ABE 307 Oct 4th, 2017
Date: Oct 11th, 2017.

Steady Flow in a Long Circular Tube Using Navier Stoke's Equation

Assume no radial flow, $v_r = 0$ (No tangential velocity $v_\theta = 0$)

$v_z$

Recall the problem of steady state flow in circular pipe. Obtain the velocity profile equation using the Equation of Continuity and Equation of Motion. Assume constant ρ and μ. Write all the assumptions and conditions for the physical situation to be used.

Eqn of Continuity

**Cylindrical coordinates (r, θ, z):**

0 → const ρ

$$\cancel{\frac{\partial \rho}{\partial t}} + \cancel{\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)} + \cancel{\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)} + \frac{\partial}{\partial z}(\rho v_z) = 0$$

$v_r = 0$  $v_\theta = 0$  $\frac{\partial}{\partial z}(\rho v_z) = 0 \rightarrow v_z \neq f(z)$

$\rho \frac{\partial v_z}{\partial z} = 0$  $v_z(r)$

Equation of Motion:

**In terms of velocity gradients for a Newtonian fluid with constant ρ and μ:**

*r-component* ($v_r = 0$)

$$\rho\left(\cancelto{0}{\frac{\partial v_r}{\partial t}} + \cancelto{0}{v_r\frac{\partial v_r}{\partial r}} + \cancelto{0}{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}} - \cancelto{0}{\frac{v_\theta^2}{r}} + \cancelto{0}{v_z\frac{\partial v_r}{\partial z}}\right) = -\frac{\partial p}{\partial r}$$

$$+ \mu\left[\cancelto{0}{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right)} + \cancelto{0}{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}} - \cancelto{0}{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}} + \cancelto{0}{\frac{\partial^2 v_r}{\partial z^2}}\right] + \cancelto{0}{\rho g_r} \quad (g_r = 0)$$

$\Rightarrow -\frac{\partial p}{\partial r} = 0$

$p \neq f(r)$

*θ-component*

$$\rho\left(\cancelto{0}{\frac{\partial v_\theta}{\partial t}} + \cancelto{0}{v_r\frac{\partial v_\theta}{\partial r}} + \cancelto{0}{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}} + \cancelto{0}{\frac{v_r v_\theta}{r}} + \cancelto{0}{v_z\frac{\partial v_\theta}{\partial z}}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta}$$

$$+ \mu\left[\cancelto{0}{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)} + \cancelto{0}{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}} + \cancelto{0}{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}} + \cancelto{0}{\frac{\partial^2 v_\theta}{\partial z^2}}\right] + \cancelto{0}{\rho g_\theta}$$

$-\frac{1}{r}\frac{\partial p}{\partial \theta} = 0$

$p \neq f(\theta)$

*z-component*

$$\rho\left(\cancelto{0}{\frac{\partial v_z}{\partial t}} + \cancelto{0}{v_r\frac{\partial v_z}{\partial r}} + \cancelto{0}{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}} + \cancelto{0}{v_z\frac{\partial v_z}{\partial z}}\right) = -\frac{\partial p}{\partial z}$$

$\rightarrow \frac{\partial v_z}{\partial z} = 0$

$v_r = 0$

$$+ \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \cancelto{0}{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}} + \cancelto{0}{\frac{\partial^2 v_z}{\partial z^2}}\right] + \rho g_z$$

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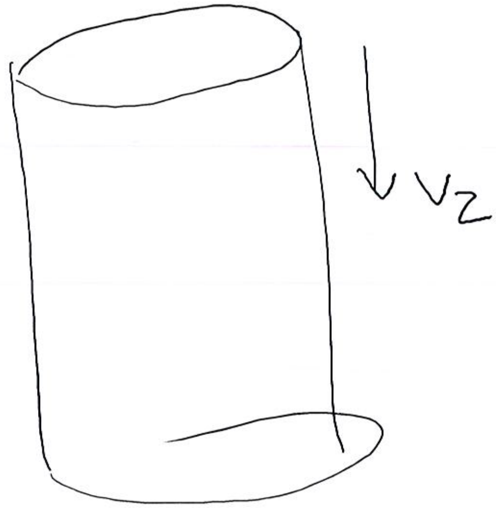
$$0 = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right] + \rho g_z$$

$v_r = 0$
$v_\theta = 0$




r-component.

$$-\frac{\partial p}{\partial r} = 0$$

$\Rightarrow p \neq P(r)$.

