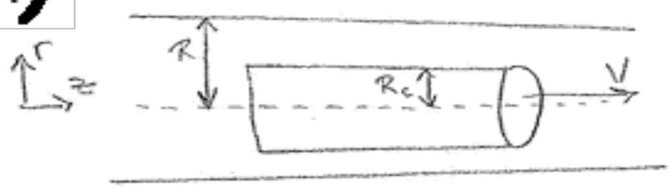
HW 5 Solution:

1)



$V_r = V_\theta = 0 \quad V_z \neq 0,\ V_z(r)$

(a) <u>Continuity</u>

$$\cancel{\frac{\partial \rho}{\partial t}} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r V_r) + \cancel{\frac{1}{r}\frac{\partial}{\partial \theta}(\rho V_\theta)} + \frac{\partial}{\partial z}(\rho V_z) = 0$$ 2

$$0 = 0$$

(b) z - component equation of motion will cancel down to:

$$0 = -\frac{dp}{dz} + \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{dV_z}{dr}\right)$$ 3

(c) Boundary conditions:

#1) $r = R$, $V_z = 0$ #2) $r = R_c$, $V_z = V$ 5

(d)

$$\mu \frac{1}{r}\frac{d}{dr}\left(r\frac{dV_z}{dr}\right) = \frac{dp}{dz} = \frac{P_2 - P_1}{L} = -\frac{\Delta P}{L}$$

$$\frac{d}{dr}\left(r\frac{dV_z}{dr}\right) = -\frac{\Delta P}{\mu L} r$$

$$r\frac{dV_z}{dr} = -\frac{\Delta P}{2\mu L} r^2 + C_1$$

$$\frac{dV_z}{dr} = -\frac{\Delta P}{2\mu L} r + \frac{C_1}{r}$$

$$V_z = -\frac{\Delta P}{4\mu L} r^2 + C_1 \ln(r) + C_2$$

Now apply B.C.s to obtain C_1 & C_2

Apply B.C.#1 $\rightarrow$ $\quad 0 = -\frac{\Delta P}{4\mu L} R^2 + C_1 \ln(R) + C_2 \qquad (1)$

Apply B.C.#2 $\rightarrow$ $\quad V = -\frac{\Delta P}{4\mu L} R_c^2 + C_1 \ln(R_c) + C_2 \qquad (2)$

Subtract (1) from (2) to get:

$$V = \frac{\Delta P}{4\mu L}\left(R^2 - R_c^2\right) + C_1\left(\ln(R_c) - \ln(R)\right)$$

$$V - \frac{\Delta P}{4\mu L}\left(R^2 - R_c^2\right) = C_1 \ln\left(R_c/R\right)$$

$$C_1 = \frac{V - \frac{\Delta P}{4\mu L}\left(R^2 - R_c^2\right)}{\ln\left(R_c/R\right)} = \frac{\frac{\Delta P}{4\mu L}\left(R^2 - R_c^2\right) - V}{\ln\left(R/R_c\right)}$$

Substitute C_1 into either equation, but I will choose (1)

$$0 = -\frac{\Delta P}{4\mu L} R^2 + \frac{\frac{\Delta P}{4\mu L}\left(R^2 - R_c^2\right) - V}{\ln\left(R/R_c\right)} \ln(R) + C_2$$

$$C_2 = \frac{\Delta P}{4\mu L} R^2 - \frac{\frac{\Delta P}{4\mu L}\left(R^2 - R_c^2\right) - V}{\ln\left(R/R_c\right)} \ln(R)$$

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$$\boxed{V_z = -\frac{\Delta P}{4\mu L} r^2 + \frac{\frac{\Delta P}{4\mu L}\left(R^2 - R_c^2\right) - V}{\ln\left(R/R_c\right)} \ln(r) + \frac{\Delta P}{4\mu L} R^2 - \frac{\frac{\Delta P}{4\mu L}\left(R^2 - R_c^2\right) - V}{\ln\left(R/R_c\right)} \ln(R)}$$

(e)

$$\tau_{rz} = -\mu \frac{dv_z}{dr}$$

Recall that earlier in the problem that $\frac{dv_z}{dr} = -\frac{\Delta P}{2\mu L} r + \frac{C_1}{r}$. Therefore,

$$-\mu \frac{dv_z}{dr} = \tau_{rz} = \frac{\Delta P}{2L} r - \mu \frac{C_1}{r}$$

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Now plug in C_1 and evaluate at $r = R$

$$\boxed{\tau_{rz}\Big|_{r=R} = \frac{\Delta P}{2L}R - \frac{\mu}{R}\left(\frac{\frac{\Delta P}{4\mu L}(R^2 - R_c^2) - V}{\ln(R/R_c)}\right)}$$

