Boundary value ODE's

Note Title 11/14/2007

For initial value problems (IVP) which change as a function of 1 variable (time), you can use predictor-corrector (P-C) numerical methods, e.g. Euler, Heun, Runge-Kutta, etc. to obtain solutions. These methods can also be used to solve any 1 dimensional ODE, e.g. where the variable is positional instead of time.

However, many practical problems involve higher order differential eqn's, often in more than 1 dimension, e.g. heat/mass transfer, unsteady state, etc. These can be numerically solved using some of the techniques you have studied in this course.

Decomposing n-dimensional ODE to systems of 1 dimensional ODE's

Any higher order ODE can be decomposed into a set of 1st order ODE's.

For example,

$y'' = f(x, y, y')$ with conditions

To uniquely solve this 2nd order ODE, 2 boundary/initial conditions must be known, e.g.

$$\left.\begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y'_0 \end{aligned}\right\} \text{known values}$$

Note: The y'' is a function of y', y, and x.

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This 2nd order ODE can be resolved into 2 first order ODE's by creating a new variable defined as y', e.g. $W = y' = dy/dx$

$W = y' \Rightarrow W' = y''$

So, re-writing the above ODE,

$W' = f(x, y, W)$ } two 1st order
$y' = W$ } ODE's

Conditions $y(x_0) = y_0$ } known
$W(x_0) = y'_0$ } values

You can now use the predictor-corrector methods to solve these, since W' (or y'') is simply the slope of the W' eqn.

Example:

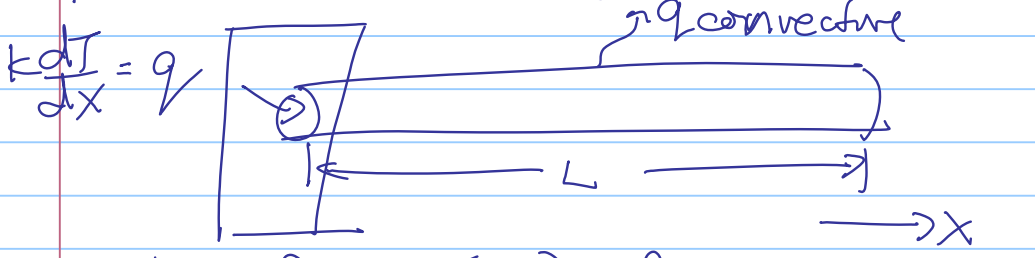
Assume you have a 2nd order ODE,

$$\frac{d^2T}{dx^2} - \alpha(T - T_s) = 0$$

Boundary Conditions $T(0) = 500$
$T'(0) = -3600$

(α, T_s are constants)

FYI, this eqn models a heat transfer fin that conducts heat down its length (L) and loses heat through its peripheral surface by convection.



Looking for $T(x)$ for $0 \leq x \leq L$

Note Title 11/14/2007

Define $W = T'$, which gives 2 first order ODE's

$\alpha = 400$, $T_s = 300$

$W = T'$ $\quad T(0) = 500$

$W' - \alpha(T - T_s) = 0$ $\quad W(0) = -3600$

Note that T'' or W' is simply the slope of the 2nd eqn, so you can use use the initial point (x_0) and the slope $W'(0)$ to predict the next step value.

Let's assume we use step size of $\Delta x = 0.01$, $0 \le x \le 0.1$. You can use any P-C method to solve this. We'll use Euler's method as an example.

Using the slope at $x=0$ $[W(0) = -3600]$ and the initial point $T(0) = 500$, calculate $T(0.01)$

$T(0.01) = (W(0)[\Delta x] + T(0)$

$= (-3600)(0.01) + 500 = 464$

Now use the new value of $T(0.01)$ to estimate the 2nd derivative $[T'' = W']$ at $x = 0.01$.

using $W' - \alpha(T - T_s) = 0$ or

$W'(T) = \alpha(T - T_s)$

$= 400(464 - 200) = 65,600$

Recognize that $W'(T) = T''(x)$ is the slope of the slope (W), so we can re-apply Euler's method to get a value for $W(0.01)$, the slope of $T(x)$.

