Date : Sep 20th, 2017 & Sep 22nd, 2017

Application of Shell Momentum Balance
Flow of Fluid through a Circular Tube

Consider a circular tube and fluid is flowing downwards under the effect of gravity and pressure difference. Density of fluid is ρ and viscosity of fluid is μ. Assume steady state, laminar flow. The length of the tube is L and radius is R. Assume that the length of tube is large compared to radius so we can ignore entrance effect i.e. we can ignore the fact that at tube entrance and exit the flow may not necessarily be parallel to the tube wall and can be unsteady.

We will use cylindrical coordinates for this problem.

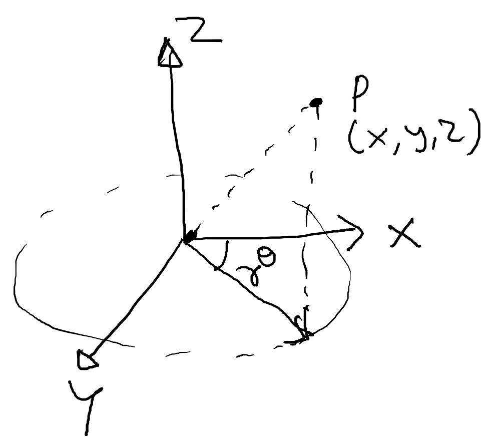
$v = v(r)$

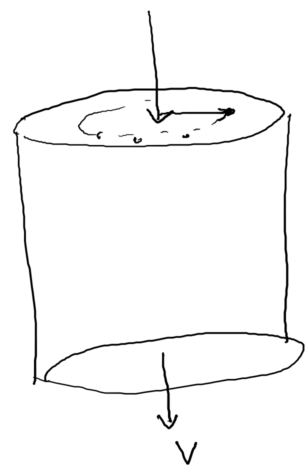
cartesian

$P = P(x, y, z)$

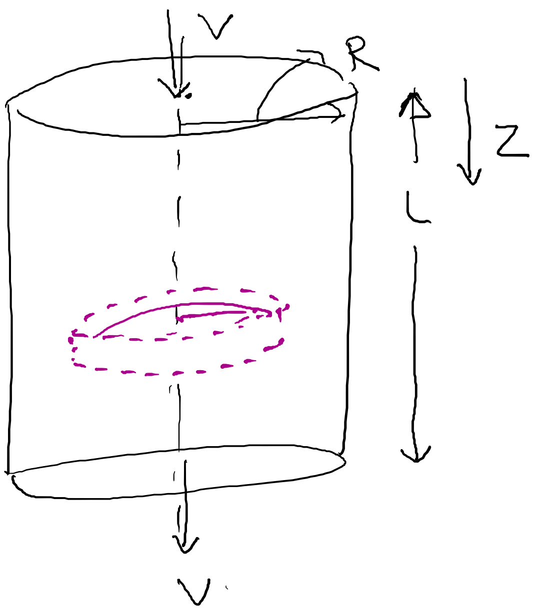
cylindrical

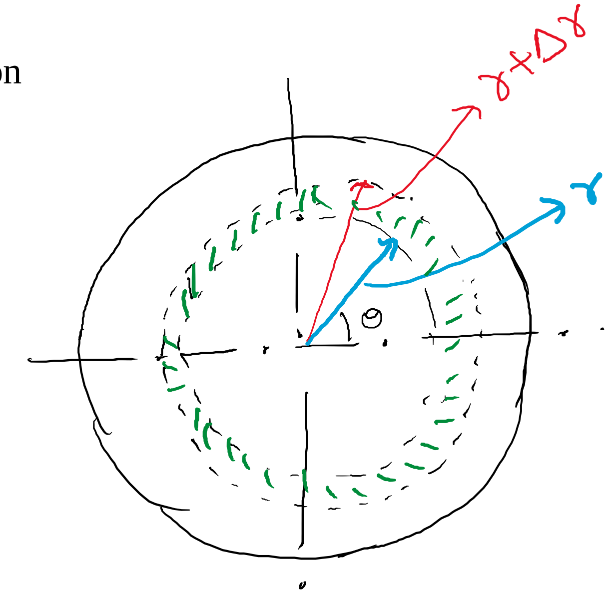
$P = P(r, \theta, z)$





Physical Description of the Flow Situation





Step 1: Identify the non-vanishing velocity components in cylindrical co-ordinates and write down the expression for velocity components.

