Macroscopic Mechanical Energy Balance

ABE 307

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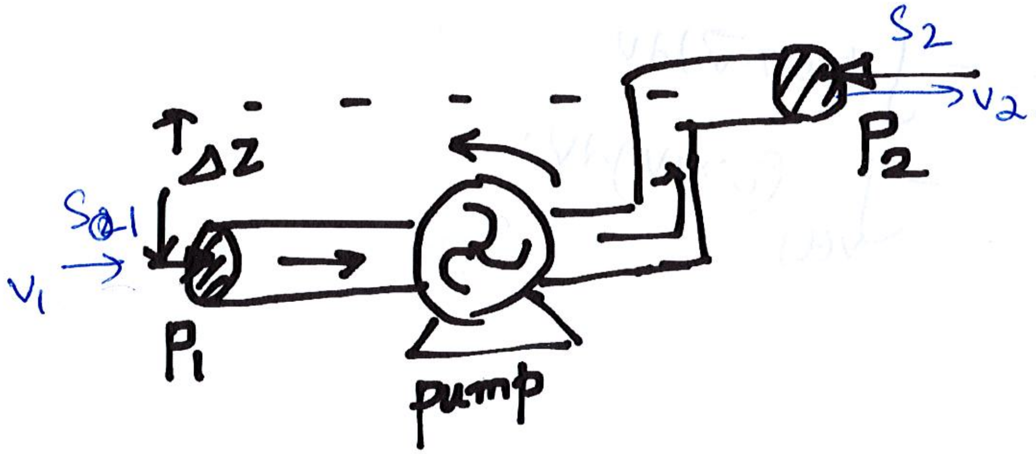
Mechanical energy refers to the energy that can be converted to work. Unlike total energy, mechanical energy is not conserved. This includes Kinetic and Potential Energy of the fluid. In addition, mechanical energy can be added to the systems by pumps (which has moving parts ie rotating impeller which inputs energy). Fluid can also do work by moving turbines. There is always loss of mechanical energy in the system due to viscous dissipation which is related to the frictional losses.



$\hat{\Phi}_1$ = per unit mass potential energy.

Unsteady State Macroscopic Mechanical Energy Balance (Engineering Bernoulli Equation)

$$\frac{d}{dt}\left(KE_{total} + PE_{total}\right) = \left(\frac{1}{2}\rho_1 v_1^2 v_1 S_1 - \frac{1}{2}\rho_2 v_2^2 v_2 S_2\right)$$

$$+ \left[\rho_1 \hat{\Phi}_1 v_1 S_1 - \rho_2 \hat{\Phi}_2 v_2 S_2\right] + W_m$$

$$\cancel{+ \; p_1 S_1} \rightarrow p$$

$$+ \underbrace{\left[p_1 S_1 \langle v_1 \rangle - p_2 S_2 \langle v_2 \rangle\right]}_{\text{work done per unit time by pressure force}}.$$

W_m = mechanical work done on the fluid.

$$+ \underbrace{\int_{V(t)} p(\nabla\cdot\vec{v})\, dV}_{\text{work done due to expansion or contraction in the system}} + \underbrace{\int_{V(t)} (\tau : \nabla v)\, dV}_{\text{viscous dissipation}}$$

Define : $K_{total} = \int_V \frac{1}{2} \rho v^2 dV$ → v = average velocity in whole system.

$\Phi_{total} = \int \rho \Phi dV$ → Φ = average potential energy

$$\frac{d}{dt}(K_{total} + \Phi_{total}) = \left[\frac{1}{2}\rho_1 \langle v_1^3 \rangle S_1 - \frac{1}{2}\rho_2 \langle v_2^3 \rangle S_2\right] + [\hat{\Phi}_1 w_1 - \hat{\Phi}_2 w_2] + \left[\frac{p_1}{\rho_1} w_1 - \frac{p_2}{\rho_2} w_2\right] + W_m + E_c + E_v.$$

$w_1 = \rho_1 \langle v_1 \rangle S_1$
$w_2 = \rho_2 \langle v_2 \rangle S_2$

$$= -\Delta\left[\frac{1}{2}\frac{\langle v^3 \rangle}{\langle v \rangle} + \hat{\Phi} + \frac{p}{\rho}\right] w + W_m \cancel{+} E_c \cancel{+} E_v$$

$E_c = -\int p(\nabla \cdot \underset{\sim}{v})\, dV$ → compression term
+ve → compression
−ve → expansion.
0 → incompressible fluid

$E_v = -\int (\tau : \nabla v) dV$. → viscous dissipation, always +ve for Newtonian fluids.

