Solving ODE's Numerically

Note Title 10/29/2007

Given $f'(t) = \dfrac{df(t)}{dt}$ and $f(0) = f_0$

the objective is to numerically estimate $f(t)$.

Euler's method

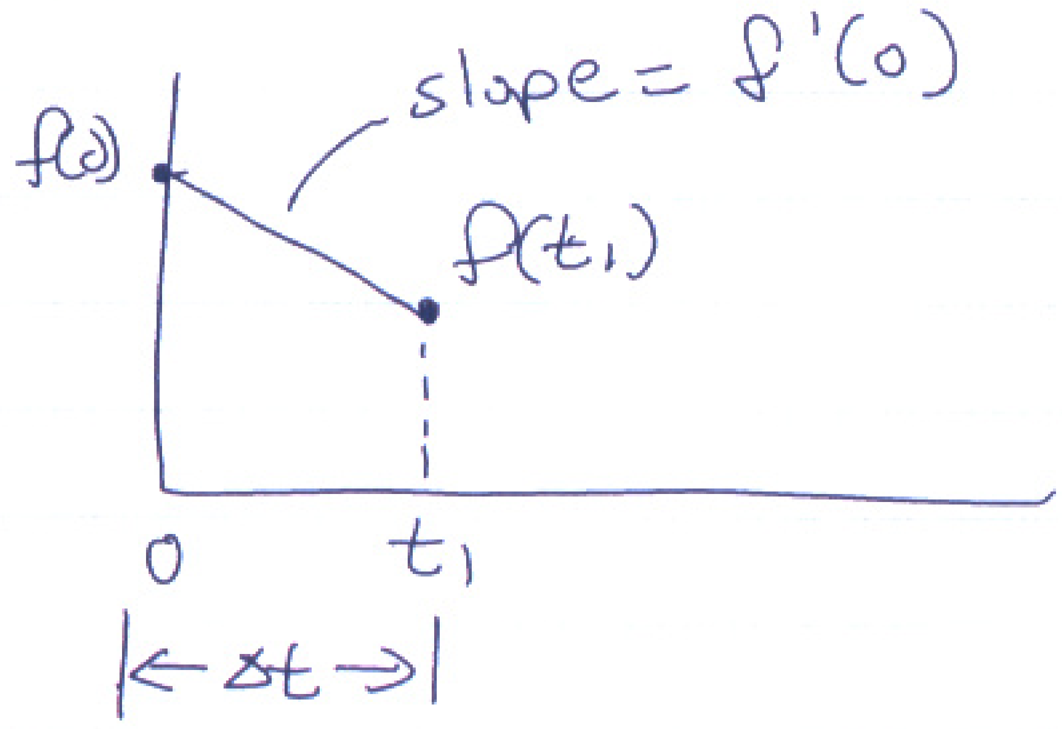
Recognize that $f'(t)$ is geometrically the slope, so we can use this and the initial point to estimate a new value for $f(t)$.

First, select a stepsize in t, eg. Δt

Now, predict the value of $f(t+\Delta t)$ by using a linear prediction model

$$f(0+\Delta t) = \underbrace{f'(0)}_{\text{slope}}\Delta t + \underbrace{f(0)}_{\text{initial point}}$$

Graphically,



Repeat this process to obtain sequential values of $f(t)$, ie,

$$f(t+\Delta t) = f'(t)\Delta t + f(t)$$

Note that as Δt gets smaller, the numerical solution becomes more accurate, however this means more calculations.

