

## LEARNING OBJECTIVES

In this section, you will:

- View vectors geometrically.
- Find magnitude and direction.
- Perform vector addition and scalar multiplication.
- Find the component form of a vector.
- Find the unit vector in the direction of  $v$ .
- Perform operations with vectors in terms of  $i$  and  $j$ .
- Find the dot product of two vectors.
- Identify orthogonal vectors.

## 9.1 VECTORS

An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of  $140^\circ$ . A north wind (from north to south) is blowing at 16.2 miles per hour, as shown in **Figure 1**. What are the ground speed and actual bearing of the plane?

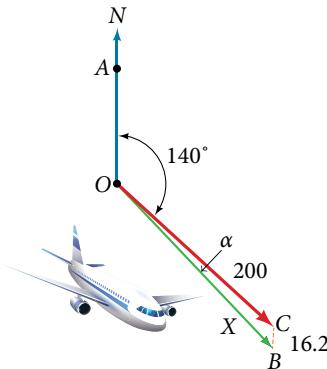


Figure 1

Ground speed refers to the speed of a plane relative to the ground. Airspeed refers to the speed a plane can travel relative to its surrounding air mass. These two quantities are not the same because of the effect of wind. Later in this section, we will find the airplane's ground speed and bearing, while investigating another approach to problems of this type. First, however, let's examine the basics of vectors.

### A Geometric View of Vectors

A **vector** is a specific quantity drawn as a line segment with an arrowhead at one end. It has an **initial point**, where it begins, and a **terminal point**, where it ends. A vector is defined by its **magnitude**, or the length of the line, and its direction, indicated by an arrowhead at the terminal point. Thus, a vector is a directed line segment. There are various symbols that distinguish vectors from other quantities:

- Lower case, boldfaced type, with or without an arrow on top such as  $v$ ,  $\mathbf{u}$ ,  $w$ ,  $\vec{v}$ ,  $\overrightarrow{\mathbf{u}}$ ,  $\overrightarrow{w}$ .
- Given initial point  $P$  and terminal point  $Q$ , a vector can be represented as  $\overrightarrow{PQ}$ . The arrowhead on top is what indicates that it is not just a line, but a directed line segment.
- Given an initial point of  $(0, 0)$  and terminal point  $(a, b)$ , a vector may be represented as  $\langle a, b \rangle$ .

This last symbol  $\langle a, b \rangle$  has special significance. It is called the **standard position**. The position vector has an initial point  $(0, 0)$  and a terminal point  $\langle a, b \rangle$ . To change any vector into the position vector, we think about the change in the  $x$ -coordinates and the change in the  $y$ -coordinates. Thus, if the initial point of a vector  $\overrightarrow{CD}$  is  $C(x_1, y_1)$  and the terminal point is  $D(x_2, y_2)$ , then the position vector is found by calculating

$$\overrightarrow{CD} = \langle x_2 - x_1, y_2 - y_1 \rangle = \langle a, b \rangle$$

In **Figure 2**, we see the original vector  $\overrightarrow{CD}$  and the position vector  $\overrightarrow{AB}$ .

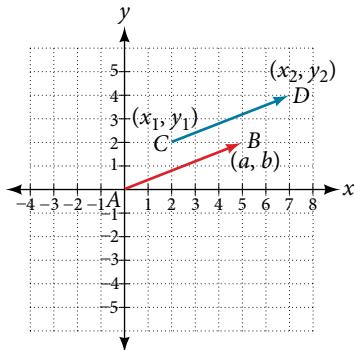


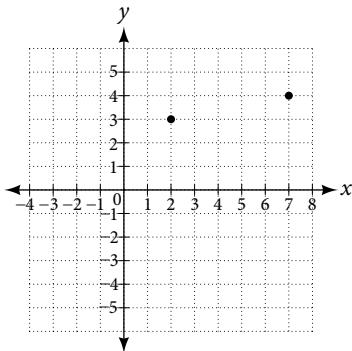
Figure 2

**Properties of Vectors**

- i) A vector is a directed line segment with an initial point and a terminal point.
- ii) Vectors are identified by magnitude i.e. the length of the line, and by direction represented by the arrowhead pointing toward the terminal point.
- iii) The position vector  $\langle a, b \rangle$  has an initial point at  $(0, 0)$  and is identified by its terminal point  $(a, b)$ .

