Chapter 5 Isolated singularities

Let f be a complex function which is **holomorphic** at a point $a \in \mathbb{C}$. In chapter 4, we have seen that the behavior of f near a is described by the **Taylor series of f at a**. In this chapter, we are going to study the behavior of a holomorphic function f near a point at which f is perhaps not holomorphic, i.e. near a **singularity** of f.

1. Classification of singularities

Definition 5.1 Let $a \in \mathbb{C}$. We say that a function f has an ***isolated singularity*** at a if f is not holomorphic at a but is holomorphic on some "punctured disk" centered at a, i.e. if there exists $r > 0$ such that f is holomorphic on $D(a; r) \setminus \{a\}$.

Example 5.2 Let

$$f(z) = \frac{1}{\sin \frac{\pi}{z}}$$

whenever it is defined. Then f has an isolated singularity at $\frac{1}{n}$ for every $n \in \mathbb{Z} \setminus \{0\}$, and f has a non-isolated singularity at 0.

Definition 5.3 Let $a \in \mathbb{C}$ and let f be a function which has an isolated singularity at a.

- ⦿ We say that f has a ***removable singularity*** at a if it extends to be holomorphic at a, i.e. if there exist $r > 0$ and a holomorphic function $F: D(a; r) \to \mathbb{C}$ such that $F(z) = f(z)$ for every $z \in D(a; r) \setminus \{a\}$.
- ⦿ We say that f has a ***pole*** at a if $\lim_{z \to a} f(z) = \infty$ (i.e. $\lim_{z \to a} \frac{1}{f(z)} = 0$).
- ⦿ We say that f has an ***essential singularity*** at a if f has neither a removable singularity nor a pole at a.

Corollary 5.4 *Let $a \in \mathbb{C}$ and let f be a function which has an isolated singularity at a. Then f has a removable singularity at a if and only if $\lim_{z \to a} f(z)$ exists as a finite complex number.*

Proof. The (⇒) part is trivial. The (⇐) part follows from Theorem 4.23. ∎

Theorem 5.5 (Riemann extension) *Let $a \in \mathbb{C}$ and let f be a function which has an isolated singularity at a. Then f has a removable singularity at a if and only if there exists $r > 0$ such that f is bounded on $D(a;r) \setminus \{a\}$.*

Proof. The (⇒) part is trivial. To prove the (⇐) part, we let $F\colon D(a;r) \to \mathbb{C}$ be defined by

$$F(z) = \begin{cases} (z-a)f(z) & \text{if } z \neq a \\ 0 & \text{if } z = a \end{cases}.$$

Since f is bounded on $D(a;r) \setminus \{a\}$, it follows that $\lim_{z\to a}(z-a)f(z) = 0$ and so F is continuous.

Since f is holomorphic on $D(a;r) \setminus \{a\}$, F is also holomorphic on $D(a;r) \setminus \{a\}$. So by Theorem 4.23, F is holomorphic on $D(a;r)$ and is differentiable at a in particular, i.e.

$$\lim_{z\to a} f(z) = \lim_{z\to a} \frac{F(z) - F(a)}{z-a} = F'(a)$$

exists. Therefore f has a removable singularity at a by Corollary 5.4. ■

Theorem 5.6 (Casorati-Weierstrass) *Let $a \in \mathbb{C}$ and let f be a function which has an isolated singularity at a. Then f has an essential singularity at a if and only if for every $r > 0$,*

$$\overline{f(D(a;r) \setminus \{a\})} = \mathbb{C},$$

*i.e. for each $w \in \mathbb{C}$ and each $\varepsilon > 0$, there exists $z \in D(a;r) \setminus \{a\}$ such that $|f(z) - w| < \varepsilon$. (This means that for **each** $w \in \mathbb{C}$, there exists z arbitrarily near a such that $f(z)$ is near w.)*

Proof. The (⇐) part is already covered by Definition 5.3 and Theorem 5.5. To prove the (⇒) part, we suppose on the contrary that there exist $w \in \mathbb{C}$ and $r, \varepsilon > 0$ such that

$$|f(z) - w| \geq \varepsilon$$

for every $z \in D(a;r) \setminus \{a\}$. Then the function $g\colon D(a;r) \setminus \{a\} \to \mathbb{C}$ defined by

$$g(z) = \frac{1}{f(z) - w}$$

has an isolated singularity at a and is bounded on $D(a;r) \setminus \{a\}$, so g has a removable singularity at a by Theorem 5.5. In other words, there exists a holomorphic function $G\colon D(a;r) \to \mathbb{C}$ such that $G(z) = g(z)$ for every $z \in D(a;r) \setminus \{a\}$. Now,

- If $G(a) \neq 0$, then

  $$\lim_{z\to a} f(z) = w + \lim_{z\to a} \frac{1}{g(z)} = w + \lim_{z\to a} \frac{1}{G(z)} = w + \frac{1}{G(a)},$$

  so f has a removable singularity at a, which is a contradiction;
- If $G(a) = 0$, then

  $$\lim_{z\to a} f(z) = w + \lim_{z\to a} \frac{1}{g(z)} = w + \lim_{z\to a} \frac{1}{G(z)} = \infty,$$

  so f has a pole at a, which is also a contradiction. ■

Example 5.7 Show that there exists a complex number b such that

$$\left|\cos\frac{1}{b} + b^{4023} - e^{-b^2+1}\right| < 0.01.$$

Proof: [We first aim to show that 0 is an essential singularity of the function $f(z) = \cos\frac{1}{z}$, and then apply Casorati-Weierstrass.]

Let $f: \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be the function

$$f(z) = \cos\frac{1}{z},$$

and consider the sequences of complex numbers $\{a_n\}$ and $\{b_n\}$ given by

$$a_n = \frac{1}{n} \quad \text{and} \quad b_n = \frac{i}{n}$$

for every $n \in \mathbb{N}$. $\{a_n\}$ and $\{b_n\}$ both converge to 0, but $\{f(a_n)\} = \{\cos n\}$ is bounded while $\{f(b_n)\} = \{\cosh n\}$ diverges to $+\infty$. This shows that $\lim_{z\to 0} f(z)$ does not exist, so 0 is an essential singularity of f. Now the function $g: \mathbb{C} \to \mathbb{C}$ defined by

$$g(z) = -z^{4023} + e^{-z^2+1}$$

is continuous at 0, so there exists $\delta > 0$ such that

$$|g(z) - g(0)| < 0.005$$

for every $z \in D(0;\delta)$. On the other hand, by applying Casorati-Weierstrass Theorem to f, there exists $b \in D(0;\delta) \setminus \{0\}$ such that

$$|f(b) - g(0)| < 0.005,$$

so

$$\begin{aligned}\left|\cos\frac{1}{b} + b^{4023} - e^{-b^2+1}\right| &= |f(b) - g(b)| \le |f(b) - g(0)| + |g(b) - g(0)| \\ &< 0.005 + 0.005 = 0.01.\end{aligned}$$

■

Remark 5.8 In fact there is a much stronger result called ***Picard's Great Theorem*** regarding the behavior of a function near an essential singularity. It states that the image of any punctured disk centered at a not only is dense in $\mathbb{C}$ but actually "covers $\mathbb{C}$ infinitely many times", except for at most one point.

We will have a clear picture about the classification of the three types of isolated singularities after we study **Laurent series** in the next section.

2. Laurent series

Definition 5.9 A ***two-sided sequence*** of complex numbers is a function from $\mathbb{Z}$ to $\mathbb{C}$. It is usually denoted as $\{a_n\}_{n\in\mathbb{Z}}$ or

$$(\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots).$$

Definition 5.10 Let $\{a_n\}_{n\in\mathbb{Z}}$ be a two-sided sequence. A ***two-sided series*** denoted by

$$\sum_{k=-\infty}^{+\infty} a_k$$

is the ordered pair of series $(\sum_{k=0}^{+\infty} a_k, \sum_{k=1}^{+\infty} a_{-k})$. We say that the two-sided series $\sum_{k=-\infty}^{+\infty} a_k$ ***converges*** if both $\sum_{k=0}^{+\infty} a_k$ and $\sum_{k=1}^{+\infty} a_{-k}$ converge. The ***limit*** of the two-sided series is the sum of these two limits. As usual, we use the same notation $\sum_{k=-\infty}^{+\infty} a_k$ to denote the limit of this two-sided series.

A **Laurent series** is a "two-sided" analogue of a power series. It includes **negative powers** of $(z-a)$ as well.

Definition 5.11 Let $a \in \mathbb{C}$ and let $\{a_n\}_{n\in\mathbb{Z}}$ be a two-sided sequence of complex numbers. A ***Laurent series*** is a series of functions of the form

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-a)^k.$$

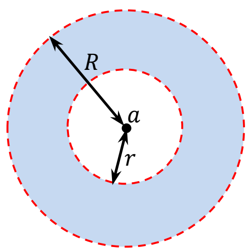
The number a is called the ***center*** of the Laurent series f, and the a_n's are called the ***coefficients*** of f.

Definition 5.12 Let $a \in \mathbb{C}$ and let $0 \le r < R \le +\infty$. The ***open annulus*** centered at a with inner radius r and outer radius R is the set

$$A(a; r, R) \coloneqq \{z \in \mathbb{C} : r < |z-a| < R\}.$$

In particular, if $0 < r < R < +\infty$, then

$$A(a; r, R) = D(a; R) \setminus \overline{D(a; r)},$$
$$A(a; 0, R) = D(a; R) \setminus \{a\},$$
$$A(a; r, +\infty) = \mathbb{C} \setminus \overline{D(a; r)},$$
$$A(a; 0, +\infty) = \mathbb{C} \setminus \{a\}.$$



Definition 5.13 Let $a \in \mathbb{C}$ and let $\{a_n\}_{n\in\mathbb{Z}}$ be a two-sided sequence of complex numbers. Let $R_0, R_1 \in [0, +\infty]$ be the extended real numbers defined by

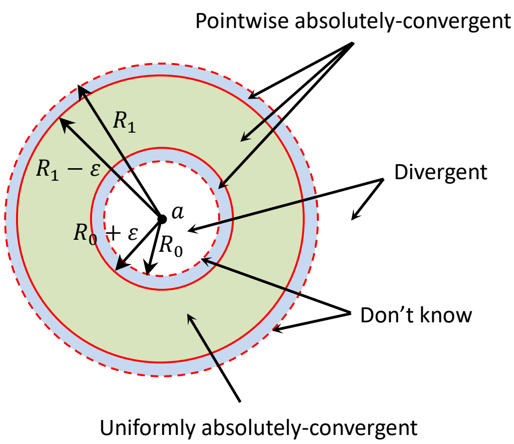
$$R_0 := \limsup_{n\in\mathbb{N}} |a_{-n}|^{\frac{1}{n}} \qquad \text{and} \qquad R_1 := \frac{1}{\limsup_{n\in\mathbb{N}} |a_n|^{\frac{1}{n}}}.$$

Then the ***annulus of convergence*** of the Laurent series $\sum_{k=-\infty}^{+\infty} a_k(z-a)^k$ is $A(a; R_0, R_1)$.

Lemma 5.14 *Let $a \in \mathbb{C}$ and let $\{a_n\}_{n\in\mathbb{Z}}$ be a two-sided sequence of complex numbers. Let $A(a; R_0, R_1)$ be the annulus of convergence of the Laurent series $\sum_{k=-\infty}^{+\infty} a_k(z-a)^k$ where $0 \le R_0 < R_1 \le +\infty$. Then*

(i) *this Laurent series is uniformly absolutely-convergent on every compact subset of $A(a; R_0, R_1)$, and its uniform limit is a holomorphic function $f: A(a; R_0, R_1) \to \mathbb{C}$;*

(ii) *this Laurent series diverges at every point in the interior of $\mathbb{C} \setminus A(a; R_0, R_1)$.*



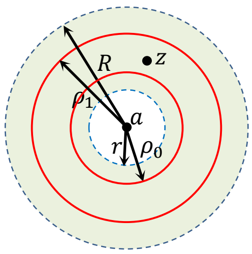
Proof. Same as Theorem 2.88 – 2.89. ∎

Theorem 5.15 (Laurent) *Let $a \in \mathbb{C}$ and $0 \le r < R \le +\infty$, and let $f: A(a; r, R) \to \mathbb{C}$ be a holomorphic function. Then there exists a two-sided sequence of complex numbers $\{a_n\}$ such that*

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k(z-a)^k$$

for every $z \in A(a; r, R)$.

