Chapter 7.

Macroscopic Balances For Isothermal Flow Systems

ABE 307
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In earlier part of course the equation of continuity and equation of motion were derived based on mass balance and momentum balance respectively over a "microscopic system" or small volume element. This led to partial differential equations. In the case of microscopic system the fluid element did not have a solid boundary, and the interactions of fluid with solid surfaces was accounted by using boundary conditions on the differential equations.

For Macroscopic Balances, we write balance equations or conservation equations for whole system together. These equations will therefore, include the fluid interaction with solid boundary surfaces. For **unsteady state** these balance equations are "ordinary differential equation" and for **steady state systems** these will be simple algebraic equations.

Forces include the forces and torque exerted by the fluid on the solid surface and the surroundings can do work W_m on the fluid by moving surfaces.

Relationships Between Equations of Change and Macroscopic Balances

directly write.

$\int_V$ eqn. of continuity dV = Eqn of Macroscopic Mass Balance.

$\int_V$ eqn of motion dV = Macroscopic momentum balance.

$\int$ (eqn of mechanical energy) dV = Macroscopic Mechanical energy balance.

Path dependent

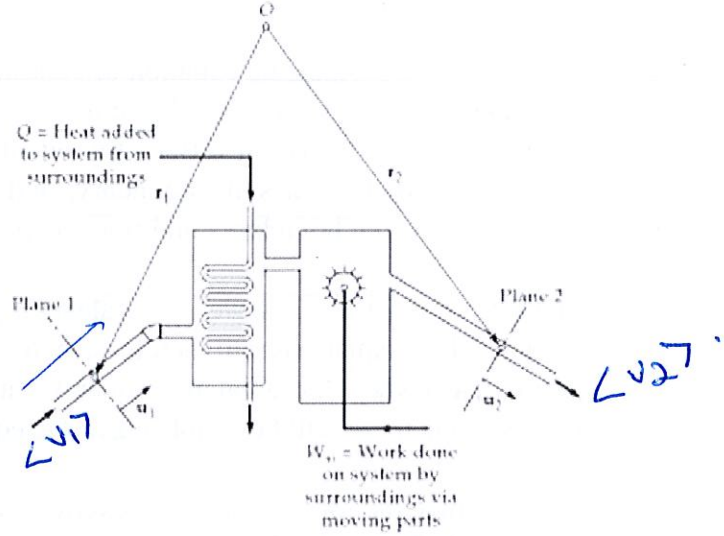
Why do We Use Macroscopic Balances?

- Simplicity of use, provides description of large systems without going through the tedious process of writing equation of change for each part of system and identifying boundary conditions to solve.
- Used to derive approximate relations that can be augmented with experimental data.
- Initial check for engineering problems
- Order of magnitude estimates of various quantities.

The Macroscopic Mass Balance

Fluid enters the plane 1 with cross section area S1 and leaves at plane 2 with cross section area S2.

Average velocity at S1 = $\langle v_1 \rangle$
Average velocity at S2 = $\langle v_2 \rangle$



(m_{total})

Assumptions for the System

i. At the planes 1 and 2, the time smoothed velocity is perpendicular to the respective planes.
ii. The properties of fluid such as density and other physical properties are uniform.

Law of Conservation of Mass

$$\frac{d(m_{total})}{dt} = \text{Mass flow rate entering at plane 1} - \text{Mass flowrate leaving at plane 2}$$

$$= \rho_1 \langle v_1 \rangle S_1 - \rho_2 \langle v_2 \rangle S_2 .$$

$\rho v S = w$ mass flow rate.

Unsteady State Macroscopic Mass Balance

$$\frac{d}{dt}(m_{total}) = w_1 - w_2$$
$$= -\Delta w$$

$\Delta w = w_2 - w_1 .$

Steady State Macroscopic Mass Balance

$$\Delta w = 0 .$$

Even though we write a steady state balance for whole system, within the system the flow can still be unsteady in various parts.

