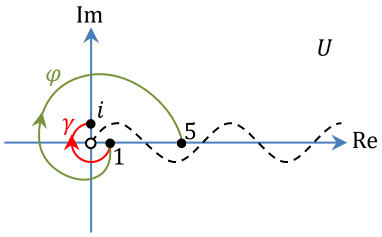
Part A: Brief responses (27 points)

1. Let $U=\mathbb{C}\setminus\{x+iy\in\mathbb{C}: x\geq 0 \text{ and } y=\sin x\}$, which is a simply connected region that does not contain 0. Let $\log: U\to\mathbb{C}$ be the holomorphic branch of complex logarithm such that
$$\log 1 = 0.$$
(2 points)



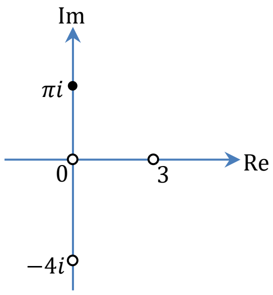
(a) What is the value of $\log i$?

Let $\gamma:\left[0,\frac{3\pi}{2}\right]\to U$ be the curve $\gamma(t)=e^{-it}$. Then γ is a curve in U joining 1 to i, so
$$\log i = \log 1 + \int_\gamma \frac{1}{z}dz = 0 + \int_0^{3\pi/2}\frac{1}{e^{-it}}\cdot -ie^{-it}dt = -i\int_0^{3\pi/2}dt = -\frac{3\pi}{2}i.$$

(b) What is the value of 5^i?

Let $\varphi:[0,2\pi]\to U$ be the curve $\varphi(t)=\left(\frac{4t}{2\pi}+1\right)e^{-it}$. Then φ is a curve in U joining 1 to 5, so
$$\log 5 = \log 1 + \int_\varphi \frac{1}{z}dz = 0 + \int_0^{2\pi}\frac{1}{(4t/2\pi+1)e^{-it}}\cdot\left[\frac{4}{2\pi}e^{-it}+\left(\frac{4t}{2\pi}+1\right)\left(-ie^{-it}\right)\right]dt$$
$$= \ln 5 - 2\pi i.$$
Therefore $5^i = e^{i\log 5} = e^{i(\ln 5 - 2\pi i)} = e^{2\pi}e^{i\ln 5}$.

2. Let f be the complex function
$$f(z)=\frac{1}{z^2(z-3)(z+4i)^3}.$$
(4 points)



(a) Write down all the possible **annuli of convergence** for all Laurent series of f of the form $\sum_{k=-\infty}^{+\infty}a_k(z-3)^k$.

$A(3;0,3)$, $A(3;3,5)$ and $A(3;5,+\infty)$. The distances from 3 to the poles 3, 0 and $-4i$ of the function f are $|3-3|=0$, $|3-0|=3$ and $|3+4i|=5$ respectively.

(b) What is the **radius of convergence** of the **Taylor series of f at πi**?

π. The distance from πi to the nearest pole 0 of the function f is $|\pi i - 0| = \pi$.

3. Consider the following complex functions:

$$f(z) = e^{\frac{1}{\cos z}}, \qquad g(z) = \frac{z}{\sin^2 z}, \qquad h(z) = \frac{(z-i)^2}{z^2+1}.$$

For **each** of these functions,

(i) write down all its **isolated singularities** in $\mathbb{C}$;

(ii) classify each isolated singularity as a **removable singularity**, a **pole**, or an **essential singularity**; if it is a pole, also state the **order of the pole**.

(6 points)

It is clear that f has isolated singularities at $n\pi + \pi/2$ for every $n \in \mathbb{Z}$, the zeros of $\cos z$. Now for each $n \in \mathbb{Z}$, the limit

$$\lim_{z\to n\pi+\frac{\pi}{2}} f(z) = \lim_{z\to n\pi+\frac{\pi}{2}} e^{\frac{1}{\cos z}} = \lim_{z\to 0} e^{\frac{(-1)^n}{\sin z}} = \lim_{z\to 0} e^{\frac{1}{z}\cdot\frac{(-1)^n z}{\sin z}}$$

does not exist (because otherwise $\lim_{z\to 0} e^{\frac{1}{z}} = \lim_{z\to 0}\left(e^{\frac{1}{z}\cdot\frac{(-1)^n z}{\sin z}}\right)^{\frac{\sin z}{(-1)^n z}} = \left(\lim_{z\to 0} e^{\frac{1}{z}\cdot\frac{(-1)^n z}{\sin z}}\right)^{(-1)^n}$ would exist, a contradiction), so f has an <u>essential singularity at $n\pi + \pi/2$ for every $n \in \mathbb{Z}$</u>.

