Lecture 5
PHYS 2B

Electric Fields

Structure of charge distribution determines decay of field with distance!

1) point charge/sphere $$E_{\text{point charge}} = k_e \frac{q}{r^2}$$ Coulomb's law

2) electric dipole $$\left|\vec{E}_{dipole}\right| = \frac{1}{2\pi\varepsilon_o}\frac{p}{z^3}$$ superposition (sum)

3) infinite line of charge $$E_y = \frac{1}{2\pi\varepsilon_o}\frac{\lambda}{z}$$ superposition (integral)

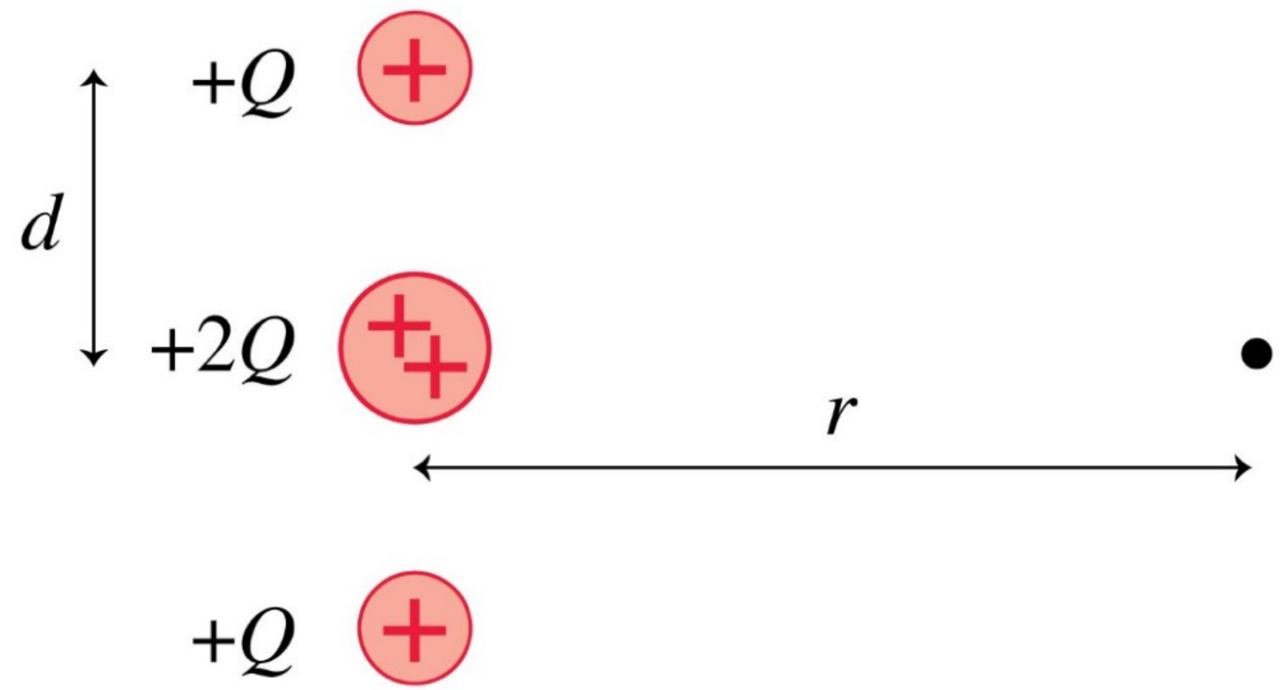
4) annular ring of charge along axis $$E_{ring} = \frac{1}{4\pi\varepsilon_o}\frac{qz}{\left(z^2+R^2\right)^{3/2}}$$

5) circular disk of charge along axis $$E_{disk} = \frac{\sigma}{2\varepsilon_o}\left(1-\frac{z}{\sqrt{z^2+R^2}}\right)$$

6) infinite sheet of charge $$E_{plane} = \frac{\sigma}{2\varepsilon_o}$$

iClicker question 5-1

When $r >> d$, the electric field strength at the dot is



A. $\dfrac{Q}{4\pi\epsilon_0 r^2}$

B. $\dfrac{2Q}{4\pi\epsilon_0 r^2}$

C. $\dfrac{4Q}{4\pi\epsilon_0 r^2}$

D. $\dfrac{4Q}{4\pi\epsilon_0 (r^2 + d^2)}$

E. $\dfrac{4Q}{4\pi\epsilon_0 r}$

iClicker question 5-1

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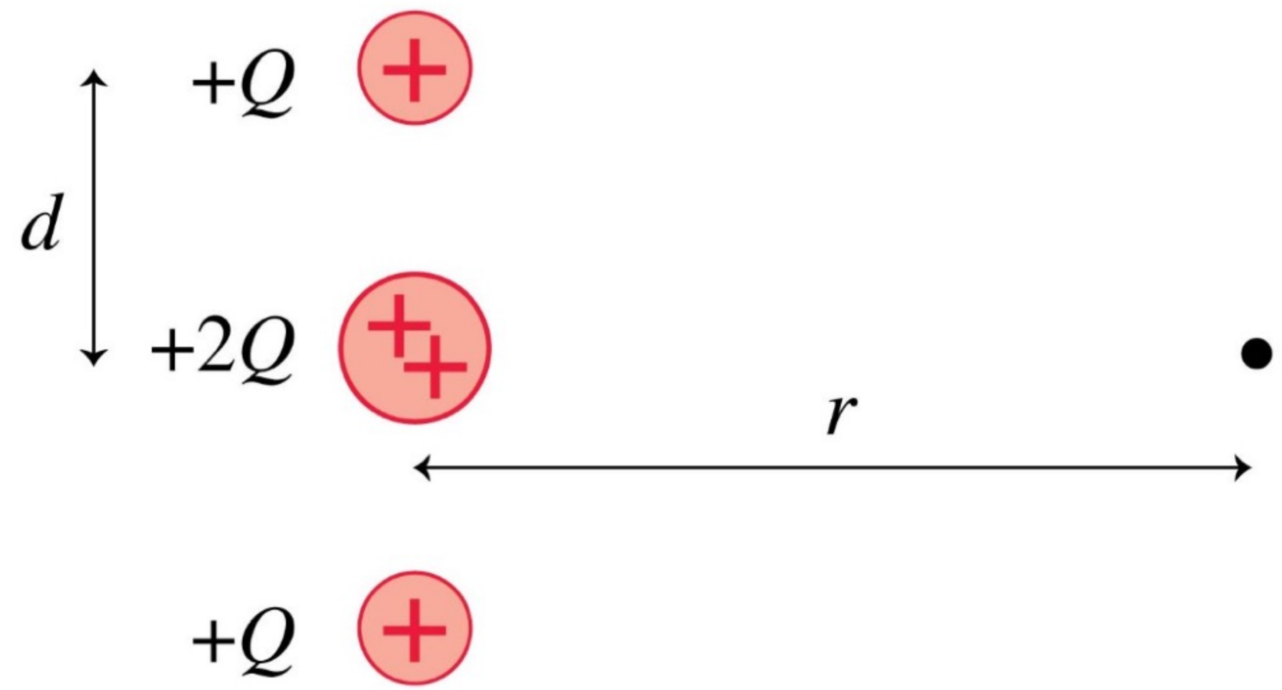


A. $\dfrac{Q}{4\pi\epsilon_0 r^2}$

B. $\dfrac{2Q}{4\pi\epsilon_0 r^2}$

✔ C. $\dfrac{4Q}{4\pi\epsilon_0 r^2}$ Looks like a point charge $4Q$ at the origin.

D. $\dfrac{4Q}{4\pi\epsilon_0 (r^2 + d^2)}$

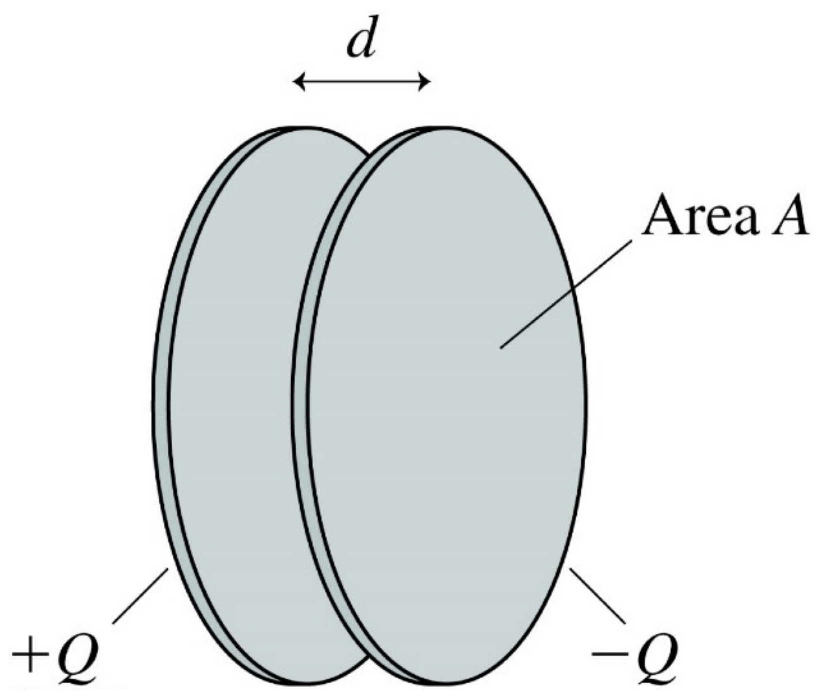
E. $\dfrac{4Q}{4\pi\epsilon_0 r}$

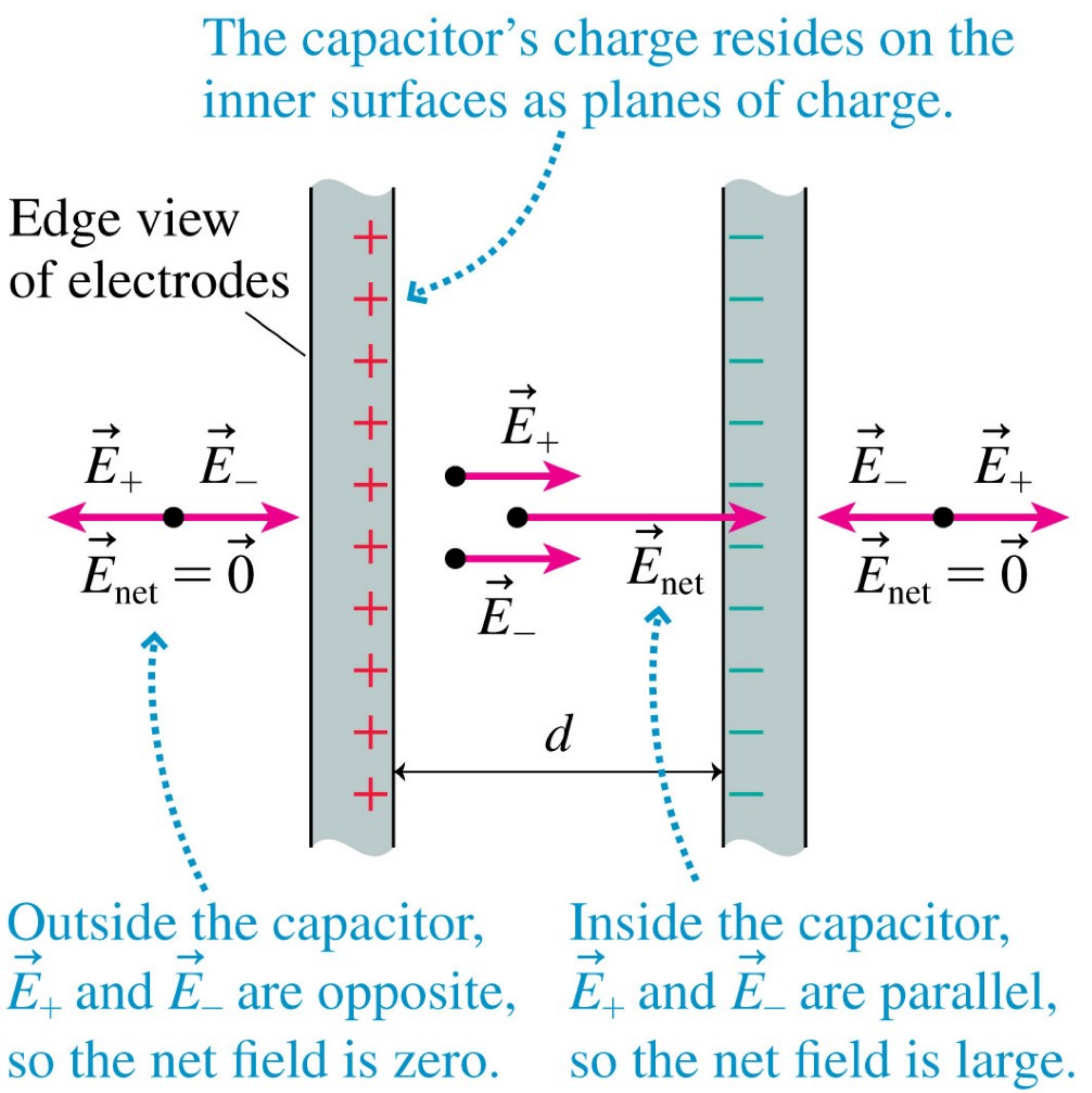
How about a Sphere of Charge?

- A sphere of charge Q and radius R, be it a **uniformly charged sphere** or just a **spherical shell**, has an electric field *outside* the sphere that is exactly the same as that of a point charge Q located at the center of the sphere:

$$\vec{E}_{\text{sphere}} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} \qquad \text{for } r \geq R$$

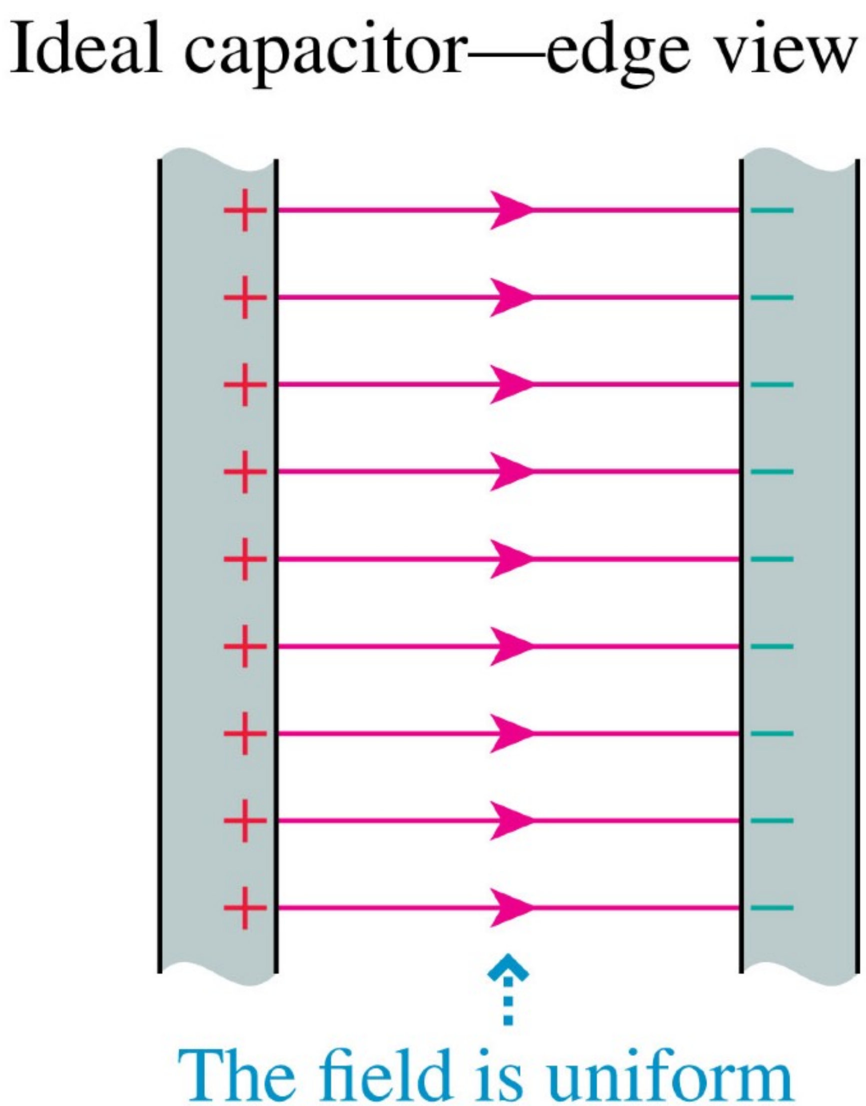
The Parallel-Plate Capacitor



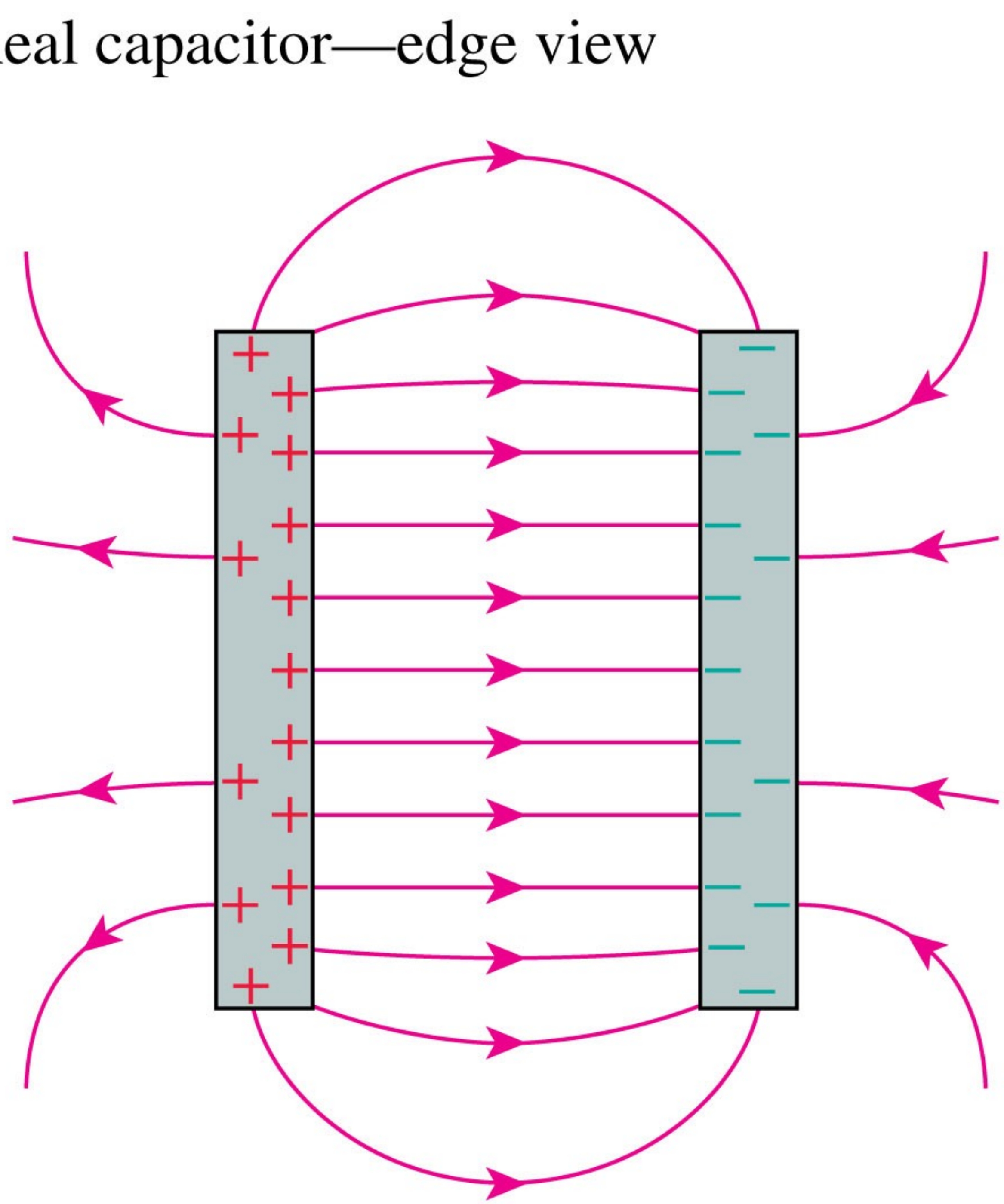


$$\vec{E}_{capacitor} = \begin{cases} \left(\dfrac{Q}{\epsilon_0 A}, \text{ from positive to negative}\right) & \text{inside} \\ \vec{0} & \text{outside} \end{cases}$$

The Ideal Capacitor



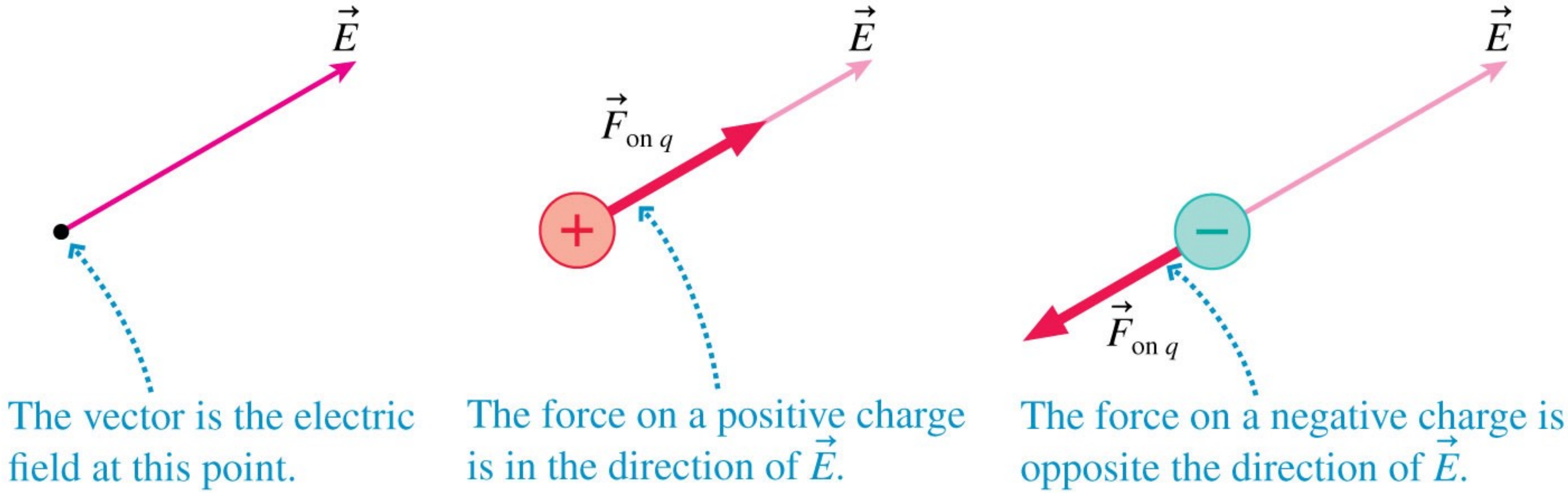
- **Uniform electric field:** *same* in strength and direction at every point in a region of space.



weak **fringe field**

Motion of a Charged Particle in an Electric Field

- Consider a particle of charge q and mass m at a point where an electric field $\vec{E}$ has been produced by *other* charges, the source charges.
- The electric field exerts a force $\vec{F}_{\text{on } q} = q\vec{E}$.



- If this is the only force acting on q, it causes the charged particle to accelerate with

$$\vec{a} = \frac{\vec{F}_{\text{on } q}}{m} = \frac{q}{m}\vec{E}$$

- In a uniform field

$$a = \frac{qE}{m} = \text{constant}$$

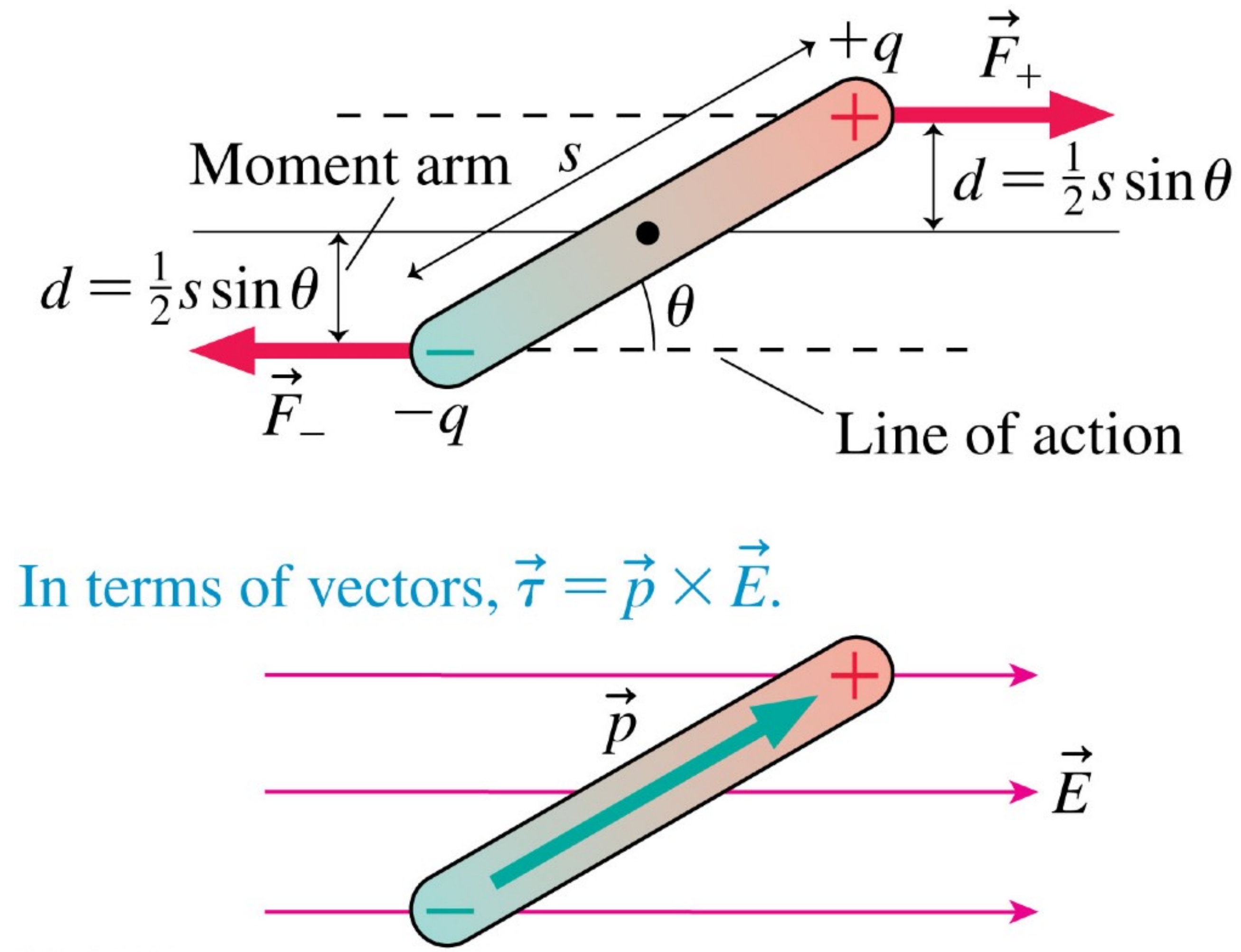
Motion of a Charged Particle in an Electric Field

- "**DNA fingerprints**" are measured with the technique of *gel electrophoresis*.
- A solution of negatively charged DNA fragments migrate through the gel when placed in a uniform electric field.
- Because the gel exerts a drag force, the fragments move at a terminal speed inversely proportional to their size.

HW review: The Torque on a Dipole

- The torque on a dipole placed in a uniform external electric field is

$$\tau = 2 \times dF_+$$
$$= 2(\tfrac{1}{2}s\sin\theta)(qE) = pE\sin\theta$$



- and the rest is PHYS 2A ;)

General principles to find E for a charge distribution

Sources of $\vec{E}$

Electric fields are created by charges.

Multiple point charges

MODEL Model objects as point charges.

VISUALIZE Establish a coordinate system and draw field vectors.

SOLVE Use superposition: $\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \cdots$

Continuous distribution of charge

MODEL Model objects as simple shapes.

