ABE 307, Fall 2017
Homework 3

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Problem 1

1.) $V_z = \frac{\rho g}{2\mu}\delta^2\left(1-\left(\frac{x}{\delta}\right)^2\right)$ (derived in class)

$$V_{avg} = \frac{\int_0^w\int_0^\delta V_z\,dx\,dy}{\int_0^w\int_0^\delta dx\,dy} = \frac{1}{\delta}\int_0^\delta V_z\,dx.$$

$$= \frac{\rho g \delta}{2\mu}\int_0^\delta\left(1-\left(\frac{x}{\delta}\right)^2\right)dx$$

$$= \frac{\rho g \delta}{2\mu}\left(x-\frac{x^3}{3\delta^2}\right]_0^\delta$$

$$= \frac{\rho g \delta}{2\mu}\left[\delta-\frac{\delta^3}{3\delta^2}\right] = \frac{\rho g \delta}{2\mu}\left[\delta-\frac{\delta}{3}\right] = \frac{2\rho g \delta^2}{6\mu}$$

$$= \frac{\rho g \delta^2}{3\mu}.$$

w = mass flow rate $= \rho w \delta V_{avg}$

$$= \frac{\rho^2 g w \delta^3}{3\mu} \qquad \nu = \frac{\mu}{\rho}$$

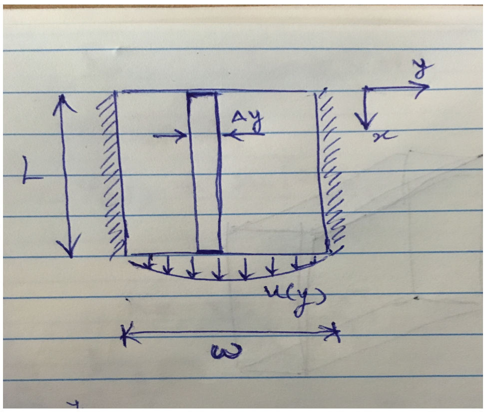
$$= \frac{\rho g w \delta^3}{3\nu}.$$

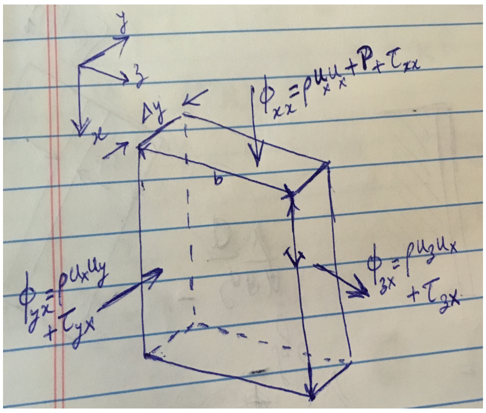
$$= \frac{(0.8\times10^3)(9.81)\,w(0.003)^3}{3(2\times10^{-4})}.$$

$$= 0.3528\ w\ \text{kg/s}.$$

Problem 2

Consider a thin shell as shown in the figure below:





Assume the thickness of the shell and the width of the shell to be Δy and b respectively. Let the velocity in the flow be $u(y)$. Assuming the flow to be steady and fully developed, we have:
(Momentum Flux entering the shell)−(Momentum Flux leaving the shell)+Forces along X direction $= 0$. As shown in the figure, we have:

$$\phi_{xx} = \rho u_x u_x + P + \tau_{xx} \tag{1}$$

$$\phi_{yx} = \rho u_y u_x + \tau_{yx} \tag{2}$$

$$\phi_{zx} = \rho u_z u_x + \tau_{zx} \tag{3}$$

Force due to gravity:

$$F_g = \rho g \Delta y b L. \tag{4}$$

Since the flow is fully developed we have $\frac{\partial u_x}{\partial x} = 0$, and $u_y = u_z = 0$. Hence, we have:

$$\phi_{xx} = \rho u_x u_x + P \tag{5}$$

$$\phi_{yx} = \tau_{yx} = -\frac{\mu \partial u_x}{\partial y} \tag{6}$$

$$\phi_{zx} = 0 \tag{7}$$

Plugging into the momentum conservation equation, we have:

$$(\phi_{xx,0} - \phi_{xx,L}).\Delta y.b + (\phi_{yx,y} - \phi_{yx,y+\Delta y}).L.b + F_g = 0. \tag{8}$$

$$(\rho u_x^2 + P0 - \rho u_x^2 - P1).\Delta y.b + \left(-\mu \frac{\partial u_x}{\partial y}_y + \mu \frac{\partial u_x}{\partial y}_{y+\Delta y} \right).L.b + \rho g \Delta y b L = 0. \tag{9}$$

