Definition. A subset $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ of $\mathbb{R}^n$ is $\boxed{\textbf{orthogonal}}$ if $\vec{u}_i \cdot \vec{u}_j = 0$ whenever $i \neq j$.

Example. Show that $\left\{ \overset{u_1}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \overset{u_2}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}, \overset{u_3}{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}} \right\}$ is orthogonal.

$(u_i \cdot u_j = u_j \cdot u_i)$

$u_1 \cdot u_2 = 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 = 0$

$u_2 \cdot u_3 = 1 \cdot 1 + 0 \cdot 0 + 1 \cdot (-1) = 0$

$u_3 \cdot u_1 = 1 \cdot 0 + 0 \cdot 1 + (-1) \cdot 0 = 0$

orthogonal $\Rightarrow$ L.I

L.I $\not\Rightarrow$ orthogonal.

$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is L.I in $\mathbb{R}^2$

but not orthogonal.

$u_i \cdot u_i \neq 0$.

> **Theorem.** If $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent.

$c_1\vec{u}_1 + c_2\vec{u}_2 \cdots + c_p\vec{u}_p = \vec{0}$: show $c_1 = 0, c_2 = 0 \cdots c_p = 0$

To get value of c_1: take dot product with $\vec{u}_1$ on both sides of the equation:

$c_1(\vec{u}_1 \cdot \vec{u}_1) + c_2(\underbrace{\vec{u}_2 \cdot \vec{u}_1}_{0}) + \cdots + c_p(\underbrace{\vec{u}_p \cdot \vec{u}_1}_{0}) = \underbrace{\vec{0} \cdot \vec{u}_1}_{0} \rightsquigarrow c_1(\underbrace{\vec{u}_1 \cdot \vec{u}_1}_{\neq 0}) = 0 \rightsquigarrow c_1 = 0$.

Similarly, $c_2 = 0, c_3 = 0 \cdots$

Definition. Let W be a subspace of $\mathbb{R}^n$. An $\boxed{\textbf{orthogonal basis}}$ for W is a basis for W that is also orthogonal.

Example. Explain why $S = \left\{ \overset{u_1}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \overset{u_2}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}, \overset{u_3}{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}} \right\}$ is an orthogonal basis for $\mathbb{R}^3$.

(1) basis $\to$ S is L.I ✓
$\quad\searrow$ span $S = \mathbb{R}^3$

(2) orthogonal: $u_i \cdot u_j = 0$ $(i \neq j)$ ✓ (orthogonal $\Rightarrow$ L.I)

From previous eg: $\{u_1, u_2, u_3\}$ is orthogonal.

So, $\{u_1, u_2, u_3\}$ is also L.I

Span $\{u_1, u_2, u_3\} = \mathbb{R}^3$ since $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ has a pivot in every row.

> **Theorem.** Let $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. Then for each $\vec{y} \in W$, if $\vec{y} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_p\vec{u}_p$, then $c_j = \dfrac{\vec{y} \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j}$ or $\dfrac{\vec{y} \cdot \vec{u}_j}{\|\vec{u}_j\|^2}$ for $j = 1, 2, \ldots, p$.

$\vec{y} = c_1\vec{u}_1 + c_2\vec{u}_2 \cdots + c_p\vec{u}_p$

To find c_1: take dot product with $\vec{u}_1$ on both sides of this equation.

$\vec{y} \cdot \vec{u}_1 = c_1(\vec{u}_1 \cdot \vec{u}_1) + c_2(\underbrace{\vec{u}_2 \cdot \vec{u}_1}_{0}) \cdots + c_p(\underbrace{\vec{u}_p \cdot \vec{u}_1}_{0})$ $\quad$ $(\vec{u}_i \cdot \vec{u}_j = 0$ if $i \neq j)$

$\vec{y} \cdot \vec{u}_1 = c_1(\vec{u}_1 \cdot \vec{u}_1)$

$\boxed{c_1 = \dfrac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}}$ $\rightsquigarrow$ in general $c_j = \dfrac{\vec{y} \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j}$ $\quad$ $(\vec{u}_j \neq 0$, so $\vec{u}_j \cdot \vec{u}_j \neq 0)$.