VISUALIZE

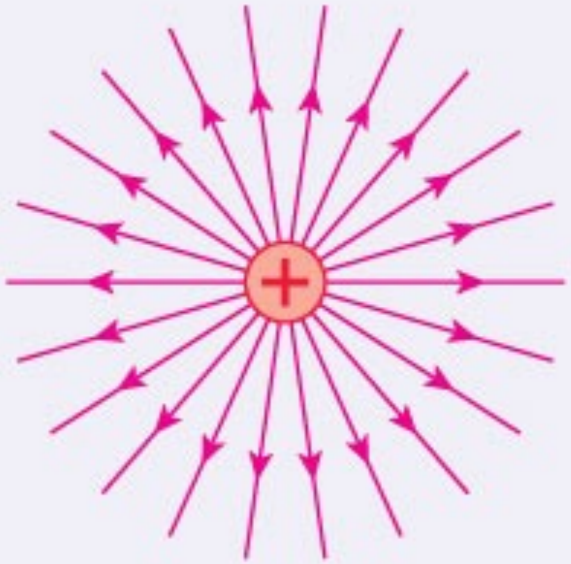
- Establish a coordinate system.
- Divide the charge into small segments ΔQ.
- Draw a field vector for one or two pieces of charge.

SOLVE

- Find the field of each ΔQ.
- Write $\vec{E}$ as the sum of the fields of all ΔQ. Don't forget that it's a *vector* sum; use components.
- Use the charge density (λ or η) to replace ΔQ with an integration coordinate, then integrate.

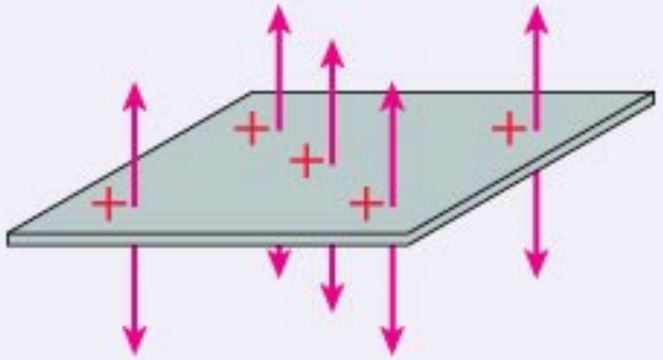
Four Key Electric Field Models

Point charge with charge q



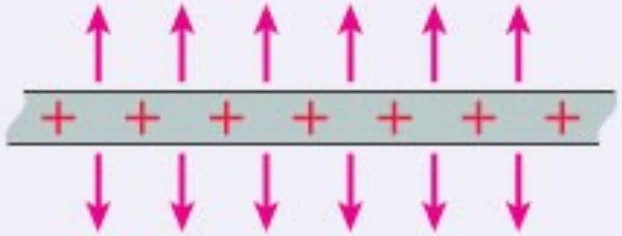
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

Infinite plane of charge with surface charge density η



$$\vec{E}_{\text{plane}} = \left(\frac{|\eta|}{2\epsilon_0}, \begin{cases}\text{away if } + \\ \text{toward if } -\end{cases}\right)$$

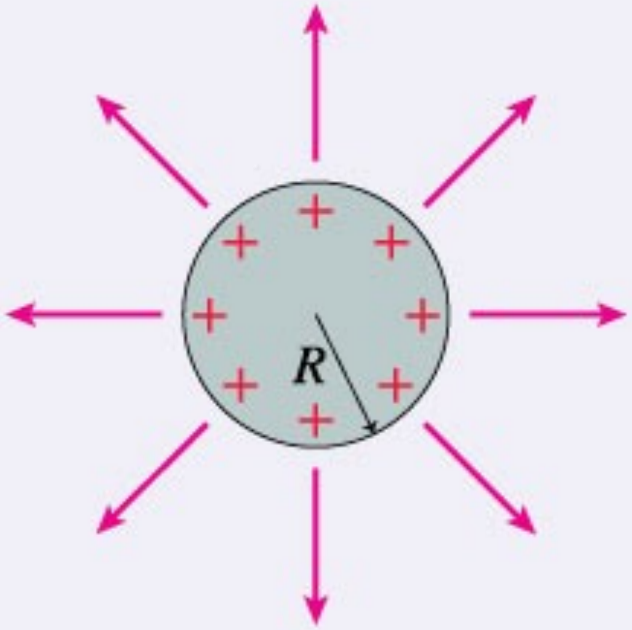
Infinite line of charge with linear charge density λ

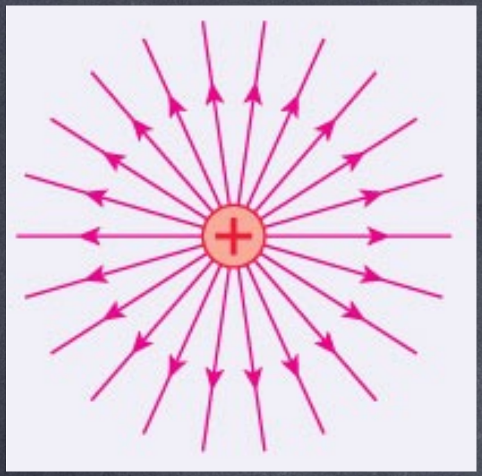


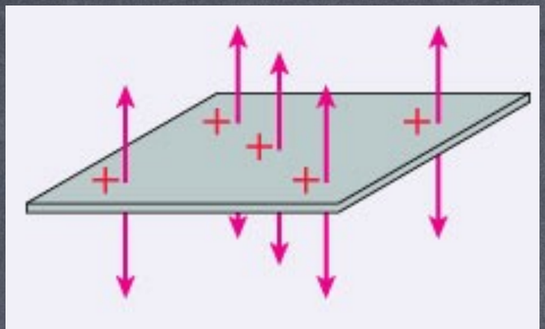
$$\vec{E}_{\text{line}} = \left(\frac{1}{4\pi\epsilon_0}\frac{2|\lambda|}{r}, \begin{cases}\text{away if } + \\ \text{toward if } -\end{cases}\right)$$

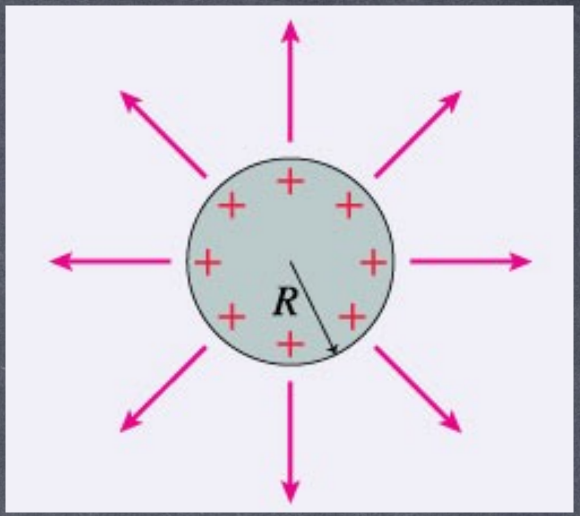
Sphere of charge with total charge Q

Same as a point charge Q for $r > R$







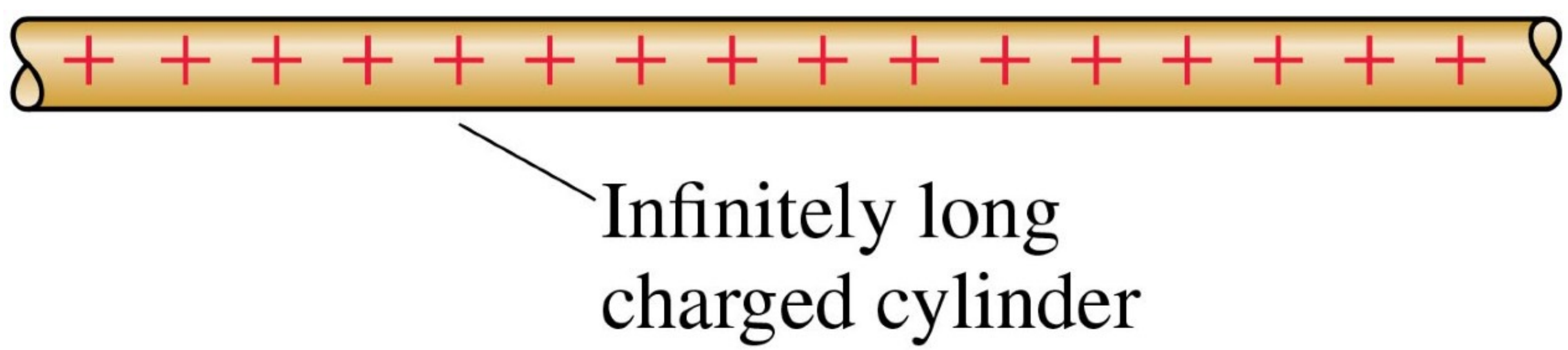


Is there an easier way to find the electric field due to such symmetric charge distributions?

Electric Field of a Charged Cylinder

- Suppose we knew only two things about electric fields:
  1. The field points away from positive charges, toward negative charges.
  2. An electric field exerts a force on a charged particle.

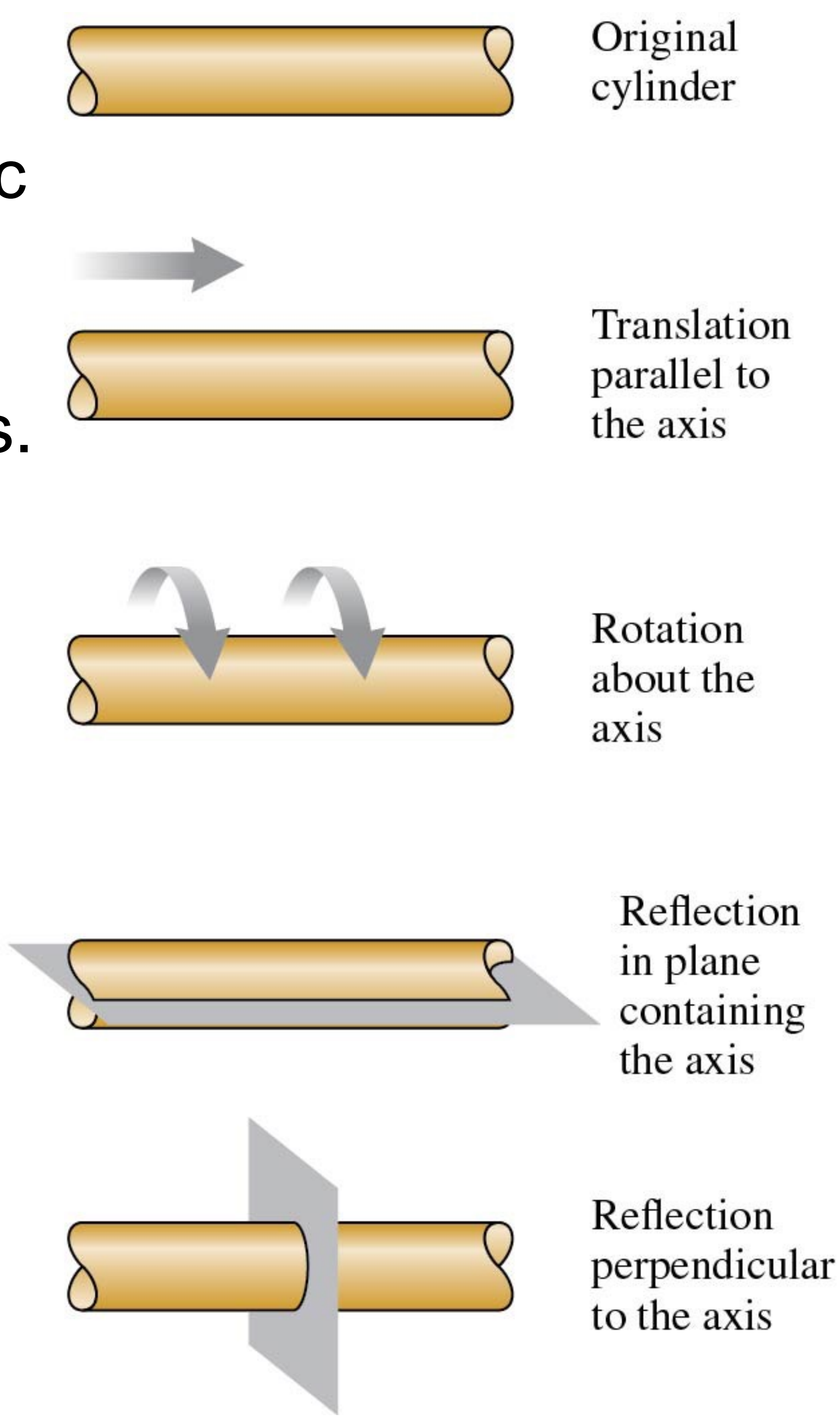
  - From this information alone, what can we deduce about the electric field of an infinitely long charged cylinder?



  - All we know is that this charge is positive, and that it has *cylindrical symmetry*.

Cylindrical Symmetry

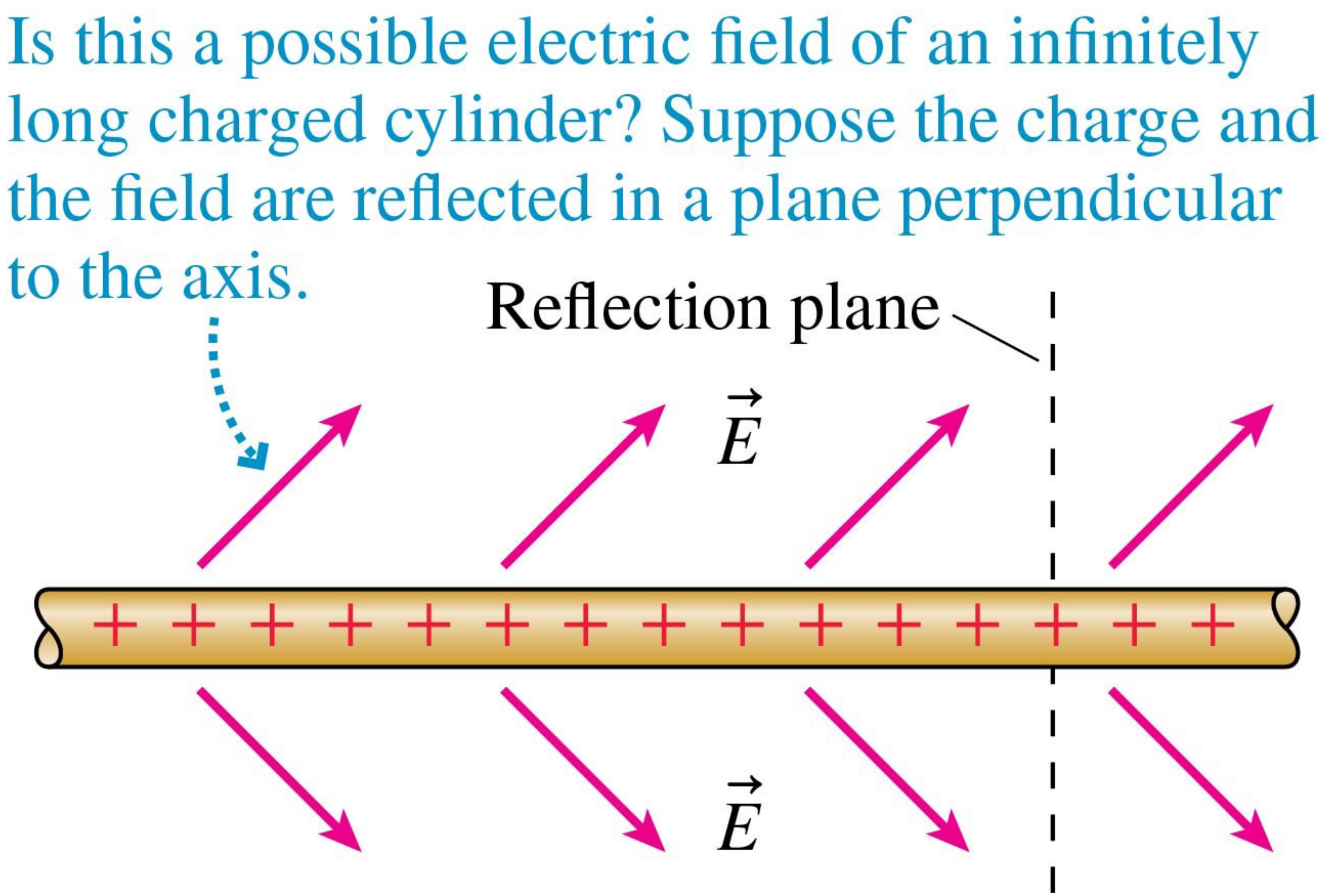
- An infinitely long charged cylinder is symmetric with respect to
  - *Translation* parallel to the cylinder axis.
  - *Rotation* by an angle about the cylinder axis.
  - *Reflections* in any plane containing or perpendicular to the cylinder axis.



- The **symmetry** of the electric *field* **must match** the symmetry of the *charge distribution*.

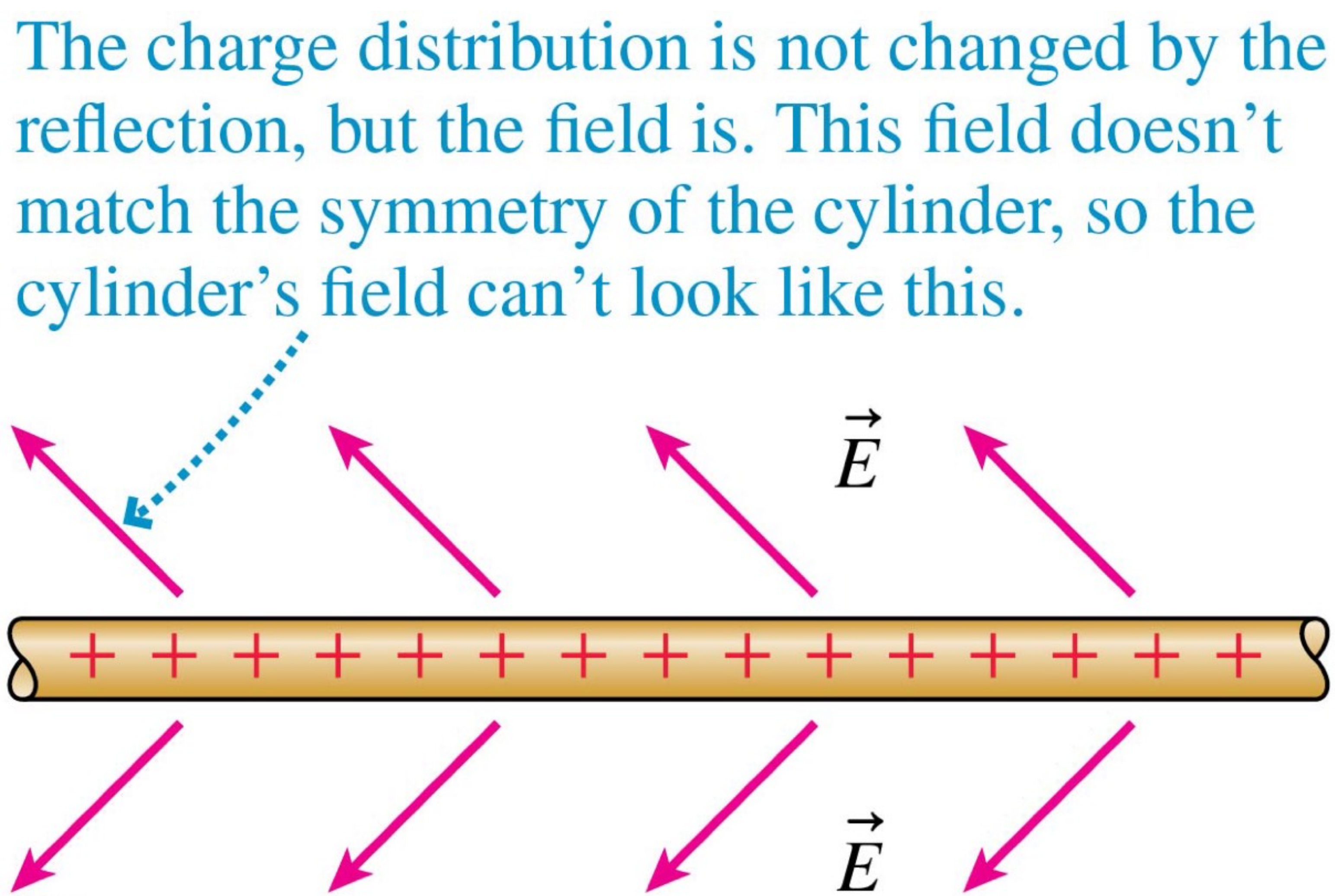
Electric Field of a Charged Cylinder

- Could the field look like the figure below? (Imagine this picture rotated about the axis.)
- The next slide shows what the field would look like reflected in a plane perpendicular to the axis (left to right).



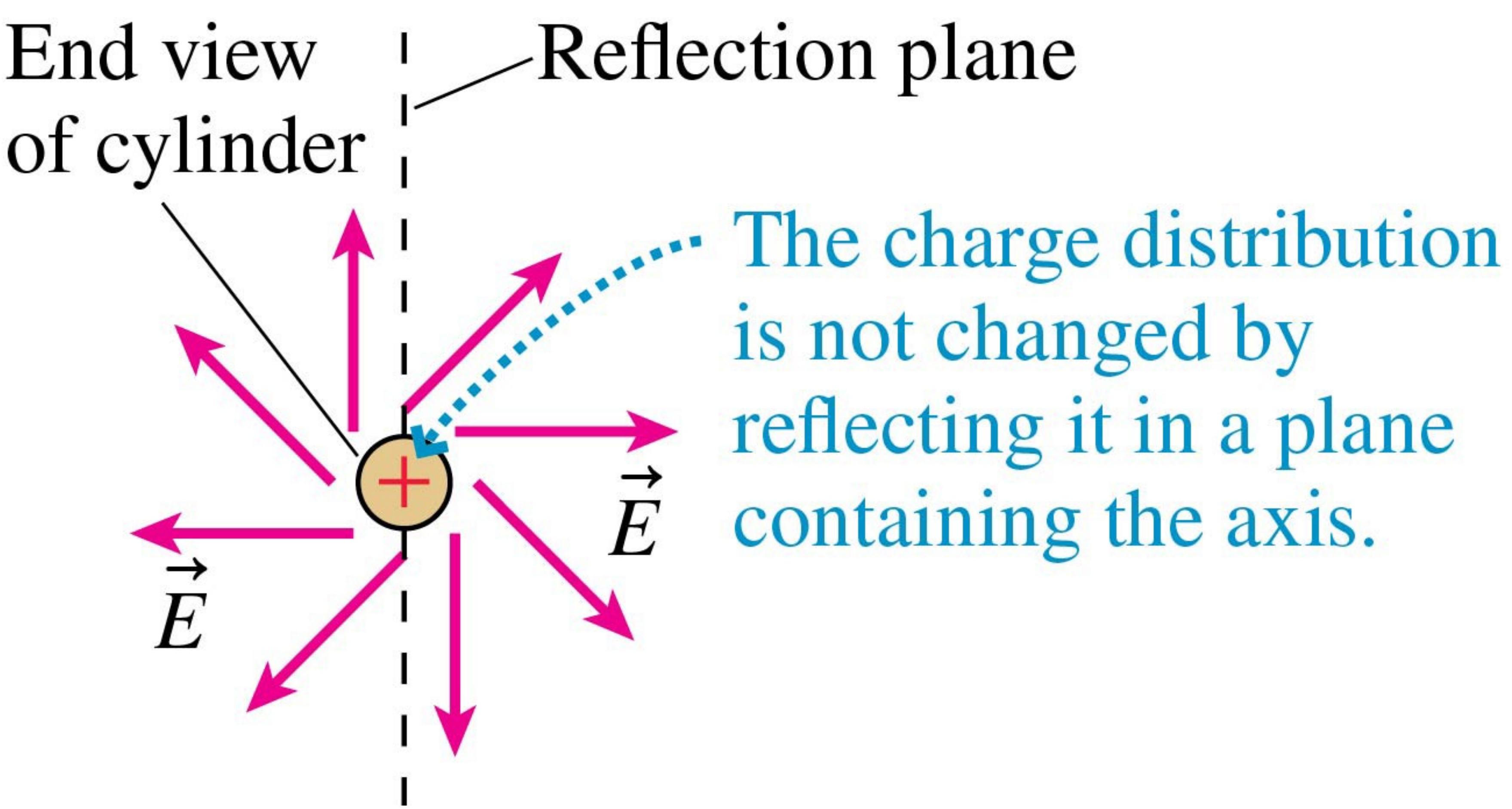
Electric Field of a Charged Cylinder

- This reflection, which does not make any change in the charge distribution itself, *does* change the electric field.
- Therefore, the electric field of a cylindrically symmetric charge distribution **cannot have a component parallel to the cylinder axis**.



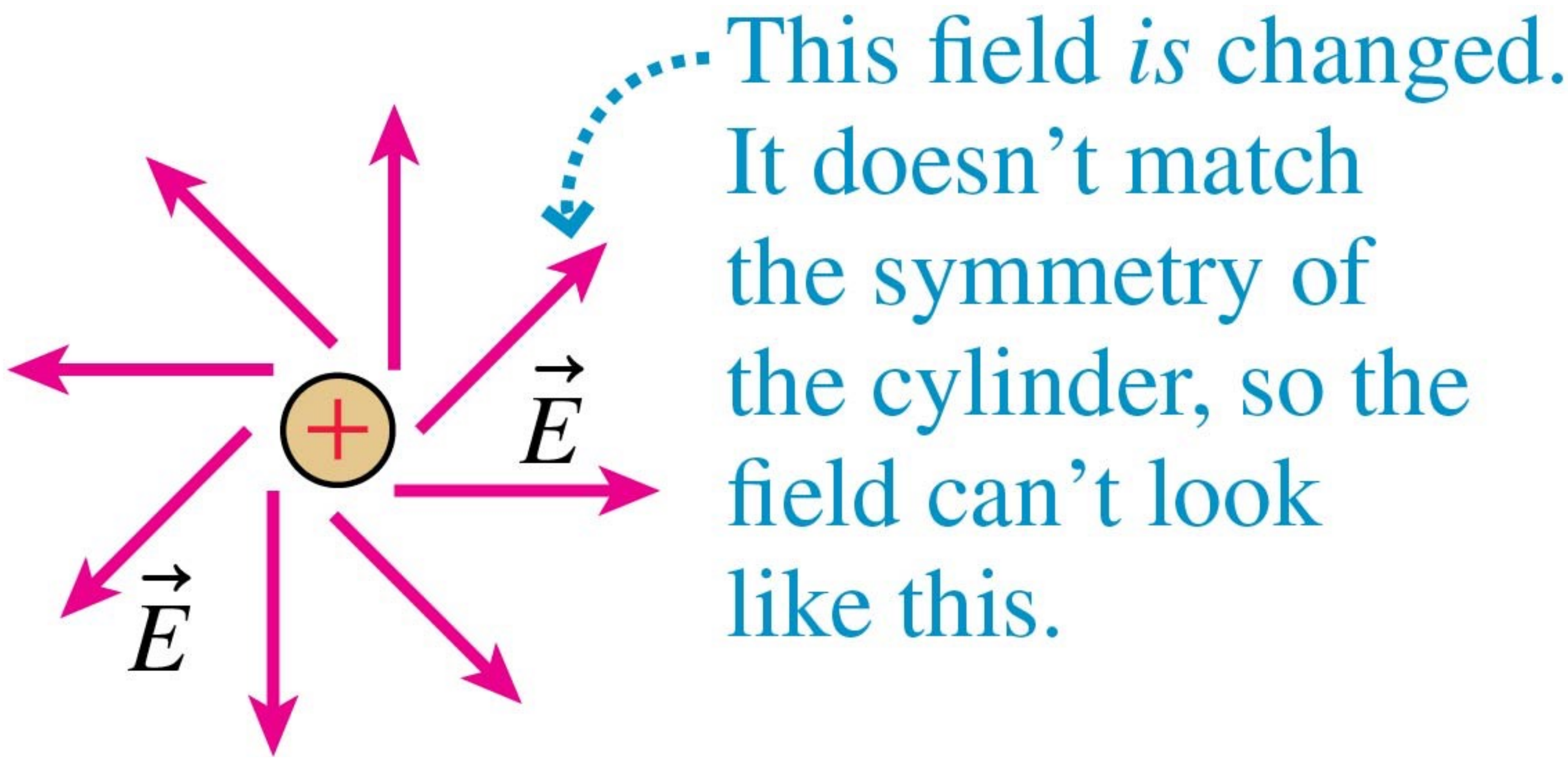
Electric Field of a Charged Cylinder

- Could the field look like the figure below? (Here we're looking down the axis of the cylinder.)
- The next slide shows what the field would look like reflected in a plane containing the axis (left to right).