It is clear that g has isolated singularities at $n\pi$ for every $n \in \mathbb{Z}$, the zeros of $\sin^2 z$. Now

$$g(z) = \frac{1}{z}\cdot \underbrace{\frac{z^2}{\sin^2 z}}_{\to 1 \text{ as } z\to 0} \qquad \text{and} \qquad g(z) = \frac{1}{(z-n\pi)^2}\cdot \underbrace{\frac{z(z-n\pi)^2}{\sin^2 z}}_{\to(-1)^n n\pi \text{ as } z\to n\pi}$$

for each $n \in \mathbb{Z}\setminus\{0\}$, so g has a <u>simple pole at 0</u> and a <u>double pole at $n\pi$ for every non-zero integer n</u>.

It is clear that h has isolated singularities at i and $-i$, the zeros of z^2+1. Now

$$\lim_{z\to i} h(z) = \lim_{z\to i}\frac{z-i}{z+i} = 0 \in \mathbb{C} \qquad \text{and} \qquad h(z) = \frac{1}{z+i}\cdot \underbrace{(z-i)}_{\substack{\text{holomorphic,}\\ \text{nonzero near } -i}},$$

so h has a <u>removable singularity at i</u> and a <u>simple pole at $-i$</u>.

4. Let f be the function

$$f(z) = z^3 \cos\frac{1}{z}.$$

(3 points)

(a) Compute the residue $\operatorname{Res}(f; 0)$.

The Laurent series of f at 0 is

$$f(z) = z^3\left(1 - \frac{z^{-2}}{2!} + \frac{z^{-4}}{4!} - \frac{z^{-6}}{6!} + \cdots\right) = z^3 - \frac{1}{2!}z + \frac{1}{4!}z^{-1} - \frac{1}{6!}z^{-3} + \cdots.$$

The residue of f at 0 is the coefficient of z^{-1} in this series, so

$$\operatorname{Res}(f; 0) = \frac{1}{4!} = \frac{1}{24}.$$

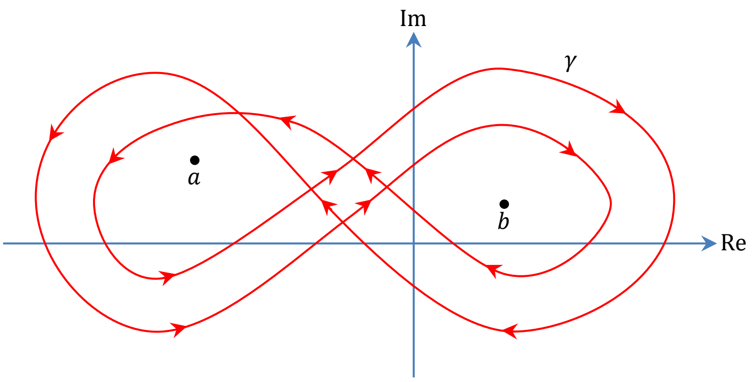
(b) Compute the line integral

$$\oint_{\partial D(0;10)} f(z)dz.$$

Since $\partial D(0; 10)$ is a simple closed curve and 0 is the only isolated singularity of f in the interior of $\partial D(0; 10)$, Cauchy's residue theorem gives

$$\oint_{\partial D(0;10)} f(z)dz = 2\pi i \operatorname{Res}(f; 0) = 2\pi i \cdot \frac{1}{24} = \frac{\pi i}{12}.$$

5. The following diagram shows the image of a closed C^1 curve γ in $\mathbb{C}$, together with the positions of two complex numbers a and b.