$c_3 = \frac{\vec{y}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3} = \frac{1\cdot 1 + 2\cdot 0 + 3(-1)}{1\cdot 1 + 0\cdot 0 + (-1)(-1)} = \frac{-2}{2} = -1$

Example. $\beta = \left\{ \begin{array}{c} u_1 \\ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{array}, \begin{array}{c} u_2 \\ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \end{array}, \begin{array}{c} u_3 \\ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \end{array} \right\}$ is an orthogonal basis for $\mathbb{R}^3$. Find $[\vec{y}]_\beta$ if $\vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

$[\vec{y}]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

$c_i = \frac{\vec{y}\cdot\vec{u}_i}{\vec{u}_i\cdot\vec{u}_i}$

$c_1 = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1} = \frac{1\cdot 0 + 2\cdot 1 + 3\cdot 0}{0\cdot 0 + 1\cdot 1 + 0\cdot 0} = \frac{2}{1} = 2$

$c_2 = \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2} = \frac{1\cdot 1 + 2\cdot 0 + 3\cdot 1}{1\cdot 1 + 0\cdot 0 + 1\cdot 1} = \frac{4}{2} = 2$

$[\vec{y}]_\beta = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$

$u_i\cdot u_j = 0$
$u_i\cdot u_i = 1$

Definition. A subset $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ of $\mathbb{R}^n$ is $\boxed{\textbf{orthonormal}}$ if it is orthogonal and every vector $\vec{u}_i$ in the subset is a unit vector. (length = 1) → length = 1

Let W be a subspace of $\mathbb{R}^n$. An $\boxed{\textbf{orthonormal basis}}$ (ONB) for W is a basis for W that is also orthonormal.

$u_i\cdot u_j = 0$ if $i \neq j$; $u_i\cdot u_i = 1$, $\|u_i\| = 1$; L.I + spans W

$\|u_i\| = \sqrt{u_i\cdot u_i}$

$\|u_i\| = 1 = \vec{u}_i\cdot\vec{u}_i$

Example. Determine whether the basis for $\mathbb{R}^3$ is orthonormal. (assume basis.)

1. $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$

① orthogonal: $\vec{e}_1\cdot\vec{e}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1\cdot 0 + 0\cdot 1 + 0\cdot 0 = 0$; $\vec{e}_2\cdot\vec{e}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$, $\vec{e}_3\cdot\vec{e}_1 = 0$.

② normal: $\|\vec{e}_1\| = \sqrt{\vec{e}_1\cdot\vec{e}_1} = \sqrt{1\cdot 1 + 0\cdot 0 + 0\cdot 0} = 1$, $\|\vec{e}_2\| = \sqrt{0\cdot 0 + 1\cdot 1 + 0\cdot 0} = 1$, $\|\vec{e}_3\| = 1$.

YES, it is ONB.

2. $\left\{ \begin{array}{c} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ u_1 \end{array}, \begin{array}{c} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ u_2 \end{array}, \begin{array}{c} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\ u_3 \end{array} \right\}$ not ONB.

① orthogonal: $u_1\cdot u_2 = 0\cdot 1 + 1\cdot 0 + 0\cdot 1 = 0$; $u_2\cdot u_3 = 0$, $u_3\cdot u_1 = 0$

② normal: $\|u_1\| = \sqrt{u_1\cdot u_1} = \sqrt{0\cdot 0 + 1\cdot 1 + 0\cdot 0} = 1$

$\|u_2\| = \sqrt{1\cdot 1 + 0\cdot 0 + 1\cdot 1} = \sqrt{2}$ ✗

$\|u_3\| = \sqrt{2}$ ✗

$\{u_1, u_2, u_3\}$ not ONB.

$\left\{\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|}\right\}$ - ONB

$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \right\}$ is an ONB.

Definition. An $\boxed{\textbf{orthogonal matrix}}$ is a square matrix U such that $U^T = U^{-1}$.

→ $U^{-1}U = I = UU^{-1}$
$U^TU = I = UU^T$

Columns of an orthogonal matrix form an orthonormal set.