**Example 1** Consider the vector whose initial point is  $P(2, 3)$  and terminal point is  $Q(7, 4)$ . Find the position vector.

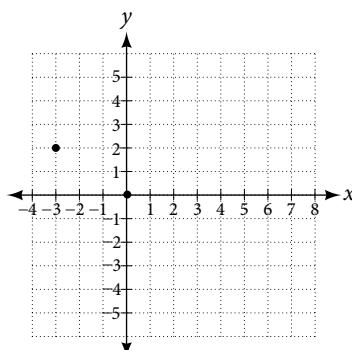
**Solution**



**Example 2** Find the position vector given that vector  $v$  has an initial point at  $(-3, 2)$  and a terminal point at  $(4, 5)$ ,

then graph both vectors in the same plane.

**Solution**



## Finding Magnitude and Direction

To work with a vector, we need to be able to find its magnitude and its direction. We find its magnitude using the Pythagorean Theorem or the distance formula, and we find its direction using the inverse tangent function.

### Magnitude and Direction of a Vector

Given a position vector  $v = \langle a, b \rangle$ , the magnitude is found by  $|v| = \sqrt{a^2 + b^2}$ .

The direction is equal to the angle formed with the  $x$ -axis, or with the  $y$ -axis, depending on the application.

For a position vector, the direction is found by  $\tan \theta = \frac{b}{a} \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$ , as illustrated in **Figure 5**.

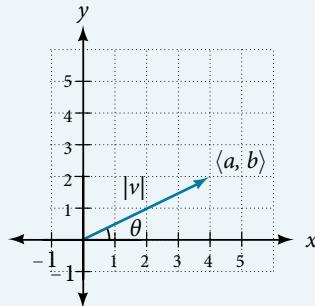


Figure 5

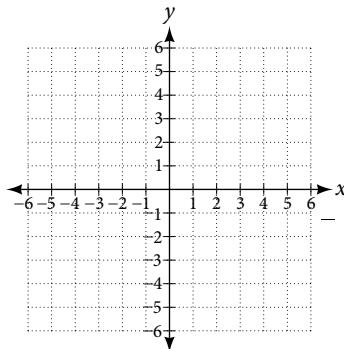
Two vectors  $v$  and  $u$  are considered equal if they have the same magnitude and the same direction.

Additionally, if both vectors have the same position vector, they are equal.

**Example 3** Find the magnitude and direction of the vector with initial point  $P(-8, 1)$  and terminal point

$Q(-2, -5)$ . Draw the vector.

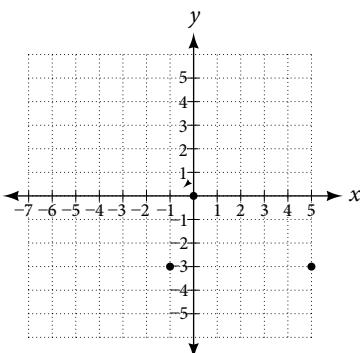
**Solution**



**Example 4** Show that vector  $v$  with initial point at  $(5, -3)$  and terminal point at  $(-1, 2)$  is equal to vector  $u$  with initial point at  $(-1, -3)$  and terminal point at  $(-7, 2)$ .

Draw the position vector on the same grid as  $v$  and  $u$ . Next, find the magnitude and direction of each vector.

**Solution**



## Performing Vector Addition and Scalar Multiplication

The sum of two vectors  $u$  and  $v$ , or **vector addition**, produces a third vector  $u + v$ , the **resultant** vector.

To find  $u + v$ , we have the initial point of  $v$  meet the terminal end of  $u$ .

This position corresponds to the notion that we move along the first vector and then, from its terminal point, we move along the second vector.

The sum  $u + v$  is the resultant vector because it results from addition or subtraction of two vectors.

The resultant vector travels directly from the beginning of  $u$  to the end of  $v$  in a straight path.

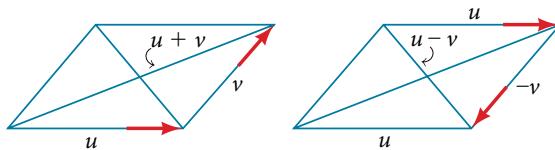


**Vector subtraction** is similar to vector addition. To find  $u - v$ , view it as  $u + (-v)$ .

Adding  $-v$  is reversing direction of  $v$  and adding it to the end of  $u$ .

