Null Spaces

Definition: Null space. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear. The null space of T or the kernel of T (denoted $\ker(T)$).
$\ker(T) := \{\vec{v} \in \mathbb{R}^n : T(\vec{v}) = \vec{0}\}$
We know that $\ker(T) \neq \emptyset$ as $T(\vec{v}) = 0$

Proposition: $T : \mathbb{R}^n \to \mathbb{R}^n$, linear
$ker(T)$ is a linear subspace of $\mathbb{R}^n$.
Proof: Pick $\vec{u}, \vec{v} \in \ker(T), \alpha, \beta \in \mathbb{R}$
TBA

Proposition: $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear then $\ker(T) = \{\vec{0}\}$ if and only if T is one-one.
Proof: Pick $\vec{u}, \vec{v}$ such that $T(\vec{u}) = T(\vec{v})$
$= T(\vec{u} - \vec{v}) = \vec{0}$, as T is a linear map
So, $\vec{u}, \vec{v} \in \ker(T) = \{\vec{0}\}$
Therefore, $\vec{u} - \vec{v} = 0, \vec{u} = \vec{v}$
Hence, T is one to one.
Other side: Note, $\vec{0} \in \ker(T)$
Pick $\vec{v} \in |(T)\rangle$
Then, $T(\vec{v}) = \vec{0} = T(\vec{0})$
Since T is one to one, $\vec{v} = 0$
$\ker(T) \subset \{\vec{0}\}$
$\ker(T) = \{\vec{0}\}$

- So essentially, we have a lot of vectors in the domain that map to zero
  - But we also have a lot of vectors that map to non-zero vectors, which gives the transform its non-triviality
- $*$Say some $S \subset \mathbb{R}^n$ is linear $\implies T(S)$ is also linear, and a subspace
  - What follows is that is you have transform $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then $T(\mathbb{R}^n)$ is a subspace
- In conclusion, the vectors in $\ker(T)$ all map to $\vec{0}$, and the rest of the vectors span $T(R^n)$

Example: $T : \mathbb{R}^2 \to \mathbb{R}^4$, $\begin{bmatrix} x \\ y \end{bmatrix} \quad \begin{bmatrix} x \\ 0 \\ y \\ x+y \end{bmatrix}$

$$ker(T) = \{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \}$$

$$T(R^2) = \{ \begin{bmatrix} x \\ 0 \\ y \\ x+y \end{bmatrix} : x, y \in \mathbb{R}\}$$

$$= \{x \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} : x, y \in \mathbb{R}\}$$

$$= L(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix})$$

$$T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Definition: Rank-nullity. Rank is the dimension of $T(\mathbb{R}^n)$, nullity is the dimension of $ker(T)$

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then, $\ker(T) + T(\mathbb{R}^n) = n$

Proof: Since $ker(T)$ is a subspace, let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be a basis of $ker(T), (k < n)$.

Pick $\vec{w}_1 \quad ker(T), T(\vec{w}_1) \neq \vec{0}$

$\{\vec{v}_1, \dots, \vec{v}_k, \vec{w}_1\}$ is linearly independent.

$c_1\vec{w}_1 + b_1\vec{v}_1 + \dots + b_k\vec{v}_k = \vec{0}$

as $\{\vec{v}_i\}_{ki}$ is a basis, $b_i = 0$,

Pick $\vec{w} \quad L(\{\vec{w}_1, \vec{v}_1, \ldots, \vec{v}_k\})$ and complete it to a basis of $\mathbb{R}^n$
Remains to show $\{T(\vec{w}_i), \ldots, T(\vec{w}_{n_0k}, \vec{v}_1)\}$

Proof of rank nullity: Note that $T(\mathbb{R}^n)$ is a linear subspace of the codomain $\mathbb{R}^m$. Thus, there exists $\{w_1, \ldots, w_l\} \subset \mathbb{R}^n$ such that $\{T(w_1), \ldots, T(w_l)\}$, which is a basis of $T(\mathbb{R}^n)$.
Now, we shall show that $\{w_1, \ldots, w_l\} \subset \mathbb{R}^n$ is linearly independent. Suppose $c_1 w\vec{w}_1 + \cdots + c_l\vec{w}_l = \vec{0} \in R^n$. This implies that $T(c_1\vec{w}_1 + \cdots + c_l\vec{w}_l) = \vec{0} \in \mathbb{R}^n$. We can apply the linear property such that $c_1 T(\vec{w}_1) + \cdots + c_l T(\vec{w}_l) = \vec{0} \in \mathbb{R}^n$. Therefore, $c_1 = c_2 = \cdots = c_l = 0 \in \mathbb{R}$
Since $\ker(T) \subset \mathbb{R}^n$ is a subspace, say, $\{v_1, \ldots, v_k\} \subset \mathbb{R}^n$ is a basis of $\ker(T)$.
Claim: $\{v_1, \ldots, v_k\} r \{w_1, \ldots, w_l\} = B$ is a basis of $\mathbb{R}^n$. We know this because the union of two linearly independent sets is linearly independent from our homework.
Pick $\vec{u} \in \mathbb{R}^n$. Either $T(\vec{u} = \vec{0}) \implies \vec{u} \in \ker(T) \implies \vec{u} \in L(B)$, or $T(\vec{u})(\neq \vec{0}) \in T(\mathbb{R}^n)$
This implies that $T(\vec{u}) = \alpha_1 T(\vec{w}_1) + \cdots + \alpha_l T(w\vec{w}_l)$, given $\alpha_i \in \mathbb{R}$.
This implies that $T(\vec{u}) = T(\alpha_1\vec{w}_1 + \ldots \alpha_l\vec{w}_l)$
which implies that $T(\alpha_1\vec{w} + \cdots + \alpha_l\vec{w}_l) - T(\vec{u}) = \vec{0} \in \mathbb{R}^m$
Which implies that $T(\alpha_1\vec{w} + \cdots + \alpha_l\vec{w}_l - \vec{u}) = \vec{0} \in \mathbb{R}^m$
$\alpha_1\vec{w} + \cdots + \alpha_l\vec{w}_l - \vec{u} \in \ker(T)$
$\alpha_1\vec{w} + \cdots + \alpha_l\vec{w}_l - \vec{u} = \beta_1 v_1 + \cdots + \beta_2 v_k$
$\vec{u} = \alpha_1\vec{w}_1 + \cdots + \alpha_l\vec{w}_l - \beta_1\vec{v}_1 - \cdots - \beta_k v_k \in L(B)$ So, B is a basis of R^n.
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Example: $T : \mathbb{R}^3 \to \mathbb{R}^2$ as $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \begin{bmatrix} x \\ y+z \end{bmatrix}$

$$\ker T = \{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{bmatrix} x \\ y+z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$$

$= L(\{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\})$. Therefore, $\ker(T) = 1$

$$T(\mathbb{R}^3) := \{\begin{bmatrix} x \\ y+z \end{bmatrix} : x, y, z \in \mathbb{R}\} = L\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\} = \mathbb{R}^n$$

Therefore, $T(\mathbb{R}^3) = 2$

Thus, $\ker(T) + T(\mathbb{R}^3) = 3$