Fluid Statics and One-Dimensional Laminar Flow

2

In this chapter we shall examine the two simplest types of fluid motion: fluids at rest and steady, one-dimensional, laminar flow. This will serve to introduce the student to some of the fundamental concepts of fluid mechanics without getting involved in the mathematical details required to examine many of the more complex flows encountered in practice. We will first present a general formulation of the linear momentum principle and then use this result to derive the *equations of fluid statics*, which are simply a special form of the more general *equations of motion* to be presented in Chaps. 4 and 5. The second topic covered in this chapter, one-dimensional laminar flow, is another example of a special form of the general equations of motion.

*2.1 The Material Volume

As previously discussed, the linear momentum principle states that

$$\begin{Bmatrix}\text{the time rate of change}\\ \text{of momentum of the body}\end{Bmatrix} = \begin{Bmatrix}\text{the force acting}\\ \text{on the body}\end{Bmatrix} \tag{2.1-1}$$

The "body" referred to in Eq. 2.1-1 consists of some fixed quantity of material, such as a steel ball moving under the action of the earth's gravitational field, or all the asparagus and butter sauce contained in a sealed polyethylene bag, or a weather balloon filled with helium being bounced around in the turbulent atmosphere. The material within these three systems, or bodies, remains the same; therefore, they may be called material volumes. It follows immediately from the principle of conservation of mass that the mass of a material volume is constant.

In applying Eq. 2.1-1 to a fluid, some conceptual difficulties arise. If we define our system at some time, $t = 0$, as all the material within a sphere of radius r_0, we might imagine that at some other time we could not locate the system with a smooth closed surface, owing to the random motion of the molecules. If we wish to work within the realm of continuum mechanics, and we do, we must restrict ourselves to cases where this random molecular motion can be neglected. As we mentioned in Chap. 1, we may do so when the gross dimensions of the system under consideration are much larger than the mean free path of the molecules in the fluid. Under these conditions, we may correctly speak of a material volume which may be continuously changing size and shape but exchanging no mass with its surroundings. We denote a material volume as $\mathscr{V}_m(t)$, and the surface of a material volume as $\mathscr{A}_m(t)$. The subscript m reminds us that we are discussing a "material" volume and surface, both of which may be functions of time. Later in the text we shall have occasion to discuss volumes fixed in space, designated by $\mathscr{V}$, and volumes which move through space in an arbitrary manner, designated by $\mathscr{V}_a(t)$. The surface areas associated with these volumes are indicated by $\mathscr{A}$ and $\mathscr{A}_a(t)$, respectively. These volumes and the surfaces associated with them are designated by capital script letters to alert the student to their special significance. In addition, it allows us to use the symbol A to indicate cross-sectional areas and *portions of closed surfaces*, while closed surfaces are clearly distinguished by the use of $\mathscr{A}$, $\mathscr{A}_m(t)$, and $\mathscr{A}_a(t)$.

The linear momentum principle

Consider now the differential volume of fluid dV illustrated in Fig. 2.1-1. The mass, dM, contained in this differential volume is given by

$$dM = \rho\, dV \tag{2.1-2}$$

and the momentum (mass times velocity) of the differential element is, therefore,

$$dM\,\mathbf{v} = \rho\mathbf{v}\, dV \tag{2.1-3}$$

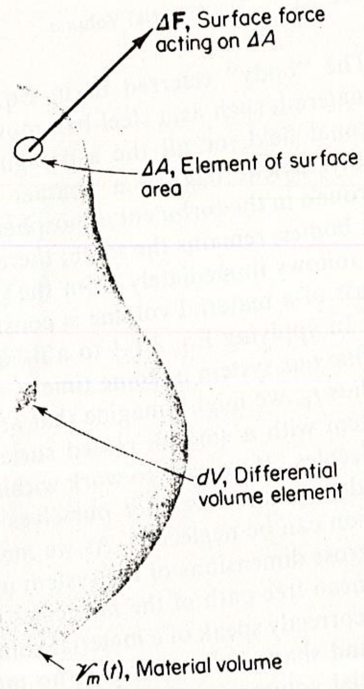

Fig. 2.1-1. A material volume.

We may now write the momentum of the material volume† as,

$$\{\text{momentum of the material volume}\} = \int_{\mathscr{V}_m(t)} \rho \mathbf{v}\, dV \qquad (2.1\text{-}4)$$

and Eq. 2.1-1 may be written

$$\frac{D}{Dt}\int_{\mathscr{V}_m(t)} \rho\mathbf{v}\,dV = \begin{Bmatrix}\text{force acting}\\ \text{on the material}\\ \text{volume}\end{Bmatrix} \qquad (2.1\text{-}5)$$

The derivative, D/Dt, is called the *material derivative* and will be discussed i detail along with the *total* and *partial time derivatives* in Chap. 3. For th present, it is sufficient to say the material derivative simply indicates that w are taking the time rate of change of a material volume.

The forces acting on a material volume of fluid consist of body forces— such as gravitational, electrostatic, and electromagnetic forces—that act o the mass as a whole and surface forces that act on the bounding surface of th material volume. If the body force per unit mass is represented by **g**, th total body force is

$$\begin{Bmatrix}\text{body force exerted}\\ \text{on the material}\\ \text{volume}\end{Bmatrix} = \int_{\mathscr{V}_m(t)} \rho\mathbf{g}\,dV \qquad (2.1\text{-}6)$$

† This type of terminology will be used throughout the text instead of the corre ...tement *the momentum of the body having a configuration* $\mathscr{V}_m(t)$.

We shall consider only the gravitational body force; **g** will therefore represent the gravity vector. In treating the surface force it will be appropriate to work in terms of the stress vector, $\mathbf{t}_{(\mathbf{n})}$, defined by Eq. 2.1-7.

$$\mathbf{t}_{(\mathbf{n})} = \lim_{\Delta A \to 0}\left(\frac{\Delta \mathbf{F}}{\Delta A}\right) \qquad (2.1\text{-}7)$$

where $\Delta\mathbf{F}$ is the force exerted by the surroundings on the area ΔA.