The new vector begins at the start of  $u$  and stops at the end point of  $-v$ .

See **Figure below** for a visual that compares vector addition and vector subtraction **using parallelograms**.



### Vector Addition / Subtraction

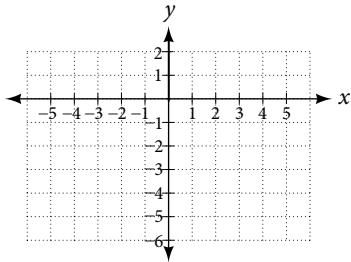
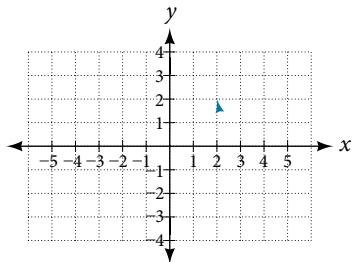
Given two vectors  $u = \langle a, b \rangle$  and  $v = \langle c, d \rangle$ , then

the sum of these two vectors is  $u + v = \langle a + c, b + d \rangle$ , and

the difference of these two vectors is  $u - v = u + (-v) = \langle a - c, b - d \rangle$

**Example 5** Given  $u = \langle 3, -2 \rangle$  and  $v = \langle -1, 4 \rangle$ , find two new vectors  $u + v$ , and  $u - v$ .

**Solution**



## Multiplying By a Scalar

Adding and Subtracting vectors gives a new vector with a different magnitude and direction. The process of multiplying a vector by a **scalar**/constant, changes only the magnitude of the vector. Scalar multiplication has no effect on the direction unless the scalar is negative, in which case the direction of the resulting vector is opposite the direction of the original vector.

### Scalar Multiplication

**Scalar multiplication** involves the product of a vector and a scalar.

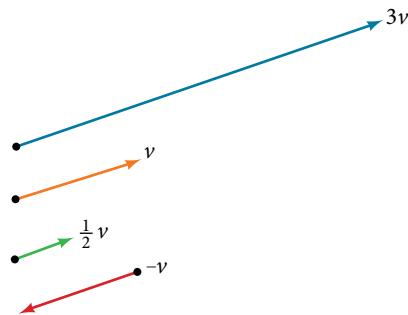
Each component of the vector is multiplied by the scalar. Thus, to multiply  $v = \langle a, b \rangle$  by  $k$ , we have

$$kv = \langle ka, kb \rangle$$

Only the magnitude changes, unless  $k$  is negative, and then the vector reverses direction.

**Example 6** Given vector  $v = \langle 3, 1 \rangle$ , find  $3v$ ,  $\frac{1}{2}v$ , and  $-v$ .

**Solution**



Example 7 Given  $u = \langle 3, -2 \rangle$  and  $v = \langle -1, 4 \rangle$ , find a new vector  $w = 3u + 2v$ .

Solution

## Finding Component Form

In some applications, it is helpful for us to be able to break a vector down into its components.

Vectors are comprised of two components:

the horizontal component is *the x* direction, and the vertical component is the *y* direction.

For example, think of the position vector  $\langle 2, 3 \rangle$  as a sum of the vectors  $v_1 = \langle 2, 0 \rangle$  and  $v_2 = \langle 0, 3 \rangle$ .

$$v = \langle 2 + 0, 3 + 0 \rangle = \langle 2, 3 \rangle$$

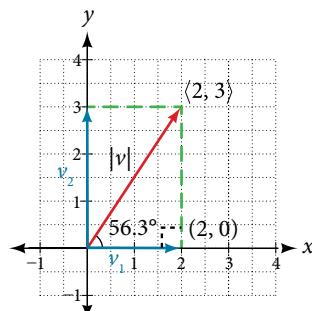
Using the Pythagorean Theorem, the magnitude of  $v_1$  is 2, and the magnitude of  $v_2$  is 3.

To find the magnitude of  $v$ , use the formula with the position vector.

$$\begin{aligned} |v| &= \sqrt{|v_1|^2 + |v_2|^2} \\ &= \sqrt{2^2 + 3^2} \\ &= \sqrt{13} \end{aligned}$$

The magnitude of  $v$  is  $\sqrt{13}$ . To find the direction, we use the tangent function  $\tan \theta = \frac{y}{x}$ .

