Finite Difference Numerical Solutions

Note Title 11/16/2007

Many important problems in engineering & Science involve more than 1 dimension. While the numerical methods you have learned with predictor corrector methods, these are somewhat limited when it comes to multi-dimensional and/or transient system. For these, it is most efficient to use finite difference methods.

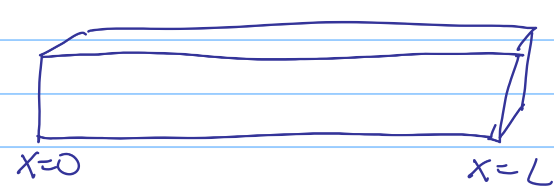
Example 1D, transient system, heat transfer

$$\frac{\partial^2 T}{\partial x^2} = \alpha \frac{\partial T}{\partial t} \qquad t<0 \quad T(x) = T_0 \text{ for all } x$$

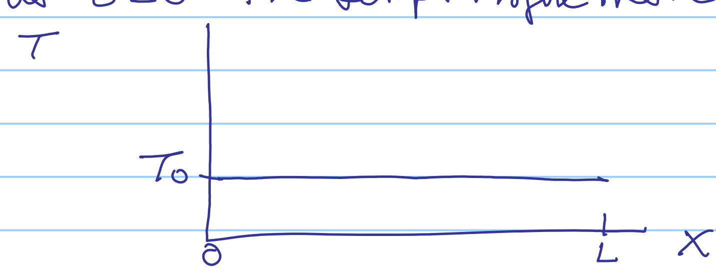
$$t \geq 0 \quad T(0) = T_L$$

$$\alpha = \text{constant} = \frac{\rho C_p}{k} \qquad T(L) = T_0$$

This equation describes the conductive ($q = -k\frac{dT}{dx}$) transfer of heat in 1D as a function of time! Initially the system is at a uniform temperature T_0, at all positions of X. At $t=0$, the temperature at $X=0$ is suddenly raised to T_L.
The temperature inside the block changes as energy enters and is conducted in the x direction



at $t=0$ the temp. Profile in the block is



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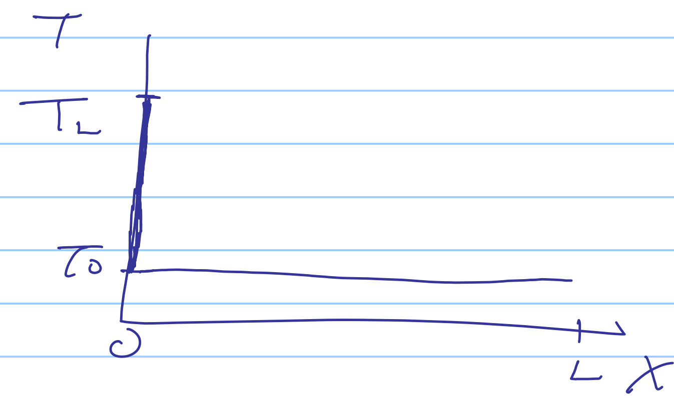
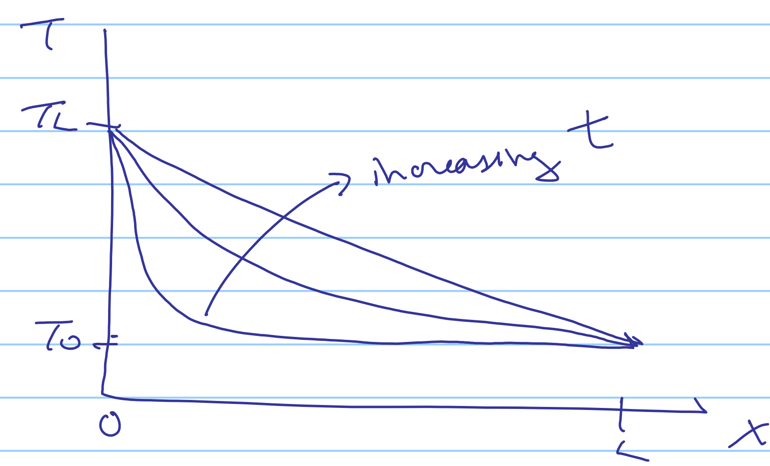
for $t>0$, the temperature at $x=0$ is suddenly raised to T_L, so the profile would look like

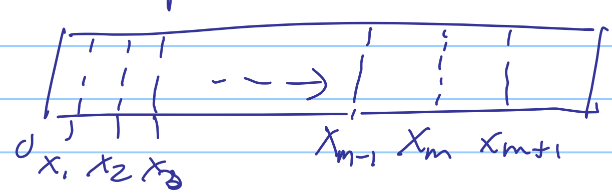


As time advances, the thermal wave would move down the length of the block, so the thermal profile would change with time



Our task is to create a quantitative way to solve for $T(x,t)$.

First, divide the length of the block into small steps in x



where m is the mth position in the x direction.

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$\frac{dT}{dx}$ can be approximated as

$$\frac{dT}{dx} \simeq \frac{T_{m+1}^{p} - T_m^{p}}{\Delta x} \qquad \Delta x = (x_{m+1} - x_m)$$

where p refers to time, since $T(x,t)$.

