ABE 307
Date:

Rotational Viscometers

Viscometers are instruments used for measurement of viscosity in different processing industries. We have seen the use of capillary viscometer. A capillary viscometer uses various measurements of pressure differences and mass flow rate to estimate viscosity of a fluid (recall the problem based on Hagen-Poiuessile eqn)

Another instrument that is used by industries is rotational viscometer.

Concept of Rotational Viscometer: The resistance to rotational motion of fluid is measured by torque generated which is related to the geometry of viscometer and fluid properties to calculate viscosity.

Categories of Rotational Viscometer: Two major types of viscometers

(Note: This information has been taken from viscometer information provided by OFITE. OFITE is an instrumentation company based in Texas that builds and supplies various instruments for industrial use. http://www.ofite.com/couette-viscometers)

a) Stormer Type: Instrument measures shear rate against constant torque.
b) Searle and Couette Type: Instrument measures torque with defined shear rate.

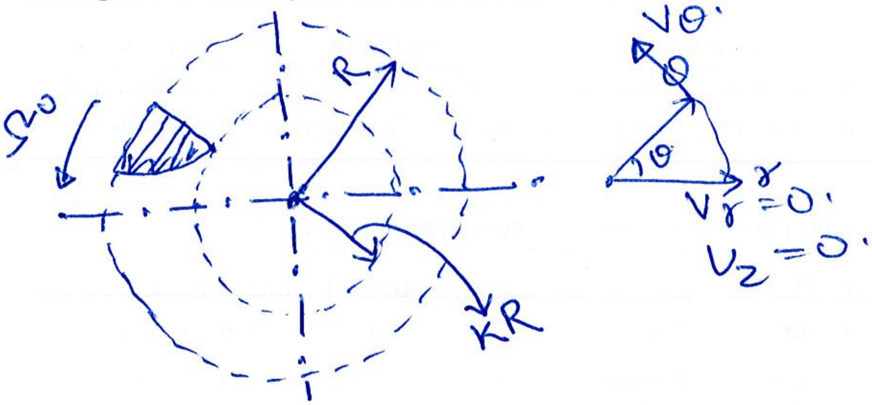
We will derive equations for Couette Viscometer and Searle viscometer. Both Searle and Couette viscometers are in "cup and bob" type viscometer category.

Problem 1: Derive the equations to be used for viscosity measurement using Couette Viscometer
Problem 2: Derive the equations to be used for viscosity measurement using Searle (when inner cup is rotating) viscometer.

Couette viscometer generally consists of two concentric cylinders. Outer cylinder rotates with an angular velocity Ω and the inner cylinder is stationary. The angular momentum gets transferred to the stationary cylinder producing a torque which is measured by a sensing device. From force balances the torque applied by momentum transfer is equal to the torque measured by the devices. This theory is used to develop relationship between torque measured and other parameters of the device that is finally used to estimate viscosity μ.

Use the above information and general equations (Eqn of continuity and Eqn of motion for constant ρ and μ to get the equation for measuring viscosity using Couette viscometer.

Set up and Geometry of the viscometer



Objective:

To obtain an expression that relates the torque to fluid property and geometry of the instrument.

Flow Condition:

Steady state, incompressible fluid, μ = cott. no radial flow, no flow in z-direction; laminar flow.

Approach:

Use the eqn. of continuity & motion to come up with relationship for velocity, τ (torque) & geometry.

Assumptions:

ρ & μ = cott, $v_z = 0$, $v_r = 0$. $v_\theta = v_\theta(r)$.

$P = P(z, r)$ P is function of r, because of centrifugal force
P is " " z " of gravity force.

Eqn of Continuity

Cylindrical coordinates (r, θ, z):

$$\cancelto{0}{\frac{\partial \rho}{\partial t}} + \cancelto{0}{\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)} + \cancel{\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)} + \cancelto{0}{\frac{\partial}{\partial z}(\rho v_z)} = 0$$

Steady state $\quad v_r = 0 \quad v_\theta \neq f(\theta)$.

trivial equation
no new information.

