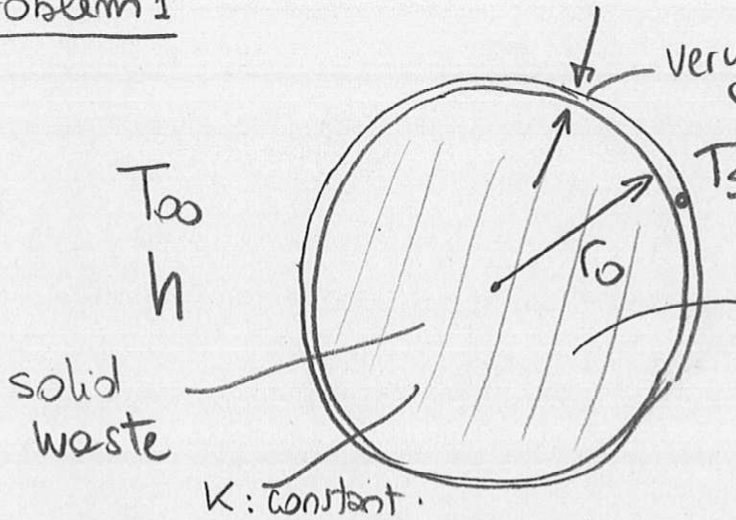
ABE 308 EXAMPLE PROBLEMS EXAM 1

SOLUTIONS

Problem 1



$Q = Q_0\left(1-\left(\frac{r}{r_0}\right)^2\right)$

Q_0: constant

r_0: radius of the container

Let's assume steady state, no convection, i.e. only radial conduction through the solid waste and heat generation

For a sphere the governing equation is

$$K\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+Q=0$$

Boundary conditions are symmetry at the sphere center, and convection outside, i.e at $r=r_0$ (assuming a very thin wall container. Thus, the mathematical model is:

$$\begin{cases}\dfrac{1}{r^2}\dfrac{d}{dr}\left(r^2\dfrac{dT}{dr}\right)=-\dfrac{Q_0}{K}\left[1-\left(\dfrac{r}{r_0}\right)^2\right] & (1)\\[2ex] \dfrac{dT}{dr}=0 \text{ at } r=0 & (1a)\ [\text{symmetry condition}]\\[2ex] -K\dfrac{dT}{dr}=h(T_s-T_\infty) \text{ at } r=r_0 & (1b)\end{cases}$$

Condition (1b) assumes that the container wall is very thin, i.e $r_{container}\cong r_0$

By integrating Eq. (1) once

$$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = -\frac{Q_0}{K}\left(r^2 - \frac{r^4}{r_0^2}\right)$$

$$r^2\frac{dT}{dr} = -\frac{Q_0}{K}\left[\frac{r^3}{3} - \frac{r^5}{5r_0^2}\right] + C_1$$

and

$$\frac{dT}{dr} = -\frac{Q_0}{K}\left(\frac{r}{3} - \frac{r^3}{5r_0^2}\right) + \frac{C_1}{r^2} \qquad (2)$$

using Eq. (1a)

$$\left.\frac{dT}{dr}\right|_{r=0} = 0 = -\frac{Q_0}{K}\left(\frac{0}{3} - \frac{0}{5r_0^2}\right) + \frac{C_1}{0^2}$$

To avoid $\frac{dT}{dr} \rightarrow \infty \Longrightarrow C_1 = 0$

so

$$\frac{dT}{dr} = -\frac{Q_0}{K}\left(\frac{r}{3} - \frac{r^3}{5r_0^2}\right) \qquad (3)$$

By integrating Eq. (3) once

$$T(r) = -\frac{Q_0}{K}\left[\frac{r^2}{6} - \frac{r^4}{20r_0^2}\right] + C_2 \qquad (4)$$

To get C_2 we can use boundery condition given by Eq. (1b) by using Eq.(3)

$$\left.\frac{dT}{dr}\right|_{r=r_0} = -\frac{Q_0}{K}\left(\frac{r_0}{3}-\frac{r_0}{5}\right) \quad (5)$$

substituting Eq.(5) into Eq.(1b) we obtain

$$-K\cdot\left[-\frac{Q_0}{K}\,\frac{5r_0-3r_0}{15}\right] = h\left(T_5 - T_\infty\right)$$

or

$$\frac{2Q_0r_0}{15} = h\left(T_5 - T_\infty\right) \quad (6)$$

$$\text{but } T_s = T(r=r_0) = -\frac{Q_0}{K}\left(\frac{r_0^2}{6}-\frac{r_0^2}{20}\right) + C_2$$

$$T_s = -\frac{Q_0}{K}\left(\frac{10r_0^2-3r_0^2}{60}\right) + C_2 = -\frac{Q_0}{K}\,\frac{7r_0^2}{60} + C_2 \quad (7)$$

substituting Eq.(7) into Eq.(6)

$$\frac{2Q_0r_0}{15} = h\left[-\frac{Q_0}{60K}7r_0^2 + C_2 - T_\infty\right] \quad (8)$$