(f)

$R = 0.17\,cm \quad R_c = 0.15\,cm \quad V = 10\,cm/s \quad \mu = 0.03\,\frac{g}{cm \cdot s} =$

$$\frac{\Delta P}{L} = 100\,\frac{dyne}{cm^3} = 100\,\frac{\frac{g \cdot cm}{s^2}}{cm^3} = 100\,\frac{g}{cm^2 \cdot s^2}$$

$$\tau_{rz}\Big|_{r=R} = \frac{1}{2}(100)(0.17) - \frac{0.03}{0.17}\left[\frac{\frac{1}{4}\left(\frac{100}{0.03}\right)(0.17^2 - 0.15^2) - 10}{\ln(0.17/0.15)}\right]$$

Unit check: $\frac{g}{cm^{\cancel{2}} \cdot s^2} \cdot \cancel{cm}$ $\qquad \frac{g}{cm^2 \cdot s}\left[\frac{\frac{g}{\cancel{cm^2 \cdot s^2}}}{\frac{g}{\cancel{cm \cdot s}}} \cdot cm^{\cancel{2}} - \frac{cm}{s}\right]$

$$\frac{g}{cm \cdot s^2} - \frac{g}{cm^2 \cdot s} \cdot \frac{cm}{s} \checkmark$$

$$\tau_{rz}\Big|_{r=R} = 8.5 + 6.580$$

$$\boxed{\tau_{rz}\Big|_{r=R} = 15.08\,\frac{g}{cm \cdot s^2} = 15.08\,\frac{dyne}{cm^2}}$$

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2) a.) Unsteady state Navier Stokes Equation

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x \cancelto{0}{\frac{\partial v_z}{\partial x}} + v_y \cancelto{0}{\frac{\partial v_z}{\partial y}} + v_z \cancelto{0}{\frac{\partial v_z}{\partial z}}\right)$$

$$= -\cancelto{0}{\frac{\partial p}{\partial z}} + \mu\left(\frac{\partial^2 v_z}{\partial y^2} + \cancelto{0}{\frac{\partial^2 v_z}{\partial x^2}} + \cancelto{0}{\frac{\partial^2 v_z}{\partial z^2}}\right)$$

$$\rho \frac{\partial v_z}{\partial t} = \mu \frac{\partial^2 v_z}{\partial y^2}$$

b.) Initial condition

$t=0 \quad v_z(y,0) = V_{ss}(y)$

Boundary conditions

$y = \pm \frac{h}{2} \quad v_z(y,t) = 0$

$y=0 \quad \frac{\partial v_z}{\partial y} = 0.$

c.) Steady state.

$$0 = -\frac{\partial p}{\partial z} + \mu \frac{d^2}{dy^2}(v_{ss}(y))$$

$$-\frac{(P_0 - P_L)}{L} = \mu \frac{\partial^2}{\partial y^2} v_{ss}(y)$$

$$\left(\frac{-\Delta p}{\mu L}\right) y + c_1 = \frac{\partial}{\partial y} v_{ss}(y)$$

$$\left(\frac{-\Delta p}{\mu L}\right) \frac{y^2}{2} + c_1 y + c_2 = v_{ss}(y)$$

$v_{ss}\left(\frac{h}{2}\right) = 0$

$v_{ss}\left(-\frac{h}{2}\right) = 0.$

$\frac{\partial v_z}{\partial y} = 0$ at $y = 0$

$\Rightarrow c_1 = 0$

$$-\frac{\Delta p}{\mu L}\left(\frac{h^2}{8}\right) + c_2 = 0$$

$$c_2 = \frac{\Delta p}{\mu L}\left(\frac{h^2}{8}\right)$$

$$v_{ss}(y) = \frac{\Delta p h^2}{2\mu L}\left(\frac{1}{4} - \left(\frac{y}{h}\right)^2\right)$$

c.)
&d.) ~~$V_z(y,t)$~~

~~$e\partial V$~~

$$y^* = \frac{y}{h}$$

$$V^* = \frac{V_z(y,t)}{V_{ss,max}}$$

$$t^* = \frac{\mu t}{e h^2}.$$

Substituting dimensionless variables in Navier stokes equation

$$\frac{\partial V_z^*}{\partial t^*} = \frac{\partial^2 V_z^*}{\partial^2 y^*}$$