$v_z = v_z(r)$

$v_r = 0$ (no radial flow)

$v_\theta = 0$ (no rotation)

[Note: momentum still gets transferred in r direction]

Step 2 : What are the non-vanishing momentum transfer components in cylindrical co-ordinates?

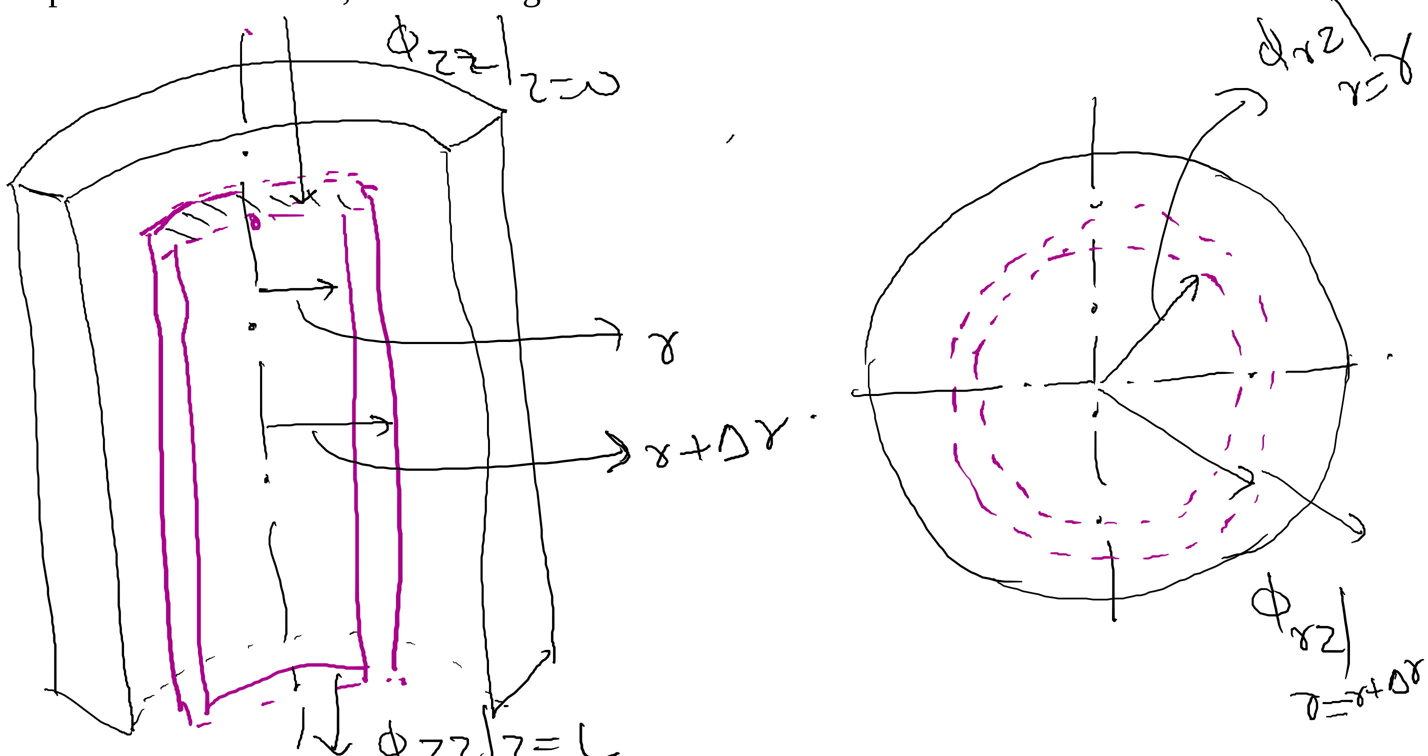
$$\boxed{\phi_{rz} \quad \phi_{zz} \quad \cancel{\phi_{\theta z}}} \rightarrow \cancel{\tau_{\theta z}}^{\,0} + \cancel{\rho v_\theta v_z}^{\,0}$$

$\frac{\partial v_z}{\partial \theta} = 0$

neglecting ϕ_{zr} $\quad \cancel{\phi_{rr}}^{\,0} \quad \cancel{\phi_{\theta r}}^{\,0}$