$\underbrace{\begin{array}{c} u_1 \\ u_2 \\ \vdots \\ u_n \end{array}\begin{bmatrix} \cdots \\ \cdots \\ \cdots \end{bmatrix}}_{U^T} \underbrace{\begin{array}{c} u_1 \; u_2 \cdots u_n \\ \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \end{bmatrix}_{n\times n} \end{array}}_{U} = \underbrace{\begin{bmatrix} u_1\cdot u_1 & u_1\cdot u_2 & \cdots & u_1\cdot u_n \\ u_2\cdot u_1 & u_2\cdot u_2 & \cdots & \\ \cdots & & & u_n\cdot u_n \end{bmatrix}}_{U^TU} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{I} \Leftrightarrow$

$u_1\cdot u_1 = 1$
$u_2\cdot u_2 = 1$
$u_3\cdot u_3 = 1$
$u_i\cdot u_j = 0 \; (i\neq j)$.

$\{u_1, u_2, \cdots u_n\}$ form an orthonormal set.

Section 6.2 - Orthogonal Sets

may not be square.

Theorem. Let U be any matrix. Then

- U has orthonormal columns iff $U^TU = I$.
- If U is square, then U is orthogonal iff its columns are orthonormal.
- If U is square, then U is orthogonal iff its rows are orthonormal.

inverse is defined

$U^T = U^{-1}$

Example. Determine whether $A = \begin{array}{ccc} u_1 & u_2 & u_3 \\ \end{array}$ $\begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$ is orthogonal.

$(A^T = A^{-1})$

Method 1: $A^TA = I$; $AA^T = I$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Method 2: show $\{u_1, u_2, u_3\}$ is an orthonormal set.

$u_1 \cdot u_1 = 1$ $\quad u_1 \cdot u_2 = 0$

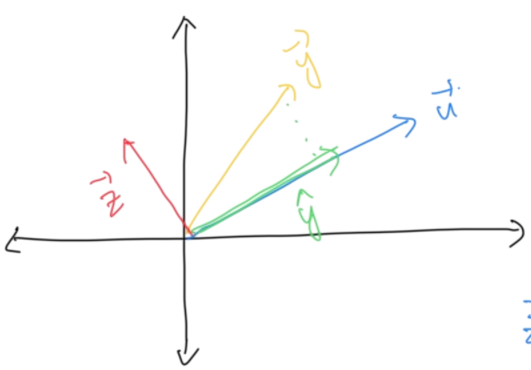
$u_2 \cdot u_2 = 1$ $\quad u_2 \cdot u_3 = 0$

$u_3 \cdot u_3 = 1$ $\quad u_1 \cdot u_3 = 0$.

Orthogonal Projections

Fix a nonzero vector $\vec{u} \in \mathbb{R}^n$. Can we decompose a vector $\vec{y} \in \mathbb{R}^n$ into a sum of a scalar multiple of $\vec{u}$ and a vector orthogonal to $\vec{u}$?

$$\boxed{\vec{y} = \hat{y} + \vec{z}}$$



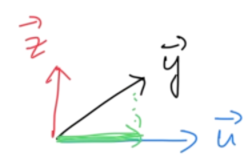
$\hat{y}$ is parallel to $\vec{u}$

$\hat{y} \in \text{span}\{\vec{u}\}$

$\hat{y} = c \cdot \vec{u}$, where $c \in \mathbb{R}$.

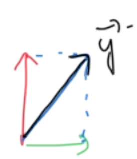
$\vec{z}$ is orthogonal to $\vec{u}$.

so, $\vec{z} \cdot \vec{u} = 0$



$\text{Proj}_{\vec{u}}(\vec{y})$

$\vec{y} = c\vec{u} + \vec{z}$



$\vec{y} = \boxed{c \cdot \vec{u}} + \boxed{\vec{z}}$

$\vec{y} \cdot \vec{u} = c(\vec{u} \cdot \vec{u}) + (\vec{z} \cdot \vec{u})$.