example: $f'(t) = -2f(t)$ $f(0) = 10$

Select a step size of $\Delta t = 0.1$

$f(0) = 10$

$f(1) = (-2)(10)(0.1) + 10 = 8.0$

$f(2) = (-2)(8)(0.1) + 8 = 6.4$

$f(3) = (-2)(6.4)(0.1) + 6.4 = 5.12$

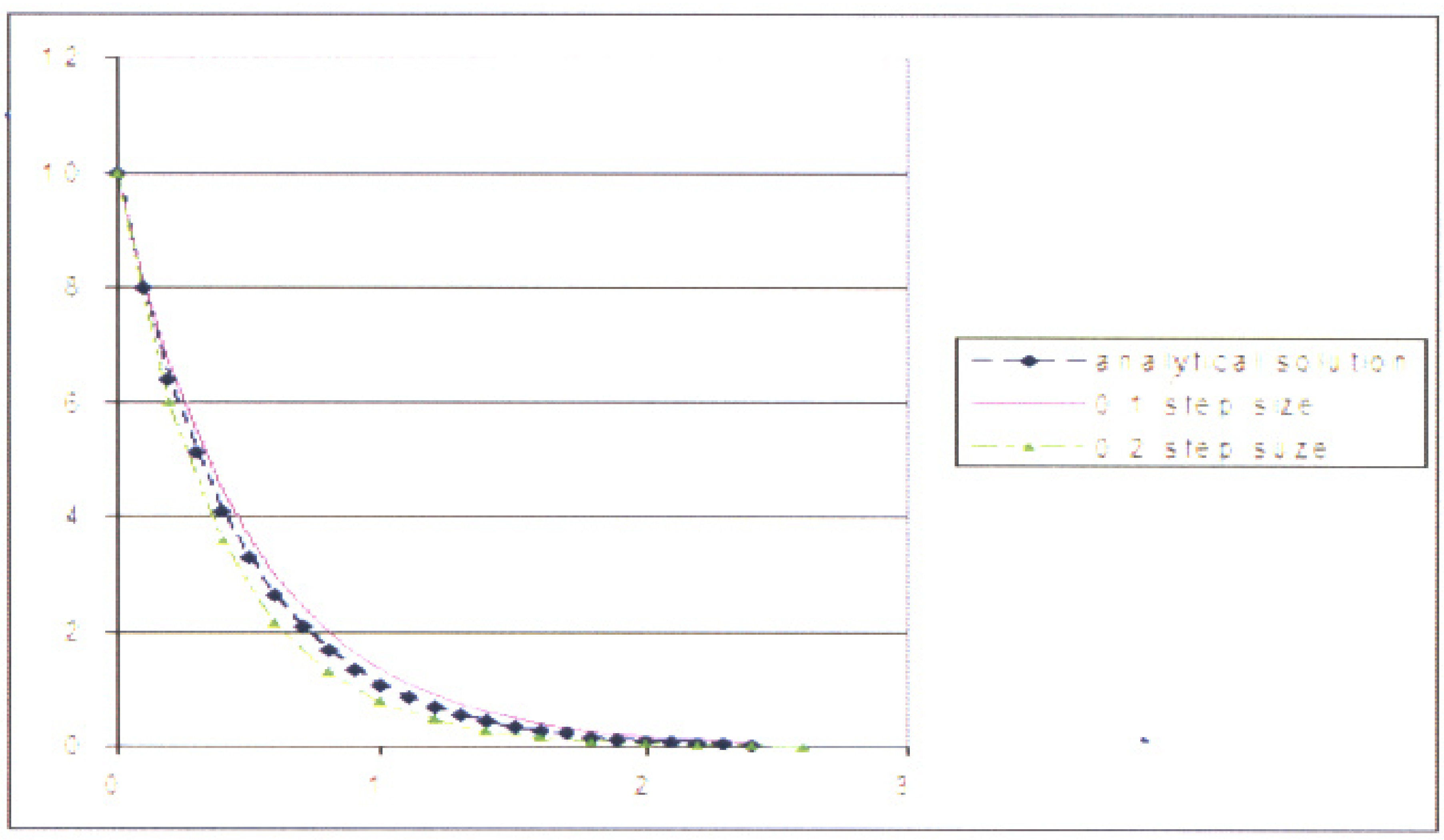
$f(4) = (-2)(5.12)(0.1) + 5.12 = 4.096$

etc.

