an orthonormal set $\mathcal{B}$ which are polynomials in x^2. Furthermore, span$(\mathcal{B})$ is the set of all polynomials in x^2.

By Stone–Weierstrass Theorem, the set of polynomials in x^2 (separate points, vanish at no points) is dense in $C^0[0,1]$ with the sup-norm.

Alternatively. By Stone–Weierstrass Theorem, polynomials are dense in $C^0[0,1]$ with the sup-norm. If $p(x)$ approximate $g(x)$ uniformly, then

$$|p(x^2) - g(x^2)| \leq \|p-g\|_\infty$$

Hence $p(x^2)$ approximate $g(x^2)$ uniformly also. Note that any continuous function $f(x)$ can be written as $g(x^2)$ where $g(x) = f(\sqrt{x})$.

If a subset is dense in sup-norm, then it is also dense in L^2-norm, since

$$\|f\|_{L^2} = \left(\int_0^1 |f|^2\right)^{\frac{1}{2}} \le \|f\|_\infty$$

(also follows from Q1(c)).

Since $C^0[0,1]$ is dense in $L^2[0,1]$, we conclude that polynomials in x^2 is dense in $L^2[0,1]$, hence $\mathcal{B}$ is a complete orthonormal basis.

[5] (b) Prove that for any $x \in [0,1]$,

$$\sum_{k=1}^{\infty} \left| \int_0^x \phi_k(t)dt \right|^2 = x.$$

Solution: Since ϕ_k is an orthonormal basis, by Parsevel's identity,

$$\sum_{k=1}^{\infty} \left| \int_0^x \phi_k(t)dt \right|^2 = \sum_{k=1}^{\infty} \left| \int_0^1 \mathbf{1}_{[0,x]}\phi_k(t)dt \right|^2 = \sum_{k=1}^{\infty} \left|(\mathbf{1}_{[0,x]}, \phi_k)\right|^2$$
$$= \|\mathbf{1}_{[0,x]}\|_{L^2}^2 = \int_0^1 |\mathbf{1}_{[0,x]}(t)|^2 dt = \int_0^1 \mathbf{1}_{[0,x]}(t)dt = x$$

[10] (c) Given a sequence of functions $f_n \in L^2[0,1]$ with $\|f_n\|_{L^2} = 1$, prove that there exists a subsequence $\{f_{n_k}\}$ and a function $f \in L^2[0,1]$ such that

$$\lim_{k\to\infty} (f_{n_k}, \phi_j) = (f, \phi_j) \qquad \forall j \ge 1.$$

Solution: Denote by $a_{ij} := (f_i, \phi_j)$. By assumption $|a_{ij}|^2 \le \sum_{j=1}^{\infty} |a_{ij}|^2 = \|f_i\|_{L^2}^2 = 1$ is bounded.

Consider the sequence $\{a_{11}, a_{21}, a_{31}, ...\}$. By Bolzano–Weierstrass Theorem, it has a convergent subsequence $a_{n_k,1}$. Denote the limit by a_1.

Now consider $\{a_{n_1,2}, a_{n_2,2}, a_{n_3,2}, ...\}$. Again it has a convergent subsequence $a_{n_{k_l},2}$ Denote the limit by a_2.

Continuing the construction, by diagonal argument, we pick the k-th element from the k-th subsequences and form a subsequence f_{n_k}. By construction

$$\lim_{k\to\infty} (f_{n_k}, \phi_j) = a_j, \qquad \forall j = 1, 2, 3, ...$$

Fix $N > 0$. Then for $0 < \epsilon < 1$, there exists f_{n_K} such that $|(f_{n_K}, \phi_j) - a_j| < \frac{\epsilon}{2^j}$ for $j = 1, ..., N$, and hence

$$\begin{aligned}
\sum_{j=1}^{N} |a_j|^2 &\leq \sum_{j=1}^{N} |(f_{n_K}, \phi_j) + \frac{\epsilon}{2^j}|^2 \\
&\leq 2\sum_{j=1}^{N} |(f_{n_K}, \phi_j)|^2 + 2\sum_{j=1}^{N} \frac{\epsilon^2}{2^{2j}} \qquad \text{since } (a+b)^2 \leq 2a^2 + 2b^2 \\
&\leq 2\|f_{n_K}\|^2 + \frac{2\epsilon^2}{3} < 3
\end{aligned}$$

Taking limit $N \to \infty$, we conclude that $\sum_{j=1}^{\infty} |a_j|^2 \leq 3$ and hence

$$f = \sum_{k=1}^{\infty} a_k \phi_k \in L^2[0,1]$$

is our required element.

Note. Bolzano–Weierstrass Theorem only works for *finite dimensional* spaces! $\|f_n\|$ bounded does not imply the existence of a converging subsequence f_{n_k}!

PROBLEM 4. (25 points)

Let $X \subset \mathbb{R}^N$ be a bounded measurable set. Assume $\mathcal{B}$ is a covering of X by open balls such that every $x \in X$ is the center of some balls $B(x, r) \in \mathcal{B}$, and $\sup_{B \in \mathcal{B}} r(B) < \infty$.

[10] (a) Prove that there exists a (finite or countably infinite) subcollection of balls $B_k := B(x_k, r_k) \in \mathcal{B}$ such that

- $|x_k - x_i| > r_i$ whenever $i < k$,
- $B(x_k, \frac{r_k}{3})$ are all disjoint,
- $\{\overline{B_k}\}$ covers X.