In general, a scalar or a vector can be specified simply in terms of its spatial coordinates and time; however, this is not the case with the stress vector, for it also depends on the orientation (given by the normal vector **n**) of the surface in question. The subscript (**n**) is thus used in denoting the stress vector. As an example of this dependence upon **n**, we might consider the solid bar under compression illustrated in Fig. 2.1-2. It is appropriate to

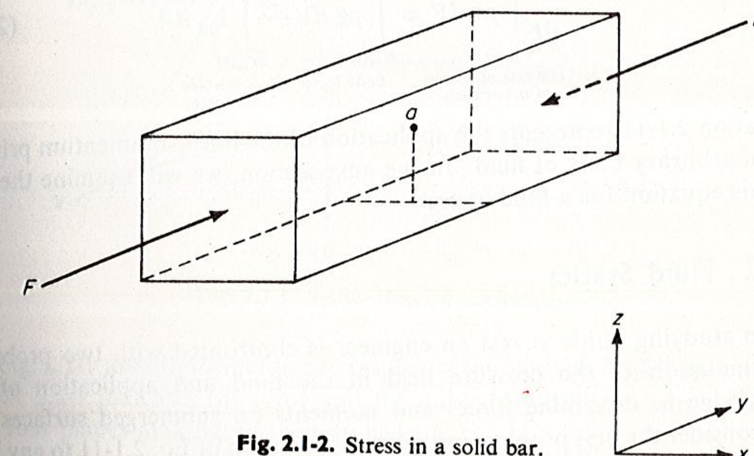

Fig. 2.1-2. Stress in a solid bar.

ask the following questions. What is the temperature in the bar at point a? What is the velocity at point a? What is the density at point a? However, the question—What is the stress at point a?—is incomplete. We must ask: What is the stress at point a for a surface having a normal **n**?

To illustrate this point, we shall consider the state of stress for the plane at point a having a normal $\pm\mathbf{j}$. If the total compressive force acting on the bar is F and the cross-sectional area of the bar is A, then the stress vector at point a for a surface having a normal $\pm\mathbf{j}$ is given by

$$\mathbf{t}_{(\pm\mathbf{j})} = \pm\mathbf{j}\left(\frac{F}{A}\right) \qquad (2.1\text{-}8)$$

where the plus or minus sign on the right-hand side of Eq. 2.1-8 is used depending on whether the normal to the surface is chosen as $\pm\mathbf{j}$. What about the stress at point a acting on the surface having a normal $\pm\mathbf{k}$? In the absence of any ambient pressure, it is obvious that this stress is zero.

$$\mathbf{t}_{(\pm\mathbf{k})} = 0 \tag{2.1-9}$$

When discussing the forces exerted by one phase upon another, we can easily become confused as to which phase we are referring. To eliminate this difficulty, we shall adhere to the following convention: $\mathbf{t}_{(\mathbf{n})}$ at a phase interface will refer to the force per unit area exerted *by* the phase into which the normal $\mathbf{n}$ points, *on* the phase for which $\mathbf{n}$ is the outwardly directed normal.

The net surface force acting on the material volume may be represented in terms of the stress vector

$$\left\{\begin{matrix}\text{surface force exerted}\\ \textit{by}\text{ the surroundings}\\ \textit{on}\text{ the material volume}\end{matrix}\right\} = \int_{\mathscr{A}_m(t)} \mathbf{t}_{(\mathbf{n})}\, dA \tag{2.1-10}$$

and Eq. 2.1-1 becomes

$$\underbrace{\frac{D}{Dt}\int_{\mathscr{V}_m(t)} \rho\mathbf{v}\, dV}_{\text{Time rate of change of momentum}} = \underbrace{\int_{\mathscr{V}_m(t)} \rho\mathbf{g}\, dV}_{\text{Body force}} + \underbrace{\int_{\mathscr{A}_m(t)} \mathbf{t}_{(\mathbf{n})}\, dA}_{\text{Surface force}} \tag{2.1-11}$$

Equation 2.1-11 represents the application of the linear momentum principle to an arbitrary body of fluid. In the next section, we will examine the form of this equation for a fluid at rest.

*2.2 Fluid Statics

In studying fluids at rest an engineer is confronted with two problems: determination of the pressure field in the fluid and application of this knowledge to determine forces and moments on submerged surfaces. We will consider the first problem initially. Application of Eq. 2.1-11 to any body of fluid at rest yields†

$$0 = \int_{\mathscr{V}} \rho\mathbf{g}\, dV + \int_{\mathscr{A}} \mathbf{t}_{(\mathbf{n})}\, dA \tag{2.2-1}$$

Note that in this case the limits of integration for the volume and surface integrals are independent of time.

The stress vector for a static fluid

The commonly accepted definition of a fluid is that it will deform *continuously* under the application of a *shear* stress. In light of this definition,

† A scalar zero in a vector equation should be read as the null vector.

the stress on a fluid element at rest must always act *normal* to the surface under consideration. We now wish to apply Eq. 2.2-1 to the tetrahedron shown in Fig. 2.2-1, subject to the restriction that all shear stresses are zero, to prove that the static stress in a fluid is isotropic. The vector forces shown

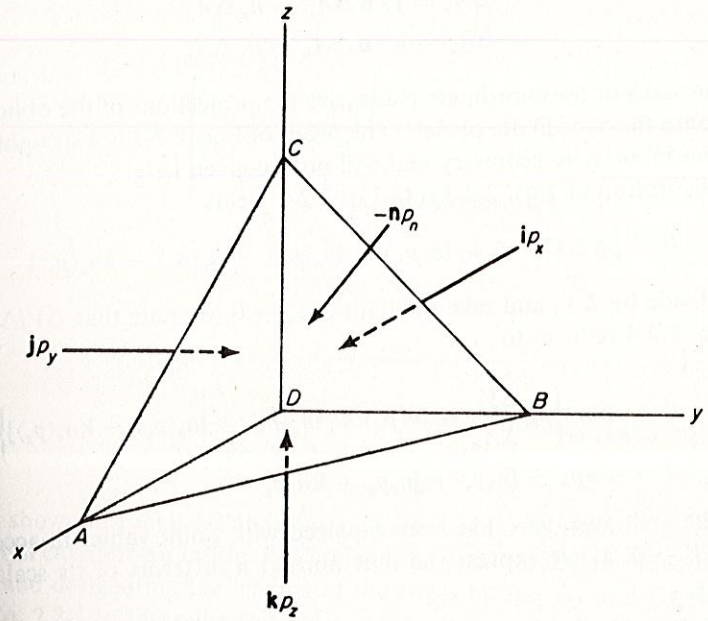

Fig. 2.2-1. Static stress on a tetrahedron.

in Fig. 2.2-1 are all normal to the respective surfaces on which they act, and Table 2.2-1 indicates the force and the outwardly directed unit normal for the four planes making up the tetrahedron.