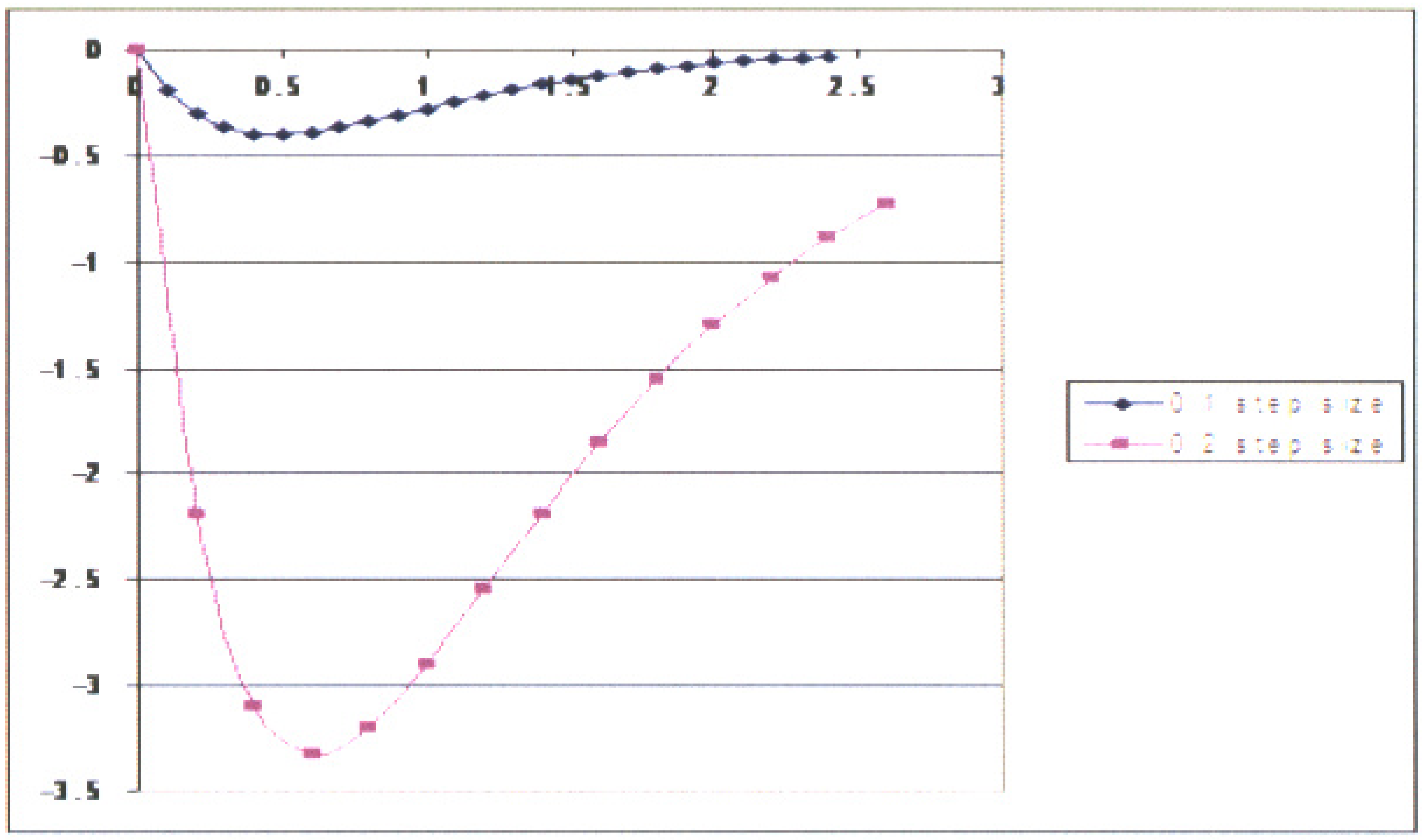
Note: The analytical soln for this $f'(t) = -2t$ is $f(t) = e^{-2t}$

Plotting this vs the numerical soln,

The numerical solns for stepsizes of 0.1 and 0.2 are presented below compared to the analytical soln.