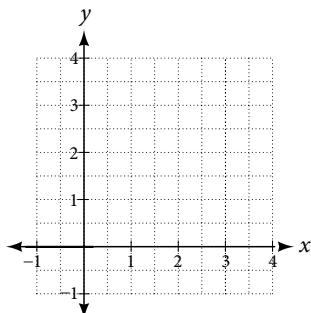
$$\begin{aligned} \tan \theta &= \frac{v_2}{v_1} \\ \tan \theta &= \frac{3}{2} \\ \theta &= \tan^{-1} \left( \frac{3}{2} \right) = 56.3^\circ \end{aligned}$$



Thus, the magnitude of  $v$  is  $\sqrt{13}$  and the direction is  $56.3^\circ$  off the horizontal.

**Example 8** Find the components of the vector  $v$  with initial point  $(3, 2)$  and terminal point  $(7, 4)$ .

**Solution**



### Finding the Unit Vector in the Direction of $v$

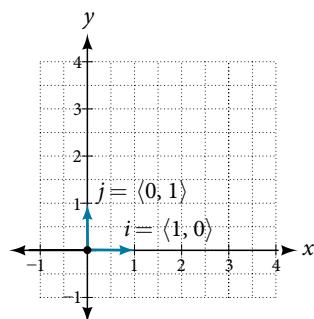
In addition to finding a vector's components, it is also useful in solving problems to find a vector in the same direction as the given vector, but of magnitude 1. We call a vector with a magnitude of 1 a **unit vector**.

We can then preserve the direction of the original vector while simplifying calculations.

Unit vectors are defined in terms of components:

The horizontal unit vector is written as  $i = \langle 1, 0 \rangle$  and is directed along the positive horizontal axis.

The vertical unit vector is written as  $j = \langle 0, 1 \rangle$  and is directed along the positive vertical axis.

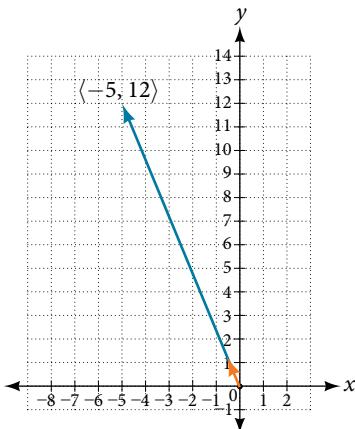


#### The Unit Vectors

If  $v$  is a nonzero vector, then  $\frac{v}{|v|}$  is a unit vector in the direction of  $v$ . Any vector divided by its magnitude is a unit vector. Notice that magnitude is always a scalar, and dividing by a scalar is the same as multiplying by the reciprocal of the scalar.

**Example 9** Find a unit vector in the same direction as  $v = \langle -5, 12 \rangle$ .

**Solution**



## Performing Operations with Vectors in Terms of $i$ and $j$

So far, we have investigated the basics of vectors:

magnitude and direction, vector addition and subtraction, scalar multiplication, the components of vectors, and the representation of vectors geometrically.

Now, we will represent vectors in rectangular coordinates in terms of  $i$  and  $j$ .

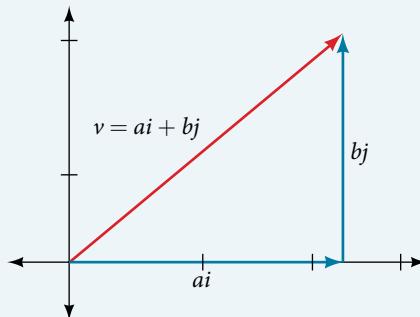
### Vectors in the Rectangular Plane

Given a vector  $v$  with initial point  $P = (x_1, y_1)$  and terminal point  $Q = (x_2, y_2)$ ,  $v$  is written as

$$v = (x_2 - x_1)i + (y_2 - y_1)j$$

The position vector from  $(0, 0)$  to  $(a, b)$ , where  $(x_2 - x_1) = a$  and  $(y_2 - y_1) = b$ , is written as  $v = ai + bj$ . This vector sum is called a linear combination of the vectors  $i$  and  $j$ .

The magnitude of  $v = ai + bj$  is given as  $|v| = \sqrt{a^2 + b^2}$ . See **Figure 16**.



Example 10 Given a vector  $v$  with initial point  $P = (2, -6)$  and terminal point  $Q = (-6, 6)$ ,

write the vector in terms of  $i$  and  $j$ .