Table 2.2-1

STATIC STRESS ON A TETRAHEDRON

Plane	Area	Normal	Stress vector
ABC	ΔA_n	$\mathbf{n}$	$-\mathbf{n}p_n$
BCD	ΔA_x	$-\mathbf{i}$	$\mathbf{i}p_x$
ADC	ΔA_y	$-\mathbf{j}$	$\mathbf{j}p_y$
ABD	ΔA_z	$-\mathbf{k}$	$\mathbf{k}p_z$

Application of Eq. 2.2-1 to the tetrahedron, and expression of the area integrals of the stresses in terms of an average stress and an area, yields

$$0 = \rho\mathbf{g}\,\Delta V - \mathbf{n}\langle p_n\rangle\,\Delta A_n + \mathbf{i}\langle p_x\rangle\,\Delta A_x + \mathbf{j}\langle p_y\rangle\,\Delta A_y + \mathbf{k}\langle p_z\rangle\,\Delta A_z \tag{2.2-2}$$

To proceed with our analysis and prove that $p_n = p_x = p_y = p_z$ we must express the areas ΔA_x, ΔA_y, ΔA_z in terms of ΔA_n. These expressions are given by,

$$\Delta A_x = \mathbf{i} \cdot \mathbf{n}\, \Delta A_n = n_x\, \Delta A_n \tag{2.2-3a}$$

$$\Delta A_y = \mathbf{j} \cdot \mathbf{n}\, \Delta A_n = n_y\, \Delta A_n \tag{2.2-3b}$$

$$\Delta A_z = \mathbf{k} \cdot \mathbf{n}\, \Delta A_n = n_z\, \Delta A_n \tag{2.2-3c}$$

i.e., the areas of the coordinate planes are the projections of the oblique area, ΔA_n, onto the coordinate planes. The proof of Eqs. 2.2-3 is a straightforward problem in analytic geometry and will not be given here.

Substitution of Eqs. 2.2-3 into Eq. 2.2-2 yields

$$0 = \rho\mathbf{g}\, \Delta V - \Delta A_n\, [\mathbf{n}\langle p_n\rangle - \mathbf{i}n_x\langle p_x\rangle - \mathbf{j}n_y\langle p_y\rangle - \mathbf{k}n_z\langle p_z\rangle] \tag{2.2-4}$$

If we divide by ΔA_n and take the limit $\Delta A_n \to 0$, we note that $\Delta V/\Delta A_n \to 0$, and Eq. 2.2-4 reduces to

$$\begin{aligned} 0 &= \lim_{\Delta A_n \to 0} \left\{ \rho\mathbf{g} \frac{\Delta V}{\Delta A_n} - [\mathbf{n}\langle p_n\rangle - \mathbf{i}n_x\langle p_x\rangle - \mathbf{j}n_y\langle p_y\rangle - \mathbf{k}n_z\langle p_z\rangle] \right\} \\ &= -\mathbf{n}p_n + \mathbf{i}n_x p_x + \mathbf{j}n_y p_y + \mathbf{k}n_z p_z \end{aligned} \tag{2.2-5}$$

Here the area averages have been replaced with point values in accordance with $\Delta V \to 0$. If we express the unit normal $\mathbf{n}$ in terms of its scalar components,

$$\mathbf{n} = \mathbf{i}n_x + \mathbf{j}n_y + \mathbf{k}n_z \tag{2.2-6}$$

we may write Eq. 2.2-5 as

$$0 = \mathbf{i}n_x(p_n - p_x) + \mathbf{j}n_y(p_n - p_y) + \mathbf{k}n_z(p_n - p_z) \tag{2.2-7}$$

Now, if a vector is equal to zero, it readily follows (by forming the dot product with $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, respectively) that the scalar components of the vector are zero. Thus, Eq. 2.2-7 leads us to the conclusion that $p_x = p_n$, $p_y = p_n$, and $p_z = p_n$, and our proof is complete.

Dropping the subscript n, we express the stress vector acting on any arbitrary surface as

$$\mathbf{t}_{(\mathbf{n})} = -\mathbf{n}p \tag{2.2-8}$$

where $\mathbf{n}$ is the outwardly directed unit normal. Equation 2.2-8 indicates that the magnitude of the stress vector is given by the pressure, and the direction is opposite to that of the unit normal. The minus sign in Eq. 2.2-8 is a manifestation of the convention that $\mathbf{t}_{(\mathbf{n})}$ represents the stress exerted *on* the system *by* the surroundings when $\mathbf{n}$ is the outwardly directed normal for the system.

We now wish to determine the pressure in the fluid illustrated in Fig. 2.2-2. Our first step is to apply Eqs. 2.2-1 and 2.2-8 to the differential volume element shown in Fig. 2.2-2 and illustrated in detail in Fig. 2.2-3. Taking the differential cube shown in Fig. 2.2-3 to have its sides parallel to the coordinate planes, and designating the lengths of the edges by Δx, Δy and Δz, we may apply Eq. 2.2-1 to this cube to yield

$$0 = \rho\mathbf{g}\, \Delta x\, \Delta y\, \Delta z - \int_{\mathscr{A}} \mathbf{n}p\, dA \tag{2.2-9}$$

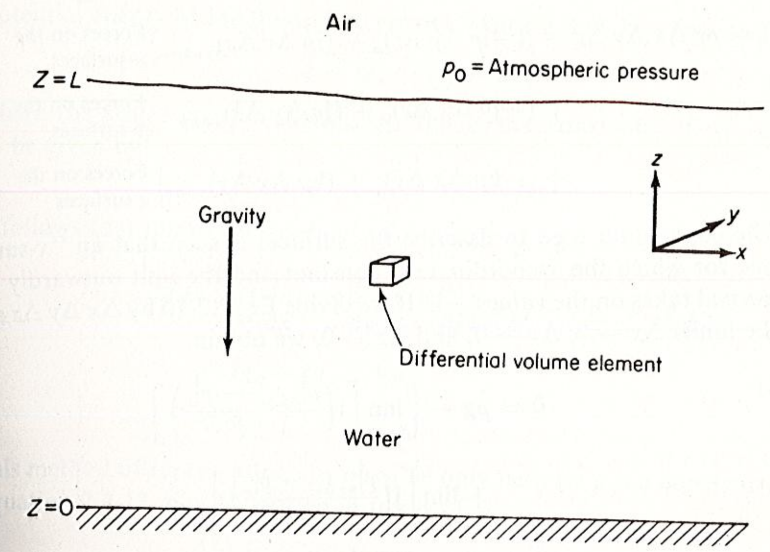

Fig. 2.2-2. A fluid at rest.