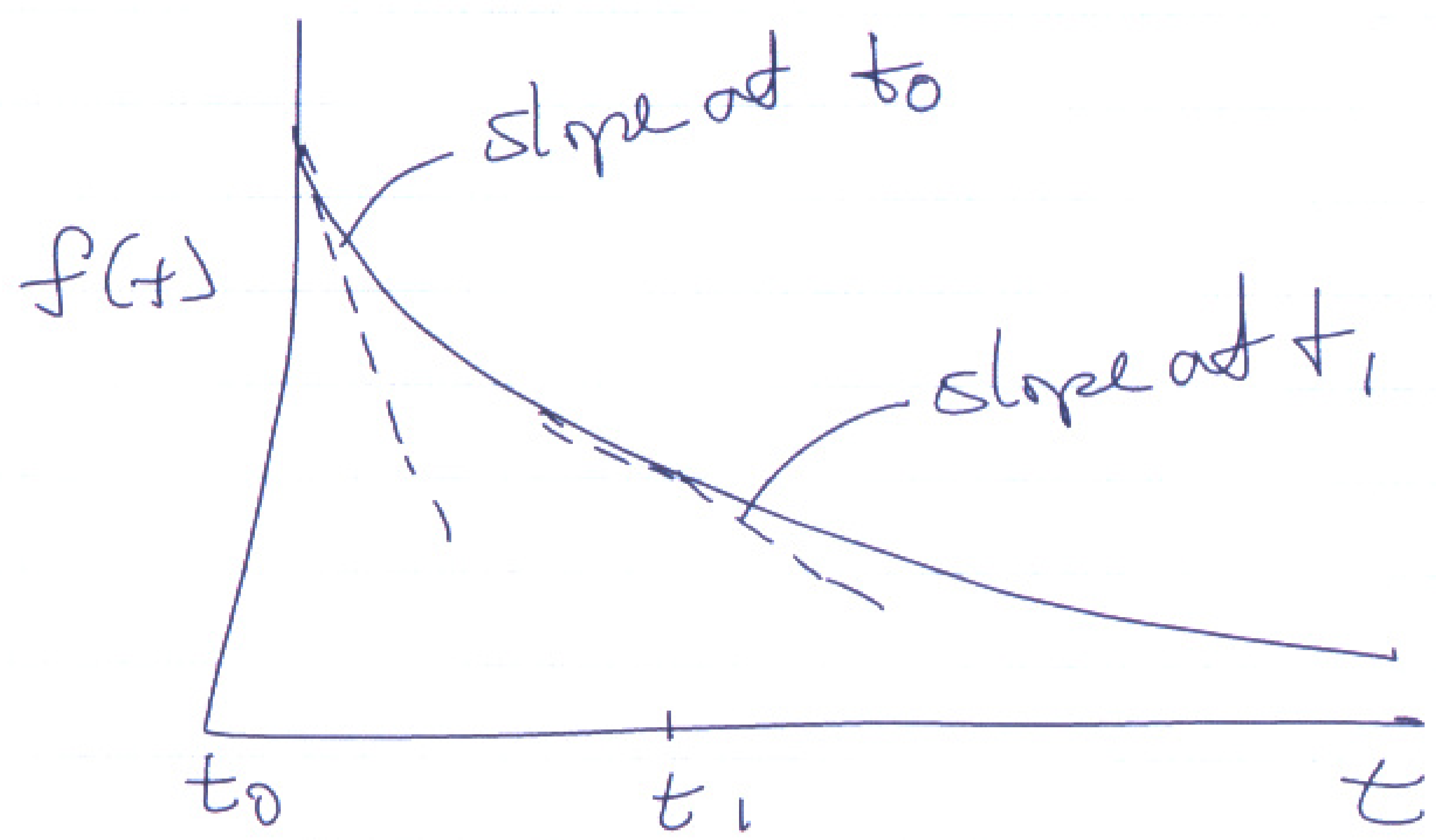


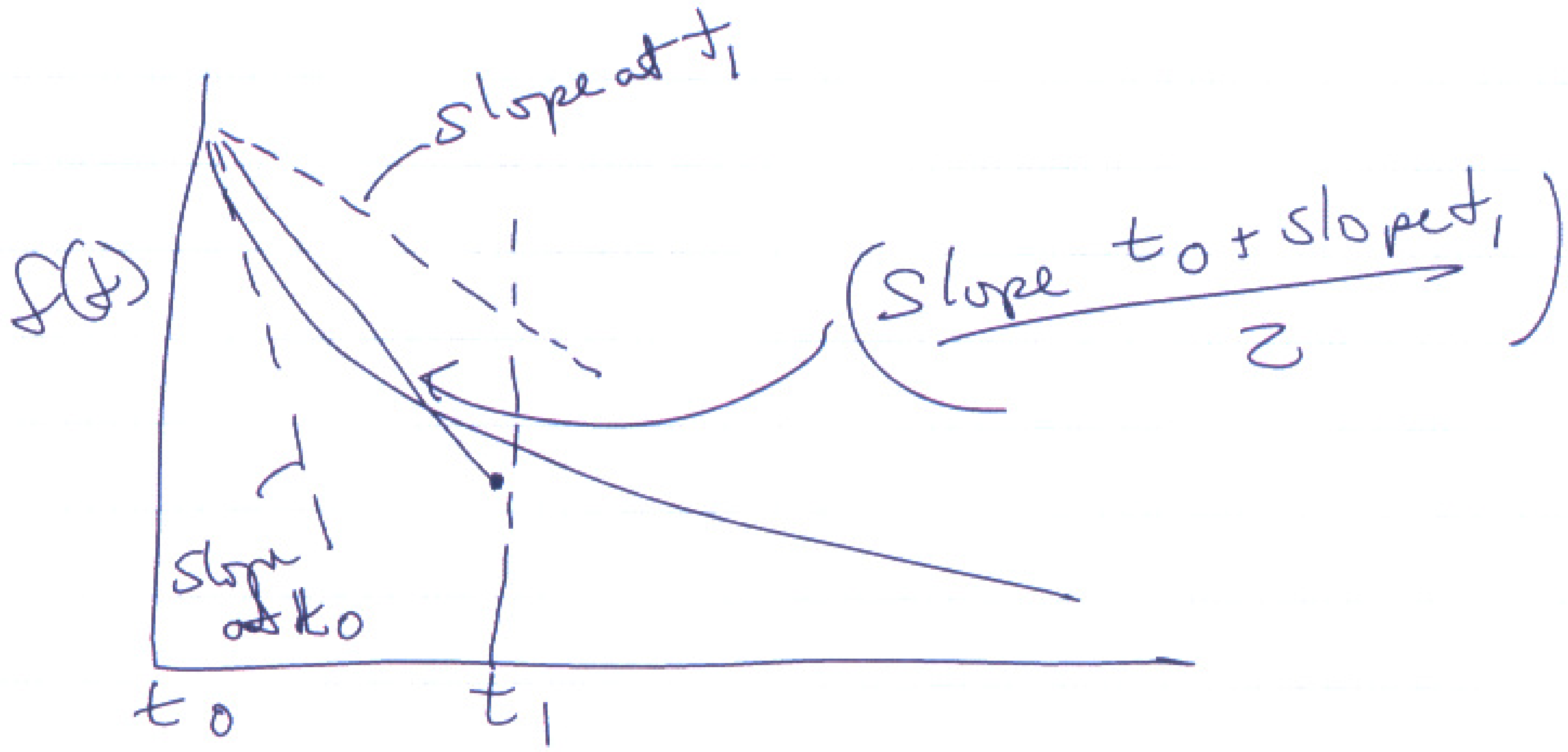
The plot of error for the step sizes is given below. As you can see, stepsize affects model accuracy.



