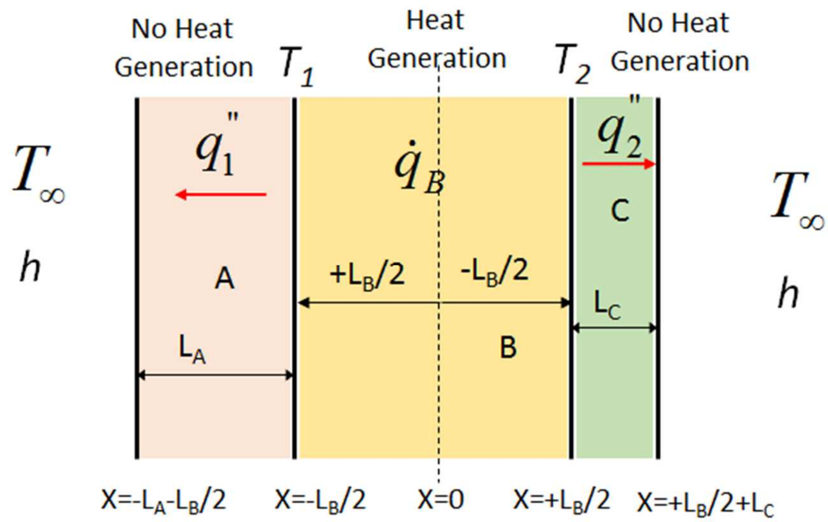
Homework 3 – Solution

Problem 2



L_A = 30mm	k_A = 25 W/m.K	h = 800 W/m^2.K
L_B = 60mm	k_B = 50 W/m.K	$T_\infty = 25^o C$
L_C = 20mm	k_C = ?	T_1 = 250^{o}C
		T_2 = 210^{o}C

Assume

- Steady State
- One 1D heat transfer
- Negligible contact resistances

Let's solve **<u>first the layer B</u>** where heat generation occurs. The relevant equations arisen from the energy transfer model and the assumptions simplify to:

$$\begin{cases} \dfrac{d^2T_B}{dx^2} = -\dfrac{Q_B}{k_B} & & (1) \\ at\ x = -L_B/2 & T_b(x = -L_B/2) = T_1 & (1a) \\ at\ x = +L_B/2 & T_b(x = +L_B/2) = T_2 & (1b) \end{cases}$$

This is the model that has to be solved. By integrating Eq. (1) twice the solution is:

$$T_B(x) = -\frac{Q_B}{2k_B}x^2 + C_1x + C_2 \qquad (2)$$

It is important consider that the units of Q_B (not given in the problem) should be in W/m^3 to get consistent units. For instance in Eq.(2) we have the following term $\frac{Q_B}{2k_B}x^2$, which should have units of temperature. Let's check the units: $\frac{Q_B}{2k_B}\cdot x^2 = \frac{\frac{W}{m^3}}{\frac{W}{m.K}}\cdot m^2 = K$. So to be consistent with the equations given in the Equations summary let's assume that $Q_B = \dot{q}_B$

Eq. (2) provides the temperature profile in the layer B. In order to have the complete description of that temperature profile we need to estimate the value of the integrations constants C_1 and C_2 that can be obtained using the boundary conditions (1a) and (1b). This is done below:

$$T_B(x=-\frac{L_B}{2}) = T_1 = -\frac{\dot{q}_B}{2k_B}\frac{L^2_B}{4} - C_1\frac{L_B}{2} + C_2 \quad (3)$$

$$T_B(x=+\frac{L_B}{2}) = T_2 = -\frac{\dot{q}_B}{2k_B}\frac{L^2_B}{4} + C_1\frac{L_B}{2} + C_2 \quad (4)$$

By combining Eqs. (3) and (4) expressions for C_1 and C_2 can be obtained, which are given below:

$$C_1 = -\frac{T_1 - T_2}{L_B} \quad \textit{and} \quad C_2 = \left(\frac{T_1 + T_2}{2}\right) + \frac{\dot{q}_B}{2k_B}\frac{L_B^2}{4} \quad (5)$$