(8 points)

(a) Compute the **winding numbers** $n(\gamma; a)$ and $n(\gamma; b)$.

$n(\gamma; a) = 2$ and $n(\gamma; b) = -2.$

(b) Let $f: \mathbb{C} \setminus \{b\} \to \mathbb{C}$ be a holomorphic function, whose Laurent series at b is

$$\sum_{k=-\infty}^{+\infty} a_k (z-b)^k .$$

Compute the following **line integrals** in terms of b and the Laurent series coefficients.

(i) $\oint_\gamma f(z)dz$ (ii) $\oint_\gamma zf(z)dz$

The point b is the only singularity of f, and from (a) we have $n(\gamma; b) = -2$. Since $\operatorname{Res}(f; b) = a_{-1}$, by Cauchy's residue theorem we have

$$\oint_\gamma f(z)dz = 2\pi i \cdot n(\gamma; b) \cdot \operatorname{Res}(f; b) = 2\pi i(-2)(a_{-1}) = -4\pi i a_{-1}.$$

Next since the coefficient of $(z-b)^{-1}$ in the Laurent series

$$(z-b)f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-b)^{k+1}$$

is $\operatorname{Res}\big((z-b)f; b\big) = a_{-2}$, by Cauchy's residue theorem we have

$$\begin{aligned}\oint_\gamma zf(z)dz &= \oint_\gamma (z-b)f(z)dz + b\oint_\gamma f(z)dz \\ &= 2\pi i(-2)(a_{-2}) + b(-4\pi i a_{-1}) \\ &= -4\pi i(a_{-2} + b a_{-1}).\end{aligned}$$

(c) Let φ be a counterclockwise oriented simple closed C^1 curve in $\mathbb{C}$, and let g be a meromorphic function on $\mathbb{C}$ such that g has no zeros or poles on $\operatorname{image}\varphi$. Suppose that g has **8 poles in the interior of $\boldsymbol{\varphi}$**, counting multiplicities. If $g \circ \varphi$ is the closed C^1 curve γ whose image is shown above,

(i) **how many zeros in the interior of $\boldsymbol{\varphi}$** does g have, counting multiplicities?

(ii) **how many zeros in the interior of $\boldsymbol{\varphi}$** does $g - a$ have, counting multiplicities?

By the argument principle, the number of zeros of g minus the number of poles of g in the interior of φ is given by $n(g \circ \varphi; 0) = n(\gamma; 0) = -1$. Now g has 8 poles in the interior of φ, so g has 7 zeros in the interior of φ.

Similarly, the number of zeros of $g - a$ minus the number of poles of $g - a$ in the interior of φ is given by $n(g \circ \varphi; a) = n(\gamma; a) = 2$ as found in (a). Now $g - a$ has 8 poles in the interior of φ, so $g - a$ has 10 zeros in the interior of φ.

6. Give an explicit example of each of the following:

(4 points)

(a) **Two power series** f and g in the complex variable z, both centered at 0, such that
   - f and g both have radius of convergence 1, but
   - the power series $f + g$ has radius of convergence strictly greater than 1.

   Possible example:

   $$f(z) = \sum_{k=0}^{+\infty} z^k \qquad \text{and} \qquad g(z) = \sum_{k=0}^{+\infty} -z^k.$$

   f and g both have radius of convergence 1, but $f + g$ is the zero power series which has radius of convergence $+\infty$.

(b) **A function** $f: \mathbb{C} \setminus \{i, -i\} \to \mathbb{C}$ which
   - has simple zeros at 0 and at 1,
   - has double poles at i and at $-i$,
   - is holomorphic on $\mathbb{C} \setminus \{i, -i\}$.