$\cancel{\phi_{r\theta}}^{\,0} \quad \cancel{\phi_{z\theta}}^{\,0} \quad \cancel{\phi_{\theta\theta}}^{\,0}$

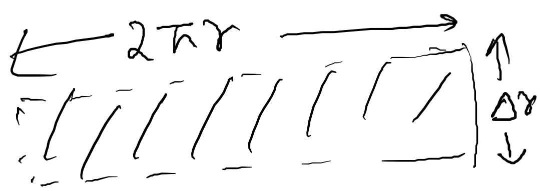
Step 3 : Choose a shell, draw the figure of shell selected



Step 4 : After you have identified which momentum components will be needed for momentum balance equation, write the expression for momentum transfer in and out of the shell.

Rate of z momentum balance components:

ϕ_{rz} ϕ_{zz}



z momentum in at $z=0$ $\left(\phi_{zz}|_{z=0}\right)(2\pi r\Delta r)$.

out at $z=L$ $\rightarrow$ $\left(\phi_{zz}|_{z=L}\right)(2\pi r\Delta r)$

Momentum in at $r=r$ $\left(\phi_{rz}|_{r=r}\right)(2\pi rL)$

" out at $r=r+\Delta r$ $\left(\phi_{rz}|_{r=r+\Delta r}\right)(2\pi rL)$

Step 5 : General Momentum Balance on Shell

$$\phi_{zz}\big|_{z=0}(2\pi r\Delta r) - \phi_{zz}\big|_{z=L}(2\pi r\Delta r)$$
$$+ \phi_{rz}\big|_{r=r}(2\pi r L) - \phi_{rz}\big|_{r+\Delta r}(2\pi r L)$$
$$+ \underbrace{(2\pi r\Delta r L)}_{\text{volume of shell}}\rho g = 0$$

Step 6 : Evaluate the components in Momentum Balance Equation to check if any other term goes to zero or can be cancelled out at input and output. For this

$$\phi_{zz} = p + \cancelto{0}{\tau_{zz}} + \rho v_z v_z$$

$$\phi_{rz} = \tau_{rz} + \rho \cancelto{0}{v_r} v_z$$

$(v_r = 0)$ is assumed

$$\tau_{zz} = -\mu\left[\cancelto{0}{\frac{2\partial v_z}{\partial z}}\right]$$

$$\tau_{rz} = -\mu\left[\frac{\partial v_z}{\partial r}\right]$$

Flow is fully developed v_z at $z=0$ $z=L$ is same

Rewrite Momentum Balance Equation after Applying Assumptions:

$$\left(p + \rho \cancel{v_z v_z}\right)\Big|_{z=0} (2\pi r \Delta r) - \left(p + \rho \cancel{v_z v_z}\right)\Big|_{z=L} (2\pi r \Delta r)$$

$$+ (\tau_{rz})(2\pi r L)\Big|_{r=r} - (\tau_{rz})(2\pi r L)\Big|_{r=r+\Delta r}$$

$$+ \rho g (2\pi r \Delta r L) = 0$$

Step 7 : Differential Equation for the Shear Stress

Divide by $(2\pi \Delta r L)$

$$\frac{\left(r p|_{z=0} - r p|_{z=L}\right)}{L} + \frac{r\tau_{rz}|_{r=r} - r\tau_{rz}|_{r+\Delta r}}{\Delta r} + \rho g r = 0$$

$$\frac{r p|_{z=0} - r p|_{z=L} + \rho g r L}{L} = \frac{r\tau_{rz}|_{r+\Delta r} - r\tau_{rz}|_{r=r}}{\Delta r}$$

$$\frac{r\left[(p|_{z=0} - (p|_{z=L} - \rho g L)\right]}{L} = \frac{d}{dr}(r\tau_{rz})$$

$$r\left[\frac{P_0 - P_L}{L}\right] = \frac{d}{dr}(r\tau_{rz})$$

$$P = p + \rho g (h)$$

$$P_0 = p - (\rho g (0))$$

$$P_L = p - \rho g L$$

(since z is in opposite direction)