Substituting Eq. (5) into Eq. (2) the temperature profile for the layer B is obtained:

$$\begin{aligned} T_B(x) &= -\frac{\dot{q}_B}{2k_B}x^2 - \frac{T_1 - T_2}{L_B}x + \frac{T_1 + T_2}{2} + \frac{\dot{q}_B}{2k_B}\frac{L_B^2}{4} \\ T_B(x) &= \frac{\dot{q}_B}{2k_B}\left(\frac{L_B^2}{4} - x^2\right) - \frac{T_1 - T_2}{L_B}x + \frac{T_1 + T_2}{2} \end{aligned} \quad (6)$$

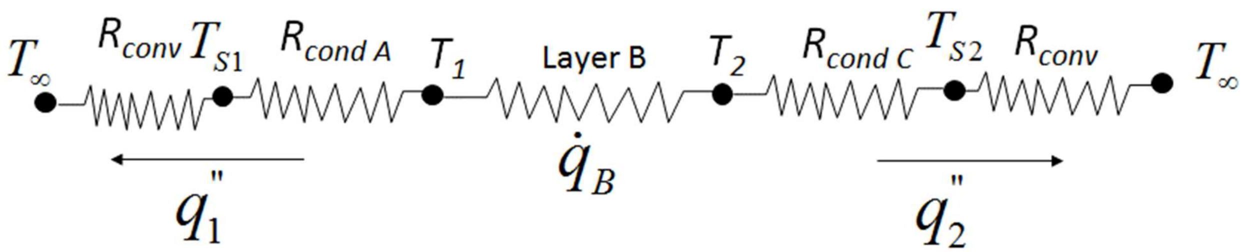
Eq. (6) can be used to describe the temperature profile in the layer B provided the generation of energy $\dot{q}_B$ is known. In fact, that has to be calculated in the first question of this problem. To calculate it, we need to use the assumptions initially done when we start to solve the problem. An important assumption in this problem **<u>is the steady state</u>**, which is not only useful to simplify the microscopic balance of energy but also to estimate $\dot{q}_B$. Under **<u>steady state conditions</u>** all the heat generated in the layer B has to be dissipated through layers A and C, otherwise the condition would be violated and the material will start to either heat or cool down over time somewhere in the composite wall. Thus, a condition arising from the steady state assumption is:

$$\dot{q}_B = q_1^{"} + q_2^{"} \quad (7)$$

It is important to check the units involved in Eq. (7), $q_1^{"}$ *and* $q_2^{"}$ are heat flux with units of W/m^2, whereas as demonstrated above $\dot{q}_B$ has units of W/m^3. Thus, if multiply $\dot{q}_B$ by L_B, that product will have units of W/m^2, and the correct equation to estimate the value of $\dot{q}_B$ will be:

$$\dot{q}_B \cdot L_B = q_1^{"} + q_2^{"} \tag{8}$$

Thus, if we calculate $q_1^{"}$ *and* $q_2^{"}$ it would be possible to estimate $\dot{q}_B$. In order to calculate these heat fluxes we can again use the **<u>assumption of steady state</u>** and use the concept of **<u>thermal resistances</u>** and <u>their electrical analogy</u> to estimate these fluxes. For instance, the flux going to the left, i.e. $q_1^{"}$, has a resistance due to conduction in the layer A and convection, whereas the heat flux going to the right, i.e. $q_2^{"}$, has a resistance associated to the layer C and convection. Schematically we can have the analog "electrical circuit" depicted below:



From the figure, we can calculate the heat fluxes through the layer A and the layer C

<u>Layer A</u>

$$\boxed{q_1^{"} = \frac{T_1 - T_\infty}{R_{cond\ A} + R_{conv}} = \frac{T_1 - T_\infty}{\frac{L_A}{k_A} + \frac{1}{h}} = \frac{(250-25)K}{\frac{30x10^{-3}m}{25\frac{W}{m\cdot K}} + \frac{1}{800\frac{W}{m^2.K}}} = 91.8\frac{kW}{m^2}}$$

<u>Layer C</u>

$$\boxed{q_2^{"} = \frac{T_2 - T_\infty}{R_{cond\_C} + R_{conv}} = \frac{T_2 - T_\infty}{\frac{L_c}{k_c} + \frac{1}{h}} = \frac{(210-25)K}{\frac{20x10^{-3}m}{50\frac{W}{m\cdot K}} + \frac{1}{800\frac{W}{m^2.K}}} = 112.1\frac{kW}{m^2}}$$