At $t=0$ $V_z^* = 1-4y^{*2}$

$$V_z^*(y^*, t^*) = T(t^*)\,Y(y^*)$$

$$Y \frac{dT}{dt^*} = T \frac{dY^2}{dy^2}$$

$$\frac{1}{T}\frac{dT}{dt^*} = \frac{1}{Y}\frac{d^2Y}{dy^{*2}} = -\alpha^2.$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -\alpha^2.$$

$$Y = A\sin\alpha y^* + B\cos\alpha y^*$$

$$\frac{\partial v_z^*}{\partial y} = 0 \quad \text{at} \quad y^* = 0.$$

$$A\alpha \cos \alpha y^* - B\alpha \overset{0}{\cancel{\sin}} \alpha y^* = 0$$

$$\Rightarrow A = 0.$$

$$\therefore \quad Y(y^*) = B \cos \alpha y^*$$

$$\frac{1}{T} \frac{dT}{dt^*} = -\alpha^2.$$

$$T(t^*) = C e^{-\alpha^2 t^*}$$

$$y^* = \pm 1/2 \quad Y(y^*) = 0.$$

$$B \cos \frac{\alpha}{2} = 0$$

$$\frac{\alpha_n}{2} = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow \alpha_n = (2n+1)\pi$$

eigen values.

$$Y(y^*) = \sum_{n=0}^{\infty} B \exp(-\alpha_n^2 t^*) \cos(\alpha_n y^*)$$

I.C

$t^* = 0$

$$v_z^*(x,0) = 1 - 4y^{*2}$$

$$= \sum_{n=0}^{\infty} B_n \cos\{(2n+1)\pi y^*\}$$

multiply by $\cos (2n+1)\pi y^*$ and integrate with y from $-\frac{h}{2}$ to $\frac{h}{2}$

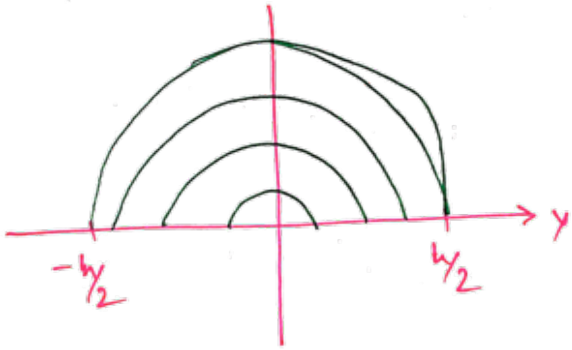
since $\int_{-1/2}^{1/2} (\cos (2n+1)\pi \cancel{y}^*)^2 dy^* = 0$

$$B_n = \frac{(-1)^n \cancel{32} y^{*2}}{(2n+1)^2 \pi^3}$$

$$v_y^*(y^*,t^*) = \sum_{n=0}^{\infty} \frac{32(-1)^n \exp(-\alpha_n^2 t^*) \cos(\alpha_n y^*)}{(2n+1)^2 \pi^3}$$

$$v_y(y,t) = \frac{\Delta p h^2}{8\mu L} \sum_{n=0}^{\infty} \frac{32(-1)^n \exp\left(-\alpha_n^2 \frac{\nu t}{h^2}\right) \cos\left(\alpha_n \frac{y}{h}\right)}{(2n+1)^2 \pi^3}.$$

e.)



3) To solve problem 3, make a simple modification to the 'transient flow in semi-infinite fluid' problem solved in class by changing the boundary conditions:

$\emptyset = 0 \; at \; \eta = 0$,
$\emptyset = 1 \; at \; \eta = \infty$.

$$\frac{d\phi}{d\eta} = \exp(-\eta^2) \cdot C_1$$

exp(co) = C_1.

$$\phi = \int_0^{\eta} C_1 \exp(-\bar{\eta}^2)\, d\bar{\eta} + C_2$$

η → change of variable

B.C

$\phi = 0$ at $\eta = 0$

$\phi = 1$ at $\eta \to \infty$.

$C_2 = 0$,

$$C_1 = \frac{+1}{\int_0^{\infty} \exp(-\bar{\eta}^2) \cdot d\eta}$$

$$\phi = \frac{\int_0^{\eta} \exp(-\bar{\eta}^2) \cdot d\eta}{\int_0^{\infty} \exp(-\bar{\eta}^2) \cdot d\eta}$$

Error function. erf

$\phi = \frac{v_x}{v_0}$

$\eta = \frac{y}{\sqrt{4\nu t}}$

erf(0) = 0

erf(∞) = 1

=> $\emptyset = \text{erf}(\eta)$.