$\phantom{\vec{y} \cdot \vec{u} = c(\vec{u} \cdot \vec{u}) + (} 0$

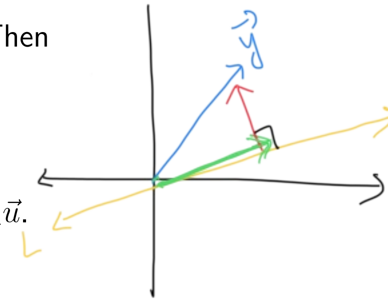
$\vec{y} \cdot \vec{u} = c \cdot (\vec{u} \cdot \vec{u})$

$$\boxed{c = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}}$$

$$\vec{y} = \underbrace{\left(\frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\right)\vec{u}}_{} + \vec{z}.$$

$\vec{z}$ ⇝ orthogonal to $\vec{u}$

Projection of $\vec{y}$ onto $\vec{u}$.



Definition. Let $\vec{u}, \vec{y} \in \mathbb{R}^n$ where $\vec{u} \neq \vec{0}$. Let L be the line through $\vec{0}$ and $\vec{u}$. Then

- The $\boxed{\textbf{orthogonal projection of } \vec{y} \textbf{ onto } L}$ is $\hat{y} = \text{proj}_L \vec{y} = \dfrac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$.
- The $\boxed{\textbf{component of } \vec{y} \textbf{ orthogonal to } L}$ is $\vec{z} = \vec{y} - \text{proj}_L \vec{y} = \vec{y} - \dfrac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$.

Example. Let $\vec{u} = \begin{bmatrix} 3 \\ -9 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$.

1. Express $\vec{y}$ as a sum of two orthogonal vectors, one in $\underbrace{\text{Span}\{\vec{u}\}}_{L}$ and the other orthogonal to $\vec{u}$.

$$\vec{y} = c\vec{u} + \vec{z} = \text{proj}_L(\vec{y}) + \vec{z}$$

parallel to $\vec{u}$.

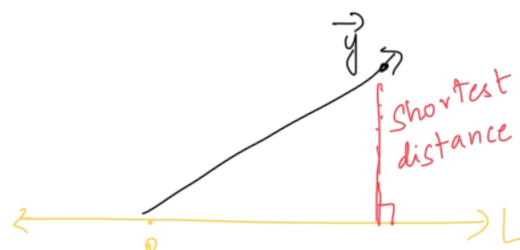
$$\text{Proj}_L(\vec{y}) = \left(\frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\right)\vec{u} = \left(\frac{9 \cdot 3 + 5(-9)}{3 \cdot 3 + (-9)(-9)}\right)\begin{bmatrix} 3 \\ -9 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 3 \\ -9 \end{bmatrix} = \begin{bmatrix} -3/5 \\ 9/5 \end{bmatrix}$$

$$\vec{z} = \vec{y} - \text{Proj}_L(\vec{y}) = \begin{bmatrix} 9 \\ 5 \end{bmatrix} - \begin{bmatrix} -3/5 \\ 9/5 \end{bmatrix} = \begin{bmatrix} 48/5 \\ 16/5 \end{bmatrix} \longrightarrow \text{orthogonal to } \vec{u}$$

$$\boxed{\vec{y} = \text{proj}_L(\vec{y}) + \vec{z}}$$

$$\begin{bmatrix} 3 \\ -9 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 3 \\ -9 \end{bmatrix} + \begin{bmatrix} 48/5 \\ 16/5 \end{bmatrix}$$

2. Find the $\overset{\text{shortest distance}}{\text{distance}}$ between $\vec{y}$ and the line $L = \text{Span}\{\vec{u}\}$.



Find length of $\vec{z} = \|\vec{y} - \text{proj}_L(\vec{y})\|$.

$$\|\vec{z}\| = \sqrt{\vec{z} \cdot \vec{z}} = \left\| \begin{bmatrix} 48/5 \\ 16/5 \end{bmatrix} \right\|$$

$$= \sqrt{\left(\frac{48}{5}\right)\left(\frac{48}{5}\right) + \left(\frac{16}{5}\right)\left(\frac{16}{5}\right)}$$

$$= \frac{16\sqrt{10}}{5} \text{ units.}$$

Shortest distance between $\vec{y}$ & L is the length of the (perpendicular line) component of $\vec{y}$ orthogonal to L.