# Finite Difference Numerical Solutions



Many important problems in engineering & science involve more than 1 dimension. While the numerical methods you have learned with predictor corrector methods, these are somewhat limited when it comes to multi-dimensional and/or transient system. For these, it is most efficient to use finite difference methods.

Example 1D, transient system, heat transfer

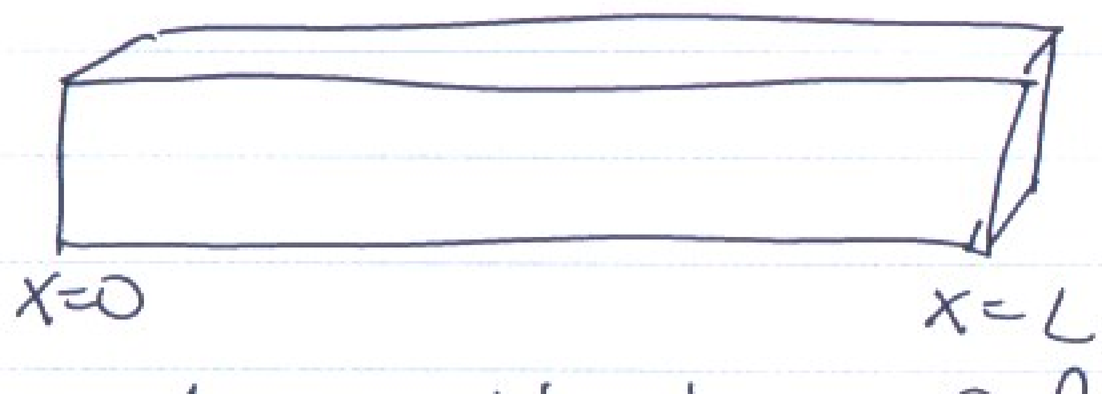
$$\frac{d^2T}{dx^2} = \alpha \frac{dT}{dt} \qquad t<0 \quad T(x) = T_0 \text{ for all } x$$

$$t \geq 0 \quad T(0) = T_L$$

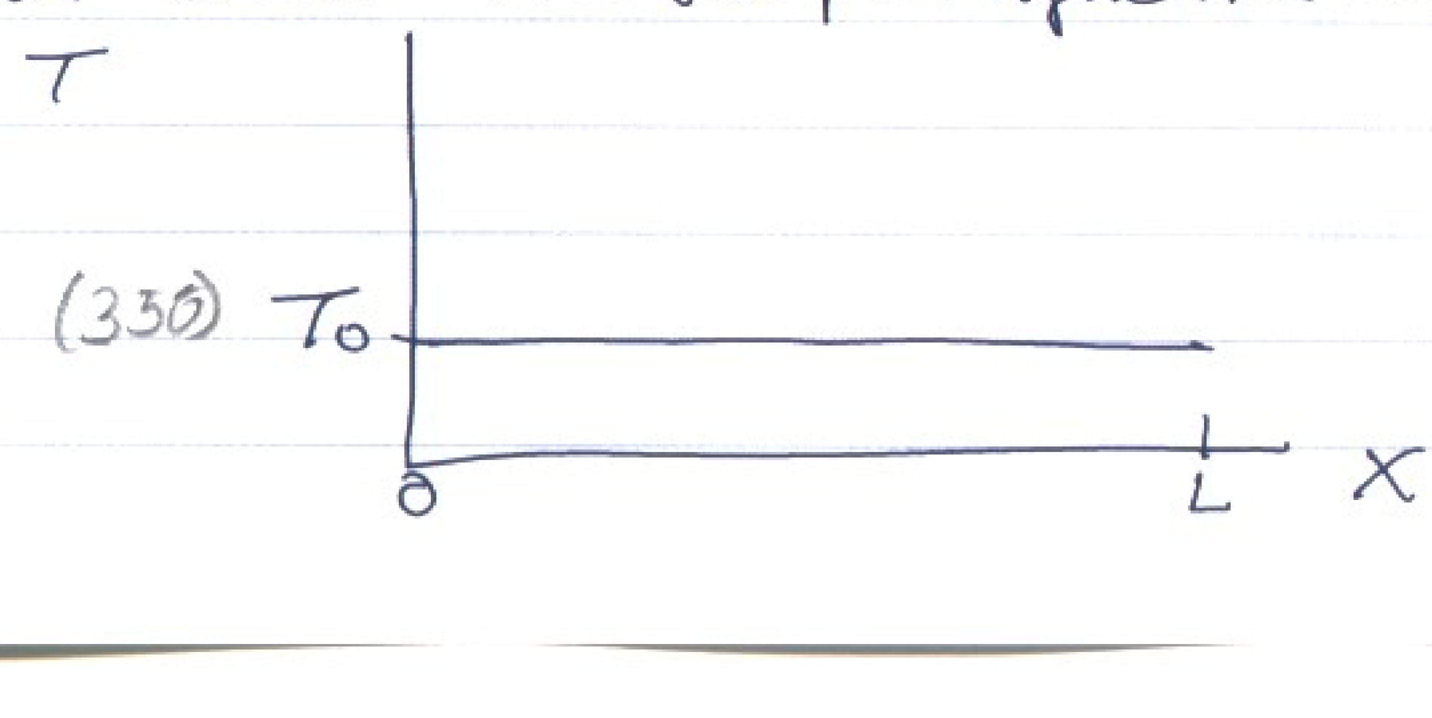
$$\alpha = \text{constant} = \frac{\rho C_p}{k} \qquad T(L) = T_0$$

