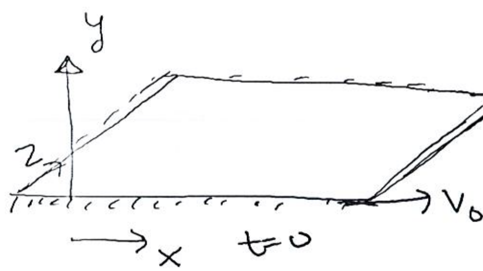
Unsteady State : Laminar Flow Near a Wall Suddenly Set in Motion
(Semi-Infinite Body of Fluid)

→ Semi-infinite fluid → only bounded at lower end

→ The plate is set to motion at $t=0$ with velocity v_0



→ Fluid moves due to momentum transfer.

* ρ & μ = const.

*No Pressure gradient in direction of motion
& no gravitational force in direction of motion.

$\boxed{\text{Step 1}}$ Start with NVS Equation
Cartesian coordintes [Plate geometry]

$$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}$$

$\nu = \frac{\mu}{\rho}$ = kinematic viscosity
gamma.

$\boxed{\text{Step 2}}$ Initial conditions (IC)
$t=0, \quad v_x = 0 =$

Boundary conditions (B.C) (No slip)
$v_x = V_0, \quad y=0 \quad t \geq 0$
$v_x = 0 \quad y \to \infty \quad t > 0$

$$\phi = \frac{v_x}{v_0} \rightarrow \text{Dimensionless velocity}$$

$$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}$$

$$\frac{\partial v_x}{\partial \phi} \cdot \frac{\partial \phi}{\partial t} = v_0 \nu \frac{\partial^2 \phi}{\partial y^2}$$

$$\cancel{v_0} \frac{\partial \phi}{\partial t} = \cancel{v_0} \nu \cdot \frac{\partial^2 \phi}{\partial y^2}$$

$$\boxed{\frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial y^2}} \text{ — I}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left[\frac{\partial v_x}{\partial \phi} \cdot \frac{\partial \phi}{\partial y}\right]$$

$$= \frac{\partial}{\partial y}\left[v_0 \cdot \frac{\partial \phi}{\partial y}\right]$$

$$= \left(\frac{\partial^2 \phi}{\partial y^2}\right) v_0$$

$$v_x = v_x(y,t) \qquad \phi = (y,t)$$

Change B.Cs in form of dimensionless variables

$v_x = v_0$ at $y = 0$ $t > 0$

$v_x = 0$ $y \to \infty$ $t > 0$

$\phi = 1$ at $y = 0$, $t > 0$

$\phi = 0$ at $y \to \infty$ $t > 0$

$$\phi = \frac{v_x}{v_0}$$

$$\phi = \phi(\eta)$$

ϕ, y, t, ν $\boxed{\eta = \frac{y}{\sqrt{4 \nu t}}} \rightarrow$ trial & error. to form a dimensionless variable

$\rightarrow$ We want to convert equn I to ODE, hence choose a combination of variable method.

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Now convert PDE to ODE

$$\frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial y^2}$$

L.H.S

$$\frac{\partial \phi}{\partial t} \quad \frac{\partial \phi}{\partial \eta} \cdot \frac{\partial \eta}{\partial t}$$

$$\frac{\partial \phi}{\partial t} = \frac{-1}{2} \frac{\eta}{t} \cdot \frac{\partial \phi}{\partial \eta}$$

$$\eta = \frac{y}{\sqrt{4\nu t}}$$

$$\frac{\partial \eta}{\partial t} = \frac{y}{\sqrt{4\nu}} \left(-\frac{1}{2}\right) t^{-1/2-1}$$

$$= -\frac{1}{2} \frac{y}{\sqrt{4\nu}} t^{-3/2}$$

$$= \frac{-1}{2} \frac{y}{\sqrt{4\nu t}\, t}$$

$$= \frac{-1}{2} \frac{\eta}{t}$$

R.H.S

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}\right)$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{1}{4\nu t} \cdot \frac{\partial^2 \phi}{\partial \eta^2}$$

$$\eta = \frac{y}{\sqrt{4\nu t}}$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{\sqrt{4\nu t}}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial \eta} \cdot \frac{1}{\sqrt{4\nu t}}\right)$$

$$= \frac{\partial^2 \phi}{\partial \eta^2} \frac{1}{\sqrt{4\nu t}} \cdot \frac{\partial \eta}{\partial y}$$

$\phi = \phi(\eta)$

$\phi \rightarrow$ dimensionless function

$\eta \rightarrow$ dimensionless variable.