So from Eq.(8)

$$\dot{q}_B = \frac{q_1^{"} + q_2^{"}}{L_B} = \frac{91.8 + 112.1}{60x10^{-3}m}\ kW/m^2 = 3{,}398\frac{kW}{m^3} \approx 3.4\frac{MW}{m^3}$$

$$\boxed{\dot{q}_B = 3.4\frac{MW}{m^3} = 3.4x10^3\frac{kW}{m^3}}$$

We need to determine now the value of k_B, but for that, we will need to estimate either the temperature profile in the layer *C* or the layer *A.*

Temperature Profile in Layer C

In layer C there is no heat generation so the Microscopic Balance of Energy with the assumptions used becomes:

$$\begin{cases} \dfrac{d^2T_C}{dx^2}=0 & (9) \\ at\ x=+L_B/2 \quad T_C(x=L_B/2)=T_2 & (9a) \\ at\ x=+L_B/2 \quad q_2^{"}=-k_C\dfrac{dT_C}{dx} & (9b) \end{cases}$$

By integrating twice Eq. (9) the following is obtained:

$$T_C(x)=B_1x+B_2 \qquad (10)$$

Where B_1 and B_2 are integration constants that can be determined from the boundary conditions (9a) and (9b), so by using Eq. (9b)

$$-k_C\left.\frac{dT_C}{dx}\right|_{x=+\frac{L_B}{2}}=-k_CB_1=q_2^{"} \quad \Rightarrow \quad B_1=-\frac{q_2^{"}}{k_C} \qquad (11)$$

And by using Eq. (9a) the following is obtained:

$$T_C\big|_{x=+\frac{L_B}{2}}=T_2=B_1\frac{L_B}{2}+B_2 \quad \Rightarrow \quad B_2=T_2-B_1\frac{L_2}{2}=T_2+\frac{q_2^{"}}{k_C}\frac{L_B}{2} \qquad (12)$$

By substituting Eqs (11) and (12) into Eq. (10), the temperature profile for the layer C can be obtained as:

$$T_C(x)=-\frac{q_2^{"}}{k_C}x+T_2+\frac{q_2^{"}}{k_C}\frac{L_B}{2}=T_2+\frac{q_2^{"}}{k_C}\left(\frac{L_B}{2}-x\right) \qquad for \qquad x\geq\frac{L_B}{2} \qquad (13)$$

Temperature Profile in Layer A

In layer C there is no heat generation so the Microscopic Balance of Energy with the assumptions used becomes:

$$\begin{cases} \dfrac{d^2T_A}{dx^2}=0 & (14) \\ at\ x=-L_B/2 \quad T_A(x=-L_B/2)=T_1 & (14a) \\ at\ x=-L_B/2 \quad k_A\dfrac{dT_A}{dx}=q_1^{"} & (14b) \end{cases}$$

By integrating twice Eq. (9) the following is obtained:

$$T_A(x)=D_1x+D_2 \qquad (15)$$

Where D_1 and D_2 are integration constants that can be determined from the boundary conditions (14a) and (14b), so by using Eq. (14b)

$$k_A\left.\frac{dT_A}{dx}\right|_{x=-\frac{L_B}{2}}=k_AD_1=q_1^{"} \quad \Rightarrow \quad D_1=\frac{q_1^{"}}{k_A} \qquad (16)$$

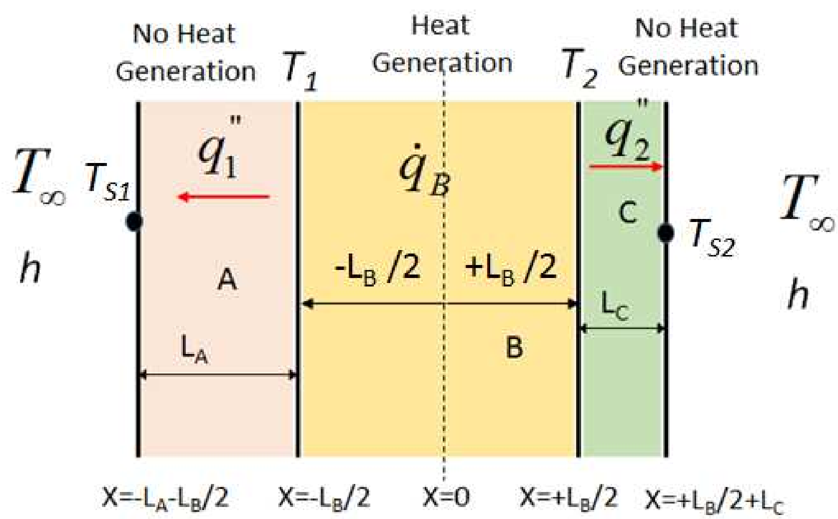
And by using Eq. (9a) the following is obtained:

$$T_A\Big|_{x=-\frac{L_B}{2}} = T_1 = D_1\frac{-L_B}{2} + D_2 \quad \Rightarrow \quad D_2 = T_1 + D_1\frac{L_B}{2} = T_1 + \frac{q_1^{"}}{k_A}\frac{L_B}{2} \qquad (17)$$

By substituting Eqs (16) and (17) into Eq. (15), the temperature profile for the layer A can be obtained as:

$$T_A(x) = \frac{q_1^{"}}{k_A}x + T_1 + \frac{q_1^{"}}{k_A}\frac{L_B}{2} = T_1 + \frac{q_1^{"}}{k_A}\left(\frac{L_B}{2} + x\right) \qquad for \quad x \le \frac{-L_B}{2} \quad (18)$$

The temperatures T_{S1} and T_{S2}, indicated in the figure below, can be calculated as:



$$q_1^{"} = \frac{T_{S1} - T_\infty}{\frac{1}{h}} \quad \Rightarrow \quad T_{S1} = T_\infty + \frac{q_1^{"}}{h} = 25 + \frac{91.8kW/m^2}{0.8\frac{kW}{m^2K}} = 139.8^oC$$

And

$$q_2^{"} = \frac{T_{S2} - T_\infty}{\frac{1}{h}} \quad \Rightarrow \quad T_{S2} = T_\infty + \frac{q_2^{"}}{h} = 25 + \frac{112.1kW/m^2}{0.8\frac{kW}{m^2K}} = 165.1^oC$$

To determine k_B we can use the steady state approximation in the interface B-C, i.e. in that interface the flux does not change. Mathematically, this can be expressed as following:

$$at \;\; x = +\frac{L_B}{2} \qquad -k_B\frac{dT_B(x)}{dx} = -k_C\frac{dT_C(x)}{dx} \qquad (19)$$

Or we can use a similar condition at the interface A-C.

$$at \;\; x = -\frac{L_B}{2} \qquad -k_B\frac{dT_B(x)}{dx} = -k_A\frac{dT_A(x)}{dx} \qquad (20)$$

Regardless the condition we choose, the solution will be finally the same, here let us use Eq. (19), which combined with the temperature profiles for T_B and T_C (Eqs.6 and 13) yields:

$$k_B \frac{dT_B(x)}{dx}\bigg|_{x=+\frac{L_B}{2}} = -k_B\left(\frac{\dot{q}_B}{k_B}\frac{L_B}{2} + \frac{T_1 - T_2}{L_B}\right) \tag{21}$$

And

$$k_C \frac{dT_C(x)}{dx}\bigg|_{x=+\frac{L_B}{2}} = -q_2^{''} \tag{22}$$

By substituting Eqs. (21) and (22) in Eq. (19) we obtain:

$$-k_B\left(\frac{\dot{q}_B}{k_B}\frac{L_B}{2} + \frac{T_1 - T_2}{L_B}\right) = -q_2^{''} \quad \Rightarrow \quad q_2^{''} = \dot{q}_B\frac{L_B}{2} + k_B\frac{T_1 - T_2}{L_B} \tag{23}$$

From where k_B can be determined:

$$k_B = \frac{L_B}{T_1 - T_2}\left(q_2^{''} - \frac{\dot{q}_B L_B}{2}\right) = \frac{60x10^{-3}}{250-210}\left(112.1 - \frac{204}{2}\right) = 0.015\frac{kW}{m.K}$$

Or

$$\boxed{k_B = 0.015\frac{kW}{m.K} = 15.0\frac{W}{m.K}}$$

Summarizing and substituting values the temperature profiles in layers A, B and C are:

Layer A

$$T_A(x) = T_1 + \frac{q_1^{''}}{k_A}\left(\frac{L_B}{2} + x\right) = 250 + \frac{91.8x10^3 W/m^2}{25 W/m.K}\left(\frac{60x10^{-3}}{2} + x\right)$$

$$\boxed{T_A(x) = T_1 + \frac{q_1^{''}}{k_A}\left(\frac{L_B}{2} + x\right) = 360.2 + 3672\,x \qquad for \quad x \le -\frac{L_B}{2}} \tag{24}$$

