Note Title 10/29/2008

Runge Kutta concepts

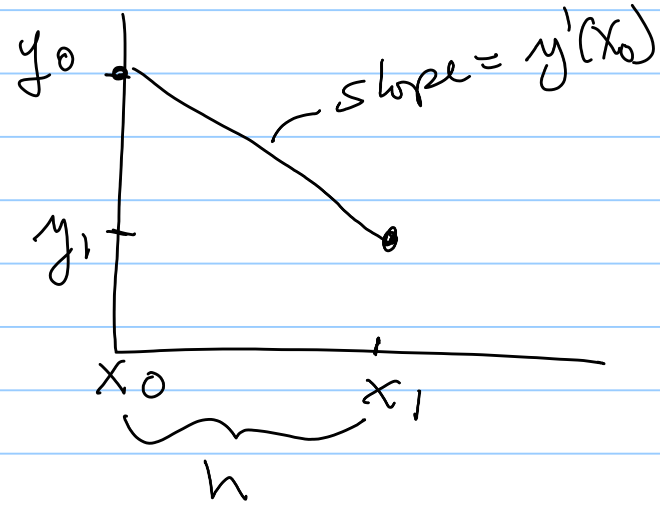
Given a 1st order differential eqn,

$$y'(x) = f(x, y) \qquad y(x_0) = y_0$$

Using Euler's method to predict the next value of y, some distance h from x_0

$$x_1 = x_0 + h$$

$$y_1 = y_0 + y'(x_0) \cdot h$$



Heun's method

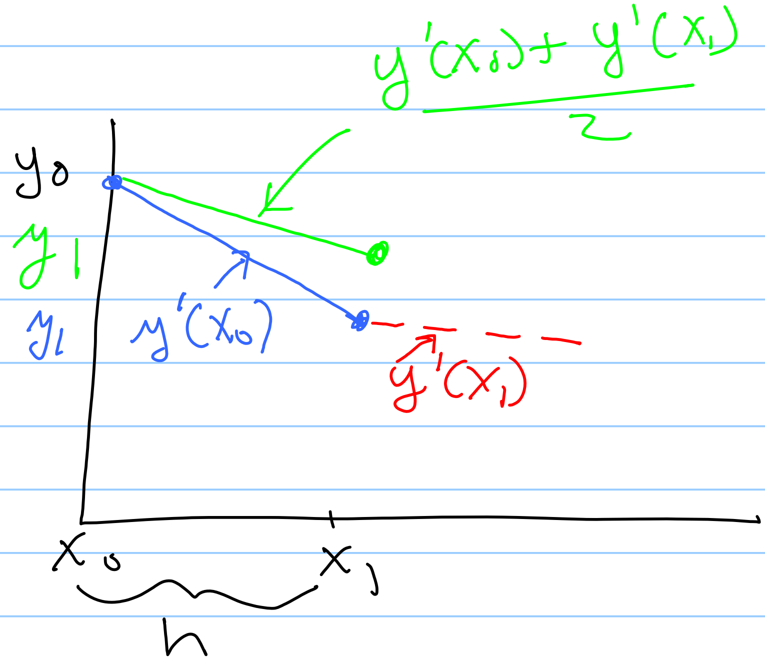
Estimate $y'(x_1)$ and combine w/ $y'(x_0)$ to give better estimate of y' to predict y_1

Predict y_1 using Euler $\Rightarrow$ $y_1 = y_0 + y'(x_0) \cdot h$

Calculate $y'(x_1) = f(x_1, y_1)$

Correct prediction of y_1 using combined slope

$$y_1 = y_0 + \frac{y'(x_0) + y'(x_1)}{2} \cdot h$$



Note that y_1 is a better prediction of $y(x_1)$ than y_1

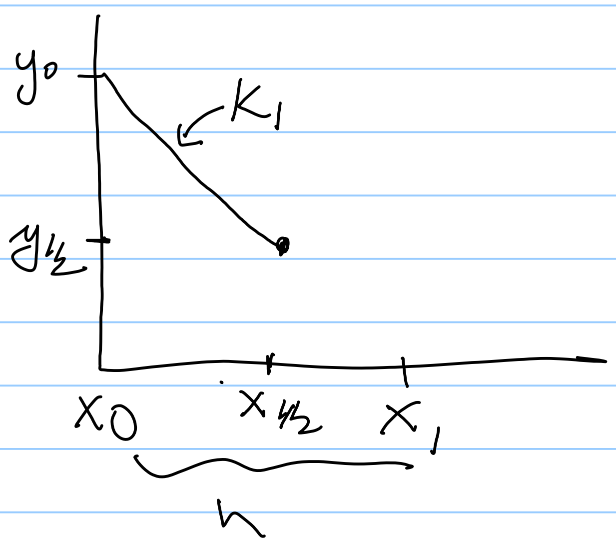
Runge Kutta (4th order)