Proof. The proof is similar to that of Taylor's Theorem (Theorem 4.1). For each $z \in A(a; r, R)$, we let $\rho_0 \in (r, |z-a|)$ and $\rho_1 \in (|z-a|, R)$. Then Cauchy integral formula together with Theorem 3.45 gives



$$f(z) = \frac{1}{2\pi i}\left(\oint_{\partial D(a;\rho_1)} \frac{f(w)}{w-z}\,dw - \oint_{\partial D(a;\rho_0)} \frac{f(w)}{w-z}\,dw\right).$$

Since $\left|\frac{z-a}{w-a}\right| < 1$ for every $w \in \partial D(a; \rho_1)$, we have

$$\oint_{\partial D(a;\rho_1)} \frac{f(w)}{w-z}\,dw = \oint_{\partial D(a;\rho_1)} \frac{f(w)}{w-a}\frac{1}{1-\frac{z-a}{w-a}}\,dw = \oint_{\partial D(a;\rho_1)} \left[\sum_{k=0}^{+\infty} \frac{f(w)}{(w-a)^{k+1}}(z-a)^k\right] dw$$

$$= \sum_{k=0}^{+\infty} \left[\oint_{\partial D(a;\rho_1)} \frac{f(w)}{(w-a)^{k+1}} dw \right] (z-a)^k$$

by Weierstrass' M-test and Corollary 3.20 (details similar to the proof of Theorem 4.1). Similarly, since $\left|\frac{w-a}{z-a}\right| < 1$ for every $w \in \partial D(a;\rho_0)$, we have

$$-\oint_{\partial D(a;\rho_0)} \frac{f(w)}{w-z} dw = \oint_{\partial D(a;\rho_0)} \frac{f(w)}{z-a} \frac{1}{1-\frac{w-a}{z-a}} dw = \oint_{\partial D(a;\rho_0)} \left[\sum_{k=0}^{+\infty} \frac{f(w)}{(z-a)^{k+1}} (w-a)^k \right] dw$$

$$= \sum_{k=0}^{+\infty} \left[\oint_{\partial D(a;\rho_0)} f(w)(w-a)^k dw \right] \frac{1}{(z-a)^{k+1}}$$

$$= \sum_{k=1}^{+\infty} \left[\oint_{\partial D(a;\rho_0)} f(w)(w-a)^{k-1} dw \right] \frac{1}{(z-a)^k}$$

by Weierstrass' M-test and Corollary 3.20 again. Hence

$$f(z) = \sum_{k=0}^{+\infty} \left[\oint_{\partial D(a;\rho_1)} \frac{f(w)}{(w-a)^{k+1}} dw \right] (z-a)^k + \sum_{k=1}^{+\infty} \left[\oint_{\partial D(a;\rho_0)} f(w)(w-a)^{k-1} dw \right] \frac{1}{(z-a)^k}.$$

Moreover, for each $n \in \mathbb{N}$, the line integral $\oint_{\partial D(a;\rho)} \frac{f(w)}{(w-a)^{n+1}} dw$ does not depend on the choice of $\rho \in (r, R)$ (according to the general version of Cauchy-Goursat, Theorem 3.45). So one may take $a_n \coloneqq \frac{1}{2\pi i} \oint_{\partial D(a;\rho)} \frac{f(w)}{(w-a)^{n+1}} dw$ to finish the proof, i.e.

$$f(z) = \sum_{k=-\infty}^{+\infty} \left[\frac{1}{2\pi i} \oint_{\partial D(a;\rho)} \frac{f(w)}{(w-a)^{k+1}} dw \right] (z-a)^k.$$

Note that this Laurent series is unique for each given f. ∎

Definition 5.16 Let $a \in \mathbb{C}$ and $0 \le r < R \le +\infty$, and let $f: A(a; r, R) \to \mathbb{C}$ be a holomorphic function. The Laurent series

$$\sum_{k=-\infty}^{+\infty} a_k (z-a)^k$$

centered at a, where

$$a_n = \frac{1}{2\pi i} \oint_{\partial D(a;\rho)} \frac{f(w)}{(w-a)^{n+1}} dw$$

and $\rho \in (r, R)$, is called the ***Laurent series of f in the annulus $A(a; r, R)$***.

Although the coefficients of the Laurent series of a function are given by line integrals in Definition 5.16, we usually do not compute the coefficients this way. For **rational functions**, the Laurent series can be obtained by the same routine as before:

$$\text{Long division} \to \text{Partial fractions} \to \text{Apply } \frac{1}{1-w} = \sum_{k=0}^{+\infty} w^k \text{ for } |w| < 1.$$

Example 5.17 Let $f: \mathbb{C} \setminus \{0, 1, 2\} \to \mathbb{C}$ be the function

$$f(z) = \frac{2}{z(z-1)(z-2)}.$$

Find the Laurent series of f in the following regions.

(a) $A(0; 0, 1)$,

(b) $A(0; 1, 2)$,

(c) $A(2; 1, 2)$, and

(d) $A(1; 1, +\infty)$.

Solution:

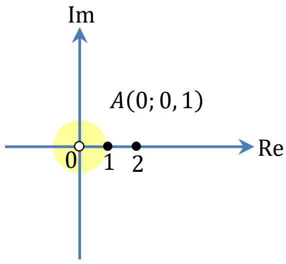
Expanding f into partial fractions, we have

$$f(z) = \frac{2}{z(z-1)(z-2)} = \frac{1}{z} - \frac{2}{z-1} + \frac{1}{z-2}.$$

(a) For every $z \in A(0; 0, 1)$, we have $|z| < 1$ and $\left|\frac{z}{2}\right| < 1$. So,

$$f(z) = \frac{1}{z} + \frac{2}{1-z} - \frac{1}{2} \cdot \frac{1}{1-\frac{z}{2}} = \frac{1}{z} + 2\sum_{k=0}^{+\infty} z^k - \frac{1}{2}\sum_{k=0}^{+\infty} \left(\frac{z}{2}\right)^k$$

$$= z^{-1} + \sum_{k=0}^{+\infty} \left(2 - \frac{1}{2^{k+1}}\right) z^k.$$

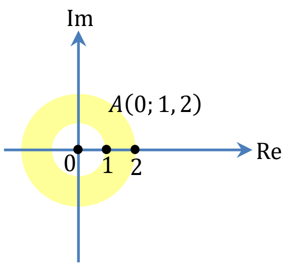


(b) For every $z \in A(0; 1, 2)$, we have $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$. So,

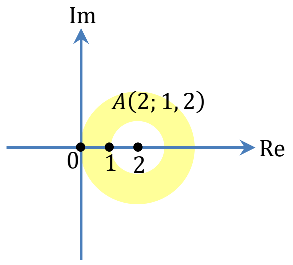
$$f(z) = \frac{1}{z} - \frac{2}{z} \cdot \frac{1}{1-\frac{1}{z}} - \frac{1}{2} \cdot \frac{1}{1-\frac{z}{2}}$$

$$= \frac{1}{z} - \frac{2}{z}\sum_{k=0}^{+\infty} \left(\frac{1}{z}\right)^k - \frac{1}{2}\sum_{k=0}^{+\infty} \left(\frac{z}{2}\right)^k$$

$$= -\sum_{k=0}^{+\infty} \frac{1}{2^{k+1}} z^k - z^{-1} - \sum_{k=2}^{+\infty} 2z^{-k}.$$



(c) For every $z \in A(2; 1, 2)$, we have $\left|\frac{z-2}{2}\right| < 1$ and $\left|\frac{1}{z-2}\right| < 1$. So,

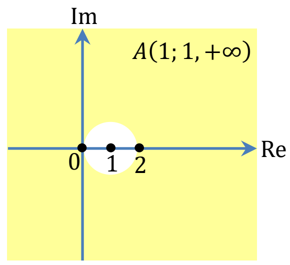


$$f(z) = \frac{1}{2} \cdot \frac{1}{1 + \frac{z-2}{2}} - \frac{2}{z-2} \cdot \frac{1}{1 + \frac{1}{z-2}} + \frac{1}{z-2}$$

$$= \frac{1}{2} \sum_{k=0}^{+\infty} \frac{(-1)^k}{2^k} (z-2)^k - \frac{2}{z-2} \sum_{k=0}^{+\infty} (-1)^k \left(\frac{1}{z-2}\right)^k + \frac{1}{z-2}$$

$$= \sum_{k=0}^{+\infty} \frac{(-1)^k}{2^{k+1}} (z-2)^k - (z-2)^{-1} + \sum_{k=2}^{+\infty} 2(-1)^k (z-2)^{-k}.$$

(d) For every $z \in A(1; 1, +\infty)$, we have $\left|\frac{1}{z-1}\right| < 1$. So,



$$f(z) = \frac{1}{z-1} \cdot \frac{1}{1 + \frac{1}{z-1}} - \frac{2}{z-1} + \frac{1}{z-1} \cdot \frac{1}{1 - \frac{1}{z-1}}$$

$$= \frac{1}{z-1} \cdot \sum_{k=0}^{+\infty} (-1)^k \left(\frac{1}{z-1}\right)^k - \frac{2}{z-1} + \frac{1}{z-1} \cdot \sum_{k=0}^{+\infty} \left(\frac{1}{z-1}\right)^k$$

$$= \sum_{k=2}^{+\infty} [(-1)^{k-1} + 1](z-1)^{-k} = \sum_{k=1}^{+\infty} 2(z-1)^{-2k-1}.$$

Example 5.18 Let $f: \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be the function

$$f(z) = z^2 \sin \frac{1}{z^2}.$$

Find the Laurent series of f in the region $\mathbb{C} \setminus \{0\}$.

Solution:

For every $z \in \mathbb{C} \setminus \{0\}$, we have

$$f(z) = z^2 \sin \frac{1}{z^2} = z^2 \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{1}{z^2}\right)^{2k+1}$$

$$= \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!} \frac{1}{z^{4k}} = 1 + \sum_{k=1}^{+\infty} \frac{(-1)^k}{(2k+1)!} z^{-4k}.$$

Given a Laurent series of a function f, to determine its region of convergence, we look for the **largest** disk or annulus which **does not contain any (non-removable) singularity of f**.

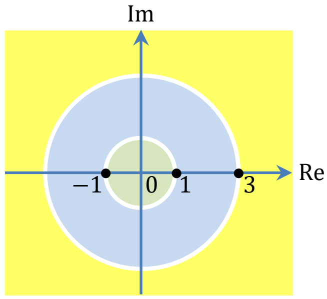
Example 5.19 Let f be the function

$$f(z) = \frac{1}{1-z^2} + \frac{1}{3-z}$$

wherever it is defined. Find all the Laurent series of f in the form $\sum_{k=-\infty}^{\infty} a_k z^k$. For each Laurent series, state the corresponding region in which the series converges to f.

Solution:

We are asked to find all the Laurent series of f centered at 0. Clearly the three points $-1,\ 1,\ 3$ are the only singularities of f in $\mathbb{C}$. So we consider the regions $D(0;1)$, $A(0;1,3)$ and $A(0;3,+\infty)$, as illustrated in the diagram on the right.



(i) For every $z \in D(0;1)$, we have $|z^2| < 1$ and $\left|\frac{z}{3}\right| < 1$. So,

$$f(z) = \frac{1}{1-z^2} + \frac{1}{3}\cdot\frac{1}{1-\frac{z}{3}} = \sum_{k=0}^{+\infty} (z^2)^k + \frac{1}{3}\sum_{k=0}^{+\infty}\left(\frac{z}{3}\right)^k$$

$$= \sum_{k=0}^{+\infty} z^{2k} + \sum_{k=0}^{+\infty}\left(\frac{1}{3^{k+1}}\right)z^k = \sum_{k=0}^{+\infty}\left[\frac{1+(-1)^k}{2} + \frac{1}{3^{k+1}}\right]z^k.$$

(ii) For every $z \in A(0;1,3)$, we have $\left|\frac{1}{z^2}\right| < 1$ and $\left|\frac{z}{3}\right| < 1$. So,

$$f(z) = -\frac{1}{z^2}\cdot\frac{1}{1-\frac{1}{z^2}} + \frac{1}{3}\cdot\frac{1}{1-\frac{z}{3}} = -\frac{1}{z^2}\sum_{k=0}^{+\infty}\left(\frac{1}{z^2}\right)^k + \frac{1}{3}\sum_{k=0}^{+\infty}\left(\frac{z}{3}\right)^k$$

$$= \sum_{k=0}^{+\infty}\frac{1}{3^{k+1}}z^k - \sum_{k=1}^{+\infty} z^{-2k}.$$

(iii) For every $z \in A(0;3,+\infty)$, we have $\left|\frac{1}{z^2}\right| < 1$ and $\left|\frac{3}{z}\right| < 1$. So,

$$f(z) = -\frac{1}{z^2}\cdot\frac{1}{1-\frac{1}{z^2}} - \frac{1}{z}\cdot\frac{1}{1-\frac{3}{z}} = -\frac{1}{z^2}\sum_{k=0}^{+\infty}\left(\frac{1}{z^2}\right)^k - \frac{1}{z}\sum_{k=0}^{+\infty}\left(\frac{3}{z}\right)^k$$

$$= -\sum_{k=0}^{+\infty}\left(\frac{1}{z}\right)^{2k+2} - \sum_{k=0}^{+\infty}3^k\left(\frac{1}{z}\right)^{k+1} = \sum_{k=1}^{+\infty}\left[-\frac{1+(-1)^k}{2} - 3^{k-1}\right]z^{-k}.$$

Laurent series is a useful tool for **classifying** the three types of **isolated singularities**.