This equation describes the conductive ($q = -k \frac{dT}{dx}$) transfer of heat in 1D as a function of time! Initially the system is at a uniform temperature $T_0$, at all positions of $X$. At $t=0$, the temperature at $x=0$ is suddenly raised to $T_L$.

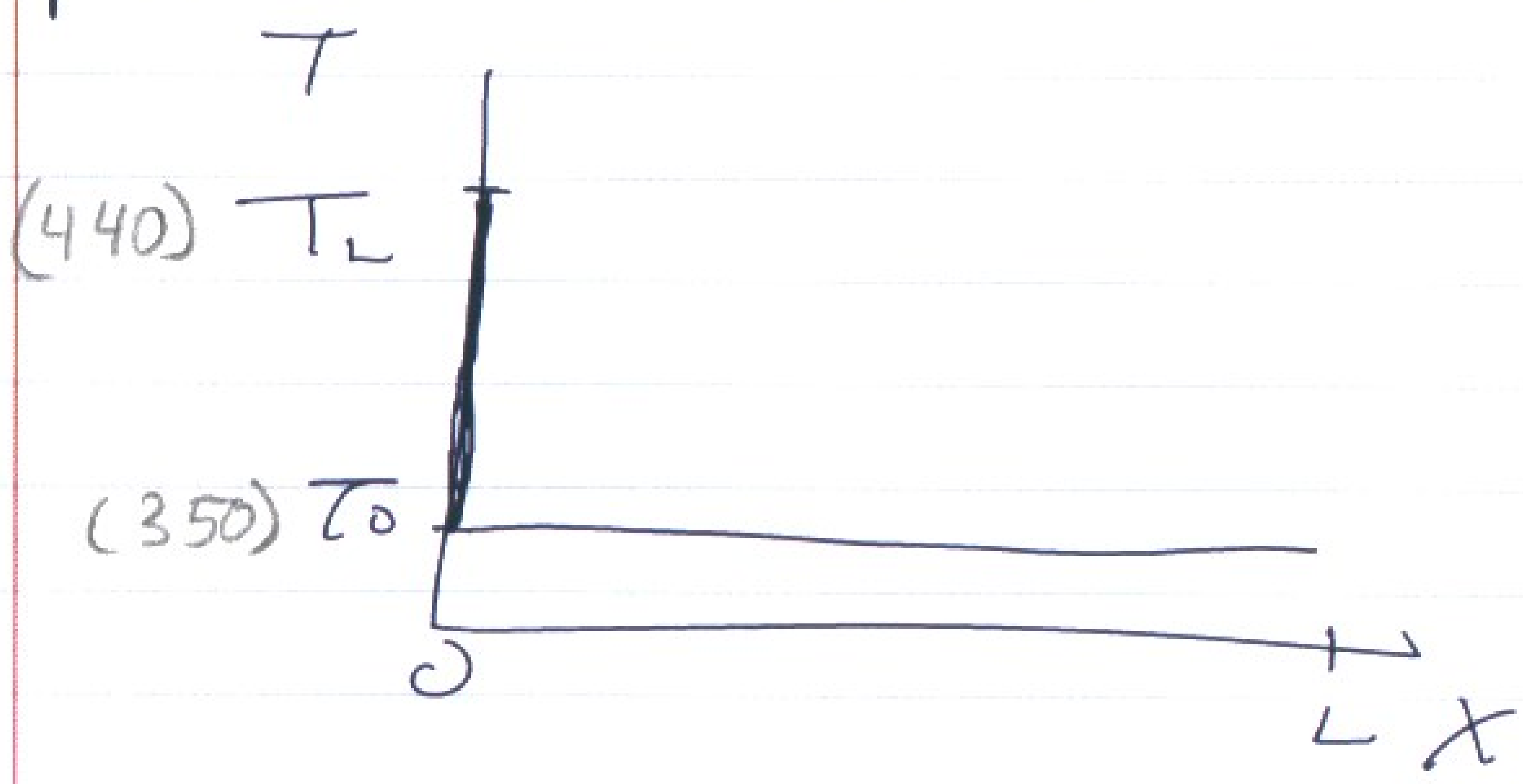
The temperature inside the block changes as energy enters and is conducted in the $x$ direction



