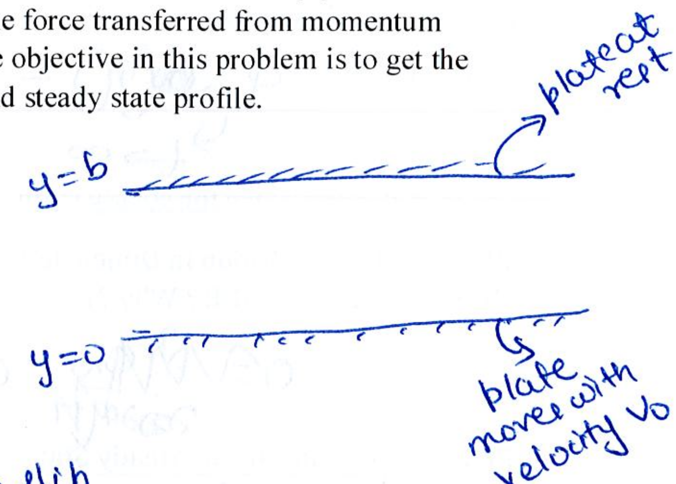
ABE 307
Date:

Unsteady State: Laminar Flow in Between Parallel Plate

Consider a situation where fluid is at rest and the bottom plate starts moving with velocity v0. As $t \rightarrow \infty$, the velocity profile of fluid reaches a steady state. This is because the top plate at rest provides a resistance to the fluid flow which balances the force transferred from momentum imparted by the bottom flow over a long time scale. The objective in this problem is to get the velocity profile for the fluid that reflects the transient and steady state profile.



Equation of Motion:

$$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}$$

ν $\nu = \frac{\mu}{\rho}$.

Boundary Conditions:

Initial Condition (I.C.) = $V_x = 0 \quad t < 0$.

B.C. 1 = $V_x = V_0 \ , \ y = 0, \ t \geq 0$ } No slip condition.

B.C.2 = $V_x = 0 \quad y = b \ ; t > 0$

Dimensionless Variable: Choose dimensionless variables that can change BCs from 0 to 1

For choosing dimensionless variable you need to choose variables representative of each variable in the original equation such as velocity, distance, time.

Dimensionless Velocity $\phi = \frac{V_x}{V_0}$; Dimensionless distance $\eta = \frac{y}{b}$;

Dimensionless time τ = $\frac{\nu t}{b^2}$. ν, t, b, y.

Convert original Eqn of Motion in terms of dimensionless variable:

$$\frac{\partial \phi}{\partial \tau} = \frac{\partial^2 \phi}{\partial \eta^2}. \qquad \phi = \phi(\eta, \tau)$$

Change I.C and Boundary Conditions in form of dimensionless variables:

I.C = $\phi = 0 \ , \tau < 0$.

B.C 1 = $\phi = 1, \ \eta = 0 \ ; \tau \geq 0$.

B.C. 2 = $\phi = 0, \ \eta = 1 \ ; \tau > 0$.

$\phi = 0$ at $\eta = 1$

$\phi_\infty = 0$ at $\eta = 1$

$\phi = \phi_\infty - \phi_t$

$\phi = 0$ at $\eta = 1$

$\phi_\infty = 0$ at $\eta = 1$

$\phi_t = \phi - \phi_\infty$

$= 0$ at $\eta = 1$

$$\frac{\partial V_x}{\partial t} = \nu \frac{\partial^2 V_x}{\partial y^2}$$

$$\boxed{\phi = \frac{V_x}{V_0} \Rightarrow \frac{\partial V_x}{\partial \phi} = V_0}$$

$$\boxed{\tau = \frac{\nu t}{b^2} \qquad \frac{\partial \tau}{\partial t} = \frac{\nu}{b^2}}$$

L.H.S

$$\frac{\partial V_x}{\partial t} = \frac{\partial V_x}{\partial \tau} \cdot \frac{\partial \tau}{\partial t}$$

$$= \frac{\nu}{b^2} \frac{\partial V_x}{\partial \tau}$$

$$= \frac{\nu}{b^2} \cdot \frac{\partial V_x}{\partial \phi} \cdot \frac{\partial \phi}{\partial \tau}$$

$$\boxed{\frac{\partial V_x}{\partial t} = \frac{V_0 \nu}{b^2} \cdot \frac{\partial \phi}{\partial \tau}}$$

$$\eta = \frac{y}{b}$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{b}$$

R.H.S

$$\nu \cdot \frac{\partial^2 V_x}{\partial y^2} = \nu \cdot \frac{\partial}{\partial y} \left(\frac{\partial V_x}{\partial y} \right)$$

$$= \nu \frac{\partial}{\partial y} \left[\frac{\partial V_x}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \right]$$

$$= \nu \frac{\partial}{\partial y} \left[\frac{1}{b} \cdot \frac{\partial V_x}{\partial \eta} \right]$$

$$= \nu \cdot \frac{\partial}{\partial y} \left[\frac{1}{b} \cdot \frac{\partial V_x}{\partial \phi} \cdot \frac{\partial \phi}{\partial \eta} \right]$$