$W(0.01) = W'(0.01)(\Delta x) + W(0)$

$= 65600(0.01) + -3600$

$= -2944$

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Now we can use $W(0.01)$ to predict the next value of $T(0.02)$.

$$T(0.02) = w(0.01)(\Delta x) + T(0.01)$$
$$= (-2944)(0.01) + 464$$
$$= 434.56$$

Calculate $W'(0.02) = \alpha(T(0.02) - T_s)$
$$= 400(434.56 - 300)$$
$$= 53824$$

Calculate $W(0.02) = (53824)(0.01) - 2944$
$$= -2405.76$$

Repeat until $x = L$

So process steps are:

1) Estimate $T(x_{i+1})$ using $W(x_i)$ & $T(x_i)$
2) Estimate $W'(x_{i+1})$ using $T(x_{i+1})$
3) Estimate $W(x_{i+1})$ using $W'(x_{i+1})$

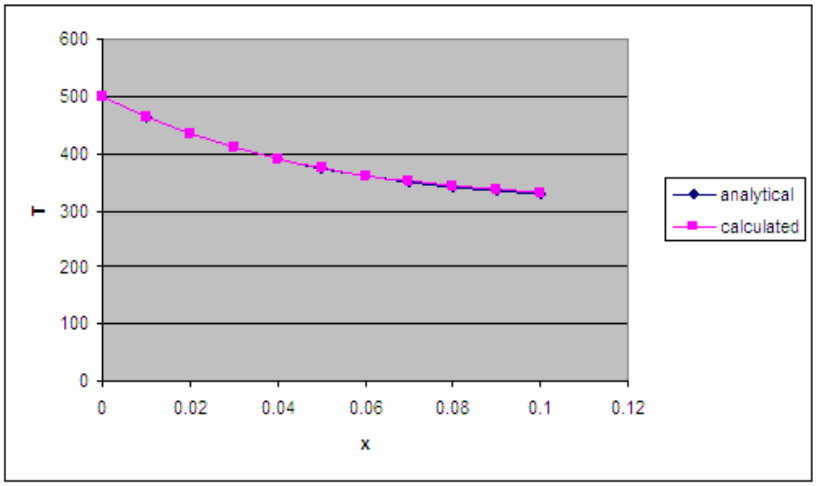
Steps 1 & 3 require a P-C method, so you can use R-K or Euler, etc. Step 2 requires situational ODE, e.g. $W' = \alpha(T - T_s)$ from this example

As a matter of completeness, this ODE has a simple analytical soln.

$$T(x) = (T(0) - T_s)\, e^{-\sqrt{\alpha}\, x} + T_s$$

Using the Euler method and stepsize 0.01 as in the example gives the soln shown below.

To			500	Ts	300	
alpha			400			
Wo			-3600			
x	W	W '	T		T anal	Error
0	-3600		500		500	0
0.01	-2944	65600	464		463.7462	-0.2538
0.02	-2405.76	53824	434.56		434.064	-0.4959
0.03	-1963.75	44200.96	410.5024		409.7623	-0.7400
0.04	-1600.29	36345.96	390.8649		389.8658	-0.9991
0.05	-1300.84	29944.8	374.862		373.5759	-1.2861
0.06	-1053.43	24741.42	361.8536		360.2388	-1.6147



Note that this method can also be used if the initial slope is not known, but a second boundary condition is known instead.
Assume the same conditions (α, $T(0)$) but now you want to design the fin length such that the tip is at the same temperature as the environment, i.e. T_s.
In this situation, you do not know $W(0)$, but instead you know $T(L) = T_s$.
To solve this numerically, we will guess initial values of $W(0)$ and solve for $T(L)$ until $T(L) = 300$.
This is called the shooting method.
This an analogy to shooting at a target and correcting your aim until you hit the bullseye.

For example, guess that $W(0) = -3600$ as in the previous example.
This would predict the length should be approx 0.11. However, the soln rapidly diverges beyond this value.
(See data + plot in appendix 1).

If we guess $W(0) = -3400$, the solution diverges to positive values of T and never reaches $T = 300$.
(See data + plot in appendix 2).

If we guess $W(0) = -3500$, the solution diverges to negative values of T again.
(See data + plot in appendix 3).

After iterating a bit, we find that a value of $\dot{W}(0) = -3493.25$ fits the physical situation appropriately as well as the analytical solution.
(see data + plot in appendix 4).