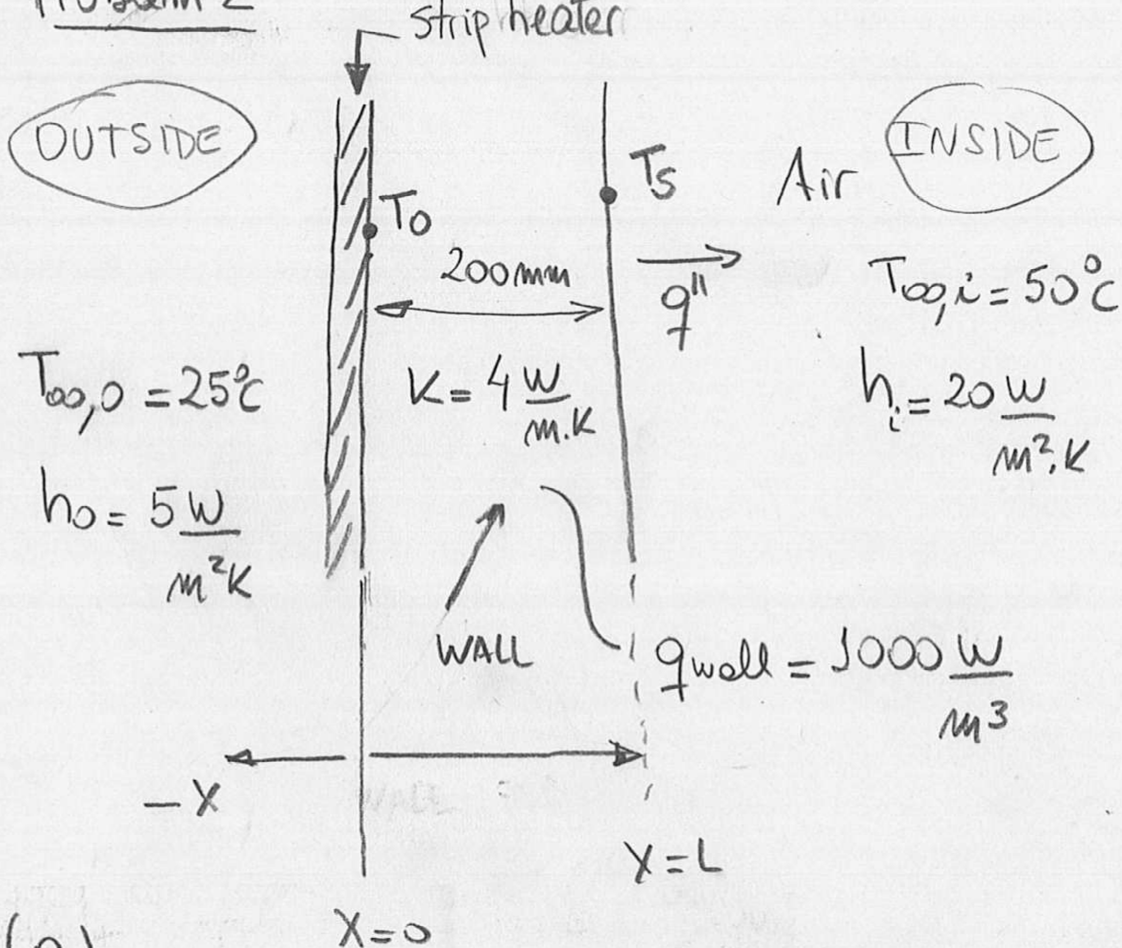
C_2 can be determined from Eq.(8)

$$C_2 = \frac{2Q_0r_0}{15h} + \frac{7Q_0r_0^2}{60K} + T_\infty \quad (9)$$

substituting Eq.(9) into Eq.(4)

$$\boxed{T(r) = T_\infty + \frac{2Q_0r_0}{15h} + \frac{Q_0r_0^2}{K}\left[\frac{7}{60} - \frac{1}{6}\left(\frac{r}{r_0}\right)^2 + \frac{1}{20}\left(\frac{r}{r_0}\right)^4\right]}$$

Problem 2



(a)

Since it could be difficult to sketch the temperature profile $T(X)$ versus X, it is better to get the temperature profile.

Let's assume steady state, constant thermal conductivity, heat flow in a direction (x), no convection in the wall (only convection at the boundary condition $X = L$).