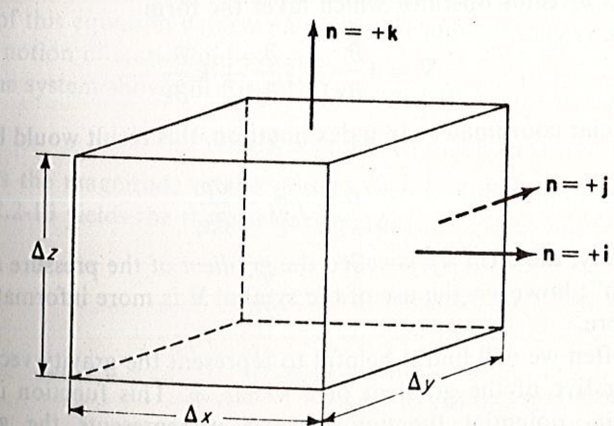

Fig. 2.2-3. Differential volume element.

Evaluation of the area integral for the six sides of the cube gives

$$\begin{aligned} 0 = \rho \mathbf{g}\, \Delta x\, \Delta y\, \Delta z - \Big\{ &(-\mathbf{i}p\, \Delta y\, \Delta z)_x + (\mathbf{i}p\, \Delta y\, \Delta z)_{x+\Delta x} && \text{Forces on the } x\text{-surfaces} \\ &+ (-\mathbf{j}p\, \Delta x\, \Delta z)_y + (\mathbf{j}p\, \Delta x\, \Delta z)_{y+\Delta y} && \text{Forces on the } y\text{-surfaces} \\ &+ (-\mathbf{k}p\, \Delta x\, \Delta y)_z + (\mathbf{k}p\, \Delta x\, \Delta y)_{z+\Delta z} \Big\} && \text{Forces on the } z\text{-surfaces} \end{aligned} \quad (2.2\text{-}10)$$

The convention used to describe the surfaces is such that an "x-surface" is one for which the x-coordinate is constant and the unit outwardly directed normal takes on the values $\pm\mathbf{i}$. If we divide Eq. 2.2-10 by $\Delta x\, \Delta y\, \Delta z$ and take the limits $\Delta x \to 0$, $\Delta y \to 0$, and $\Delta z \to 0$, we obtain

$$\begin{aligned} 0 = \rho \mathbf{g} - \Bigg\{ &\lim_{\Delta x \to 0} \left[\mathbf{i} \left(\frac{p|_{x+\Delta x} - p|_x}{\Delta x} \right) \right] \\ &+ \lim_{\Delta y \to 0} \left[\mathbf{j} \left(\frac{p|_{y+\Delta y} - p|_y}{\Delta y} \right) \right] \\ &+ \lim_{\Delta z \to 0} \left[\mathbf{k} \left(\frac{p|_{z+\Delta z} - p|_z}{\Delta z} \right) \right] \Bigg\} \end{aligned} \quad (2.2\text{-}11)$$

Each of the limits is the definition of a partial derivative, and Eq. 2.2-12 results from the limiting process.

$$0 = \rho \mathbf{g} - \left(\mathbf{i} \frac{\partial p}{\partial x} + \mathbf{j} \frac{\partial p}{\partial y} + \mathbf{k} \frac{\partial p}{\partial z} \right) \quad (2.2\text{-}12)$$

This equation may be written in the more compact form,

$$0 = \rho \mathbf{g} - \nabla p \quad (2.2\text{-}13)$$

where ∇ is a vector operator which takes the form

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \quad (2.2\text{-}14)$$

in rectangular coordinates. In index notation, this result would be expressed

$$0 = \rho g_i - \frac{\partial p}{\partial x_i} \quad (2.2\text{-}15)$$

Traditionally, the term ∇p is called the *gradient* of the pressure and denoted by "grad p"; however, the use of the symbol ∇ is more informative and will be used here.

Very often we will find it helpful to represent the gravity vector in terms of the negative of the gradient of a scalar, ϕ. This function is called the gravitational potential function because it represents the gravitational

potential energy of the fluid. The gravity vector $\mathbf{g}$ may be expressed

$$\mathbf{g} = \mathbf{i}g_x + \mathbf{j}g_y + \mathbf{k}g_z \quad (2.2\text{-}16)$$

where the scalar components g_x, g_y, and g_z are constants. If we choose ϕ to be given by

$$\phi = -(xg_x + yg_y + zg_z) \quad (2.2\text{-}17)$$

it follows that minus the gradient of ϕ is equal to $\mathbf{g}$.

$$\begin{aligned} -\nabla\phi &= \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right)(xg_x + yg_y + zg_z) \\ &= \mathbf{i}g_x + \mathbf{j}g_y + \mathbf{k}g_z \\ &= \mathbf{g} \end{aligned} \quad (2.2\text{-}18)$$

This method of representing $\mathbf{g}$ is possible only because $\mathbf{g}$ is a constant vector. Equation 2.2-13 may be written in terms of ϕ.

$$0 = \rho \nabla \phi + \nabla p \quad (2.2\text{-}19)$$

If the density is constant, this equation may be written as

$$0 = \nabla(\rho\phi + p) \quad (2.2\text{-}20)$$

Since this result holds everywhere in the fluid, the term $\rho\phi + p$ must be a constant, and we may write

$$p = C - \rho\phi \quad (2.2\text{-}21)$$

Here we have integrated the equations of fluid statics, and C is the constant of integration. Use of the potential function to obtain the integrated form of Eq. 2.2-13 is satisfying from the mathematical point of view, but the solution of this equation deserves a treatment more closely connected to the student's notion of static fluid systems.

For the system shown in Fig. 2.2-2, the gravity vector is

$$\mathbf{g} = -\mathbf{k}g \quad (2.2\text{-}22)$$

where g is the magnitude of the gravity vector. Substitution of Eq. 2.2-22 into Eq. 2.2-13 yields the three scalar equations

$$\frac{\partial p}{\partial x} = 0 \quad (2.2\text{-}23a)$$

$$\frac{\partial p}{\partial y} = 0 \quad (2.2\text{-}23b)$$

$$\frac{\partial p}{\partial z} = -\rho g \quad (2.2\text{-}23c)$$

components of the original vector equation are qui
d by the index notation of Eq. 2.2-15, where $g_1 = g_2 =$
gration of Eq. 2.2-23c gives

$$p = -\rho g z + C(x, y) \qquad (2.2\text{-}2$$

t of integration may be a function of x and y; howev
indicate that the pressure is neither a function of x nor
2.2-24

$$p = -\rho g z + C \qquad (2.2\text{-}$$

y condition is needed in order to determine the constan
the pressure everywhere in the fluid. It is obtained
the pressure at $z = L$ is the atmospheric pressure p_0.