**Solution**

Example 11 Given initial point  $P_1 = (-1, 3)$  and terminal point  $P_2 = (2, 7)$ , write the vector  $v$  in terms of  $i$  and  $j$ .

**Solution**

### Performing Operations on Vectors in Terms of $i$ and $j$

When vectors are written in terms of  $i$  and  $j$ , we can carry out addition, subtraction, and scalar multiplication by performing operations on corresponding components.

#### ***Adding and Subtracting vectors in rectangular Coordinates***

Given  $v = ai + bj$  and  $u = ci + dj$ , then

$$v + u = (a + c)i + (b + d)j$$

$$v - u = (a - c)i + (b - d)j$$

Example 12 Find the sum of  $v_1 = 2i - 3j$  and  $v_2 = 4i + 5j$ .

**Solution**

## Calculating the Component Form of a Vector: Direction

We have seen how to draw vectors according to their initial and terminal points and how to find the position vector. We have also examined notation for vectors drawn specifically in the Cartesian coordinate plane using  $i$  and  $j$ . For any of these vectors, we can calculate the magnitude.

Now, we want to combine the key points, and look further at the ideas of magnitude and direction.

Calculating direction follows the same straightforward process we used for polar coordinates.

We find the direction of the vector by finding the angle to the horizontal.

We do this by using the basic trigonometric identities, but with  $|v|$  replacing  $r$ .

### ***Vector Components in terms of Magnitude and Direction***

Given a position vector  $v = \langle x, y \rangle$  and a direction angle  $\theta$ ,

$$\cos \theta = \frac{x}{|v|} \quad \text{and} \quad \sin \theta = \frac{y}{|v|}$$

$$x = |v| \cos \theta \qquad y = |v| \sin \theta$$

Thus,  $v = xi + yj = |v|\cos \theta i + |v|\sin \theta j$ , and magnitude is expressed as  $|v| = \sqrt{x^2 + y^2}$ .

**Example 13** Write a vector with length 7 at an angle of  $135^\circ$  to the positive  $x$ -axis in terms of magnitude and direction.

**Solution**

## Finding the Dot Product of Two Vectors

Scalar multiplication involves multiplying a vector by a scalar, and the result is a vector. As we have seen, multiplying a vector by a number is called scalar multiplication.

If we multiply a vector by a vector, there are two possibilities: the **dot product** and the **cross product**. We will only examine the dot product here; you may encounter the cross product in more advanced mathematics courses.

The dot product of two vectors involves multiplying two vectors together, and the result is a scalar.

### Dot Product

The **dot product** of two vectors  $v = \langle a, b \rangle$  and  $u = \langle c, d \rangle$  is the sum of the product of the horizontal components and the product of the vertical components.

$$v \cdot u = ac + bd$$

To find the angle between the two vectors, use the formula below.

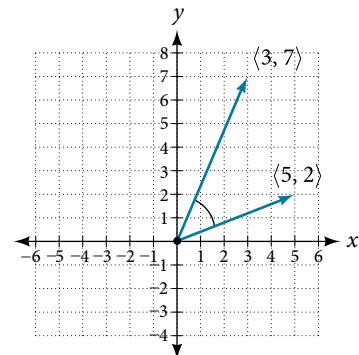
$$\cos \theta = \frac{v}{|v|} \cdot \frac{u}{|u|}$$

Example 14 Find the dot product of  $v = \langle 5, 12 \rangle$  and  $u = \langle -3, 4 \rangle$ .

Solution

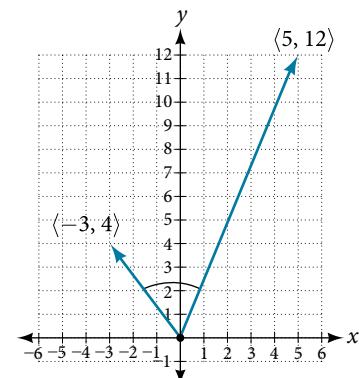
Example 15 Find the dot product of  $v_1 = 5i + 2j$  and  $v_2 = 3i + 7j$ . Then, find the angle between the two vectors.

Solution



Example 16 Find the angle between  $u = \langle -3, 4 \rangle$  and  $v = \langle 5, 12 \rangle$ .