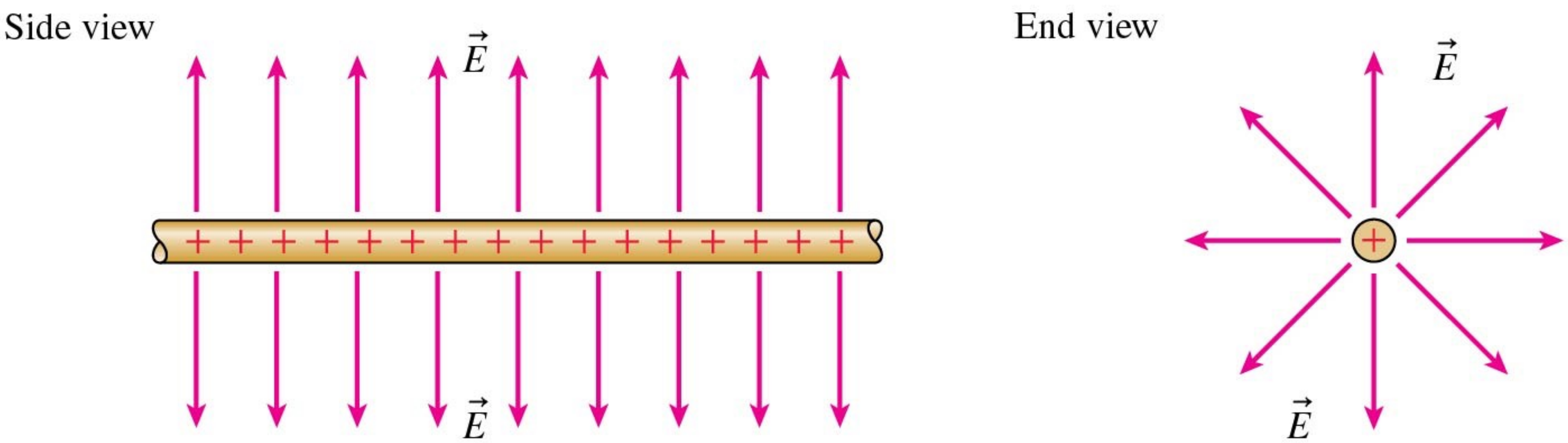
Electric Field of a Charged Cylinder

- This reflection, which does not make any change in the charge distribution itself, *does* change the electric field.
- Therefore, the electric field of a cylindrically symmetric charge distribution **cannot have a component tangent to the circular cross section**.



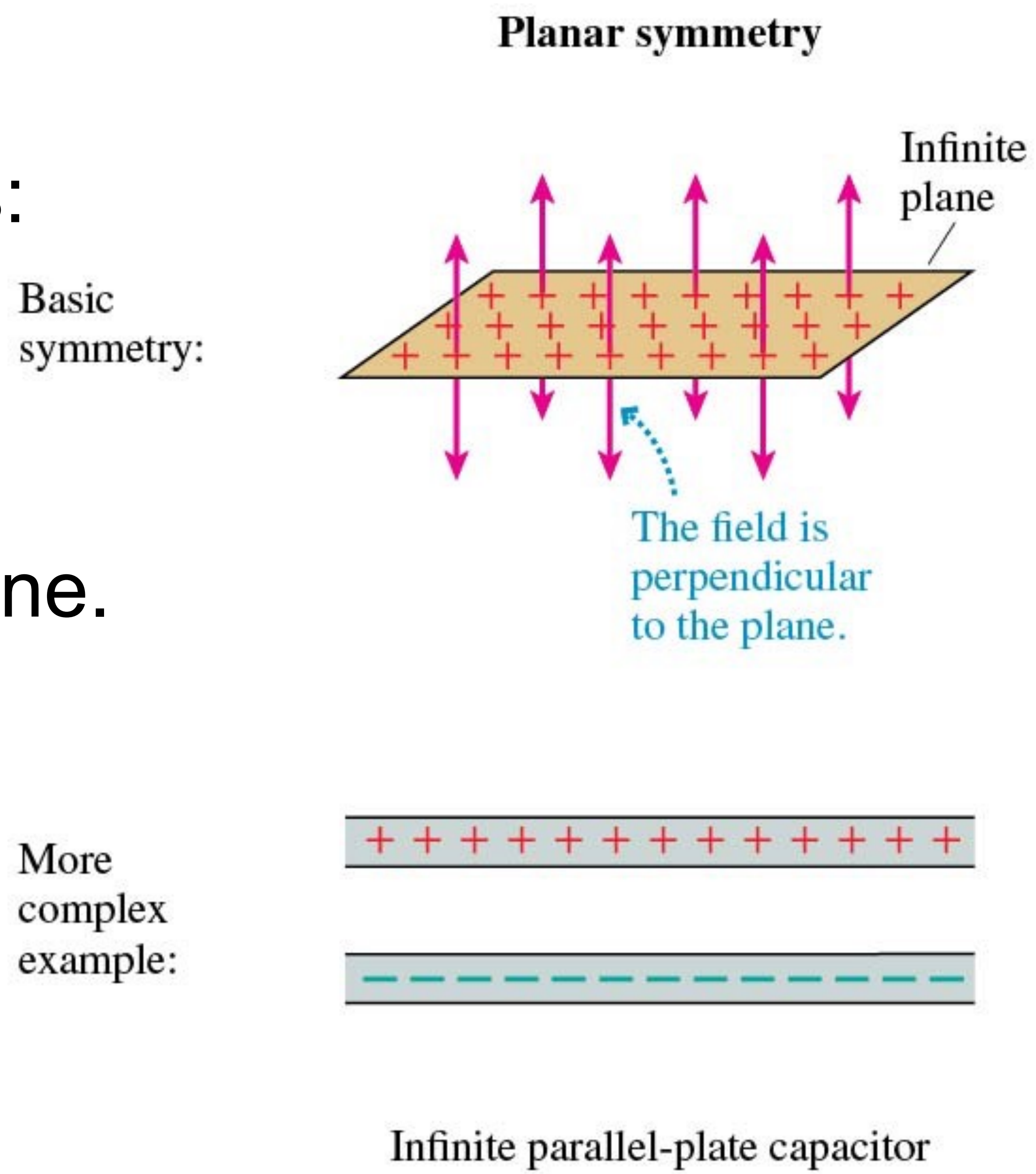
Electric Field of a Charged Cylinder

- Based on symmetry arguments alone, an infinitely long charged cylinder *must* have a radial electric field, as shown below.
- This is the one electric field shape that matches the symmetry of the charge distribution.



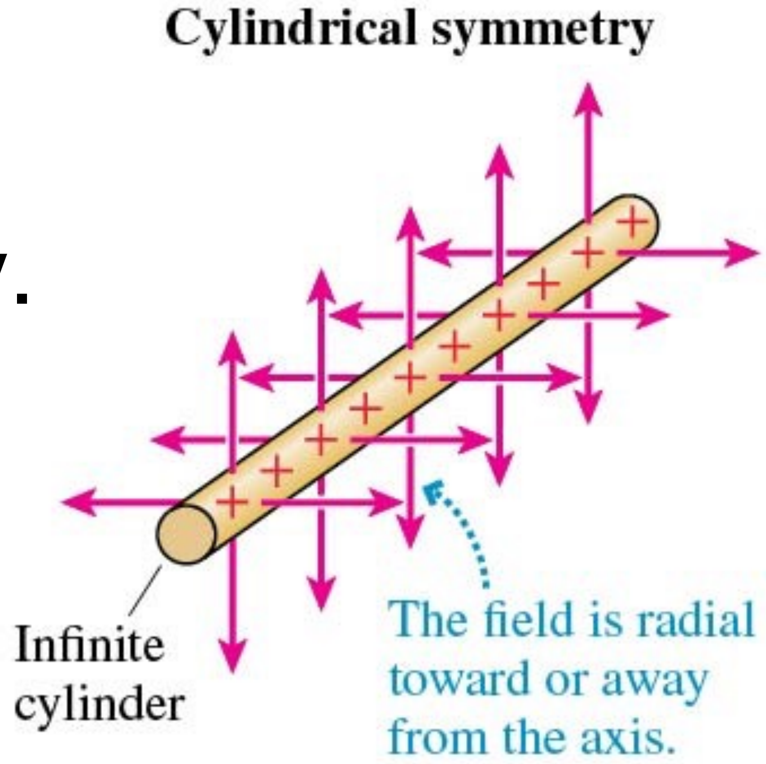
Fundamental symmetry: **Planar**

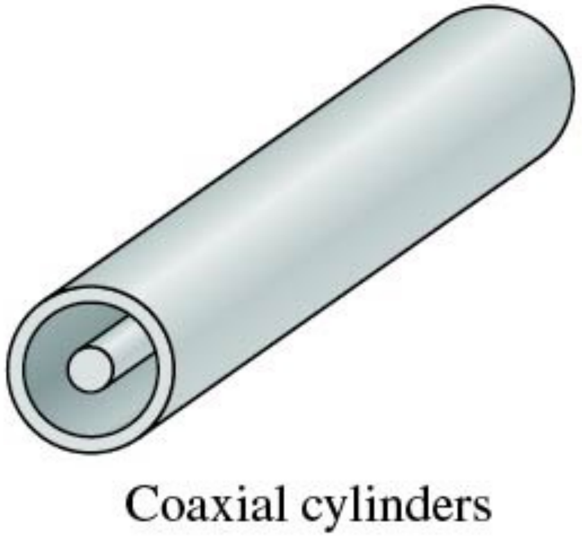
- Three fundamental symmetries: the first is **planar symmetry**.
- Planar symmetry involves symmetry with respect to:
  - *Translation* parallel to the plane.
  - *Rotation* about any line perpendicular to the plane.
  - *Reflection* in the plane.



Fundamental symmetry: **Cylindrical**

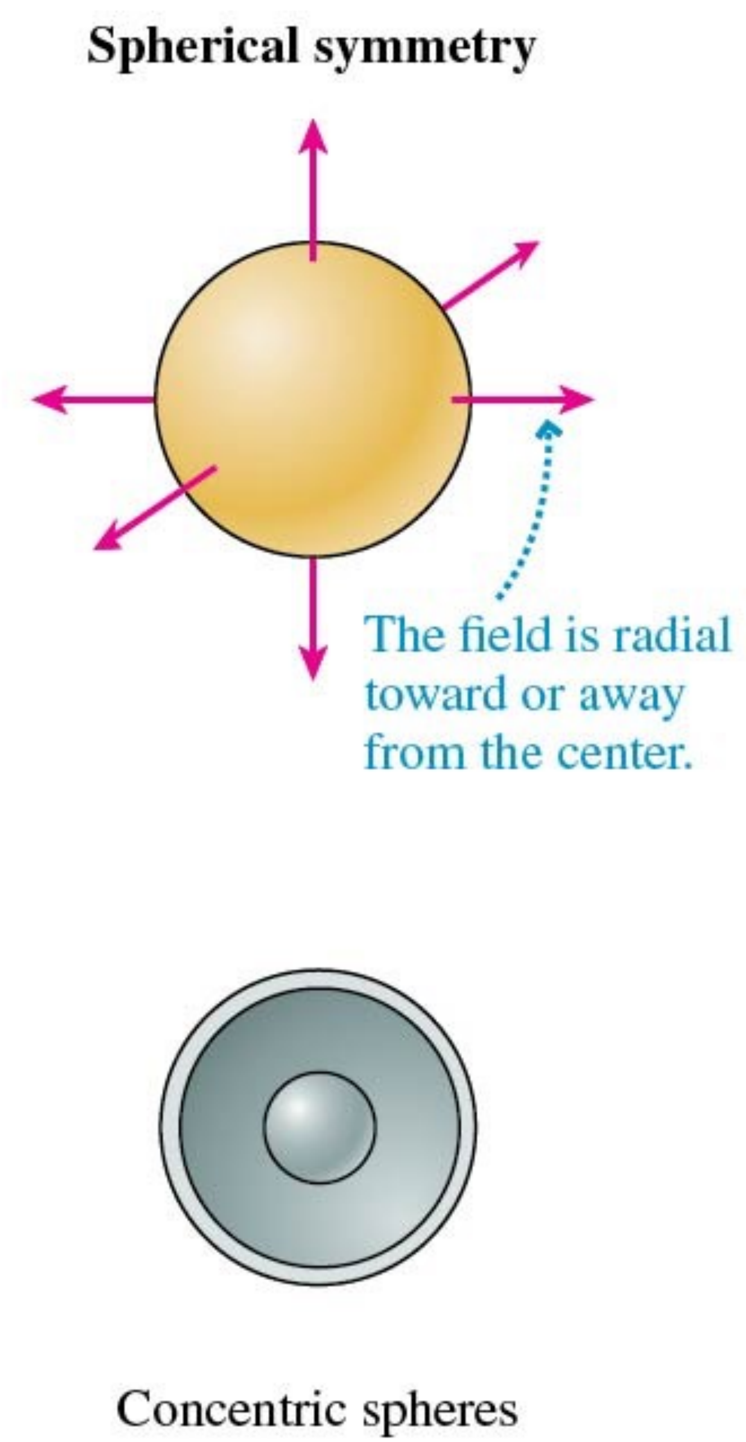
- Three fundamental symmetries the second is **cylindrical symmetry**.
- Cylindrical symmetry involves symmetry with respect to
  - *Translation* parallel to the axis.
  - *Rotation* about the axis.
  - *Reflection* in any plane containing or perpendicular to the axis.





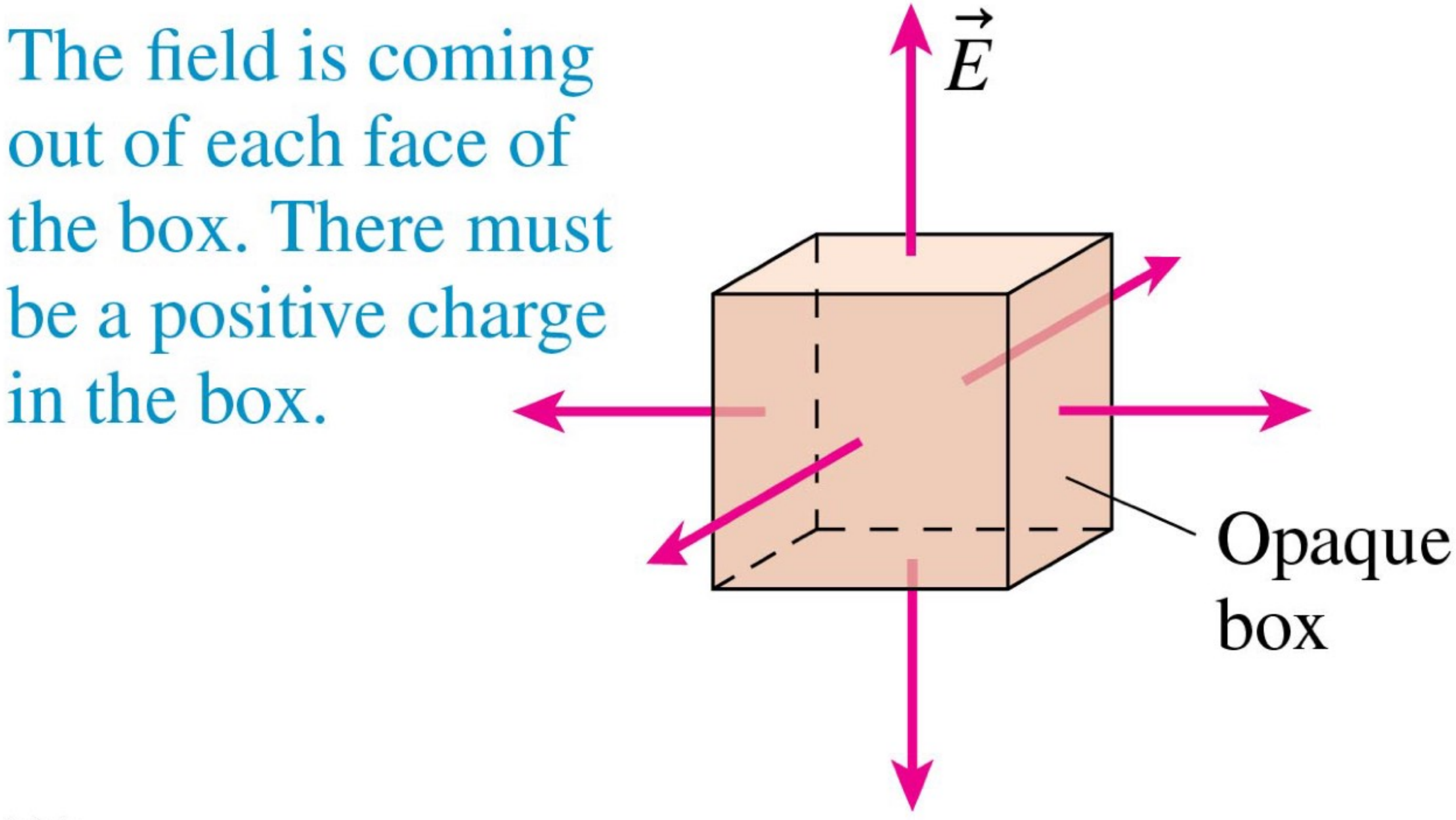
Fundamental symmetry: **Spherical**

- Three fundamental symmetries: the third is **spherical symmetry**.
- Spherical symmetry involves symmetry with respect to
  - *Rotation* about any axis that passes through the center point.
  - *Reflection* in any plane containing the center point.



The Concept of Flux

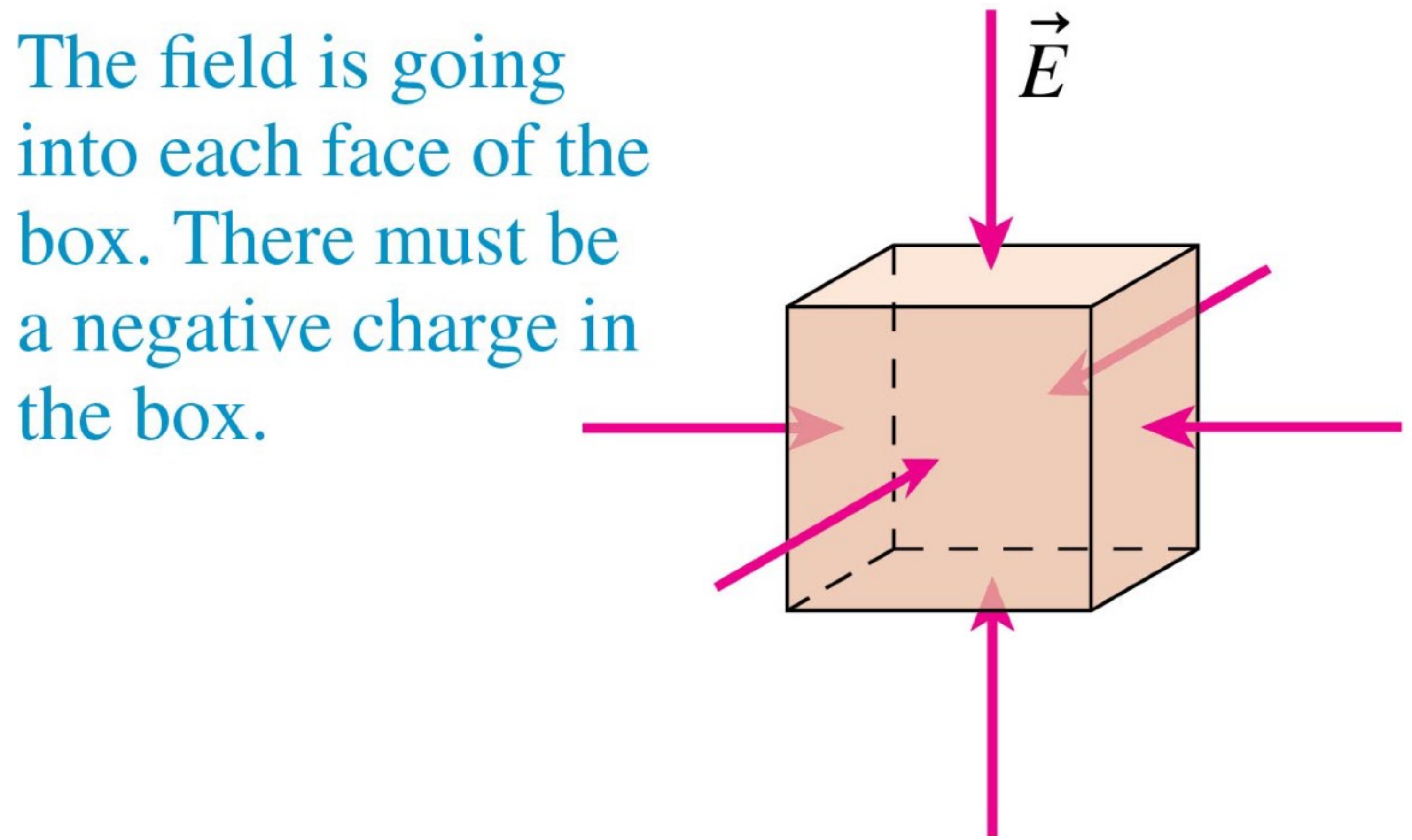
- Consider a box surrounding a region of space.
- We can't see into the box, but we know there is an ***outward-pointing*** electric field passing through every surface.
- Since electric fields point away from positive charges, we can conclude that the box must contain net *positive* electric charge.



The Concept of Flux

- Consider a box surrounding a region of space.
- We can’t see into the box, but we know there is an ***inward-pointing*** electric field passing through every surface.

- Since electric fields point toward negative charges, we can conclude that the box must contain net *negative* electric charge.

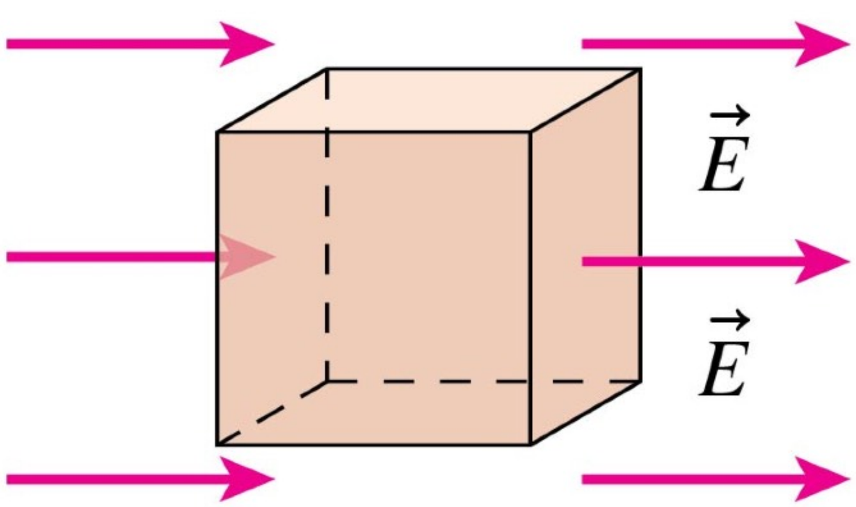


The Concept of Flux

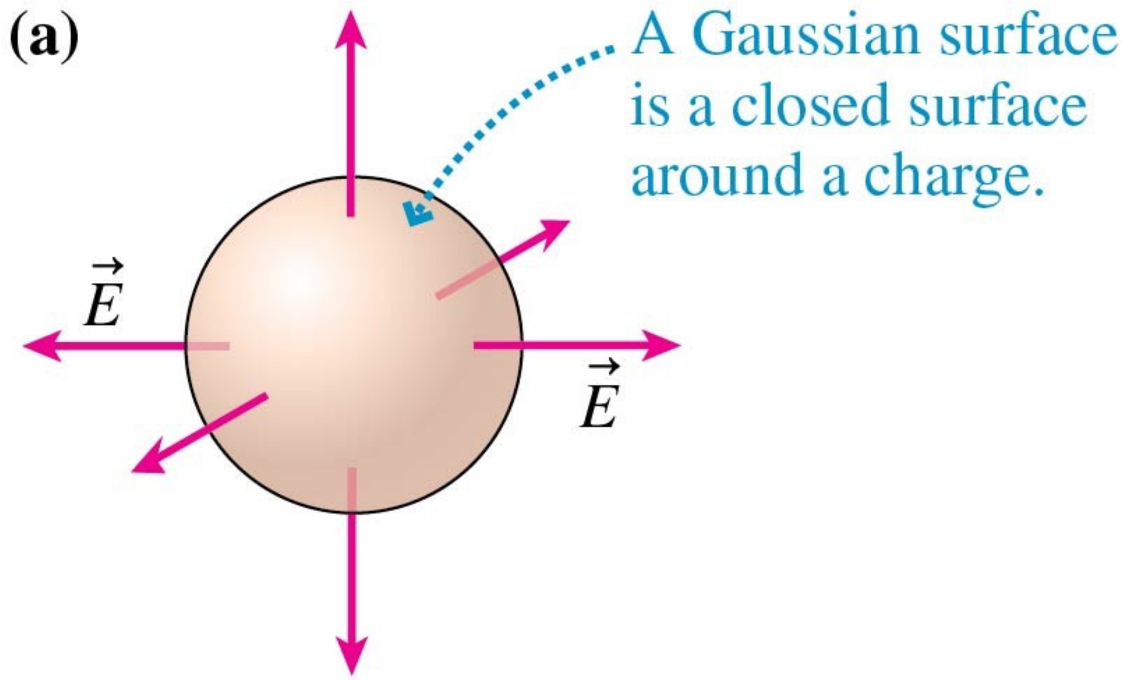
- Consider a box surrounding a region of space.
- We can't see into the box, but we know that the electric field points into the box on the left, and an equal electric field points out of the box on the right.

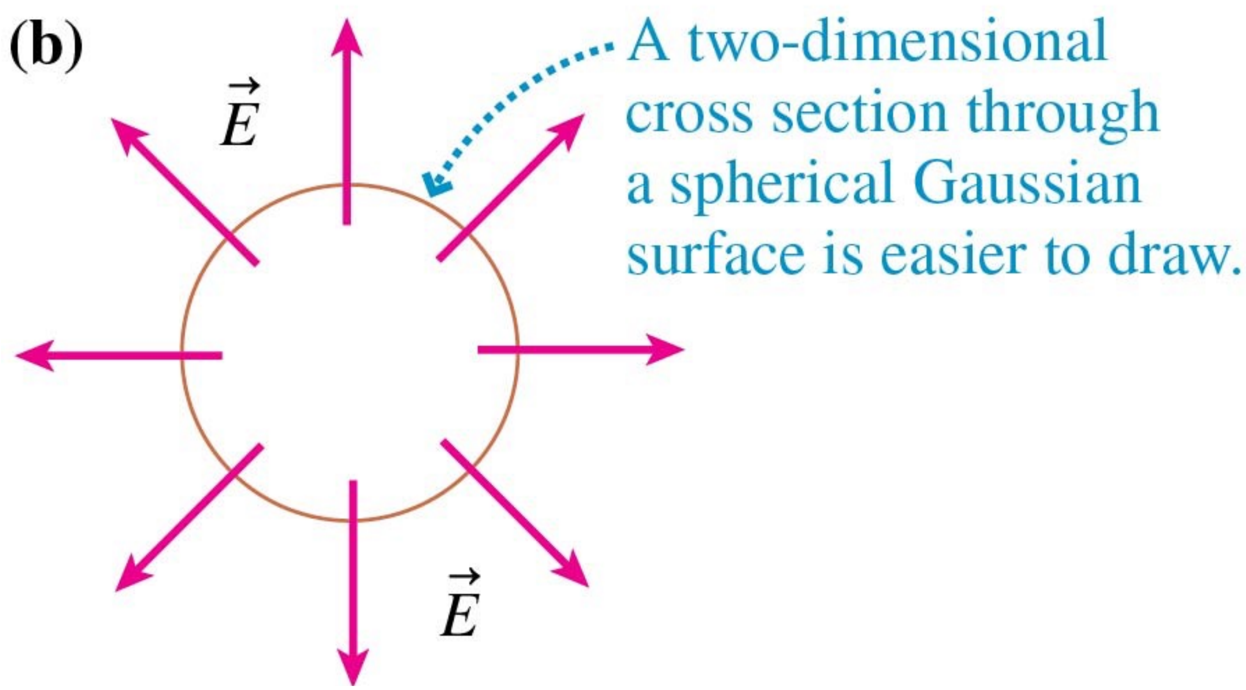
- Since this external electric field is not altered by the contents of the box, the box must contain *zero* net electric charge.

