Conversion of Equations of Change (Continuity & Motion)
From Eulerian to Lagrangian

① Eqn. of continuity

$$\frac{\partial \rho}{\partial t} = -(\nabla \cdot \rho \vec{v}) = -[\rho \nabla \cdot \vec{v} + v \cdot \nabla \rho]$$

$\nabla \cdot s\vec{v} = (\nabla s \cdot v) + s(\nabla \cdot \vec{v})$

$$\frac{\partial \rho}{\partial t} + v \cdot \nabla \rho = -\rho(\nabla \cdot \vec{v})$$

$\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$

$\left(\frac{\partial}{\partial t} + v \cdot \nabla\right)\rho \rightarrow D\rho/Dt$

$$\boxed{\frac{D\rho}{Dt} = -\rho(\nabla \cdot \vec{v})}$$

② Equation of Motion

Assume a generic function f

$$\left(\frac{\partial}{\partial t} + \nabla \cdot \vec{v}\right)\rho f$$

$$= \frac{\partial}{\partial t}(\rho f) + \frac{\partial}{\partial x}(\rho v_x f) + \frac{\partial}{\partial y}(\rho v_y f) + \frac{\partial}{\partial z}(\rho v_z f)$$

$$= \rho\left[\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} + v_z \frac{\partial f}{\partial z}\right] + f\underbrace{\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)\right]}_{0}$$

$$= \boxed{\rho \frac{Df}{Dt}}$$

$\frac{\partial \rho}{\partial t} = -(\nabla \cdot \rho \vec{v})$ From continuity equation.

$\rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0$

Eqn. of motion

$$\frac{\partial}{\partial t}(\rho \vec{v}) = -[\nabla \cdot \rho \vec{v}\vec{v}] - \nabla p - [\nabla \cdot \tau] + \rho \vec{g}$$

$$\frac{\partial}{\partial t}(\rho \vec{v}) + [\nabla \cdot \rho \vec{v}\vec{v}] = -\nabla p - [\nabla \cdot \tau] + \rho \vec{g}$$

$$\boxed{\rho \frac{DV}{Dt} = -\nabla p - [\nabla \cdot \tau] + \rho \vec{g}}$$

Interpreting fluid motion as Newton's Law.

L.H.S → acceleration $\left(\frac{DV}{Dt}\right) \times \underbrace{\rho}_{\text{mass per unit volume}}$

(Left hand side of Eqn).

R.H.S → Sum of pressure forces, viscous forces $(\nabla \cdot \vec{\tau})$ + Body forces $(\rho \vec{g})$

(Right hand side of Eqn).