Definition 5.20 Let $a \in \mathbb{C}$ and f be a function which has an isolated singularity at a, i.e. there exists $r > 0$ such that $f: D(a;r) \setminus \{a\} \to \mathbb{C}$ is a holomorphic function. Then the Laurent series

$$\sum_{k=-\infty}^{+\infty} a_k (z-a)^k$$

of f in the punctured disk $A(a;0,r) = D(a;r) \setminus \{a\}$ is particularly important, and is called the ***Laurent series of f at a***. The part $\sum_{k=0}^{+\infty} a_k(z-a)^k$ is called the ***analytic part*** of this Laurent series, while the remaining part $\sum_{k=-\infty}^{-1} a_k(z-a)^k$ is called the ***principal part*** of this Laurent series.

Theorem 5.21 *Let $a \in \mathbb{C}$ and f be a function which has an isolated singularity at a. Then*

(i) *f has a removable singularity at a if and only if the principal part of the Laurent series of f at a is zero;*

(ii) *f has a pole at a if and only if the principal part of the Laurent series of f at a is a non-zero finite sum;*

(iii) *f has an essential singularity at a if and only if the principal part of the Laurent series of f at a is an infinite sum.*

Proof. We only need to prove (i) and (ii).

(i) The ($\Leftarrow$) part is trivial. To prove the ($\Rightarrow$) part, we suppose that f has a removable singularity at a, so there exists $r > 0$ such that f extends to a holomorphic function $F: D(a;r) \to \mathbb{C}$. On $D(a;r) \setminus \{a\}$, the Taylor series of F at a is the same as the Laurent series of f at a, so in particular the Laurent series of f at a has zero principal part.

(ii) To prove the ($\Rightarrow$) part, we suppose that f has a pole at a, i.e. $\lim_{z\to a} f(z) = \infty$. Then there exists $r > 0$ such that $|f(z)| > 1$ whenever $z \in D(a;r) \setminus \{a\}$. In other words, $\frac{1}{f}$ has an isolated singularity at a and is bounded above by 1 on $D(a;r) \setminus \{a\}$. Thus $\frac{1}{f}$ has a removable singularity at a, and it extends to a holomorphic function $G: D(a;r) \to \mathbb{C}$ such that $G(a) = 0$. Suppose that G has a zero of order m at a. Then by Factor Theorem (Lemma 4.15), there exists a holomorphic function $g: D(a;r) \to \mathbb{C}$ such that $g(a) \neq 0$ and

$$G(z) = (z-a)^m g(z)$$

for every $z \in D(a;r)$. Since g is continuous at a, there exists $\rho \in (0, r]$ such that

$g(z) \neq 0$ for any $z \in D(a;\rho)$, i.e. $\frac{1}{g}$ is holomorphic on $D(a;\rho)$. Suppose that the Taylor series of $\frac{1}{g}$ on $D(a;\rho)$ is given by $\frac{1}{g(z)} = \sum_{k=0}^{+\infty} a_k(z-a)^k$. Then

$$f(z) = \frac{1}{G(z)} = \frac{1}{(z-a)^m}\frac{1}{g(z)} = \frac{1}{(z-a)^m}\sum_{k=0}^{+\infty} a_k(z-a)^k = \sum_{k=-m}^{+\infty} a_{k+m}(z-a)^k$$

for every $z \in D(a;\rho) \setminus \{a\}$, whose principal part $\sum_{k=-m}^{-1} a_{k+m}(z-a)^k$ is a sum of at most m terms.

To prove the ($\Leftarrow$) part, we suppose that the Laurent series of f on $D(a;r) \setminus \{a\}$ is

$$f(z) = \sum_{k=-m}^{+\infty} a_k(z-a)^k.$$

with $m \geq 1$ and $a_{-m} \neq 0$. Let $g: D(a;r) \to \mathbb{C}$ be the power series defined by

$$g(z) = \sum_{k=0}^{+\infty} a_{k-m}(z-a)^k.$$

Then $g(a) \neq 0$. Since g is continuous at a, there exists $\rho \in (0,r]$ such that $|g(z)| \geq \frac{|g(a)|}{2}$ for every $z \in D(a;\rho)$, so

$$|f(z)| = \left|\frac{g(z)}{(z-a)^m}\right| \geq \frac{|g(a)|}{2|z-a|^m}$$

for every $z \in D(a;\rho) \setminus \{a\}$, which implies that $\lim_{z\to a} f(z) = \infty$. ■

Definition 5.22 Let $a \in \mathbb{C}$ and let f be a function which has a pole at a, so that the principal part of the Laurent series $\sum_{k=-\infty}^{+\infty} a_k(z-a)^k$ of f at a is a non-zero finite sum. We say that f has a ***pole of order*** $\boldsymbol{n}$ at a if

$$a_{-n} \neq 0 \qquad \text{but} \qquad a_k = 0 \text{ for all } k < -n,$$

i.e. if n is the greatest natural number such that $a_{-n} \neq 0$. We also say that f has a ***simple pole*** (resp. ***double pole***) at a if f has a pole of order 1 (resp. 2) at a.

Example 5.23 Let $a \in \mathbb{C}$ and let

$$f(z) = \frac{1}{(z-a)^n}.$$

Then f has a pole of order n at a.

Corollary 5.24 (Factor Theorem) *Let $a \in \mathbb{C}$ and let f be a function which has an isolated singularity at a. Then f has a pole of order n at a if and only if there exist $r > 0$ and a holomorphic function $g : D(a; r) \to \mathbb{C}$ such that $g(a) \neq 0$ and*

$$f(z) = \frac{g(z)}{(z-a)^n}$$

for every $z \in D(a; r) \setminus \{a\}$.

Example 5.25 Let f be a function with some isolated singularities that are either removable or poles. Show that the function $\frac{f'}{f}$ has simple poles precisely at all the zeros and poles of f.

Proof: It is clear that $\frac{f'}{f}$ has isolated singularities only at the zeros and poles of f, and $\frac{f'}{f}$ is holomorphic everywhere else.

(i) Suppose that f has a zero of order m at a. Then by Factor Theorem, there exist $r > 0$ and a holomorphic function $g : D(a; r) \to \mathbb{C}$ such that $g(z) \neq 0$ for any $z \in D(a; r)$ and

$$f(z) = (z-a)^m g(z)$$

for every $z \in D(a; r) \setminus \{a\}$. Thus

$$f'(z) = (z-a)^m g'(z) + m(z-a)^{m-1} g(z)$$

and so

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} + \frac{m}{z-a}$$

for every $z \in D(a; r) \setminus \{a\}$. But since $g(z) \neq 0$ for any $z \in D(a; r)$, g'/g is holomorphic on $D(a; r)$. So f'/f has a simple pole at a.

(ii) Suppose that f has a pole of order n at b. Then by Factor Theorem, there exist $R > 0$ and a holomorphic function $h : D(b; R) \to \mathbb{C}$ such that $h(z) \neq 0$ for any $z \in D(b; R)$ and

$$f(z) = \frac{h(z)}{(z-a)^n}$$

for every $z \in D(b; R) \setminus \{b\}$. Thus

$$f'(z) = \frac{h'(z)}{(z-b)^n} - \frac{nh(z)}{(z-b)^{n+1}}$$

and so

$$\frac{f'(z)}{f(z)} = \frac{h'(z)}{h(z)} - \frac{n}{z-b}$$

for every $z \in D(b; R) \setminus \{b\}$. But since $h(z) \neq 0$ for any $z \in D(b; R)$, h'/h is holomorphic on $D(b; R)$. So f'/f also has a simple pole at b. ■

Remark 5.26 The following is a summary of the **classification of isolated singularities**.

Type of isolated singularity at a	***Removable singularity***	***Pole***	***Essential singularity***
Limit behavior near a	$\lim_{z \to a} f(z) \in \mathbb{C}$	$\lim_{z \to a} f(z) = \infty$	$\lim_{z \to a} f(z)$ does not exist
Direct image of punctured disks centered at a	There exists $r > 0$ so that $f(D(a;r) \setminus \{a\})$ is bounded	Given any disk, there exists $r > 0$ so that $f(D(a;r) \setminus \{a\})$ is in the exterior of the disk	For every $r > 0$, $\overline{f(D(a;r) \setminus \{a\})} = \mathbb{C}$
Principal part of Laurent series at a	Principal part is zero (All a_{-k}'s are zero)	Principal part is non-zero finite sum (Finitely many non-zero a_{-k}'s)	Principal part is an infinite sum (Infinitely many nonzero a_{-k}'s)
Is $(z-a)^n f$ holomorphic at a?	f extends to be holomorphic at a	$(z-a)^n f$ extends to be holomorphic at a when $n \in \mathbb{N}$ is large	$(z-a)^n f$ is still not holomorphic at a for any $n \in \mathbb{N}$

Example 5.27 Find all the isolated singularities of the following functions, and determine whether they are removable, essential, or poles. Also determine the order of any pole.

(a) $f(z) = \frac{e^{2z}-1}{z}$

(b) $g(z) = \frac{\operatorname{Log} z}{(z-1)^3}$, where $\operatorname{Log}: \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ is the principal branch of logarithm

(c) $h(z) = z^2 \sin \frac{1}{z}$

Solution: [Let's do the classification using Laurent series this time.]

(a) f is clearly holomorphic on $\mathbb{C} \setminus \{0\}$, and 0 is the only isolated singularity of f. For every $z \in \mathbb{C} \setminus \{0\}$, we have

$$f(z) = \frac{e^{2z}-1}{z} = \frac{1}{z} \sum_{k=1}^{+\infty} \frac{(2z)^k}{k!} = \sum_{k=0}^{+\infty} \frac{2^{k+1}}{(k+1)!} z^k .$$

The Laurent series of f at 0 has zero principal part, so f has a removable singularity at 0.

(b) Since Log is holomorphic on $\mathbb{C}\setminus(-\infty,0]$, we see that g is holomorphic on $(\mathbb{C}\setminus(-\infty,0])\setminus\{1\}$, and 1 is the only **isolated** singularity of g. For every $z\in D(1;1)\setminus\{1\}$, we have

$$g(z)=\frac{\operatorname{Log} z}{(z-1)^3}=\frac{1}{(z-1)^3}\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{k}(z-1)^k$$

$$=\sum_{k=0}^{+\infty}\frac{(-1)^k}{k+3}(z-1)^k-\frac{1}{2}(z-1)^{-1}+(z-1)^{-2}.$$

The principal part of the Laurent series of g at 1 has nonzero terms up to the exponent -2, so g has a double pole at 1.

(c) The function h is holomorphic on $\mathbb{C}\setminus\{0\}$, and 0 is the only isolated singularity of h. For every $z\in\mathbb{C}\setminus\{0\}$, we have

$$h(z)=z^2\sin\frac{1}{z}=z^2\sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{1}{z}\right)^{2k+1}=z+\sum_{k=1}^{+\infty}\frac{(-1)^k}{(2k+1)!}z^{1-2k}.$$

The principal part of the Laurent series of h at 0 has infinitely many nonzero terms, so h has an essential singularity at 0.

3. Residue

Generalized Cauchy integral formula only deals with line integrals whose integrand is of the form

$$\frac{f(z)}{(z-a)^n},$$

where $n\in\mathbb{N}$ and the function f is holomorphic at $a\in\mathbb{C}$. Such an integrand has a pole of order at most n at a. In this section, our goal is to find a more general formula that handles line integrals whose integrands are **holomorphic except possibly at some isolated singularities**. We will need two ingredients, namely

- **Residues** (regarding the integrand) and
- **Winding numbers** (regarding the path of integration).

Remark 5.28 Consider a function f which is holomorphic except at some isolated singularities, e.g. $f(z)=z^2e^{\frac{1}{z}}$ with a singularity at 0. Then the Laurent series of f at 0,

$$z^2e^{\frac{1}{z}}=z^2+z+\frac{1}{2}+\frac{1}{3!}z^{-1}+\frac{1}{4!}z^{-2}+\cdots,$$

converges uniformly on compact subsets of a punctured disk centered at 0 ($\mathbb{C}\setminus\{0\}$ for this particular example).

