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ABE 307
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## Boundary Layer Velocity Profile

Consider a flow of fluid over a flat plate. Far from the plate let us say that the flow is uniform with a constant velocity $V_\infty$. This is called: Free stream Velocity.

No slip condition, plate is stationary.

So what is the fluid flow like over the plate?

Velocity increases from 0 to approx. $V_\infty$ from $y=0$, to $y=\delta$

Boundary Layer: The thin layer near the vicinity of the plate in which there is a velocity profile, after which the fluid flows with free stream velocity.

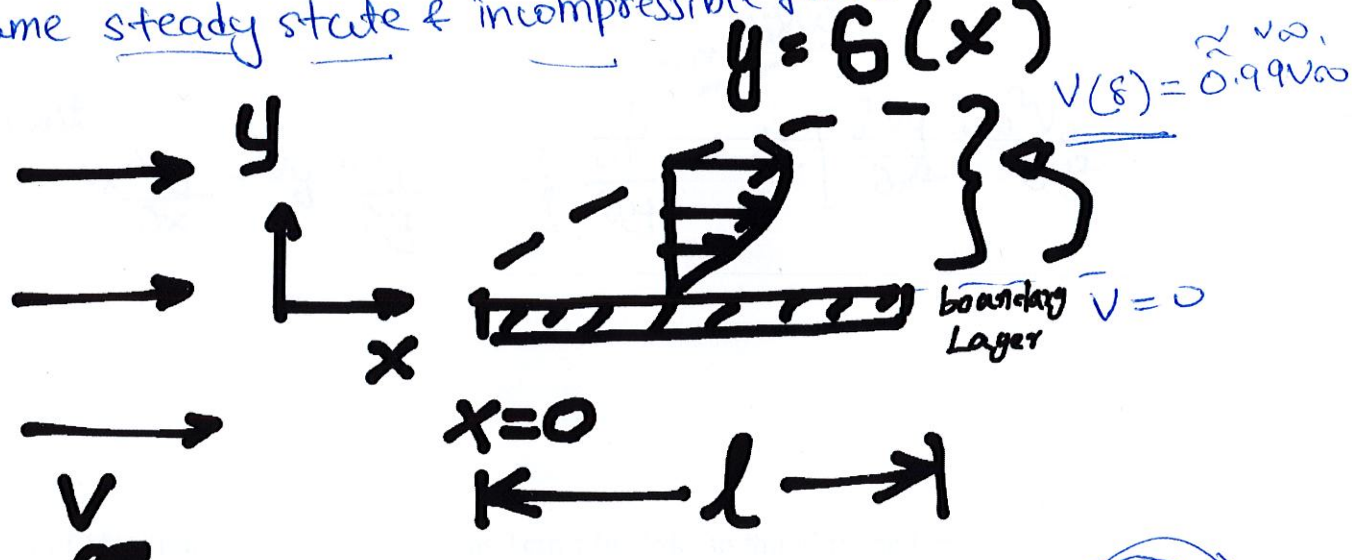
Why are we interested in velocity profile in boundary layer?

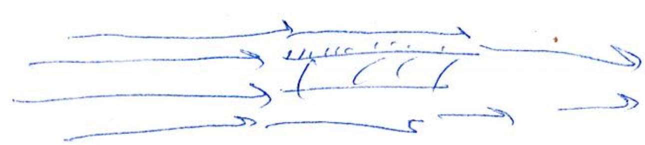
To be able to calculate the drag forces or shear forces of fluid on the solid object.

Dimension of Problem?

Planar flow,
2D $(V_x, V_y)$.

→ Assume steady state & incompressible fluid.



$v_z = 0.$ (2)

Equation of Continuity:

$\nabla \cdot \vec{V} = 0.$ (for incompressible flow).

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \cancelto{0}{\frac{\partial V_z}{\partial z}} = 0$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0. \quad \text{— (1)}$$

Equation of Motion: These equation of motions are written to describe flow within the boundary layer.