A field passing through the box implies there's no net charge in the box.



Gaussian Surfaces

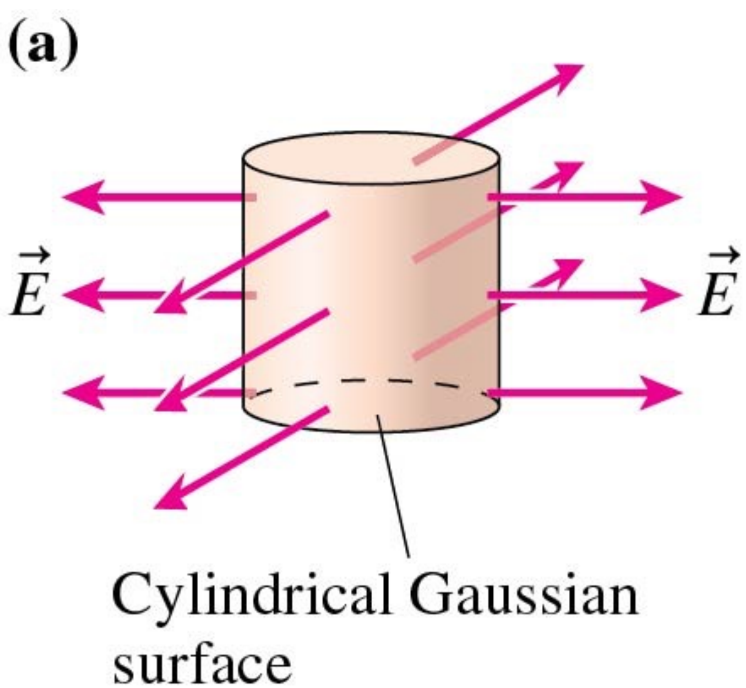


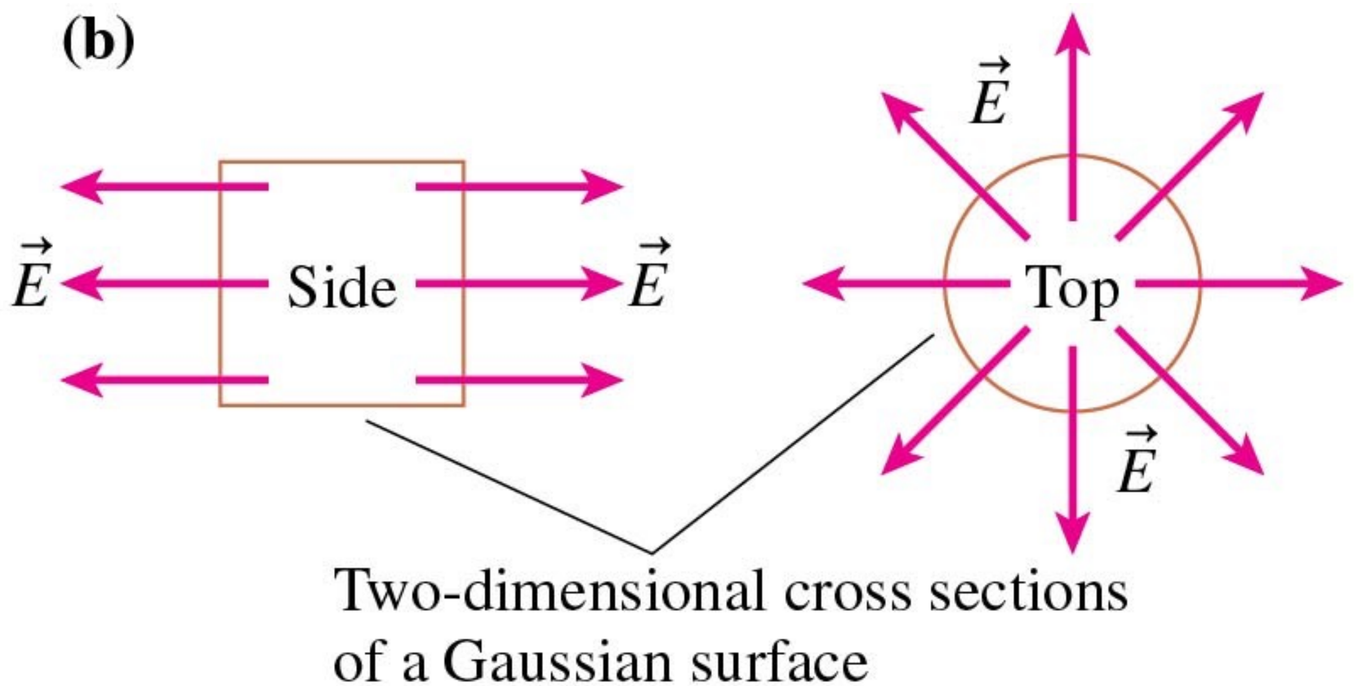


- A closed surface through which an electric field passes is called a **Gaussian surface**.
- This is an imaginary, mathematical surface, not a physical surface.

Gaussian Surfaces

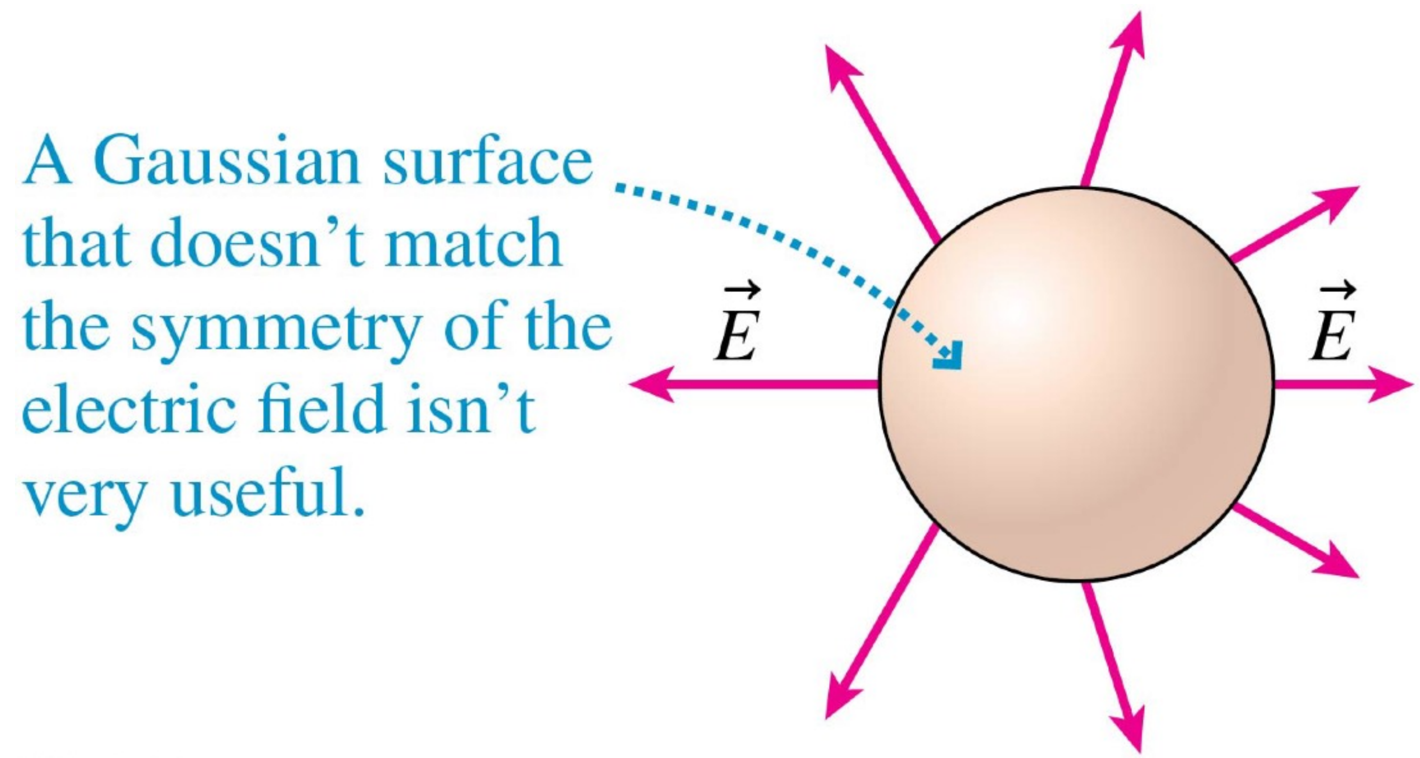
- A Gaussian surface is most useful when it matches the shape and symmetry of the field.
- Figure (a) below shows a *cylindrical* Gaussian surface.
- Figure (b) simplifies the drawing by showing two-dimensional end and side views.
- The electric field *in this example* is everywhere *perpendicular* to the side wall and no field passes through the top and bottom surfaces.





Gaussian Surfaces

- Not every surface is useful for learning about charge.
- Consider the spherical surface in the figure.

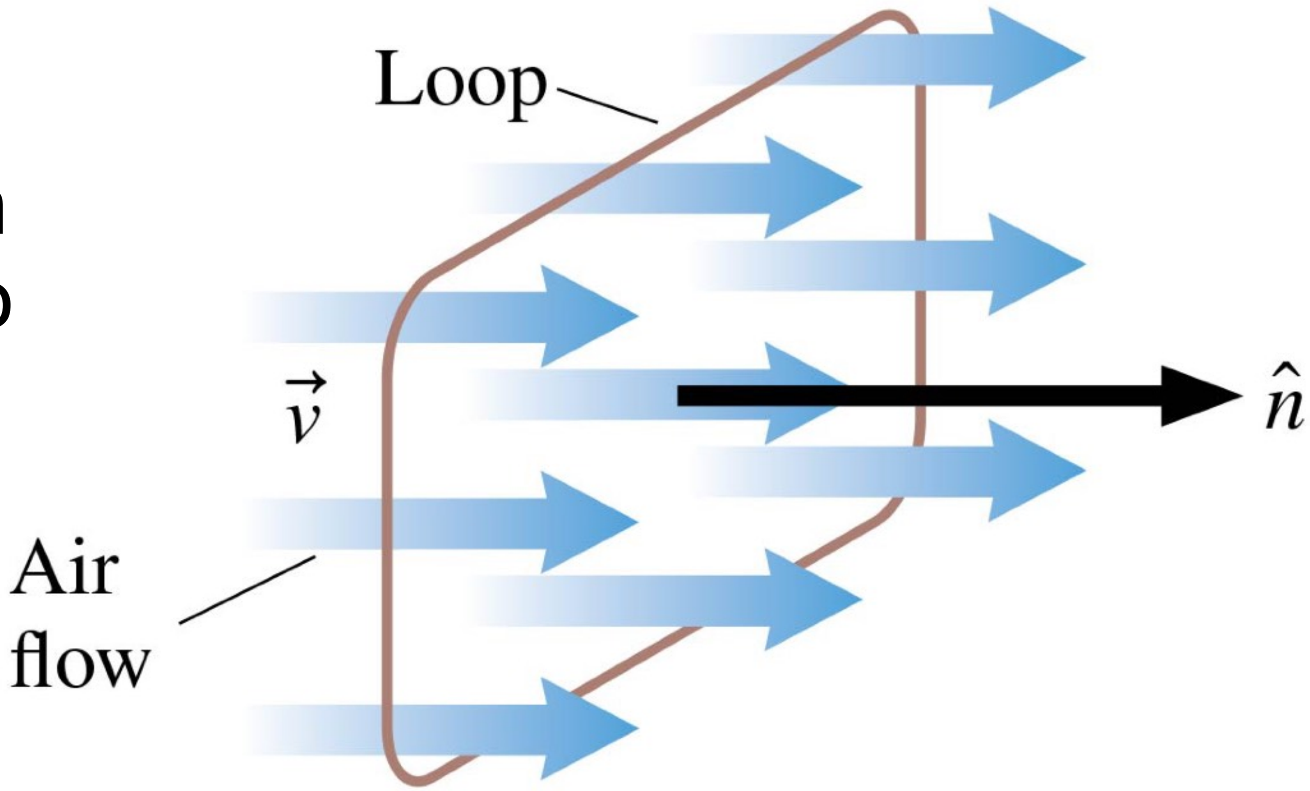


- This is a Gaussian surface, and the protruding electric field tells us there's a positive charge inside.
- It might be a point charge located on the left side, but we can't really say.
- A Gaussian surface that **doesn't match** the symmetry of the charge distribution **isn't terribly useful**.

The Basic Definition of Flux

- Imagine holding a rectangular wire loop of area A in front of a fan.
- The volume of air flowing through the loop each second depends on the angle between the loop and the direction of flow.

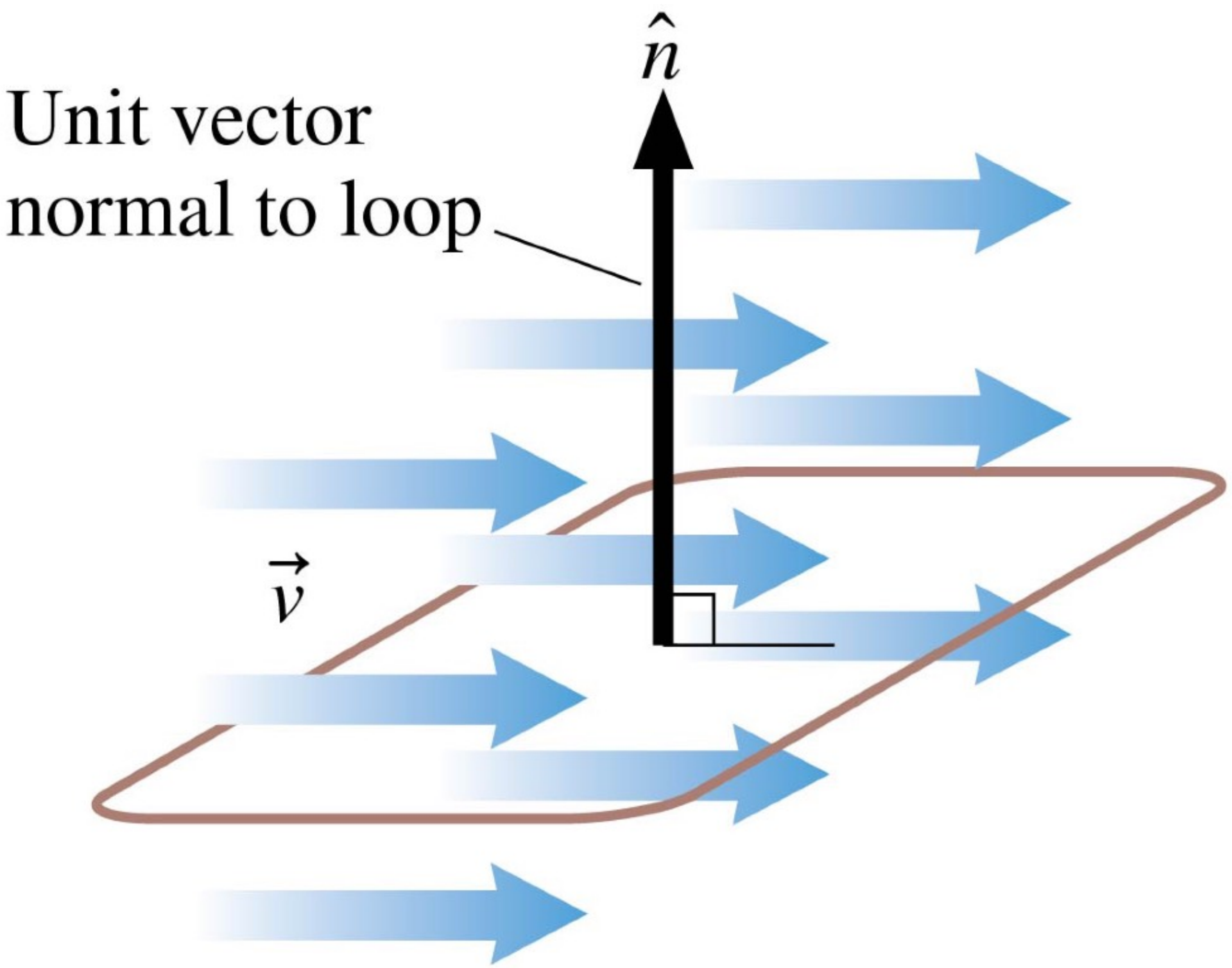
  - The flow is *maximum* through a loop that is perpendicular to the airflow.



The air flowing through the loop is maximum when $\theta = 0°$.

The Basic Definition of Flux

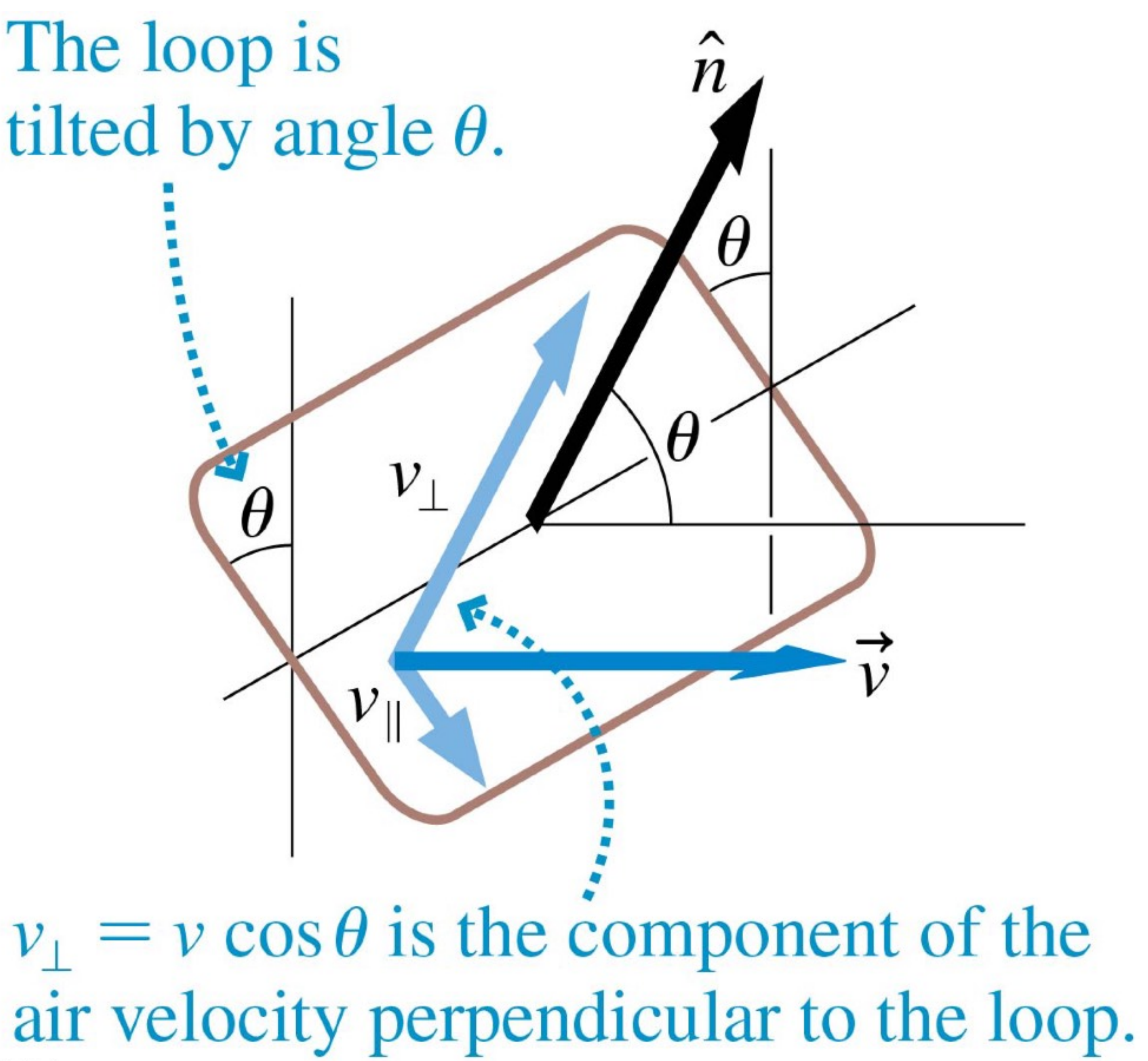
- Imagine holding a rectangular wire loop of area A in front of a fan.
- No air goes through the same loop if it lies parallel to the flow.



No air flows through the loop when $\theta = 90°$.

The Basic Definition of Flux

- Imagine holding a rectangular wire loop of area A in front of a fan.
- The volume of air flowing through the loop each second depends on the angle θ between the loop normal and the velocity of the air:



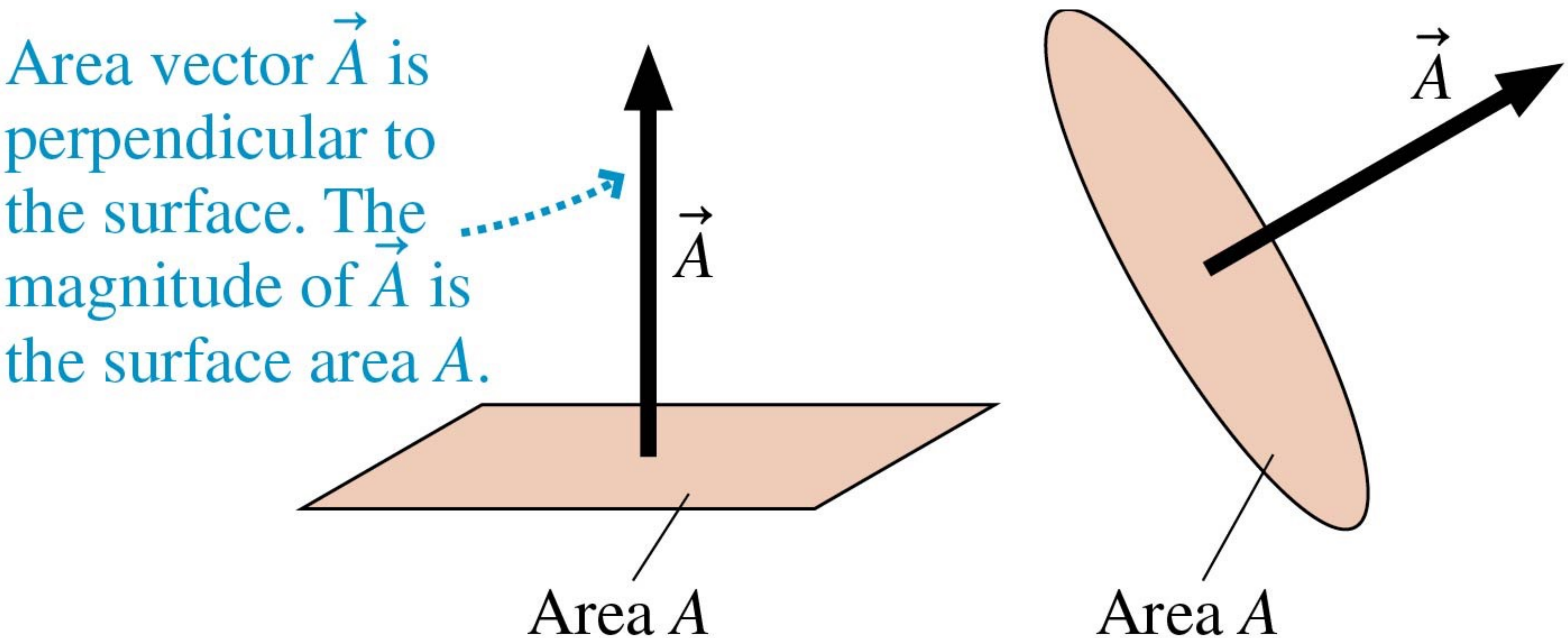
$$\text{volume of air per second (m}^3\text{/s)} = v_\perp A = vA\cos\theta$$

Electric Flux

- As we saw, **flux** is the amount of something that flows through a given area.
- Electric flux, $\mathscr{D}_E$, measures the amount of electric field lines that passes through a given area.
- If I were to measure the flux originating from a charged ball through this surface, what would I have to take into account?
    - **How big** the area is.
    - **How strong** the electric field is.
    - The **angle** between the area and the electric field.

The Area Vector

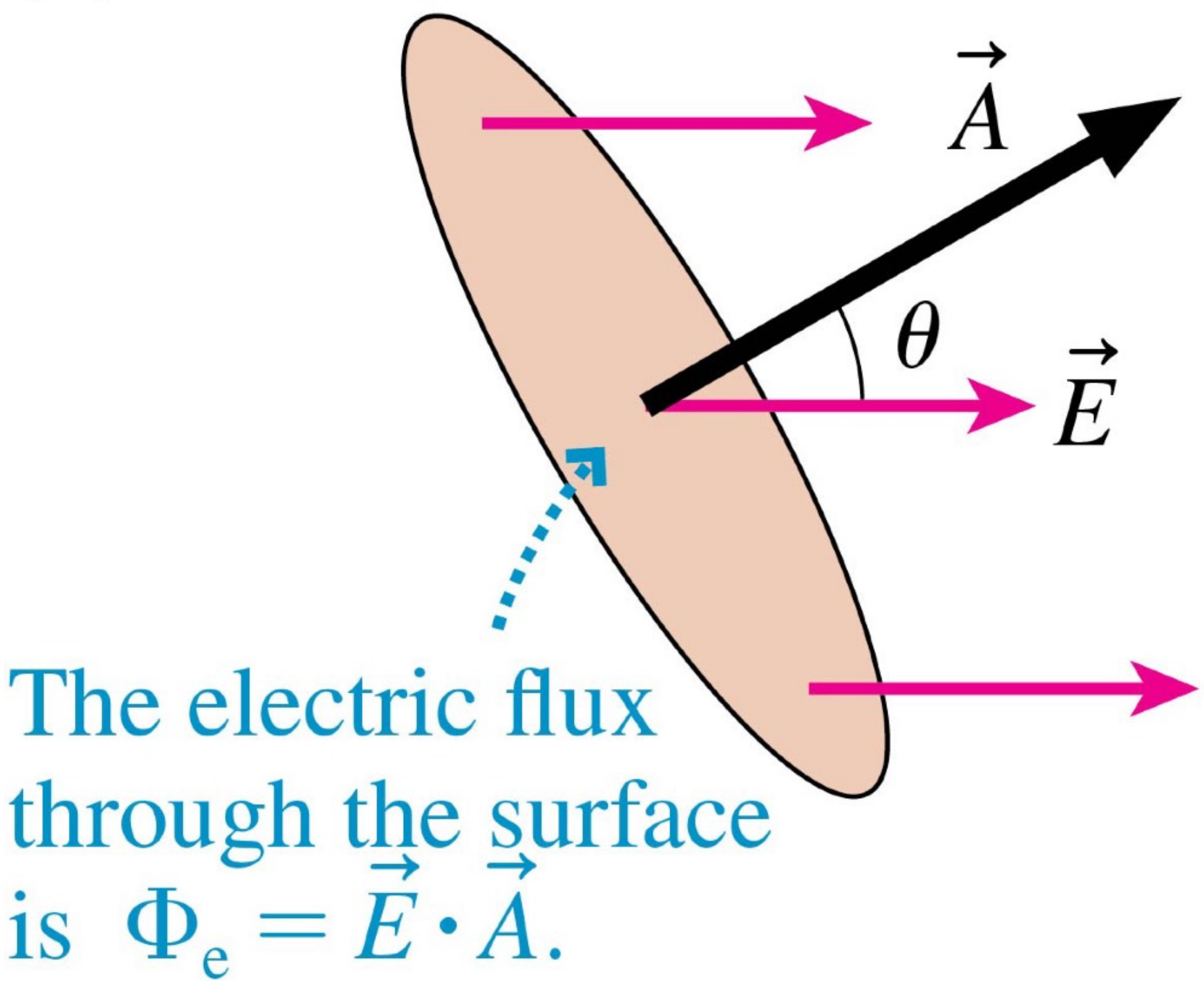
- Let's define an area vector $\vec{A} = A\hat{n}$ to be a vector in the direction of $\hat{n}$, perpendicular to the surface, with a magnitude A equal to the area of the surface.
- Vector $\vec{A}$ has units of m^2.



