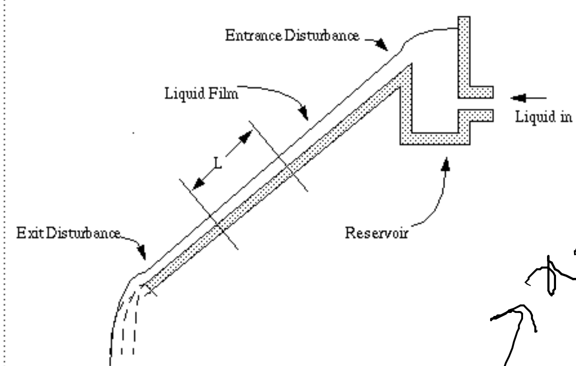
Date : Sep 15th
& Sep 18th, 2017.

Application of Shell Momentum Balance
Thin Film of Falling Fluid

Consider a thin film of falling fluid with thickness δ as shown on a reservoir. We will ignore the initial disturbance and are interested in velocity profile of fluid in the section with steady flow profile. Use shell momentum balance to :

a. Find the Velocity Profile
b. Find the maximum Velocity
c. Find the average velocity
d. Calculate Mass flow rate
e. Force exerted by fluid on solid surface

Fig. 2.2-1



$\delta \rightarrow$ Thickness of fluid.

Direction of fluid $\rightarrow v_z$

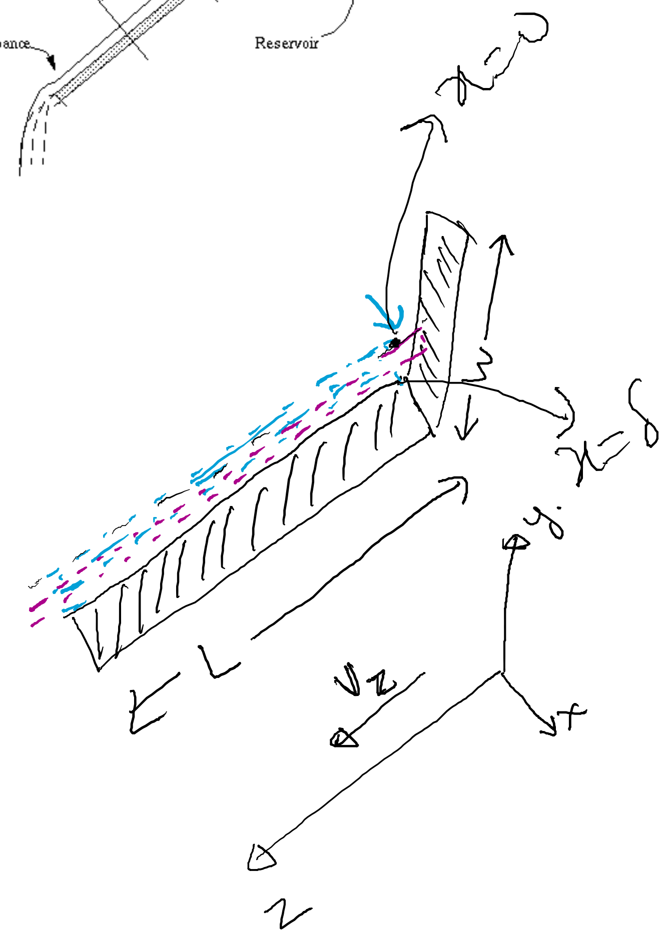
v_z = function of x-direction or varies as depth of liquid.

$v_x = 0$; $v_y = 0$
(No flow in x direction or y direction).

Width of the plate $\rightarrow$ W
length of section for steady flow $\rightarrow$ L

$v_z \neq f(z) \Rightarrow \frac{\partial v_z}{\partial z} = 0$

$v_z \neq f(y) \Rightarrow \frac{\partial v_z}{\partial y} = 0$



(Make assumptions based on geometry and physical situation).

Here, Width of the plate >> δ i.e. (thickness of fluid).