The Macroscopic Momentum Balance

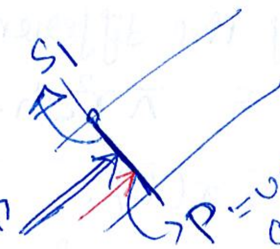
Assumptions for the System

i. At the planes 1 and 2, the time smoothed velocity is perpendicular to the respective planes.

ii. The properties of fluid such as density and other physical properties are uniform.

iii. Forces associate with stress tensor τ are neglected on planes because : pressure forces dominating where fluid is entering & leaving.

iv. The Pressure does not vary over the cross section



P = uniform across the cross section.

$$\frac{d}{dt}\left(P_{momentum-total}\right) = \left[\underbrace{\rho_1 \langle v_1\rangle S_1}_{m_1} \underbrace{\langle v_1\rangle}_{\langle v_1\rangle_1} \vec{u}_1 - \rho_2 \langle v_2\rangle S_2 \langle v_2\rangle \vec{u}_2 \right] + \left[P_1 S_1 \vec{u}_1 - P_2 S_2 \vec{u}_2 \right] + (m_{total})\vec{g} + F_{s\to f}$$

Momentum = $P_{momentum}$.

$\vec{u}_1$ = unit vector normal to cross section S_1.

$\vec{u}_2$ = unit vector normal to cross section S_2.

$F_{s\to f}$ = Workdone by solid surface on the fluid.

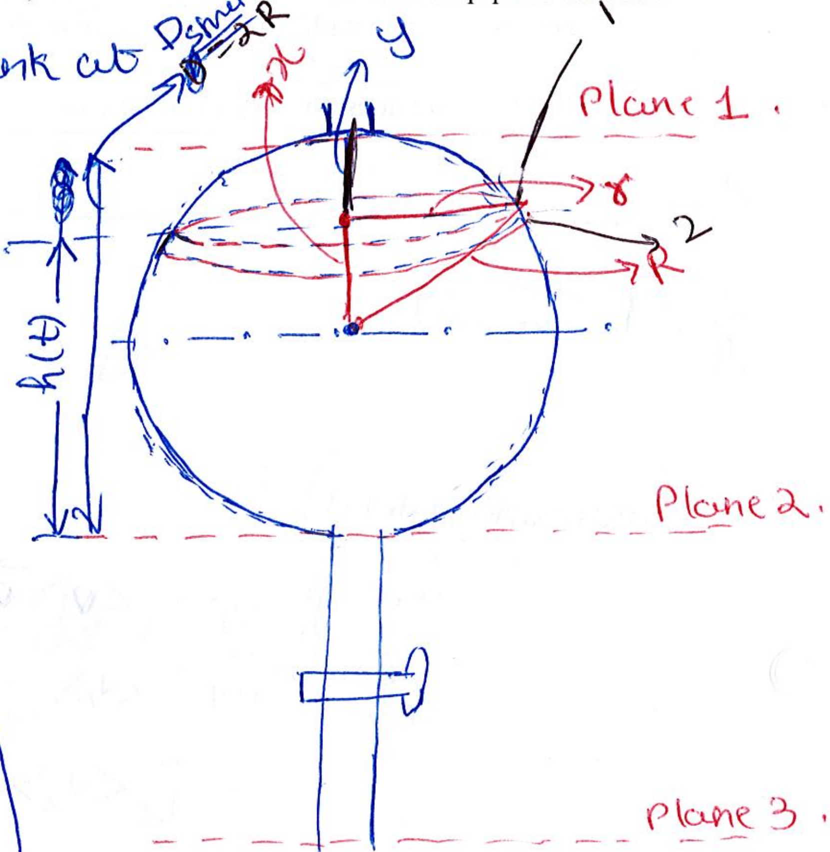
Steady state: $\frac{d}{dt}\left(P_{momentum-T}\right) = 0$.

$$\boxed{\Rightarrow F_{f\to s} = \rho_1 \langle v_1^2\rangle S_1 \vec{u}_1 - \rho_2 \langle v_2^2\rangle S_2 \vec{u}_2 + \left[P_1 S_1 \vec{u}_1 - P_2 S_2 \vec{u}_2\right] + m_{total}\vec{g}}$$

