Derive the requested differential equation models for a baking cookie as a function of time (t) and cookie temperature (T) (along with situational parameters such as area, volume, etc., and the assumptions given below). Please read the assumptions before starting your solutions.

a) (4 points) Draw an appropriate picture of the system describing the baking process. Clearly define any parameters used in the model, including their dimensions (so you can check for dimensional consistency in your models).
b) (20 points) Derive the differential equation model for density of the cookie as a function of time, $d\rho/dt$, including the assumptions below.
c) (20 points) Derive the differential equation model for temperature of the cookie as a function of time, dT/dt, including the assumptions below and the density model from part b, as needed.
d) (6 points) Demonstrate that your models are dimensionally consistent.

Assumptions:

1. Assume a cookie is a flat cylinder (radius R, thickness x) which is baked by putting it into a hot oven environment at a constant temperature, Th. The initial density of the cookie is ρo and the cookie's initial temperature is To.

2. During the baking process, the cookie loses water (moisture).

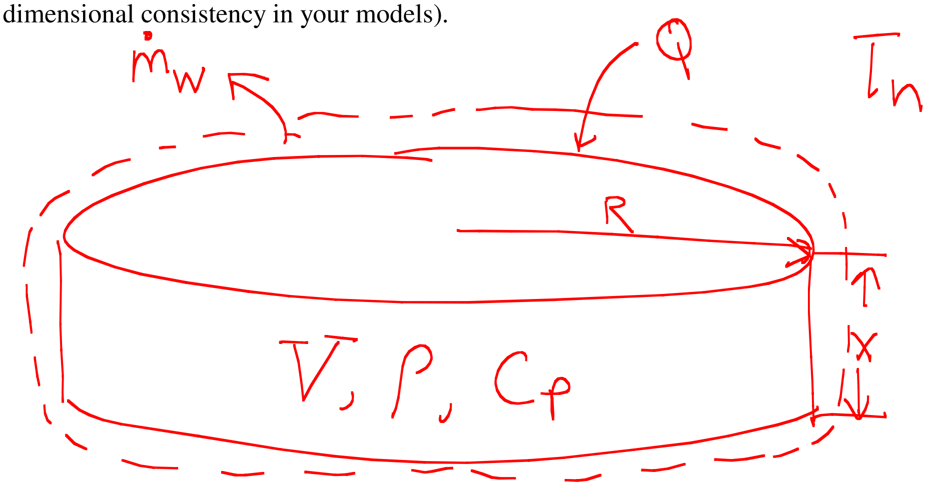
3. Assume the cookie's internal temperature, T, is uniform within the cookie, i.e. the cookie temperature does not vary spatially within the cookie.

4. During the baking process, heat enters the cookie. Assume this heat transfer occurs from the oven environment to the cookie following Newton's law for convective heat transfer, i.e. $Q = h A (T - Th)$, where A is the area of the cookie, h is the heat transfer coefficient, and Q is the rate of heat transfer (energy/time). (Note: You should determine the units of h, so you can to check for dimensional consistency.)

5. You may assume that the cookie volume is constant as it bakes. Note, however, that the density changes since water is lost as the baking process occurs.

6. Since we do not know the details of how water migration and evaporation from the cookie surface occurs, assume that the rate of water loss is proportional to the temperature of the cookie, i.e. αT. (Note: You should determine the dimensions of α so you can check for dimensional

a) (4 points) Draw an appropriate picture of the system describing the baking process. Clearly define any parameters used in the model, including their dimensions (so you can check for dimensional consistency in your models).



$\dot{m}_w$ = rate of water loss [≡] $\frac{m}{L^2 - t}$

Q = rate of heat loss [≡] $\frac{E}{t}$

V = cookie volume [≡] $L^3 = 2R^2X$

A = cookie area = $2\pi R^2 + 2\pi RX$

ρ = cookie density [≡] $\frac{m}{L^3}$

C_p = cookie heat capacity [≡] $\frac{E}{m - T}$
assumed constant

Note:
Assumptions
$U = H$ [≡] $\frac{E}{m}$
$\dot{m}_w = f(T) = \alpha T$

b) (20 points) Derive the differential equation model for density of the cookie as a function of time, dρ/dt, including the assumptions below.

mass balance

$\cancel{In}^{0} - out = acc$

$$-\dot{m}_w = \frac{d}{dt}(\rho V)_{sys}$$

$$-f(T) = V_{sys}\frac{d(\rho)_{sys}}{dt}$$

b) $$\frac{-\alpha T}{V_{sys}} = \frac{d(\rho)_{sys}}{dt}$$

note that $\rho(t) = \rho(0) - \frac{\alpha}{V}\int T\,dt$

c) (20 points) Derive the differential equation model for temperature of the cookie as a function of time, dT/dt, including the assumptions below and the density model from part

