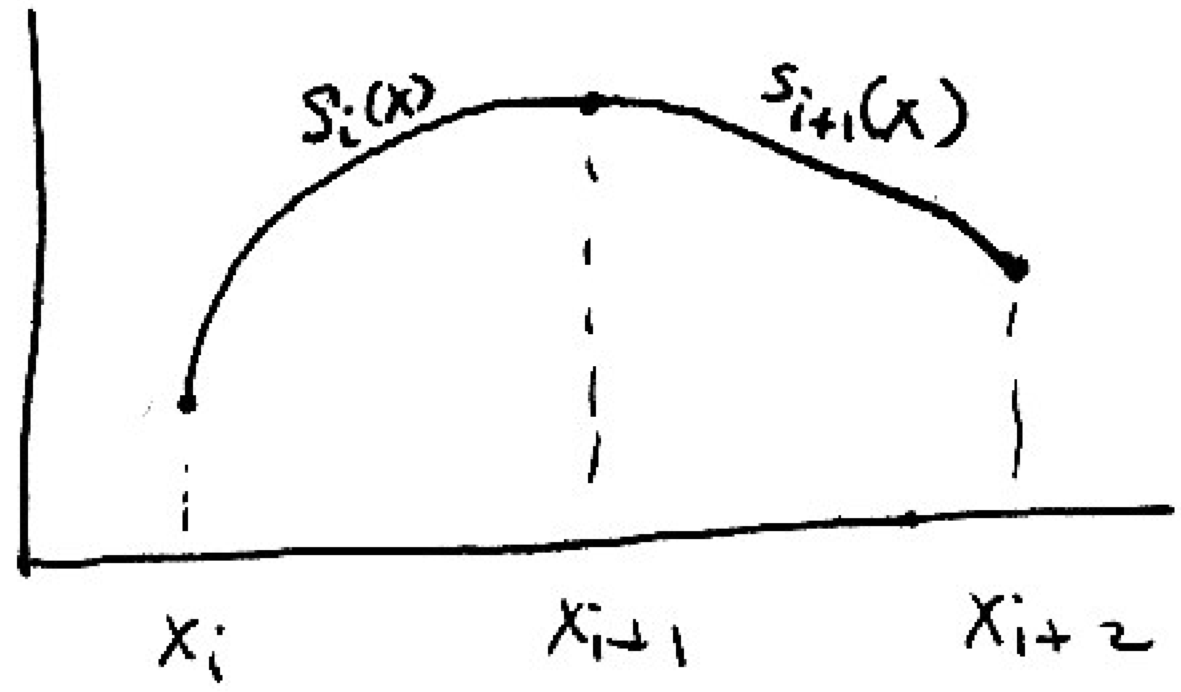

$$y(x) = a + bx + cx^2 + dx^3$$

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$
$$S_{i+1}(x) = a_{i+1} + b_{i+1}(x - x_{i+1}) + c_{i+1}(x - x_{i+1})^2 + d_{i+1}(x - x_{i+1})^3$$

at $x = x_{i+1}$
$$\left.\begin{aligned} S_i(x_{i+1}) &= S_{i+1}(x_{i+1}) \\ S'(x_{i+1}) &= S'_{i+1}(x_{i+1}) \\ S''(x_{i+1}) &= S''_{i+1}(x_{i+1}) \end{aligned}\right\}$$ requirements for continuity & smoothness

$$S_i(x_{i+1}) = S_{i+1}(x_{i+1})$$
$$a_i + b_i(x_{i+1} - x_i) + c_i(x_{i+1} - x_i)^2 + d_i(x_{i+1} - x_i)^3 = a_{i+1}$$

Define $(x_{i+1} - x_i) = h_i$ (interval size)

$$a_i + b_i h_i + c_i h_i^2 + d_i h_i^3 = a_{i+1} \qquad (1)$$

$$S'_i(x_{i+1}) = S'_{i+1}(x_{i+1})$$
$$b_i + 2c_i h_i + 3 d_i h_i^2 = b_{i+1} \qquad (2)$$

$$S''_i(x_{i+1}) = S''_{i+1}(x_{i+1})$$
$$\therefore \; c_{i+1} = c_i + 3 d_i h_i \qquad (3)$$