Equation of Motion:

In terms of velocity gradients for a Newtonian fluid with constant ρ and μ:

r-component[a]

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] + \rho g_r$$

θ-component[b]

$$\rho\left(\cancelto{0}{\frac{\partial v_\theta}{\partial t}} + \cancel{v_r\frac{\partial v_\theta}{\partial r}} + \cancelto{0}{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}} + \cancelto{0}{\frac{v_r v_\theta}{r}} + \cancelto{0}{v_z\frac{\partial v_\theta}{\partial z}}\right) = \cancelto{0}{-\frac{1}{r}\frac{\partial p}{\partial \theta}} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}} + \cancelto{0}{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}} + \cancelto{0}{\frac{\partial^2 v_\theta}{\partial z^2}}\right] + \cancelto{0}{\rho g_\theta}$$

$$\frac{\partial}{\partial \theta}\left(\cancelto{0}{\frac{\partial v_\theta}{\partial \theta}}\right)$$

z-component

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

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r-component

$$-\frac{\rho v_\theta^2}{r} = -\frac{\partial p}{\partial r}$$ → pressure variation in r component

θ-component

$$0 = \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)\right]$$ ✓

z-component

$$0 = -\frac{\partial p}{\partial z} + \rho g_z.$$ → Pressure relation with z-~~comp~~ direction.

velocity profile.

$$0 = \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)\right]$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) = C_0$$

$$\frac{\partial}{\partial r}(r v_\theta) = C_0 r$$

$$r v_\theta = \frac{C_0 r^2}{2} + C_1$$

$$\boxed{v_\theta = \frac{C_0 r}{2} + \frac{C_1}{r}}$$

Use B.C.

① at $r = R$, $\quad v_\theta = \Omega_0 R$

② $r = kR \quad v_\theta = 0$

① $\left\{\Omega_0 R = \frac{C_0 R}{2} + \frac{C_1}{R}\right\}$ ~~xk~~

② $0 = \frac{C_0 kR}{2} + \frac{C_1}{kR} \Rightarrow$

$$C_0 = \left(-\frac{C_1}{kR}\right)\frac{kR}{2}$$

$$= \frac{-2C_1}{kR^2}$$

$(1 \times K) - 2$

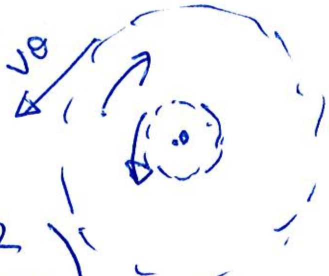
$$\Rightarrow K\Omega_0 R = \frac{C_1 K}{R} - \frac{C_1}{KR}$$

$$\Rightarrow C_1 = \frac{k\Omega_0 R}{\frac{k}{R} - \frac{1}{kR}}$$

$$v_\theta = \frac{-\Omega_0 R\left[\frac{r}{KR} - \frac{KR}{r}\right]}{\left(1/K - K\right)}.$$

$$\cancel{\tau_{r\theta} = -\mu \frac{v_\theta}{r}}$$

$$\tau_{r\theta} = -\mu r \frac{d}{dr}\left(\frac{v_\theta}{r}\right)$$



$$\tau_{r\theta} = -2\mu\Omega_0\left(\frac{R}{r}\right)^2\left(\frac{K^2}{1-K^2}\right)$$

$$T = \underbrace{-\tau_{r\theta}\Big|_{r=KR}}_{\text{Torque force.}} (KR)\, \underbrace{(2\pi KRL)}_{\text{surface area of the inner cylinder.}}$$

simplify

Expression from fluid flow $\leftarrow$ $$T = 4\pi\mu\Omega_0 R^2 L\left(\frac{K^2}{1-K^2}\right).$$

Expression from instrument measuring torque. $\leftarrow$ $$T' = \underbrace{k_t}_{\text{instrument const.}}\ \theta_b \rightarrow \text{deflection angle}$$ } This is what your instrument measures.

→ On equating the two expressions

$$\boxed{\mu = \frac{k_t\,\theta_b}{4\pi\Omega_0 R^2 L\left(\frac{K^2}{1-K^2}\right)}}$$

since this expression was developed using laminar steady flow assumption, always maintain the flow situation to match assumption.