θ-component

$$-\frac{1}{r}\cdot\frac{\partial p}{\partial \theta} = 0$$

$p \neq f(\theta)$. $\Rightarrow p = f(z)$

z-component,

$$0 = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial V_z}{\partial r}\right)\right] + \rho g_z$$

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we solve z-component to get velocity profile. since p & g are dependent on z and $V_z$ is function of r, the two terms need to be equal to a const

$f(x)$
$f(y)$
$f(x) + f(y) = 0$.
$f(x) = f(y) = C_0$ with opposite sign.

$P = (p - \rho g z)$ → modified pressure.

$$0 = -\frac{\partial P}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_z}{\partial r}\right)\right].$$

$$+\frac{\partial P}{\partial z} = C_0$$

$$P = C_0 z + C_1$$

(Pressure equation).

$$\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_z}{\partial r}\right)\right] = C_0.$$

$$\frac{\partial}{\partial r}\left(r\cdot\frac{\partial V_z}{\partial r}\right) = \frac{C_0 r}{\mu}$$

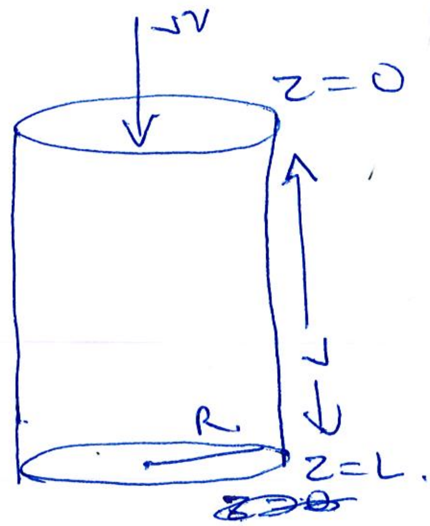
$$r\cdot\frac{\partial V_z}{\partial r} = \frac{C_0 r^2}{2\mu} + C_1'$$

$$\frac{\partial V_z}{\partial r} = \frac{C_0 r}{2\mu} + \frac{C_1'}{r}$$

$$V_z = \frac{C_0 r^2}{4\mu} + C_1' \ln r + C_2$$

$$P = C_0 z + C_1$$

at $z=0$ $\quad P = P_0$.
at $z=L$ $\quad P = P_L$.




$$C_1 = P_0.$$

$$P_L = C_0 L + C_1$$

$$\boxed{C_0 = \frac{P_L - C_1}{L}}$$

$$= \frac{P_L - P_0}{L}$$

$$\boxed{P = \left(\frac{P_L - P_0}{L}\right) z + P_0}$$

$$v_z = \frac{C_0 r^2}{4\mu} + \cancel{C_1' \ln r} + C_2$$

at $r=0$ we need velocity to be finite., ($(\ln r \to \infty)$ as $r \to 0$). Hence, $C_1' = 0$.

at $r = R$, $v_z = 0$ $\rightarrow$ No slip.

$$0 = \frac{C_0 R^2}{4\mu} + C_2$$

$$C_2 = \frac{-C_0 R^2}{4\mu}$$

$$v_z = \left(\frac{P_L - P_0}{L}\right)\frac{r^2}{4\mu} - \left(\frac{P_L - P_0}{L}\right)\frac{R^2}{4\mu}$$

$$\boxed{v_z = \left(\frac{P_L - P_0}{4\mu L}\right)\left[r^2 - R^2\right].}$$

$\rightarrow$ This is the same velocity profile we got from shell momentum balance.