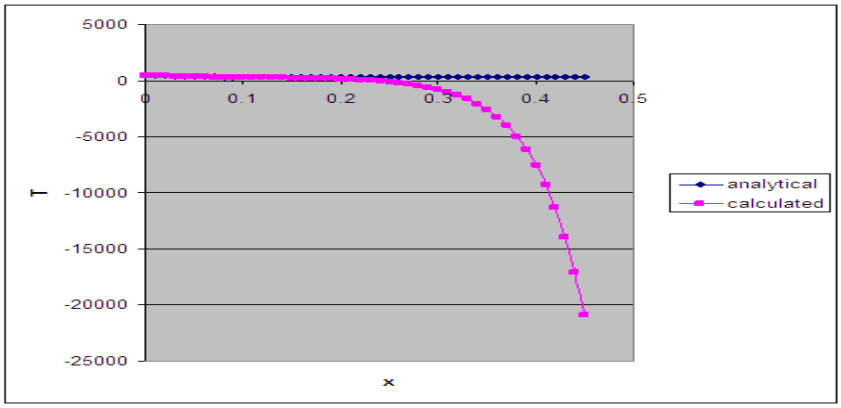
Note that this method requires guessing different values for $\dot{W}(0)$. By using a systematic means of guessing, e.g. linearly interpolate, you can refine your guesses more rapidly.

This method can be used with any P-C estimation method and can be extended to higher order ODE's in the same fashion.
Since it does not depend on the variables used, it can to solve either IVP or BVP problems.

Appendix 1.
Guess W(0) = −3600

To	493	Ts	300
alpha	400		
Wo	-3600		

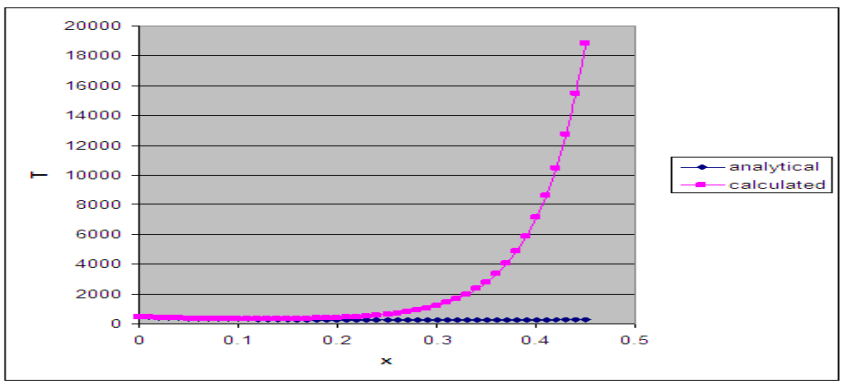
x	W	W '	T	T anal	error
0	-3600		493	493	0
0.01	-2972	62800	457	458.015	1.015035
0.02	-2462.88	50912	427.28	429.3718	2.091769
0.03	-2052.28	41060.48	402.6512	405.9206	3.269446
0.04	-1723.76	32851.38	382.1284	386.7205	4.592042
0.05	-1464.2	25956.33	364.8908	371.0007	6.109898
0.06	-1263.2	20099.54	350.2489	358.1305	7.88163
0.07	-1112.74	15046.73	337.6168	347.5932	9.976387
0.08	-1006.78	10595.79	326.4895	338.966	12.47655
0.09	-941.091	6568.679	316.4217	331.9027	15.48099
0.1	-913.047	2804.317	307.0108	326.1197	19.10892
0.11	-921.526	-847.873	297.8803	321.385	23.50469
0.12	-966.866	-4533.98	288.6651	317.5086	28.84351
0.13	-1050.88	-8401.44	278.9964	314.3348	35.33841
0.14	-1176.93	-12605	268.4876	311.7363	43.24875
0.15	-1350.06	-17312.7	256.7183	309.6089	52.89062
0.16	-1577.19	-22712.9	243.2177	307.8671	64.64939
0.17	-1867.4	-29021.7	227.4459	306.441	78.99518
0.18	-2232.32	-36491.3	208.7718	305.2735	96.50164
0.19	-2686.52	-45420.5	186.4487	304.3176	117.8689
0.2	-3248.19	-56166.6	159.5835	303.5349	143.9514
0.21	-3939.78	-69159.4	127.1016	302.8941	175.7925
0.22	-4788.96	-84918.5	87.70381	302.3695	214.6657
0.23	-5829.71	-104074	39.81416	301.94	262.1258
0.24	-7103.64	-127393	-18.4829	301.5883	320.0713
0.25	-8661.72	-155808	-89.5193	301.3004	390.8198
0.26	-10566.3	-190455	-176.137	301.0647	477.2012
0.27	-12893.5	-232720	-281.799	300.8717	582.6708
0.28	-15736.4	-284293	-410.734	300.7137	711.4474
0.29	-19208.8	-347239	-568.098	300.5843	868.682
0.3	-23449.5	-424074	-760.186	300.4784	1060.664
0.31	-28628.3	-517872	-994.681	300.3917	1295.072
0.32	-34952.1	-632385	-1280.96	300.3207	1581.284
0.33	-42674	-772194	-1630.48	300.2626	1930.747
0.34	-52102.9	-942890	-2057.22	300.215	2357.44
0.35	-63616	-1151302	-2578.25	300.176	2878.43
0.36	-77673.6	-1405766	-3214.41	300.1441	3514.558
0.37	-94838.2	-1716460	-3991.15	300.118	4291.268
0.38	-115796	-2095813	-4939.53	300.0966	5239.629
0.39	-141386	-2558998	-6097.5	300.0791	6397.575
0.4	-172632	-3124543	-7511.36	300.0647	7811.423
0.41	-210782	-3815070	-9237.68	300.053	9537.729
0.42	-257364	-4658200	-11345.5	300.0434	11645.54
0.43	-314241	-5687658	-13919.1	300.0355	14219.18
0.44	-383687	-6944622	-17061.6	300.0291	17361.59
0.45	-468481	-8479371	-20898.4	300.0238	21198.45



