Date: 23rd oct 2017.

Flow in Rectangular channel.

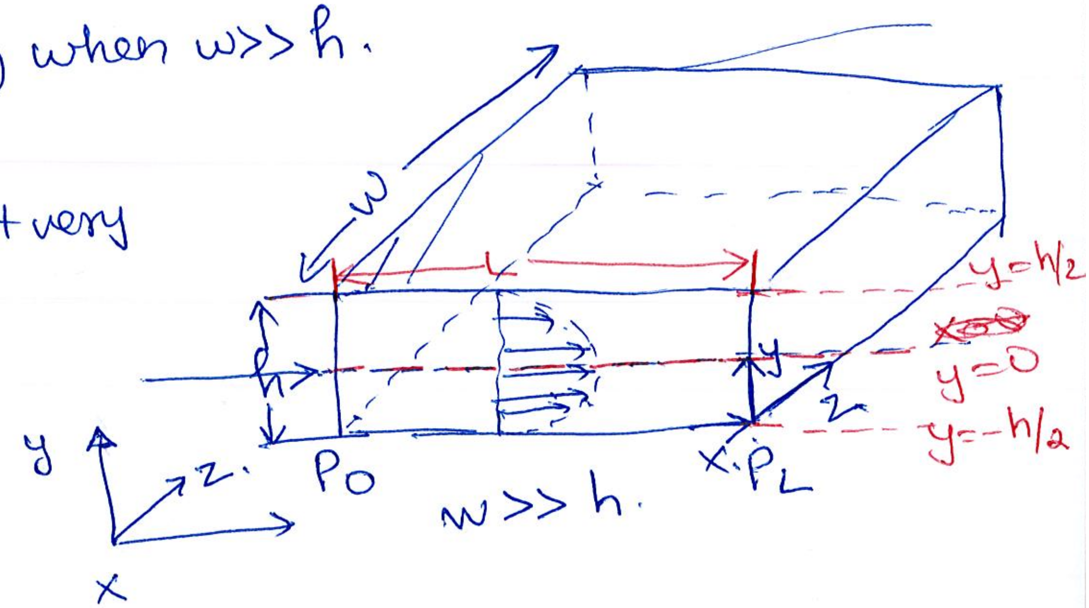
(1)

$v_x = v_x(y)$ when $w >> h$.

We assume w is not very large as h.

$v_x = v_x(y, z)$.

Assume steady state profile.



Start with NVS.

x-component in cartesian coordinate

$$0 = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) \rightarrow (1) \rightarrow$$ Verify that we will get this equation using NVS.

Boundary conditions

$y = \pm h/2$, $v_x = 0$. $v_x(\pm h/2, z) = 0$. → (2) → for all values of z.

$z = \pm w/2$, $v_x(y, \pm w/2) = 0$. ~~(no slip)~~. → (3) ↳ No cross flow at ends, flow is only within channel.

- From empirical observation, we will have parabolic velocity profile, with maxima at center.

$$\Rightarrow \frac{\partial v_x}{\partial y} = 0 \quad \text{at } y = 0. \quad - 4$$

$$\frac{\partial v_x}{\partial z} = 0 \quad \text{at } z = 0. \quad - 5$$

} symmetrical flow.

$$-\frac{\partial p}{\partial x} = \frac{\Delta P}{L}$$

$$0 = \frac{\Delta P}{L} + \mu\left(\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right). \quad (*) \quad (2)$$

$v_x(y)$ for $w >> h$.

$$v_x = \underbrace{v_x(y)}_{\text{velocity when } w>>h} + \underbrace{\phi(y,z)}_{\text{Deviation from velocity profile } v_x(y) \text{ due to } w>>h \text{ not true.}} \quad -(6)$$

Substitute velocity profile in original equation of motion.

(*)

$$0 = \frac{\Delta P}{L} + \mu\left[\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right] \quad -(7).$$

$$\boxed{\phi = \phi(y,z).}$$

For $w>>h$, $\phi = 0 \Rightarrow$ No effect of width.

$$\left.\begin{aligned} &\Rightarrow \quad 0 = \frac{\Delta P}{L} + \mu\frac{\partial^2 v_x}{\partial y^2} \quad (8) \\ &\& \quad \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad (9) \end{aligned}\right\} \text{For } w>>h.$$

Boundary conditions for modified equation. (10)

$y = \pm h/2$ $\quad v_x(y) = 0 \quad \phi = 0.$ (comes from B.C 2)

$y = 0$ $\quad \frac{\partial v_x}{\partial y} = 0. \Rightarrow \frac{dv_x}{dy} = 0 \quad \frac{\partial \phi}{\partial y} = 0 \quad -(11).$

$z = \pm w/2$ $\quad v_x = 0 \Rightarrow v_x(y) = -\phi$ (from B.C 3) (12)

$z = 0$ $\quad \frac{\partial v_x}{\partial z} = 0 \Rightarrow \frac{\partial \phi}{\partial z} = 0. \quad -(13).$