   Possible example:

   $$f(z) = \frac{z(z-1)}{(z^2+1)^2}.$$

Part B: Short problems (73 points)

7. Let $f:\overline{D(0;1)} \to \mathbb{C}$ be a continuous function which is holomorphic on $D(0;1)$, and let M, N be non-negative real numbers. If

$$|f(e^{i\theta})| \le M \qquad \text{for every } \theta \in [0,\pi] \qquad \text{and}$$
$$|f(e^{i\theta})| \le N \qquad \text{for every } \theta \in [\pi, 2\pi],$$

show that

$$|f(0)| \le \sqrt{MN}.$$

(6 points)

Let $g:\overline{D(0;1)} \to \mathbb{C}$ be the function

$$g(z) = f(z)f(-z).$$

Then g is also continuous on $\overline{D(0;1)}$ and holomorphic on $D(0;1)$. Applying mean value property (Theorem 4.37) to g, we have $g(0) = \frac{1}{2\pi}\int_0^{2\pi} g(e^{it})dt$, i.e.

$$\begin{aligned}(f(0))^2 &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})f(-e^{it})dt \\ &= \frac{1}{2\pi}\int_0^{\pi} f(e^{it})f(e^{i(t+\pi)})dt + \frac{1}{2\pi}\int_\pi^{2\pi} f(e^{it})f(e^{i(t-\pi)})dt.\end{aligned}$$

Therefore

$$\begin{aligned}|f(0)|^2 &\le \frac{1}{2\pi}\int_0^{\pi} |f(e^{it})||f(e^{i(t+\pi)})|dt + \frac{1}{2\pi}\int_\pi^{2\pi} |f(e^{it})||f(e^{i(t-\pi)})|dt \\ &\le \frac{1}{2\pi}\cdot MN\cdot\pi + \frac{1}{2\pi}\cdot NM\cdot\pi \\ &= MN,\end{aligned}$$

i.e. $|f(0)| \le \sqrt{MN}$.

8. Let $f: \mathbb{C} \to \mathbb{C}$ be an entire function such that

$$\operatorname{Re} f(z) \neq \operatorname{Im} f(z)$$

for any $z \in \mathbb{C}$. Show that f is a constant function.

(8 points)

The given condition implies that $f(\mathbb{C}) \subseteq \{w \in \mathbb{C} : \operatorname{Re} w \neq \operatorname{Im} w\}$. Since f is continuous and $\mathbb{C}$ is connected, it follows that $f(\mathbb{C})$ is also **connected**. This implies that either

$$f(\mathbb{C}) \subseteq \{w \in \mathbb{C} : \operatorname{Re} w < \operatorname{Im} w\} \qquad \text{or} \qquad f(\mathbb{C}) \subseteq \{w \in \mathbb{C} : \operatorname{Re} w > \operatorname{Im} w\}.$$

Without loss of generality, suppose that $f(\mathbb{C}) \subseteq \{w \in \mathbb{C} : \operatorname{Re} w < \operatorname{Im} w\}$. Let $g: \mathbb{C} \to \mathbb{C}$ be the function defined by

$$g(z) = e^{(1+i)f(z)}.$$

Since f is entire, g is also entire. Now since

$$\begin{aligned} |g(z)| &= \left|e^{(1+i)f(z)}\right| \\ &= e^{\operatorname{Re}[(1+i)f(z)]} \\ &= e^{\operatorname{Re} f(z) - \operatorname{Im} f(z)} \\ &\leq 1 \end{aligned}$$

for every $z \in \mathbb{C}$, g is also bounded. Therefore by Liouville's Theorem, g is a constant function, i.e. there exists $c \in \mathbb{C} \setminus \{0\}$ such that

$$g(z) = c$$

for every $z \in \mathbb{C}$. Now differentiating both sides we get

$$(1+i)f'(z)e^{(1+i)f(z)} = 0,$$

and so $f'(z) = 0$ for every $z \in \mathbb{C}$. Therefore f is a constant function.

9. Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be a holomorphic function such that

$$f(z) = f\left(\frac{1}{z}\right)$$

for every $z \in \mathbb{C} \setminus \{0\}$. If $f(z) \in \mathbb{R}$ for every $z \in \partial D(0;1)$, show that $f(z) \in \mathbb{R}$ for every $z \in \mathbb{R} \setminus \{0\}$.

