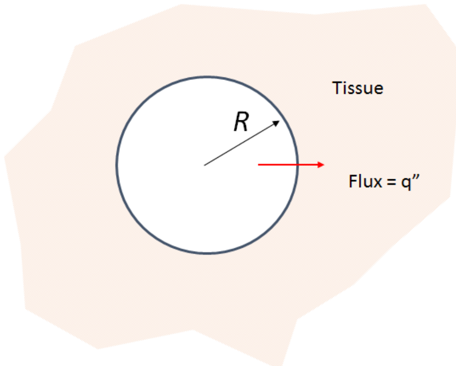
Homework 3 - Spring 2018

Question 1

(a)



(b) Estimating the heat transfer in the tissuue and a assuming
- no convection in the tissue
- no heat generation in the tissue
- steady state
- spherical geometry and only radial flow
- constant properties (density, heat capacity and thermal conductivity)

$$\frac{k}{\rho c}\frac{1}{r^2}\frac{d}{dr}(r^2\frac{dT}{dt})=0 \qquad (1)$$

$$T=37^oC \quad at \quad r\rightarrow\infty \qquad (1a)$$

$$-k\frac{dT}{dr}=q" \quad at \quad r=R \qquad (1b)$$

$$r^2\frac{dT}{dr}=C_1 \quad \Rightarrow \quad \frac{dT}{dr}=\frac{C_1}{r^2}$$

$$T(r)=-\frac{C_1}{r}+C_2$$

$$-k\frac{dT}{dr}\bigg|_{r=R}=-k\frac{C_1}{R^2}=q" \quad \Rightarrow \quad C_1=-\frac{q"R^2}{k}$$

$$T\left(r\rightarrow\infty\right)=37^oC=C_2$$

$$T(r)=\frac{q"R^2}{k}\frac{1}{r}+37$$

Integrating Eq. (1) once

$$r^2 \frac{dT}{dr} = C_1 \quad \Rightarrow \quad \frac{dT}{dr} = \frac{C_1}{r^2} \qquad (2)$$

Integrating Eq. (2) again

$$T(r) = -\frac{C_1}{r} + C_2 \qquad (3)$$

By using Eq. (2) and the boundary condition (1b)

$$-k\left.\frac{dT}{dr}\right|_{r=R} = -k\frac{C_1}{R^2} = q'' \quad \Rightarrow \quad C_1 = -\frac{q''R^2}{k} \qquad (4)$$

Substituting Eq.(3) and (1a) in Eq. (3) we can obtain:

$$T(r \to \infty) = 37^o C = C_2 \qquad (5)$$

Substituting Eqs (4) and (5) into Eq.(3) we can obtain

$$T(r) = \frac{q''R^2}{k}\frac{1}{r} + 37 \qquad (6)$$

From Eq.(6) the maximum temperature is obtained when r=R, which by replacing into Eq.(6) becomes:

$$T(r=R) = T_{max} = \frac{q''R^2}{k}\frac{1}{R} + 37 = 37 + \frac{q''R}{k} \qquad (7)$$

$q_{imp} := 1 \cdot W \qquad D_{imp} := 40 \cdot mm \qquad R_{imp} := \frac{D_{imp}}{2} \qquad k_{tissue} := 0.5 \cdot \frac{W}{m \cdot K}$

Replacing values in the above equation, q" will be called q_{imp_2p}

$q_{imp\_2p} := \frac{q_{imp}}{4\pi \cdot R_{imp}^2} \qquad T_{max} := 37 \cdot K + \frac{q_{imp\_2p} \cdot R_{imp}}{k_{tissue}}$

$\boxed{T_{max} = 45\,K}$

Question 2

See solution in a separate document, too many eqautions to include in MathCad

Question 3

Data $\qquad kJ := 1000 \cdot J$

$T_{air} := 250 \cdot K \qquad V_{bean} := 50mm^3 \qquad A_{bean} := 60 \cdot mm^2 \qquad R_{bean} := \frac{3 \cdot V_{bean}}{A_{bean}}$

$R_{bean} = 2.5\cdot mm$ $h_{air} := 15\cdot\frac{W}{m^2\cdot K}$ $T_{ini\_bean} := 25\cdot K$

$k_{bean} := 0.18\cdot\frac{W}{m\cdot K}$ $c_{bean} := 2.5\cdot\frac{kJ}{K\cdot kg}$ $\rho_{bean} := 600\cdot\frac{kg}{m^3}$ $T_{end\_bean} := 200\cdot K$

(a)