Heun's method (Predictor-Corrector method)

An improvement over Euler's method is to use the average of 2 slopes to predict the slope of a step.
In this method, the slope's at t and t_1 are used to get a better estimate of the slope.





so

$$f(t_1) = \left[\frac{f'(t_0) + f'(t_1)}{2}\right]\Delta t + f(t_0)$$

Now the challenge is how to calculate $f'(t_1)$.

To do this we first use Euler's method to predict $f(t_1)$, then we use this estimate to correct to get the new averaged slope

Since we know $f'(t_1)$, first use Euler's method to predict $f(t_1)$

$$f(t_1) = f'(t_0)\Delta t + f(t_0)$$

Using this value of $f(t_1)$ we apply the differential eqn $f'(t)$ to get $f'(t_1)$

$$f'(t_1) = f'(f(t_1), t_1)$$

We then use this value to 'correct' the slope for the step

$$f(t_1) = \frac{f'(t_0) + f'(f(t_1), t_1)}{2}\Delta t + f(t_0)$$

Using the example from before,
$f'(t) = -2f(t)$ $\quad f(0) = 10$ $\quad \Delta t = 0.1$
Using Euler's method
$f'(t_0) = -2(10) = -20$
$f(t_1) = (-2)(10)(0.1) + 10 = 8$
Using this value of $f(t_1)$, calc. $f'(t_1)$
$f'(t_1)_{est} = (-2)(8) = -16$
Using this as an estimate of $f'(t_1)$ average it w/ $f(t_0)$ to get corrected slope