Example 7.1.1: Draining of a Spherical Tank

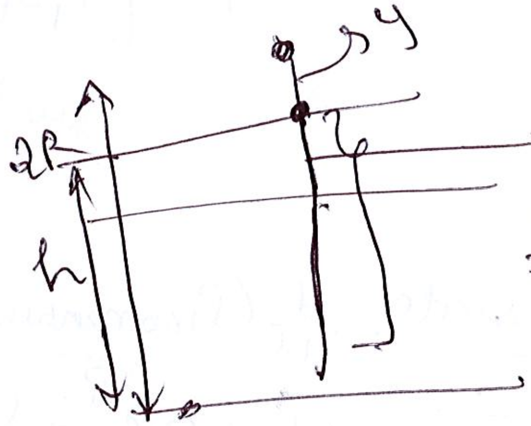
A spherical tank of radius R and its drainpipe of length L and diameter D are completely filled with a heavy oil. At time t=0 the valve at the bottom of the drainpipe is opened. How long will it take to drain the tank? There is an air vent at the very top of the spherical tank. Ignore the amount of oil that clings to the inner surface of the tank, and assume that the flow in drainpipe is laminar.

$h(t)$ = height of liquid in tank at any instant t.



Volume of the differential disk $= \pi r^2 dh$.

$$\boxed{\begin{aligned} r &= r(t). \\ y &= 2R - h. \\ x &= R - y \\ &= R - (2R-h) \\ x. &= h - R \\ r^2 &= R^2 - x^2 \\ &= R^2 - (h-R)^2 \\ r^2 &= 2Rh - h^2. \end{aligned}}$$



$y = 2R - h$

$x = R - y$

$= R - (2R - h)$

$$dV = \pi (2Rh - h^2) \cdot dh.$$

$$dM = \rho dV$$

$$dM. = \rho \pi (2Rh - h^2) \cdot dh.$$

$$m_{total} = \int_0^{h} \rho \pi (2Rh - h^2) \cdot dh.$$

$$\frac{dM}{dt} = \rho \frac{d}{dt}\left[\;\right].$$

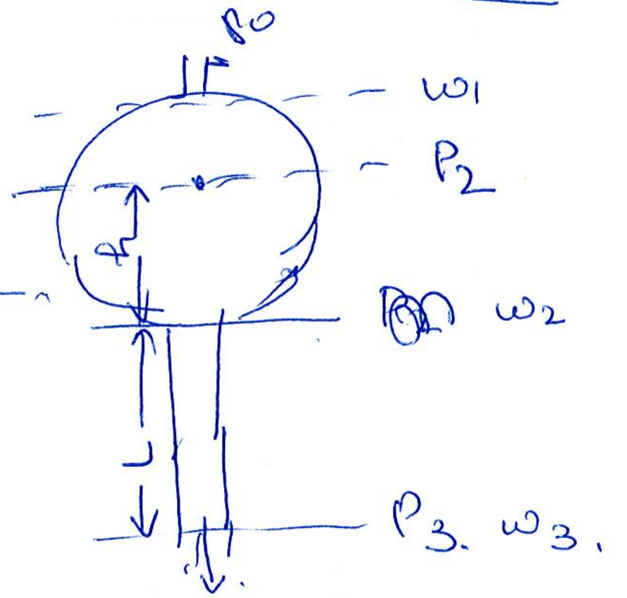
Since, the mass change in tank is flowing out of pipe

~~mass flowrate~~ rate of change of mass in tank = mass flowrate out of pipe.

Hagen Poiseuille Equation

$$\Delta P = \frac{128 \mu L Q}{\pi D^4}$$

Q = volumetric flow rate
D = Diameter of pipe.



$$P_2 - P_3 = \frac{128 \mu L Q}{\pi D^4}.$$

$P_2 = P_0$
$P_3 = \rho g h + \rho g L + P_0$
$\Rightarrow P_2 - P_3 = \rho g (h+L)$.

$\Delta w = \cancel{w_1}^{0} - w_2$.

$\Delta w = -w_2$

w_3 = mass flow rate exiting the pipe

$= Q \rho$

$$w_3 = \frac{(P_2 - P_3)\pi D^4 \rho}{128 \mu L}$$

$\Rightarrow -w_2 = w_3$.