**Equation of Motion** for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates

$$\rho\left(\cancelto{0}{\frac{\partial v_x}{\partial t}} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + \cancelto{0}{v_z\frac{\partial v_x}{\partial z}}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \cancelto{0}{\frac{\partial^2 v_x}{\partial z^2}}\right) + \rho g_x$$

$$\rho\left(\cancelto{0}{\frac{\partial v_y}{\partial t}} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + \cancelto{0}{v_z\frac{\partial v_y}{\partial z}}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \cancelto{0}{\frac{\partial^2 v_y}{\partial z^2}}\right) + \rho g_y$$

(neglect) z-component

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z$$

x-component

$$\left(V_x\frac{\partial V_x}{\partial x} + V_y \cdot \frac{\partial V_x}{\partial y}\right) = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\mu}{\rho}\left[\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2}\right].$$

y-component:

$$V_x\frac{\partial V_y}{\partial x} + V_y \cdot \frac{\partial V_y}{\partial y} = \frac{-1}{\rho} \cdot \underbrace{\frac{\partial P}{\partial y}}_{\text{modified}} + \frac{\mu}{\rho}\left[\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2}\right].$$

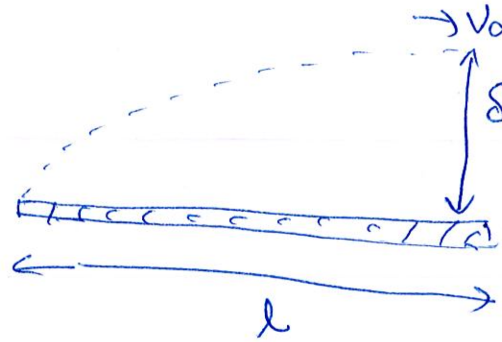
Order of Magnitude Analysis for Various Terms In Order to Simplify the Equation

## Order of magnitude.



$$v_x \frac{\partial v_x}{\partial x} + v_y \cdot \frac{\partial v_x}{\partial y} = -\frac{1}{\rho} \cdot \frac{\partial P}{\partial x} + \frac{\mu}{\rho}\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right].$$

$O \rightarrow$ for order of magnitude.




$$\frac{\partial v_x}{\partial x} \sim O\left(\frac{V_\infty}{l}\right)$$

$$\frac{\partial v_x}{\partial y} \sim O\left(\frac{V_\infty}{\delta}\right).$$

$$v_x \frac{\partial v_x}{\partial x} \sim O\left(\frac{V_\infty^2}{l}\right)$$

$$v_y \cdot \frac{\partial v_x}{\partial y} \sim O\left(? \ \frac{V_\infty}{\delta}\right).$$

$$\sim O\left(\frac{V_\infty \cdot \delta}{l} \cdot \frac{V_\infty}{\delta}\right)$$

$$v_y \frac{\partial v_x}{\partial y} \sim \left(V_\infty^2/l\right)$$

continuity

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$\Rightarrow \frac{\partial v_y}{\partial y} = \frac{-\partial v_x}{\partial x}$$

$$v_y = \int^{\delta} \frac{\partial v_x}{\partial x} \cdot \partial y$$

$$= {}^{o}\frac{V_\infty}{l} \cdot \delta.$$

$$\frac{\partial^2 v_x}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x}\right) \sim O\left(\frac{1}{l} \cdot \frac{V_\infty}{l}\right)$$

$$\sim O\left(\frac{V_\infty}{l^2}\right)$$

$$\frac{\partial^2 v_x}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial y}\right) \sim O\left[\frac{1}{\delta} \cdot \frac{V_\infty}{\delta}\right) \sim O\left(\frac{V_\infty}{\delta^2}\right).$$

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} \sim O\left(\frac{V_\infty^2}{l}\right)$$

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$$v_x \frac{\partial v_x}{\partial x} + v_y \cdot \frac{\partial v_x}{\partial y} = \frac{-1}{\rho}\frac{\partial P}{\partial x} + \frac{\mu}{\rho}\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right]$$