Choose a shell $\perp^{ar}$ to x direction

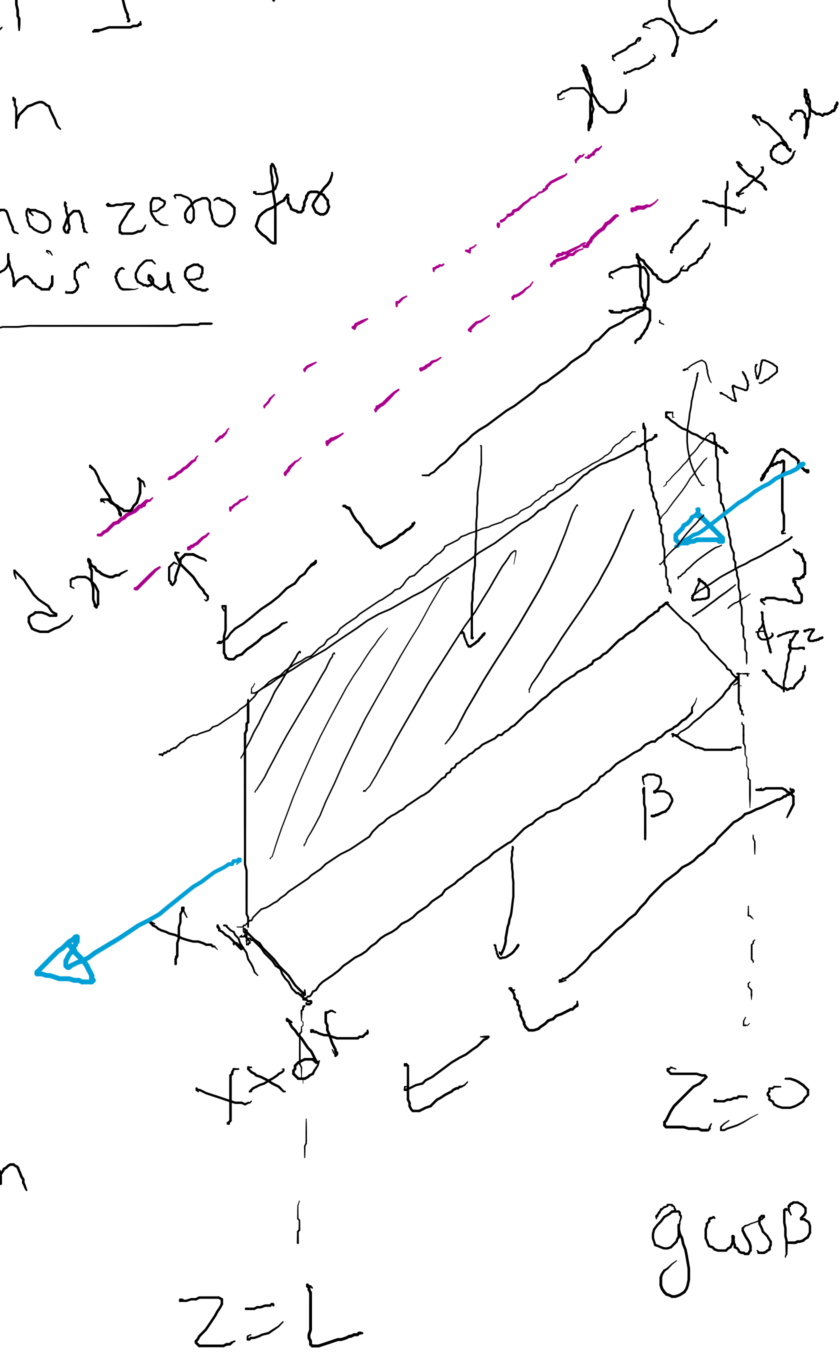
Components that are non zero for this case

ϕ_{xz} ϕ_{yz} ϕ_{zz}

ϕ_{zz} = flux of z momentum in z direction

ϕ_{xz} = flux of z momentum in x-direction

ϕ_{yz} = flux of z momentum in y-direction.



$\phi_{yz} = 0$, since no velocity gradient in y direction.

W >> δ.

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Write momentum balance equation for the components that are non-zero.

→ Rate of z-momentum in at $z=0$

$$\left(\phi_{zz}\big|_{z=0}\right) W\Delta X$$

→ Rate of z-momentum out at $z=L$

$$\left(\phi_{zz}\big|_{z=L}\right) W\Delta X$$

→ Rate of z-momentum in at $x=x$

$$\left(\phi_{xz}\big|_{x=x}\right) WL$$

→ Rate of z-momentum out at $x=x+\Delta x$

$$\left(\phi_{xz}\big|_{x=x+\Delta x}\right) WL$$

<u>Momentum Balance Eqn:</u>

(Momentum In) − (Momentum out) + Body forces = 0

$$\left\{\left(\phi_{xz}\big|_{x=x}\right)LW - \left(\phi_{xz}\big|_{x+\Delta x}\right)LW\right\} \quad \text{—— (I)}$$

$$+ \left\{\cancel{\left(\phi_{zz}\big|_{z=0} W\Delta X\right) - \phi_{zz}\big|_{z=L} W\Delta X}\right\}$$

$$+ \rho g \cos\beta \,(LW\Delta X) = 0$$

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$$\phi_{zz} = p + \cancel{\tau_{zz}}^{\,0} + \rho v_z v_z$$

$$\boxed{\phi_{xz} = \tau_{xz} + \cancel{\rho v_x v_z}^{\,0} \Rightarrow v_x = 0}$$

$p = p(x)$.