- An electric field passes through a surface of area A.
- The electric flux can be defined as the dot-product:

$$\Phi_e = \vec{E} \cdot \vec{A} \quad \text{(electric flux of a constant electric field)}$$

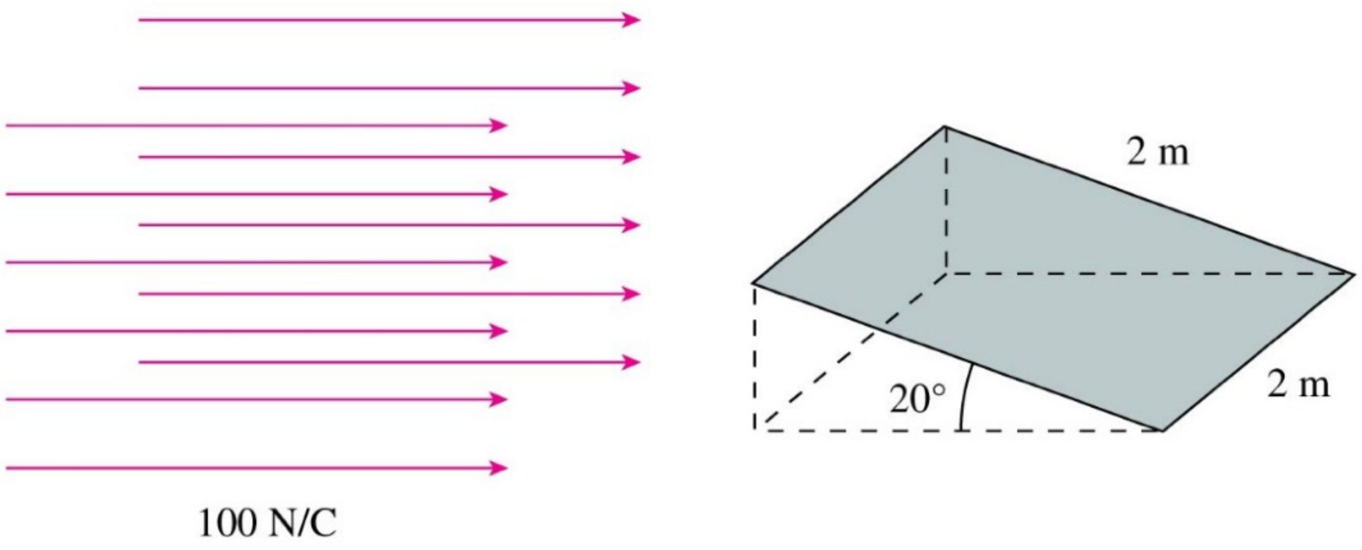
$$\Phi_E = \vec{E} \cdot \vec{A} = EA\cos\theta$$



iClicker question 5-2

The electric flux through the shaded surface is

A. 0
B. 400 cos(20º) N m²/C
C. 400 N m²/C
D. 400 cos(70º) N m²/C
E. Some other value.



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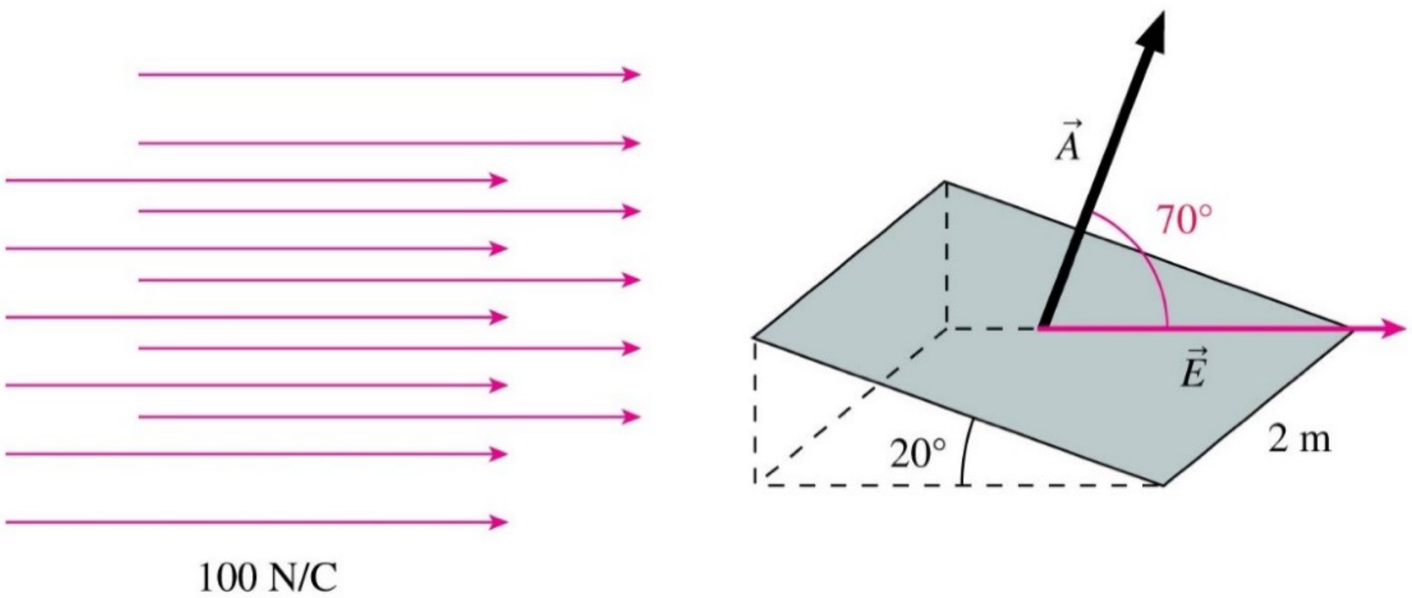
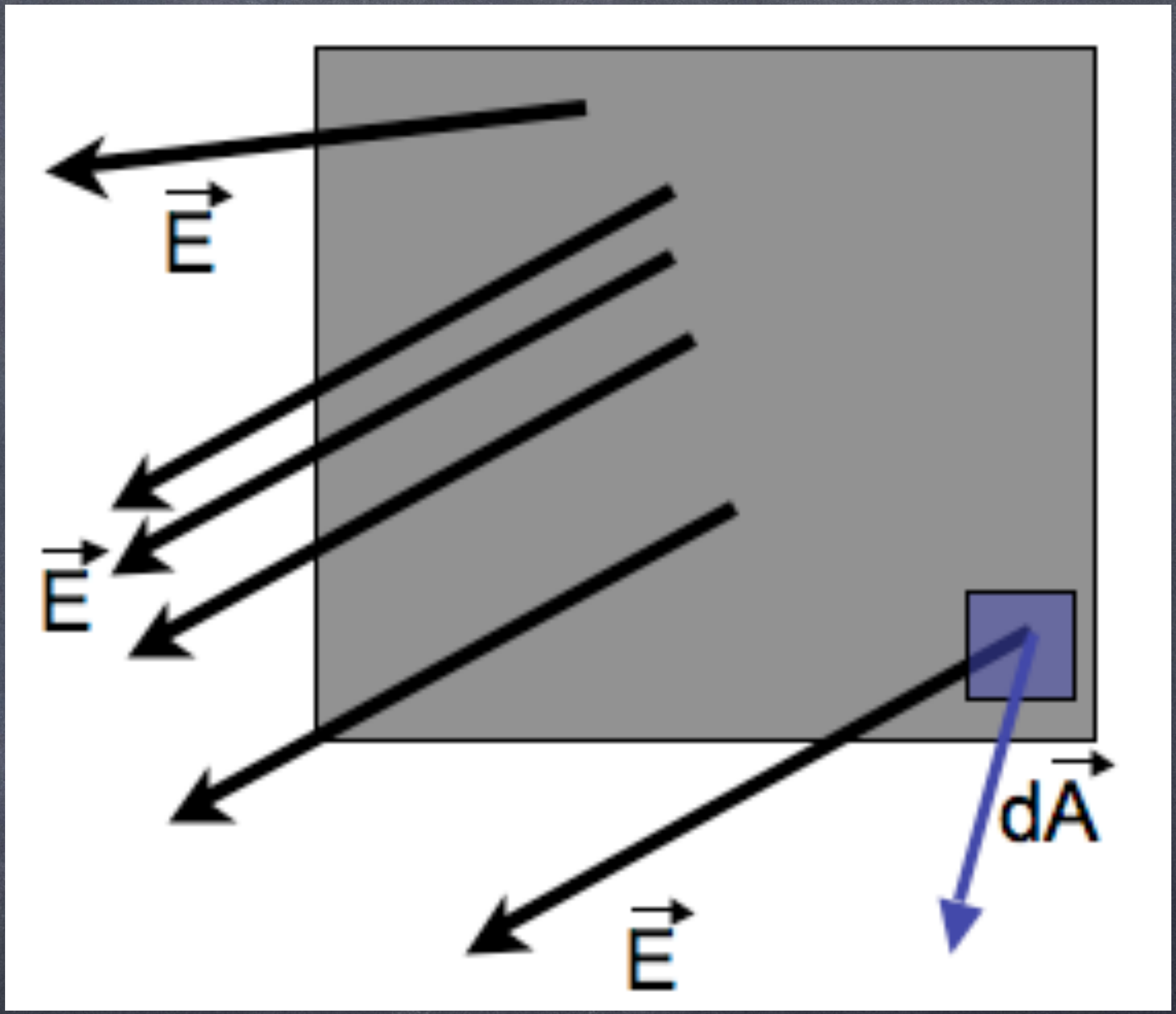


Electric Flux

- Unfortunately, not all electric fields are uniform:
- In this case, we look at a small area element dA and measure the electric flux through it.



- To get the entire flux over the surface we need to sum up all the small area elements (integrate):

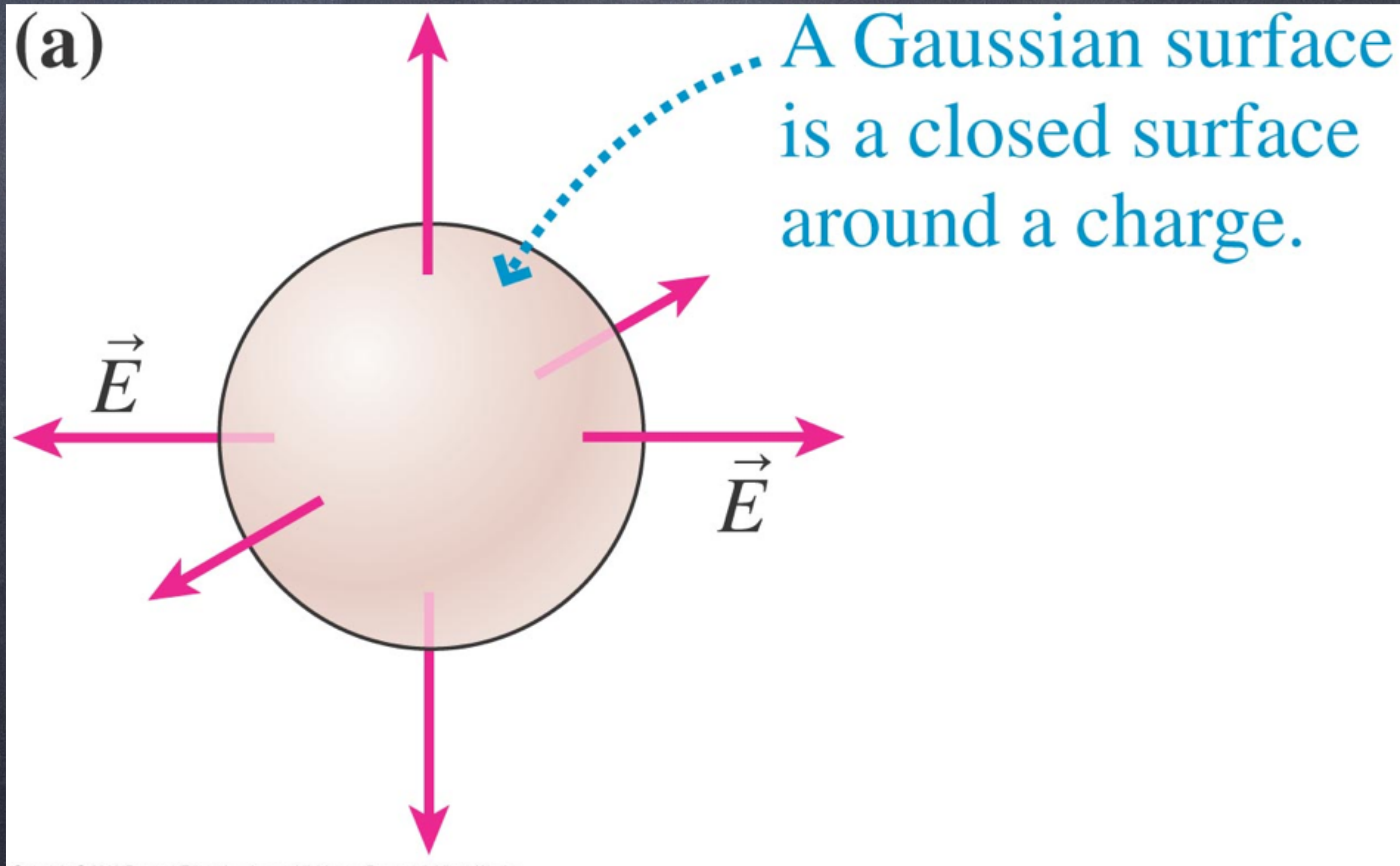
$$\Phi_E = \int \vec{E} \cdot d\vec{A}$$

Electric Flux

- An imaginary sheet is known as an open surface. This is because nothing can possibly be enclosed by it.
- It is far more useful to deal with the electric flux from what is known as a **closed surface**.
- A closed surface is one that encloses a small volume & has no openings (it distinctly divides a space into inside and outside regions).
- Examples of a closed surface are spheres, boxes, and cylinders.

Electric Flux

- In electrostatics, a closed surface that an electric field passes through is known as a Gaussian surface.
- Note: a Gaussian is an imaginary surface; it does not necessarily coincide with a physical object.
- The convention becomes that lines "exiting" the surface are positive and the lines entering are considered negative.



Gauss' Law

- If we try to find the electric flux through the Gaussian surface we change our integral to:

$$\Phi_E = \oint \vec{E} \cdot d\vec{A}$$

- where the loop on the integral sign means we take the integration over the entire closed surface.
- Gauss' Law states that the electric flux through any closed surface is equal to the net charge, Q_{inside}, inside the surface divided by the constant ε_o.

$$\Phi_E = \frac{Q_{inside}}{\varepsilon_o}$$

- where $\varepsilon_o = 8.85 \times 10^{-12} C^2/(Nm^2)$ and is called the permittivity of free space.

Gauss' Law

$$\Phi_E = \oint \vec{E} \cdot d\vec{A}$$

integral of E over surface

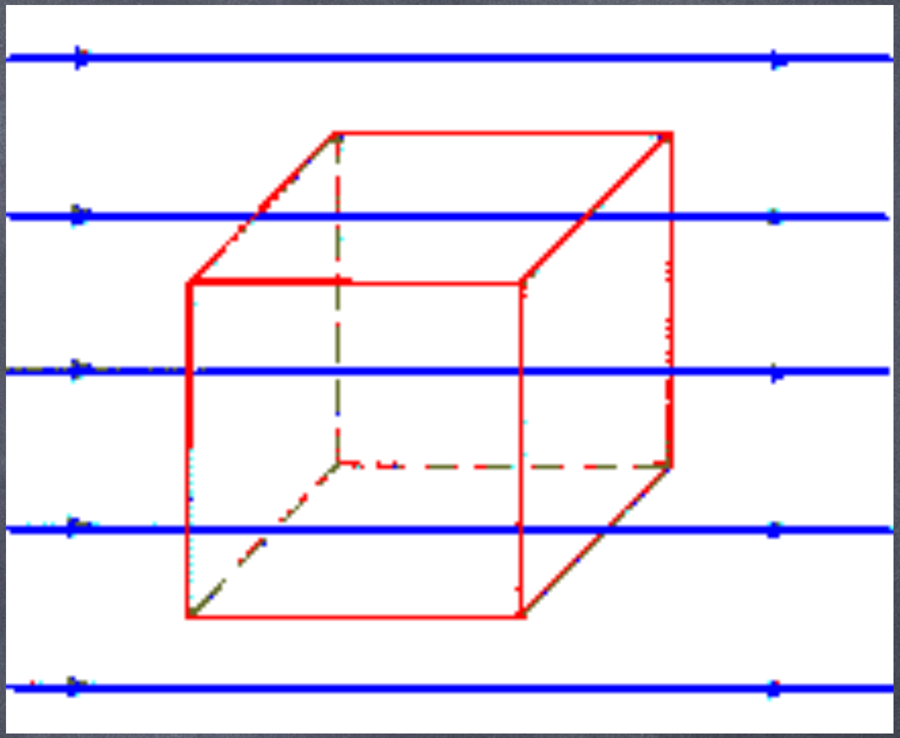
=

$$\Phi_E = \frac{Q_{inside}}{\varepsilon_o}$$

charge enclosed

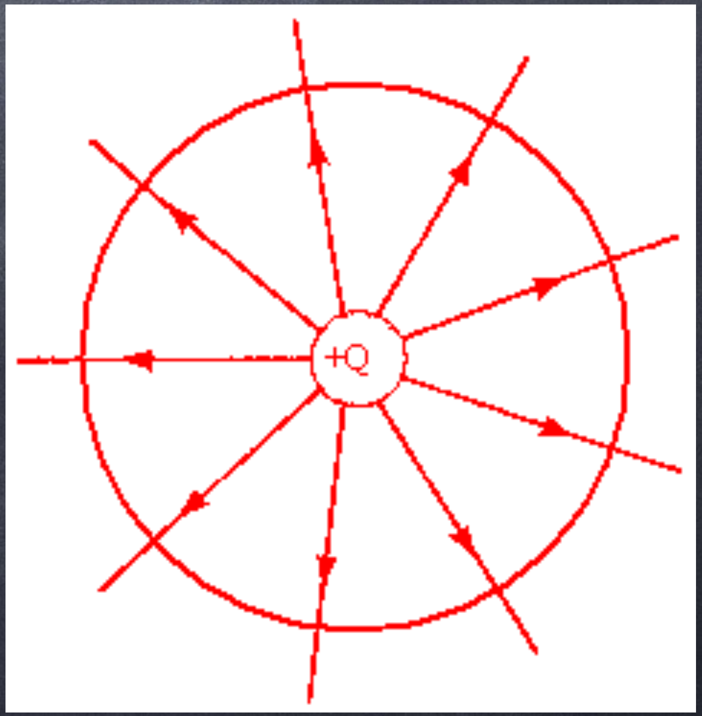
Gauss' Law

- What if there were no charges present inside the closed Gaussian surface, what would the electric flux be?



- The amount of field lines that enter the surface equal the amount of field lines that leave the surface. This leads to a net electric flux of zero.

    - But if a net positive charge is present, then there is a source of electric field lines and the electric flux cannot be zero.



Applications of Gauss' Law

- The power of Gauss' Law is to use the net charge enclosed in a surface to find the electric field at a certain distance (or vice versa).
- The trick comes in finding a proper Gaussian surface.
- If you are looking for the electric field from a configuration of charges, you would like to choose a surface in which the electric field is constant (in magnitude) at all points on the surface.
- This allows you to pull E out of the integral.

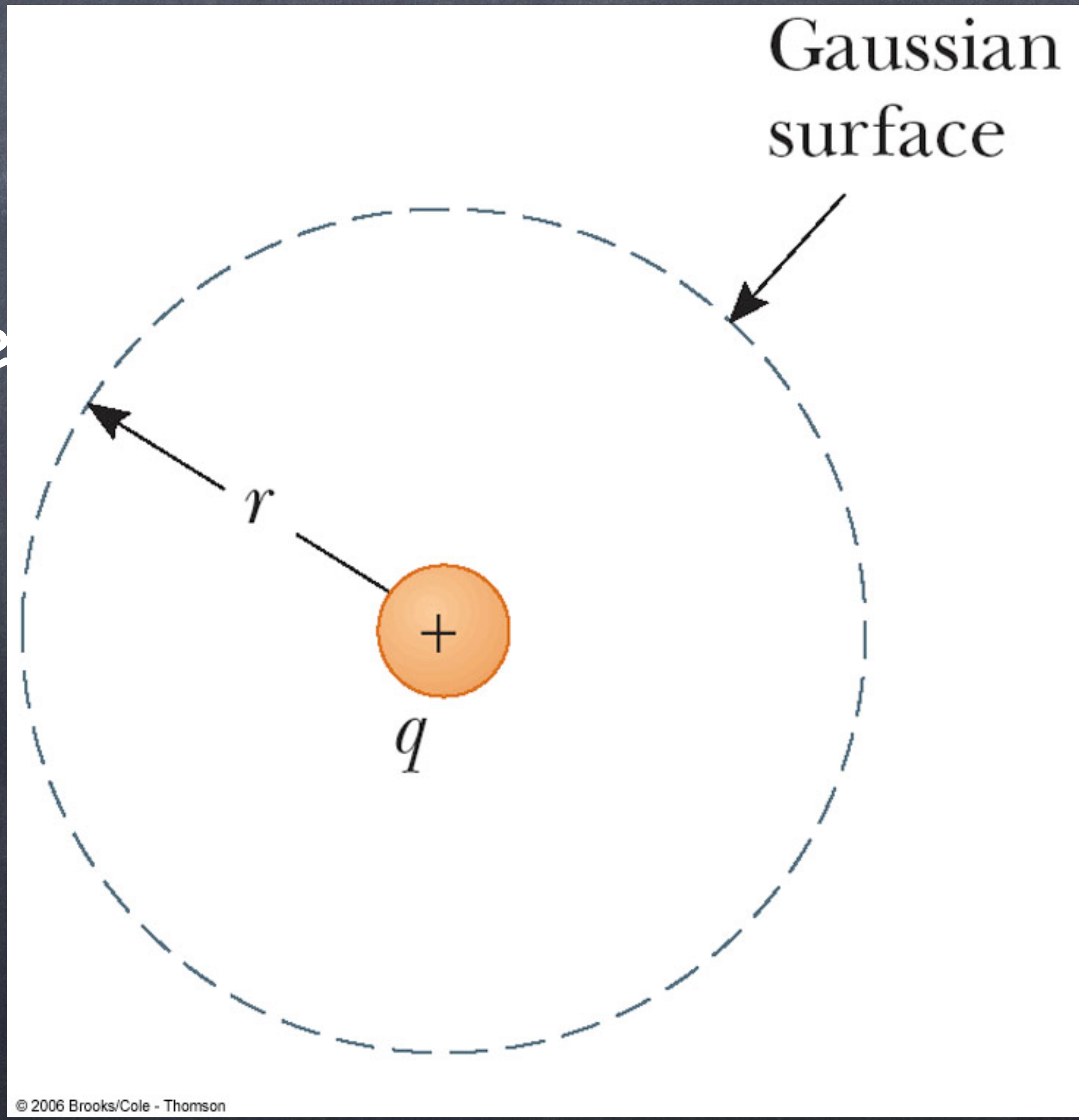
Gauss' Law

- Example
- Use Gauss' Law to find the electric field a distance r from a lone positive point charge +q.

- Answer
- Define a coordinate system.
- Choose the point charge as r = 0. Any outward distance would be positive.

Gauss' Law

- Answer
- If we are going to use Gauss' Law we must choose a Gaussian surface.
- Since this problem has spherical symmetry (it doesn't matter which direction you travel outward), use a spherical Gaussian surface.



- Next, apply Gauss's Law.
- How much charge is located inside the Gaussian sphere?
- +q.

$$\Phi_E = \frac{Q_{inside}}{\varepsilon_o} = \frac{+q}{\varepsilon_o}$$

Gauss' Law

- Answer
- Next turning to the definition of electric flux.

$$\Phi_E = \oint \vec{E} \cdot d\vec{A}$$

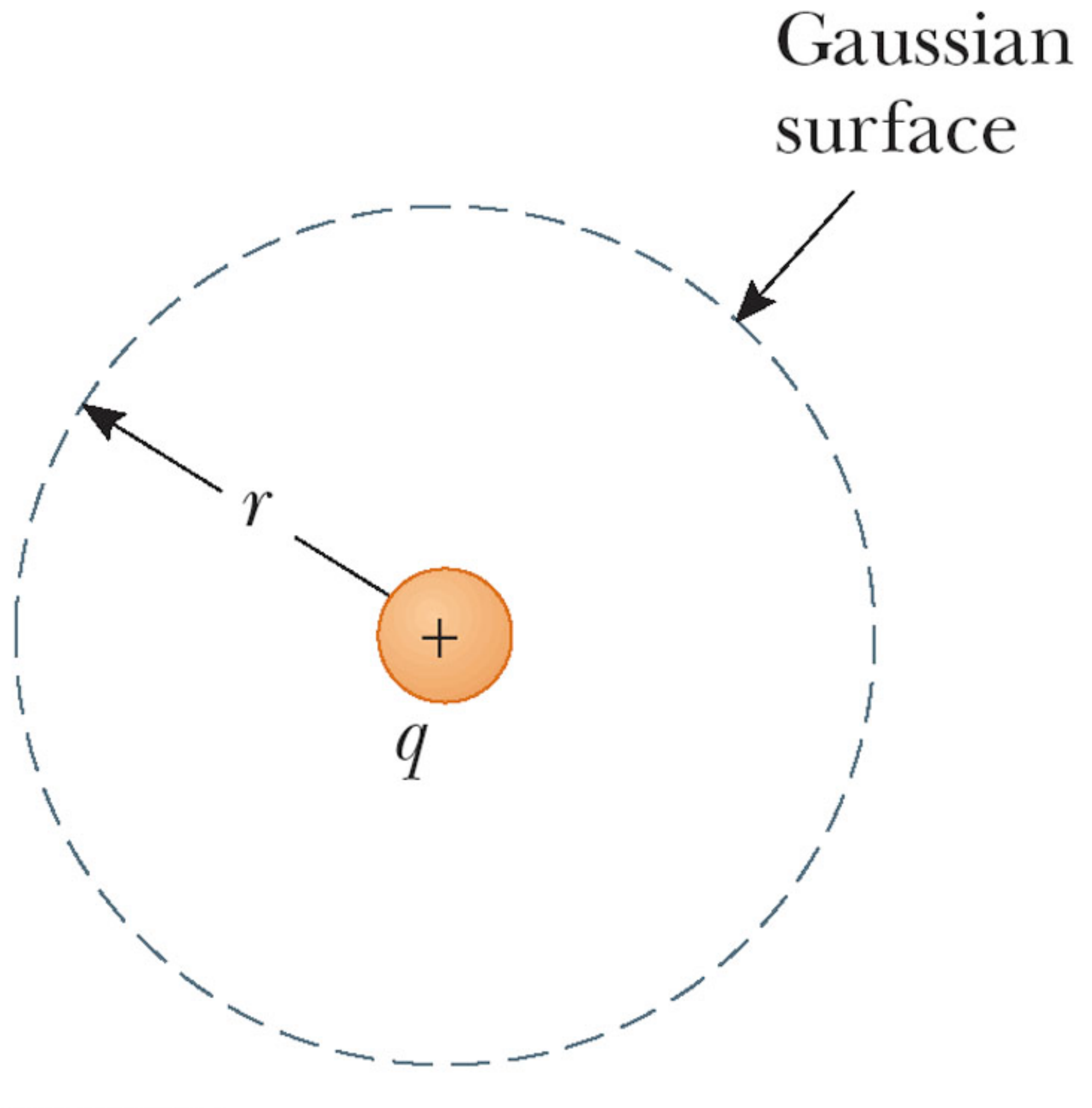
- Since the electric field is constant in magnitude at all points on the Gaussian surface and dA is always parallel to E, we can pull it out of the integral.

