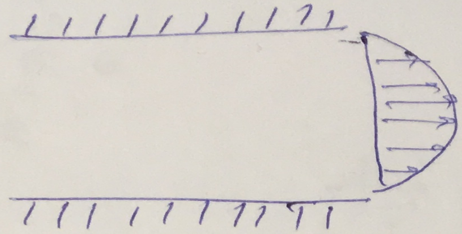
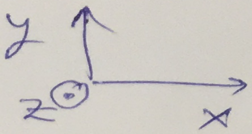
## Assumptions:






① Fully developed flow : $\frac{\partial(\cdot)}{\partial x} = 0$

② Steady flow : $\frac{\partial(\,)}{\partial t} = 0$ $\left[ \frac{D(\,)}{Dt} \neq 0 \right]$

③ $u_z \approx 0$

$$\mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x} + \rho g_x = 0$$

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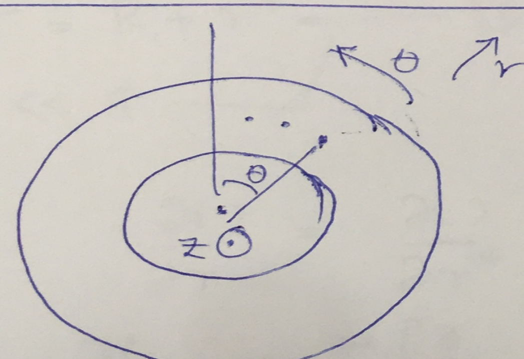


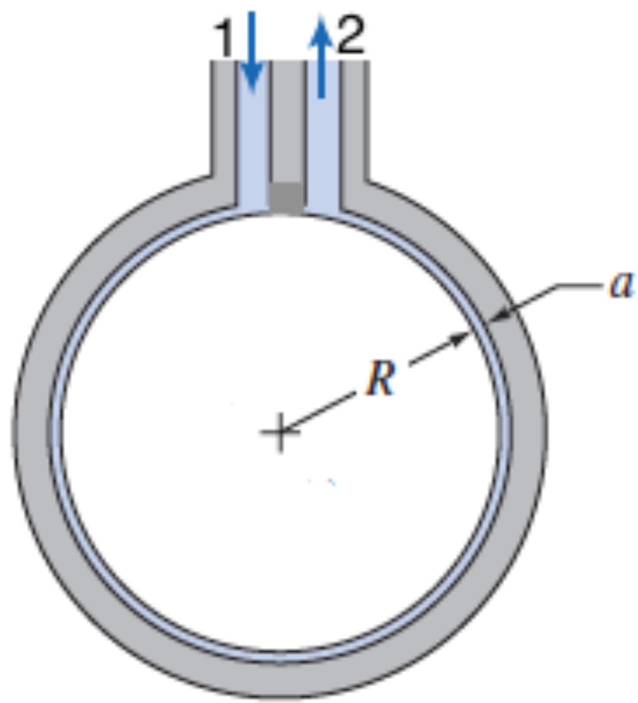

① Axisymmetric flow ; $\frac{\partial(\,)}{\partial \theta} = 0$

② S.F $\frac{\partial(\cdot)}{\partial t} = 0$

③ $\frac{\partial(\,)}{\partial z} = 0$

# Problem 1:

Consider a closed housing with a close fitting drum inside. The clearance is small compared to the diameter. A fluid is pumped through the annular space between the drum and the housing. Find $P_1 - P_2$ as a function of Volume flow rate per unit length Q.



**Incompressible continuity equation:**

$$\cancel{\frac{1}{r}\frac{\partial(ru_r)}{\partial r}}+\cancel{\frac{1}{r}\frac{\partial(u_\theta)}{\partial\theta}}+\cancel{\frac{\partial u_z}{\partial z}}=0 \qquad \textbf{eq a)}$$

*r-component:*

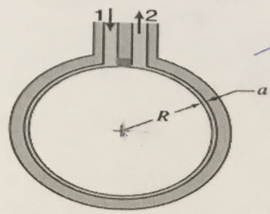
$$\rho\left(\cancel{\frac{\partial u_r}{\partial t}}+\cancel{u_r\frac{\partial u_r}{\partial r}}+\cancel{\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}}-\frac{u_\theta^2}{r}+\cancel{u_z\frac{\partial u_r}{\partial z}}\right)$$