First let's find out if the lumped parameter model can be used

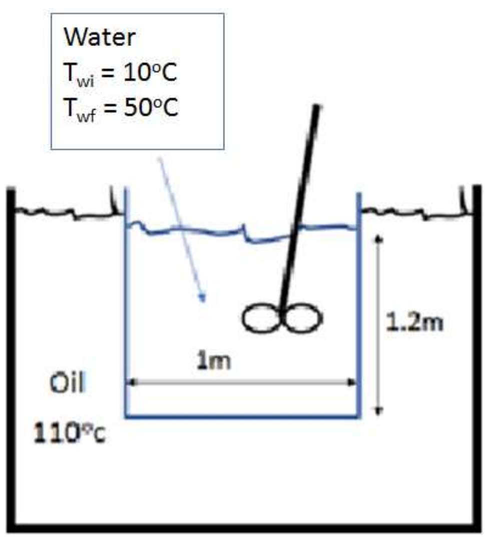
$$B_{i\_bean} := \frac{h_{air}\cdot R_{bean}}{k_{bean}} \qquad B_{i\_bean} = 0.208$$

A little bit larger than 0.1 so it could be fine to use the lump parameter. So we need to estimate the resistance to heat transfer, which is outside the bean, i.e. it is only the convection, so U = h_{air}

$$m_{bean} := \rho_{bean}\cdot V_{bean} \qquad m_{bean} = 0.03\cdot gm$$

$$t_{roast} := -\ln\left(\frac{T_{end\_bean} - T_{air}}{T_{ini\_bean} - T_{air}}\right)\cdot\frac{m_{bean}\cdot c_{bean}}{h_{air}\cdot A_{bean}} \qquad \boxed{t_{roast} = 125.34\,s}$$

Question 4



$$\frac{T(x,t)-T_i}{T_\infty - T_i} = 1 - erf\left[\frac{x}{2\sqrt{\alpha t}}\right] - \exp\left(\frac{h\,x}{k} + \frac{h^2\,\alpha\,t}{k^2}\right)\cdot\left(1 - erf\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right)$$

$T_{wi} := 10 \cdot K$ $\quad T_{wf} := 50 \cdot K$ $\quad T_{oil} := 110 \cdot K$ $\quad D_{cyl} := 1 \cdot m$ $\quad H_{cyl} := 1.2 \cdot m$

$A_{jacket} := 4.2 \cdot m^2$ $\quad U := 250 \cdot \frac{W}{m^2 \cdot K}$ $\quad V_{cyl} := \pi \cdot \left(\frac{D_{cyl}}{2}\right)^2 \cdot H_{cyl}$

$V_{cyl} = 0.942 \cdot m^3$

Since the tank is well agitated the temperature of the water is the same in the tank so we can use the lumped parameter model, from where we can estimate the time.

Let's assume that the density and thermal capacity of water as

$\rho_{water} := 1000 \cdot \frac{kg}{m^3}$ $\quad c_{water} := 4.18 \cdot \frac{kJ}{kg \cdot K}$

$m_{water} := \rho_{water} \cdot V_{cyl}$ $\quad m_{water} = 942.478\,kg$

$$t := \frac{-m_{water} \cdot c_{water}}{U \cdot A_{jacket}} \cdot \ln\left(\frac{T_{wf} - T_{oil}}{T_{wi} - T_{oil}}\right) \qquad \boxed{t = 31.943 \cdot min}$$

<u>Question 5</u>

(1) Value of x where T = -2C

$T_{soil\_ini} := 18 \cdot K$ $\quad T_{cold\_air} := -15 \cdot K$ $\quad h_{cold\_air} := 25 \cdot \frac{W}{m^2 K}$ $\quad t_{freez} := 4 \cdot hr$

$\alpha_{soil} := 6 \cdot 10^{-7} \cdot \frac{m^2}{s}$ $\quad k_{soil} := 0.65 \cdot \frac{W}{m \cdot K}$ $\quad T_{soil} := -2 \cdot K$

$$\theta := \frac{T_{soil} - T_{soil\_ini}}{T_{cold\_air} - T_{soil\_ini}} \qquad \theta = 0.606$$

$$T1 := \frac{1}{2\sqrt{\alpha_{soil} \cdot t_{freez}}} \qquad T1 = 5.379 \frac{1}{m}$$

$$T2 := \frac{h_{cold\_air}}{k_{soil}} \qquad T2 = 38.462 \frac{1}{m}$$

$$T3 := \frac{h_{cold\_air}^{2} \cdot \alpha_{soil} \cdot t_{freez}}{k_{soil}^{2}} \qquad T3 = 12.781$$

$$T4 := h_{cold\_air} \cdot \frac{\sqrt{\alpha_{soil} \cdot t_{freez}}}{k_{soil}} \qquad T4 = 3.575$$

$$f(x) := \theta - 1 + \mathrm{erf}(x \cdot T1) + \exp(T2 \cdot x + T3) \cdot (1 - \mathrm{erf}(T1 \cdot x + T4))$$