$$\Phi_E = (E)\oint dA$$

- Taking the surface integral:

$$\oint dA = 4\pi r^2$$

$$\Phi_E = E\left(4\pi r^2\right)$$



Gauss' Law

- Answer
- Equate the two terms:

$$\Phi_E = \Phi_E$$

$$E\left(4\pi r^2\right) = \frac{+q}{\varepsilon_o}$$

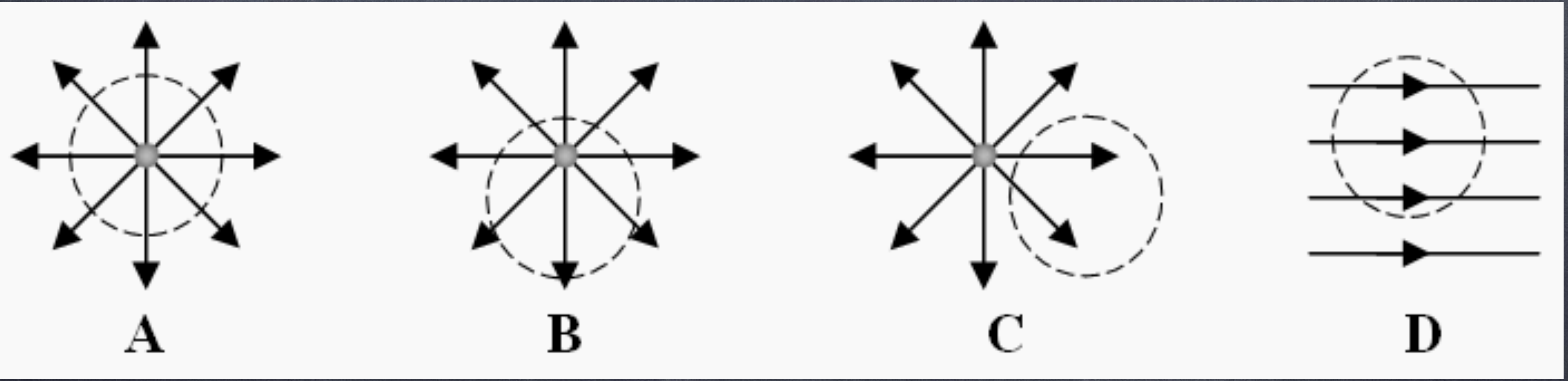
$$E = \frac{1}{4\pi\varepsilon_o}\frac{+q}{r^2}$$

$$E = k_e\frac{+q}{r^2}$$

- This corresponds to our original Coulomb's Law and validates the results.

iClicker Question 5-3

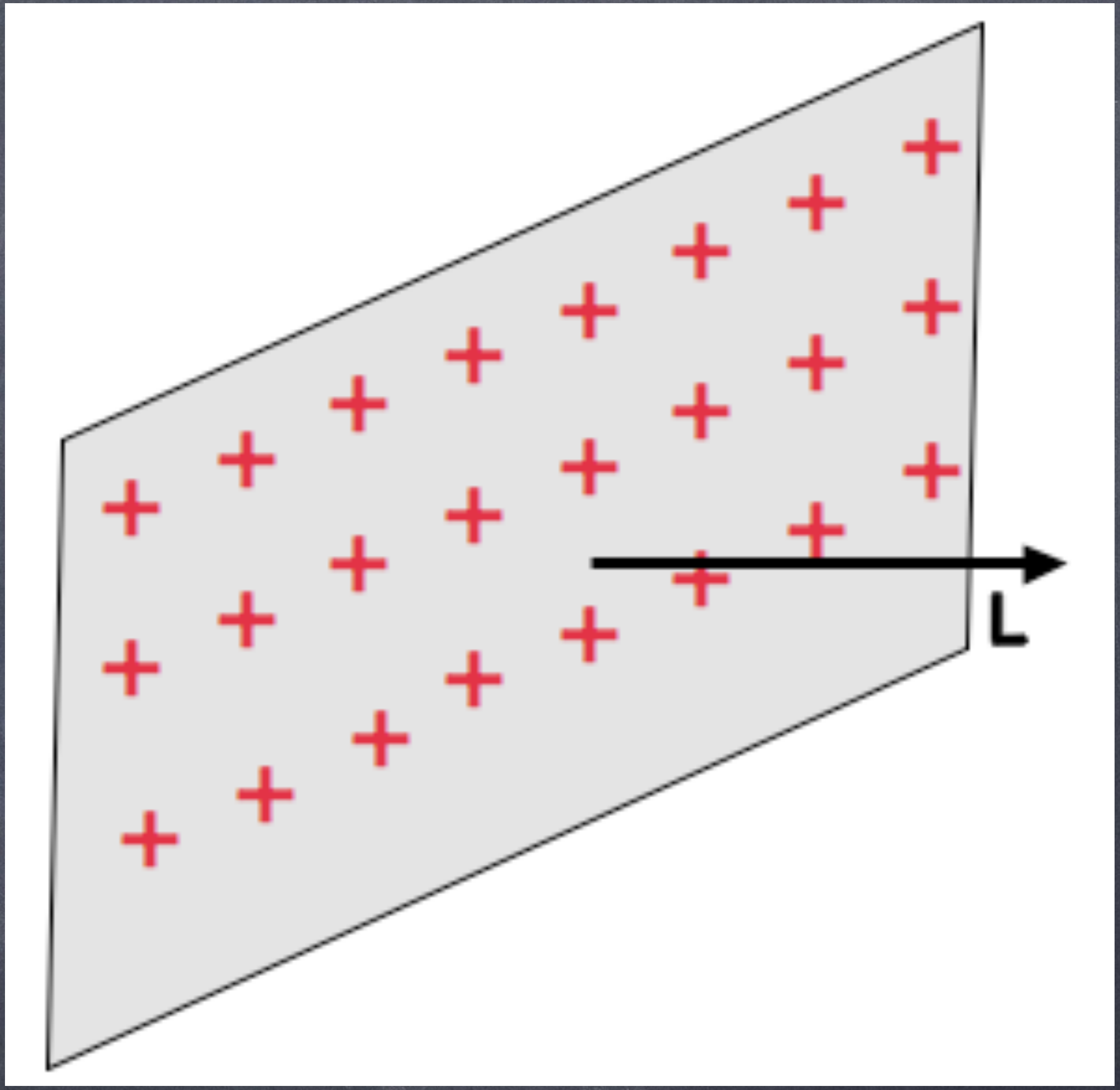
- In each of the four cases below, a Gaussian circle is represented by the dashed line circle and the arrows represent electric field lines. In which of the four cases is the flux through the Gaussian circle **not** equal to zero?



- A) A and B only.
- B) C and D only.
- C) A only.
- D) D only.
- E) A, B, C, and D.

Gauss' Law

- <u>Example</u>
- What is the electric field, E, a distance L above an infinite thin sheet of charge with surface density, σ.



- <u>Answer</u>
- Choose an appropriate Gaussian surface.
- Take a cylindrical surface with length 2L and faces of area A.

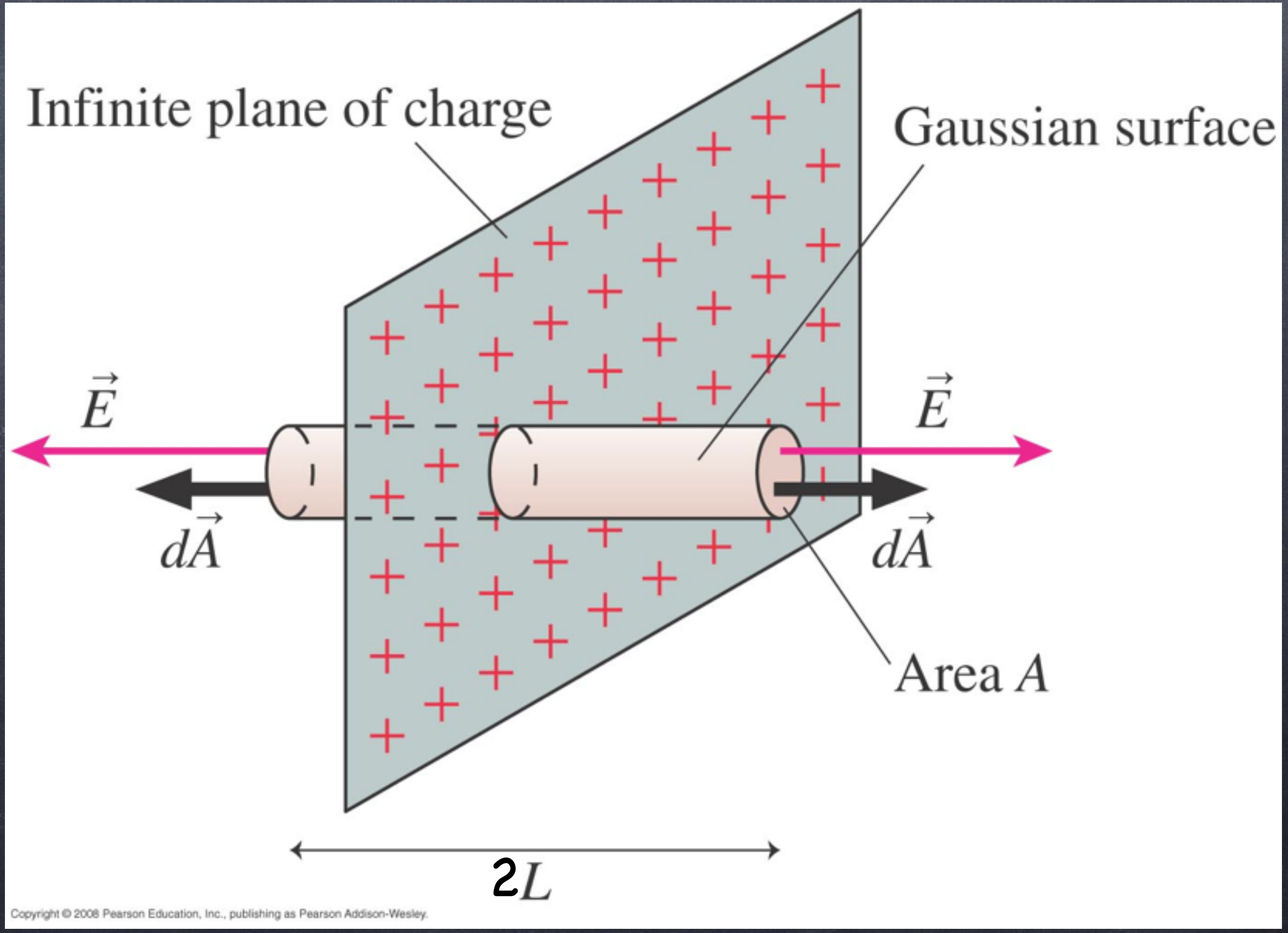
Gauss' Law

- Answer

- The electric field is perpendicular to both faces of the cylinder.
- So the total flux through both faces is: $$\Phi_E = 2EA$$

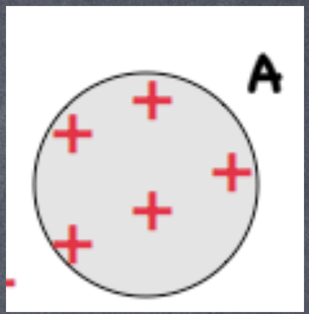
- There is no flux through the wall of the cylinder because the field vectors are tangent to the wall.
- Leaving the net flux to be:

$$\Phi_E = 2EA$$



Gauss' Law

- <u>Answer</u>



- Next, we should find the amount of charge contained in the Gaussian surface.
- The surface charge density will be:

$$\sigma = \frac{Q}{A}$$

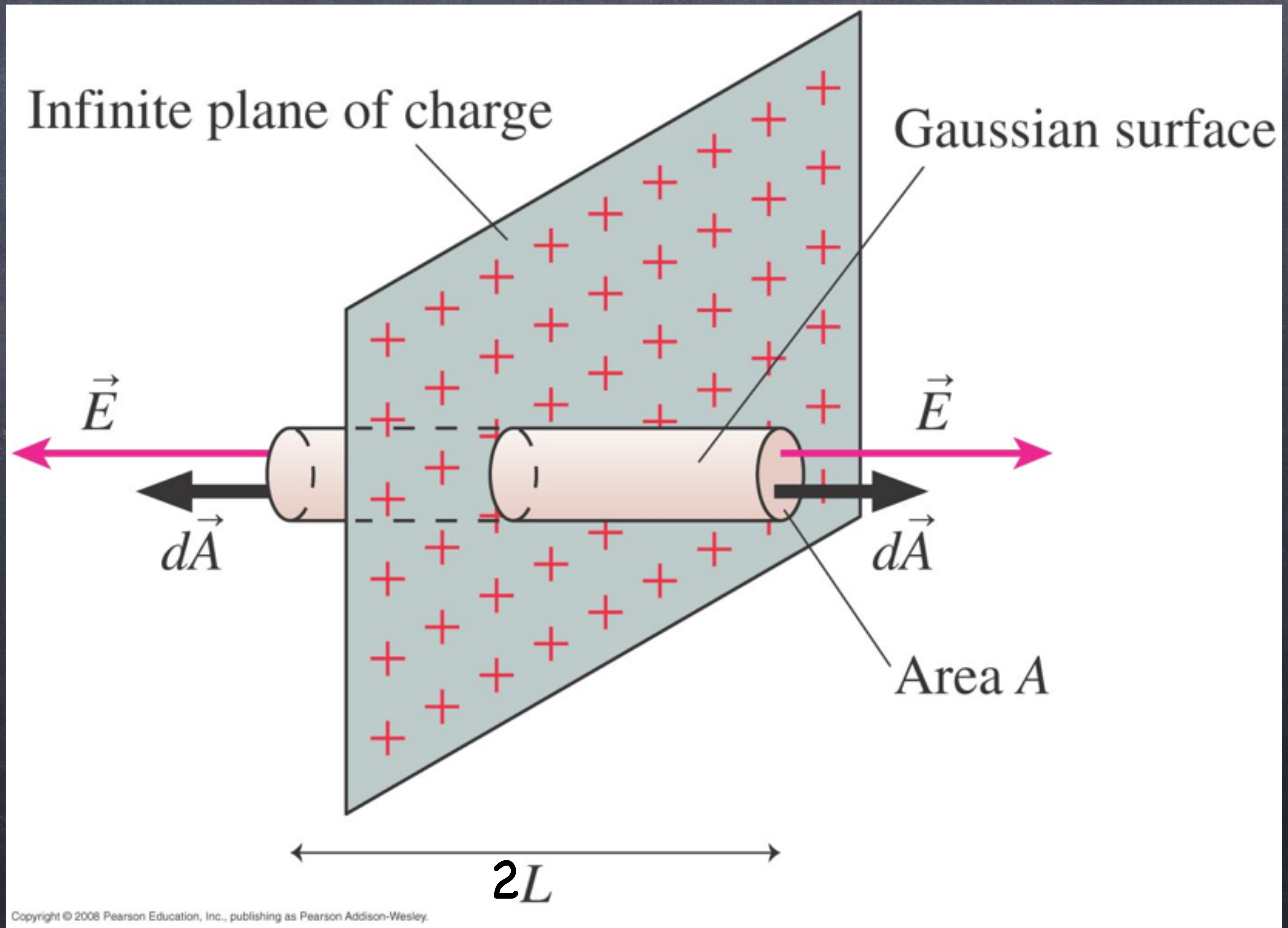
$$Q = \sigma A$$

- That is the charge enclosed by the Gaussian surface.

$$\Phi_E = \Phi_E$$

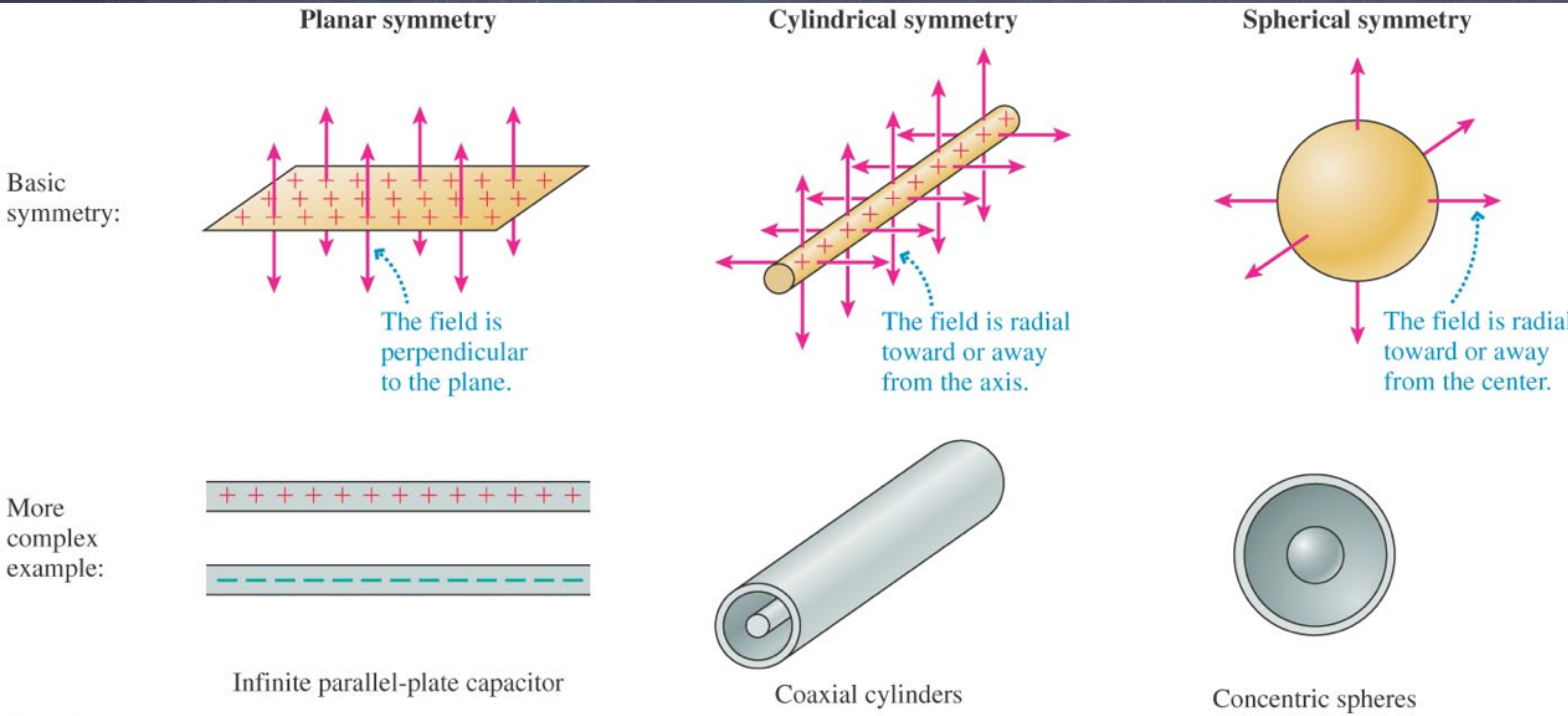
$$2EA = \frac{Q}{\varepsilon_o} = \frac{\sigma A}{\varepsilon_o}$$

$$E = \frac{\sigma}{2\varepsilon_o}$$



Gauss' Law

- Remember: There are three fundamental symmetries that you exploit in order to use Gauss' Law to solve for a given electric field: Planar, Cylindrical, and Spherical.



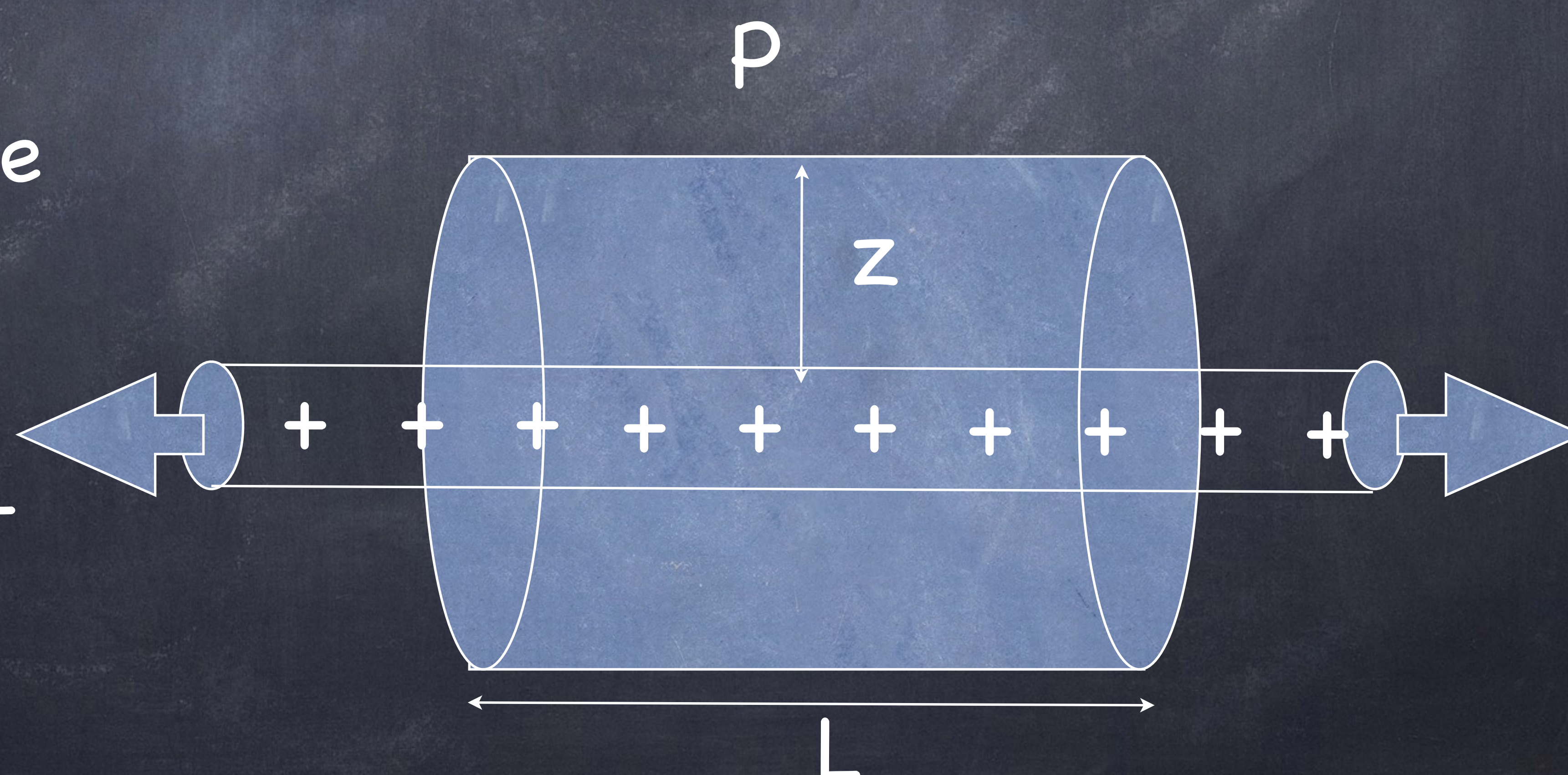
Gauss' Law

- Again let's determine the magnitude of the electric field at a point P which is at a distance z from a very long (nearly infinite) wire of uniformly distributed charge. Assume z is much smaller than the length of the wire and let ∂ be the charge per unit length of the wire.

- <u>Solution</u>:

Choose an appropriate Gaussian surface.

Take a cylindrical surface with length L and radius z.



Gauss' Law

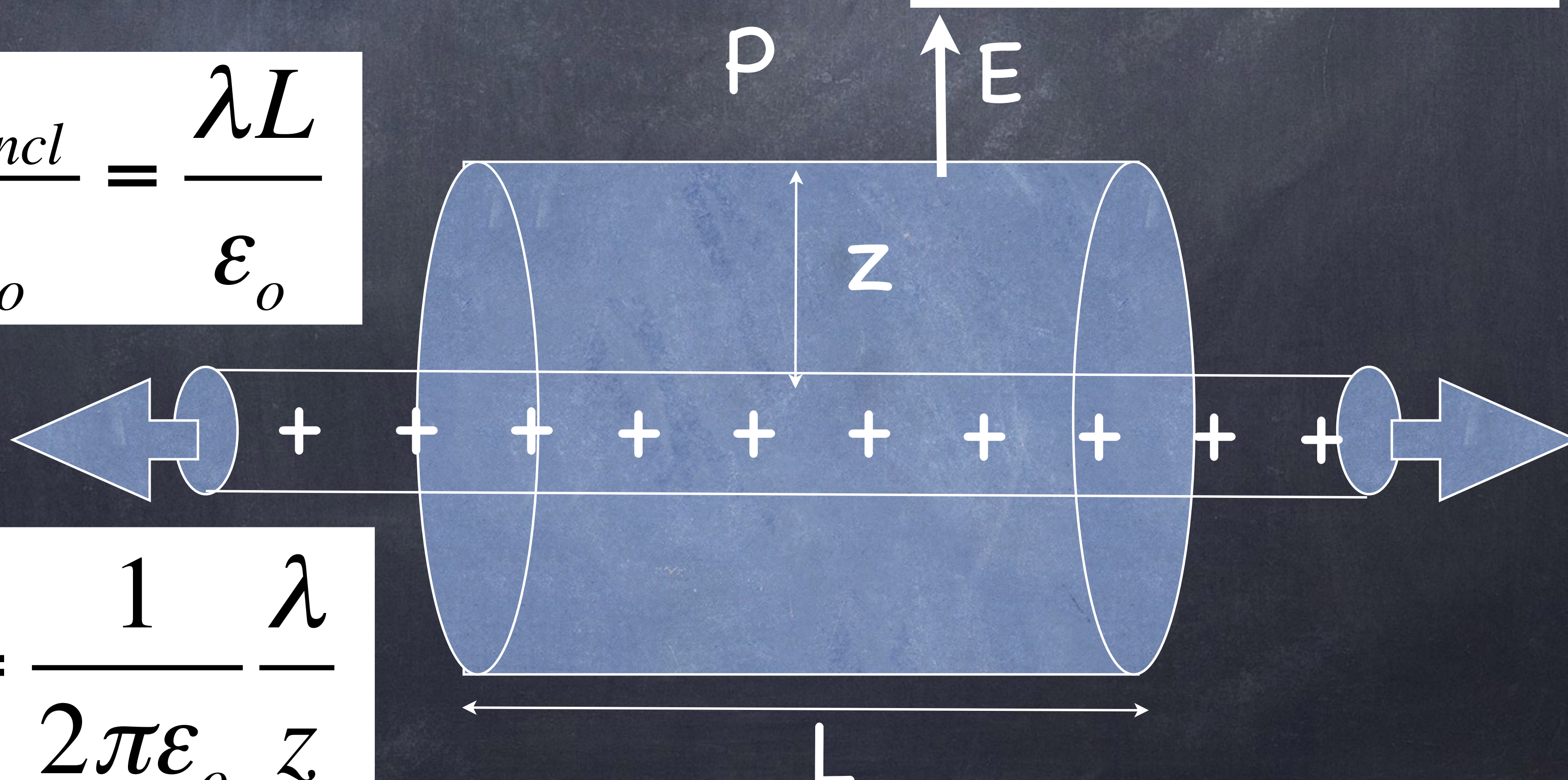
- The electric field is perpendicular to the entire cylindrical part of the surface (radially outward with the same magnitude at each point).
- The electric field is parallel to both faces of the cylinder.
- So the total flux through the cylindrical surface is:

$$\Phi_E = EA_{cylindrical\ part}\cos\theta = E\left(2\pi zL\right)\cos 0$$

$$\Phi_E = E\left(2\pi zL\right)$$

Via Gauss' Law:

$$\Phi_E = \frac{q_{encl}}{\varepsilon_o} = \frac{\lambda L}{\varepsilon_o}$$



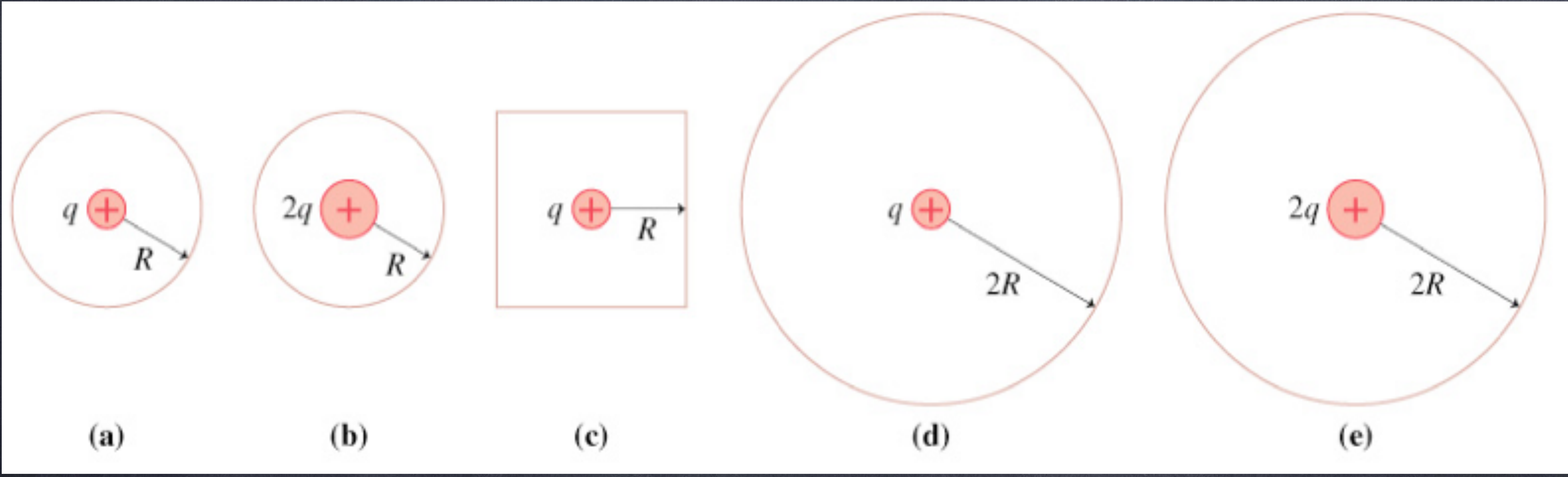
Equating the two:

$$E\left(2\pi zL\right) = \frac{\lambda L}{\varepsilon_o}$$

$$E = \frac{1}{2\pi\varepsilon_o}\frac{\lambda}{z}$$

iClicker Question 5-4

- These are two-dimensional cross sections through three dimensional closed spheres and a cube. Rank the order, from largest to smallest, the electric fluxes Φ_a to Φ_e through surfaces a to e.



