ABE 307
Date Nov 10^{th} 2017
Friday.

Dimensionless Variables and Buckingham's Pi Theory

In many practical cases of fluid flow problems the number of independent parameters are exceedingly larger than the basic dimensions, which leads to more unknown indices to be solved than the number of equations available.

In most cases, three basic dimensions are used M, L and T. Thus, there will be 3 simultaneous equations to be solved. But, if you have 4 independent indices to be evaluated to find the functional dependence then it will be tough to find the relationship.

Example: The pressure drop per unit length 'p' due to friction in a pipe depends upon the diameter 'D', the mean velocity 'v', the density 'ρ' and the dynamic viscosity 'μ'. Find the relationship between pressure drop and other variables.

$$\frac{\Delta P}{L},\ D,\ v,\ \rho,\ \mu \rightarrow n=5 \quad m=3$$

$$\Delta P' = \frac{\Delta P}{L} \propto f(D, v, \rho, \mu).$$

$$\frac{\Delta P}{L} \propto D v \rho \mu$$

$$\frac{\Delta P}{L} = c\, D^a v^b \rho^c \mu^d. \quad \text{— I}$$

$$\left[\frac{\Delta P}{L}\right] = \frac{MLT^{-2}}{L^2 . L} = ML^{-2}T^{-2}$$

$$[D] = L$$

$$[v] = LT^{-1}$$

$$[\rho] = ML^{-3}$$

$$[\mu] = ML^{-1}T^{-1}$$

$$ML^{-2}T^{-2} = L^a (LT^{-1})^b (ML^{-3})^c (ML^{-1}T^{-1})^d$$

$$= L^{a+b-3c-d} M^{c+d} T^{-b-d}.$$

$$c+d = 1 \quad \Big| \quad a+b-3c-d = -2$$

$$-b-d = -2.$$

$$C = 1-d \quad \text{(sub in I)}$$

$$b = 2-d$$

$$a + (2-d) - 3(1-d) - d = -2$$

$$\Rightarrow a = (\ \_\_\_\ )$$

$$\frac{\Delta P}{L} = c\left(\frac{\rho v^2}{D}\right)\left(\frac{\mu}{\rho v D}\right)^{d}.$$

Dimensional homogeneity test does not give exact formula!
(No way to find d).

Buckingham's π (Pi) Theorem

Buckingham's pi theory provides a systematic way to identify dimensionless numbers for developing functional relationships. So, you do not have to randomly select indices that should not be solved in the simultaneous equations.

Statement: A relationship between n dimensional parameters can be reduced to a relationship between (n-m) dimensionless parameters, where m is the number of independent parameters.

Steps for Applying Buckingham's π (Pi) Theorem:

Step 1 – List all the dimensional parameters involved. Let n be the number of parameters.

Step 2 – Choose a set of fundamental dimensions. Generally, for fluid flow problems MLT, r = # of primary dimensions, r = 3

Step 3 – List the dimensions of all parameters in terms of primary (fundamental) dimensions

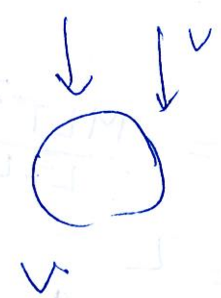
Step 4 – Select m independent or repeating parameters, m = r. Use these repeating parameters to come up with dimensionless numbers.

Step 5 - Set up dimensional equations combining the parameters selected as independent parameters with other (n-m) parameters to form dimensionless numbers for your systems.

Application of Buckingham's π (Pi) Theorem

Drag force (F_D) on a sphere is dependent on Diameter (D), Velocity (V), density (ρ) and viscosity (μ). Find the dimensionless numbers that can be used for designing experiments to quantify the drag force for fluid flowing past a sphere.

F_D, D, V, ρ, μ.



Step 1 ⇒ $n = 5$.

Step 2 ⇒ $r = 3$, M, L, T

Step 3 ⇒ $[F_D] = MLT^{-2}$.

$[D] = L$

$[V] = LT^{-1}$

$[\rho] = ML^{-3}$

$[\mu] = ML^{-1}T^{-1}$

Step 4 → $m = r = 3$ repeating parameters.

D, V, ρ.

Dimensionless Variables and Buckingham's Pi Theory

In many practical cases of fluid flow problems the number of independent parameters are exceedingly larger than the basic dimensions, which leads to more unknown indices to be solved than the number of equations available.

In most cases, three basic dimensions are used M, L and T. Thus, there will be 3 simultaneous equations to be solved. But, if you have 4 independent indices to be evaluated to find the functional dependence then it will be tough to find the relationship.

Example: The pressure drop per unit length 'p' due to friction in a pipe depends upon the diameter 'D', the mean velocity 'v', the density 'ρ' and the dynamic viscosity 'μ'. Find the relationship between pressure drop and other variables.