Integrate and use Boundary Conditions:

$$r\tau_{rz} = \frac{r^2}{2}\frac{(P_0 - P_L)}{L} + C_2$$

$$\tau_{rz} = \frac{r}{2}\left(\frac{P_0 - P_L}{L}\right) + \frac{C_2}{r}$$

Because at $r = 0$, τ_{rz} must be finite

$$\Rightarrow C_2 = 0$$

$$\boxed{\tau_{rz} = \frac{r}{2}\left(\frac{P_0 - P_L}{L}\right)}$$

Step 8 : Use Newton’s Law of Viscosity and Obtain Velocity Profile

$$\tau_{rz} = -\mu\frac{\partial v_z}{\partial r}$$

$$-\mu\frac{\partial v_z}{\partial r} = \frac{r}{2}\left(\frac{P_0 - P_L}{L}\right)$$

$$\frac{\partial v_z}{\partial r} = \frac{-r}{2\mu}\left(\frac{P_0 - P_L}{L}\right)$$

$$v_z = \frac{-r^2}{4\mu}\left(\frac{P_0 - P_L}{L}\right) + C_3$$

at the tube wall, $v_z = 0$, solid-liquid interface
no slip condition (tube is stationary).

at $r = R$, $v_z = 0$

$$\Rightarrow \quad C_3 = \frac{R^2}{4\mu}\left(\frac{P_0 - P_L}{L}\right)$$

$$v_z = \frac{1}{4\mu}\left(\frac{P_0 - P_L}{L}\right)\left[R^2 - r^2\right]$$

$$\boxed{v_z = \frac{(P_0 - P_L)R^2}{4\mu L}\left[1 - \frac{r^2}{R^2}\right]}$$

Answer the following questions based on your derivations:

1) What is the maximum velocity for fluid flowing in circular pipe?
2) What is the average velocity for fluid flowing in circular pipe?
3) What is the mass flow rate of fluid flowing in circular pipe?
4) What is the force applied by the fluid on the walls of pipe?

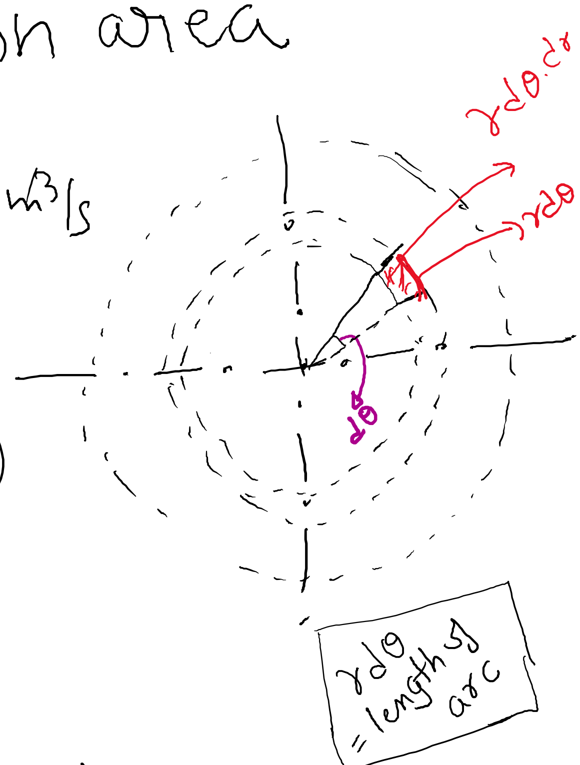
(1) $\frac{\partial v_z}{\partial r} = 0$ (Shows stationary point)

$\frac{\partial}{\partial r}\left(\frac{\partial v_z}{\partial r}\right) = \frac{\partial^2 v_z}{\partial r^2} < 0$ (maxima)

(2) Average velocity

$$= \frac{\text{volumetric flow rate}}{\text{cross section area}}$$

$$\langle v_z \rangle = \frac{\int_0^R \int_0^{2\pi} \underset{(m/s)}{v_z} \, \underbrace{r.d\theta.dr}_{m^2} = m^3/s}{\int_0^R \int_0^{2\pi} r\,d\theta.dr \quad (m^2)}$$



(3) Mass flow rate

$$w = \rho \int_0^R \int_0^{2\pi} v_z \, r.d\theta.dr$$

$$\boxed{w = \frac{\rho (P_0 - P_L) \pi R^4}{8 \mu L}}$$ → Hagen Poisuelli Equation.

$$Re = \frac{\rho \langle v \rangle D}{\mu}$$

$Re < 2100$ (Laminar flow).