So in order to compute

$$\oint_{\partial D(0;1)} z^2 e^{\frac{1}{z}} dz,$$

we may integrate the Laurent series of f at 0 **term-by-term**:

$$\oint_{\partial D(0;1)} z^2 e^{\frac{1}{z}} dz = \oint_{\partial D(0;1)} \left(z^2 + z + \frac{1}{2} + \frac{1}{3!} z^{-1} + \frac{1}{4!} z^{-2} + \cdots\right) dz$$

$$= \oint_{\partial D(0;1)} z^2 dz + \oint_{\partial D(0;1)} z dz + \oint_{\partial D(0;1)} \frac{1}{2} dz + \oint_{\partial D(0;1)} \frac{1}{3!} z^{-1} dz + \oint_{\partial D(0;1)} \frac{1}{4!} z^{-2} dz + \cdots$$

$$= 0 + 0 + 0 + \frac{1}{3!}(2\pi i) + 0 + \cdots = \frac{\pi i}{3}.$$

In conclusion, thanks to the fact that for each integer n and each small $r > 0$,

$$\oint_{\partial D(a;r)} (z-a)^n dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{if } n \neq -1 \end{cases},$$

we see that **the coefficient of $(z-a)^{-1}$** of a Laurent series at a is particularly important.

Definition 5.29 Let $a \in \mathbb{C}$ and let f be a function which has an isolated singularity at a. The ***residue of f at a*** is the coefficient of $(z-a)^{-1}$ in the Laurent series of f at a, i.e.

$$\operatorname{Res}(f; a) := a_{-1}$$

if $f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-a)^k$ for every $z \in D(a;r) \setminus \{a\}$.

Example 5.30 Let

$$f(z) = \frac{\sin z}{z} \qquad \text{and} \qquad g(z) = \frac{1}{(z-2)^2}.$$

Find the residues of f and g at each of their isolated singularities in $\mathbb{C}$.

Solution:

(i) The only isolated singularity of f is the removable singularity at 0. The Laurent series of f at 0 has zero principal part, so $\operatorname{Res}(f; 0) = 0$.

(ii) The only isolated singularity of g is the double pole at 2. The Laurent series of g at 2 is just

$$g(z) = \frac{1}{(z-2)^2}.$$

The coefficient of $(z-2)^{-1}$ of this Laurent series is 0, so $\operatorname{Res}(g; 2) = 0$.

Example 5.31 Let

$$f(z) = \frac{4}{(z+1)^2(z^2+1)}.$$

Find the residues of f at each of its isolated singularities in $\mathbb{C}$.

Solution:

The only isolated singularities of f in $\mathbb{C}$ are -1, i and $-i$. For every z near -1, we have

$$f(z) = \frac{2}{z+1} + \frac{2}{(z+1)^2} - \frac{2z}{(z^2+1)} = \qquad ,$$

so $\mathrm{Res}(f; -1) =$. For every z near i, we have

$$f(z) = \frac{2}{z+1} + \frac{2}{(z+1)^2} - \frac{1}{z+i} - \frac{1}{z-i} = \qquad ,$$

so $\mathrm{Res}(f; i) =$. For every z near $-i$, we have

$$f(z) = \frac{2}{z+1} + \frac{2}{(z+1)^2} - \frac{1}{z+i} - \frac{1}{z-i} = \qquad ,$$

so $\mathrm{Res}(f; -i) =$.

It may sometimes be too tedious to find the residue by computing the Laurent series. The following **computational formula** is sometimes helpful in this situation.

Lemma 5.32 *Let $a \in \mathbb{C}$ and let f be a function which has an isolated singularity at a.*

- *If f has a removable singularity at a, then*
$$\mathrm{Res}(f; a) = 0.$$
- *If f has a simple pole at a, then*
$$\mathrm{Res}(f; a) = \lim_{z \to a} (z-a) f(z).$$
- *In general, if f has a pole of order n at a, then*
$$\mathrm{Res}(f; a) = \frac{1}{(n-1)!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} [(z-a)^n f(z)].$$

Proof. If f has a pole of order (at most) n at a, then there exists $r > 0$ such that

$$f(z) = \sum_{k=-n}^{+\infty} a_k (z-a)^k$$

for every $z \in D(a; r) \setminus \{a\}$. Then $(z-a)^n f(z) = \sum_{k=0}^{+\infty} a_{k-n} (z-a)^k$, and so

$$\frac{d^{n-1}}{dz^{n-1}} [(z-a)^n f(z)] = \sum_{k=0}^{+\infty} \frac{(k+n-1)!}{k!} a_{k-1} (z-a)^k$$

for every $z \in D(a; r) \setminus \{a\}$. $\frac{d^{n-1}}{dz^{n-1}} [(z-a)^n f(z)]$ has (at most) a removable singularity at a, so

$$\mathrm{Res}(f; a) = a_{-1} = \frac{1}{(n-1)!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} [(z-a)^n f(z)].$$

■

Example 5.31 Let

$$f(z) = \frac{4}{(z+1)^2(z^2+1)}.$$

Find the residues of f at each of its isolated singularities in $\mathbb{C}$.

Solution:
The only isolated singularities of f in $\mathbb{C}$ are -1, i and $-i$. f has a double pole at -1, and

$$\mathrm{Res}(f; -1) =$$

f has a simple pole at i, and

$$\mathrm{Res}(f; i) =$$

f has a simple pole at $-i$, and

$$\mathrm{Res}(f; -i) =$$

Example 5.33 Let

$$g(z) = \frac{e^z}{\sin z}.$$

Find the residues of g at each of its isolated singularities in $\mathbb{C}$.

Solution: The only isolated singularities of g are the simple poles at .

$$\mathrm{Res}(g; \quad) =$$

Example 5.34 Let

$$h(z) = (z+1)^3 e^{-\frac{1}{z}}.$$

Find the residues of h at each of its isolated singularities in $\mathbb{C}$.

Solution: [At essential singularities, Lemma 5.32 does not work. We must use Laurent series.]
The only isolated singularity of h is the essential singularity at 0. For every $z \in \mathbb{C} \setminus \{0\}$, we have

$$h(z) = (z+1)^3 e^{-\frac{1}{z}} =$$

The residue of h at 0 is the coefficient of z^{-1} in this Laurent series expansion, i.e.

$$\mathrm{Res}(h; 0) =$$

Next we study the **winding number**, which is an integer that describes how many times a closed curve "wraps around" a certain point.

Theorem 5.35 *Let $\gamma\colon [0,1] \to \mathbb{C}$ be a closed C^1 curve and let $a \in \mathbb{C} \setminus (\operatorname{image}\gamma)$, i.e. γ does not pass through a. Then*

$$\frac{1}{2\pi i}\oint_\gamma \frac{1}{z-a}\,dz$$

is an integer.

Proof. Let $\phi\colon [0,1] \to \mathbb{C}$ be defined by

$$\phi(t) = \exp\left(\int_0^t \frac{\gamma'(s)}{\gamma(s)-a}\,ds\right).$$

Differentiating ϕ using Fundamental Theorem of Calculus (the MATH1013 version), we have

$$\phi'(t) = \phi(t) \cdot \frac{\gamma'(t)}{\gamma(t)-a}$$

for every $t \in (0,1)$. Now this implies that

$$\frac{d}{dt}\frac{\phi(t)}{\gamma(t)-a} = \frac{\phi'(t)}{\gamma(t)-a} - \frac{\phi(t)\gamma'(t)}{(\gamma(t)-a)^2} = 0$$

for every $t \in (0,1)$, so $\frac{\phi}{\gamma-a}$ is a constant function on $[0,1]$. In particular, we have

$$\frac{\phi(1)}{\gamma(1)-a} = \frac{\phi(0)}{\gamma(0)-a} = \frac{1}{\gamma(1)-a},$$

so $1 = \phi(1) = \exp\left(\int_0^1 \frac{\gamma'(t)}{\gamma(t)-a}\,dt\right)$, i.e.

$$\frac{1}{2\pi i}\oint_\gamma \frac{1}{z-a}\,dz = \frac{1}{2\pi i}\int_0^1 \frac{\gamma'(t)}{\gamma(t)-a}\,dt$$

is an integer. ∎

Definition 5.36 Let $\gamma\colon [0,1] \to \mathbb{C}$ be a closed C^1 curve and let $a \in \mathbb{C} \setminus (\operatorname{image}\gamma)$, i.e. γ does not pass through a. The ***winding number of γ around a*** is the integer defined by

$$n(\gamma; a) := \frac{1}{2\pi i}\oint_\gamma \frac{1}{z-a}\,dz.$$

Example 5.37 Let $a \in D(0;1)$. Then the winding number of $\partial D(0;1)$ around a is 1.

This example seems to suggest that the winding number $n(\gamma;\,\cdot)$ is constant on each connected component of $\mathbb{C} \setminus (\operatorname{image}\gamma)$. Let's prove this claim.

Lemma 5.38 *Let* $\gamma: [0,1] \to \mathbb{C}$ *be a closed* C^1 *curve. Then the function*

$$n(\gamma;\ \cdot): \mathbb{C} \setminus (\operatorname{image} \gamma) \to \mathbb{Z}$$

is continuous.

Proof. Let $L > 0$ be the arc-length of γ. Let $a \in \mathbb{C} \setminus (\operatorname{image} \gamma)$. Since $[0,1]$ is compact, $\delta = \min\{|\gamma(t) - a| : t \in [0,1]\} > 0$ exists. Then for every $w \in D\left(a, \frac{\delta}{2}\right)$, we have

$$|n(\gamma; w) - n(\gamma; a)| = \left| \frac{1}{2\pi i} \oint_\gamma \left(\frac{1}{z-w} - \frac{1}{z-a} \right) dz \right| = \frac{|w-a|}{2\pi} \left| \oint_\gamma \frac{1}{(z-w)(z-a)} dz \right|$$
$$\leq \frac{|w-a|}{2\pi} \cdot \frac{2}{\delta^2} \cdot L = \frac{L}{\pi \delta^2} |w-a|,$$

so $n(\gamma;\ \cdot)$ is continuous at a. ■

Corollary 5.39 *Let* $\gamma: [0,1] \to \mathbb{C}$ *be a closed* C^1 *curve. Then the function*

$$n(\gamma;\ \cdot): \mathbb{C} \setminus (\operatorname{image} \gamma) \to \mathbb{Z}$$

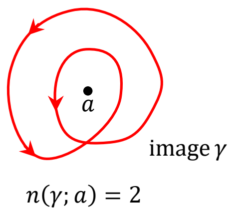
is constant on each connected component of $\mathbb{C} \setminus (\operatorname{image} \gamma)$*.*

Corollary 5.40 *Let* $\gamma: [0,1] \to \mathbb{C}$ *be a* ***simple*** *closed* C^1 *curve oriented counterclockwise, and let* $a \in \mathbb{C} \setminus (\operatorname{image} \gamma)$*. Then*

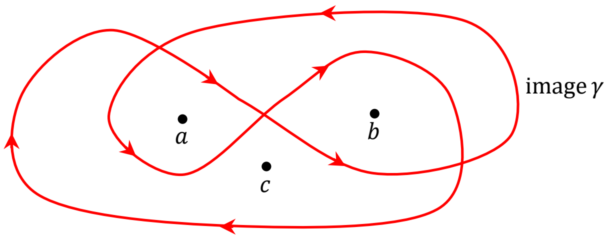
$$n(\gamma; a) = \begin{cases} 0 & \text{if } a \text{ is in the exterior of } \gamma \\ 1 & \text{if } a \text{ is in the interior of } \gamma \end{cases}.$$

Proof. Cauchy integral formula. ■

Remark 5.41 Let $\gamma: [0,1] \to \mathbb{C}$ be a closed C^1 curve and let $a \in \mathbb{C} \setminus (\operatorname{image} \gamma)$. The winding number $n(\gamma; a)$ describes how many times the curve γ "**wraps around**" the point a in the **counterclockwise** sense, as t increases from 0 to 1.



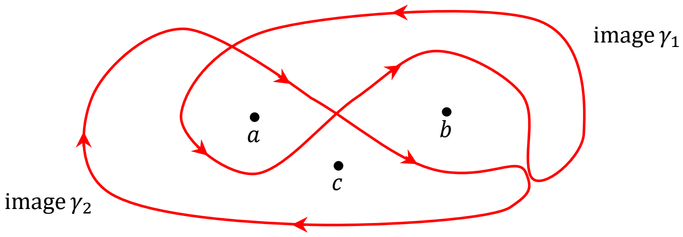
Example 5.42 The following shows the image of the ***Pochhammer contour*** γ. It is a closed C^1 curve in $\mathbb{C}$ that goes around its image once.



Find the winding numbers of γ around the points a, b and c respectively.

Solution:

We decompose γ into two **simple** closed piecewise C^1 curves γ_1 and γ_2, as shown in the diagram below.