Ignore this term due to much lower magnitude.

$$\boxed{\delta << l}$$

$$\because \quad \frac{\partial^2 v_x}{\partial x^2} \sim O\left(\frac{v_\infty}{l^2}\right) << \frac{\partial^2 v_x}{\partial y^2} \sim O\left(\frac{v_\infty}{\delta^2}\right)$$

$$v_x \cdot \frac{\partial v_x}{\partial x} + v_y \cdot \frac{\partial v_x}{\partial y} = -\frac{1}{\rho} \cdot \frac{\partial P}{\partial x} + \nu \frac{\partial^2 v_x}{\partial y^2}.$$

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y-component

$$v_x \cdot \frac{\partial v_y}{\partial x} + v_y \cdot \frac{\partial v_y}{\partial y} = \frac{-1}{\rho}\frac{\partial P}{\partial y} + \frac{\mu}{\rho}\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right].$$

$$v_x \cdot \frac{\partial v_y}{\partial x} \sim O\left(v_\infty \cdot \frac{\delta v_\infty}{l} \cdot \frac{1}{l}\right) \sim O\left(\frac{\delta v_\infty^2}{l^2}\right).$$

$$v_y \cdot \frac{\partial v_y}{\partial y} \sim O\left(\frac{\delta v_\infty}{l} \cdot \frac{\delta v_\infty}{l} \cdot \frac{1}{\delta}\right) \sim O\left(\frac{v_\infty^2 \delta}{l^2}\right).$$

$$\frac{-1}{\rho}\frac{\partial P}{\partial y} \sim O\left(\frac{\delta v_\infty^2}{l^2}\right).$$

$$\text{since } O\left(-\frac{1}{\rho} \cdot \frac{\partial P}{\partial y}\right) << O\left(\frac{-1}{\rho} \cdot \frac{\partial P}{\partial x}\right)$$

we can ignore, y-component of equation of motion.

So, our final equations that we need to solve for getting velocity profile in Boundary layer are: (5)

(1) $\dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} = 0$ (continuity equation).

(2) $v_x \dfrac{\partial v_x}{\partial x} + v_y \cdot \dfrac{\partial v_x}{\partial y} = -\dfrac{1}{\rho} \cdot \dfrac{\partial P}{\partial x} + \nu \left[ \dfrac{\partial^2 v_x}{\partial y^2} \right]$

] Prandtl Boundary layer equations.

Further simplification, eliminate $\dfrac{\partial P}{\partial x}$ using Potential flow, since outside the boundary layer, potential flow theory is applicable.

B.C $v_x = 0, \quad y = 0$
$v_y = 0, \quad y = 0$ } No slip condition.

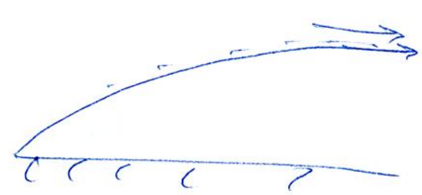
$\frac{1}{2} \rho (v_x^2 + v_y^2) + P = \text{const}$.

$\frac{1}{2} \rho (v_e^2 + 0) + P = \text{const}$

$\Rightarrow \frac{1}{2} \times 2 \rho v_e \cdot \dfrac{dv_e}{dx} + \dfrac{dP}{dx} = 0$

$\Rightarrow -\dfrac{1}{\rho} \cdot \dfrac{dP}{dx} = v_e \cdot \dfrac{dv_e}{dx}$.

$v_e$ = velocity outside the boundary layer
$= V_\infty$ (Potential flow velocity).



(2) $\boxed{v_x \dfrac{\partial v_x}{\partial x} + v_y \cdot \dfrac{\partial v_x}{\partial y} = v_e \dfrac{dv_e}{dx} + \nu \dfrac{\partial^2 v_x}{\partial y^2}}$ 4.4.11

From Prandtl's equations → we derive the Von-Karman momentum integral equation.