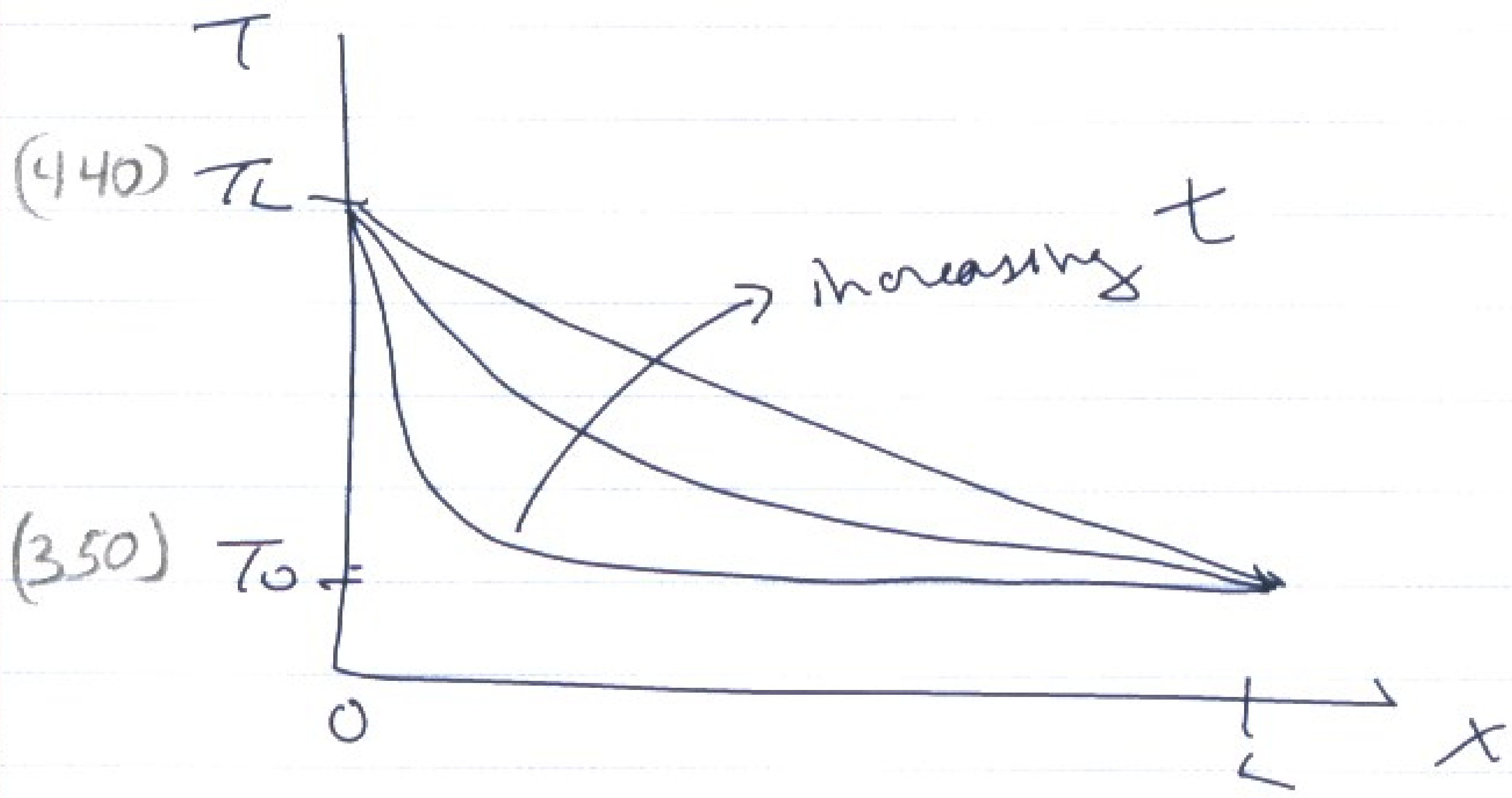

at $t=0$ the temp. profile in the block is



For $t > 0$, the temperature at $x = 0$ is suddenly raised to $T_L$, so the profile would look like

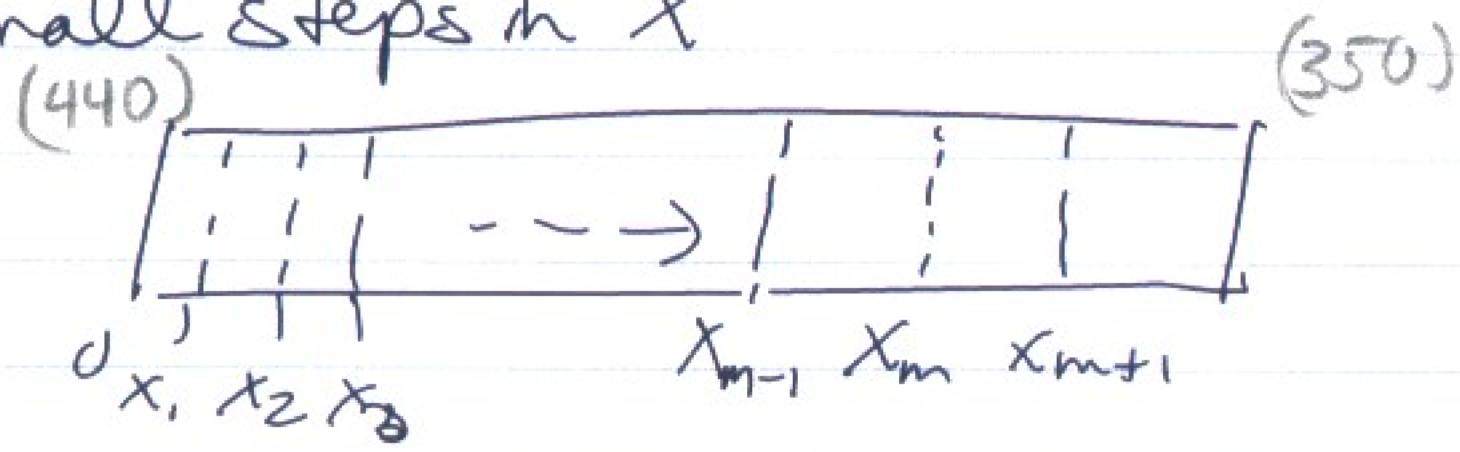



As time advances, the thermal wave would move down the length of the block, so the thermal profile would change with time




Our task is to create a quantitative way to solve for $T(x,t)$.

First, divide the length of the block into small steps in $x$




where $m$ is the $m$th position in the $x$ direction.

Note Title 11/16/2007

$\frac{dT}{dx}$ can be approximated as

$$\frac{dT}{dx} \cong \frac{T_{m+1}^{p} - T_{m}^{p}}{\Delta x} \qquad \Delta x = (x_{m+1} - x_m)$$

where p refers to time, since $T(x,t)$.

Similarly, the 2nd derivative with respect to x would be

$$\frac{d}{dx}\left(\frac{dT}{dx}\right) \cong \frac{d}{dx}\left(\frac{T_{m+1}^{p} - T_{m}^{p}}{\Delta x}\right)$$

$$= \frac{\left(\frac{T_{m+1}^{p} - T_{m}^{p}}{\Delta x}\right) \left(\frac{T_{m}^{p} - T_{m-1}^{p}}{\Delta x}\right)}{\Delta x}$$

$$= \frac{T_{m+1}^{p} - 2T_{m}^{p} + T_{m-1}^{p}}{(\Delta x)^2}$$

For the time derivative

$$\frac{dT}{dt} \cong \frac{T_m^{p+1} - T_m^{p}}{\Delta t}$$

Note that $\Delta x$ is the position increment size and $\Delta t$ is the time increment size.