Layer B

$$T_B(x) = \frac{\dot{q}_B}{2k_B}\left(\frac{L_B^2}{4} - x^2\right) - \frac{T_1 - T_2}{L_B}x + \frac{T_1 + T_2}{2} = \frac{3.4x10^6 W/m^2}{2x15 W/m.K}\left(\frac{\left(60x10^{-3}\right)^2}{4} - x^2\right) - \frac{250-210}{60x10^{-3}}x + \frac{250+210}{2}$$

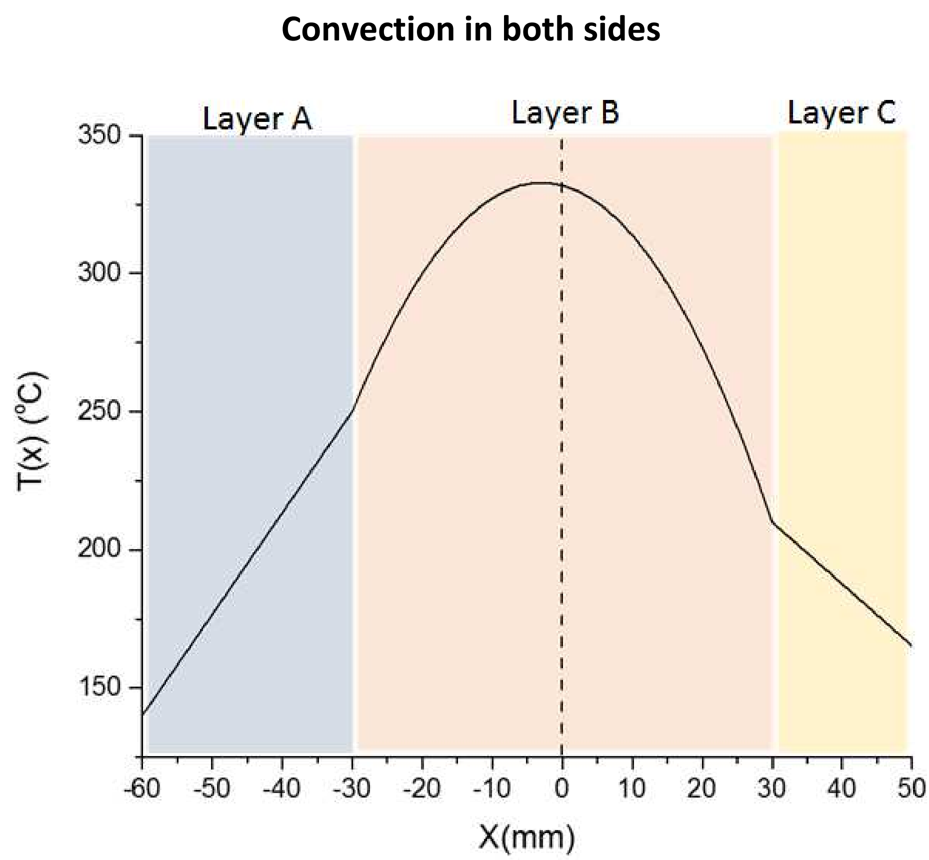
$$\boxed{T_B(x) = -113,333.3x^2 - 666.7x + 332 \qquad -\frac{L_B}{2} \le x \le \frac{L_B}{2}}$$

Layer C

$$T_C(x) = T_2 + \frac{q_2^{"}}{k_C}\left(\frac{L_B}{2} - x\right) = 210 + \frac{112.1x10^3 W/m^2}{50W/m.K}\left(\frac{60x10^{-3}}{2} - x\right) \quad for \quad \frac{L_B}{2} \le x \le \frac{L_B}{2} + L_C$$

$$\boxed{T_C(x) = 277.3 - 2242x \quad for \quad \frac{L_B}{2} \le x \le \frac{L_B}{2} + L_C}$$

A plot of the temperature profiles in each layer is illustrated in the next page



(c) If the coolant is lost on the layer A, the value of h = 0, and the profile of temperature in the layer A is described by the same model than before (i.e. the same differential equation) but different boundary conditions, which are given below

$$\begin{cases} \dfrac{d^2T_A}{dx^2} = 0 & (25) \\ at\ x = -\dfrac{L_B}{2} - L_A \qquad k_A \dfrac{dT_A(x)}{dx} = q_1^{"} = 0 & (25a) \\ at\ x = -L_B/2 \qquad T_A(x = -\dfrac{L_B}{2}) = T_1 & (25b) \end{cases}$$