$$= \nu \cdot \frac{\partial}{\partial y} \left[\frac{1}{b} \cdot V_0 \cdot \frac{\partial \phi}{\partial \eta} \right]$$

$$= \frac{\nu V_0}{b} \frac{\partial}{\partial y} \left[\frac{\partial \phi}{\partial \eta} \right]$$

$$= \frac{\nu V_0}{b} \frac{\partial}{\partial \eta} \left[\frac{\partial \phi}{\partial \eta} \right] \cdot \frac{\partial \eta}{\partial y}$$

$$\boxed{= \frac{\nu V_0}{b^2} \frac{\partial^2 \phi}{\partial \eta^2}}$$

$$\cancel{\frac{v_0}{b^2}\frac{\partial\phi}{\partial t}}$$

$$\frac{\nu \cancel{v_0}}{\cancel{b^2}}\frac{\partial\phi}{\partial\tau} = \frac{\nu \cancel{v_0}}{\cancel{b^2}}\frac{\partial^2\phi}{\partial\eta^2}$$

$$\boxed{\frac{\partial\phi}{\partial\tau} = \frac{\partial^2\phi}{\partial\eta^2}}$$ → Dimensionless form for your eqn. of motion

Finding Solution to the Problem

We aim to convert the PDE into ODE through use of BCs and form of velocity profile assumed based on expectations about the dependence of velocity profile on certain parameters.

We know that velocity will have two components: steady state and transient state. We can postulate the velocity as

$\Phi(\eta, \tau) =$ $\underbrace{\phi_\infty(\eta)}_{t\to\infty} - \phi_t(\eta, \tau).$

a. Let's first solve for steady state

$$\frac{\partial \phi}{\partial \tau} = \frac{\partial^2 \phi}{\partial \eta^2}$$

$$\cancel{\frac{\partial \phi_\infty}{\partial \tau}}^{\,0} = \frac{\partial^2 \phi_\infty}{\partial \eta^2}$$

Write the Eqn of Motion in Dimensionless variable form for steady state
(Is this eqn PDE or ODE ? Why ?)

$$0 = \frac{\partial^2 \phi_\infty}{\partial \eta^2}.$$

$$\left(\begin{array}{l} \phi = 1 \text{ at } \eta = 0 \\ \phi = 0 \text{ at } \eta = 0.1 \end{array} \right)$$

Boundary Conditions for Steady State:

$\phi_\infty = 1$ at $\eta = 0$

$\phi_\infty = 0$ at $\eta = 1$

$\phi = \phi_\infty^{-bt}$

$\phi\ t=0.$

Solve the steady state velocity profile eqn :

$$\frac{\partial^2 \phi_\infty}{\partial \eta^2} = 0$$

$$\phi_\infty = c_1 \eta + c_2.$$

at $\eta = 0$, $\phi_\infty = 1$

$\Rightarrow c_2 = 1$

at $\eta = 1$, $\phi_\infty = 0$

$0 = c_1 + 1 \Rightarrow c_1 = -1$

$\boxed{\phi_\infty = 1 - \eta}$ $\rightarrow$ steady state velocity profile.

$$\phi(\eta, \tau) = (1-\eta) - \phi_t(\eta, \tau).$$

b. Finding Solution for the transient state velocity profile

Substitute the overall velocity function in the dimensionless form of eqn of motion to obtain a differential eqn for transient state velocity profile.

$$\cancel{+}\frac{\partial \phi_t}{\partial \tau} = \frac{\cancel{f}\partial^2 \phi_t}{\partial \eta^2} \qquad \phi_t = f(\eta, \tau) \qquad \boxed{\frac{\partial \phi}{\partial \tau} = \frac{\partial^2 \phi}{\partial \eta^2}}$$

$$\frac{\partial \phi_t}{\partial \tau} = \frac{\partial^2 \phi_t}{\partial \eta^2} \quad \text{— I} \qquad \phi = 0 \text{ at } \eta = 1$$

Write the boundary conditions specifically for transient state velocity component

$$\phi_t = 0, \quad \eta = 0, 1 \qquad \phi = 1 \text{ at } \eta = 0.$$

$$\phi_t = \phi_\infty \quad \tau = 0 \qquad \phi_\infty = 1 \text{ at } \eta = 0.$$

$$\phi = \phi_\infty - \phi_t.$$

This is where the method of separation of variables will be useful. We now postulate the transient state velocity profile to be product of two functions [f(η) ; g (τ)]which are only dependent on one variable independently.

