

Triangle inequality

Proposition: Triangle inequality. Let $v, w \in \mathbb{R}^n$

Then, $\|v + w\| \leq \|v\| + \|w\|$

Proof:

$$\begin{aligned} & \|v + w\|^2 - \langle v + w, v + w \rangle \\ &= \langle v, v + w \rangle + \langle w, v + w \rangle \\ &= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle \\ &\leq \|v\|^2 + 2|\langle v, w \rangle| + \|w\|^2, \text{ where with the Cauchy Schwarz Inequality, we can} \\ &\text{rewrite this as:} \end{aligned}$$

$$\begin{aligned} &\leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 \\ &= (\|v\| + \|w\|)^2 \\ &\implies \|v + w\|^2 \leq (\|v\| + \|w\|)^2 \\ &\implies \|v + w\| \leq (\|v\| + \|w\|) \end{aligned}$$