Dot product

Definition: Dot Product. Let $v, w \in \mathbb{R}^n$ with $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$, with $v_n, w_n \in \mathbb{R}$. The dot product of v and w is denoted as $v \cdot w$ and is defined by

$$\begin{bmatrix} v_1 & \dots & v_n \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = v_1 w_1 + \dots + v_n w_n$$

So, dot product in $\mathbb{R}^n$ is a map $: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

- The geometric interpretation of the dot product is that it shows orthogonality between vectors in $\mathbb{R}^n$
  - When two vectors are orthogonal, the dot product will be zero
  - When they are aligned, it will be equal to the product of their magnitudes
- The computation of the dot product is very simple
  - simply line up the components, multiply, and sum
- Length relates to dot product in the following way (note that $\langle , \rangle$ is another notation for the dot product)
  - $v \cdot v = \langle v, v \rangle = ||v||^2$. We know that $||v||$ is the length of v

Bilinearity

Definition: Bilinear map. A map of $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called bilinear if f is linear map in one component when the other component is fixed.
$f : (v, w) \mapsto f(v, w)$
$f : (v_0, \cdot) \mapsto f(v_0, \cdot)$
$f(v_1, \lambda w_1 + w_2) = \lambda f(v_1, w_1) + f(v_1, w_2)$

- Essentially, this is saying that the map acts linear for one component at a time
  - Meaning that the map is consistent under scalar multiplication and addition when one of the values is held constant

Example: Bilinearity for the dot product in
$\langle , \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $v_0 \in \mathbb{R}^n$

> Then, $\langle v_0, \rangle : \mathbb{R}^n \to \mathbb{R}$ is linear (as the fist element is fixed to v_0)
> $\langle , v_0 \rangle : \mathbb{R}^n \to \mathbb{R}$ is also linear.

Inner product

> Example: $v, w \in \mathbb{R}^n$
> $\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$
> $= v_1 w_2 + v_2 w_2$
> Say instead of I, we have
> $\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$
> This would be similar to dot product, but a bit scaled in one of the axes

- This introduces the idea of the *inner product*, a concept that is introduced in class but not fully developed
  - From Wikipedia: Inner products allow formal definitions of intuitive geometric notions, such as lengths, angles, and orthogonality (zero inner product) of vectors

> Remark: Inner product in $\mathbb{R}^n$. All dot products are a class of inner products. All inner products are bilinear, but not all bilinear maps are inner products.

> Remark: There are a class of matrices called positive definite matrices. If $A \in M_2(\mathbb{R})$ is positive definite, then $\begin{bmatrix} v_1 & v_2 \end{bmatrix} A \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ defines an inner product and after a choice of basis, all inner products are of this above form.
> Positive definite means that for $v^T A v \geq 0$ and $v^T A v = 0$ only if $v = \vec{0} \in \mathbb{R}^n$.