(8 points)

Since f is holomorphic on $D(0;1) \setminus \{0\}$ and $f(z) \in \mathbb{R}$ for every $z \in \partial D(0;1)$, by Schwarz reflection principle (Theorem 4.27) the function $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by

$$g(z) = \begin{cases} f(z) & \text{if } z \in \overline{D(0;1)} \setminus \{0\} \\ \overline{f(1/\bar{z})} & \text{if } z \in \mathbb{C} \setminus D(0;1) \end{cases}$$

is well-defined and holomorphic on $\mathbb{C} \setminus \{0\}$. Now f, g are both holomorphic on $\mathbb{C} \setminus \{0\}$ and their values agree on $D(0;1) \setminus \{0\}$, so by identity theorem we must have $f(z) = g(z)$ for every $\mathbb{C} \setminus \{0\}$. Hence for every $z \in \mathbb{R} \setminus \{0\}$, if $|z| \geq 1$ we have

$$\begin{aligned} f(z) = g(z) &= \overline{f(1/\bar{z})} && (\text{since } |z| \geq 1) \\ &= \overline{f(1/z)} && (\text{since } z \in \mathbb{R}) \\ &= \overline{f(z)} && (\text{since } f(z) = f(1/z)), \end{aligned}$$

i.e. $f(z) \in \mathbb{R}$; while if $0 < |z| < 1$, then $\left|\frac{1}{z}\right| \geq 1$, so $f(z) = f\left(\frac{1}{z}\right) \in \mathbb{R}$ also.

<u>Alternative solution</u>:

Let $\sum_{k=-\infty}^{+\infty} a_k z^k$ be the Laurent series of f at 0. Then since

$$\sum_{k=-\infty}^{+\infty} a_k z^k = f(z) = f\left(\frac{1}{z}\right) = \sum_{k=-\infty}^{+\infty} a_k \left(\frac{1}{z}\right)^k = \sum_{k=-\infty}^{+\infty} a_k z^{-k} = \sum_{k=-\infty}^{+\infty} a_{-k} z^k$$

for every $z \in \mathbb{C} \setminus \{0\}$, we have $a_{-n} = a_n$ for every $n \in \mathbb{Z}$. Now for every $n \in \mathbb{Z}$ we have

$$\overline{a_n} = \overline{\frac{1}{2\pi i} \oint_{\partial D(0;1)} \frac{f(z)}{z^{n+1}} dz} = \overline{\frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{f(e^{it})}{(e^{it})^{n+1}} i e^{it} dt} = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{\overline{f(e^{it})}}{(e^{-it})^{-n+1}} i e^{-it} dt$$

$$= \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{\overline{f(e^{-iu})}}{(e^{iu})^{-n+1}} i e^{iu} du = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{f(e^{-iu})}{(e^{iu})^{-n+1}} i e^{iu} du = \frac{1}{2\pi i} \oint_{\partial D(0;1)} \frac{f(1/z)}{z^{-n+1}} dz$$

$$= \frac{1}{2\pi i} \oint_{\partial D(0;1)} \frac{f(z)}{z^{-n+1}} dz = a_{-n} = a_n,$$

so $a_n \in \mathbb{R}$. This implies that $f(z) \in \mathbb{R}$ for every $z \in \mathbb{R} \setminus \{0\}$.

10. Let $f:\mathbb{C}\setminus\{0,2,3\}\to\mathbb{C}$ be the function

$$f(z)=\frac{1}{z}+\frac{1}{(z-2)^2}+\frac{1}{z-3}.$$

(a) Compute the Taylor series of f at 1. What is its disk of convergence?

(7 points)

For every $z\in D(1;1)$, we have $|z-1|<1$, so

$$f(z)=\frac{1}{1+(z-1)}+\frac{1}{\big(1-(z-1)\big)^2}-\frac{1}{2}\cdot\frac{1}{1-\frac{z-1}{2}}$$

$$=\sum_{k=0}^{+\infty}(-1)^k(z-1)^k+\sum_{k=0}^{+\infty}(k+1)(z-1)^k-\frac{1}{2}\sum_{k=0}^{+\infty}\left(\frac{z-1}{2}\right)^k$$

$$=\sum_{k=0}^{+\infty}\left[(-1)^k+k+1-\frac{1}{2^{k+1}}\right](z-1)^k.$$

The disk of convergence of this Taylor series is $D(1;1)$.