Guess --> $x := 0.1 \cdot m$

$$depth\_2 := root(f(x), x) \qquad \boxed{depth\_2 = 4.3 \cdot cm}$$

(2) The temperature at the surface

$$x_{surface} := 0$$

$$T_{surface} := T_{cold\_air} - (T_{cold\_air} - T_{soil\_ini}) \cdot [\mathrm{erf}(x \cdot T1) + \exp(T2 \cdot x + T3) \cdot (1 - \mathrm{erf}(T1 \cdot x + T4))]$$

$$\boxed{T_{surface} = 6.6 K}$$

(3) Flux

$$Flux := h_{cold\_air} \cdot (T_{surface} - T_{cold\_air}) \qquad \boxed{Flux = 538.8 \cdot \frac{W}{m^2}}$$

Question 6

We will assume an infinite slab heated from both sides. Data of the problem are given below

$$T_{meat\_ini} := 25 \cdot K \qquad Thickness := 2.5 \cdot cm \qquad L_{meat} := \frac{Thickness}{2} \qquad \rho_{meat} := 1280 \cdot \frac{kg}{m^3}$$

$$c_{meat} := 4184 \cdot \frac{J}{kg \cdot K} \qquad k_{meat} := 0.5 \cdot \frac{W}{m \cdot K} \qquad h_{oven} := 10 \cdot \frac{W}{m^2 \cdot K} \qquad T_{steril} := 121 \cdot K$$

We could use the chart but let's find out if rather using the charts we could use the approximate equations that are accurate for Fourier Number (Fo_{meat}) larger than 0.2. Since in this case we know the time (t_{ster}) the calculation of the Fourier number is straight forward

$$t_{ster} := 20 \cdot min \qquad \alpha_{meat} := \frac{k_{meat}}{\rho_{meat} \cdot c_{meat}} \qquad \alpha_{meat} = 9.34 \times 10^{-8} \frac{m^2}{s}$$

and

$$Fo_{meat} := \frac{\alpha_{meat} \cdot t_{ster}}{L_{meat}^{2}} \qquad Fo_{meat} = 0.717$$

so, I can use the approximate solution for the slab

The values of C_1 and γ_1 are function of the Biot number, which has to be calculated

$$Bi_{meat} := \frac{h_{oven} \cdot L_{meat}}{k_{meat}} \qquad Bi_{meat} = 0.25$$

It is important to recognize that the hotter temperature will be the surface of the meat, which cannot reach a temperature higher than 155C. We can use 155K because we are dealing with temperature differences. And we prefer to us 155K because it is allowing us to use the units of MathCad without much problems. We also know that the coldest temperature is the center of the meat, which has to be 121C (or 121K) at least 21 minutes. So we need to estimate the Temperature in the oven to teach those conditions

From Tables for Bimeat = 0.25 C_1 = 1.0382 and γ_1 = 0.4802, please note that MathCad units for radis are the default so nothing is included

$$C_1 := 1.0382 \qquad \gamma_1 := 0.4802$$

let's calculate first the time for the center of the meat to reach a temperature of 121C if the temperature of the oven is 250C

Iteration 1 $\quad T_{oven\_1} := 250 \cdot K$

$$x_c := 0 \qquad n_c := \frac{x_c}{L_{meat}} \qquad x_s := L_{meat} \qquad n_s := \frac{x_s}{L_{meat}}$$

$$\theta_{steril\_1\_c} := \frac{T_{steril} - T_{oven\_1}}{T_{meat\_ini} - T_{oven\_1}} \qquad \theta_{steril\_1\_c} = 0.573$$

$$Fo_{meat\_1} := \frac{-1}{\gamma_1^{\,2}} \cdot \ln\left(\frac{\theta_{steril\_1\_c}}{C_1 \cdot \cos(\gamma_1 \cdot n_c)}\right) \quad \text{--->} \quad t_{st\_1} := \frac{Fo_{meat\_1} \cdot L_{meat}^{\,2}}{\alpha_{meat}} \qquad t_{st\_1} = 71.8 \cdot min$$

Now for that time we can calculate the temperature in the surface of the meat.

$$\theta_{steril\_1\_s} := C_1 \cdot \exp\left(-\gamma_1^{\,2} \cdot Fo_{meat\_1}\right) \cos(\gamma_1 \cdot n_s) \qquad \theta_{steril\_1\_s} = 0.508$$

$$T_{s\_1} := T_{oven\_1} - \theta_{steril\_1\_s} \cdot \left(T_{oven\_1} - T_{meat\_ini}\right) \qquad T_{s\_1} = 135.6\,K \qquad \text{(Or Celcius)}$$