Deriving the Von-Karman Momentum Integral Equation

The equation obtained for Boundary Layer:

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = v_e \frac{d v_e}{dx} + \nu \frac{\partial^2 v_x}{\partial y^2} \quad - \underline{4.4.11}$$

$$\nu = \frac{\mu}{\rho}.$$

**Step 1:** Transform this equation into PDE with single dependent variable

Hint: Use equation of continuity. & eliminate $v_y$.

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \Rightarrow v_y = -\int_0^y \frac{\partial v_x}{\partial x} \cdot dy.$$

Sub

$$v_x \cdot \frac{\partial v_x}{\partial x} - \int_0^y \left(\frac{\partial v_x}{\partial x}\right) dy \cdot \frac{\partial v_x}{\partial y} = v_e \cdot \frac{d v_e}{dx} + \nu \cdot \frac{\partial^2 v_x}{\partial y^2}.$$

$\int$ to get to an eqn that allows for calculation of shear stresses.

This is now a PDE in single dependent variable that can be integrated to get Von-Karman equation.

**Step 2:** Multiply the new Eqn ( ) with density ρ and integrate from $y = 0$ to $y = \infty$

$$\underbrace{\int_0^\infty \rho v_x \cdot \frac{\partial v_x}{\partial x} \cdot dy}_{\mathrm{I}} - \underbrace{\int_0^\infty \rho \int_0^y \left(\frac{\partial v_x}{\partial x} \cdot dy\right) \cdot \frac{\partial v_x}{\partial y} dy}_{\mathrm{II}} = \underbrace{\int_0^\infty \rho v_e \cdot \frac{d v_e}{dx} \cdot dy}_{\mathrm{III}} + \underbrace{\int_0^\infty \mu \frac{\partial^2 v_x}{\partial y^2} \cdot dy}_{\mathrm{IV}}.$$

Your equations now has four different terms to be integrated.

$$\mathrm{I} = \int_0^\infty \rho v_x \cdot \frac{\partial v_x}{\partial x} \cdot dy$$

$$\mathrm{II} = \int_0^\infty \rho \int_0^y \left(\left(\frac{\partial v_x}{\partial x}\right) dy\right) \cdot \frac{\partial v_x}{\partial y} \cdot dy.$$

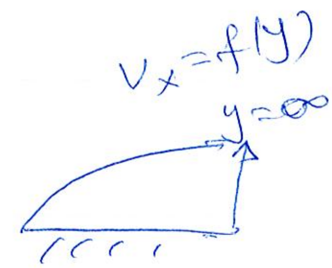
$$\mathrm{III} = \int_0^\infty \rho v_e \cdot \frac{d v_e}{dx} \cdot dy$$

$$\mathrm{IV} = \int_0^\infty \mu \cdot \frac{\partial^2 v_x}{\partial y^2} \cdot dy.$$

$$\left.\frac{\partial v_x}{\partial y}\right|_{y=\infty} = 0$$

**Step 3:** Integrate Term IV using $\frac{\partial v_x}{\partial y} = 0\ at\ y = \infty$

$$\int_0^\infty \mu \cdot \frac{\partial^2 v_x}{\partial y^2} \cdot dy = \cancel{\left.\mu \frac{\partial v_x}{\partial y}\right|_{y=\infty}}^{\,0} - \left.\mu \frac{\partial v_x}{\partial y}\right|_{y=0}$$




Why is this boundary condition true? –

$\left.\frac{\partial v_x}{\partial y}\right|_{y=\infty} = 0$ ; because outside boundary layer your velocity is not dependent on $y$.