(b) Compute the Laurent series of f **centered at 3** which **converges at 1**. What is its annulus of convergence?

(6 points)

Note that $1<|1-3|<3$. Now for every $z\in A(3;1,3)$, we have $1<|z-3|<3$, so

$$f(z)=\frac{1}{3}\cdot\frac{1}{1+\frac{z-3}{3}}+\frac{1}{(z-3)^2}\cdot\frac{1}{\left(1+\frac{1}{z-3}\right)^2}+\frac{1}{z-3}$$

$$=\frac{1}{3}\sum_{k=0}^{+\infty}(-1)^k\left(\frac{z-3}{3}\right)^k+\frac{1}{(z-3)^2}\sum_{k=0}^{+\infty}(k+1)(-1)^k\left(\frac{1}{z-3}\right)^k+\frac{1}{z-3}$$

$$=\sum_{k=0}^{+\infty}\frac{(-1)^k}{3^{k+1}}(z-3)^k+\frac{1}{z-3}+\sum_{k=2}^{+\infty}(k-1)(-1)^k(z-3)^{-k}.$$

The annulus of convergence of this Laurent series is $A(3;1,3)$.

11. Let $a \in (0, \pi)$. Evaluate the integral

$$\int_{-\infty}^{+\infty} \frac{\cos x}{x^2 - 2x\cos a + 1} dx$$

in terms of a, **by considering a line integral of a relevant complex holomorphic function**.

(12 points)

Let $R > 1$ and define $\gamma_1: [-R, R] \to \mathbb{C}$ by $\gamma_1(x) = x$; and $\gamma_2: [0, \pi] \to \mathbb{C}$ by $\gamma_2(t) = Re^{it}$. Let $\gamma = \gamma_1 * \gamma_2$, which is a counterclockwise oriented simple closed curve.

Let $f: \mathbb{C} \setminus \{e^{ia}, e^{-ia}\} \to \mathbb{C}$ be the holomorphic function $f(z) = \frac{e^{iz}}{z^2 - 2z\cos a + 1}$. Then the only isolated singularity of f in the interior of γ is the simple pole at e^{ia}, with

$$\mathrm{Res}(f; e^{ia}) = \lim_{z \to e^{ia}} \frac{e^{iz}}{z - e^{-ia}} = \frac{e^{i(\cos a + i\sin a)}}{e^{ia} - e^{-ia}} = \frac{e^{-\sin a} e^{i\cos a}}{2i \sin a}.$$

So we have

$$\int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz = \oint_{\gamma} f(z) dz = 2\pi i \,\mathrm{Res}(f; e^{ia}) = \frac{\pi e^{-\sin a} e^{i\cos a}}{\sin a}$$

by Cauchy's residue theorem. On the other hand, we have

$$\int_{\gamma_1} f(z) dz = \int_{-R}^{R} \frac{e^{ix}}{x^2 - 2x\cos a + 1} dx\,;$$

and we also have

$$\left| \int_{\gamma_2} f(z) dz \right| \leq \int_0^{\pi} \left| \frac{e^{iRe^{it}}}{R^2 e^{2it} - 2Re^{it}\cos a + 1} \cdot iRe^{it} \right| dt$$

$$\leq \int_0^{\pi} \frac{Re^{-R\sin t}}{R^2 - 2R - 1} dt \leq \frac{\pi R}{R^2 - 2R - 1}$$

which tends to zero as $R \to +\infty$. Therefore taking limits on both sides of

$$\int_{-R}^{R} \frac{e^{ix}}{x^2 - 2x\cos a + 1} dx + \int_{\gamma_2} f(z) dz = \frac{\pi e^{-\sin a} e^{i\cos a}}{\sin a}$$

as $R \to +\infty$, we obtain

$$\lim_{R \to +\infty} \int_{-R}^{R} \frac{e^{ix}}{x^2 - 2x\cos a + 1} dx = \frac{\pi e^{-\sin a} e^{i\cos a}}{\sin a}.$$

Comparing the real parts on both sides, we have $\lim_{R \to +\infty} \int_{-R}^{R} \frac{\cos x}{x^2 - 2x\cos a + 1} dx = \frac{\pi e^{-\sin a}\cos(\cos a)}{\sin a}$.