We see that

⊙ γ_1 is oriented and γ_2 is oriented ;

⊙ a is in the of γ_1, while b and c are in the of γ_1; and

⊙ a and c are in the of γ_2, while b is in the of γ_2.

So we have

$$\begin{cases} n(\gamma_1; a) = \\ n(\gamma_1; b) = \\ n(\gamma_1; c) = \end{cases} \quad \text{and} \quad \begin{cases} n(\gamma_2; a) = \\ n(\gamma_2; b) = \\ n(\gamma_2; c) = \end{cases} .$$

Since $\gamma = \gamma_1 * \gamma_2$, we have

$$n(\gamma; a) = n(\gamma_1; a) + n(\gamma_2; a) =$$
$$n(\gamma; b) = n(\gamma_1; b) + n(\gamma_2; b) =$$
$$n(\gamma; c) = n(\gamma_1; c) + n(\gamma_2; c) =$$

Now we come to the key theorem of this section.

Theorem 5.43 (Cauchy's residue theorem) *Let $U \subseteq \mathbb{C}$ be a simply connected region, let γ be a closed piecewise C^1 curve in U, and let f be a function holomorphic on U except at some isolated singularities $z_1, z_2, \ldots \in U \setminus (\mathrm{image}\,\gamma)$. Then*

$$\oint_\gamma f(z)dz = 2\pi i \sum_j n(\gamma; z_j) \operatorname{Res}(f; z_j).$$

In case γ is a **simple** closed curve, we have the following simpler version thanks to Corollary 5.40.

Corollary 5.44 (Cauchy's residue theorem) *Let $U \subseteq \mathbb{C}$ be a simply connected region, let γ be a simple closed piecewise C^1 curve in U oriented counterclockwise, and let f be a function holomorphic on U except at some isolated singularities in $U \setminus (\mathrm{image}\,\gamma)$, such that $z_1, z_2, \ldots, z_n$ are the only singularities of f in the interior of γ. Then*

$$\oint_\gamma f(z)dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}(f; z_j).$$

Proof of Theorem 5.43. Since the singularities $\{z_n\}$ of f are isolated, there are only finitely many z_j's for which $n(\gamma; z_j)$ is non-zero. So without loss of generality, we may assume that f has only finitely many singularities $z_1, z_2, \dots, z_n \in U$. Now for each $j \in \{1, 2, \dots, n\}$, we let $p_j\left(\frac{1}{z-a_j}\right)$ be the principal part of the Laurent series of f at a_j (which is not necessarily a finite sum). Then the function $f(z) - \sum_{j=1}^{n} p_j\left(\frac{1}{z-a_j}\right)$ has removable singularities only, so it extends to a holomorphic function on U. Thus $\oint_\gamma \left[f(z) - \sum_{j=1}^{n} p_j\left(\frac{1}{z-a_j}\right)\right] dz = 0$ by Cauchy-Goursat, i.e.

$$\oint_\gamma f(z)dz = \sum_{j=1}^{n} \oint_\gamma p_j\left(\frac{1}{z-a_j}\right) dz.$$

Now each series $p_j\left(\frac{1}{z-a_j}\right) = \sum_{k=1}^{+\infty} b_{-k}(z-a_j)^{-k}$ converges uniformly on the image of γ, so

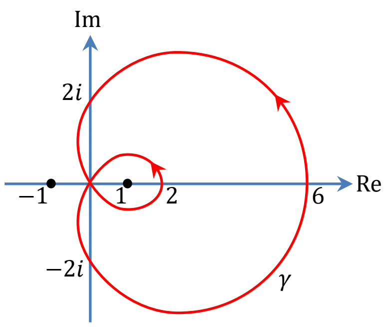
$$\oint_\gamma p_j\left(\frac{1}{z-a_j}\right) dz = \sum_{k=1}^{+\infty} b_{-k} \oint_\gamma (z-a_j)^{-k} dz = b_{-1} \oint_\gamma \frac{1}{z-a_j} dz = \operatorname{Res}(f; a_j) \cdot 2\pi i \, n(\gamma; a_j)$$

for each $j \in \{1, 2, \dots, n\}$, which completes the proof. ■

Example 5.45 Let $\gamma: [0, 2\pi] \to \mathbb{C}$ be the curve $\gamma(t) = (2 + 4\cos t)e^{it}$. Evaluate the integral

$$\oint_\gamma \frac{e^z}{z^2-1} dz.$$

Solution:



The only isolated singularities of $\frac{e^z}{z^2-1}$ are 1 and -1.

- ⦿ The winding number of γ around these isolated singularities are $n(\gamma; 1) =$ and $n(\gamma; -1) = 0$.
- ⦿ The residue of the integrand at 1 is

$$\operatorname{Res}\left(\frac{e^z}{z^2-1}; 1\right) =$$

So by Cauchy's residue theorem, we have

$$\oint_\gamma \frac{e^z}{z^2-1} dz = 2\pi i \Big[\qquad\qquad + \qquad\qquad \Big]$$

$$=$$

Example 5.46 Evaluate the line integral

$$\oint_{\partial D(0;2)} e^{e^{\frac{1}{z}}} dz.$$

Solution:

$\partial D(0;2)$ is a **simple** closed curve. The only isolated singularity of $e^{e^{1/z}}$ is 0, which lies in the **interior** of $\partial D(0;2)$. Since the Laurent series of $e^{e^{1/z}}$ at 0 is

$$e^{e^{\frac{1}{z}}} =$$

the residue of $e^{e^{1/z}}$ at 0 is the coefficient of z^{-1} in the above series, which is

$$\operatorname{Res}\left(e^{e^{\frac{1}{z}}}; 0\right) =$$

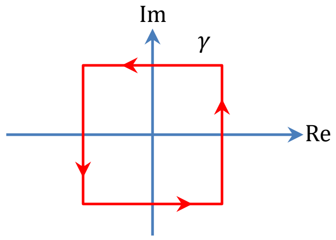
So by Cauchy's residue theorem (Corollary 5.43), we have

$$\oint_{\partial D(0;2)} e^{e^{\frac{1}{z}}} dz =$$

Example 5.47 Let γ be the counterclockwise oriented boundary of the rectangle in $\mathbb{C}$ with vertices $-2-i$, $2-i$, $2+i$ and $-2+i$. Evaluate

$$\oint_\gamma \tan \pi z \, dz.$$

Solution:



γ is a simple closed curve. The only isolated singularities of $\tan \pi z$ which lie in the interior of γ are the simple poles at

. The residues of $\tan \pi z$ at these singularities are

$\operatorname{Res}(\tan \pi z, \quad) =$, $\operatorname{Res}(\tan \pi z, \quad) =$,

$\operatorname{Res}(\tan \pi z, \quad) =$, $\operatorname{Res}(\tan \pi z, \quad) =$.

So by Cauchy's residue theorem (Corollary 5.44), we have

$$\oint_\gamma \tan \pi z \, dz =$$

4. Evaluation of real integrals

We have already seen before (cf. Example 3.46) that complex line integrals turn out to be helpful for **evaluating difficult real integrals** that physicists and engineers may encounter. Now having the powerful tool of **Cauchy's residue theorem**, we study more techniques in this aspect.

Remark 5.48 (Rational functions of cos and sin) To evaluate integrals of the type

$$\int_0^{2\pi} R(\cos t, \sin t)dt$$

where R is a rational function, it is useful to consider the line integral

$$\oint_{\partial D(0;1)} \frac{1}{z} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) dz$$

along the **unit circle** centered at the origin.

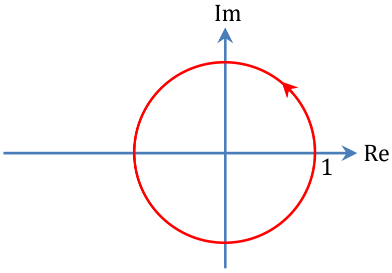
Example 5.49 Let a be a real number with $|a| > 1$. Evaluate the integral

$$\int_0^{2\pi} \frac{1}{a+\cos t} dt.$$

Solution:

Let $f: \mathbb{C} \setminus \{ \quad , \quad \} \to \mathbb{C}$ be the holomorphic function

$$f(z) =$$



Then

$$\oint_{\partial D(0;1)} f(z)dz = \qquad = i\int_0^{2\pi} \frac{1}{a+\cos t} dt.$$

On the other hand, the only singularities of f are the simple poles at , and among these two simple poles, only lies in the interior of $\partial D(0;1)$. The residue of f at is

$$\mathrm{Res}\left(f; \qquad\right) =$$

So by Cauchy's residue theorem we have

$$\oint_{\partial D(0;1)} f(z)dz =$$

Therefore

$$\int_0^{2\pi} \frac{1}{a+\cos t} dt =$$

The line integral can also be easily computed using **Cauchy integral formula**.

Example 5.50 Let n be a positive integer. Evaluate

$$\int_0^{2\pi} \sin^{2n} t\, dt.$$

Solution:

Let $f: \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be the holomorphic function

$$f(z) =$$

Then

$$\oint_{\partial D(0;1)} f(z)dz = \qquad\qquad = i\int_0^{2\pi} \sin^{2n} t\, dt.$$

On the other hand, the only singularity of f is the pole at 0, which lies in the interior of $\partial D(0;1)$.

The residue of f at 0 is the coefficient of z^{-1} in , i.e. the constant term in ,

$$\operatorname{Res}(f;0) =$$

So by Cauchy's residue theorem, we have

$$\oint_{\partial D(0;1)} f(z)dz =$$

Therefore

$$\int_0^{2\pi} \sin^{2n} t\, dt =$$

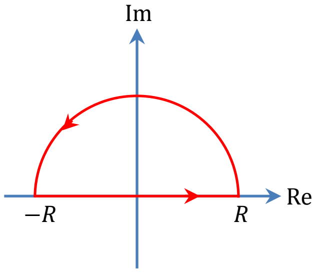
Remark 5.51 (Rational functions) To evaluate integrals of the type

$$\int_{-\infty}^{+\infty} \frac{p(t)}{q(t)} dt$$

where p and q are polynomials with $\deg q \geq \deg p + 2$, it is useful to consider the line integral

$$\oint_\gamma \frac{p(z)}{q(z)} dz$$

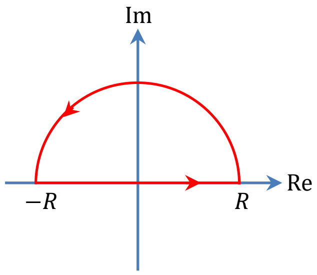
where γ is the counterclockwise oriented **boundary of a large semicircular disk** centered at the origin, whose diameter coincides with the real axis.



Large semicircle

Example 5.52 Evaluate the improper Riemann integral

$$\int_{-\infty}^{+\infty} \frac{1}{(x^2+x+1)^2}\,dx.$$



Solution: Let $R > 2$ and define

- $\gamma_1 : [-R, R] \to \mathbb{C}$ by $\gamma_1(x) = x$;
- $\gamma_2 : [0, \pi] \to \mathbb{C}$ by $\gamma_2(t) = Re^{it}$.

Let $\gamma = \gamma_1 * \gamma_2$, which is a counterclockwise oriented simple closed curve.

Let $f : \mathbb{C} \setminus \{\qquad , \qquad\} \to \mathbb{C}$ be the holomorphic function $f(z) = \frac{1}{(z^2+z+1)^2}$. Then the only singularity of f in the interior of γ is the at , and

$$\operatorname{Res}(f; \qquad) =$$

So by Cauchy's residue theorem, we have

$$\int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz = \oint_{\gamma} f(z)dz =$$

On the other hand, we have

$$\int_{\gamma_1} f(z)dz = \int_{-R}^{R} \frac{1}{(x^2+x+1)^2}\,dx\,;$$

It can also be easily computed using **generalized Cauchy integral formula**.

and by ML-estimate, we have

$$\left|\int_{\gamma_2} f(z)dz\right| \le$$

which tends to zero as $R \to +\infty$. Therefore taking limits on both sides of

$$\int_{-R}^{R} \frac{1}{(x^2+x+1)^2}\,dx + \int_{\gamma_2} f(z)dz =$$

as $R \to +\infty$, we obtain

$$\lim_{R\to+\infty} \int_{-R}^{R} \frac{1}{(x^2+x+1)^2}\,dx =$$

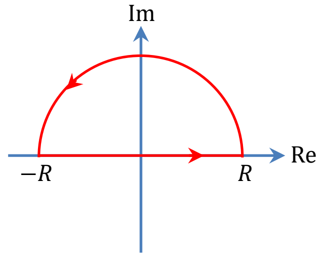
Since $|f|$ is improper Riemann integrable on $(-\infty, 0]$ and on $[0, +\infty)$ (e.g. by comparison test), the required improper Riemann integral of f indeed equals the above Cauchy principal value, i.e.

$$\int_{-\infty}^{+\infty} \frac{1}{(x^2+x+1)^2}\,dx =$$

Remark 5.53 The improper Riemann integral $\int_{-\infty}^{+\infty} f(x)dx = \lim\limits_{M\to-\infty} \lim\limits_{N\to+\infty} \int_M^N f(x)dx$ is different from its **Cauchy principal value** $\mathrm{P.V.}\int_{-\infty}^{+\infty} f(x)dx = \lim\limits_{R\to+\infty} \int_{-R}^{R} f(x)dx$ in general. They are the same when $|f|$ is Riemann integrable both on $(-\infty, 0]$ and on $[0, +\infty)$.