The solution is similar than the one obtained before, i.e.

$$T_A(x) = D_1 x + D_2 \tag{26}$$

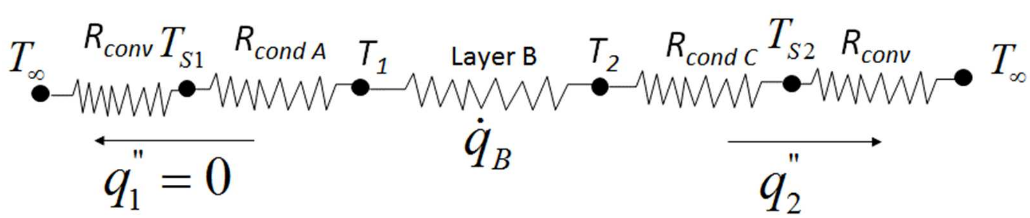
From Eq. (25a) $D_1 = 0$, and from Eq. (25b) $T_A = T_1$

$$T_A = T_1 \tag{27}$$

The generation of heat in the layer B remains the same but $q_1^{"} = 0$ so from Eq. (8):

$$\dot{q}_B \cdot L_B = q_2^{"} = 204 \frac{kW}{m^2} \tag{28}$$

We can now use the electrical analog diagram



Thus, the heat flux $q_2^{"}$ is calculated as:

$$q_2^{"} = 204 \frac{W}{m^2} = \frac{T_2 - T_\infty}{R_{condC} + R_{conv}} = \frac{T_2 - T_\infty}{\dfrac{L_C}{k_C} + \dfrac{1}{h}} \quad \Rightarrow \quad T_2 = T_\infty + q_2^{"} \left(\frac{L_C}{k_C} + \frac{1}{h} \right) \tag{29}$$

Substituting values in Eq. (29):

$$T_2 = 25 + 204 \frac{kW}{m^2} \left(\frac{20x10^{-3}\, m}{50x10^{-3}\ kW/m.K} + \frac{1}{0.8\, kW/m^2.K} \right) = 361.6^o C$$

Since $T_A = T_1$ =constant, we can use the boundary condition at $x = -L_B/2$ that there is not net flux, mathematically can be written as:

$$at\ x = -\frac{L_B}{2} \qquad k_B \frac{dT_B}{dx}\bigg|_{x=-\frac{L_B}{2}} = \frac{\dot{q}_B x}{k_B}\bigg|_{x=-\frac{L_B}{2}} - \frac{T_1 - T_2}{L_B} = 0 \tag{30}$$

From Eq. (30), we can obtain:

$$T_1 = \ T_2 + \frac{\dot{q}_B L_B^2}{2k_B} = 361.6 + \frac{3.4x10^3\,kW\,/\,m^3\left(60x10^{-3}\right)^2 m^2}{2\,x\,0.015\,kW\,/\,m.K} = 769.6^oC$$

In summary the Equations for the temperature profile in each layer are:

Layer A

$$\boxed{T_A = T_1}$$

Layer B

$$T_B(x) = \frac{\dot{q}_B}{2k_B}\left(\frac{L_B^2}{4} - x^2\right) - \frac{T_1 - T_2}{L_B}x + \frac{T_1 + T_2}{2} = \frac{3.4x10^6 W\,/\,m^3}{2x15W\,/\,m.K}\left(\frac{\left(60x10^{-3}\right)^2}{4} - x^2\right) - \frac{769.6 - 361.6}{60x10^{-3}}x + \frac{769.6 + 361.6}{2}$$

$$T_B(x) = -113,333.3x^2 - 6800x + 667.6$$

Layer C

$$T_C(x) = T_2 + \frac{q_2^{"}}{k_C}\left(\frac{L_B}{2} - x\right) \ = 361.6 + \frac{204\,x10^3 W\,/\,m^2}{50W\,/\,m.K}\ \left(\frac{60x10^{-3}m}{2} - x\right)$$

$$\boxed{T_C(x) = 484 - 4080x}$$

The following plot illustrates the temperature profile in the system