Now the time need to be 20min more with a temperature at at least 121C or more, so new time t_{new_1} = 71.8+20 min

$$t_{hold} := 20 \cdot min \qquad t_{new\_1} := t_{st\_1} + t_{hold} \qquad t_{new\_1} = 91.8 \cdot min$$

$$Fo_{meat\_new\_1} := \frac{\alpha_{meat} \cdot t_{new\_1}}{L_{meat}^{2}} \qquad Fo_{meat\_new\_1} = 3.292$$

$$\theta_{steril\_new\_1\_c} := C_1 \cdot \exp\left(-\gamma_1^{2} \cdot Fo_{meat\_new\_1}\right) \cos\left(\gamma_1 \cdot n_c\right) \qquad \theta_{steril\_new\_1\_c} = 0.486$$

$$T_{c\_new\_1} := T_{oven\_1} - \theta_{steril\_new\_1\_c} \cdot \left(T_{oven\_1} - T_{meat\_ini}\right)$$

$T_{c\_new\_1} = 140.7\,K$ Which is fine, but new let'c check what is the temperature in the surface of the meat

$$\theta_{steril\_new\_1\_s} := C_1 \cdot \exp\left(-\gamma_1^{2} \cdot Fo_{meat\_new\_1}\right) \cos\left(\gamma_1 \cdot n_s\right) \qquad \theta_{steril\_new\_1\_s} = 0.431$$

and the temperature in the surface of the meat is:

$$T_{s1} := T_{oven\_1} - \theta_{steril\_new\_1\_s} \cdot \left(T_{oven\_1} - T_{meat\_ini}\right) \qquad T_{s1} = 153.025\,K$$

Which is OK, so an oven temperature of 250C will be fine. Let's repeat the calculation for an oven temperature of 350C to see what happens

Iteration 2 $T_{oven\_2} := 350 \cdot K$

$$\theta_{steril\_2\_c} := \frac{T_{steril} - T_{oven\_2}}{T_{meat\_ini} - T_{oven\_2}} \qquad \theta_{steril\_2\_c} = 0.705$$

$$Fo_{meat\_2} := \frac{-1}{\gamma_1^{2}} \cdot \ln\left(\frac{\theta_{steril\_2\_c}}{C_1 \cdot \cos\left(\gamma_1 \cdot n_c\right)}\right) \quad \text{--->} \quad t_{st\_2} := \frac{Fo_{meat\_2} \cdot L_{meat}^{2}}{\alpha_{meat}} \qquad t_{st\_2} = 46.9 \cdot min$$

Now for that time we can calculate the temperature in the surface of the meat.

$$\theta_{steril\_2\_s} := C_1 \cdot \exp\left(-\gamma_1^{2} \cdot Fo_{meat\_2}\right) \cos\left(\gamma_1 \cdot n_s\right) \qquad \theta_{steril\_2\_s} = 0.625$$

$$T_{s\_2} := T_{oven\_2} - \theta_{steril\_2\_s} \cdot \left(T_{oven\_2} - T_{meat\_ini}\right) \qquad T_{s\_2} = 146.9\,K \qquad (\text{Or Celcius})$$

Now the time need to be 20min more with a temperature at at least 121C or more, so new time t_{new_2} = 46.9+20 min

$$t_{new\_2} := t_{st\_2} + t_{hold} \qquad t_{new\_2} = 66.9 \cdot min$$

$$Fo_{meat\_new\_2} := \frac{\alpha_{meat} \cdot t_{new\_2}}{L_{meat}^{2}} \qquad Fo_{meat\_new\_2} = 2.398$$

$$\theta_{steril\_new\_2\_c} := C_1 \cdot \exp\left(-\gamma_1^{2} \cdot Fo_{meat\_new\_2}\right) \cos\left(\gamma_1 \cdot n_c\right) \qquad \theta_{steril\_new\_2\_c} = 0.597$$

$$T_{c\_new\_2} := T_{oven\_2} - \theta_{steril\_new\_2\_c} \cdot \left(T_{oven\_2} - T_{meat\_ini}\right)$$

$T_{c\_new\_2} = 155.9\,K$

It is a little over, but new let'c check what is the temperature in the surface of the meat

$$\theta_{steril\_new\_2\_s} := C_1 \cdot \exp\left(-\gamma_1^{\,2} \cdot Fo_{meat\_new\_2}\right) \cos\left(\gamma_1 \cdot n_s\right) \qquad \theta_{steril\_new\_2\_s} = 0.53$$

and the temperature in the surface of the meat is:

$$T_{s2} := T_{oven\_2} - \theta_{steril\_new\_2\_s} \cdot \left(T_{oven\_2} - T_{meat\_ini}\right) \qquad T_{s2} = 177.9\,K$$

Which is over the limit of 155K, so an oven temperature of 350C will not work.