Example 5.54 Let n be a positive integer. Evaluate the improper Riemann integral

$$\int_{-\infty}^{+\infty} \frac{1}{x^{2n}+1}\,dx.$$



Solution: Let $R > 1$ and define

- $\gamma_1 : [-R, R] \to \mathbb{C}$ by $\gamma_1(x) = x$;
- $\gamma_2 : [0, \pi] \to \mathbb{C}$ by $\gamma_2(t) = Re^{it}$.

Let $\gamma = \gamma_1 * \gamma_2$, which is a counterclockwise oriented simple closed curve.

Let $f : \mathbb{C} \setminus \{ \qquad \} \to \mathbb{C}$ be the holomorphic function $f(z) = \frac{1}{z^{2n}+1}$.

Now for each $k \in \{ \qquad \}$ (the other n simple poles are in the exterior of γ), we have

$$\operatorname{Res}\left(f; \qquad \right) =$$

by l'Hôpital's rule. So by Cauchy's residue theorem, we have

$$\int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz = \oint_{\gamma} f(z)dz =$$

On the other hand, we have

$$\int_{\gamma_1} f(z)dz = \int_{-R}^{R} \frac{1}{x^{2n}+1}\,dx\,;$$

and by ML-estimate, we have

$$\left|\int_{\gamma_2} f(z)dz\right| \le$$

which tends to zero as $R \to +\infty$. Therefore taking limits on both sides of

$$\int_{-R}^{R} \frac{1}{x^{2n}+1}\,dx + \int_{\gamma_2} f(z)dz =$$

as $R \to +\infty$, we obtain

$$\lim_{R\to+\infty} \int_{-R}^{R} \frac{1}{x^{2n}+1}\,dx =$$

Since $|f|$ is improper Riemann integrable on $(-\infty, +\infty)$, the required improper Riemann integral of f indeed equals the above Cauchy principal value, i.e.

$$\int_{-\infty}^{+\infty} \frac{1}{x^{2n}+1}\,dx =$$

Remark 5.55 (Rational function times $\cos$ or $\sin$) To evaluate integrals of the type

$$\int_{-\infty}^{+\infty} \frac{p(t)}{q(t)} \cos t \, dt \qquad \text{or} \qquad \int_{-\infty}^{+\infty} \frac{p(t)}{q(t)} \sin t \, dt$$

where p and q are polynomials with $\deg q > \deg p$, it is useful to consider the line integral

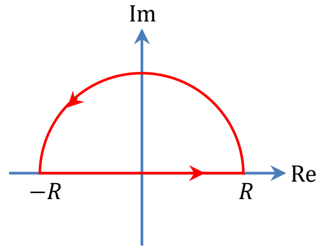
$$\oint_\gamma \frac{p(z)}{q(z)} e^{iz} dz$$

where γ is a **large semicircle** in the upper half-plane centered at the origin whose diameter coincides with the real axis, because the function e^{iz} is bounded in the upper half-plane (why?). When estimating the integral along the large circular arc, we simply use ML-estimate in case $\deg q \geq \deg p + 2$; but in case $\deg q = \deg p + 1$, we often need to use **Jordan's inequality**:

$$\sin t \geq \frac{2}{\pi} t \qquad \text{for every } t \in \left[0, \frac{\pi}{2}\right].$$

Example 5.56 Evaluate the integral

$$\int_{-\infty}^{+\infty} \frac{x \sin x}{x^2 + 1} dx.$$



Solution: Let $R > 1$ and define

- $\gamma_1 \colon [-R, R] \to \mathbb{C}$ by $\gamma_1(x) = x$;
- $\gamma_2 \colon [0, \pi] \to \mathbb{C}$ by $\gamma_2(t) = Re^{it}$.

Let $\gamma = \gamma_1 * \gamma_2$, which is a counterclockwise oriented simple closed curve. Let $f \colon \mathbb{C} \setminus \{ \quad \} \to \mathbb{C}$

be the holomorphic function $f(z) =$. Then the only singularity of f in the interior of γ is

the at , at which the residue of f is $\mathrm{Res}(f; \ ) =$. So we have

$$\int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz = \oint_\gamma f(z) dz =$$

by Cauchy's residue theorem. On the other hand, we have

It can also be easily computed using **Cauchy integral formula**.

$$\int_{\gamma_1} f(z) dz = \qquad\qquad = i \int_{-R}^{R} \frac{x \sin x}{x^2 + 1} dx;$$

and by Jordan's inequality $\sin t \geq \frac{2}{\pi} t$ for every $t \in \left[0, \frac{\pi}{2}\right]$, we have

$$\left| \int_{\gamma_2} f(z) dz \right| \leq \int_0^\pi \left| \qquad\qquad \right| dt \leq$$

which tends to zero as $R \to +\infty$.

Therefore taking limits on both sides of

$$i\int_{-R}^{R}\frac{x\sin x}{x^2+1}dx+\int_{\gamma_2}f(z)dz=$$

as $R\to+\infty$, we obtain

$$\lim_{R\to+\infty}\int_{-R}^{R}\frac{x\sin x}{x^2+1}dx=$$

Note that $|f|$ is **not** improper Riemann integrable on $(-\infty,+\infty)$, so the required improper Riemann integral of f does not exist, and what we have obtained above is just the Cauchy principal value, i.e.

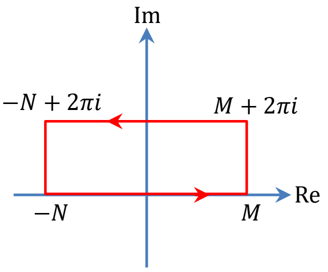
$$\mathrm{P.V.}\int_{-\infty}^{+\infty}\frac{x\sin x}{x^2+1}dx=$$

In some other situations, we may consider **other kinds of paths of integration** as appropriate.

Example 5.57 Let $a\in(0,1)$. Evaluate the improper Riemann integral

$$\int_{-\infty}^{+\infty}\frac{e^{ax}}{1+e^x}dx.$$

Solution: Let $M>0$, $N>0$, and let γ be the boundary of the rectangle in $\mathbb{C}$ with vertices M, $M+2\pi i$, $-N+2\pi i$ and $-N$, oriented counterclockwise. Let $f:\mathbb{C}\setminus\{\qquad\}\to\mathbb{C}$ be the holomorphic function $f(z)=\frac{e^{az}}{1+e^z}$. Then the only singularity of f in the interior of γ is the at , and



$$\mathrm{Res}(f;\quad)=$$

by l'Hôpital's rule. So by Cauchy's residue theorem, we have

$$\int_{-N}^{M}\frac{e^{ax}}{1+e^x}dx+\int_0^{2\pi}\frac{e^{a(M+it)}}{1+e^{M+it}}idt-\int_{-N}^{M}\frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}}dx-\int_0^{2\pi}\frac{e^{a(-N+it)}}{1+e^{-N+it}}idt$$

$$=\oint_\gamma f(z)dz=$$

On the other hand, we have

$$\left|\int_0^{2\pi}\frac{e^{a(M+it)}}{1+e^{M+it}}idt\right|\le$$

which tends to 0 as $M\to+\infty$, and

$$\left|\int_0^{2\pi}\frac{e^{a(-N+it)}}{1+e^{-N+it}}idt\right|\le$$

which also tends to zero as $N\to+\infty$.

So taking limits on both sides of

$$\int_{-N}^{M} \frac{e^{ax}}{1+e^x}dx + \int_0^{2\pi} \frac{e^{a(M+it)}}{1+e^{M+it}} idt - \int_{-N}^{M} \frac{e^{a(x+2\pi i)}}{1+e^x}dx - \int_0^{2\pi} \frac{e^{a(-N+it)}}{1+e^{-N+it}} idt =$$

as $M \to +\infty$ and $N \to +\infty$, we obtain

Therefore

$$\int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^x}dx =$$

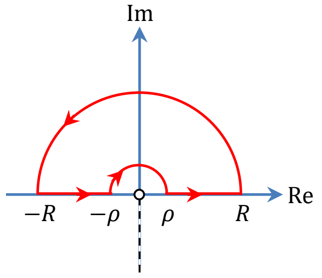
Remark 5.58 (Logarithms) To evaluate integrals of the type

$$\int_0^{+\infty} R(t)\ln t\, dt \qquad \text{or} \qquad \int_0^{+\infty} R(t)t^a dt$$

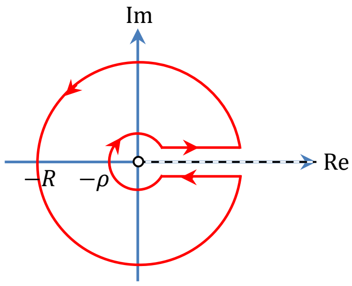
where R is a rational function and a is not an integer, it is useful to consider line integrals

$$\oint_\gamma R(z)\log z\, dz \qquad \text{or} \qquad \oint_\gamma R(z)e^{a\log z}dz.$$

The branch of complex logarithm has to be defined on a simply connected region not containing 0, so the simple closed curve γ has to be chosen in a way that avoids the point 0. The following "**indented path**" or "**key-hole path**" are natural choices of γ.



Indented path: Branch of logarithm defined on $\mathbb{C}\setminus\{iy\in\mathbb{C}: y\le 0\}$



Key-hole path: Branch of logarithm defined on $\mathbb{C}\setminus[0,+\infty)$

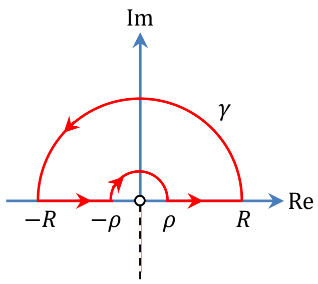
Example 5.59 Evaluate the improper Riemann integral

$$\int_0^{+\infty} \frac{\ln x}{(1+x^2)^2}dx.$$

Solution:

Let $U = \mathbb{C}\setminus\{iy\in\mathbb{C}: y\le 0\}$. Let $R>1$ and $\rho\in(0,1)$, and define

- $\gamma_1: [0,\pi]\to\mathbb{C}$ by $\gamma_1(t) = Re^{it}$,
- $\gamma_2: [-R,-\rho]\to\mathbb{C}$ by $\gamma_2(x) = x$,
- $\gamma_3: [0,\pi]\to\mathbb{C}$ by $\gamma_3(t) = \rho e^{i(\pi-t)}$, and
- $\gamma_4: [\rho,R]\to\mathbb{C}$ by $\gamma_4(x) = x$.



Let γ be the "indented path" $\gamma = \gamma_1 * \gamma_2 * \gamma_3 * \gamma_4$, which is a counterclockwise oriented simple closed curve in U. Let $\log: U \to \mathbb{C}$ be the branch of logarithm such that

$$\log(re^{i\theta}) = \ln r + i\theta \qquad \text{for } r > 0 \text{ and } -\frac{\pi}{2} < \theta < \frac{3\pi}{2}$$

(i.e. $\log 1 = 0$). Let f: $\to \mathbb{C}$ be the holomorphic function

$$f(z) = \frac{\log z}{(1+z^2)^2}.$$

Then the only isolated singularity of f in the interior of γ is the at , and

$$\mathrm{Res}(f;\ ) =$$

So by Cauchy's residue theorem, we have

$$\int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz + \int_{\gamma_4} f(z)dz = \oint_{\gamma} f(z)dz =$$

On the other hand, we have

$$\int_{\gamma_2} f(z)dz = \qquad = \int_{\rho}^{R} \frac{\ln x + i\pi}{(1+x^2)^2} dx$$

It can also be easily computed using **generalized Cauchy integral formula**.

and

$$\int_{\gamma_4} f(z)dz = \int_{\rho}^{R} \frac{\ln x}{(1+x^2)^2} dx\,;$$

while

$$\left|\int_{\gamma_1} f(z)dz\right| \leq$$

which tends to 0 as $R \to +\infty$, and

$$\left|\int_{\gamma_3} f(z)dz\right| \leq$$

which also tends to 0 as $\rho \to 0^+$. So taking limits on both sides of

$$\int_{\gamma_1} f(z)dz + \int_{\rho}^{R} \frac{\ln x + i\pi}{(1+x^2)^2} dx + \int_{\gamma_3} f(z)dz + \int_{\rho}^{R} \frac{\ln x}{(1+x^2)^2} dx =$$

as $R \to +\infty$ and $\rho \to 0^+$, we obtain

Therefore comparing the real parts of both sides, we have

$$\int_0^{+\infty} \frac{\ln x}{(1+x^2)^2} dx = -\frac{\pi}{4}.$$

Note that by comparing the imaginary parts of both sides, we obtain another result as a by-product:

$$\int_0^{+\infty} \frac{1}{(1+x^2)^2} dx = \frac{\pi}{4}.$$

The method of evaluating improper integrals using Cauchy's residue theorem still has applications in some other situations. Interested students may refer to Supplementary Note G for details.