$$f'(t_1) = \frac{f'(t_0) + f'(t_1)_{est}}{2} = -18$$

So predicted / corrected value of $f(t_1)$

$$f(t_1) = \frac{f'(t_0)+f'(t_1)}{2}\Delta t + f(t_0)$$

$$= \frac{-20+(-16)}{2}\Delta t + f(t_0)$$

$$= \frac{-36}{2}(0.1) + 10$$

$$= 8.2$$

Repeating this process,

$f(t_1) = 8.2 \quad \Delta t = 0.1$

$f'(t_1) = (-2)(8.2) = -16.4$

$f'(t_2)_{est} = f'(t_1)\Delta t + f(t_1)$

$= (-16.4)(0.1) + 8.2$

$= 6.56$

$f'(t_2) = (-2)(6.56) = -13.12$

corrected slope $= \frac{-16.4 - 13.12}{2} = -14.76$

Corrected $f(t_2) = (-14.76)(0.1) + 8.2$

$= 6.724$

repeating for next step

$f'(t_2) = (-2)(6.724) = -13.448$

$f(t_3) = (-2)(6.724)(0.1) + 6.724$

$= 5.3792$

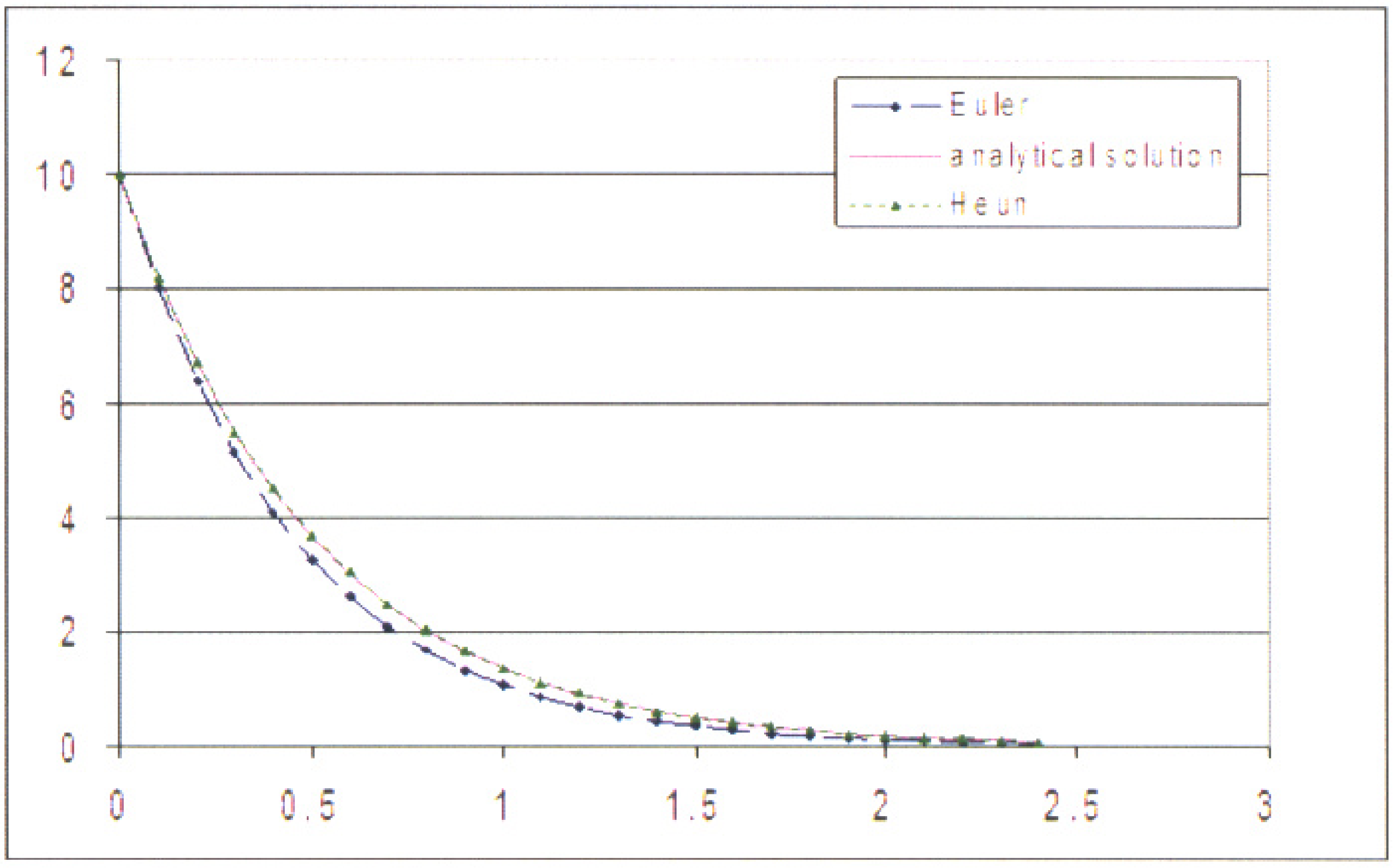
$f'(t_3)_{est} = (-2)(5.3792) = -10.7584$