Energy balance $\quad In - out = acc$

$$Q - \dot{m}_w H_w = \frac{d}{dt}(\rho V H)_{sys}$$

$$H_w = C_{p_w}(T - T_{vap}) + \Delta H_{vap}$$

$$Q - \dot{m}_w\left[C_{p_w}(T - T_{vap}) + \Delta H_{vap}\right] = \frac{d}{dt}(\rho V H)_{sys}$$

Inserting assumptions relationships,

$$hA(T - T_h) - (\alpha T)\left[C_{p_w}(T - T_{vap}) + \Delta H_{vap}\right]$$

$$= V_{sys}C_{p_{sys}}\frac{d}{dt}(\rho T)_{sys}$$

$$\frac{hA(T-T_h)}{V_{sys}C_{p_{sys}}} - \frac{(\alpha T)\left[C_{pw}(T-T_{vap})+\Delta H_{vap}\right]}{V_{sys}C_{p_{sys}}}$$

$$= \rho\frac{dT}{dt} + T\frac{d\rho}{dt}$$

Since $\frac{d\rho}{dt} = -\frac{\alpha T}{V}$ and dividing by ρ

$$\frac{hA(T-T_h)}{\rho V_{sys}C_{p_{sys}}} - \frac{(\alpha T)\left[C_{pw}(T-T_{vap})+\Delta H_{vap}\right]}{\rho V_{sys}C_{p_{sys}}}$$

$$-\frac{(T)(-\alpha T)}{V\rho} = \frac{dT}{dt}$$

c)

$$\frac{dT}{dt} = \frac{hA(T-T_h)}{\rho V_{sys}C_{p_{sys}}} - \frac{(\alpha T)\left[C_{pw}(T-T_{vap})+\Delta H_{vap}\right]}{\rho V_{sys}C_{p_{sys}}} + \frac{\alpha T^2}{V\rho}$$

note that the energy and mass balances are coupled for these models, eg. must solve simultaneously

d) (6 points) Demonstrate that your models are dimensionally consistent.

b) $$\frac{-\alpha T}{V_{sys}} = \frac{d(\rho)_{sys}}{dt}$$

$$\dot m_w = \alpha T \Rightarrow \alpha = \frac{\dot m_w}{T} [=] \frac{m}{t \cdot T}$$

$$\frac{\left(\frac{m}{t-\cancel{T}}\right)(\cancel{T})}{L^3} = \left(\frac{1}{t}\right)\left(\frac{m}{L^3}\right) \quad ✓$$

$$\frac{m}{t-L^3} = \frac{m}{t-L^3}$$

c) $$\frac{dT}{dt} = \frac{hA(T-T_h)}{\rho V_{sys} C_{p_{sys}}} - \frac{(\alpha T)[C_{pw}(T-T_{vap}) + \Delta H_{vap}]}{\rho V_{sys} C_{p_{sys}}} + \frac{\alpha T^2}{V \rho}$$

$$Q [=] \frac{E}{t} = hA(T-T_h) [=] h(L^2)(T)$$

$$\therefore h [=] \frac{E}{L^2-t-T}$$

$$\Delta H_{vap} [=] \frac{E}{m}$$

$$\frac{T}{t} = \frac{\left(\frac{E}{\cancel{L^2}-t-\cancel{T}}\right)(\cancel{L^2})(\cancel{T})}{\left(\frac{\cancel{m}}{\cancel{L^3}}\right)(\cancel{L^3})\left(\frac{\cancel{E}}{\cancel{m}-T}\right)} + \frac{\left(\frac{\cancel{m}}{t-\cancel{T}}\,\cancel{T}\right)\left[\frac{\cancel{E}}{\cancel{m}-\cancel{T}}(\cancel{T}) + \frac{\cancel{E}}{\cancel{m}}\right]}{\left(\frac{\cancel{m}}{\cancel{L^3}}\right)(\cancel{L^3})\left(\frac{\cancel{E}}{\cancel{m}-T}\right)} + \frac{\left(\frac{\cancel{m}}{t\,\cancel{T}}\right)(T^{\cancel{2}})}{\cancel{L^3}\left(\cancel{m}/\cancel{L^3}\right)}$$

$$= \frac{T}{t} + \frac{T}{t} + \frac{T}{t} \quad ✓$$