Orthogonality

> Definition: Orthogonality:
> Let $v \in \mathbb{R}^n$. A vector $w \in \mathbb{R}^n$ is called orthogonal to v if $\langle v, w \rangle = 0$.
> For all $\lambda \in \mathbb{R}$
> $\langle \lambda v, w \rangle = \lambda \langle v, w \rangle = 0$

- This is saying no matter how we scale the vectors, they stay orthogonal to each other
  - This applied the property of bilinearity that the dot product was shown to have

> Definition:
> Let $S \subset \mathbb{R}^n$ be non-empty. Define, S-perp as
> $S^\perp := \{v \in \mathbb{R}^n : \langle v, s \rangle = 0, s \in S\}$

- This can be show to be a linear subspace.
  - This means that means that any linear combination of orthogonal vectors will also be orthogonal

> Proposition:
> Let $S(\neq_0) \subset \mathbb{R}^n$. Then, $S^\perp$ is a linear subspace of $\mathbb{R}^n$.
> Proof:
> Pick $v, w \in S^\perp, \alpha, \beta \in \mathbb{R}$.
> Then, $\langle v, s \rangle = 0$ and $\langle w, s \rangle = 0$, for all $s \in S$
> $\implies \alpha \langle v, s \rangle + \beta \langle w, s \rangle = 0$, for all $s \in S$
> $\implies \langle \alpha v + \beta w, s \rangle = 0$, for all $s \in S$
> $\implies \alpha v + \beta w \in S^\perp$. Hence, $S^\perp$ is a a linear subspace.

> Definition: Let S be a linear subspace of $\mathbb{R}^n$. Then, $S^\perp$ is called the orthogonal complement of S.
> $\mathbb{R}^n = S \oplus S^\perp$
> From this, this means that $S + S^\perp = \mathbb{R}^n$ and that $S \cap S^\perp = \{\vec{0}\}$

> Remark:
> Let $S(\neq 0) \subset \mathbb{R}^n$ and $S^\perp$ be the collection of all vectors orthogonal to the elements

of S. Then the linear span of the set is the same as set: $((S))^{\perp} = S^{\perp}$.

Pythagorean identity:
let $v_1, v_2, \ , v_k \in \mathbb{R}$ be such that $\langle v, v \rangle = 0$, whenever $\ \neq \ $.
Then, $||v_1 + \ + v_k||^2 = ||v_1||^2 + ||v_2||^2 + \ + ||v_k||^2$
Proof:
Base case: $||v_1||^2 = ||v_1||^2$
for $k = 2$. $\langle v_1, v_2 \rangle = 0, v \in \mathbb{R}^n$
$||v_1 + v_2||^2 = \langle v_1 + v_2, v_1 + v_2 \rangle$
$= \langle v_1, v_1 + v_2 \rangle + \langle v_2, v_1 + v_2 \rangle$
$||v_1||^2 + 2\langle v_1, v_2 \rangle + ||v_2||^2$. Since we know that the dot product between the two elements must be zero, we can eliminate $2\langle v_1, v_2 \rangle$
$= ||v_1||^2 + ||v_2||^2$. This proves $k = 2$
Assume, for $k = \ $,
$||v_1 + \ + v||^2 = ||v_1||^2 + \ + ||v_2||^2$, whenever $\langle v, v \rangle = 0, \ \neq \ $ for $k = \ + 1$,
suppose,
$\{v_1, \ , v, v_{+1}\}$ bet such that $\langle v, v \rangle = 0, \ \neq$
$||v_1 + \ + v + v_{+1}||^2$. Say that $w = v_1 + \ + v$
$= ||w + v_{+1}||$
$= ||w||^2 + 2\langle w, v_{+1} \rangle + ||v_{+1}||^2$
$= ||w||^2 + ||v_{+1}||^2$
By the inductive assumption,
$||w||^2 = ||v_1 + \ + v||^2$
$= ||v_1||^2 + \ + ||v||^2$
and hence $||v_1 + \ + v_{+1}||^2$
$= ||w||^2 + ||v_{+1}||^2$
$= ||v_1||^2 + \ + ||v||^2 + ||v_{+1}||^2$