Simplifying the above equation and using $\left(-\frac{\partial u_x}{\partial y}_y + \frac{\partial u_x}{\partial y}_{y+\Delta y}\right) = \frac{\partial^2 u_x}{\partial y^2}.\Delta y$, we have:

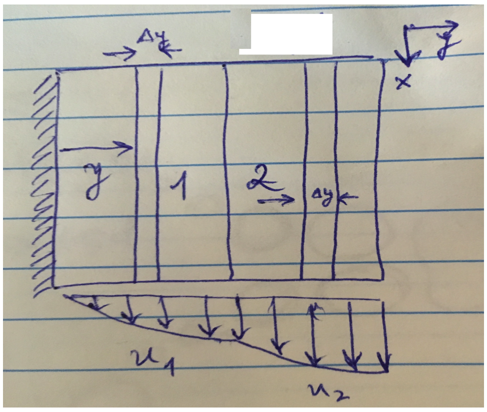
$$\mu\frac{\partial^2 u}{\partial y^2} + \rho g + \frac{(P0 - P1)}{L} = 0. \tag{10}$$

We solve the above equation with the following boundary conditions:
$u(y = 0) = 0$,
$u(y = W) = 0$, to arrive at:

$$u(y) = \frac{1}{2\mu}\left(\frac{P1 - P0}{L} - \rho g\right)\left[y^2 - Wy\right] \tag{11}$$

Problem 3

We consider 2 thin shells (as done in Problem 2) in Oils 1 and 2 of thickness Δy, length L, and width b. Following the steps laid down in Equations 1 through 9, and using the fact that in the present problem there is no applied pressure gradient we have for each shell:



$$(\rho u_x^2 - \rho u_x^2).\Delta y.b + \left(-\mu\frac{\partial u_x}{\partial y}_y + \mu\frac{\partial u_x}{\partial y}_{y+\Delta y}\right).L.b + \rho g\Delta y b L = 0. \tag{12}$$

Which simplifies to:

$$\mu\frac{\partial^2 u}{\partial y^2} + \rho g = 0. \tag{13}$$

Since we have two different fluids with different properties, we assume two different velocities $u_1(y)$ and $u_2(y)$ such that for oil 1, we have:

$$\mu_1 \frac{\partial^2 u_1}{\partial y^2} + \rho_1 g = 0, \tag{14}$$

and for oil 2:

$$\mu_2 \frac{\partial^2 u_2}{\partial y^2} + \rho_2 g = 0. \tag{15}$$

Solving the above equations, we have:

$$u_1(y) = -\frac{\rho_1 g y^2}{2\mu_1} + a_1 y + b_1, \tag{16}$$

and,

$$u_2(y) = -\frac{\rho_2 g y^2}{2\mu_2} + a_2 y + b_2, \tag{17}$$

where a_1, b_1, a_2, and, b_2 are constants to be determined using boundary conditions. We have the following boundary conditions:
a) $u_1(y = 0) = 0$.

b) $u_1(y = d) = u_2(y = d)$.
Since oil 2 is on contact with air which has negligible density and viscosity,

c) $\frac{\partial u_2}{\partial y}_{y=2d} = 0$.

Finally, since the two fluids are in contact with each other, there is stress balance across the interface, and hence,

d) $\mu_1 \frac{\partial u_1}{\partial y}_{y=d} = \mu_2 \frac{\partial u_2}{\partial y}_{y=d}$. Solving the above equations, we have:

$$a_1 = \frac{\rho_2 g d + \rho_1 g d}{\mu_1}, a_2 = \frac{2\rho_2 g d}{\mu_2}. \tag{18}$$

$$b_1 = 0, b_2 = \left[\frac{\rho_1 g d^2}{2\mu_1} + \frac{\rho_2 g d^2}{\mu_1} - \frac{3\rho_2 g d^2}{2\mu_2}\right]. \tag{19}$$

Thus, we have:

$$u_1(y) = -\frac{\rho_1 g y^2}{2\mu_1} + \left(\frac{\rho_2 g d + \rho_1 g d}{\mu_1}\right) y, \tag{20}$$

and

$$u_2(y) = -\frac{\rho_2 g y^2}{2\mu_2} + \frac{2\rho_2 g d y}{\mu_2} + \left[\frac{\rho_1 g d^2}{2\mu_1} + \frac{\rho_2 g d^2}{\mu_1} - \frac{3\rho_2 g d^2}{2\mu_2}\right]. \tag{21}$$