$$\Rightarrow -w_2 = \frac{(P_2 - P_3)\pi D^4 \rho}{128 \mu L}.$$

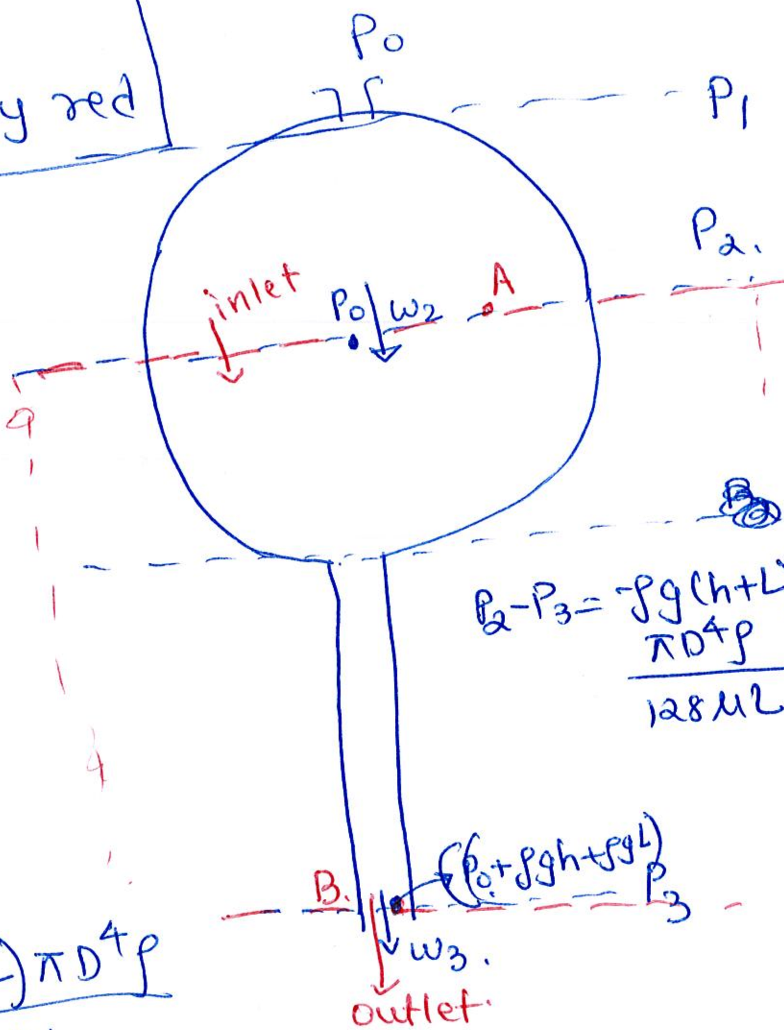
$$-w_2 = -\frac{(\rho g (h+L))\pi D^4 \rho}{128 \mu L}.$$

$$w_2 = \frac{\rho g (h+L) \pi D^4 \rho}{128 \mu L}.$$

writing steady state mass balance on control volume shown by red border

mass flow rate in − mass flow rate out = 0

$$\boxed{\begin{aligned} w_2 - w_3 &= 0 \\ w_2 &= w_3. \end{aligned}}$$



$$P_2 - P_3 = -\rho g(h+L)$$

$$\frac{\pi D^4 \rho}{128 \mu L}$$

$$w_3 = \frac{(P_2 - P_3)\pi D^4 \rho}{128 \mu L}$$

$$w_3 = \frac{-\rho g(h+L)\pi D^4 \rho}{128 \mu L}.$$

$$w_2 = w_3 = \frac{-\rho g(h+L)\pi D^4 \rho}{128 \mu L}.$$

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rate of mass change in the storage tank = mass flow rate flowing out of pipe.

D = Diameter of pipe.

R = Radius of storage tank.

$$\cancel{\pi} \rho R\left(2h - \frac{h^2}{R}\right)\frac{dh}{dt} = \frac{-\rho g(h+L)\cancel{\pi} D^4 \rho}{128 \mu L}.$$

$$\left(2h - \frac{h^2}{R}\right)\frac{dh}{dt} = \frac{-\rho g(h+L)D^4}{128 \mu L R}$$

$$\frac{(2Rh - h^2)}{\cancel{R}}\frac{dh}{dt} = \frac{-\rho g(h+L)D^4}{128 \mu L \cancel{R}}$$

$$-\frac{(2Rh-h^2)}{(h+L)}\cdot\frac{dh}{dt} = A \qquad \text{where } A = \frac{\rho g D^4}{128\mu L}.$$

Use change of variable,

$H = h + L$ at $t = 0$, $H = 2R + L \Rightarrow (h = 2R)$.