A) $\Phi_a > \Phi_c > \Phi_b > \Phi_d > \Phi_e$.

B) $\Phi_b = \Phi_e > \Phi_a = \Phi_c = \Phi_d$.

C) $\Phi_e > \Phi_d > \Phi_b > \Phi_c > \Phi_a$.

D) $\Phi_b > \Phi_a > \Phi_c > \Phi_e > \Phi_d$.

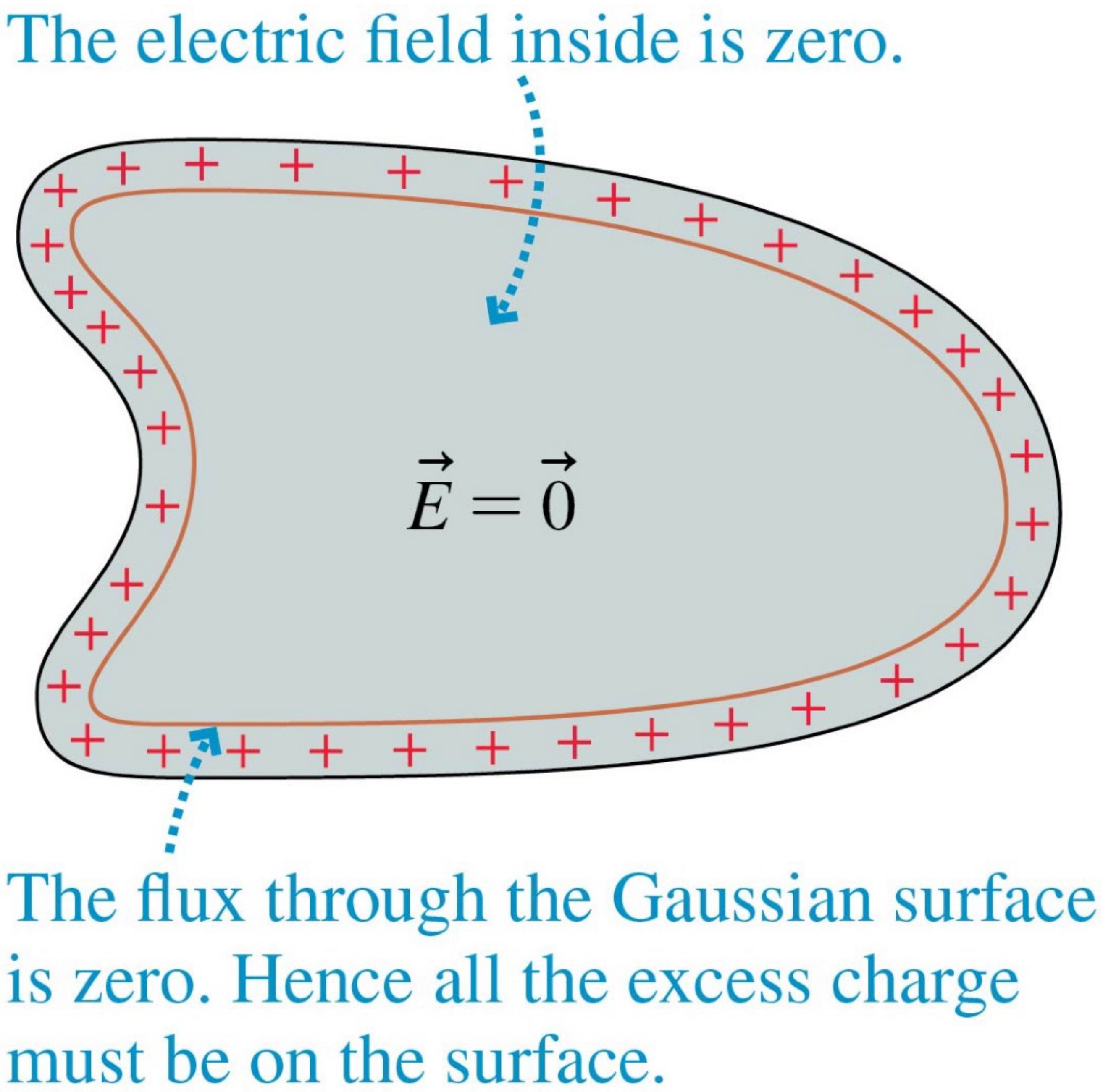
E) $\Phi_d = \Phi_e > \Phi_c > \Phi_a = \Phi_b$.

Gauss' Law

- Gauss' Law affords us the opportunity to solve very complex situations by exploiting symmetry.
- Before Gauss' Law, we had a hard time solving situations that had distributions of charges.
- After Gauss' Law, we can examine a situation for symmetry and then fully describe the electric field for that situation.
- Nowhere is this more evident, then when we try and describe real world applications, as with charged conductors.

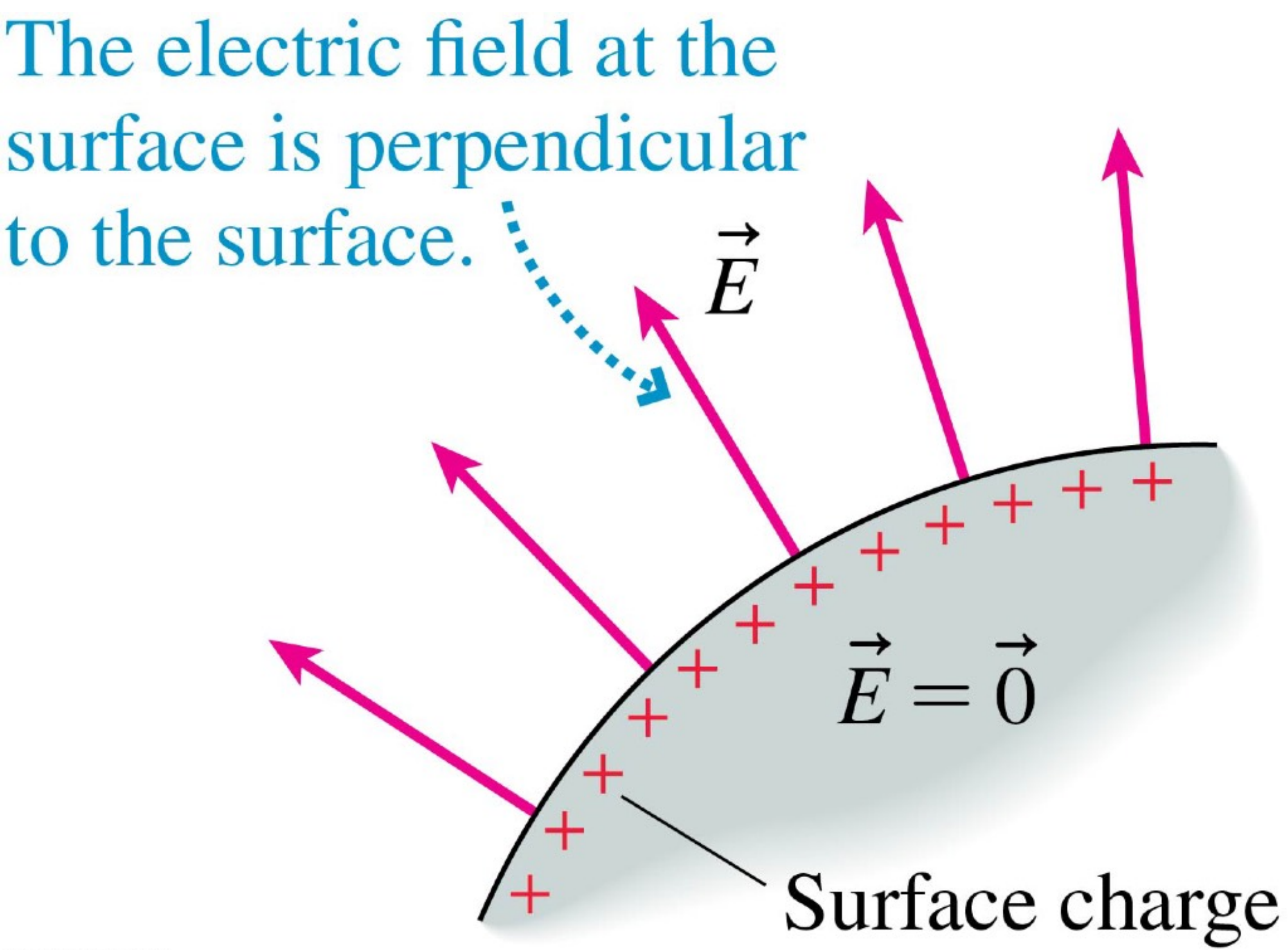
Conductors in Electrostatic Equilibrium

- The figure shows a Gaussian surface just inside a conductor’s surface.
- The electric field must be zero at all points within the conductor, or else the electric field would cause the charge carriers to move and it wouldn’t be in equilibrium.
- By Gauss’s Law, $Q_{\text{in}} = 0$

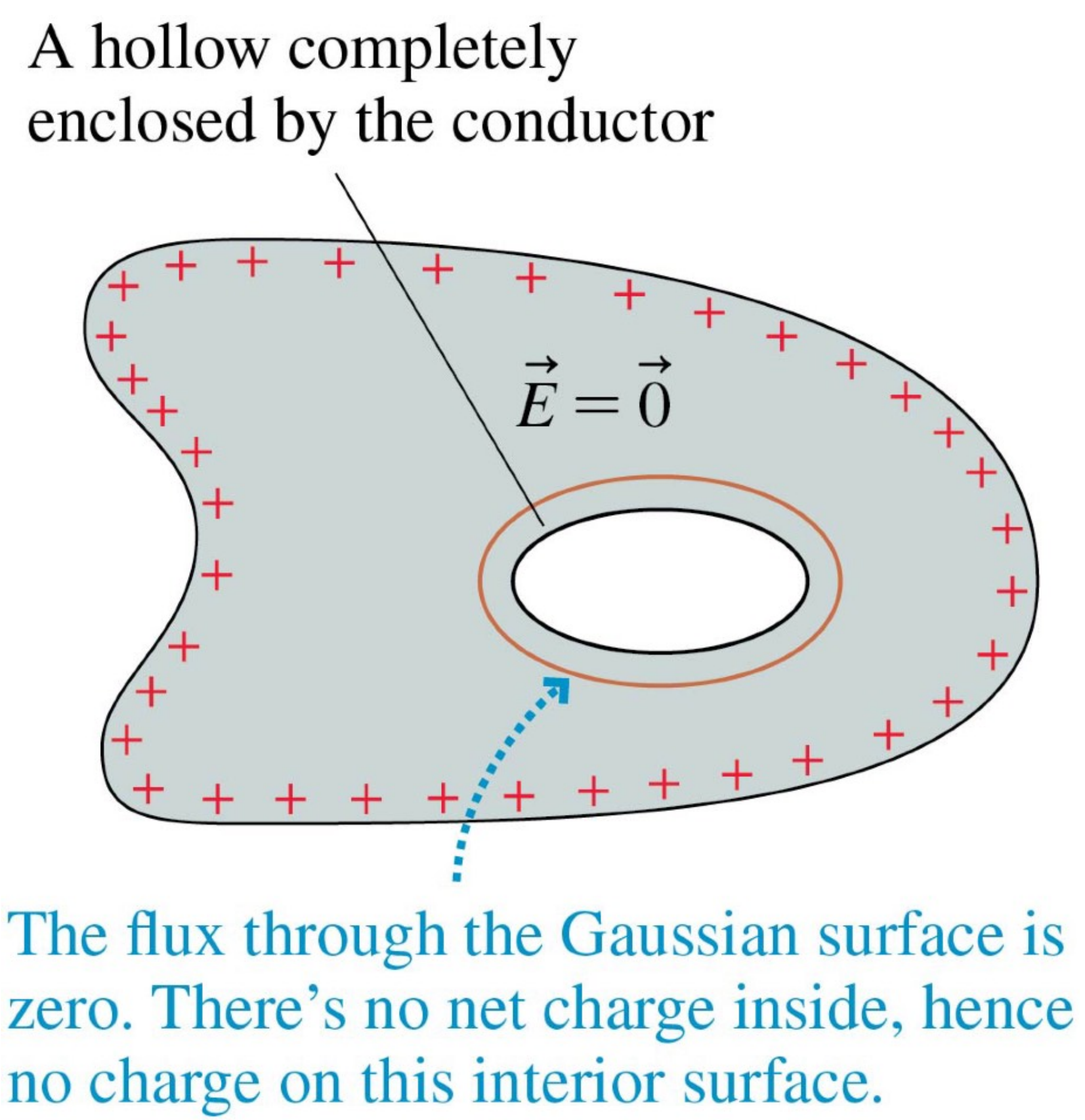


Conductors in Electrostatic Equilibrium

- **The external electric field right at the surface of a conductor must be perpendicular to that surface.**
- If it were to have a tangentia component,
  it would exert a force on the surface charges and cause a surface current, and the conductor would not be in electrostatic equilibrium.

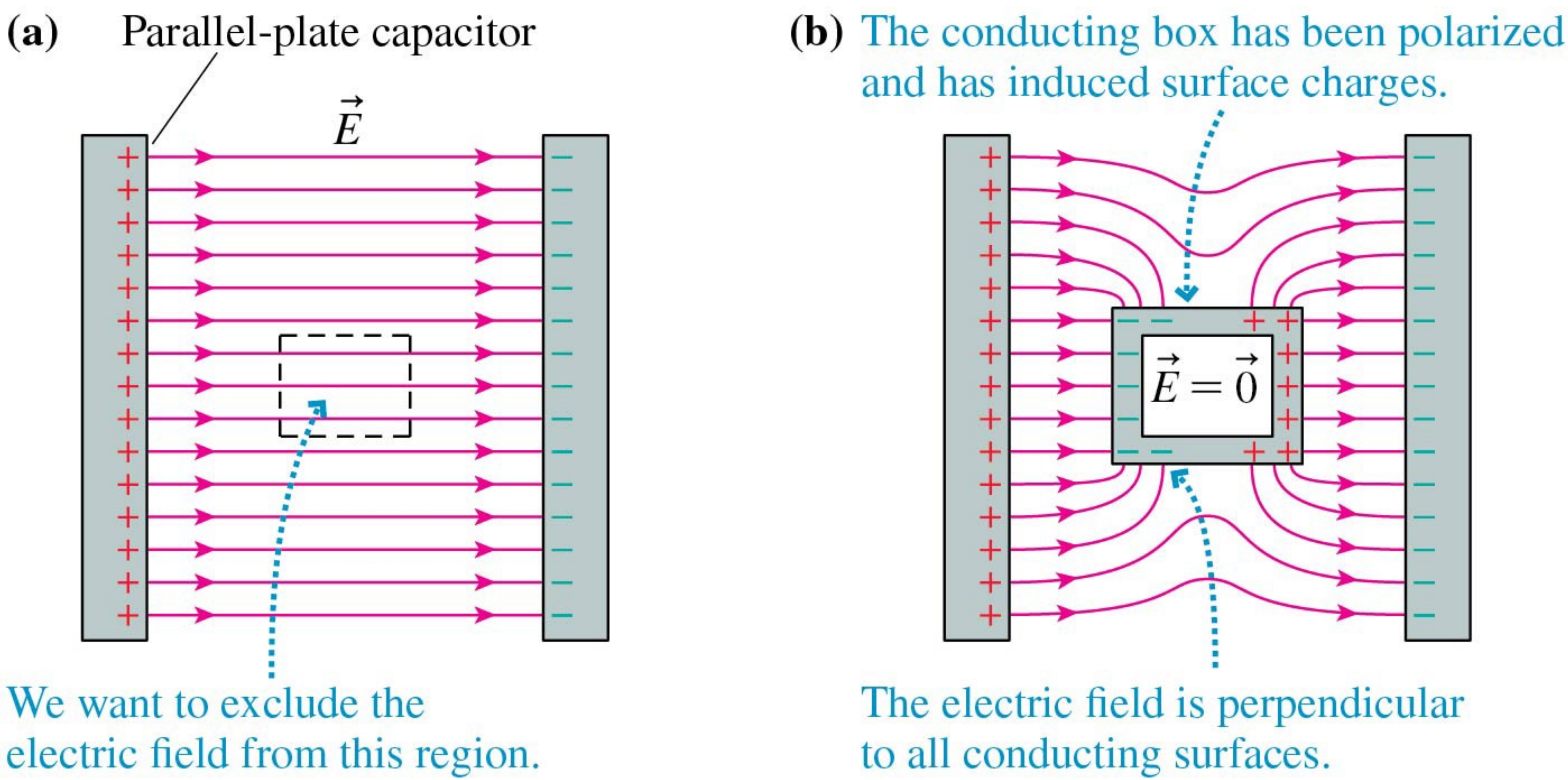


- The figure shows a charged conductor with a hole inside.
- Since the electric field is zero inside the conductor, we must conclude that $Q_{\text{in}} = 0$ for the interior surface.
- Furthermore, since there's no electric field inside the conductor and no charge inside the hole, the electric field in the hole must be zero.



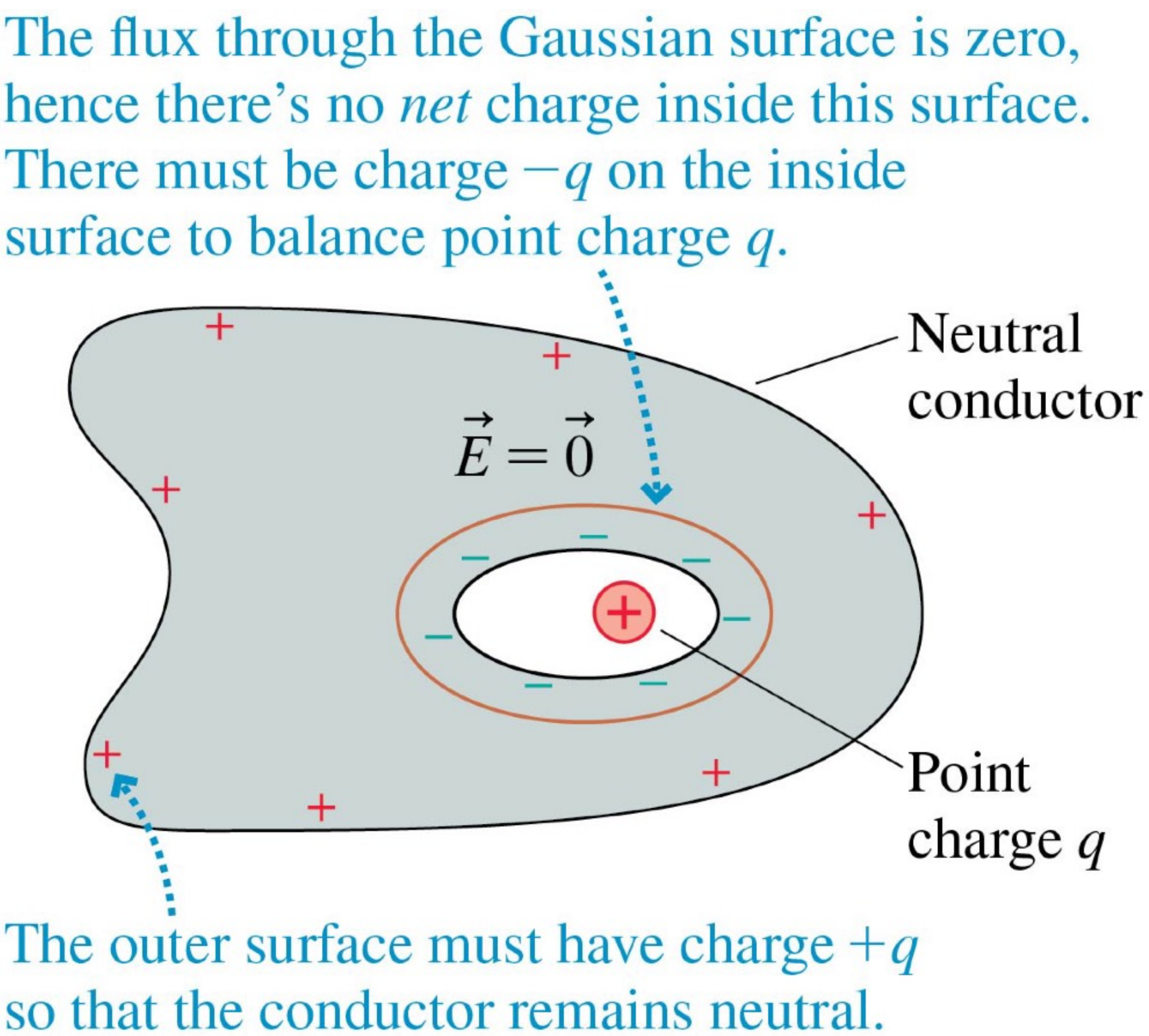
Faraday Cages

- The use of a conducting box, or *Faraday cage*, to exclude electric fields from a region of space is called **screening**.



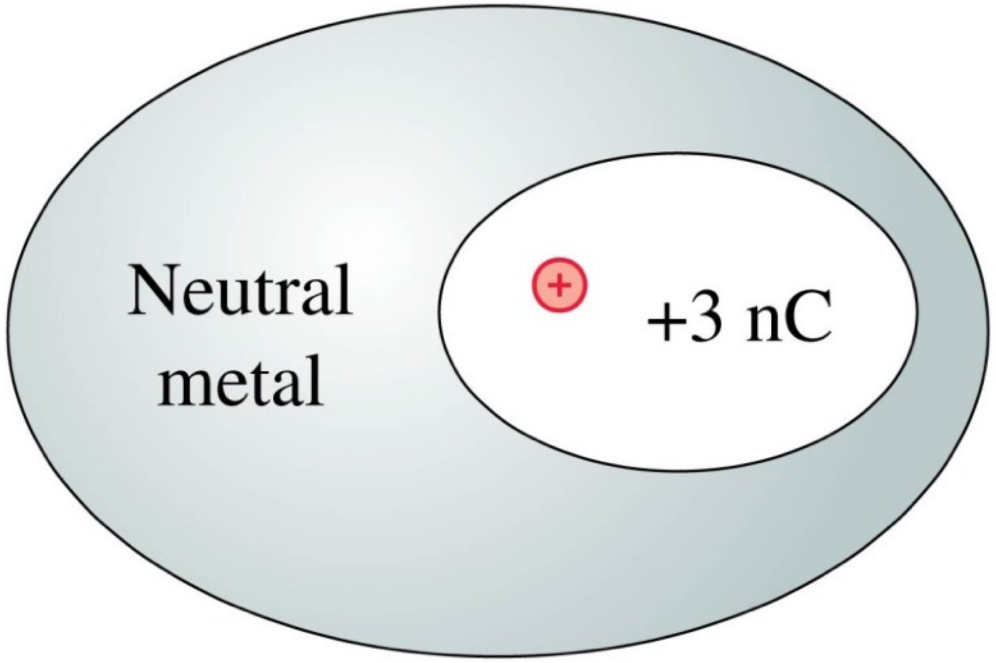
Conductors in Electrostatic Equilibrium

- The figure shows a charge q inside a hole within a neutral conductor.
- Net charge $-q$ moves to the inner surface and net charge $+q$ is left behind on the exterior surface.



iClicker question 5-5

Charge +3 nC is in a hollow cavity inside a large chunk of metal that is electrically neutral. The total charge on the exterior surface of the metal is



A. 0 nC

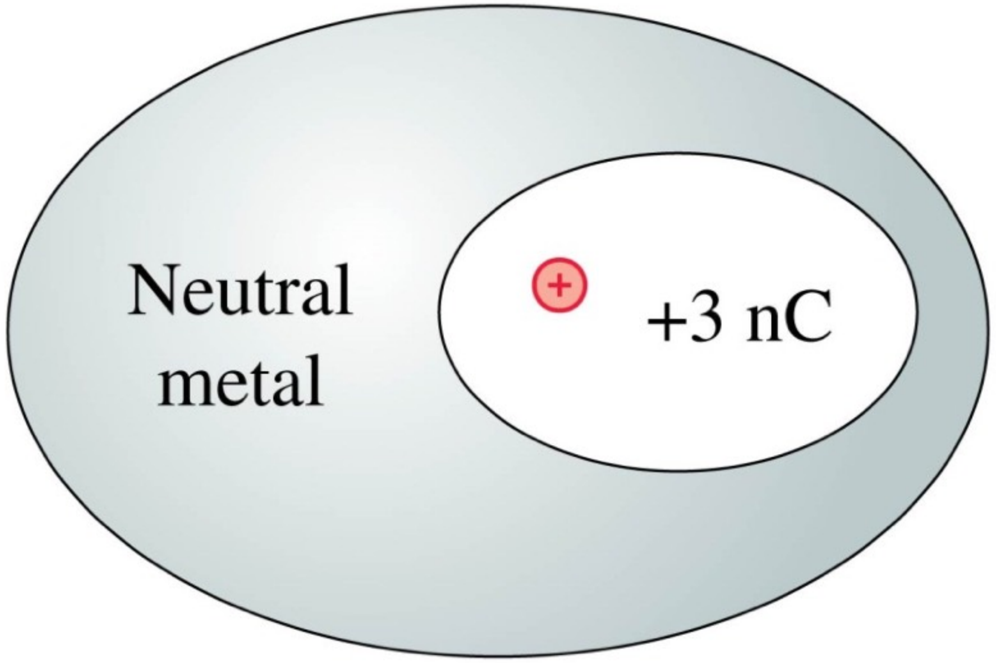
B. +3 nC

C. –3 nC

D. Can’t say without knowing the shape and location of the hollow cavity.

iClicker question 5-5

Charge $+3\text{ nC}$ is in a hollow cavity inside a large chunk of metal that is electrically neutral. The total charge on the exterior surface of the metal is