$$p = p_0, \quad z = L \qquad (2.$$

f which to Eq. 2.2-25 yields,

$$p\big|_{z=L} = p_0 = -\rho g L + C \qquad (2$$

of integration is

$$C = p_0 + \rho g L$$

C into Eq. 2.2-25 yields the final expression for the pr

$$p = p_0 + \rho g(L - z)$$

n is such a simple one that the student can easily have
of the various steps; therefore, a review will be helpf

ar momentum principle and Eq. 2.2-8 were appli
al volume element to develop the equations of flui
tions were solved to yield Eq. 2.2-25.
lary condition was specified and applied to Eq.
n expression for the pressure.

seem so to the student, but the final step in this gene
most troublesome. The differential equations of mo
e and for all, and methods of solution (for those ca
) are neatly tabulated in mathematics and fluid m
, boundary conditions are specified mainly on th
ion, and thus present a more difficult problem.
d as they may seem, for there are two rules which gui
conditions: velocity is a continuous functio

vector is a continuous functi
postulate discussed in Sec. 1.
of the second rule—i.e., pre

2.3 Barometers

A barometer is a devic
atmosphere. The one illustra
at one end and immersed i
obtained by filling the tube
closing the open end (the thu
pool of barometer fluid. Th
pressure there is the vapor p
p_{vp}. For mercury the vapor p
at room temperature is approx
3×10^{-6} atm and may be con
to be zero. The differential e
for the pressure is

$$\frac{\partial p}{\partial z} = -\rho_{Hg} g$$

which is integrated to give

$$p = -\rho_{Hg} g z + C$$

The boundary condition is

B.C. 1:

$$p = p_{vp} \approx 0, \qquad z = h$$

the application of which to Eq
gives

$$p = \rho_{Hg} g (h - z)$$

Since the pressure at $z = 0$
atmospheric pressure, p_0, we

$$p_0 = \rho_{Hg} g h$$

Because of the universal use of
pressure, it is often reported in
pressure varies from day to da

Taking the pressure to be conti

These three scalar components of the original vector equation are quite obviously expressed by the index notation of Eq. 2.2-15, where $g_1 = g_2 = 0$ and $g_3 = -g$. Integration of Eq. 2.2-23c gives

$$p = -\rho g z + C(x, y) \tag{2.2-24}$$

where the constant of integration may be a function of x and y; however, Eqs. 2.2-23a and b indicate that the pressure is neither a function of x nor y, and we write Eq. 2.2-24

$$p = -\rho g z + C \tag{2.2-25}$$

A single boundary condition is needed in order to determine the constant C and thus specify the pressure everywhere in the fluid. It is obtained by recognizing that the pressure at $z = L$ is the atmospheric pressure p_0. We may write

$$\text{B.C. 1:} \qquad p = p_0, \qquad z = L \tag{2.2-26}$$

the application of which to Eq. 2.2-25 yields,

$$p\big|_{z=L} = p_0 = -\rho g L + C \tag{2.2-27}$$

and the constant of integration is

$$C = p_0 + \rho g L \tag{2.2-28}$$

Substitution of C into Eq. 2.2-25 yields the final expression for the pressure

$$p = p_0 + \rho g (L - z) \tag{2.2-29}$$

This problem is such a simple one that the student can easily have missed the significance of the various steps; therefore, a review will be helpful.

1. The linear momentum principle and Eq. 2.2-8 were applied to a differential volume element to develop the equations of fluid statics.
2. The equations were solved to yield Eq. 2.2-25.
3. A boundary condition was specified and applied to Eq. 2.2-25 to obtain an expression for the pressure.

It may not seem so to the student, but the final step in this general process is usually the most troublesome. The differential equations of motion can be derived once and for all, and methods of solution (for those cases which can be solved) are neatly tabulated in mathematics and fluid mechanics texts; however, boundary conditions are specified mainly on the basis of physical intuition, and thus present a more difficult problem. Things are not quite as bad as they may seem, for there are two rules which guide us in specifying boundary conditions: velocity is a continuous function; the stress

vector is a continuous function. Both these ideas follow from the continuum postulate discussed in Sec. 1.1. Boundary condition 1 results from application of the second rule—i.e., pressure is a continuous function for fluids at rest.†

2.3 Barometers

A barometer is a device for measuring the absolute pressure of the atmosphere. The one illustrated in Fig. 2.3-1 consists of a single tube closed at one end and immersed in the barometer fluid. Such a system may be obtained by filling the tube with the barometer fluid (usually mercury), closing the open end (the thumb does nicely), and immersing the tube in the pool of barometer fluid. The fluid vaporizes in the closed end, and the pressure there is the vapor pressure p_{vp}. For mercury the vapor pressure at room temperature is approximately 3×10^{-6} atm and may be considered to be zero. The differential equation for the pressure is

$$\frac{\partial p}{\partial z} = -\rho_{\mathrm{Hg}} g \tag{2.3-1}$$

which is integrated to give

$$p = -\rho_{\mathrm{Hg}} g z + C \tag{2.3-2}$$

The boundary condition is

$$\text{B.C. 1:} \qquad p = p_{vp} \approx 0, \qquad z = h \tag{2.3-3}$$

the application of which to Eq. 2.3-2 gives

$$p = \rho_{\mathrm{Hg}} g (h - z) \tag{2.3-4}$$

Since the pressure at $z = 0$ is the atmospheric pressure, p_0, we write

$$p_0 = \rho_{\mathrm{Hg}} g h \tag{2.3-5}$$

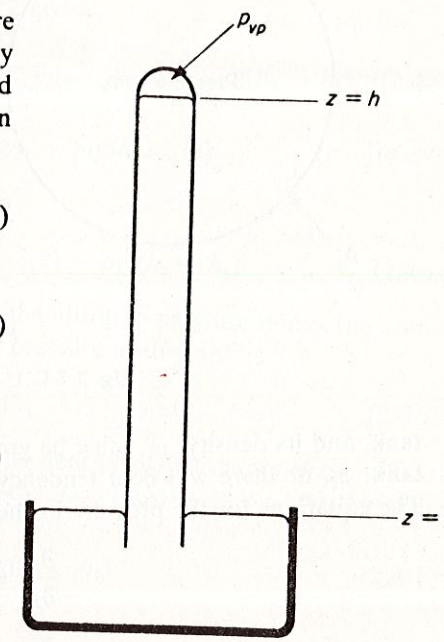

Fig. 2.3-1. Mercury barometer.

Because of the universal use of mercury barometers to measure atmospheric pressure, it is often reported in terms of h, or inches of mercury. Atmospheric pressure varies from day to day, the average being 29.92 in. Hg. This value

† Taking the pressure to be continuous neglects any effect of surface tension at the gas-liquid interface.

is referred to as one *standard atmosphere*, and we may use Eq. 2.3-5 to determine that

$$p_0\,(\text{standard}) = 14.696\ \text{lb}_f/\text{in.}^2$$

This quantity is often written 14.696 psia (pounds per square inch absolute).