Objective is to determine all a_i, b_i, c_i, & d_i for each $S_i(x)$

Solve (3) for d_i

$$d_i = \frac{c_{i+1} - c_i}{3h_i} \qquad (4)$$

Plug into (1) & (2) to eliminate d_i

$$a_{i+1} = a_i + b_i h_i + c_i h_i^2 + \left(\frac{c_{i+1} - c_i}{3h_i}\right) h_i^3$$

$$= a_i + b_i h_i + \left(\frac{c_{i+1} + 2c_i}{3}\right) h_i^2 \qquad (5)$$

$$b_{i+1} = b_i + 2c_i h_i + 3\left(\frac{c_{i+1} - c_i}{3h_i}\right) h_i^2$$

$$= b_i + (c_{i+1} + c_i) h_i \qquad (6)$$

Solving (5) for b_i

$$b_i = \frac{a_{i+1} - a_i}{h_i} - \left(\frac{c_{i+1} + 2c_i}{3}\right) h_i \qquad (7)$$

also, for b_{i-1}

$$b_{i-1} = \frac{a_i - a_{i-1}}{h_{i-1}} - \left(\frac{c_i + 2c_{i-1}}{3}\right) h_{i-1} \qquad (8)$$

a_i, h_i values are known from x_i, y_i data points

from (4) & (7), b_i and d_i values only depend on c_i values

A commonly used assumption is a "natural" cubic spline which assumes that $c_0 = c_n = 0$, or that the 2nd derivative is zero at the end points of the data range.

$a_i \to$ all known from data points

$b_i \to \quad b_i = \dfrac{a_{i+1} - a_i}{h_i} - \left(\dfrac{C_{i+1} + 2C_i}{3}\right) h_i \quad (7) \qquad h_i = (X_{i+1} - X_i)$
known from data points

$C_i \to$ natural cubic spline
assume $C_0 = 0,\ C_n = 0$

$d_i \to \quad d_i = \dfrac{C_{i+1} - C_i}{3h_i} \qquad (4)$

need to put these eqns in a form where values can be calculated easily for $i = 1, 2, 3 \ldots$, i.e. these are sets of eqns

First, re-write (6) reducing the index i

$$b_i = b_{i-1} + (C_i - C_{i-1})h_{i-1} \qquad (6a)$$

re-write (7) reducing the index i

$$b_{i-1} = \frac{a_i - a_{i-1}}{h_{i-1}} - \left(\frac{C_i + 2C_{i-1}}{3}\right) h_{i-1} \qquad (7a)$$

Insert (7) into (6a) for b_i and insert (7a) into (6a) for b_{i-1}

$$\frac{a_{i+1} - a_i}{h_i} - \left(\frac{C_{i+1} + 2C_i}{3}\right) h_i = \frac{a_i - a_{i-1}}{h_{i-1}} - \left(\frac{C_i + 2C_{i-1}}{3}\right) h_{i-1} + (C_i - C_{i-1}) h_{i-1}$$

re-arranging

$$\frac{3(a_{i+1} - a_i)}{h_i} - \frac{3(a_i - a_{i-1})}{h_{i-1}} = h_{i-1} C_{i-1} + 2(h_i + h_{i-1}) C_i + h_i C_{i+1} \qquad (9)$$

for $i = 1, 2, 3 \ldots$ (9) is a set of simultaneous eqns which can be used to solve for C_i values

=1 $$\frac{3(a_2-a_1)}{h_1} - \frac{3(a_1-a_0)}{h_0} = h_0 c_0 + 2(h_0+h_1)c_1 + h_1 c_2$$

=2 $$\frac{3(a_3-a_2)}{h_2} - \frac{3(a_2-a_1)}{h_1} = h_1 c_1 + 2(h_1+h_2)c_2 + h_2 c_3$$

=3 $$\frac{3(a_4-a_3)}{h_3} - \frac{3(a_3-a_2)}{h_2} = h_2 c_2 + 2(h_2+h_3)c_3 + h_3 c_4$$

$$\vdots$$

These simultaneous eqns can be solved to get c_i (a_i are known)

Putting these in matrix format,

$$\begin{bmatrix} 0 \\ \frac{3(a_2-a_1)}{h_1} - \frac{3(a_1-a_0)}{h_0} \\ \frac{3(a_3-a_2)}{h_2} - \frac{3(a_2-a_1)}{h_1} \\ \frac{3(a_4-a_3)}{h_3} - \frac{3(a_3-a_2)}{h_2} \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 & 0 & \cdots & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & 0 & \cdots & 0 \\ 0 & 0 & h_2 & 2(h_2+h_3) & h_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & 0 & \cdot & & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$A = H \cdot c$$

$\therefore$ $c_0 = c_n = 0$ (terminal conditions for natural cubic spline)

$A = H \cdot C \quad \therefore \quad C = A \cdot H^{-1}$

So can obtain C_i values by solving simultaneous eqns.

Once all the C_i values are known, can solve (4) and (7) to obtain d_i & b_i values.

$$d_i = \frac{c_{i+1} - c_i}{3h_i} \quad (4) \qquad b_i = \frac{a_{i+1} - a_i}{h_i} - \left(\frac{c_{i+1} + 2c_i}{3}\right) h_i \quad (7)$$

So now have all a_i, b_i, c_i & d_i for each $S_i(x)$ equation