Appendix 2
Guess W(0) = -3400

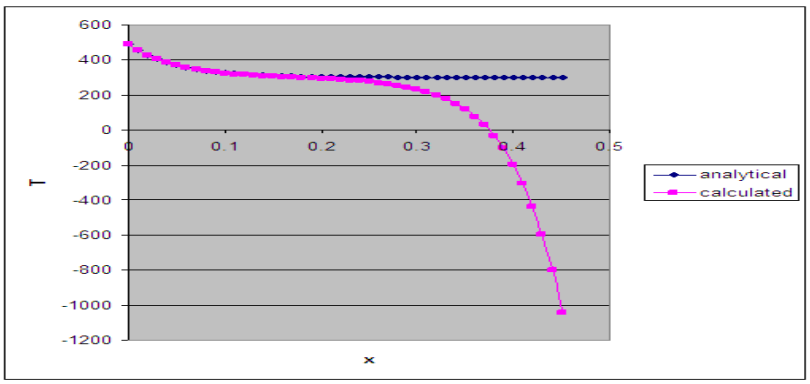
To	493	Ts	300
alpha	400		
Wo	-3400		

x	W	W '	T	T anal	error
0	-3400		493	493	0
0.01	-2764	63600	459	458.015	-0.98496
0.02	-2238.56	52544	431.36	429.3718	-1.98823
0.03	-1802.66	43589.76	408.9744	405.9206	-3.05375
0.04	-1438.87	36379.11	390.9478	386.7205	-4.22729
0.05	-1132.64	30623.63	376.5591	371.0007	-5.55833
0.06	-871.704	26093.09	365.2327	358.1305	-7.10223
0.07	-645.642	22606.27	356.5157	347.5932	-8.92246
0.08	-445.404	20023.7	350.0593	338.966	-11.0932
0.09	-262.984	18242.08	345.6052	331.9027	-13.7025
0.1	-91.0821	17190.15	342.9754	326.1197	-16.8557
0.11	77.17606	16825.82	342.0646	321.385	-20.6795
0.12	248.5213	17134.53	342.8363	317.5086	-25.3277
0.13	429.8074	18128.61	345.3215	314.3348	-30.9867
0.14	628.2858	19847.84	349.6196	311.7363	-37.8833
0.15	851.8957	22360.98	355.9025	309.6089	-46.2936
0.16	1109.581	25768.57	364.4214	307.8671	-56.5543
0.17	1411.65	30206.89	375.5172	306.441	-69.0762
0.18	1770.185	35853.49	389.6337	305.2735	-84.3603
0.19	2199.527	42934.23	407.3356	304.3176	-103.018
0.2	2716.851	51732.34	429.3309	303.5349	-125.796
0.21	3342.848	62599.75	456.4994	302.8941	-153.605
0.22	4102.56	75971.14	489.9279	302.3695	-187.558
0.23	5026.374	92381.38	530.9534	301.94	-229.013
0.24	6151.242	112486.9	581.2172	301.5883	-279.629
0.25	7522.161	137091.8	642.7296	301.3004	-341.429
0.26	9193.966	167180.5	717.9512	301.0647	-416.887
0.27	11233.53	203956.3	809.8909	300.8717	-509.019
0.28	13722.43	248890.5	922.2262	300.7137	-621.512
0.29	16760.24	303780.2	1059.451	300.5843	-758.866
0.3	20468.45	370821.1	1227.053	300.4784	-926.574
0.31	24995.4	452694.9	1431.737	300.3917	-1131.35
0.32	30522.16	552676.5	1681.691	300.3207	-1381.37
0.33	37269.81	674765.2	1986.913	300.2626	-1686.65
0.34	45508.26	823844.4	2359.611	300.215	-2059.4
0.35	55567.03	1005877	2814.694	300.176	-2514.52
0.36	67848.49	1228146	3370.364	300.1441	-3070.22
0.37	82843.88	1499540	4048.849	300.118	-3748.73
0.38	101153	1830915	4877.288	300.0966	-4577.19
0.39	123508.3	2235527	5888.818	300.0791	-5588.74
0.4	150803.9	2729560	7123.901	300.0647	-6823.84
0.41	184131.7	3332776	8631.94	300.053	-8331.89
0.42	224824.7	4069303	10473.26	300.0434	-10173.2
0.43	274510.7	4968602	12721.5	300.0355	-12421.5
0.44	335177.2	6066644	15466.61	300.0291	-15166.6
0.45	409250.7	7407353	18818.38	300.0238	-18518.4