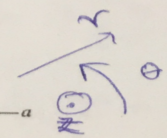
$$=-\frac{\partial P}{\partial r}+\cancel{\rho g_r}+\mu\left[\cancel{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)}-\cancel{\frac{u_r}{r^2}}+\cancel{\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}}-\cancel{\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}}+\cancel{\frac{\partial^2 u_r}{\partial z^2}}\right]$$

***θ direction***

$$\rho\left(\cancel{\frac{\partial v_\theta}{\partial t}}+\cancel{v_r\frac{\partial v_\theta}{\partial r}}+\cancel{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}}+\cancel{\frac{v_r v_\theta}{r}}+\cancel{v_z\frac{\partial v_\theta}{\partial z}}\right)$$

$$=-\frac{1}{r}\frac{\partial P}{\partial\theta}+\cancel{\rho g_\theta}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)+\cancel{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}}+\cancel{\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}}+\cancel{\frac{\partial^2 v_\theta}{\partial z^2}}\right]$$




$$\boxed{+\frac{\rho u_\theta^2}{r}} = +\frac{\partial P}{\partial r} \quad \text{——} \; (1)$$

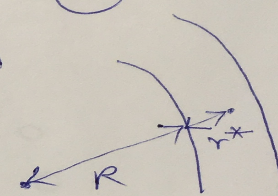
$$\frac{\partial u_\theta}{\partial \theta} = 0 \qquad \frac{\partial u_\theta}{\partial z} = 0$$

$$u_\theta = f(r)$$

$$P = \underbrace{f_1(r)}_{\text{centrifugal}} + \underbrace{f_2(\theta)}_{P} \quad \text{——} \; (a)$$

$$\rightarrow \frac{1}{r}\frac{\partial P}{\partial \theta} = \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right)\right] \quad \text{——} \; (2)$$

$$\frac{a}{R} \ll 1$$




$$r = R + r^* \quad \text{——} \; (4)$$

$$\frac{r^*}{R} \ll 1$$

$$\frac{\partial(\;)}{\partial r} = \frac{\partial(\;)}{\partial r^*} \quad \text{——} \; (3)$$

$$\frac{1}{R+r^*}\frac{\partial P}{\partial \theta} = \mu\left[\frac{\partial}{\partial r^*}\left(\frac{1}{R+r^*}\left(\frac{\partial}{\partial r^*}((R+r^*)u_\theta)\right)\right)\right]$$

$$R + r^* \simeq R$$

$$\frac{1}{R}\frac{\partial P}{\partial \theta} = \mu\left[\frac{\partial}{\partial r^*}\left[\frac{1}{R}\frac{\partial(R u_\theta)}{\partial r^*}\right]\right]$$

$$\boxed{\frac{1}{R}\frac{\partial P}{\partial \theta} = \mu\frac{\partial}{\partial r^*}\left(\frac{\partial u_\theta}{\partial r^*}\right) = \mu\frac{\partial^2 u_\theta}{\partial r^{*2}}}$$

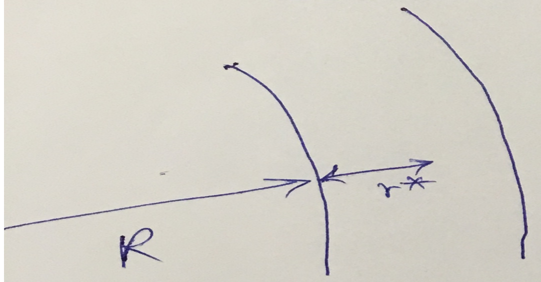
$$\frac{1}{R}\left(\frac{\partial p}{\partial \theta}\right) = \mu \frac{\partial^2 u_\theta}{\partial r^{*2}} \quad \text{———} \quad (5)$$

$$P = f_1(r) + \underbrace{f_2(\theta)}$$

$$\frac{\partial p}{\partial \theta} = f_2'(\theta)$$

$$C_1 + \frac{1}{R}\left(\partial p / \partial \theta\right) r^* = \mu \frac{\partial u_\theta}{\partial r^*}$$

$$\boxed{u_\theta = \frac{1}{2\mu R}\left(\frac{\partial p}{\partial \theta}\right) r^{*2} + \frac{C_1}{\mu} r^* + C_2}$$