2.4 Manometers

Manometers are devices which make use of columns of liquid to determine pressure differences. The simplest type is the U-tube illustrated in Fig. 2.4-1. The manometer fluid must be immiscible with the fluid in the

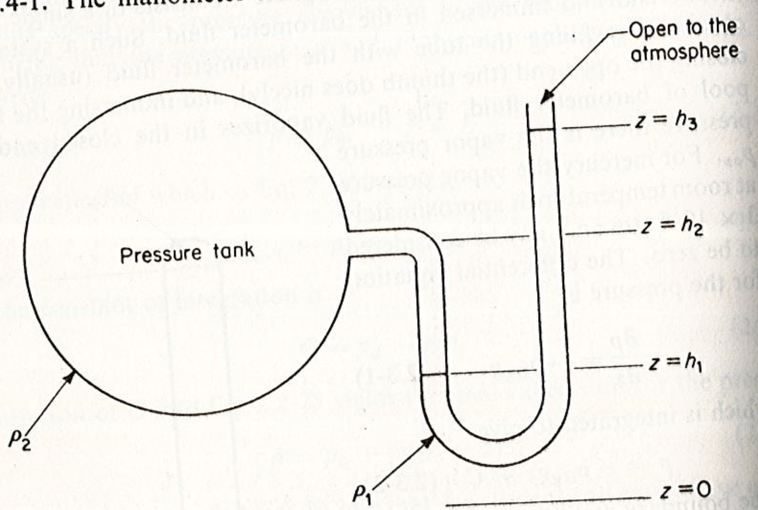

Fig. 2.4-1. U-tube manometer.

tank, and its density, ρ_1, must be greater than the density of the fluid in the tank, ρ_2, or there will be a tendency for it to replace the fluid in the tank. The equations for the pressure in the two fluids are

$$\frac{\partial p_1}{\partial z} = -\rho_1 g \tag{2.4-1}$$

$$\frac{\partial p_2}{\partial z} = -\rho_2 g \tag{2.4-2}$$

The boundary conditions for this system are

$$\text{B.C. 1:} \qquad p_1 = p_0, \qquad z = h_3 \tag{2.4-3}$$

$$\text{B.C. 2:} \qquad p_2 = p_1, \qquad z = h_1 \tag{2.4-4}$$

Both these conditions are derived from the notion that the pressure is a continuous function.† Solution of the two differential equations gives

$$p_1 = -\rho_1 g z + C_1 \tag{2.4-5}$$

$$p_2 = -\rho_2 g z + C_2 \tag{2.4-6}$$

Application of boundary condition 1 gives

$$p_1|_{z=h_3} = p_0 = -\rho_1 g h_3 + C_1 \tag{2.4-7a}$$

or

$$C_1 = p_0 + \rho_1 g h_3 \tag{2.4-7b}$$

and the pressure in fluid 1 is

$$p_1 = p_0 + \rho_1 g(h_3 - z) \tag{2.4-8}$$

Application of boundary condition 2 gives

$$p_2|_{z=h_1} = -\rho_2 g h_1 + C_2 = p_1|_{z=h_1} = p_0 + \rho_1 g(h_3 - h_1) \tag{2.4-9a}$$

or

$$C_2 = p_0 + \rho_2 g h_1 + \rho_1 g(h_3 - h_1) \tag{2.4-9b}$$

and the pressure in fluid 2 is

$$p_2 = p_0 + \rho_2 g(h_1 - z) + \rho_1 g(h_3 - h_1) \tag{2.4-10}$$

The gauge pressure, p_g, is defined as the absolute pressure minus the atmospheric pressure; therefore, the gauge pressure in the tank is

$$p_g = (p_2 - p_0) = \rho_2 g(h_1 - z) + \rho_1 g(h_3 - h_1) \tag{2.4-11}$$

If the fluid in the tank is a gas, the density of the manometer fluid will usually be much larger than that of the gas; thus, $\rho_1 \gg \rho_2$, and the gauge pressure is

$$p_g \simeq \rho_1 g(h_3 - h_1) \tag{2.4-12}$$

Manometer calculation

The fluid in the tank is a gas and a mercury manometer is used to measure the pressure. The readings on the manometer are

$$\left.\begin{aligned} h_1 &= 3.25\ \text{ft} \\ h_3 &= 5.17\ \text{ft} \end{aligned}\right\} \quad \text{or} \quad (h_3 - h_1) = 1.92\ \text{ft}$$

† Interfacial tensions are obviously being neglected here.

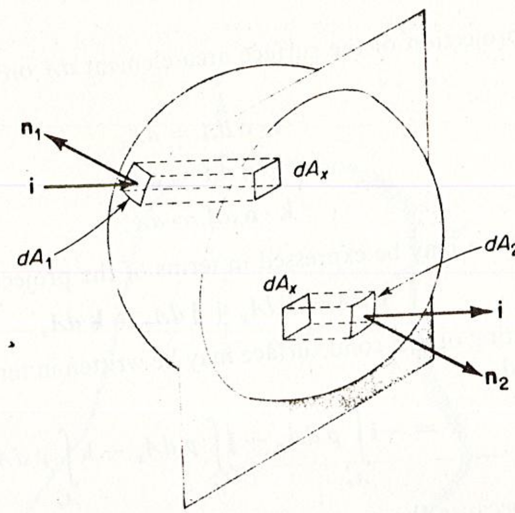

Fig. 2.6-3. Projected areas for a spherical surface.

under the same integral sign,

$$F_x = \int_{A_x} \Big\{[p_0 + \rho_1 g(L_1 - z)] - [p_0 + \rho_2 g(L_2 - z)]\Big\}\, dA_x \tag{2.6-11}$$

$$= \int_{A_x} [(\rho_1 L_1 - \rho_2 L_2)g + (\rho_2 - \rho_1)gz]\, dA_x$$

Since

$$dA_x = dy\, dz \tag{2.6-12}$$

the force may be written as

$$F_x = \int_{z=\ell-r_0}^{z=\ell+r_0} \int_{y=-\sqrt{r_0^2-(z-\ell)^2}}^{y=+\sqrt{r_0^2-(z-\ell)^2}} [(\rho_1 L_1 - \rho_2 L_2)g + (\rho_2 - \rho_1)gz]\, dy\, dz \tag{2.6-13}$$