For Steady State, the mechanical Energy balance is

$$\Delta\left[\frac{1}{2}\frac{\langle v^3 \rangle}{\langle v \rangle} + \Phi + \frac{p}{\rho}\right] w = -E_c - E_v + W_m$$

Simplifying by Finding E_c and E_v terms

continuity eqn
$\frac{\partial \rho}{\partial t} + (\nabla \cdot (\rho \underset{\sim}{v})) = 0.$
at steady state
$(\nabla \cdot \rho \underset{\sim}{v}) = 0.$

$E_c = -\int p(\nabla \cdot \underset{\sim}{v}) dV = 0$

$(\nabla \cdot \rho \underset{\sim}{v}) = 0.$

$\Rightarrow (\nabla \cdot \rho v) = (\nabla \rho \cdot \underset{\sim}{v}) + \rho(\nabla \cdot \underset{\sim}{v}) - 0$

$\Rightarrow (\nabla \cdot \underset{\sim}{v}) = -1/\rho (\nabla \rho \cdot \underset{\sim}{v}).$

$E_c = \int \frac{p}{\rho}(\nabla \rho) \cdot \underset{\sim}{v}\, dV = \frac{p}{\rho}(v \cdot \nabla \rho) dV$ → $\cancel{B}$ $(\nabla \rho) \cdot v = (v \cdot \nabla \rho)$ commutative property.

$dV = S(s)\cdot dS$

$$E_c = \frac{P}{\rho}\,(v\cdot\nabla\rho)\,dV$$

Using the directions as streamline.

$$E_c \approx \int \frac{P}{\rho}\left(\rho\, v\cdot\frac{d\rho}{ds}\right)\cdot S(s)\,ds.$$

$w = \rho v S = \text{const}$ (for steady state)

$w_1 = w_2$.

$$\frac{d}{ds}(1/\rho) = \frac{-1}{\rho^2}\cdot\frac{d\rho}{ds}.$$

$$= \int_1^2 \frac{P}{\rho^2}\, w\cdot\frac{d\rho}{ds}\cdot ds$$

$$= -w\int_1^2 P\cdot\frac{d}{ds}(1/\rho)\cdot ds$$

→ Integration by parts.

$$= -w\left[\frac{P}{\rho}\Big|_1^2 - \int_1^2 \frac{1}{\rho}\cdot\frac{dP}{ds}\,ds\right]$$

$$E_c = -w\left[\frac{\Delta P}{\rho}\right] + w\int_1^2 \frac{1}{\rho}\cdot dP$$

Steady state Macroscopic Balance.

$$\Delta\left[\frac{1}{2}\frac{\langle v^3\rangle}{\langle v\rangle} + \hat{\Phi} + \cancel{\frac{P}{\rho}}\right]w = \cancel{w\frac{\Delta P}{\rho}} - w\int_1^2\frac{1}{\rho}\cdot dP + W_m - E_v.$$

~~Define~~

$$\boxed{\Delta\left[\frac{1}{2}\frac{\langle v^3\rangle}{\langle v\rangle} + \hat{\Phi}\right]w = W_m - E_v - w\int_1^2\frac{1}{\rho}\,dP}$$

Define $\hat{W}_m = \frac{W_m}{w}$ → work done per unit mass flow rate

$\hat{E}_v = \frac{E_v}{w}$

$$\boxed{\Delta\left[\frac{1}{2}\frac{\langle v^3\rangle}{\langle v\rangle} + \hat{\Phi}\right] = \hat{W}_m - \hat{E}_v - \int_1^2\frac{1}{\rho}\cdot dP}$$

Steady state mechanical energy balance most commonly used.

