HOMEWORK 4

SOLUTION:

1) We need Lagrangian reference to observe the whole fluid velocity profile by following the fluid particle that can account for the effect of convection too. If you watch the video it explains how you get the relationship between Lagrangian derivative and Eulerian derivative. We did this in class by using Taylor series expansion instead (mathematical explanation). The video provides the visual/physical explanation.

2) Total differentials, Substantial Differential and Partial differentials. We use Eulerian frame of reference to make observations which maps to partial derivatives.

3) Steady state is an eulerian concept since we observe a particular point in space.

4) D/Dt = 0 does not mean steady flow since it can involve change in density at other places to cancel out density change term for effective D/Dt = 0. Steady state is strictly eulerian.

5) $L = 4\,cm = 0.04\,m$

$\Omega_0 = 10\,rpm = \frac{\pi}{3}\,rad/s$

$\tau = 2.25\times10^{-5}\,N/m$

$d_{in} = 4\,cm = 0.04\,m \qquad r_{in} = 0.02\,m$

$d_{out} = 4.5\,cm = 0.045\,m \qquad r_{out} = 0.0225\,m$

r component : $-\rho\frac{V_\theta^2}{r} = -\frac{\partial p}{\partial r}$

θ component : $\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rV_\theta)\right) = 0$

z component : $-\frac{\partial p}{\partial z} - \rho g = 0$.

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rV_\theta)\right) = 0$$

$$\frac{1}{r}\frac{d}{dr}(rV_\theta) = C_1$$

$$\frac{d}{dr}(rV_\theta) = C_1 r$$

$$rV_\theta = \frac{C_1 r^2}{2} + C_2$$

$$V_\theta = \frac{C_1 r}{2} + \frac{C_2}{r}.$$

B.C $\qquad r = r_{in} \quad V_\theta = 0$

$\qquad r = r_{out} \quad V_\theta = \Omega_0 r_{out}.$

$$0 = \frac{1}{2} C_1 r_{in} + \frac{C_2}{r_{in}}$$

$$\Omega_0 r_{out} = \frac{1}{2} C_1 r_{out} + \frac{C_2}{r_{out}}$$

$$C_1 = \frac{-2C_2}{r_{in}^2}.$$

$$\Omega_0 r_{out} = \frac{1}{2}\left(-\frac{2C_2}{r_{in}^2}\right) r_{out} + \frac{C_2}{r_{out}}$$

$$\Omega_0 r_{out}^2 = C_2\left(1 - \frac{r_{out}}{r_{in}}\right)^2.$$

$$C_2 = \frac{\Omega_0 r_{out}^2}{\left(1 - \left(\frac{r_{out}}{r_{in}}\right)^2\right)} \qquad C_1 = \frac{-2\Omega_0 \left(\frac{r_{out}}{r_{in}}\right)^2}{\left(1 - \left(\frac{r_{out}}{r_{in}}\right)^2\right)}$$

$$T_z = \left(-\tau_{r\theta}\right)\Big|_{r=r_{in}} \cdot 2\pi r_{in} L \cdot r_{in}.$$

$$= 4\pi\mu\Omega r_{out}^2 L \left(\frac{\left(\frac{r_{in}}{r_{out}}\right)^2}{1 - \left(\frac{r_{in}}{r_{out}}\right)^2}\right) \qquad \boxed{\frac{r_{in}}{r_{out}} = k}$$

$$\mu = \frac{T_z}{4 \times \pi \times \Omega\, r_{out}^2 L \left[\frac{k^2}{1-k^2}\right]}$$

$$= 0.022 \ \frac{kg}{m \cdot s}.$$

6) a) Assumptions:-

$V_r \neq 0, \quad V_\theta = V_z = 0$

Continuity:-

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r V_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z) = 0$$

5 points

i.e. $\frac{d}{dr}(r V_r) = 0 \; ; \quad r V_r = C_1$

$$V_r = C_1/r \qquad \text{5 pts.} \qquad (1)$$

Equation of motion:-

r:-

$$\rho\left(\cancel{\frac{\partial V_r}{\partial t}} + V_r \frac{\partial V_r}{\partial r} + \cancel{\frac{V_\theta}{r}} \frac{\partial V_r}{\partial \theta} + \cancel{V_z} \frac{\partial V_r}{\partial z} - \cancel{\frac{V_\theta^2}{r}}\right)$$
$$= -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r V_r)\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 V_r}{\partial \theta^2}} + \cancel{\frac{\partial^2 V_r}{\partial z^2}} - \cancel{\frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta}}\right] + \cancel{\rho g_r}$$

$$\rho V_r \frac{dV_r}{dr} = -\frac{dp}{dr} + \mu \frac{d}{dr}\left(\cancel{\frac{1}{r}\frac{d}{dr}(r V_r)}\right) \qquad \text{5 points}$$

since $r V_r = C_1$, the second term on R.H.S. is zero.

$$\therefore \quad \frac{1}{2}\frac{d}{dr}\rho V_r^2 = -\frac{dp}{dr}$$

$$\frac{1}{2}\rho\left(V_r^2 - V_{R_1}^2\right) = p(R_1) - p(r) \qquad \text{5 points}$$

Since $\frac{V^2(r)}{V^2(R_1)} = \left(\frac{R_1}{r}\right)^2$, we have, 2 pts

Note:- sign mistake in Answer

$$p(r) - p(R_1) = -\frac{1}{2}\rho V^2(R_1)\left[1 - \left(\frac{R_1}{r}\right)^2\right]$$

$$\therefore \quad p(R_1) - p(R_2) = \Delta p = \frac{1}{2}\rho V^2(R_1)\left[1 - \left(\frac{R_1}{R_2}\right)^2\right]$$

3 pts

e) Using $\Delta P = 15$Pa, $R_1 = 1$mm, $R_2 = 5$mm and $\rho = 1000$kg/m^3, we have $U_2(R=5\text{mm}) = 0.177$m/s.
Mass flow rate = $2\pi R_2.(1\text{m}).U_2.(1000\text{kg/m}^3)\text{x}100 \approx 556$kg/s.

7) As derived in class, we have:

θ- component

$$0 = \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_\theta)}{\partial r}\right)\right] \checkmark$$

$$0 = \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)\right]$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) = C_0$$

$$\frac{\partial}{\partial r}(rv_\theta) = C_0 r.$$

$$rv_\theta = \frac{C_0 r^2}{2} + C_1$$

$$\boxed{v_\theta = \frac{C_0 r}{2} + \frac{C_1}{r}}$$

Use boundary conditions:
$V_\vartheta(r = kR) = \Omega\text{x}(kR)$,
$V_\vartheta(r = R) = 0$.

=> $C_0 = 2\Omega k^2/(2k^2 - 1)$, and $C_1 = -\Omega k^2R^2/(2k^2-1)$.