Solution



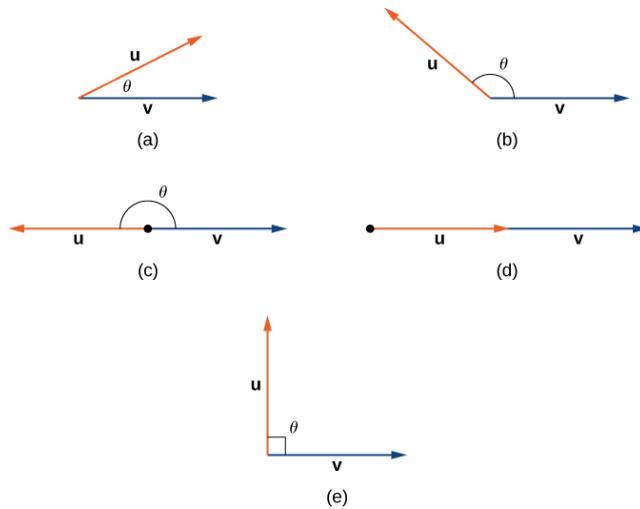
## Identifying Orthogonal Vectors

The dot product of two vectors involves multiplying two vectors together, and the result is a scalar. The dot product serves as a tool to measure the angle formed by the pair of vectors involved.

The angle formed can be acute ( $0 < \cos \theta < 1$ ), obtuse ( $-1 < \cos \theta < 0$ ), or straight ( $\cos \theta = -1$ ).

If  $\cos \theta = 1$ , then both vectors have same direction.

If  $\cos \theta = 0$ , then the vectors when placed in standard position, form a right angle.



### Orthogonal Vectors

Two given nonzero vectors  $v = \langle a, b \rangle$  and  $u = \langle c, d \rangle$  are called **orthogonal vectors**

if and only if their dot product is zero, i.e.  $v \cdot u = 0$ .

The terms *orthogonal*, *perpendicular*, and *normal* indicate that mathematical objects are intersecting at right angles. The use of each term is determined mainly by its context: We say that vectors are orthogonal and lines are perpendicular. The term *normal* is used mostly when measuring the angle made with a plane or other surface.

**Example 17** Determine whether the vectors  $v = \langle 5, 4 \rangle$  and  $u = \langle -4, 5 \rangle$  are orthogonal or not.

**Solution**

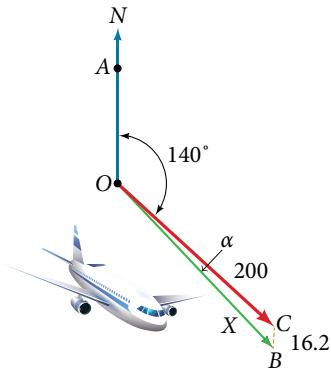
**Example 18** Find the value of x for which the vectors  $v = \langle 2x, 8 \rangle$  and  $u = \langle 4, 3 \rangle$  are orthogonal.

**Solution**

### Example 19 Finding Ground Speed and Bearing Using Vectors

We now have the tools to solve the problem we introduced in the opening of the section.

An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of  $140^\circ$ . A north wind (from north to south) is blowing at 16.2 miles per hour. What are the ground speed and actual bearing of the plane?



**Solution** The ground speed is represented by  $x$  in the diagram, and we need to find the angle  $\alpha$  in order to calculate the adjusted bearing, which will be  $140^\circ + \alpha$ .

Notice in **Figure above**, that angle  $BCO$  must be equal to angle  $AOC$  by the rule of alternating interior angles, so angle  $BCO$  is  $140^\circ$ . We can find  $x$  by the **Law of Cosines**:

$$x^2 = (16.2)^2 + (200)^2 - 2(16.2)(200)\cos(140^\circ)$$

$$x^2 = 45,226.41$$

$$x = \sqrt{45,226.41}$$

$$x = 212.7$$

The ground speed is approximately 213 miles per hour. Now we can calculate the bearing using the Law of Sines.

$$\frac{\sin \alpha}{16.2} = \frac{\sin(140^\circ)}{212.7}$$

$$\sin \alpha = \frac{16.2\sin(140^\circ)}{212.7}$$

$$= 0.04896$$

$$\sin^{-1}(0.04896) = 2.8^\circ$$

Therefore, the plane has a SE bearing of  $140^\circ + 2.8^\circ = 142.8^\circ$ . The ground speed is 212.7 miles per hour.