Similarly, the 2nd derivative with respect to x would be

$$\frac{d}{dx}\left(\frac{dT}{dx}\right) \simeq \frac{d}{dx}\left(\frac{T_{m+1}^{p} - T_m^{p}}{\Delta x}\right)$$

$$= \frac{\left(\frac{T_{m+1}^{p} - T_m^{p}}{\Delta x}\right) \left(\frac{T_m^{p} - T_m^{p}}{\Delta x}\right)}{\Delta x}$$

$$= \frac{T_{m+1}^{p} - 2T_m^{p} + T_{m-1}^{p}}{(\Delta x)^2}$$

For the time derivative

$$\frac{dT}{dt} \simeq \frac{T_m^{p+1} - T_m^{p}}{\Delta t}$$

Note that Δx is the position increment size and Δt is the time increment size.

Putting these together,

$$\frac{d^2T}{dx^2} = \alpha \frac{dT}{dt}$$

$$\frac{T_{m+1}^{p} - 2T_m^{p} + T_{m-1}^{p}}{(\Delta x)^2} = \alpha \frac{T_m^{p+1} - T_m^{p}}{\Delta t}$$

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$$\frac{T_{m+1}^{P} - 2T_{m}^{P} + T_{m-1}^{P}}{(\Delta x)^2} = \alpha \frac{T_m^{P+1} - T_m^{P}}{\Delta t}$$

Re-arranging

$$T_m^{P+1} = \frac{\Delta t}{\alpha \Delta x^2}\left[T_{m+1}^{P} + T_{m-1}^{P}\right] + \left[1 - \frac{2\Delta t}{\alpha \Delta x^2}\right] T_m^{P}$$

now, use this eqn along with initial conditions and step sizes to iterate in position and time.

Example calculations

$\Delta x = 0.1$ $\Delta t = 0.1$

$\alpha = 50$

$L = 1$ $0 \leq t \leq 4$

$T_0 = 350$ $T_L = 440$

t \ x	0	0.1	0.2	0.3	- - -
0	440	440	440	440	
0.1	350	368			
0.2	350				
0.3	350				
⋮					

$$T_{0.1}^{0.1} = \frac{0.1}{(50)(0.1)^2}\left[440 + 350\right] + \left[1 - \frac{2(0.1)}{(50)(0.1)^2}\right] 350$$

$$= 368$$

Several calculation values are shown below, along with a graph of T vs x at various times.

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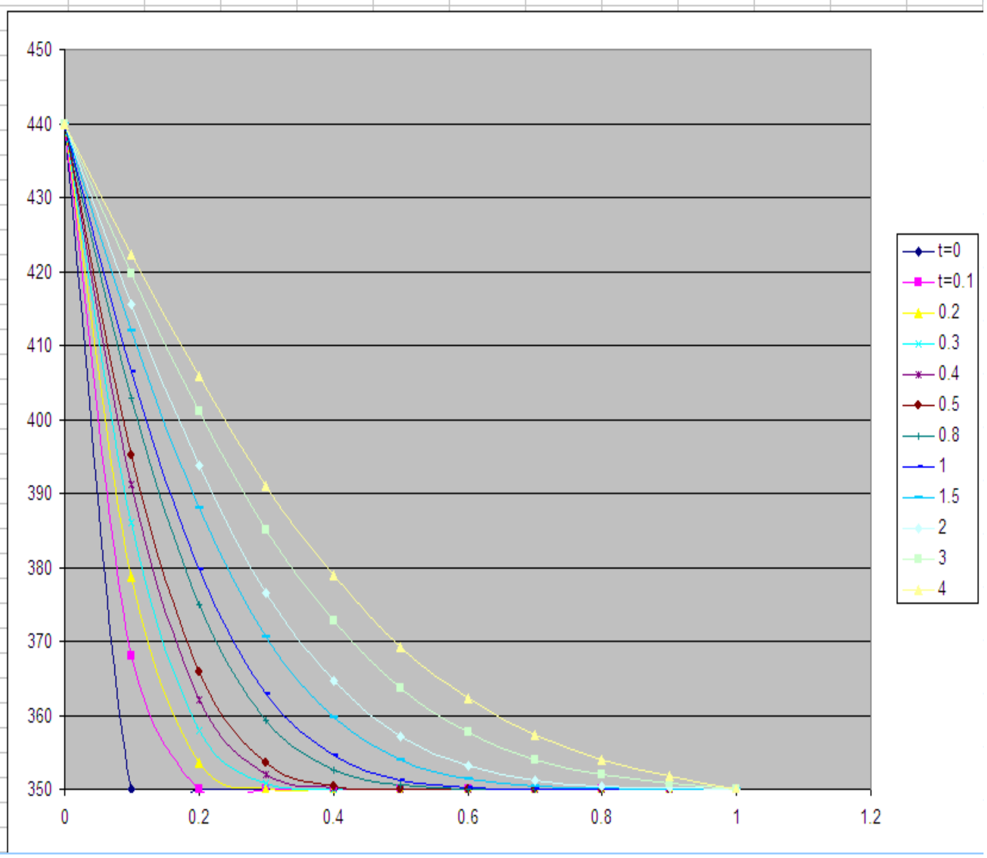
	t					
x	0	0.1	0.2	0.3	0.4	0.5
0	440	440	440	440	440	440
0.1	350	368	378.8	386	391.184	395.1296
0.2	350	350	353.6	357.92	362.096	365.8976
0.3	350	350	350	350.72	352.016	353.6576
0.4	350	350	350	350	350.144	350.4896
0.5	350	350	350	350	350	350.0288
0.6	350	350	350	350	350	350
0.7	350	350	350	350	350	350
0.8	350	350	350	350	350	350
0.9	350	350	350	350	350	350
1	350	350	350	350	350	350



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