A. 0 nC

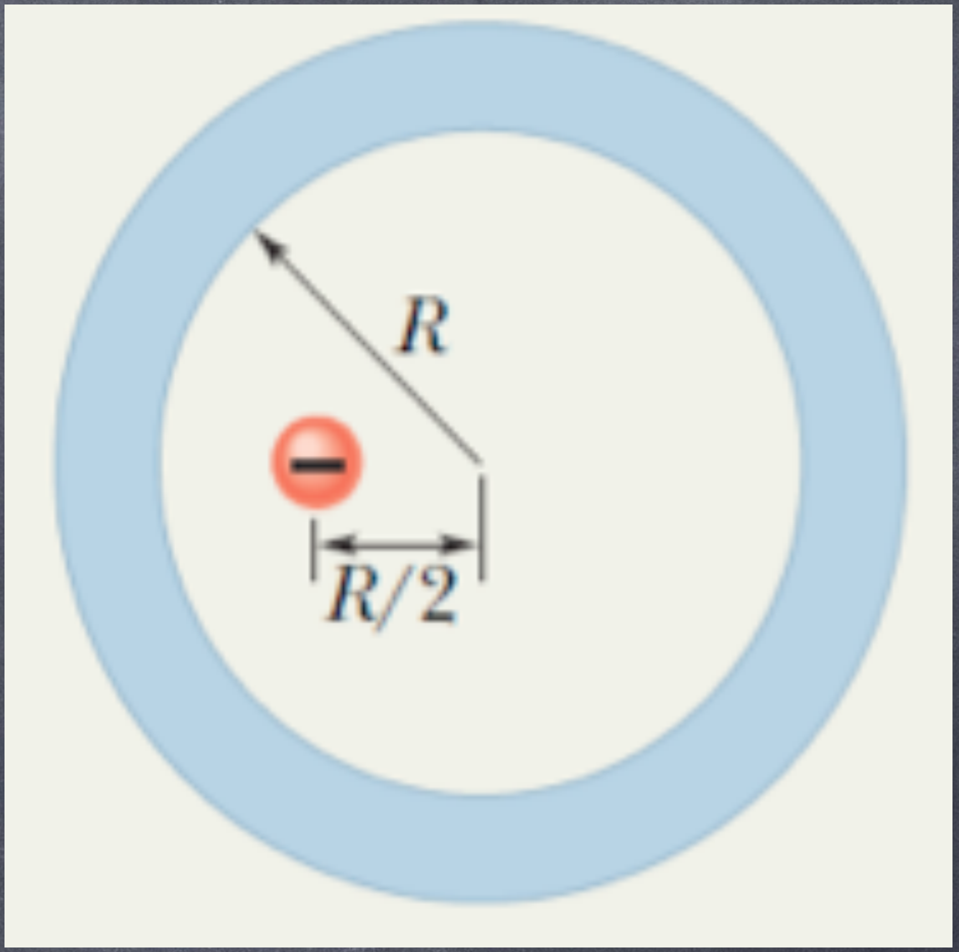
✔ **B. $+3\text{ nC}$**

C. -3 nC

D. Can't say without knowing the shape and location of the hollow cavity.

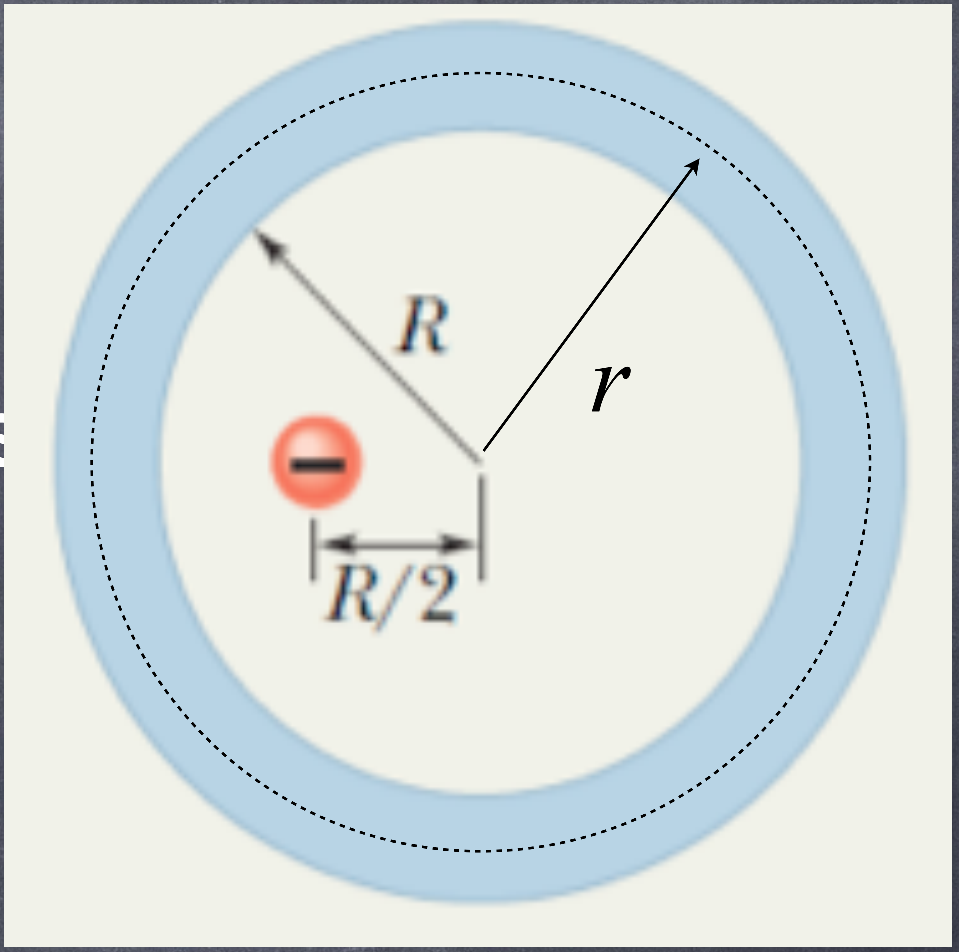
Gauss' Law

- <u>Example</u>
- The figure to the right shows a cross section of an electrically neutral spherical conducting shell of inner radius R. A point charge of -5.0μC is located at a distance R/2 from the center of the shell. What are the induced charges on the outer and inner surfaces? Are those charges uniformly distributed on the surface?



Gauss' Law

- <u>Answer</u>
- Choose an appropriate Gaussian surface.
- Take a spherical surface with radius r just slightly greater than R (so it is inside the conductor).



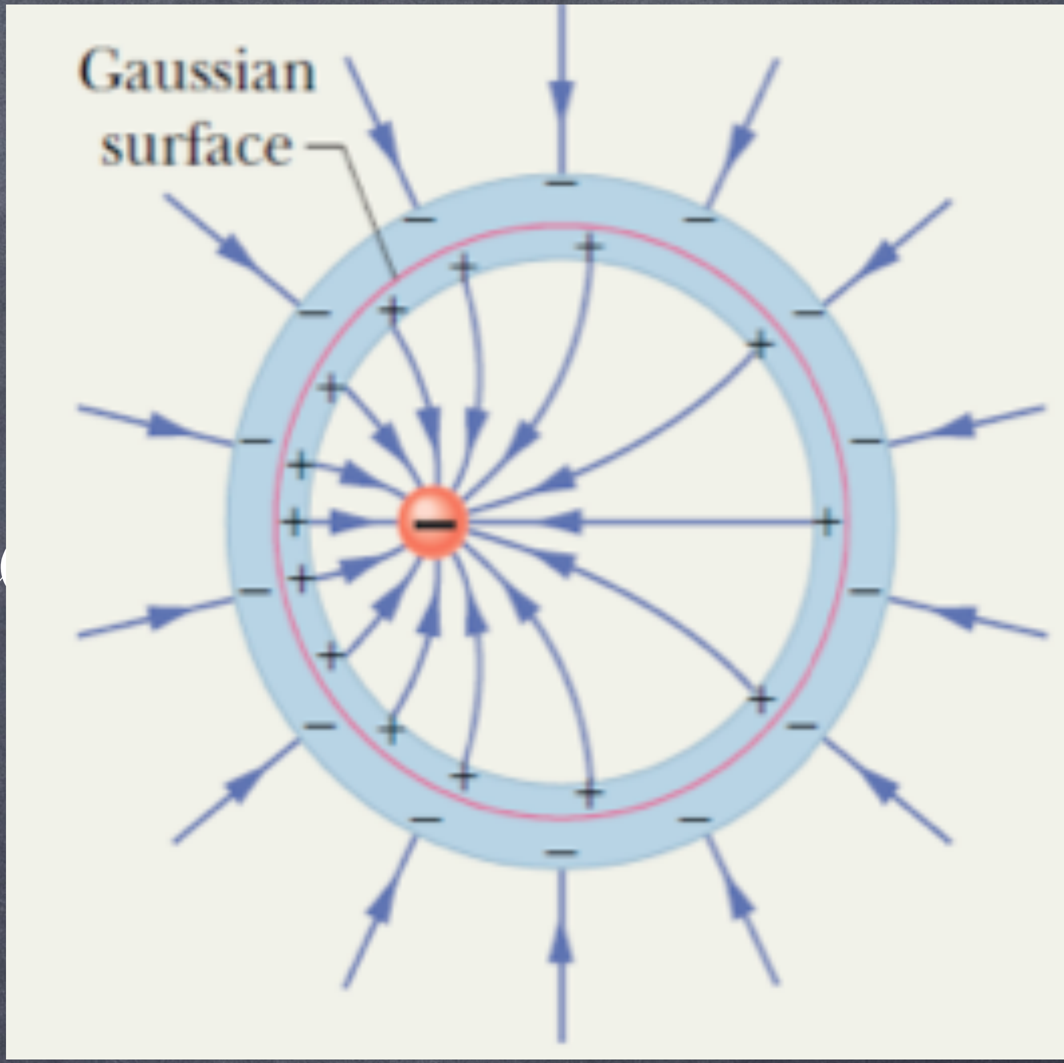
- The electric field inside any conductor is zero (assume equilibrium).
- Thus, by Gauss' Law.

$$\Phi_E = EA\cos\theta = (0)A\cos\theta = 0$$

$$\Phi_E = 0 = \frac{q_{encl}}{\varepsilon_o}$$

Gauss' Law

- Answer
- Since the amount of charge enclosed within the Gaussian surface is zero, this means that $+5.0\mu C$ must be on the inner surface (to cancel out the $-5.0\mu C$ point charge).

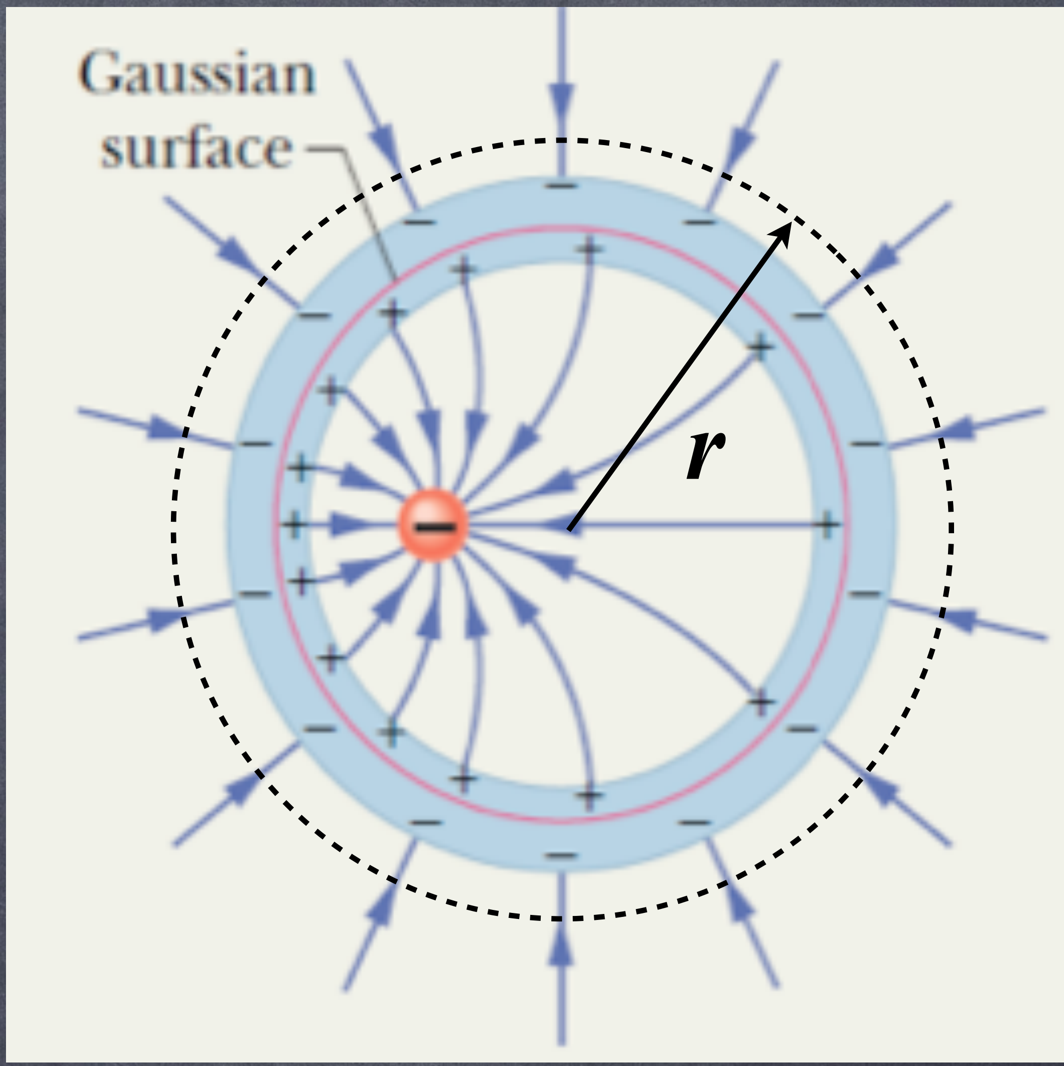


- And, since the conducting shell is neutral overall, then $-5.0\mu C$ must be located on the outer surface.
- The inner surface charges will be skewed since the center charge is off-center.

- Answer

Gauss' Law

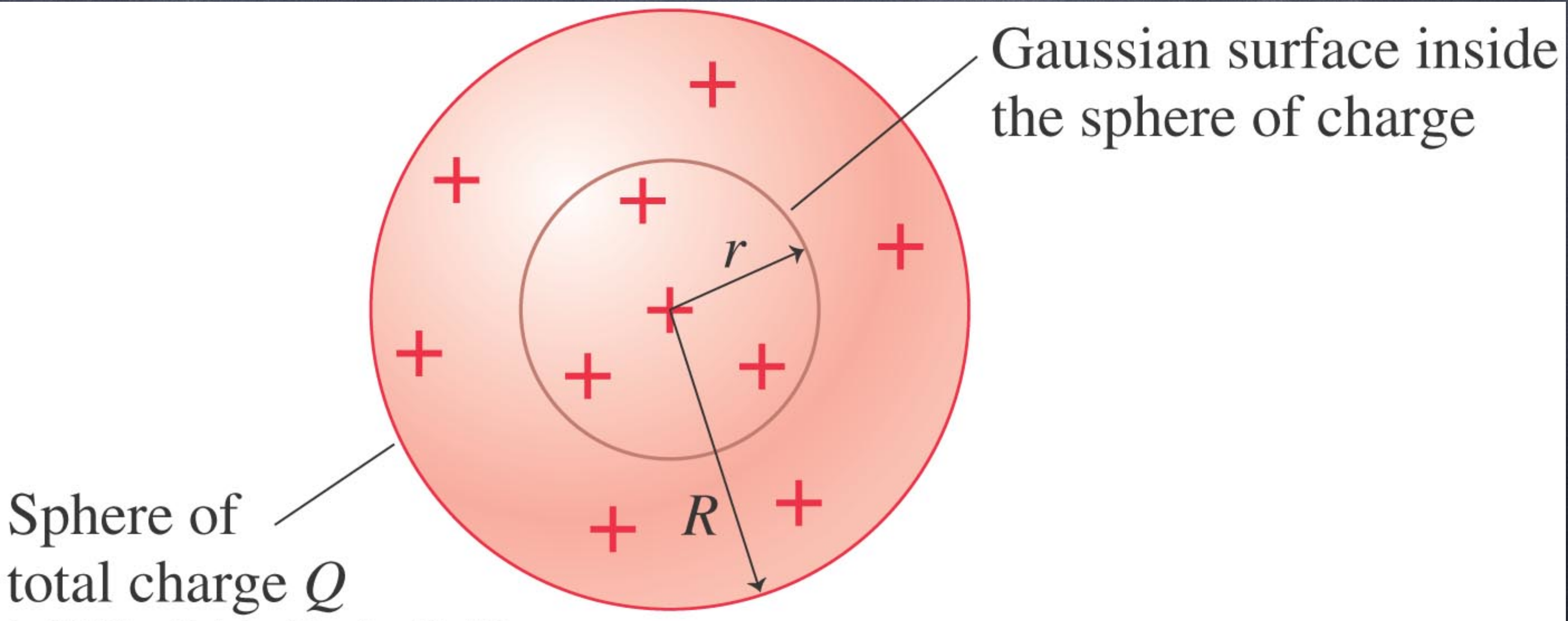
- Yet the outer surface charges will be uniformly distributed.
- This can be confirmed by taking another Gaussian surface with a larger radius, r.



- The charge enclosed by this larger Gaussian surface encloses -5.0μC of charge (the point charge).
- Note the zero electric field due to the inner charges do not affect anything on the outside of the conducting shell.

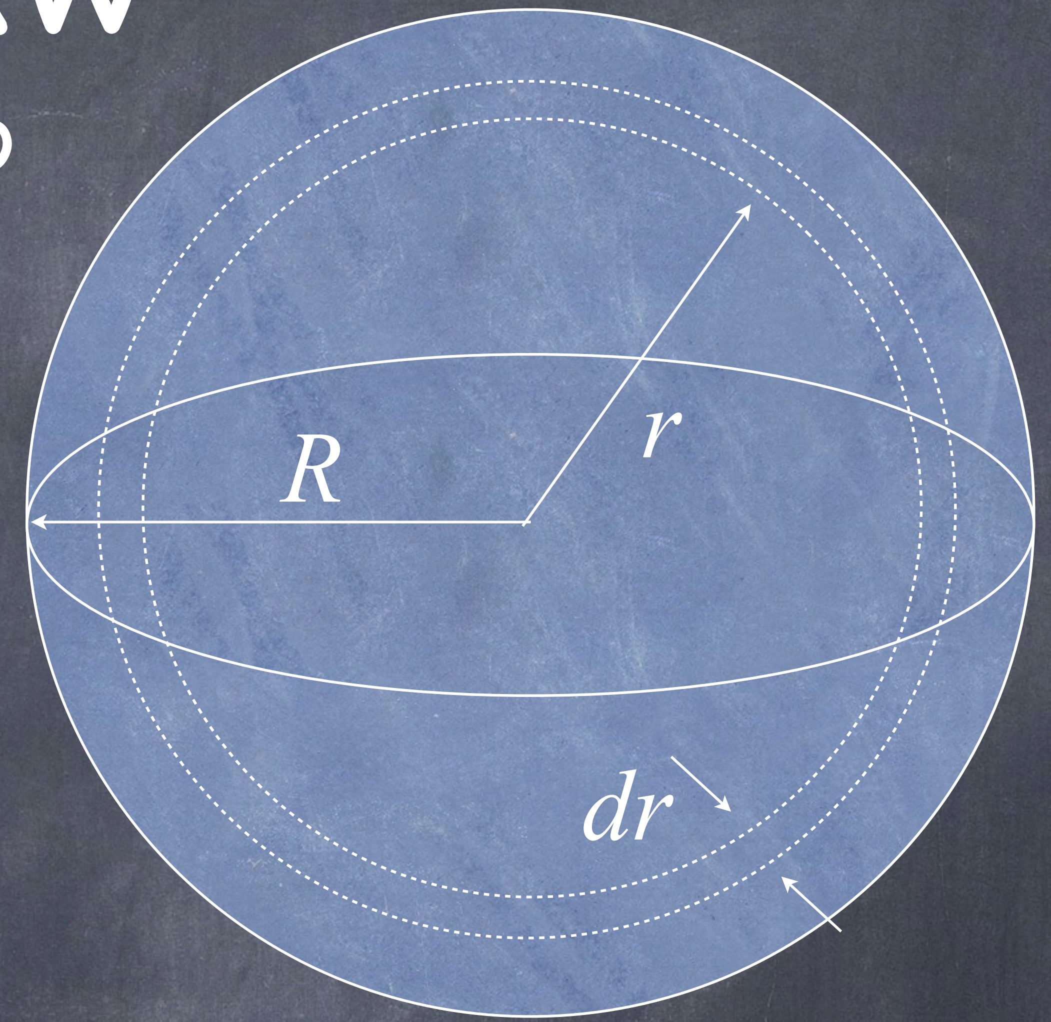
Gauss' Law

- Example
- Suppose the charge density of the solid sphere of radius R (in the figure below) is given by $\rho = \alpha r^2$ where α is a constant. (a) Find α in terms of the total charge Q on the sphere and its radius R. (b) Find the electric field as a function of r inside the sphere.



Gauss' Law

- Answer
- We can divide the sphere up into concentric thin shells of thickness, dr, and volume dV.
- In order to get the total charge Q integrate over all the thin shells from 0 to R.
- Note that every shell will have a volume dV:



$$dV = 4\pi r^2 dr$$

$$Q = \int \rho \, dV$$

$$Q = \int_0^R \left(\alpha r^2\right) 4\pi r^2 dr$$

$$Q = 4\pi\alpha \int_0^R r^4 dr$$

$$Q = 4\pi\alpha \frac{R^5}{5}$$

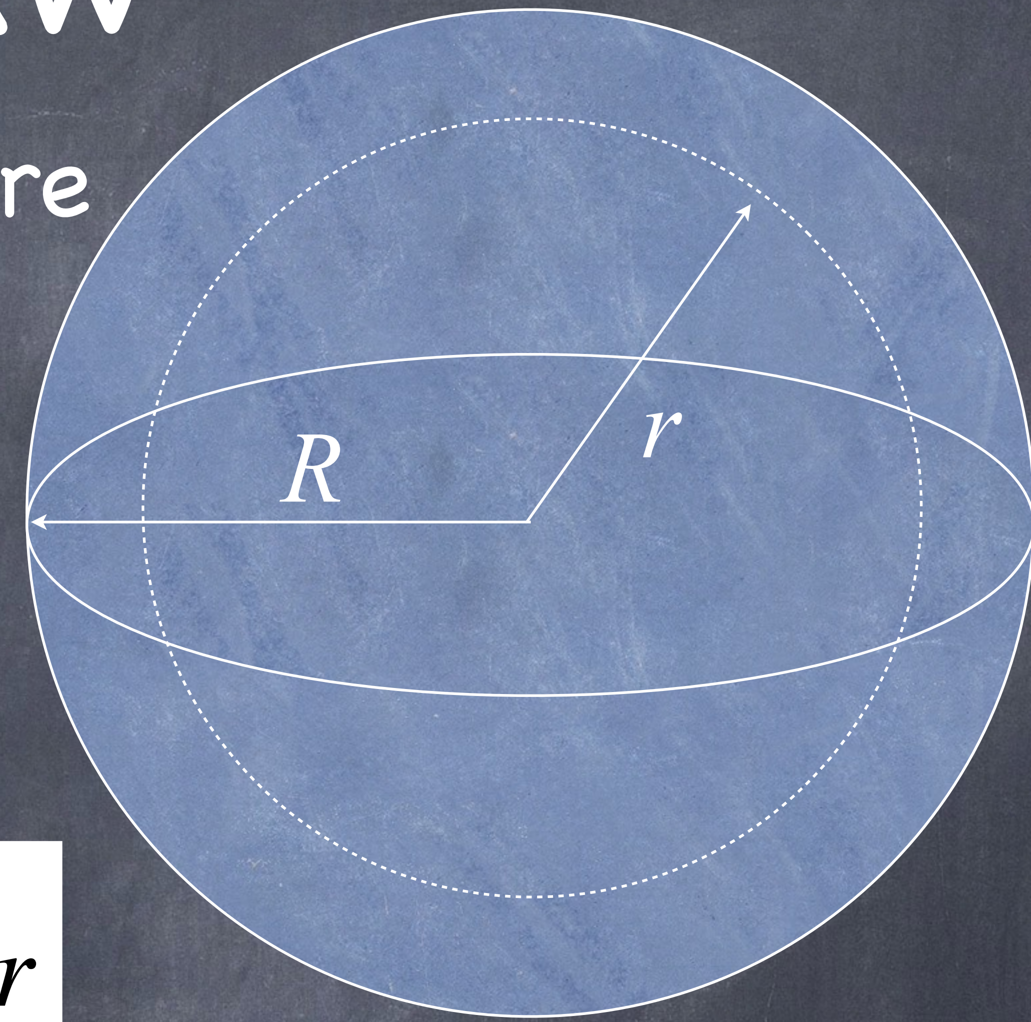
$$\alpha = \frac{5Q}{4\pi R^5}$$

Gauss' Law

- Answer

- For (b) we need a Gaussian sphere of radius r (r < R).

- We need to find how much charge we are enclosing with our Gaussian sphere.



$$Q_{encl} = \int_0^r \rho \, dV$$

$$Q_{encl} = \int_0^r \left(\alpha r^2\right) 4\pi r^2 \, dr$$

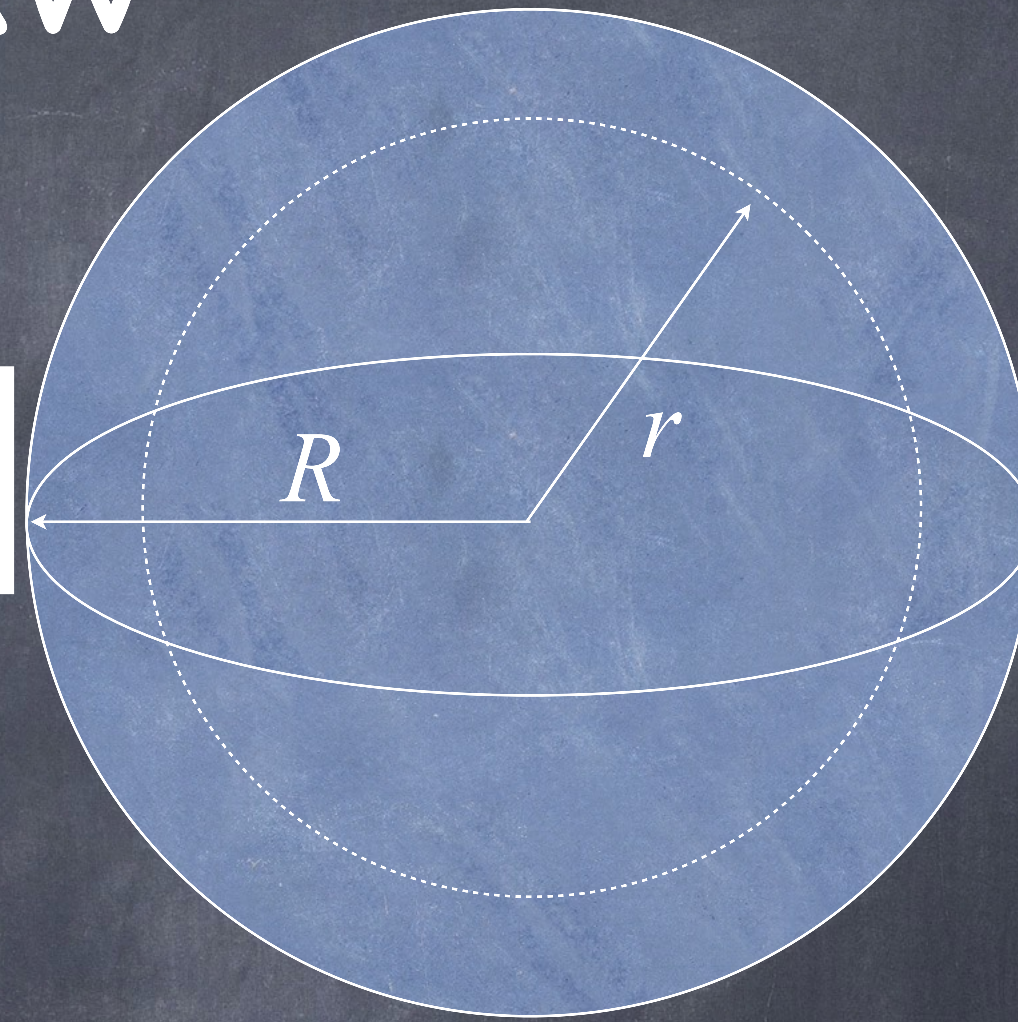
$$Q_{encl} = \int_0^r \left(\frac{5Q}{4\pi R^5} r^2\right) 4\pi r^2 \, dr$$

$$Q_{encl} = \frac{5Q}{R^5} \int_0^r r^4 \, dr$$

$$Q_{encl} = \frac{5Q}{R^5} \frac{r^5}{5} = Q \frac{r^5}{R^5}$$

Gauss' Law

- Answer
- Next to find the E field, take the surface integral:



$$\Phi_E = \oint \vec{E} \cdot d\vec{A}$$

$$\oint dA = 4\pi r^2$$

$$\Phi_E = E\left(4\pi r^2\right)$$

- Equating the two sides of Gauss' Law:

$$\Phi_E = \Phi_E$$

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon_o}$$

$$E\left(4\pi r^2\right) = \frac{Q}{\varepsilon_o}\frac{r^5}{R^5}$$

$$E = \frac{1}{4\pi r^2}\frac{Qr^5}{\varepsilon_o R^5}$$

$$E = \frac{1}{4\pi\varepsilon_o}\frac{Qr^3}{R^5}$$