Putting these together,

$$\frac{d^2T}{dx^2} = \alpha \frac{dT}{dt}$$

$$\frac{T_{m+1}^{p} - 2T_{m}^{p} + T_{m-1}^{p}}{(\Delta x)^2} = \alpha \frac{T_m^{p+1} - T_m^{p}}{\Delta t}$$

$$\frac{T_{m+1}^{p} - 2T_m^p + T_{m-1}^p}{(\Delta x)^2} = \alpha \frac{T_m^{p+1} - T_m^p}{\Delta t}$$

Re-arranging

$$T_m^{p+1} = \frac{\Delta t}{\alpha \Delta x^2}\left[T_{m+1}^p + T_{m-1}^p\right] + \left[1 - \frac{2\Delta t}{\alpha \Delta x^2}\right] T_m^p$$

now, use this eqn along with initial conditions and step sizes to iterate in position and time.

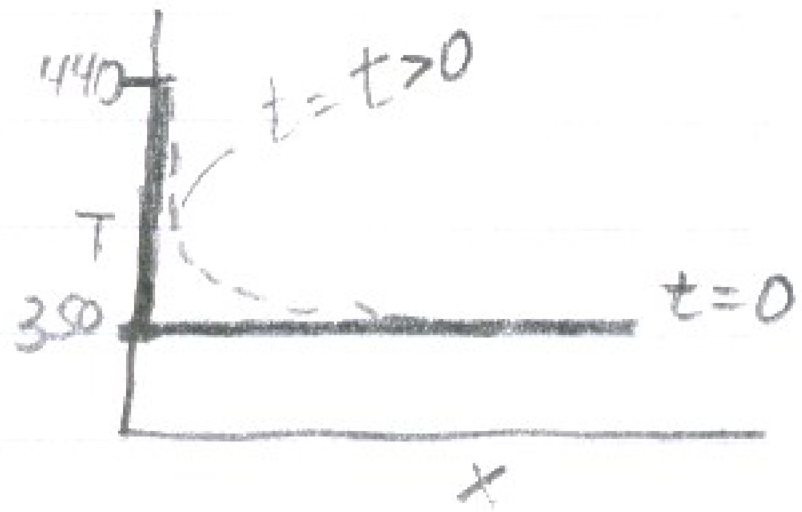
Example calculations

$\Delta x = 0.1$ $\quad \Delta t = 0.1$

$\alpha = 50$

$L = 1$ $\quad 0 \le t \le 4$

$T_0 = 350$ $\quad T_L = 440$




| x \ t | 0 | 0.1 | 0.2 | 0.3 | ... |
|---|---|---|---|---|---|
| 0 | 440 | 440 | 440 | 440 | |
| 0.1 | 350 | → 368 | | | |
| 0.2 | 350 | | | | |
| 0.3 | 350 | | | | |
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$$T_{0.1}^{0.1} = \frac{0.1}{(50)(0.1)^2}\left[440 + 350\right] + \left[1 - \frac{2(0.1)}{(50)(0.1)^2}\right] 350$$

$$= 368$$

Several calculation values are shown below, along with a graph of T vs X at various times.

| | t | | | | | |
|---|---|---|---|---|---|---|
| x | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
| 0 | 440 | 440 | 440 | 440 | 440 | 440 |
| 0.1 | 350 | 368 | 378.8 | 386 | 391.184 | 395.1296 |
| 0.2 | 350 | 350 | 353.6 | 357.92 | 362.096 | 365.8976 |
| 0.3 | 350 | 350 | 350 | 350.72 | 352.016 | 353.6576 |
| 0.4 | 350 | 350 | 350 | 350 | 350.144 | 350.4896 |
| 0.5 | 350 | 350 | 350 | 350 | 350 | 350.0288 |
| 0.6 | 350 | 350 | 350 | 350 | 350 | 350 |
| 0.7 | 350 | 350 | 350 | 350 | 350 | 350 |
| 0.8 | 350 | 350 | 350 | 350 | 350 | 350 |
| 0.9 | 350 | 350 | 350 | 350 | 350 | 350 |
| 1 | 350 | 350 | 350 | 350 | 350 | 350 |