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial \eta^2} \cdot \frac{1}{\sqrt{4\nu t}} \times \frac{1}{\sqrt{4\nu t}}$$

Original Equation

$$\frac{\partial \phi}{\partial t} = \underset{\text{gamma} = \mu/\rho}{\nu} \frac{\partial^2 \phi}{\partial y^2} \longrightarrow \text{PDE}$$

$\rightarrow$ Modified Equation

$$-\frac{1}{\cancel{2}} \frac{\eta}{\cancel{t}} \cdot \frac{\partial \phi}{\partial \eta} = \cancel{\nu} \frac{1}{\underset{2}{\cancel{4\nu t}}} \cdot \frac{\partial^2 \phi}{\partial \eta^2}$$

$$-\eta \frac{\partial \phi}{\partial \eta} = \frac{1}{2} \frac{\partial^2 \phi}{\partial \eta^2} \longrightarrow \text{ODE}$$

$$\frac{\partial^2 \phi}{\partial \eta^2} + 2\eta \cdot \frac{\partial \phi}{\partial \eta} = 0$$

B.C. in terms of ϕ, η $\qquad \phi = \frac{V_x}{V_0} \qquad \eta = \frac{y}{\sqrt{4\nu t}}$

$\phi = 1$ at $y = 0$; $\eta = 0$.

$\phi = 0$ at $y \to \infty$; $\eta \to \infty$

$$\frac{d^2\phi}{d\eta^2} + 2\eta \cdot \frac{d\phi}{d\eta} = 0. \qquad \frac{d\phi}{d\eta} = \psi$$

$$\frac{d\psi}{d\eta} + 2\eta \psi = 0$$

$$\frac{d\psi}{d\eta} = -2\eta\psi$$

$$\int \frac{d\psi}{\psi} = \int -2\eta \, d\eta$$

$$\ln \psi = -\eta^2 + C_0$$

$$\psi = \exp(-\eta^2 + C_0)$$

$$\psi = \exp(-\eta^2) \cdot \exp(C_0).$$

$$\frac{d\phi}{d\eta} = \exp(-\eta^2)\cdot C_1 \qquad \exp(C_0) = C_1.$$

$$\phi = \int_0^{\eta} C_1 \exp(-\bar{\eta}^2)\cdot d\bar{\eta} + C_2$$

$\bar{\eta} \rightarrow$ change of variable.

B.C

$\phi = 1$ at $\eta = 0$

$\phi = 0$ at $\eta \rightarrow \infty$.

$$1 = \int_0^0 c \quad) + C_2$$

$$\Rightarrow C_2 = 1$$

$$0 = C_1 \int_0^{\infty} \exp(-\bar{\eta}^2)\cdot d\eta + 1 \qquad (\bar{\eta})$$

$$C_1 = \frac{-1}{\int_0^{\infty} \exp(-\bar{\eta}^2)\cdot d\eta}$$

$$\boxed{\phi = 1 - \frac{\int_0^{\eta} \exp(-\bar{\eta}^2)\cdot d\eta}{\int_0^{\infty} \exp(-\bar{\eta}^2)\cdot d\eta}}$$

Error function. erf

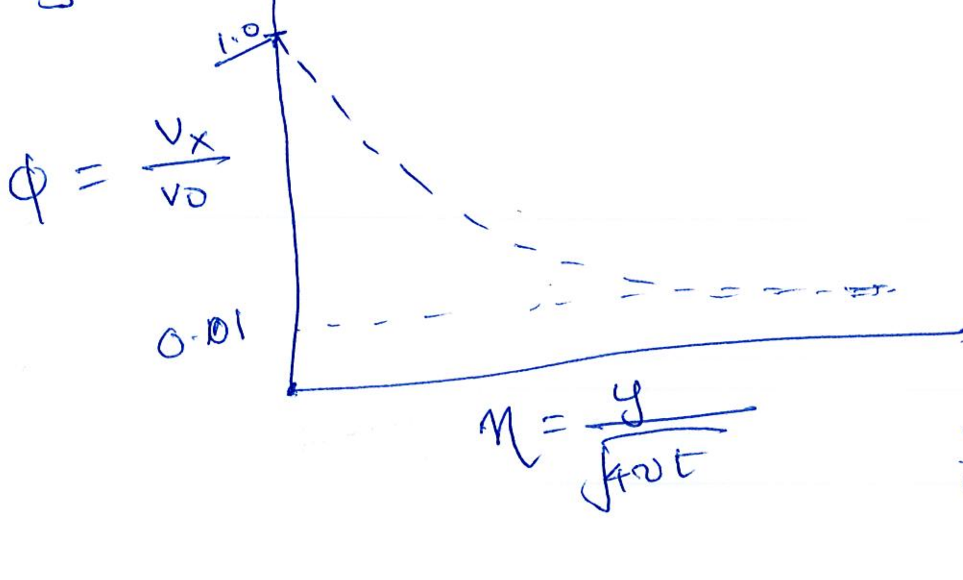
$\phi = \frac{v_x}{v_0}$

$\eta = \frac{y}{\sqrt{4\nu t}}$

$$\boxed{\begin{aligned} erf(0) &= 0 \\ erf(\infty) &= 1 \end{aligned}}$$

1-erf = ~~complem~~

1-erf = complementery error function



$$\boxed{\begin{aligned} \eta &= 2.0 \\ \phi &= 0.01 \end{aligned}}$$

Boundary layer thickness

→ distance from the plate till which we can expect the effect of moving plate by momentum transfer.

$$2.0 = \frac{y}{\sqrt{4\nu t}}$$

$$2.0 = \frac{\delta}{\sqrt{4\nu t}}$$

$$\boxed{\delta = 4\sqrt{\nu t}}$$

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