5. Argument principle

The **argument principle** focuses on functions with isolated singularities that are **either removable singularities or poles**.

Definition 5.60 Let $U \subseteq \mathbb{C}$ be a region and f be a function which is holomorphic on U except at isolated singularities. We say that f is a ***meromorphic function on U*** if it has no essential singularities, i.e. the isolated singularities of f are either removable singularities or poles. In this case, we may extend the domain of f to include its poles, and define $F: U \to \mathbb{C} \cup \{\infty\}$ by

$$F(z) = \lim_{w \to z} f(w).$$

Example 5.61 All rational functions are meromorphic functions on $\mathbb{C}$. The function

$$f(z) = e^{\frac{1}{z}}$$

is not a meromorphic function on $\mathbb{C}$ because it has an **essential singularity** at 0.

Theorem 5.62 (Argument principle) *Let $U \subseteq \mathbb{C}$ be a simply connected region, let γ be a simple closed piecewise C^1 curve in U oriented counterclockwise, and let f be a meromorphic function on U, whose zeros and poles are in $U \setminus (\text{image}\, \gamma)$. Then the line integral*

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)} dz$$

equals to the number of zeros of f in the interior of γ minus the number of poles of f in the interior of γ, both counting multiplicities (i.e. each zero / pole of order n is counted n times).

Proof. Similar to Example 5.25, we analyze the function f'/f. If f has a zero of order m at a point $a \in U$, then according to Factor Theorem, there exist $r > 0$ and a holomorphic function $g: D(a; r) \to \mathbb{C}$ such that $g(a) \neq 0$ and

$$f(z) = (z - a)^m g(z)$$

for every $z \in D(a; r)$. Then

$$\frac{f'(z)}{f(z)} = \frac{m}{z - a} + \frac{g'(z)}{g(z)}$$

for every $z \in D(a; r) \setminus \{a\}$, so a is a simple pole of f'/f and

$$\operatorname{Res}\left(\frac{f'}{f}; a\right) = m = \text{order of zero of } f \text{ at } a.$$

Similarly, if f has a pole of order n at a point $b \in U$, then by Factor Theorem, there exist $R > 0$ and a holomorphic function $h: D(b; R) \to \mathbb{C}$ such that $h(b) \neq 0$ and

$$f(z) = \frac{h(z)}{(z-b)^n}$$

for every $z \in D(b; R)$. Then

$$\frac{f'(z)}{f(z)} = \frac{-n}{z-b} + \frac{g'(z)}{g(z)}$$

for every $z \in D(a; r) \setminus \{a\}$, so b is a simple pole of f'/f and

$$\operatorname{Res}\left(\frac{f'}{f}; b\right) = -n = -(\text{order of pole of } f \text{ at } b).$$

The result then follows by applying Cauchy's residue theorem. ■

Example 5.63 Let $f: \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be the function

$$f(z) = \frac{\sin z}{z^2}.$$

Evaluate the integral

$$\oint_{\partial D(0;10)} \frac{f'(z)}{f(z)} dz.$$

Solution:
f is a meromorphic function on $\mathbb{C}$, and the circle $\partial D(0; 10)$ does not pass through any of the zeros and poles of f. In the interior of $\partial D(0; 10)$, f has zeros at , and has a simple pole at 0 (why not a double pole?). So by argument principle, we have

$$\oint_{\partial D(0;10)} \frac{f'(z)}{f(z)} dz =$$

Lemma 5.64 *Let $U \subseteq \mathbb{C}$ be a region, let $f: U \to \mathbb{C}$ be a holomorphic function, let $\gamma: [0,1] \to U$ be a closed C^1 curve, and let $a \in \mathbb{C} \setminus (\operatorname{image}(f \circ \gamma))$, i.e. $f(\gamma(t)) \neq a$ for any $t \in [0,1]$. Then*

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z) - a} dz$$

is the winding number of the curve $f \circ \gamma$ around a.

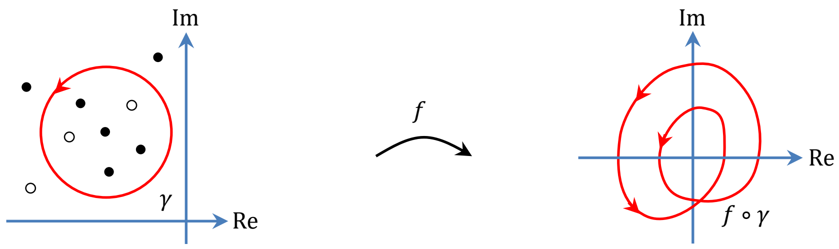
Proof.

$$n(f \circ \gamma; a) = \frac{1}{2\pi i} \oint_{f \circ \gamma} \frac{1}{z-a} dz = \frac{1}{2\pi i} \int_0^1 \frac{1}{(f \circ \gamma)(t) - a} (f \circ \gamma)'(t) dt$$
$$= \frac{1}{2\pi i} \int_0^1 \frac{f'(\gamma(t))}{f(\gamma(t)) - a} \gamma'(t) dt = \frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z) - a} dz.$$

Corollary 5.65 (Argument principle) *Let* $U \subseteq \mathbb{C}$ *be a simply connected region, let* γ *be a simple closed piecewise* C^1 *curve in* U *oriented counterclockwise, and let* f *be a meromorphic function on* U*, whose zeros and poles are in* $U \setminus (\text{image}\,\gamma)$*. Then the winding number of* $f \circ \gamma$ *around* 0*,*

$$n(f \circ \gamma; 0),$$

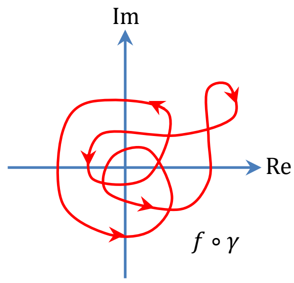
equals to the number of zeros of f *in the interior of* γ *minus the number of poles of* f *in the interior of* γ*, both counting multiplicities (i.e. each zero / pole of order* n *is counted* n *times).*



Number of zeros of f (black dots) $-$ number of poles of f (hollow dots) in the interior of γ $= 4 - 2 = 2$

The winding number $n(f \circ \gamma; 0) = 2$

Example 5.66 Let $U \subset \mathbb{C}$ be a bounded simply connected region, whose boundary ∂U is the image of a simple closed C^1 curve γ. Let f be a function which is holomorphic on $\overline{U}$ and has no zeros on ∂U. If $f \circ \gamma$ is the closed curve whose image is as shown in the diagram on the right, find the number of zeros of f in U, counting multiplicities.



Solution:

From the diagram, we see that $f \circ \gamma$ winds around the origin for three times in the counterclockwise direction, i.e.

$$n(f \circ \gamma; 0) = 3.$$

So by argument principle (Corollary 5.64), the number of zeros of f in U minus the number of poles of f in U is 3, counting multiplicities. But f has no poles in U as it is holomorphic on U. So f has 3 zeros in U, counting multiplicities.

The following (Example 5.67 – Remark 5.71) are some other **applications of the argument principle**.

Example 5.67 Show that there is no holomorphic function $f: D(0; 1) \setminus \{0\} \to \mathbb{C}$ such that

$$\big(f(z)\big)^2 = z$$

for every $z \in D(0; 1) \setminus \{0\}$. In other words, the complex "square root" function cannot be defined on the whole punctured disk $D(0; 1) \setminus \{0\}$.

Proof:
Suppose on the contrary that such a function f exists. Then differentiation gives

for every $z \in D(0;1) \setminus \{0\}$, so

$$\frac{f'(z)}{f(z)} =$$

for every $z \in D(0;1) \setminus \{0\}$. Therefore

$$\frac{1}{2\pi i} \oint_{\partial D\left(0;\frac{1}{2}\right)} \frac{f'(z)}{f(z)} dz =$$

But by argument principle, , which gives a contradiction. ■

The following is another proof of the **Fundamental Theorem of Algebra**, this time using the argument principle.

Example 5.68 Let p be a polynomial of degree n with complex coefficients. Show that p has exactly n zeros in $\mathbb{C}$, counting multiplicities.

Proof:
Without loss of generality, we assume that p is monic (i.e. its leading coefficient is 1) and let

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0,$$

where $a_0, a_1, \dots, a_{n-1} \in \mathbb{C}$. Let $R > 0$. Then we have

$$\begin{aligned}\left|\frac{1}{2\pi i}\oint_{\partial D(0;R)} \frac{p'(z)}{p(z)} dz - n\right| &= \left|\frac{1}{2\pi i}\oint_{\partial D(0;R)} \left[\frac{nz^{n-1} + (n-1)a_{n-1}z^{n-2} + \cdots + a_1}{z^n + a_{n-1}z^{n-1} + \cdots + a_0} - \frac{n}{z}\right] dz\right| \\ &= \frac{1}{2\pi}\left|\oint_{\partial D(0;R)} \frac{-a_{n-1}z^{n-1} - 2a_{n-2}z^{n-2} - \cdots - (n-1)a_1 z - na_0}{z(z^n + a_{n-1}z^{n-1} + \cdots + a_0)} dz\right| \\ &\le \frac{1}{2\pi} \cdot \frac{|a_{n-1}|R^{n-1} + 2|a_{n-2}|R^{n-2} + \cdots + n|a_0|}{R^{n+1} - |a_{n-1}|R^n - \cdots - |a_0|R} \cdot 2\pi R \\ &\le \frac{1}{2}\end{aligned}$$

whenever $R > M$, for some $M > 0$. However, according to the argument principle, $\left|\frac{1}{2\pi i}\oint_{\partial D(0;R)} \frac{p'(z)}{p(z)} dz - n\right|$ is a non-negative integer, so it must be 0. In other words,

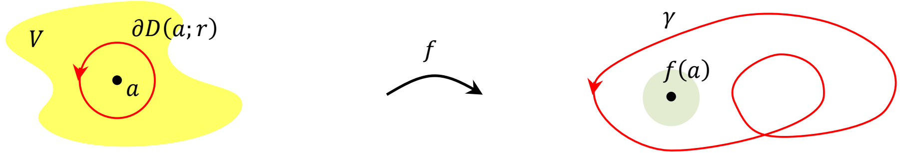
$$\frac{1}{2\pi i}\oint_{\partial D(0;R)} \frac{p'(z)}{p(z)} dz = n,$$

which implies that p has exactly n zeros in $D(0;R)$ counting multiplicities, because p has no poles. But $R > M$ was arbitrary, so p has exactly n zeros in $\mathbb{C}$, counting multiplicities. ■

The argument principle provides another slightly different proof of the **open mapping theorem**.

Theorem 4.43 (Open mapping) *The direct image of an open set via a non-constant holomorphic function is open.*

Alternative proof. Let $U \subseteq \mathbb{C}$ be a region, let $f: U \to \mathbb{C}$ be a non-constant holomorphic function, and let $V \subseteq U$ be an open set. The issue is to show that $f(V)$ is open.



For each $a \in V$, we aim to find a disk centered at $f(a)$ which contains completely in $f(V)$. Since V is open, there exists $r > 0$ such that $D(a; 2r) \subseteq V$. Let $\gamma: [0, 2\pi] \to \mathbb{C}$ be the curve

$$\gamma(t) = f\left(a + re^{it}\right)$$

(i.e. γ is the curve $f\left(\partial D(a;r)\right)$). Since f is non-constant, according to the isolated zeros theorem, we may assume without loss of generality that γ does not pass through any zeros of $f(z) - f(a)$. Then by the argument principle (Corollary 5.65),

$$n\left(\gamma; f(a)\right) = \text{number of zeros of } \left(f(z) - f(a)\right) \text{ in } D(a;r) \geq 1,$$

because a is obviously a zero of $f(z) - f(a)$. Now let $2\delta = \min\{|\gamma(t) - f(a)| : t \in [0, 2\pi]\} > 0$, which exists because $[0, 2\pi]$ is compact. Then for each $w \in D(f(a); \delta)$, $f(a)$ and w belong to the same connected component of $\operatorname{image} \gamma$, so by argument principle (Corollary 5.65) and Corollary 5.39,

$$\text{Number of zeros of } (f(z) - w) \text{ in } D(a;r) = n(\gamma; w) = n\left(\gamma; f(a)\right) \geq 1,$$

which shows that $w \in f\left(D(a;r)\right) \subseteq f(V)$. Thus $D(f(a); \delta) \subseteq f(V)$, and so $f(V)$ is open. ■

Having seen some applications of the argument principle, we study the following **generalized version of the argument principle**.