Bernoulli Engineering Equation.

$$\Delta\left[\frac{1}{2}\frac{\langle v^3\rangle}{\langle v\rangle} + \hat{\Phi}\right] + \int_1^2\frac{1}{\rho}\cdot dP = \hat{W}_m - \hat{E}_v$$

→ Eq 7.4.7 in book

$$\boxed{\hat{\Phi} = g\Delta h}$$

Steady State Macroscopic Mechanical Energy Balance for Incompressible Fluid

Estimation of E_v for Incompressible fluid.

$$E_v = -\int (\tau : \nabla v) dV$$

$$-(\tau : \nabla v) = \frac{1}{2}\mu \sum\sum \left[\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \frac{2}{3}(\nabla \cdot v)\delta_{ij}\right]^2 + k(\nabla \cdot v)^2.$$

$$= \mu \Phi_v + \underbrace{k \Phi_v}_{\text{compressibility part}} \to 0$$

For Incompressible liquid

$$E_v = \int \mu \Phi_v \cdot dV.$$

$$[E_v] = \left(\frac{v_0}{l_0}\right)^2 \to \text{since } E_v \text{ will be sum of square of velocity gradients.}$$

$dV \to$ differential volume.

$$\bar{\Phi}_v = \frac{\Phi_v}{\left(\frac{v_0}{l_0}\right)^2} \qquad d\bar{V} = \frac{dV}{(l_0)^3} \to \text{Non-dimensionality.}$$

$$E_v = \int \mu \left(\frac{v_0}{l_0}\right)^2 \bar{\Phi}_v (l_0^3) \cdot d\bar{V}$$

$$= \frac{\mu v_0^2}{l_0^2} l_0^3 \int \bar{\Phi}_v \cdot d\bar{V}$$

$$= \left(\frac{\mu}{\rho v_0 l_0}\right)(v_0^3 l_0^2 \rho) \int \bar{\Phi}_v \cdot d\bar{V}$$

$$\boxed{E_v = \rho v_0^3 l_0^2 \times \text{function of Re \& dimensionless geometrical parameters.}}$$

$$\hat{E}_v = \frac{E_v}{w} = \frac{\rho v_0^3 l_0^2 \times f()}{\rho \langle v \rangle S}$$

$$= \frac{\rho \langle v^3 \rangle S \times f()}{\rho \langle v \rangle S} = \langle v^2 \rangle \times f().$$

$$\boxed{\hat{E}_v = \frac{1}{2}\langle v^2 \rangle \times e_v} \to \text{friction loss factor.}$$

we want to be able to calculate e_v or friction loss factor for various ~~see~~ practical scenario.

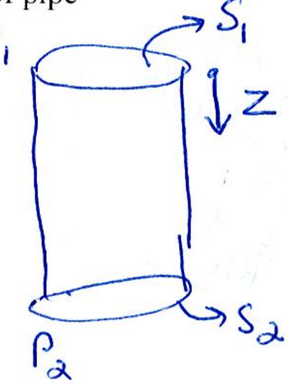
Writing Equation by Considering Friction Loss Factors in Different parts of Pipe System

e_v, f

1. For straight conduit, the friction loss factor is closely related to friction factor. Consider a steady flow of fluid with constant density in a straight section of pipe with cott cross section S and length L. Fluid is flowing in z-direction under the effect of pressure gradient and gravity.

Write Momentum Balance and Mechanical Energy Balance for this section of pipe

$$\cancel{\frac{d}{dt}(P_{total})}^{0} = -\Delta\left[\frac{\langle v^2\rangle}{\langle v\rangle} w + pS\right]\underline{u} + F_{s\to f} + m_{tot}\, g. \quad P_1$$



steady state.