The general idea behind this method is that by combining estimates of slopes (y') at predicted y_i, a better solution be obtained. Heun's method uses 2 estimates of y' to do this. The R-K 4th order method uses 4 estimates of the slope.

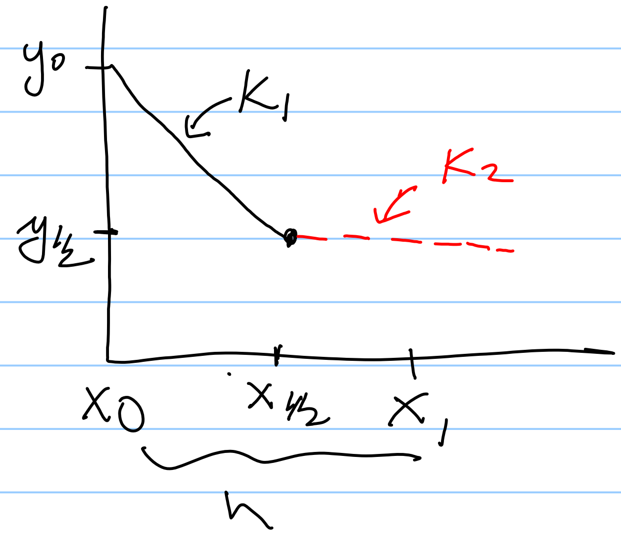
Given $y'(x) = f(x,y)$ & $y(x_0) = y_0$

first estimate of slope $\Rightarrow k_1 = y'(x_0, y_0) = f(x_0, y_0)$
Using this slope to estimate a new $y_{1/2}$ at $x = x_0 + h/2$

$$y_{1/2} = y_0 + k_1 \frac{h}{2}$$



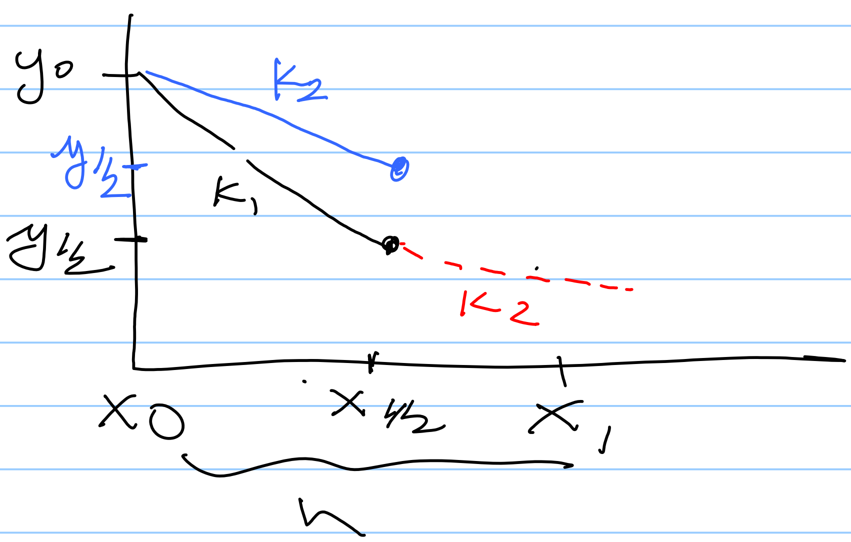
We can use this value of $y_{1/2}$ at $x_{1/2}$ to estimate y' at $x_{1/2}$