First Solve for $v_x(y)$ and the $\phi(y,z)$

$$0 = \frac{\Delta P}{L} + \mu \frac{\partial^2 v_x}{\partial y^2}$$

$$0 = \frac{\Delta P}{L} + \mu \frac{d^2 v_x}{dy^2}$$

$$\frac{dv_x}{dy} = -\frac{\Delta P}{\mu L} y + C_0$$

use B.C II, $\frac{dv_x}{dy} = 0$ at $y = 0$

$$\Rightarrow C_0 = 0$$

$$\frac{dv_x}{dy} = \frac{-\Delta P y}{\mu L}$$

$$v_x = \frac{-\Delta P y^2}{2\mu L} + C_1$$

Using B.C I

$v_x = 0,\ y = \pm h/2$

$$0 = \frac{-\Delta P}{2\mu L}\left(h/2\right)^2 + C_1$$

$$C_1 = \frac{\Delta P h^2}{8\mu L}$$

$$v_x = \Delta P\left[\frac{h^2}{8\mu L} - \frac{y^2}{2\mu L}\right]$$

$$\boxed{v_x = \frac{\Delta P}{2\mu L}\left[\frac{h^2}{4} - y^2\right]} \rightarrow \underline{v_x(y)}.$$

$(w >> h)$.

Now, we will solve for $\phi(y,z)$.

Use separation of variables.

$$\phi(y,z) = Y(y)\,Z(z)$$

$$\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0.$$

$$Z(z)\cdot\frac{\partial^2 Y}{\partial y^2} + Y(y)\cdot\frac{\partial^2 Z}{\partial z^2} = 0.$$

Divide by $Y(y)\,Z(z)$. $Z(z)$.

$$\frac{1}{Y(y)}\cdot\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z(z)}\cdot\frac{\partial^2 Z}{\partial z^2} = 0.$$

$$\frac{1}{Y(y)}\cdot\frac{\partial^2 Y}{\partial y^2} = \frac{-1}{Z(z)}\cdot\frac{\partial^2 Z}{\partial z^2} = \lambda^2 \quad -(20)$$

$$\boxed{\begin{aligned}\frac{1}{Y(y)}\cdot\frac{\partial^2 Y}{\partial y^2} &= \lambda^2 \quad -(21).\\ \frac{1}{Z(z)}\frac{\partial^2 Z}{\partial z^2} &= -\lambda^2.\end{aligned}}$$

$$\phi = Y(y)\,Z(z)$$

$$\frac{\partial\phi}{\partial y} = Z(z)\cdot\frac{\partial Y}{\partial y}.$$

B.C (11) → $y=0$ $\quad \frac{\partial\phi}{\partial y} = 0 \Rightarrow \frac{\partial Y}{\partial y} = 0$ — (A)

$$\phi = Y(y)\,Z(z).$$

$y = \pm h/2$ $\quad \phi = 0. \quad Y = 0.$ — (B)

$z = 0$ $\quad \frac{\partial\phi}{\partial z} = 0 \Rightarrow \frac{\partial Z}{\partial z} = 0$ — (C)

$Y(y) = A_1 \sin\lambda y + A_2 \cos\lambda y$ (from appendix C)

For 21

$Z(z) = B_1 \sinh\lambda z + B_2 \cosh\lambda z.$

Using (A)

$y=0, \quad \frac{\partial Y}{\partial y} = 0$

$\left.\frac{\partial Y}{\partial y}\right| = A_1 \lambda \cos\lambda y - A_2 \lambda \sin\lambda y = 0$

$\left.\frac{\partial Y}{\partial y}\right|_{y=0} = A_1 \lambda \cos\lambda - 0 = 0$

only if $A_1 = 0$.

$$\boxed{Y(y) = A_2 \cos\lambda y}.$$

Using (B)

at $y = \pm h/2, \quad Y(y) = 0$

$0 = A_2 \cos(\lambda(\pm h/2))$

$\Downarrow$

$\cos(\pm\lambda h/2) = 0.$

$\cos\left(\frac{\lambda h}{2}\right) = 0$ $\qquad \cos(-\theta) = \cos\theta$

$\sin(-\theta) = -\sin\theta.$

$\frac{\lambda h}{2} = (2n+1)\frac{\pi}{2} \qquad n = 0, 1, 2 \ldots$

$$\boxed{\lambda_n = \frac{(2n+1)\pi}{h}} \rightarrow \text{Eigen values.}$$

Using C

$z=0, \ \frac{\partial Z}{\partial z}=0$

$Z(z) = B_1 \sinh \lambda z + B_2 \cosh \lambda z.$

$\frac{\partial Z}{\partial z} = B_1 \lambda \cosh \lambda z + B_2 \lambda \sinh \lambda z.$

at $z=0$ $\frac{\partial Z}{\partial z}=0 \Rightarrow B_1 = 0$

$Z = B_2 (\cosh \lambda z).$

$\phi = \sum_{n=0}^{\infty} A_n \cos \lambda_n y \ B_n \cosh \lambda_n z$