Step 4 : Simplify Term II using integral by parts.

$$\int u\, dv = uv - \int v\, du$$

Use $dv = \frac{\partial v_x}{\partial y}\, dy$ ; $u = \int_0^y \frac{\partial v_x}{\partial x}\, dy \Big|$ *between* $y = 0$ *and* $y = \infty$

$$\int_0^\infty \rho \underbrace{\int_0^y \left(\frac{\partial v_x}{\partial x} \cdot d\bar{y}\right)}_{u} \underbrace{\frac{\partial v_x}{\partial y} \cdot dy}_{dv}.$$

Integrate dv from y = 0 to y = y to get v

$$v = \int_0^y \frac{\partial v_x}{\partial y} \cdot dy = v_x\Big|_y - \cancel{v_x\Big|_{y=0}}^{\,0} \rightarrow \text{no slip condition}$$

$$v = v_x$$

Differentiate u to get du

$$u = \int_0^y \left(\frac{\partial v_x}{\partial x} \cdot d\bar{y}\right)$$

$$du = \frac{\partial v_x}{\partial x} \cdot d\bar{y}$$

$$\rho\left[\int_0^\infty v_x \int_0^y \left(\frac{\partial v_x}{\partial x} \cdot d\bar{y}\right) dy - \int_0^\infty v_x \cdot \frac{\partial v_x}{\partial x} \cdot d\bar{y}\right].$$

$$= \rho\, v_e \int_0^y \frac{\partial v_x}{\partial x} \cdot d\bar{y} - \rho \int_0^\infty v_x \cdot \frac{\partial v_x}{\partial x} d\bar{y}$$

$v_x = 0$ at $y = 0$
$v_x = v_e$ at $y = \infty$

Substitute the modified form of II and IV to get the new equation:

$$\int_0^\infty \rho v_x \cdot \frac{\partial v_x}{\partial x} \cdot dy - \rho v_e \int_0^y \frac{\partial v_x}{\partial x} \cdot d\bar{y} + \rho \int_0^\infty v_x \cdot \frac{\partial v_x}{\partial x} \cdot d\bar{y}$$

$$= \rho \int_0^\infty v_e \cdot \frac{dv_e}{dx} \cdot dy - \mu \frac{\partial v_x}{\partial y}\Big|_{y=0}$$

Take the term that has viscosity μ on one side and rest of the term on one side of equation.

$$\underline{\mu \frac{\partial v_x}{\partial y}\Big|_{y=0}} = \rho \int_0^\infty v_e \cdot \frac{dv_e}{dx} \cdot dy - \int_0^\infty \rho v_x \cdot \frac{\partial v_x}{\partial x} \cdot dy + \rho v_e \int_0^y \frac{\partial v_x}{\partial x} d\bar{y} - \rho \int_0^\infty v_x \cdot \frac{\partial v_x}{\partial x} \cdot d\bar{y}$$

What does the term with viscosity represent?

L.H.S

Shear stress

Now to the last equation, add and subtract the term $\boxed{\rho \int_0^\infty v_x \frac{dv_e}{dx}\, dy}$

$$\mu \frac{\partial v_x}{\partial y}\Big|_{y=0} = \underline{\rho \int_0^\infty v_e \cdot \frac{dv_e}{dx} \cdot dy - \rho \int_0^\infty 2 v_x \cdot \frac{\partial v_x}{\partial x} \cdot dy} + \rho v_e \int_0^y \frac{\partial v_x}{\partial x} d\bar{y} + \rho \int_0^\infty v_x \cdot \frac{dv_e}{dx} \cdot dy$$

$$\underline{- \rho \int_0^\infty v_x \cdot \frac{dv_e}{dx} \cdot dy}$$

Simplification now involves collection of terms together.

$$= \rho \int_0^\infty (v_e - v_x) \cdot \frac{dv_e}{dx} \cdot dy$$

$$+ \rho \int_0^\infty \frac{d(v_e v_x)}{dx}\, dy - \rho \int_0^\infty \frac{d(v_x^2)}{dx} \cdot dy$$

$$\mu \cdot \frac{\partial v_x}{\partial y}\bigg|_{y=0} = \rho \int_0^\infty (v_e - v_x) \cdot \frac{dv_e}{dx} \cdot dy + \rho \int_0^\infty \frac{d}{dx}\left(v_e v_x - v_x^2\right) dy.$$

Von-karman momentum integral equation.