BC:

$$u_\theta(r^* = 0) = 0$$
$$u_\theta(r^* = a) = 0$$

$$u_\theta = \frac{1}{2\mu R}\left(\frac{\partial p}{\partial \theta}\right)\left[r^{*2} - a r^*\right]$$

$$\underline{\underline{Q}} = \int_0^a u_\theta \, dr^* = \frac{1}{2\mu R}\underbrace{\left(\frac{\partial p}{\partial \theta}\right)}_{p'} \int_0^a \left[r^{*2} - a r^*\right] dr^*$$

$$\boxed{-\frac{12\mu R Q}{\cancel{12}\, a^3} = \underline{\underline{p'}}} = \frac{\partial f_2}{\partial \theta} \quad \text{———} \quad (6)$$

$$P_2 - P_1 = \left[\cancel{f_1(r_2)} + f_2(\theta_2)\right] - \left[\cancel{f_1(r_1)} + f_2(\theta_1)\right]$$
$$= f_2(\theta_2) - f_2(\theta_1).$$

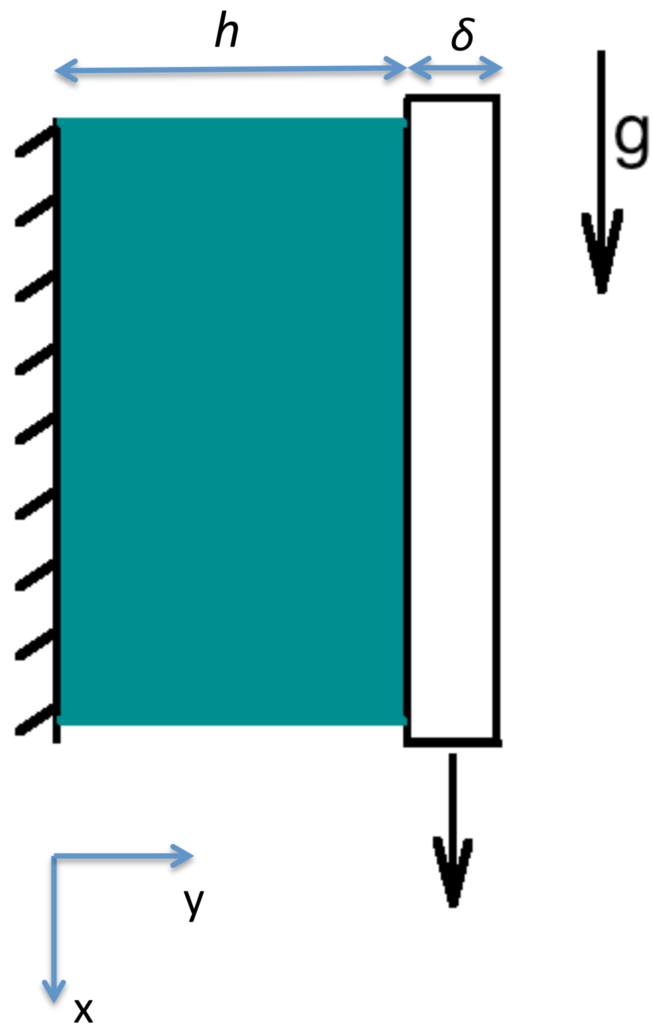
$$\Theta_2 - \Theta_1 = 2\pi$$

$$f_2\Big|_1^2 = \frac{-12}{\cancel{12}} \mu \frac{RQ}{a^3} (\Theta_2 - \Theta_1).$$

$$\boxed{\Delta P = \frac{-24 \mu \pi R Q}{\cancel{12}\, a^3} = P_2 - P_1}$$

# Problem 2:

Consider a metallic plate of thickness $\delta$ in contact with a fluid film of thickness $h$. The plate falls under the influence of gravity. Find the velocity in the film and the terminal velocity of the plate. Neglect the pressure gradient in the fluid.