Solution: Let $M_1 = \sup_{B \in \mathcal{B}}(\text{radius}(B)) < \infty$.

By definition, there exists a ball $B_1 := B(x_1, r_1)$ with $r_1 = \text{radius}(B_1) > \frac{2}{3}M_1$.

Remove from the collection all balls with center $x \in \overline{B_1}$.

Now take B_2 to be the ball with $r_2 > \frac{2}{3}M_2$ where M_2 is the sup of the radius of the remaining balls.

Remove from the collection all balls with center $x \in \overline{B_2}$.

Continue in this fashion, either we exhaust all the balls and stop, in which we have a finite subcollections, or we get a countable sequence $B_k = B(x_k, r_k)$ of balls, such that for $i < k$,

- $r_i \geq \frac{2}{3}M_i \geq \frac{2}{3}r_k$
- $|x_k - x_i| > r_i$ since $x_k \notin B_1 \cap \cdots \cap B_i$

In particular $\widetilde{B_k} := B(x_k, \frac{1}{3}r_k)$ are all disjoint, since

$$|x_i - x_k| > r_i = \frac{1}{3}r_i + \frac{2}{3}r_i \geq \frac{1}{3}r_i + \frac{4}{9}r_k > \frac{1}{3}r_i + \frac{1}{3}r_i.$$

Since X is bounded, $r_k \to 0$ and hence $M_k \leq \frac{3}{2}r_k \to 0$.

Finally we show $\{\overline{B_k}\}$ covers X. Assume not, there exists $x \in X \setminus \bigcup_{k=1}^{\infty} \overline{B_k}$. By definition of $\mathcal{B}$, there exists a ball $B(x, \rho)$ which is not removed from each step.

If $\{B_k\}$ is finite this is a contradiction since the process stopped.

If $\{B_k\}$ is infinite then $\rho \leq M_k \to 0$, which is also a contradiction.

Note. It is not enough just saying that any x eventually will be covered, since this is a countable algorithm, you may get stuck covering only a portion of X.

If $\mathcal{G} := \{B_k\}$ satisfies the conditions in (a), then it is known (by the usual geometry of $\mathbb{R}^N$ only) that there exists a positive integer b_N (depending only on the dimension) such that for any $n \geq 1$, at most b_N balls from $\{B_1, ..., B_n\}$ intersect B_n.

[5] (b) Show that we can partition $\mathcal{G} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_{b_N}$ in such a way that every balls within each $\mathcal{B}_i$ are disjoint with one another.

Solution: Initially put $B_i \in \mathcal{B}_i$ for $i = 1, ..., b_N$.

Inductively, for $j > b_N$ the ball B_j intersect with at most $b_N - 1$ balls from $1, ..., j-1$, so there exists a collection $\mathcal{B}_k$ in which this ball does not intersect with any of the members. Put B_j into $\mathcal{B}_k$.

In this way we ensure that each members within $\mathcal{B}_k$ are disjoint from each other.

This result is known as the **Besicovitch Covering Theorem**.

[10] (c) Let $f \in L^1(\mathbb{R}^N)$ and f^* be the Hardy–Littlewood maximal function.

Using (b), prove that for any $\alpha > 0$,

$$\mu(\{x : f^*(x) > \alpha\}) \leq \frac{2^N b_N}{\alpha} \int_{\mathbb{R}^N} |f| d\mu.$$

(Free Hint: If $x \in B$, then $B \subset B(x, 2r)$ where r is the radius of B.)

Remark. This result establishes the weak-type inequality of f^* for more general Borel measure μ on $\mathbb{R}^N$, and consequently the Lebesgue Differentiation Theorem, replacing the Vitali's Covering Lemma in the special case of Lebesgue measure.

Solution: The consequence of Besicovitch Covering Theorem from (b) implies that we have the inequality

$$\mathbf{1}_X \leq \sum_{B \in \mathcal{G}} \mathbf{1}_{\overline{B}} \leq b_N$$

since $\mathcal{G}$ covers X, and any point $x \in \mathbb{R}^N$ belongs to at most b_N balls from $\mathcal{G}$.

Let $E_\alpha := \{x : f^*(x) > \alpha\}$. Then by definition, for each x there exists a ball $x \in B'$ such that

$$\frac{1}{\mu(B')} \int_{B'} |f| > \alpha$$

By the Free Hint, there exists $B = B(x, 2r) \supset B'$ centered at x such that

$$\int \mathbf{1}_{\overline{B}} |f| = \int_B |f| > \int_{B'} |f| > \alpha \mu(B') = \frac{\alpha}{2^N} \mu(B)$$

Let $\mathcal{B}$ denote the collection of all such balls. Then for any compact subset $K \subset E_\alpha$, by (a) and (b) there exists a subcollection $\mathcal{G}$ covering K such that

$$\frac{\alpha}{2^N} \mu(K) \leq \frac{\alpha}{2^N} \sum_{B \in \mathcal{G}} \mu(B) \leq \sum_{B \in \mathcal{G}} \int \mathbf{1}_{\overline{B}} |f| \leq b_N \int |f|$$

Since this is true for any compact K, we have our inequality.