ABE 307

Date: 13th Oct, 2017.

Parallel Disk Viscometer

A fluid, whose viscosity is to be measured, is placed in the gap of thickness B between the two disks of radius R. One measures the torque T_z required to turn the upper disk at an angular velocity Ω. Develop the formula for deducing the viscosity from these measurements. Assume creeping flow.

steady state.

(a) Postulate that for small values of Ω the velocity profiles have the form $v_r = 0$, $v_z = 0$, and $v_\theta = rf(z)$; why does this form for the tangential velocity seem reasonable? Postulate further that $P = P(r, z)$. Write down the resulting simplified equations of continuity and motion.

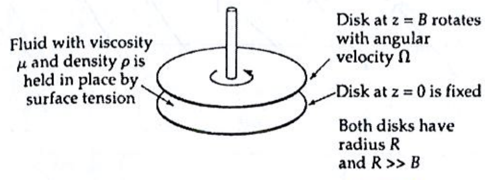


Fig. 3B.5. Parallel-disk viscometer.

(b) From the θ-component of the equation of motion, obtain a differential equation for f(z). Solve the equation for $f(z)$ and evaluate the constants of integration. This leads ultimately to the result $v_\theta = \Omega r(z/B)$. Could you have guessed this result?

(c) Show that the desired working equation for deducing the viscosity is $\mu = 2BT_z/\pi\Omega R^4$

(d) Discuss the advantages and disadvantages of this instrument.

Eqn of Continuity

Cylindrical coordinates (r, θ, z):

$$\cancel{\frac{\partial \rho}{\partial t}} + \frac{1}{r}\frac{\partial}{\partial r}(\cancel{\rho r v_r}) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\cancel{\rho v_z}) = 0 \qquad \text{(B.4-2)}$$

$v_r = 0$, $v_z = 0$.

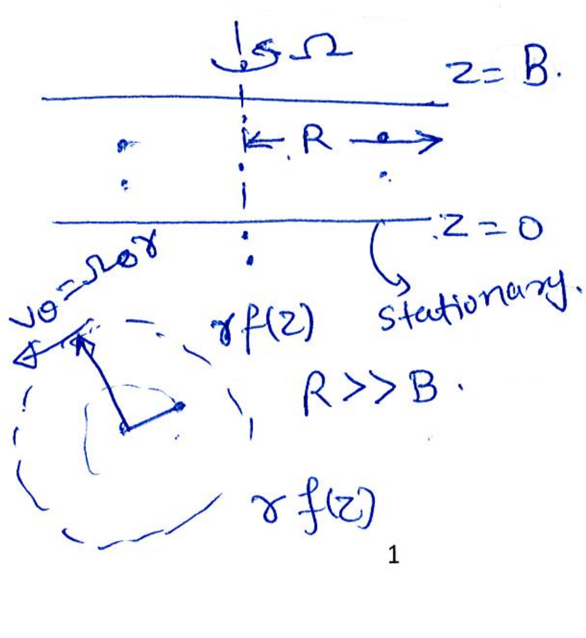
$v_\theta = r f(z)$

$= f(z, r) = r f(z)$.

$\frac{\partial v_\theta}{\partial \theta} = 0$.

$v_\theta = f(r, z)$.

$P = P(r, z)$



Equation of Motion:

Cylindrical coordinates (r, θ, z):

steady state

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \rho g_r \quad \text{(B.6-4)}$$

(from continuity).

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] + \rho g_\theta \quad \text{(B.6-5)}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \quad \text{(B.6-6)}$$

r component

$$-\frac{\rho v_\theta^2}{r} = \frac{-\partial p}{\partial r}$$

θ - component:

$$0 = \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial}{\partial r}(r v_\theta)\right)\right] + \mu\frac{\partial^2 v_\theta}{\partial z^2}$$

z - component.

$$0 = \frac{-\partial p}{\partial z} + \rho g_z$$

(b)

$$V_\theta = r f(z).$$

$$\mu \frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial}{\partial r}(r V_\theta)\right) + \mu \frac{\partial^2 V_\theta}{\partial z^2} = 0.$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial}{\partial r}(r^2 f(z)\right) + \frac{\partial^2}{\partial z^2}(r f(z)) = 0.$$

$$= \frac{\partial}{\partial r}\left[\frac{1}{\cancel{r}}\times\left(2\cancel{r}\cdot f(z) + \cancel{r^2 \frac{\partial}{\partial r} f(z)}\right)\right] + \frac{\partial^2}{\partial z^2}(r.f(z)) = 0$$

$f(z)$.

$f(z) = z$

$\frac{\partial}{\partial r}(f(n)) = 0$

$$= \frac{\partial}{\partial r}\left[2 f(z)\right] + r\frac{\partial^2 f(z)}{\partial z^2} = 0.$$

$$\frac{\partial^2 f(z)}{\partial z^2} = 0.$$

$$\boxed{\frac{d^2 f(z)}{dz^2} = 0}$$

$V_\theta = r f(z)$

Boundary conditions.

$V_\theta = r f(z) \leftarrow V_\theta = \Omega r \;, z = B$

$f(z) = \Omega$ at $z = B$

$V_\theta = 0 \;, z = 0$

$V_\theta = r f(z)$

$f(z) = 0$ at $z = 0$

B. C

$z = 0 \;, f(z) = 0$

$z = B \;, f(z) = \Omega.$

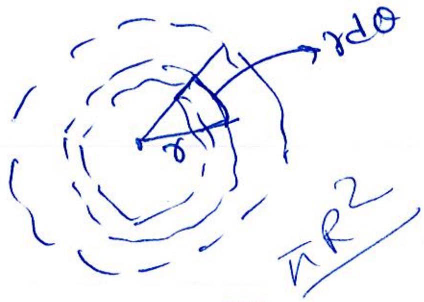
$$f(z) = C_0 z + C_1.$$

$$\boxed{f(z) = \frac{\Omega z}{B}}$$

$$\boxed{V_\theta = \frac{\Omega r z}{B}}$$

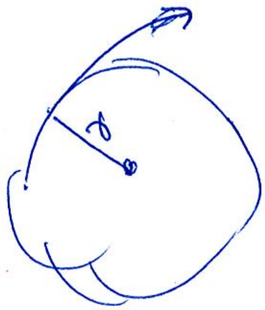
$$\tau_{z\theta} = -\mu \frac{\partial v_\theta}{\partial z}$$

$$T = \int_0^R \int_0^{2\pi} \underbrace{\tau_{z\theta}\Big|_{z=B} \times r}_{\text{Torque.}} \; \underbrace{r\,d\theta . dr}_{\text{surface area}}$$



$$\boxed{\mu = \frac{2BT_z}{\pi \Omega R^4}}$$

$\tau = F \times r$



$$\tau_{z\theta} = \frac{-\mu \Omega r}{B}.$$

$$T_z = \int_0^R \int_0^{2\pi} \frac{\mu \Omega r^3}{B} . d\theta . dr$$

$$= \frac{2\pi\mu\Omega}{B} \int_0^R r^3 . dr$$

$$T_z = \frac{\pi \mu \Omega R^4}{2B} \Rightarrow \mu = \frac{2BT_z}{\pi \Omega R^4}$$

(d) Advantage → The geometry of the instrument is easier to manufacture and clean as compared to rotational viscometer with cylinders.

Disadvantage

→ $\tau_{z\theta} = \frac{-\mu \Omega r}{B}$ → The shear stress is non-uniform and leads to uneven ~~wear~~ wear & tear of instrument.