*x component:*

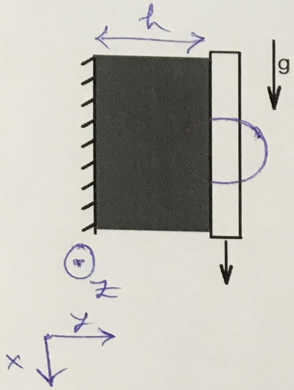
$$\rho\left(\cancel{\frac{\partial u}{\partial t}}+u\cancel{\frac{\partial u}{\partial x}}+v\cancel{\frac{\partial u}{\partial y}}+w\cancel{\frac{\partial u}{\partial z}}\right)=-\cancel{\frac{\partial P}{\partial x}}+\rho g_x+\mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}}+\frac{\partial^2 u}{\partial y^2}+\cancel{\frac{\partial^2 u}{\partial z^2}}\right)$$

*y component:*

$$\rho\left(\cancel{\frac{\partial v}{\partial t}}+u\cancel{\frac{\partial v}{\partial x}}+v\cancel{\frac{\partial v}{\partial y}}+w\cancel{\frac{\partial v}{\partial z}}\right)=-\cancel{\frac{\partial P}{\partial y}}+\cancel{\rho g_y}+\mu\cancel{\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)}$$

*z component:*

$$\rho\left(\cancel{\frac{\partial w}{\partial t}}+u\cancel{\frac{\partial w}{\partial x}}+v\cancel{\frac{\partial w}{\partial y}}+w\cancel{\frac{\partial w}{\partial z}}\right)=-\cancel{\frac{\partial P}{\partial z}}+\cancel{\rho g_z}+\mu\cancel{\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)}$$




① SF

② Fully developed flow

③ $u_z = 0$

④ $\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0$

$$\cancel{\frac{\partial u}{\partial x}}^{\,0} + \underbrace{\frac{\partial v}{\partial y}} + \cancel{\frac{\partial w}{\partial z}}^{\,0} = 0$$

$$\frac{\partial v}{\partial y} = 0 \qquad v = c$$

$$v(y=0) = 0 \ \& \ v(y=h) = 0$$

$$v = 0$$

NS:

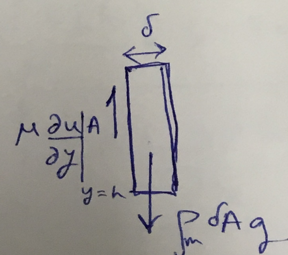
$$\underbrace{\rho g_x + \mu \frac{\partial^2 u}{\partial y^2} = 0} \quad \text{———} \quad ①$$

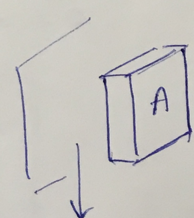
$$\mu \frac{\partial^2 u}{\partial y^2} = -\rho g_x$$

$$\frac{\partial u}{\partial y} = \frac{-\rho g_x y + c_1}{\mu} \ ; \quad u = \frac{-\rho g_x y^2}{2\mu} + c_1 y + c_2 .$$

$u(y=0) = 0 ; \longrightarrow$ BC1

$u(y=h)$




$$\mu \frac{\partial u}{\partial y}\Big|_h \cancel{A} = \rho_m \delta \cancel{A} g \quad \text{———} \quad ②$$

$$\mu \frac{\partial u}{\partial y}(y=h) = \rho_m g \longrightarrow BC_2$$

$$u = -\frac{\rho_e g y^2}{2\mu} + c_1 y + c_2$$

$$\boxed{u = -\frac{\rho_e g y^2}{2\mu} + \frac{(\rho_m \delta + \rho_e h)}{\mu} g y.}$$

$$c_1 = \frac{(\rho_m \delta + \rho_e h)}{\mu} g \quad , \quad c_2 = 0 .$$

$$@ y = h \implies \boxed{u_m = \frac{\rho_m g \delta h}{\mu_e}}$$