$$\Delta\left[\frac{\langle v^2\rangle}{\langle v\rangle} w + PS\right)\underline{u} - m_{total}\, g = F_{s\to f}.$$

$$F_{f\to s} = -\Delta\left[\cancel{\frac{\langle v^2\rangle}{\langle v\rangle} w} + PS\right]\underline{u} + m_{tot}\, g.$$

$$\boxed{\Delta w = 0, \text{ steady flow}}$$

$S_1 = S_2 = S$

$$F_{f\to s} = -\Delta[PS] + \rho S L g_z$$

$$F_{f\to s} = [P_1 - P_2]S + \rho S L g_z \quad \text{— (1).}$$

Mechanical Energy Balance:

$$\Delta\left(\frac{1}{2}\frac{\langle v^3\rangle}{\langle v\rangle} + \hat{\Phi}\right) = \hat{w}_m - \hat{E}_v - \cancel{\phi}\int_1^2 \frac{1}{\rho}\, dP.$$

$\hat{w}_m = 0 \Rightarrow$ No pump or moving parts work on the pipe section

$$\Delta\left(\frac{1}{2}\frac{\langle v^3\rangle}{\langle v\rangle} + gh\right) = -\hat{E}_v - \cancel{b}\,\frac{1}{\rho}\int_1^2 dP.$$

$w = \rho V S = $ cott.

$$\hat{E}_v = \frac{(P_1 - P_2)}{\rho} + g_z L - \cancel{\Delta\left(\frac{1}{2}\frac{\langle v^3\rangle}{\langle v\rangle}\right)}^{0}.$$

$\because S$ & $w = $ cott $\Rightarrow \langle v_1\rangle = \langle v_2\rangle$

$\therefore \Delta\langle v^3\rangle = 0$

$$\hat{E}_v = \frac{(P_1 - P_2)}{\rho} + g_z L$$

$$\hat{E}_v(\rho S) = (P_1 - P_2)S + \rho g_z S L \quad \text{— (2)}$$

comparing (1) & (2).

$$\hat{E}_v(\rho S) = F_{f\to s}$$

$$\boxed{\hat{E}_v = F_{f\to s}/\rho S};$$

$$F_{f\to s} = \cancel{\tfrac{1}{2}\rho S\langle v^2\rangle f}$$

$$\boxed{F_K = \frac{1}{2}\rho\langle v^2\rangle 2\pi R L f}$$

$$f = \frac{1}{4}\frac{D}{L}\left(\frac{P_0 - P_L}{\frac{1}{2}\rho\langle v^2\rangle}\right) \qquad F_K = F_{f\to s} = (P_0 - P_L)\pi R^2$$

$$F_{f\to s} = (P_0 - P_L) S.$$

$$\hat{E_v} = \frac{F_{f\to s}}{\rho S} = \qquad f = \frac{1}{4}\left(\frac{D}{L}\right)\frac{F_{f\to s}}{S\frac{1}{2}\rho\langle v^2\rangle}$$

$$\hat{E_v} = \frac{2Lf\langle v^2\rangle}{D} \qquad \Rightarrow F_{f\to s} = \frac{2Lf\langle v^2\rangle\rho S}{D}.$$

$\boxed{D = 4R_h}$ R_h = Hydraulic Radius.

$$\hat{E_v} = \frac{2Lf\langle v^2\rangle}{4R_h} = \frac{Lf\langle v^2\rangle}{2R_h} = \frac{1}{2}\langle v^2\rangle\left(\frac{L}{R_h}\right)f.$$

$$= \frac{1}{2}\langle v^2\rangle e_v \Rightarrow \boxed{e_v = \frac{L f}{R_h}}$$

For straight pipe $e_v = \frac{L}{R_h} f$

2. For other parts of pipe where there are bends and fittings etc,
   a. Ev can either be found by solving simultaneous momentum and mechanical energy balance
   b. Experimental Evaluation
      For case of fitting is the average velocity downstream of the bend/fitting.

Thus, approximate form of Mechanical Energy Balance for various piping system is :

$$\frac{1}{2}(v_2^2 - v_1^2) + g(h_2 - h_1) + \int_{P_1}^{P_2} \frac{1}{\rho}\, dP$$

$$= \hat{W}_m - \left(\sum_i \frac{1}{2}\langle v^2\rangle \frac{L}{R_h} f\right) - \sum_i\left(\frac{1}{2} v^2 e_v\right)_i$$

sum over all sections of straight conduits

sum over all fittings, valves meters etc.

Equ^n. 7.5-10