$dH = dh$ at $t = t_{efflux}$, $H = L \Rightarrow (h = 0)$.

$$\int -\frac{\left[2R(H-L) - (H-L)^2\right]}{H}\cdot\frac{dH}{dt} = A$$

$$\boxed{\int_{2R+L}^{L} -\frac{\left[2R(H-L) - (H-L)^2\right]}{H}\, dH = \int_{t=0}^{t_{efflux}} dt.}$$

<u>Integrate</u> this and match answer to book answer.

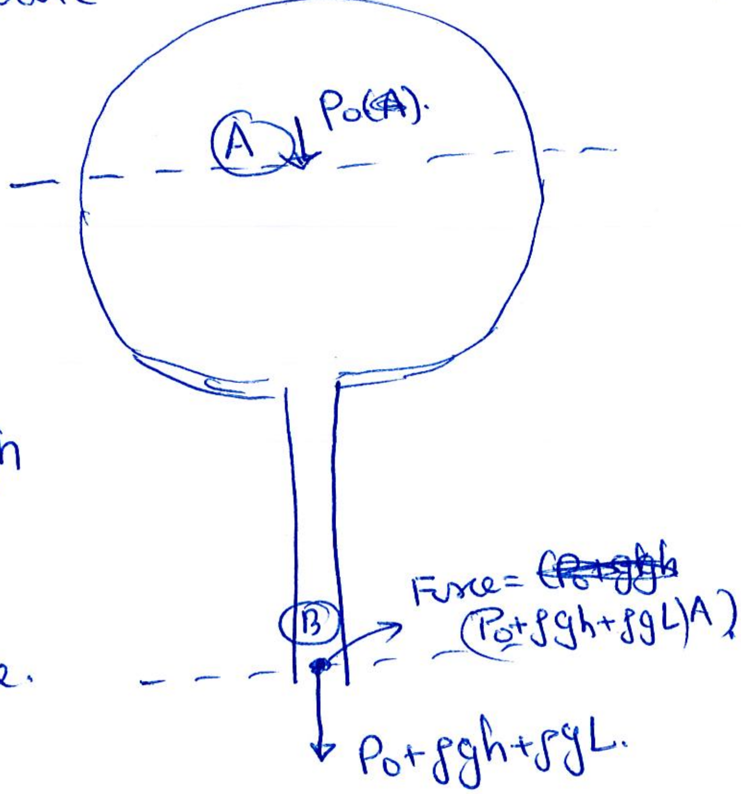
<u>Question Asked</u>

If Pressure at Plane 2 (point A) is P_0 and pressure at Plane 3 (Point B) is $(P_0 + \rho g h + \rho g L)$ which is higher than Point A, then why does the fluid not move from B to A?

* Movement of fluid (just like any body) will depend on the net direction of force acting on the fluid.

Let's draw a free body diagram for forces on fluid in this case.

At both points, the force of pressure is acting in downward direction, hence the fluid will move downward.



~~Now, the force at A is~~

The velocity of movement at A & B will be different though since ~~direction of force~~ magnitude of force will be different but direction is same.

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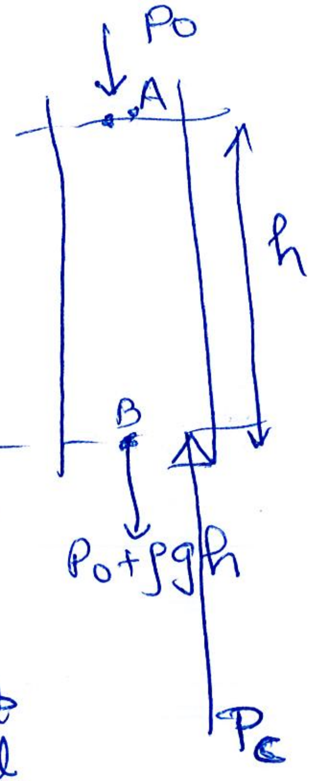
compare to situation below.



Net force at B

$= \cancel{P_0+\rho g}$

$= (P_0+\rho gh)-P_c$

if $P_c >> P_0+\rho gh$

then your net force on B is in opposite direction, hence fluid may move upward ~~but~~ due to effect of external pressure.

Remember

when in doubt draw free body diagram with all the forces on fluid

- body forces
- Pressure forces
- surface forces

or any other external field forces.