**Application of Shell Momentum Balance**

**Thin Film of Falling Fluid**

Consider a thin film of falling fluid with thickness 𝛿 as shown on a reservoir. We will ignore the initial disturbance and are interested in velocity profile of fluid in the section with steady flow profile. Use shell momentum balance to

1. Find the velocity profile
2. Find the maximum velocity
3. Find the average velocity
4. Calculate mass flow rate
5. Force exerted by fluid on solid surface

𝛿 = thickness of fluid

Vz = function of x direction (or varies as depth of liquid)

Vx = 0, vy = 0 (no flow in x or y direction)

Width of the plate = W

Length of section for steady flow = L

Vz ≠ f(z) ==> 𝛿vz / 𝛿z = 0

Vy ≠ f(y) ==> 𝛿vy / 𝛿y = 0

Make assumptions based on geometry and physical situation

Here, width of plate >> 𝛿 (thickness of fluid)

Shell Momentum balance

1. What is shell to choose?
   1. Choose thin slice of where fluid is flowing
      1. Shell is in yz plane
   2. Thickness = dx (x = x --> x + dx)
2. Components that are not zero for this case:
   1. Φxz, Φyz, Φzz
   2. Φzz = flux of z momentum in z direction
   3. Φxz = flux of z momentum in x direction
   4. Φyz = flux of z momentum in y direction = 0 (no velocity in y direction (W >> 𝛿)
3. Write momentum balance equation for the components that are non-zero.
   1. Rate of z momentum in at z = 0
      1. (Φzz | z=0)WΔx
   2. Rate of z momentum out at z = L
      1. (Φzz | z=L)WΔx
   3. Rate of z momentum in at x = x
      1. (Φxz | x=x)WL
   4. Rate of z momentum out at x = x + Δx
      1. (Φxz | x=x+Δx)WL
4. Momentum Balance Equation:
   1. (Φxz | x=x)WL - (Φxz | x=x+Δx)WL + ~~(Φ~~~~zz | z=0~~~~)WΔx - (Φ~~~~zz | z=L~~~~)WΔx~~ + ρg cos(β) (LWΔx) = 0
   2. Φxz = 𝜏xz + ~~ρv~~~~x~~~~v~~~~z~~ ==> vx = 0
   3. P = P(x)
      1. P|z = 0 & P|z = L will be same.
      2. L is large, no pressure difference in steady state.
      3. 𝜏zz|z=0 = -μ [2 𝛿vz / 𝛿z]
         1. Vz is a function of x
         2. 𝛿vz / 𝛿z = 0
   4. Φzz | z=0 = P|z = 0 + ρvzvz|z = 0
   5. Φzz | z=L = P|z =L + ρvzvz|z = L
   6. Φzz | z=0 - Φzz | z=L = 0
5. Final Momentum balance equation
   1. (Φxz | x=x)WL - (Φxz | x=x+Δx)WL + ρg cos(β) (LWΔx) = 0
   2. 𝜏xz|x=x LW - 𝜏xz|x+Δx LW + ρg cos(β) (LWΔx) = 0
6. Now we can get a differential equation.
   1. Divide by LWΔx, take limit Δx-->0
   2. (𝜏xz|x=x - 𝜏xz|x+Δx)/Δx + ρg cos(β) = 0
   3. 𝛿𝜏xz/dx = ρg cos(β)
   4. 𝜏xz = ρg cos(β) x + c1
   5. At x = 0, i.e. liquid-gas interface: 𝜏xz = 0
      1. C1 = 0
   6. 𝜏xz = ρg cos(β) x
   7. -μ dvz/dx = ρg cos(β) x
   8. Vz = -ρg cos(β) x2/ 2μ + c2
   9. At x = 𝛿, vz = 0 (no slip condition: solid-liquid interface)
      1. C2 = ρg cos(β) x2/ 2μ
   10. Vz = ρg cos(β) (𝛿2 - x2)/ 2μ
   11. Vz = ρg cos(β) / 2μ \* (1 - x2/𝛿2)
7. Max velocity
   1. 𝛿vz/𝛿x = 0, 𝛿2vz/𝛿x2 < 0
8. Average velocity
   1. Volumetric flow rate / cross sectional area = <vz>
   2. <vz> = ∫0W∫0𝛿vzdxdy/∫0W∫0𝛿dxdy
9. Force
   1. 𝜏xz|x=𝛿 (area) = ∫0L∫0W𝜏xz|x=𝛿dydz