2nd estimate of slope

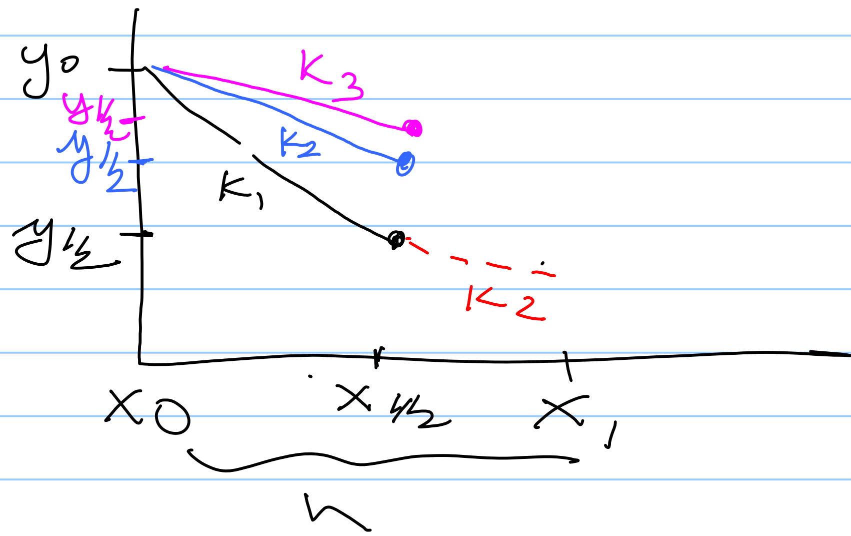
$$K_2 = y'(x_{1/2}, y_{1/2}) = f\left(x_0 + \frac{h}{2}, y_0 + K_1 \frac{h}{2}\right)$$

Using this new slope, K_2, we can get a new estimate of $y_{1/2} = y_0 + K_2 \frac{h}{2}$



Using this new $y_{1/2}$, we can get another, better estimate of the slope at $x_{1/2}$

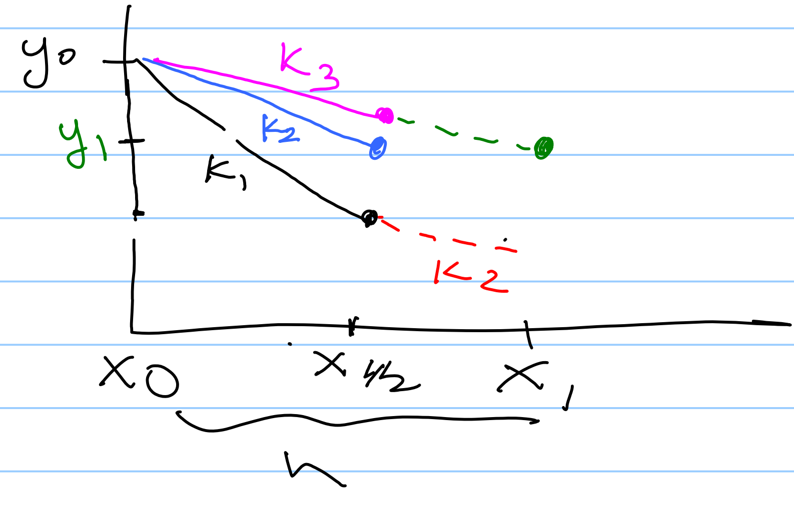
$$K_3 = y'(x_{1/2}, y_{1/2}) = f(x_0 + h/2, y_0 + K_2 \tfrac{h}{2})$$



Using this new slope, K_3, we can get a new estimate of $y_1 = y_0 + K_3 h$

Using this new y_1, we can get an estimate of the slope at $x_1 = x_0 + h$

$$K_4 = y'(x_1, y_1) = f(x_0 + h, y_0 + K_3 h)$$



Now to get a better estimate of y_1,
combine K_1, K_2, K_3, & K_4 slope estimates

The traditional R-K 4th order method uses weighting factors on K_2 & K_3, so the composite slope is

$$K = \frac{K_1 + 2K_2 + 2K_3 + K_4}{6}$$

hence, $y_1 = x_0 + Kh$

There are many variations of the R-K method using different intermediate step sizes ($h/3$, $h/4$...) and different weighting factors.

There are also adaptive methods, where the step sizes are dependent upon the rate of change of the slopes (K_1, K_2 ...). These adaptive methods can more accurately handle situations where the rate of change of the slope of $y(x)$ is high (stiff ODE).