Finally since $\left| \frac{\cos x}{x^2 - 2x\cos a + 1} \right| \leq \frac{1}{|x|^2 - 2|x| - 1} \leq \frac{2}{x^2}$ for $|x| > 5$, the integrand is improperly Riemann integrable on $(-\infty, +\infty)$ by comparison test. Therefore

$$\int_{-\infty}^{+\infty} \frac{\cos x}{x^2 - 2x\cos a + 1} dx = \frac{\pi e^{-\sin a}\cos(\cos a)}{\sin a}.$$

12. Let $\varphi: [0,1] \to \mathbb{C}$ be a closed C^1 curve, let $a \in \mathbb{C} \setminus (\operatorname{image} \varphi)$, and let $\gamma: [0,1] \to \mathbb{C}$ be a closed C^1 curve such that

$$|\gamma(t) - \varphi(t)| < |\varphi(t) - a|$$

for every $t \in [0,1]$. Show that

$$n(\gamma; a) = n(\varphi; a).$$

(8 points)

Since $a \in \mathbb{C} \setminus (\operatorname{image} \varphi)$, we may let $\psi: [0,1] \to \mathbb{C}$ be defined by

$$\psi(t) = \frac{\gamma(t) - a}{\varphi(t) - a}.$$

Then $\psi(0) = \frac{\gamma(0)-a}{\varphi(0)-a} = \frac{\gamma(1)-a}{\varphi(1)-a} = \psi(1)$, so ψ is a closed C^1 curve in $\mathbb{C}$. Observe that

$$|\psi(t) - 1| = \left|\frac{\gamma(t) - a}{\varphi(t) - a} - 1\right| = \left|\frac{\gamma(t) - \varphi(t)}{\varphi(t) - a}\right| < 1$$

for every $t \in [0,1]$, so $\operatorname{image} \psi \subset D(1;1)$. This implies that the winding number of ψ around 0 is 0. Now since

$$\frac{\psi'(t)}{\psi(t)} = \frac{\gamma'(t)}{\gamma(t) - a} - \frac{\varphi'(t)}{\varphi(t) - a}$$

for every t, we have

$$\begin{aligned} 0 = n(\psi; 0) &= \oint_\psi \frac{1}{z}\,dz = \int_0^1 \frac{\psi'(t)}{\psi(t)}\,dt \\ &= \int_0^1 \frac{\gamma'(t)}{\gamma(t) - a}\,dt - \int_0^1 \frac{\varphi'(t)}{\varphi(t) - a}\,dt \\ &= \oint_\gamma \frac{1}{z - a}\,dz - \oint_\varphi \frac{1}{z - a}\,dz \\ &= n(\gamma; a) - n(\varphi; a). \end{aligned}$$

Therefore $n(\gamma; a) = n(\varphi; a)$.

Remark: This result about winding numbers is sometimes referred to as **Rouché's Theorem for closed curves**.

13. Let $f: \mathbb{C} \to \mathbb{C}$ be the polynomial

$$f(z) = z^5 - 3z^4 + 2z - 10i.$$

How many zeros of f are there in the annulus $A(0; 1, 2)$, counting multiplicities?

(8 points)

Let $g: \mathbb{C} \to \mathbb{C}$ be the function $g(z) = -10i$. Then for every $z \in \partial D(0; 1)$, we have

$$|g(z)| = |-10i| = 10$$

$$|f(z) - g(z)| = |z^5 - 3z^4 + 2z| \le |z|^5 + 3|z|^4 + 2|z| = 6,$$

i.e. $|f(z) - g(z)| < |g(z)|$. So f and g have the same number of zeros in $D(0; 1)$ by Rouché's Theorem, i.e. f has no zeros in $D(0; 1)$. f has no zeros in $\partial D(0; 1)$ either, because $|f(z)| \ge |g(z)| - |f(z) - g(z)| \ge 10 - 6 = 4$ for every $z \in \partial D(0; 1)$.