Use this new functional form of transient velocity to get differential equn for solving :

$$\phi_t = f(\eta) \cdot g(\tau).$$

Sub. in Equn.
$$f(\eta) \cdot \cancel{g(\tau)} \frac{d g(\tau)}{d\tau} = g(\tau) \frac{d^2 f(\eta)}{d\eta^2}$$

Our goal of converting our PDEs to analytically solvable ODEs has been achieved. From here on the solution is all about using proper limits and obtaining solution to these ODEs that satisfy the requirement for velocity profile.

$$\boxed{\frac{1}{g(\tau)} \cdot \frac{dg(\tau)}{d\tau} = \frac{1}{f(\eta)} \cdot \frac{d^2 f(\eta)}{d\eta^2}} = -c^2.$$

$$\frac{d^2 y}{dx^2} + c^2 x = 0 \qquad = c^2$$

$$\frac{1}{g(\tau)} \cdot \frac{dg(\tau)}{d(\tau)} = -c^2.$$

$$\int \frac{dg(\tau)}{g(\tau)} = \int -c^2 \cdot d(\tau).$$

$$\boxed{g(\tau) = A e^{-c^2\tau}}.$$

$$\phi_t = f(\eta)\, g(\tau).$$

$$\frac{1}{f(\eta)} \cdot \frac{d^2 f(\eta)}{d\eta^2} = -c^2$$

$$\frac{1}{f(\eta)} \cdot \frac{d^2 f(\eta)}{d\eta^2} + c^2 = 0 \Rightarrow.$$

$$\boxed{f(\eta) = B\sin(c\eta) + C\cos(c\eta).}$$

Solve for constants using B.Cs.

$\phi_t = 0, \quad \eta = 0.$

$\phi(t) = f(\eta)\, g(\tau).$

$C = 0$ (because we need $f(\eta) = 0$ at $\eta = 0$

$\phi_t = 0, \quad \eta = 1.$

$f(\eta) = 0$ at $\eta = 1$

$f(\eta) = B\sin(c\eta).$

$f(\eta)\big|_{\eta=1} = B\sin(c). = 0$

$\sin(c) = 0.$

$B = 0$

But if $B = 0$ $f(\eta) \to 0$ X (not feasible solution).

$\sin(c) = 0.$

$\boxed{c = n\pi}$ $n = 0, \pm 1, \pm 2 \ldots$

$$\boxed{f(\eta) = B_n \sin(n\pi\eta)} \quad n = 0, \pm1, \pm2 \ldots$$

$$g(\tau) = A e^{-c^2\tau}.$$

$$g(\tau) = A_n e^{-n^2\pi^2\tau}.$$

$$\phi_t = \left(A_n e^{-n^2\pi^2\tau}\right)\left(B_n \sin(n\pi\eta)\right)$$

$$\phi_t = \sum_{n=0}^{\infty} D_n \left(e^{-n^2\pi^2\tau}\right) \sin(n\pi\eta)$$

(I C)

$$\phi_t = (1-\eta) \text{ at } \tau = 0.$$

$$(1-\eta) = \sum_{n=1}^{\infty} D_n \sin n\pi\eta. \text{ — (11)}$$

To find D_n, we will solve (11).

Multiply by ~~sin~~ $\sin m\pi\eta$ on both sides and integrate.

$$\int_0^1 (1-\eta) \sin m\pi\eta \cdot d\eta = \int_0^1 D_n \sin n\pi\eta \cdot \sin(m\pi\eta) \cdot d\eta.$$

$$\frac{1}{m\pi}$$

when $n = m$, $=$ ~~D_n~~ $\quad n \neq m.$

$$\int_0^1 D$$

$$\frac{1}{m\pi} = \frac{D_n}{2}$$

$$\boxed{D_n = \frac{2}{m\pi}}$$

when $n=m$

$$\int_0^1 D_n \sin^2 n\pi\eta \cdot d\eta .$$

$$= \int_0^1 D_n \left[1-\cos 2n\pi\eta\right] . d\eta .$$

$$= D_n \left[\int_0^1 d\eta - \int_0^1 \cos 2n\pi\eta \cdot d\eta\right]$$

$$= D_n \frac{1}{2}\left[\left[1 - \frac{\sin 2n\pi\eta}{2n\pi}\right]_0^1\right]$$

$$= D_n \frac{1}{2}\left[1 - \frac{\cancel{\sin(2n\pi - \sin 0)}}{2n\pi}\right]$$

$$= D_n \frac{1}{2} .$$

$$(\cancel{L.H.S}) = D_n \frac{1}{2} . = \underline{R.H.S.}$$

$$\boxed{\int_0^1 (1-\eta) \sin m\pi\eta \cdot d\eta = \frac{1}{m\pi}}$$

$$D_n = \frac{2}{m\pi} .$$

$n \neq m$ ~~solve for~~

$\sin a \sin b$

Solve at home using $\sin a \sin b$ functions, it should come out to be zero

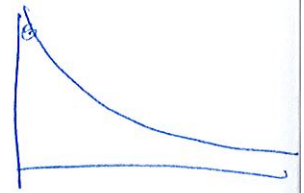
$$\sin^2(x) = \frac{1}{2}\left[1-\cos(2x)\right]$$

~~$\cos 2(x) =$~~

$$\sin^2(x) = \frac{1}{2}\left[1-\cos 2x\right].$$

$$\frac{1}{m\pi} = \sum_{n=1}^{\infty} \frac{D_n}{2}$$

$$\boxed{D_m = \frac{2}{m\pi}}$$



$$\phi(\eta,\tau) = (1-\eta) - \sum_{n=1}^{\infty} \left(\frac{2}{n\pi}\right) \exp(-n^2\pi^2\tau) \sin n\pi$$

as velocity fades out exponentes