Appendix 3
Guess W(0) = −3500

To	493	Ts	300		
alpha	400				
Wo	-3500				
x	W	W '	T	T anal	error
0	-3500		493	493	0
0.01	-2868	63200	458	458.015	0.015035
0.02	-2350.72	51728	429.32	429.3718	0.051769
0.03	-1927.47	42325.12	405.8128	405.9206	0.107846
0.04	-1581.32	34615.24	386.5381	386.7205	0.182378
0.05	-1298.42	28289.98	370.7249	371.0007	0.275784
0.06	-1067.45	23096.31	357.7408	358.1305	0.3897
0.07	-879.188	18826.5	347.0662	347.5932	0.526965
0.08	-726.091	15309.75	338.2744	338.966	0.691664
0.09	-602.037	12405.38	331.0135	331.9027	0.889231
0.1	-502.065	9997.233	324.9931	326.1197	1.126627
0.11	-422.175	7988.974	319.9724	321.385	1.412575
0.12	-359.172	6300.274	315.7507	317.5086	1.757881
0.13	-310.537	4863.584	312.159	314.3348	2.17584
0.14	-274.322	3621.438	309.0536	311.7363	2.682747
0.15	-249.081	2524.15	306.3104	309.6089	3.29853
0.16	-233.802	1527.827	303.8196	307.8671	4.047538
0.17	-227.876	592.6177	301.4815	306.441	4.959497
0.18	-231.065	-318.887	299.2028	305.2735	6.070696
0.19	-243.497	-1243.15	296.8921	304.3176	7.425427
0.2	-265.668	-2217.13	294.4572	303.5349	9.077752
0.21	-298.466	-3279.8	291.8005	302.8941	11.09366
0.22	-343.203	-4473.67	288.8158	302.3695	13.5537
0.23	-401.667	-5846.48	285.3838	301.94	16.5562
0.24	-476.199	-7453.15	281.3671	301.5883	20.22121
0.25	-569.778	-9357.94	276.6051	301.3004	24.69528
0.26	-686.149	-11637.1	270.9074	301.0647	30.15734
0.27	-829.965	-14381.7	264.0459	300.8717	36.82583
0.28	-1006.98	-17701.5	255.7462	300.7137	44.96747
0.29	-1224.27	-21729.4	245.6764	300.5843	54.90791
0.3	-1490.54	-26626.5	233.4337	300.4784	67.04474
0.31	-1816.43	-32588.7	218.5283	300.3917	81.86342
0.32	-2214.97	-39854.4	200.364	300.3207	99.9567
0.33	-2702.11	-48714.3	178.2143	300.2626	122.0483
0.34	-3297.34	-59522.7	151.1931	300.215	149.0218
0.35	-4024.46	-72712.1	118.2197	300.176	181.9563
0.36	-4912.56	-88810	77.97509	300.1441	222.169
0.37	-5997.16	-108460	28.84946	300.118	271.2685
0.38	-7321.65	-132449	-31.1222	300.0966	331.2188
0.39	-8939.01	-161735	-104.339	300.0791	404.4178
0.4	-10913.9	-197492	-193.729	300.0647	493.7935
0.41	-13325.4	-241147	-302.868	300.053	602.921
0.42	-16269.9	-294449	-436.122	300.0434	736.1654
0.43	-19865.2	-359528	-598.821	300.0355	898.8564
0.44	-24255.1	-438989	-797.472	300.0291	1097.502
0.45	-29615.1	-536009	-1040.02	300.0238	1340.047