(Step 5) $n - m = 5 - 3 = 2$

Dimensionless $\pi_1 = f(F_D, D, v, \rho)$ $\quad \pi_2 = f(D, v, \rho, \mu)$

$$M^0 L^0 T^0 = F_D^a D^b v^c \rho^d$$

$$= (MLT^{-2})^a (L)^b (LT^{-1})^c (ML^{-3})^d$$

$$M^0 L^0 T^0 = M^{a+d} L^{a+b+c-3d} T^{-2a-c}$$

Assume the non-repeating parameter to be always present, hence assume $a = 1$.

$a + d = 0$
$a = 1$
$\Rightarrow d = -1$

$a + b + c - 3d = 0$
$1 + b - 2 + 3 = 0$
$b = -2$

$-2a - c = 0$
$c = -2a$
$c = -2$

$$\pi_1 = \frac{F_D}{D^2 v^2 \rho} = \frac{F_D}{\rho v^2 D^2} = C_D = \text{Drag coefficient}$$

Dimensionless Variables and Buckingham's Pi Theory

In many practical cases of fluid flow problems the number of independent parameters are exceedingly larger than the basic dimensions, which leads to more unknown indices to be solved than the number of equations available.

In most cases, three basic dimensions are used M, L and T. Thus, there will be 3 simultaneous equations to be solved. But, if you have 4 independent indices to be evaluated to find the functional dependence then it will be tough to find the relationship.

Example: The pressure drop per unit length 'p' due to friction in a pipe depends upon the diameter 'D', the mean velocity 'v', the density 'ρ' and the dynamic viscosity 'μ'. Find the relationship between pressure drop and other variables.

$$\pi_2 = f(D, V, \rho, \mu).$$

$$\frac{\mu}{DV\rho}$$

$$= \mu^a \rho^b V^c D^d$$

$$= [ML^{-1}T^{-1}]^a [ML^{-3}]^b [LT^{-1}]^c [L]^d \qquad = \frac{1}{\rho V D/\mu} = \frac{1}{Re}$$

$$= M^{a+b} L^{-a-3b+c+d} T^{-a-c}$$

$$M^0 L^0 T^0 =$$

$$a+b=0 \qquad -a-c=0 \qquad -a-3b+c+d=0$$

$$b=-a \qquad c=-a \qquad -1+3-1+d=0$$

$$c=-1 \qquad d=-1$$

$\Rightarrow$ Assume $a=1$

$b=-1$

$$\boxed{\pi_2 = \frac{\mu}{\rho V D}} = \frac{1}{Re}$$

ABE 307
Date

Dimensionless Variables and Buckingham's Pi Theory

In many practical cases of fluid flow problems the number of independent parameters are exceedingly larger than the basic dimensions, which leads to more unknown indices to be solved than the number of equations available.

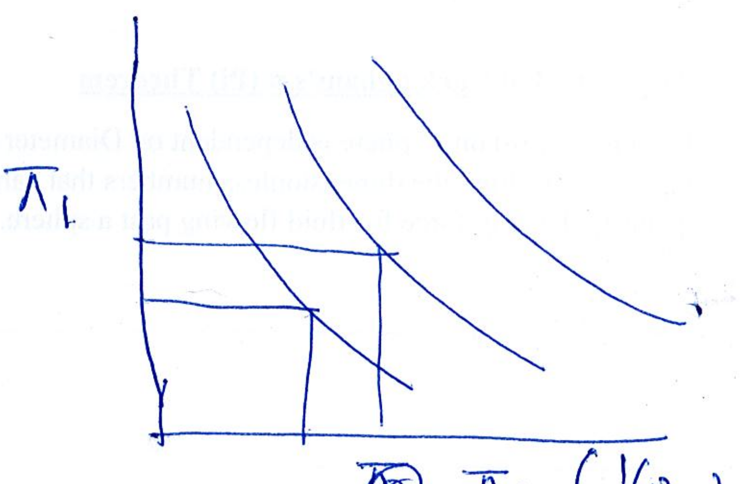
In most cases, three basic dimensions are used M, L and T. Thus, there will be 3 simultaneous equations to be solved. But, if you have 4 independent indices to be evaluated to find the functional dependence then it will be tough to find the relationship.

Example: The pressure drop per unit length 'p' due to friction in a pipe depends upon the diameter 'D', the mean velocity 'v', the density 'ρ' and the dynamic viscosity 'μ'. Find the relationship between pressure drop and other variables.

$$\pi_1 = f(\pi_2).$$

$$\pi_1 = \frac{F_D}{\rho v^2 D^2}$$

$$\pi_2 = \frac{1}{Re}$$



Limit → you can only use your dimensionless charts for the range of experiments that ~~what~~ was used to develop the chart.

Re ~~1 ≤ 20~~

$1 \le Re \le 20$