Theorem 5.69 (Generalized argument principle) *Let $U \subseteq \mathbb{C}$ be a simply connected region, let γ be a simple closed piecewise C^1 curve in U oriented counterclockwise, let f be a meromorphic function on U, whose zeros and poles are in $U \setminus (\operatorname{image} \gamma)$, and let $g: U \to \mathbb{C}$ be a holomorphic function. If in the interior of γ, f has a zero of order m_j at a_j for $j \in \{1, 2, \ldots, m\}$ and has a pole of order n_k at b_k for $k \in \{1, 2, \ldots, n\}$, then*

$$\frac{1}{2\pi i} \oint_\gamma g(z) \frac{f'(z)}{f(z)} dz = \sum_{j=1}^{m} m_j g(a_j) - \sum_{k=1}^{n} n_k g(b_k).$$

Note that Theorem 5.69 reduces to the original argument principle if we take $g \equiv 1$.

Proof. The proof is very similar to that of the argument principle. We observe that if f has a zero of order m at a point $a \in U$, then gf'/f has a simple pole at a and

$$\operatorname{Res}\left(g\frac{f'}{f};a\right) = mg(a);$$

and if f has a pole of order n at a point $b \in U$, then gf'/f also has a simple pole at b and

$$\operatorname{Res}\left(g\frac{f'}{f};b\right) = -ng(b).$$

The theorem is proved by summing terms of these types using Cauchy's residue theorem. ∎

Corollary 5.70 *Let $U, V \subseteq \mathbb{C}$ be simply connected regions, let γ be a simple closed piecewise C^1 curve in U oriented counterclockwise, and let $f: U \to V$ be a holomorphic function which has an inverse $f^{-1}: V \to U$. Then for each $w \in V$ such that $f^{-1}(w)$ is in the interior of γ, we have*

$$f^{-1}(w) = \frac{1}{2\pi i}\oint_\gamma \frac{zf'(z)}{f(z)-w}\,dz.$$

Proof. In the interior of γ, the function $f - w$ has only the simple zero $f^{-1}(w)$ and no poles, so we obtain the result by taking $g(z) = z$ in the generalized argument principle. ∎

Example 5.71 Let

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$$

be a monic polynomial. If γ is a counterclockwise oriented simple closed curve whose interior contains all the zeros of p, then

$$\frac{1}{2\pi i}\oint_\gamma \frac{zp'(z)}{p(z)}\,dz = \text{sum of zeros of } p = -a_{n-1},$$

$$\frac{1}{2\pi i}\oint_\gamma \frac{z^2p'(z)}{p(z)}\,dz = \text{sum of squares of zeros of } p = {a_{n-1}}^2 - 2a_{n-2},$$

etc.

As a final application of the argument principle, we have the following **Rouché's Theorem**, which is a very useful tool for analyzing the **number of zeros of a holomorphic function in a certain region**.

Theorem 5.72 (Rouché) *Let $U \subseteq \mathbb{C}$ be a simply connected region, let $f, g: U \to \mathbb{C}$ be holomorphic functions and let γ be a simple closed curve in U. If*

$$|f(z) - g(z)| < |f(z)| + |g(z)|$$

for every $z \in \operatorname{image}\gamma$, then f and g have the same number of zeros in the interior of γ, counting multiplicities.

Proof. The given hypothesis implies that g has no zeros on $\operatorname{image}\gamma$, and

$$\left|\frac{f(z)}{g(z)}-1\right|<\left|\frac{f(z)}{g(z)}\right|+1$$

for every $z\in\operatorname{image}\gamma$. Thus $\frac{f(z)}{g(z)}$ can never be a non-positive real number for any $z\in\operatorname{image}\gamma$, i.e. the image of $\left(\frac{f}{g}\right)\circ\gamma$ is a subset of $\mathbb{C}\setminus(-\infty,0]$, which is a simply connected region not containing 0. Choosing any branch of logarithm $\log:\mathbb{C}\setminus(-\infty,0]\to\mathbb{C}$, we see that $\log\frac{f}{g}$ is an antiderivative of $\frac{(f/g)'}{(f/g)}$ on an open set containing $\operatorname{image}\gamma$. Therefore

$$\oint_\gamma\frac{f'(z)}{f(z)}dz-\oint_\gamma\frac{g'(z)}{g(z)}dz=\oint_\gamma\frac{(f/g)'(z)}{(f/g)(z)}dz=0$$

by Corollary 3.27, which implies that f and g have the same number of zeros in the interior of γ counting multiplicities, according to the argument principle. ∎

Corollary 5.73 (Rouché) *Let $U\subseteq\mathbb{C}$ be a simply connected region, let $f,g:U\to\mathbb{C}$ be holomorphic functions and let γ be a simple closed curve in U. If*

$$|f(z)-g(z)|<|f(z)|$$

for every $z\in\operatorname{image}\gamma$, then f and g have the same number of zeros in the interior of γ, counting multiplicities.

Remark 5.74 Rouché's Theorem (in the weaker form, Corollary 5.73) says that if one subtracts a "small term" $f-g$ from a holomorphic function f on the image of a simple closed curve γ, then the number of zeros in the interior of γ (counting multiplicities) is left unchanged. In other words, the number of zeros of a holomorphic function f is the same as that of its **"dominant term"** g.

Example 5.75 Let p be the polynomial

$$p(z)=z^7+3z^2+1.$$

Find the number of zeros of p in the annulus $D(0;2)\setminus\overline{D(0;1)}$.

Solution:

Let $f:\mathbb{C}\to\mathbb{C}$ be the function $f(z)=$. Then for every $z\in\partial D(0;1)$, we have

$$|f(z)|=$$

$$|p(z)-f(z)|=$$

i.e. $|p(z)-f(z)|<|f(z)|$. So p and f have the same number of zeros in $D(0;1)$ by Rouché's Theorem, i.e. p has zeros in $D(0;1)$, and also in $\overline{D(0;1)}$ (why?).

On the other hand, let $g: \mathbb{C} \to \mathbb{C}$ be the function $g(z) =$. Then for every $z \in \partial D(0;2)$, we have

$$|g(z)| =$$
$$|p(z) - g(z)| =$$

i.e. $|p(z) - g(z)| < |g(z)|$. So p and g have the same number of zeros in $D(0;2)$ by Rouché's Theorem, i.e. p has zeros in $D(0;2)$. Consequently, p has exactly zeros in the annulus $D(0;2) \setminus \overline{D(0;1)}$.

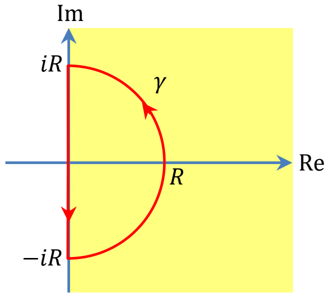
Example 5.76 Let $f: \mathbb{C} \to \mathbb{C}$ be the function

$$f(z) = z + e^{-z} - 5.$$

Show that f has exactly one zero which is a simple zero in the open right half-plane

$$\{z \in \mathbb{C}: \operatorname{Re} z > 0\},$$

and that this simple zero is in fact real.



Proof:
Let $R > 6$ and define

- ⦿ $\gamma_1: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{C}$ by $\gamma_1(t) = Re^{it}$;
- ⦿ $\gamma_2: [-R, R] \to \mathbb{C}$ by $\gamma_2(y) = -iy$.

Let γ be the semicircle $\gamma = \gamma_1 * \gamma_2$, which is a simple closed curve. Let $g: \mathbb{C} \to \mathbb{C}$ be the function $g(z) =$. Then f and g are entire. Now for every $z \in \operatorname{image} \gamma_1$, we have

$$|g(z)| =$$

$$|f(z) - g(z)| =$$

and for every $z \in \operatorname{image} \gamma_2$, we have

$$|g(z)| =$$
$$|f(z) - g(z)| =$$

i.e. $|f(z) - g(z)| < |g(z)|$ for every $z \in \operatorname{image} \gamma$. So f and g have the same number of zeros in the interior of γ by Rouché's Theorem, i.e. f has exactly 1 zero in the interior of γ which is a simple zero. But $R > 6$ was arbitrary, so f has exactly 1 zero in $\{z \in \mathbb{C}: \operatorname{Re} z > 0\}$, which is a simple zero.

It finally remains to show that this simple zero of f in $\{z \in \mathbb{C}: \operatorname{Re} z > 0\}$ is real. Since f is , and

$$f(0) = \qquad < 0 \qquad \text{and} \qquad f(5) = \qquad > 0,$$

we see that f has a zero in the open interval $(0,5)$, according to .
Thus the only simple zero of f in $\{z \in \mathbb{C}: \operatorname{Re} z > 0\}$ must be this real zero in $(0,5)$. ■

Rouché's Theorem gives yet another proof of the **Fundamental Theorem of Algebra**.

Example 5.68 Let p be a polynomial of degree n with complex coefficients. Show that p has exactly n zeros in $\mathbb{C}$, counting multiplicities.

Proof:
Without loss of generality, we assume that p is monic (i.e. its leading coefficient is 1) and let

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0,$$

where $a_0, a_1, \ldots, a_{n-1} \in \mathbb{C}$. Let $f: \mathbb{C} \to \mathbb{C}$ be the function $f(z) = z^n$. Let

$$M \coloneqq \max\left\{1, n|a_{n-1}|, n^{\frac{1}{2}}|a_{n-2}|^{\frac{1}{2}}, \ldots, n^{\frac{1}{n}}|a_0|^{\frac{1}{n}}\right\},$$

and let $R > M$ be arbitrary. Then for every $z \in \partial D(0; R)$, we have

$$\begin{aligned}|p(z) - f(z)| &= |a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0| \\ &\leq |a_{n-1}|R^{n-1} + |a_{n-2}|R^{n-2} + \cdots + |a_0| \\ &< \frac{R}{n}R^{n-1} + \frac{R^2}{n}R^{n-2} + \cdots + \frac{R^n}{n} = R^n = |f(z)|,\end{aligned}$$

so p and f have the same number of zeros in $D(0; R)$ by Rouché's Theorem, i.e. p has exactly n zeros in $D(0; R)$ counting multiplicities. But $R > M$ was arbitrary, so p has exactly n zeros in $\mathbb{C}$, counting multiplicities. ■

Summary of Chapter 5

The following are what you need to know in this chapter in order to get a pass (a distinction) in this course:

✓ **Laurent series**
- ⦿ **Laurent series, annulus of convergence**
- ⦿ To compute the **Laurent series** of a function in various annular regions
- ⦿ **Order of a pole** of a function

✓ Classification of **isolated singularities**
- ⦿ Three types of isolated singularities: **removable singularity**, **pole**, **essential singularity**
- ⦿ Classification into the three types using **limiting behavior** near the singularity
- ⦿ Classification into the three types using **image of punctured disks** centered at the singularity
- ⦿ Classification into the three types using **principal part of Laurent series**

✓ **Residues** and **winding numbers**
- ⦿ Definition of $\mathrm{Res}(f;a)$: the coefficient of $(z-a)^{-1}$ in the Laurent series of f at a
- ⦿ Formula for computing **residue at poles**
- ⦿ **Winding number** of a closed curve around a point
- ⦿ **Cauchy's residue theorem**, to **compute line integrals** using Cauchy's residue theorem

✓ Evaluation of **difficult real integrals** using complex line integrals
- ⦿ Choice of **complex integrand** → choice of **path of integration** → **estimation** of each line integral using ML-estimate or other tools
- ⦿ Integrals of **trigonometric rational functions** on $[0,2\pi]$
- ⦿ Improper integrals of **rational functions** on $(-\infty,+\infty)$ or $[0,+\infty)$
- ⦿ Improper integrals of a **rational function times $\sin$ or $\cos$** on $(-\infty,+\infty)$ or $[0,+\infty)$
- ⦿ Improper integrals of a **rational function times x^a or $\ln$** on $(0,+\infty)$
- ⦿ Improper integral and its **Cauchy principal value**

✓ **Argument principle** and its consequences
- ⦿ **Meromorphic functions**
- ⦿ **Argument principle**, generalized argument principle
- ⦿ **Rouché's Theorem**
- ⦿ To **count the number of zeros or poles** of functions on certain regions using argument principle and Rouché's Theorem