Appendix 3
Guess W(0) = −3493.25

To	493	Ts	300		
alpha	400				
Wo	-3493.25				
x	W	W '	T	T anal	error
0	-3493.25		493	493	0
0.01	-2860.98	63227	458.0675	458.015	-0.05246
0.02	-2343.15	51783.08	429.4577	429.3718	-0.08593
0.03	-1919.04	42410.48	406.0262	405.9206	-0.10556
0.04	-1571.7	34734.31	386.8358	386.7205	-0.11527
0.05	-1287.23	28447.5	371.1188	371.0007	-0.11802
0.06	-1054.24	23298.6	358.2465	358.1305	-0.11601
0.07	-863.424	19081.63	347.7041	347.5932	-0.11087
0.08	-707.145	15627.94	339.0698	338.966	-0.10382
0.09	-579.151	12799.36	331.9984	331.9027	-0.09571
0.1	-474.323	10482.76	326.2069	326.1197	-0.08718
0.11	-388.469	8585.461	321.4637	321.385	-0.07864
0.12	-318.153	7031.586	317.579	317.5086	-0.0704
0.13	-260.563	5758.973	314.3974	314.3348	-0.06263
0.14	-213.396	4716.72	311.7918	311.7363	-0.05546
0.15	-174.765	3863.136	309.6578	309.6089	-0.04894
0.16	-143.124	3164.077	307.9102	307.8671	-0.04309
0.17	-117.208	2591.581	306.479	306.441	-0.03791
0.18	-95.9807	2122.749	305.3069	305.2735	-0.03339
0.19	-78.5924	1738.826	304.3471	304.3176	-0.02951
0.2	-64.3478	1424.456	303.5611	303.5349	-0.02622
0.21	-52.6772	1167.065	302.9177	302.8941	-0.02352
0.22	-43.1136	956.3561	302.3909	302.3695	-0.02136
0.23	-35.2746	783.9016	301.9598	301.94	-0.01975
0.24	-28.8466	642.8031	301.607	301.5883	-0.01867
0.25	-23.5724	527.4168	301.3185	301.3004	-0.01812
0.26	-19.2411	433.1271	301.0828	301.0647	-0.01812
0.27	-15.6795	356.1626	300.8904	300.8717	-0.01871
0.28	-12.7451	293.4445	300.7336	300.7137	-0.01992
0.29	-10.3204	242.4642	300.6062	300.5843	-0.02184
0.3	-8.30861	201.1824	300.503	300.4784	-0.02456
0.31	-6.62913	167.948	300.4199	300.3917	-0.02819
0.32	-5.21481	141.4315	300.3536	300.3207	-0.0329
0.33	-4.00909	120.5722	300.3014	300.2626	-0.03888
0.34	-2.96373	104.5359	300.2613	300.215	-0.04638
0.35	-2.03692	92.68095	300.2317	300.176	-0.05571
0.36	-1.19159	84.53325	300.2113	300.1441	-0.06724
0.37	-0.39392	79.76689	300.1994	300.118	-0.08145
0.38	0.387991	78.19121	300.1955	300.0966	-0.09889
0.39	1.185422	79.74317	300.1994	300.0791	-0.12028
0.4	2.030271	84.48486	300.2112	300.0647	-0.14647
0.41	2.95633	92.60594	300.2315	300.053	-0.17851
0.42	4.000643	104.4313	300.2611	300.0434	-0.21768
0.43	5.204981	120.4338	300.3011	300.0355	-0.26555
0.44	6.617519	141.2538	300.3531	300.0291	-0.32404
0.45	8.294757	167.7238	300.4193	300.0238	-0.39549