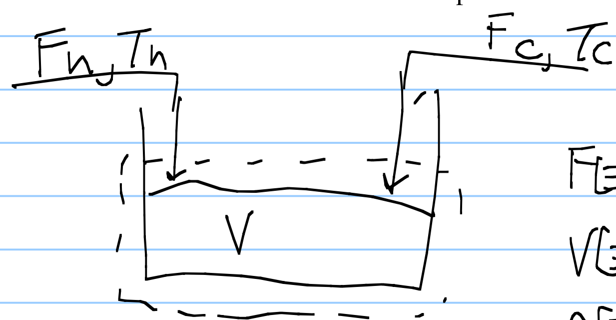
Note Title 9/11/2007

Modeling Homework
Derive the differential equation model for the given physical situation.
If you like, you are welcome to try to solve it, but you need not do so.

1. A bath tub is being filled using a cold water stream (Fc, Tc) and a hot water stream (Fh, Th). Fi is the mass flowrate of stream i and Ti is the temperature of stream i, both assumed to be constant. Assuming density ρ is constant, heat capacity, Cp is constant, and enthalpy (H) is conserved, develop a model for the volume of water in the tub as a function of time and the temperature in the tub as a function of time.



$F [\equiv] L^3/t$

$V [\equiv] L^3$

$\rho [\equiv] m/L^3$

$H [\equiv] E/L^3$

$C_p [\equiv] \frac{E}{m \cdot T}$

Mass balance

$$IN - \cancel{out}^{\,0} = ACC$$

$$\cancel{\rho} F_h + \cancel{\rho} F_c = \frac{d}{dt}(\cancel{\rho} V)_{sys}$$

$$\int (F_h + F_c)\, dt = \int d(V)_{sys}$$

$$(F_h + F_c)t = V(t) - V(0)$$

Energy balance $\qquad H = \rho C_p T$

$$In - \cancel{out}^{\,0} = ACC$$

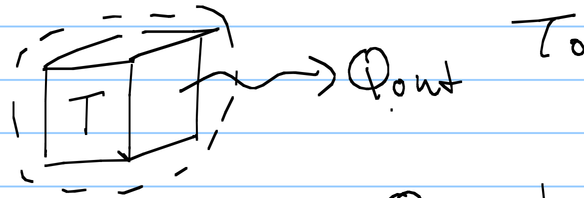
$$\cancel{\rho} F_h \cancel{C_p} T_h + \cancel{\rho} F_c \cancel{C_p} T_c = \frac{d}{dt}(V \cancel{\rho} \cancel{C_p} T)_{sys}$$

$$F_h T_h + F_c T_c = \frac{d}{dt}(VT)_{sys}$$

$$\frac{(F_h T_h + F_c T_c)t}{(F_h + F_c)t + V(0)} = T(t)$$

$$\frac{(\cancel{L^3/t}\, T)\cancel{t}}{(\cancel{L^3/t})\cancel{t} + \cancel{L^3}} = T \quad \checkmark$$

2. A cubic block of iron is being heated at a constant rate, Qin.. At the same time, the block is losing heat to the environment at a rate Qout. The environment is assumed to be at a constant temperature of To. The heat capacity and density of the iron are assumed to be constant. Derive a model of the temperature of the iron block as a function of time. (Note: for those unfamiliar with heat transfer, Qout = hA(T-To), where h is the Newton's law heat transfer coefficient {assumed constant} and A is the surface area of the block.)



$$Q_{out} = hA(T - T_o)$$

Energy balance

$$H_{sys} = \rho V C_p T$$

$$\cancel{in}^{0} - out = acc$$

$$Q_{out} = \frac{d}{dt}(H_{sys})$$

$$-hA(T - T_o) = \frac{d}{dt}(\rho V C_p T)$$

$$\int \frac{-hA}{\rho V C_p}\, dt = \int \frac{d(T)}{(T - T_o)}$$

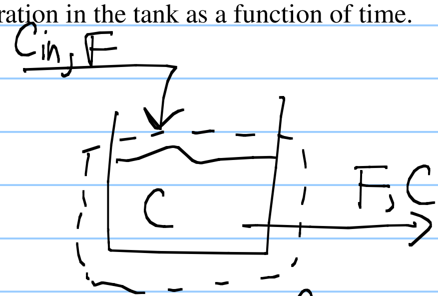
$$\frac{-hAt}{\rho V C_p} = \ln \frac{(T - T_o)}{(T(0) - T_o)}$$

$$T = (T(0) - T_o)\, e^{\frac{-hAt}{\rho V C_p}} + T_o$$

Dimensional check $\quad h [=] \frac{E}{L^2\, T\, t}$

$$\frac{hAt}{\rho V C_p} [=] \frac{\left(\frac{\cancel{E}}{\cancel{L^2}\,\cancel{T}\,\cancel{t}}\right)(\cancel{L^2})(\cancel{t})}{\left(\frac{\cancel{M}}{\cancel{L^3}}\right)(\cancel{L^3})\left(\frac{\cancel{E}}{\cancel{M}\,\cancel{T}}\right)} \quad \checkmark$$

3. A perfectly mixed reactor (homogeneous contents) has a flow of reactant A entering at a constant volumetric flowrate of F and constant composition, Cin. The volume of the reactor fluid is constant (V), so the outlet volumetric flow rate is also F. Since the reactor has a concentration of C, the outlet flow also has a concentration of C. Assuming a reaction occurs in the reactor with rate r = - kC, develop a differential equation model for the concentration in the tank as a function of time.



$r = -kC$

$[\equiv] \dfrac{\text{mass}}{t - L^3}$

Component mass balance on A

IN − out − depl. = acc

$$C_{in}F - CF - kCV = \frac{d}{dt}(CV)_{sys}$$

Divide by V

$$C_{in}\frac{F}{V} - C\frac{F}{V} - kC = \frac{dC_{sys}}{dt}$$

$$C_{in}\frac{F}{V} - \left[\frac{F}{V} + k\right]C = \frac{dC_{sys}}{dt}$$

$$\underbrace{\left\{\frac{C_{in}\left(\frac{F}{V}\right)}{\left[\frac{F}{V} + k\right]}\right\}}_{\Rightarrow \alpha} - C = \frac{dC}{dt}\,\underbrace{\frac{1}{\left(\frac{F}{V} + k\right)}}_{\Rightarrow \beta}$$

$$\alpha - C = \frac{dC}{dt}\frac{1}{\beta}$$

$$\int_0^t \beta\, dt = \int_{C(0)}^{C} \frac{dC}{\alpha - C}$$

$$\beta t = -\ln\frac{\alpha - C}{\alpha - C(0)}$$

$$(\alpha - C(0))e^{-\beta t} = \alpha - C$$

$$(\alpha - C(0))e^{-\beta t} = \alpha - C$$

$$C(t) = \alpha - (\alpha - C(0))e^{-\beta t}$$

$$= \frac{C_{in} F/V}{(F/V + k)} - \left[\frac{C_{in} F/V}{(F/V + k)} - C(0)\right] e^{-(F/V + k)t}$$

Dimensional check

$$\frac{\frac{m}{\cancel{L^3}} \frac{\cancel{L^3}}{\cancel{t} \cdot L^3}}{\left(\frac{\cancel{L^3}/\cancel{t}}{\cancel{L^3}}\right) + 1/\cancel{t}} \;[=]\; \frac{m}{L^3} \quad \checkmark$$