$p|_{z=0}$ & $p|_{z=L}$ will be same.

L is large, no pressure difference in steady state.

$$* \quad \tau_{zz} = -\mu\left[2\frac{\partial v_z}{\partial z}\right] \qquad v_z \text{ is a function of } x.$$

$$\Rightarrow \frac{\partial v_z}{\partial z} = 0.$$

$$\phi_{zz}|_{z=0} = p|_{z=0} + \rho v_z v_z|_{z=0}$$

$$\phi_{zz}|_{z=L} = p|_{z=L} + \rho v_z v_z|_{z=L}$$

$$\Rightarrow \boxed{\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L} = 0}$$

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Final momentum balance equation

(Eqn I reduces to)

$$\left(\phi_{xz}\big|_{x=x}\right)LW - \left(\phi_{xz}\big|_{x=x+\Delta x}\right)LW + \rho g \cos\beta\,(LW\Delta x) = 0$$

$$\left(\tau_{xz}\big|_{x=x}\right)LW - \left(\tau_{xz}\big|_{x+\Delta x}\right)LW + \rho g \cos\beta\,(LW\Delta x) = 0$$

Now we can get a differential eqn.

Divide by $LW\Delta x$, take limit $\Delta x \to 0$.

$$\frac{\left(\tau_{xz}\big|_{x=x} - \tau_{xz}\big|_{x+\Delta x}\right)}{\Delta x} + \rho g \cos\beta = 0$$

$$\cancel{-}\frac{d\tau_{xz}}{dx} = \cancel{-}+\rho g \cos\beta$$

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$$\frac{d\tau_{xz}}{dx} = \rho g \cos\beta$$

$$\boxed{\tau_{xz} = \rho g \cos\beta\, x + c_1}$$

at $x = 0$ ie. liquid-gas interface

$$\tau_{xz} = 0$$

$$\Rightarrow \quad c_1 = 0$$

$$\Rightarrow \quad \boxed{\tau_{xz} = \rho g \cos\beta\, x}$$

$$-\mu \frac{dv_z}{dx} = \rho g \cos\beta\, x$$

$$v_z = -\frac{\rho g \cos\beta\, x^2}{2\mu} + c_2$$

at $x = \delta$, $v_z = 0$ (No slip condition) Solid-liquid interface.

$$c_2 = \frac{\rho g \cos\beta\, \delta^2}{2\mu}$$

$$\therefore \quad v_z = \frac{\rho g \cos\beta}{2\mu}\left(\delta^2 - x^2\right)$$

$$\boxed{v_z = \frac{\rho g \cos\beta\, \delta^2}{2\mu}\left(1 - \frac{x^2}{\delta^2}\right)}$$

Max velocity $\rightarrow$ $\frac{\partial v_z}{\partial x} = 0$, $\frac{\partial^2 v_z}{\partial x^2} < 0$

(condition for maxima)

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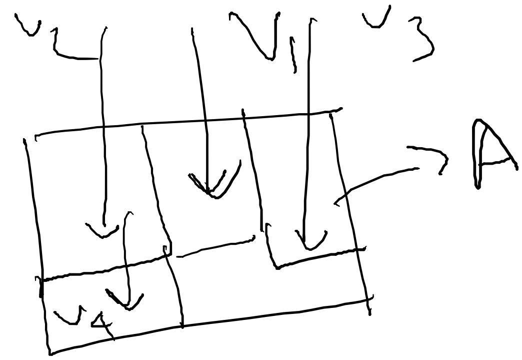
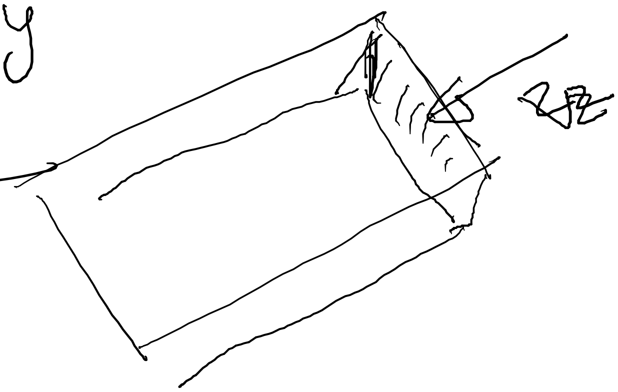
$$\langle v_2 \rangle = \frac{\text{volumetric flowrate}}{\text{cross section area}}$$



$$= \frac{\int_0^W \int_0^\delta v_2 \, dx \, dy}{\int_0^W \int_0^\delta dx \, dy}$$



<u>Force</u>

$$\tau_{xz}\big|_{x=\delta} (\text{Area}).$$

$$= \int_0^L \int_0^W \tau_{xz}\big|_{x=\delta} \, dy \, dz$$