$$f(t_3) = \frac{(-13.448 - 10.7584)}{2}(0.1) + 6.724$$

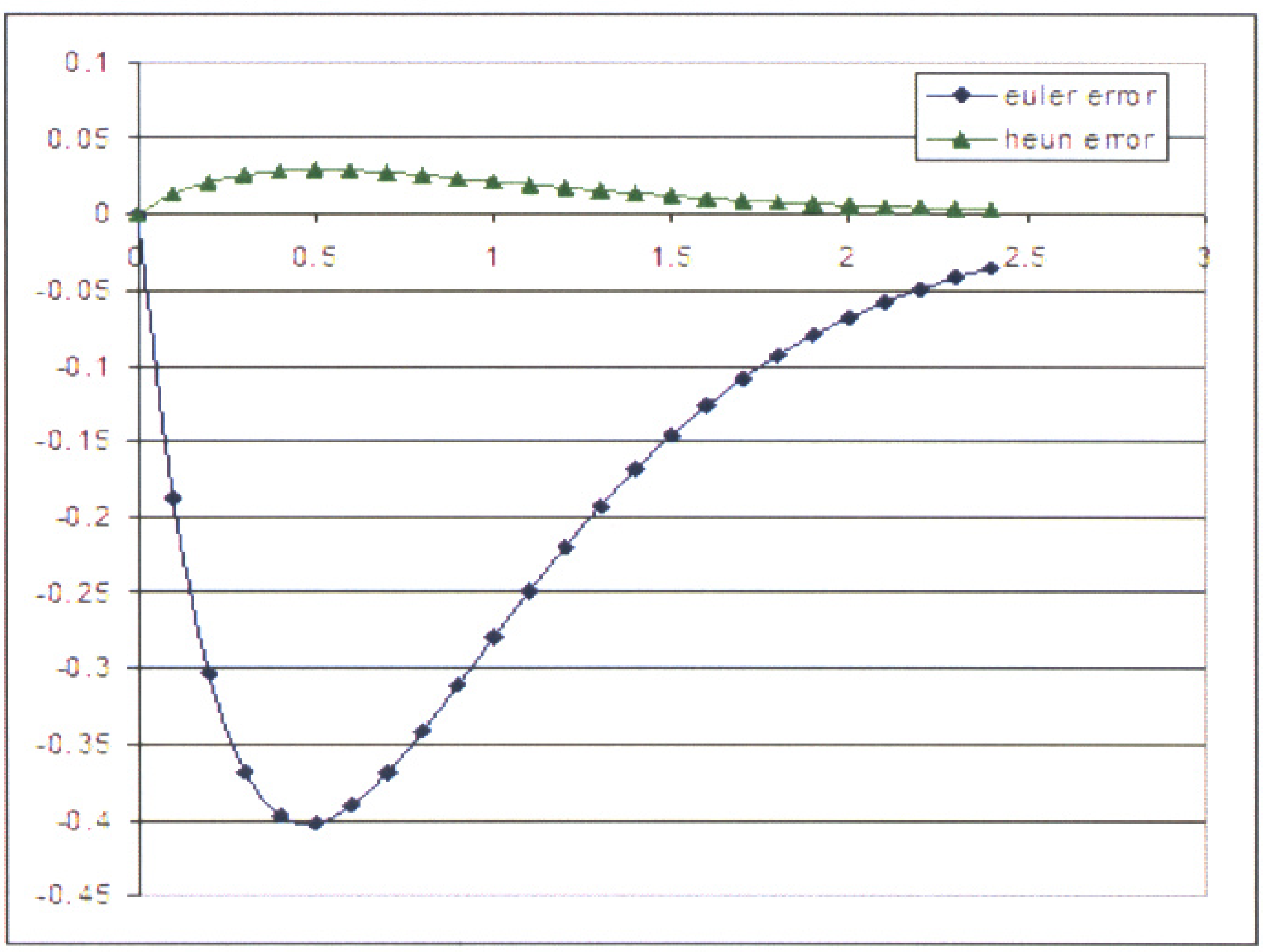
$= 5.51368$

etc.

This is a graph of the Euler vs Heun method vs the analytical soln.



The error analysis of the Euler vs Heun method shows the improved accuracy of the predictor - corrector method.



Notes

These methods are applicable to any differential eqn $f'(x)$.

The Heun method is a predictor-corrector method based on using a correction of the slope based on the average value of the predicted + corrected slopes.
Higher order predictor-corrector methods (combinations of multiple slopes) are often used also.
The Runge-Kutta methods are an example. In these, the slopes are not simply averaged, but each is weighted to have more or less impact on the final corrected slope.