On the other hand, let $h: \mathbb{C} \to \mathbb{C}$ be the function $h(z) = -3z^4$. Then for every $z \in \partial D(0; 2)$, we have

$$|h(z)| = |-3z^4| = 3|z|^4 = 3 \cdot 2^4 = 48$$

$$|f(z) - h(z)| = |z^5 + 2z + 10i| \le |z|^5 + 2|z| + 10 = 46,$$

i.e. $|f(z) - h(z)| < |h(z)|$. So f and h have the same number of zeros in $D(0; 2)$ by Rouché's Theorem, i.e. f has 4 zeros in $D(0; 2)$, counting multiplicities.

Consequently, f has exactly 4 zeros in the annulus $A(0; 1, 2)$, counting multiplicities.

14. Let n be a positive integer and let $f: \mathbb{C} \to \mathbb{C}$ be an entire function such that

$$|f(z+w)|^{\frac{1}{n}} \leq |f(z)|^{\frac{1}{n}} + |f(w)|^{\frac{1}{n}}$$

for every $z, w \in \mathbb{C}$.

(a) Using mathematical induction, show that

$$|f(kz)|^{\frac{1}{n}} \leq k|f(z)|^{\frac{1}{n}}$$

for every positive integer k and every $z \in \mathbb{C}$. (2 points)

The inequality clearly holds for $k = 1$. If it holds for some positive integer k, then

$$\left|f\big((k+1)z\big)\right|^{\frac{1}{n}} = |f(kz+z)|^{\frac{1}{n}} \leq |f(kz)|^{\frac{1}{n}} + |f(z)|^{\frac{1}{n}}$$

$$\leq k|f(z)|^{\frac{1}{n}} + |f(z)|^{\frac{1}{n}} = (k+1)|f(z)|^{\frac{1}{n}}$$

for every $z \in \mathbb{C}$, so it holds for $k+1$ as well.

(b) Using (a) or otherwise, show that f is a polynomial of degree at most n. (8 points)

Let $M = \max\{|f(z)| : z \in \overline{D(0;1)}\}$, which exists since f is continuous and $\overline{D(0;1)}$ is compact. For each $r > 0$, there exists a positive integer $k \in [r, r+1)$; thus for every $w \in \partial D(0;r)$, we have $w/k \in \overline{D(0;1)}$, which implies that

$$|f(w)| = \left|f\left(k \cdot \frac{w}{k}\right)\right| \leq k^n \left|f\left(\frac{w}{k}\right)\right| \leq k^n M \leq (r+1)^n M$$

according to the result from (a). Thus for each $a \in \mathbb{C}$ and for every $r > 0$, by Cauchy estimate we have

$$\begin{aligned}
&\left|f^{(n+1)}(a)\right| \\
&= \left|\frac{(n+1)!}{2\pi i} \oint_{\partial D(a;r)} \frac{f(z)}{(z-a)^{n+2}} dz\right| \leq \frac{(n+1)!}{2\pi} \frac{\max\{|f(z)| : z \in \partial D(a;r)\}}{r^{n+2}} \cdot 2\pi r \\
&\leq \frac{(n+1)!}{2\pi} \frac{\left(|f(a)|^{\frac{1}{n}} + [(r+1)^n M]^{\frac{1}{n}}\right)^n}{r^{n+2}} \cdot 2\pi r = \frac{(n+1)!\left(|f(a)|^{\frac{1}{n}} + M^{\frac{1}{n}}(r+1)\right)^n}{r^{n+1}}.
\end{aligned}$$

Since this inequality is true for every $r > 0$ and since

$$\lim_{r \to +\infty} \frac{(n+1)!\left(|f(a)|^{\frac{1}{n}} + M^{\frac{1}{n}}(r+1)\right)^n}{r^{n+1}} = 0,$$

it follows that $\left|f^{(n+1)}(a)\right| = 0$, i.e. $f^{(n+1)}(a) = 0$. Now we have shown that $f^{(n+1)}(a) = 0$ for every $a \in \mathbb{C}$, so f is a polynomial of degree at most n.