Integration with respect to y gives

$$F_x = 2\int_{z=\ell-r_0}^{z=\ell+r_0} [(\rho_1 L_1 - \rho_2 L_2)g + (\rho_2 - \rho_1)gz]\sqrt{r_0^2 - (z-\ell)^2}\, dz \tag{2.6-14}$$

which we may integrate in turn to give

$$F_x = [(\rho_1 L_1 - \rho_2 L_2)g + (\rho_2 - \rho_1)g\ell]\pi r_0^2 \tag{2.6-15}$$

If we add and subtract p_0 from the right-hand side and rearrange the terms we obtain

$$F_x = \underbrace{[p_0 + \rho_1 g(L_1 - \ell)]}_{\text{Mean pressure}}\underbrace{\pi r_0^2}_{\text{Area}} - \underbrace{[p_0 + \rho_2 g(L_2 - \ell)]}_{\text{Mean pressure}}\underbrace{\pi r_0^2}_{\text{Area}} \tag{2.6-16}$$

Thus, the resulting force may be written in terms of the mean pressure acting on the surface multiplied by the projected area of the surface in the direction of the force. We may use this result to quickly determine forces which act perpendicularly to the gravity vector provided the pressure is a *linear* function of z.

*2.7 Buoyancy Forces

The resultant force exerted on a body by a fluid at rest is called the buoyant force. The principle that we wish to prove here is attributed to Archimedes (287–212 BC), who supposedly discovered the phenomenon while entering the pool of a public bath.[1] The principle states that "a body is buoyed up by a force equal to the weight of the displaced fluid." We can easily prove this by applying Eq. 2.6-1 to the solid body illustrated in Fig. 2.7-1. In this case, we consider a solid body which may be separated into two regions by a curve along which

$$\mathbf{k} \cdot \mathbf{n} = 0 \tag{2.7-1}$$

Thus, the normal to the solid surface lies in the x-y plane along this curve. The analysis can be easily extended to more complex shapes simply by dividing the volume into several sections. If we designate the position of the upper surface by z_2 and the lower surface by z_1, Eq. 2.6-1 may be separated into two parts to give

$$\mathbf{F} = -\int_{A_2} \mathbf{n}_2 p\big|_{z=z_2}\, dA - \int_{A_1} \mathbf{n}_1 p\big|_{z=z_1}\, dA \tag{2.7-2}$$

Taking the dot product of Eq. 2.7-2 with the unit vector $\mathbf{k}$, we get the buoyancy force

$$\mathbf{k} \cdot \mathbf{F} = F_z = -\int_{A_2} p\big|_{z=z_2}\mathbf{n}_2 \cdot \mathbf{k}\, dA - \int_{A_1} p\big|_{z=z_1}\mathbf{n}_1 \cdot \mathbf{k}\, dA \tag{2.7-3}$$

Since $\mathbf{n} \cdot \mathbf{k}\, dA$ is the projection of the surface area on the x-y plane, we may change the variables of integration to

$$\mathbf{k} \cdot \mathbf{n}_2\, dA = dA_z, \qquad \text{upper region} \tag{2.7-4a}$$

$$\mathbf{k} \cdot \mathbf{n}_1\, dA = -dA_z, \qquad \text{lower region} \tag{2.7-4b}$$

1. H. Rouse and S. Ince, *History of Hydraulics* (New York: Dover Publications, Inc., 1963), p. 16.

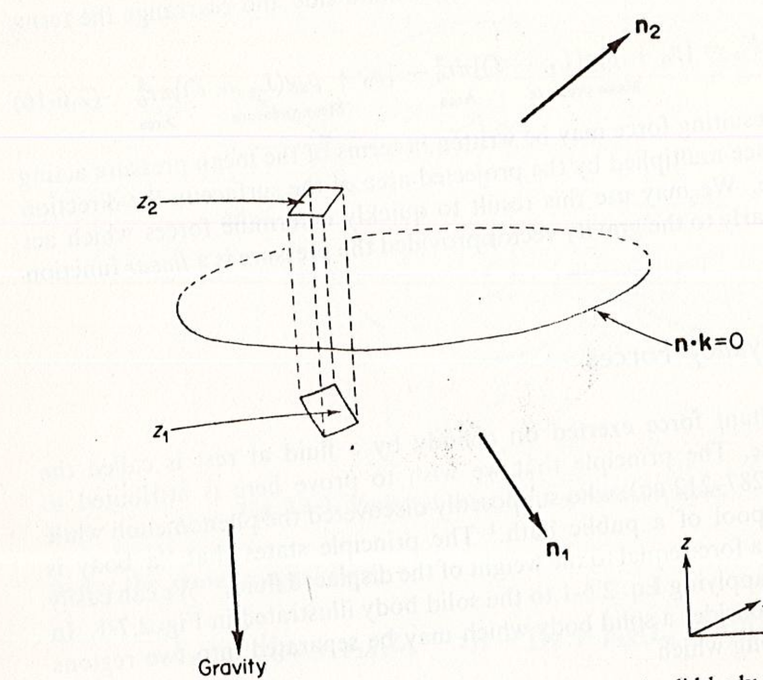

Fig. 2.7-1. Buoyancy force on a submerged solid body.

and Eq. 2.7-3 becomes

$$F_z = -\int_{A_z} p\big|_{z=z_2}\, dA_z + \int_{A_z} p\big|_{z=z_1}\, dA_z \quad \text{(2.7-}$$

Putting both terms under the same integral sign, and noting that

$$p\big|_{z=z_1} - p\big|_{z=z_2} = \rho g(z_2 - z_1) \quad \text{(2.7-}$$

we obtain

$$F_z = \int_{A_z} \rho g(z_2 - z_1)\, dA_z \quad \text{(2.7-}$$

But ρg is a constant and the volume of the solid V_s is

$$V_s = \int_{A_z} (z_2 - z_1)\, dA_z \quad \text{(2.7-}$$

and the solid body is buoyed up by the "weight" of the displaced fluid.

$$F_z = \rho g V_s \quad \text{(2.7-}$$

Fig. 2.7-2. Solid body floating at an interface.

The result is not difficult to extend to the case of a solid body at the interface between two fluids. This situation is illustrated in Fig. 2.7-2 and the proof is left to the student.

The hydrometer

A hydrometer uses the buoyancy principle to determine the ratio of densities of two fluids. In general, one of these fluids is water and the ratio of densities, ρ/ρ_{H_2O}, is called the specific gravity, γ.

$$\gamma = \frac{\rho}{\rho_{H_2O}} \quad \text{(2.7-10)}$$

A hydrometer floating first in water and then in some other fluid is shown in Fig. 2.7-3. If the mass of the hydrometer is M and the density of the air above the liquid is so small that the air contributes a negligible amount to the buoyancy force, then Archimedes' principle yields

$$Mg = \rho_{H_2O} g V_{H_2O} \quad \text{(2.7-11)}$$

$$Mg = \rho_{oil} g V_{oil} \quad \text{(2.7-12)}$$

where V_{H_2O} and V_{oil} are the volumes of the displaced water and oil respectively. Division of Eq. 2.7-12 by Eq. 2.7-11 gives

$$1 = \left(\frac{\rho_{oil}}{\rho_{H_2O}}\right)\left(\frac{V_{oil}}{V_{H_2O}}\right) \quad \text{(2.7-13)}$$

and the specific gravity of the oil is

$$\gamma_{oil} = \frac{V_{H_2O}}{V_{oil}} \quad \text{(2.7-14)}$$

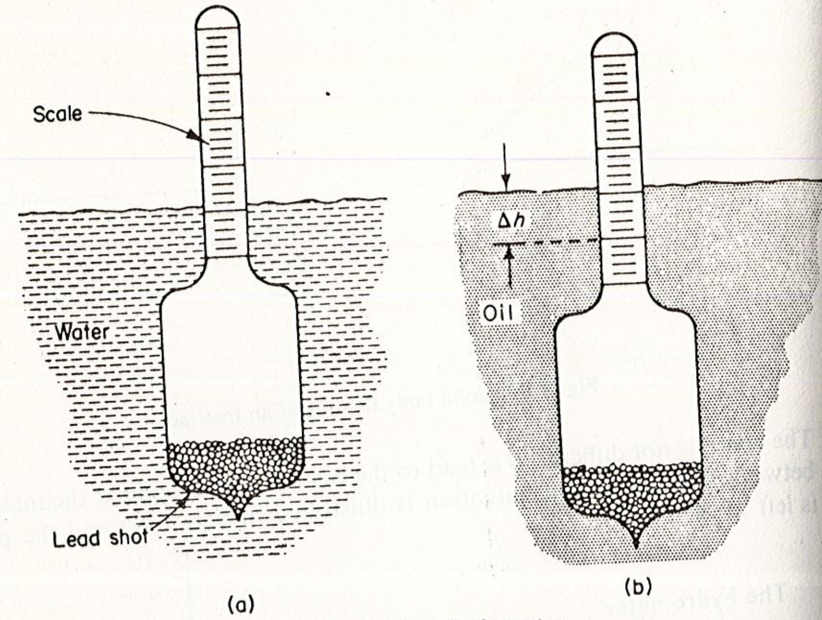

Fig. 2.7-3. A hydrometer.

If the cross-sectional area of the graduated shaft is A then

$$V_{\text{oil}} = V_{H_2O} + A\,\Delta h \tag{2.7-15}$$

and

$$\gamma_{\text{oil}} = \frac{1}{1 + \left(\dfrac{A}{V_{H_2O}}\right)\Delta h} \tag{2.7-16}$$

We need to know both the cross-sectional area A and the volume V_{H_2O} in order to determine γ_{oil}. The area A can be measured easily and V_{H_2O} may be determined from Eq. 2.7-11 since the density of water is well known and the mass of the hydrometer is measured easily. Knowing A and V_{H_2O} allows us to mark off a scale on the stem of the hydrometer giving the specific gravity directly.

2.8 One-Dimensional Laminar Flows

If only a single component of the velocity vector is nonzero, the linear momentum principle is easy to apply, provided the flow is laminar and the streamlines are straight. The subject of streamlines will be discussed in detail in Chap. 3; for now, it will suffice to say that for *steady flow*, streamlines are

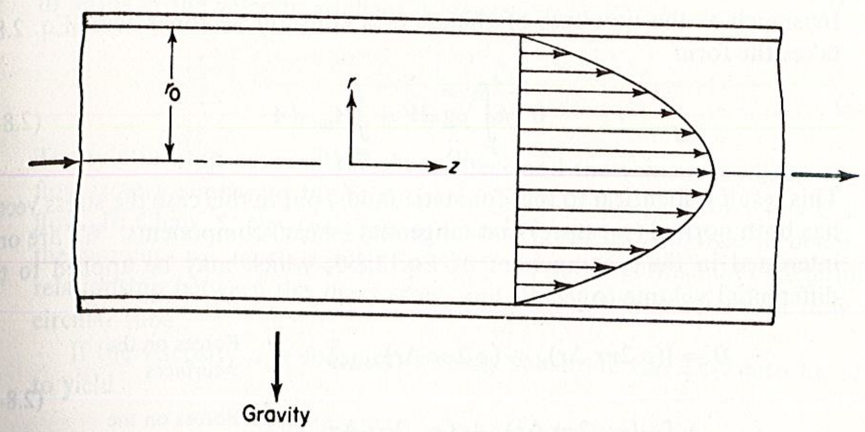

Fig. 2.8-1. Laminar flow in a circular tube.

imaginary lines traced out by particles of fluid. Thus, the streamlines in the Couette viscometer described in Chap. 1 would be concentric circles. For the flow illustrated in Fig. 2.8-1, $v_\theta = v_r = 0$, and v_z is only a function of r; thus a fluid particle would move in a straight line parallel to the z-axis.

We start the analysis with the linear momentum equation

$$\frac{D}{Dt}\int_{\mathscr{V}_m(t)} \rho \mathbf{v}\, dV = \int_{\mathscr{V}_m(t)} \rho \mathbf{g}\, dV + \int_{\mathscr{A}_m(t)} \mathbf{t}_{(\mathbf{n})}\, dA \tag{2.8-1}$$

which is to be applied to the differential cylindrical section shown in Fig. 2.8-2. This section of fluid is a material volume, and is therefore continuously deforming; however, the velocity of every fluid particle in this volume is a constant. Thus, the momentum is also constant

$$\int_{\mathscr{V}_m(t)} \rho \mathbf{v}\, dV = \text{constant vector} \tag{2.8-2}$$

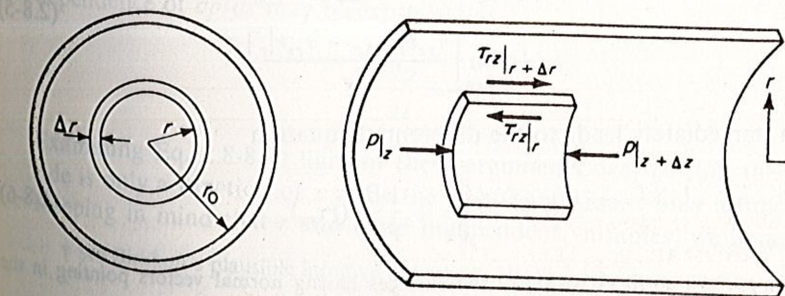

Fig. 2.8-2. Differential material volume element.