The balance of energy is:

$$\frac{d^2T}{dx^2} = -\frac{\dot{q}_{well}}{K} \quad (1)$$

$$-k\frac{dT}{dX} = 0 \quad \text{at} \quad X = 0 \quad (1a) \quad [\text{NO HEAT LOST OUTSIDE}]$$

$$-k\frac{dT}{dX} = h_i(T_s - T_{\infty,i}) \quad \text{at} \quad X = L \quad (1b)$$

By integrating Eq. (1)

$$\frac{dT}{dx} = -\frac{\dot{q}_{wall}}{k}x + C_1 \qquad (2)$$

by using boundary condition (1a) we found

$$\left.\frac{dT}{dx}\right|_{x=0} = 0 = -\frac{\dot{q}_{wall}}{k}.0 + C_1 \Longrightarrow C_1 = 0$$

$$\frac{dT}{dx} = -\frac{\dot{q}_{wall}}{k}x \qquad (3)$$

By integrating Eq. (3)

$$T(x) = -\frac{\dot{q}_{wall}x^2}{2k} + C_2 \qquad (4)$$

To determine C_2 we can use boundary condition (1b)

$$-\cancel{k}\left[-\frac{\dot{q}_{wall}}{\cancel{k}}L\right] = h_i(T_s - T_{\infty,i}) \qquad (5)$$

but $T_s = T(x=L) = -\frac{\dot{q}_{wall}L^2}{2k} + C_2 \qquad (6)$

substituting Eq. (6) into Eq. (5)

$$\frac{\dot{q}_{wall}L}{h_i} = -\frac{\dot{q}_{wall}L^2}{2k} + C_2 - T_{\infty,i}$$

$$C_2 = \frac{\dot{q}_{wall}L}{h_i} + \frac{\dot{q}_{wall}L^2}{2k} + T_{\infty,i} \qquad (7)$$

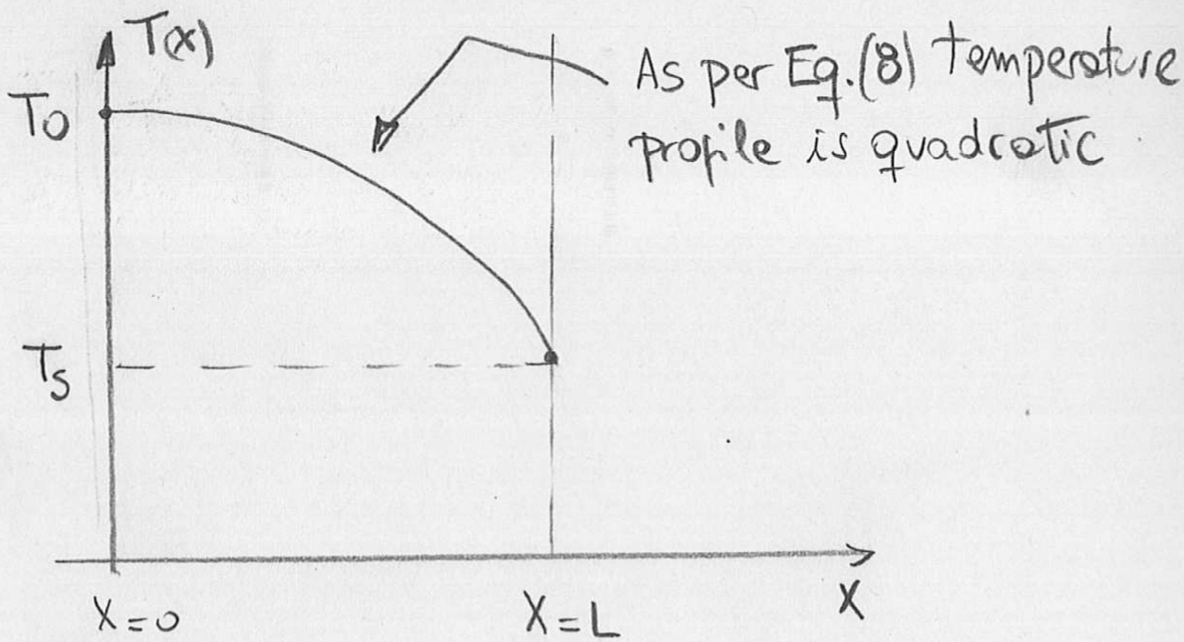
substituting Eq. (7) into Eq. (4)

$$T(x) = -\frac{\dot{q}_{wall}\, x^2}{2K} + \frac{\dot{q}_{wall}\, L}{h_i} + \frac{\dot{q}_{wall}\, L^2}{2K} + T_{\infty,i}$$

or

$$\boxed{T(x) = T_{\infty,i} + \dot{q}_{wall}\, L\left(\frac{1}{h_i} + \frac{L}{2K}\right) - \frac{\dot{q}_{wall}}{2K}\, x^2} \quad (8)$$

A T(x) versus x schematic diagram is:



(b) From Eq. (8)

$$T(x=L) = T_s = T_{\infty,i} + \frac{\dot{q}_{wall}\, L}{h_i} + \cancel{\frac{\dot{q}_{wall}\, L^2}{2K}} - \cancel{\frac{\dot{q}_{wall}\, L^2}{2K}}$$

$$T_s = 50^\circ C + \frac{1000 \times 200 \times 10^{-3}\ W/m^2}{20\ \frac{W}{m^2 \cdot {}^\circ C}} = 60^\circ C$$

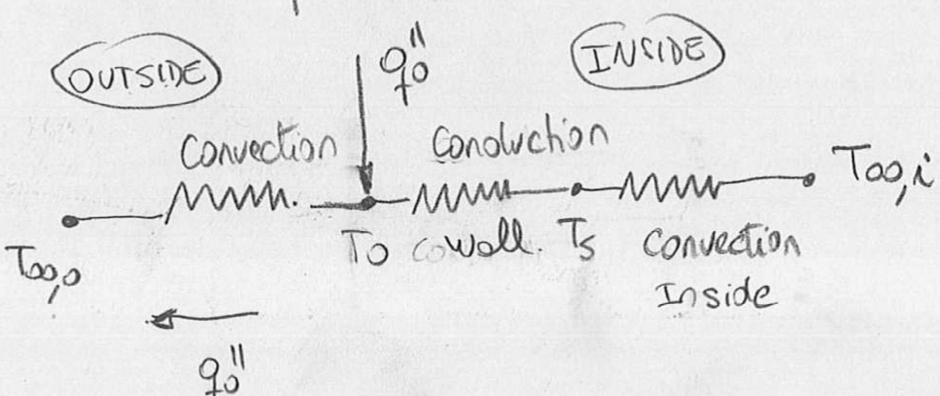
and From Eq. (8)

$$T_o = T(x=0) = T_{\infty,i} + \dot{q}_{wall} L \left[\frac{1}{h_i} + \frac{L}{2K} \right]$$

$$T_o = 50^\circ C + 1000 \times 200 \times 10^{-3} \frac{W}{m^2} \left[\frac{1}{20 \frac{W}{m^2 \cdot {}^\circ C}} + \frac{200 \times 10^{-3} m}{2 \times 4 \frac{W}{m^2 {}^\circ C}} \right]$$

$$\boxed{T_o = 65^\circ C}$$

(C) we can draw an equivalent circuit and determine q_o'' the heat flux to be entered in the strip heater



If all heat applied by the heater strip is lost by convection outside the inside of the chamber will keep the heat generated in the wall. so

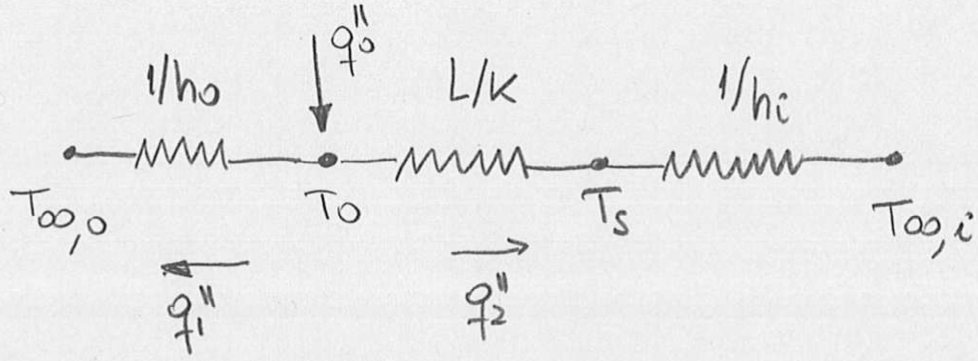
$$q_o'' = h_o \left[T_o - T_{\infty,o} \right]$$

$$q_o'' = 5 \frac{W}{m^2 {}^\circ C} \left[65 - 25 \right] {}^\circ C = 200 \frac{W}{m^2}$$

$$\boxed{q_o'' = 200 \frac{W}{m^2}}$$

(d) If the heat generation in the wall is switched off what would the steady temperature T_o $[T(x=0)]$ (8)

the equivalent electrical circuit is



$$q_o'' = q_1'' + q_2'' \qquad (9)$$

$$q_2'' = \frac{T_o - T_{\infty,i}}{\sum Res} = \frac{T_o - T_{\infty,i}}{\frac{L}{K} + \frac{1}{h_i}} \qquad (10)$$

$$q_1'' = \frac{T_o - T_{\infty,o}}{\sum Res} = \frac{T_o - T_{\infty,o}}{\frac{1}{h_o}} \qquad (11)$$

Since the stripper remain constant $q_o'' = 200 \frac{W}{m^2}$

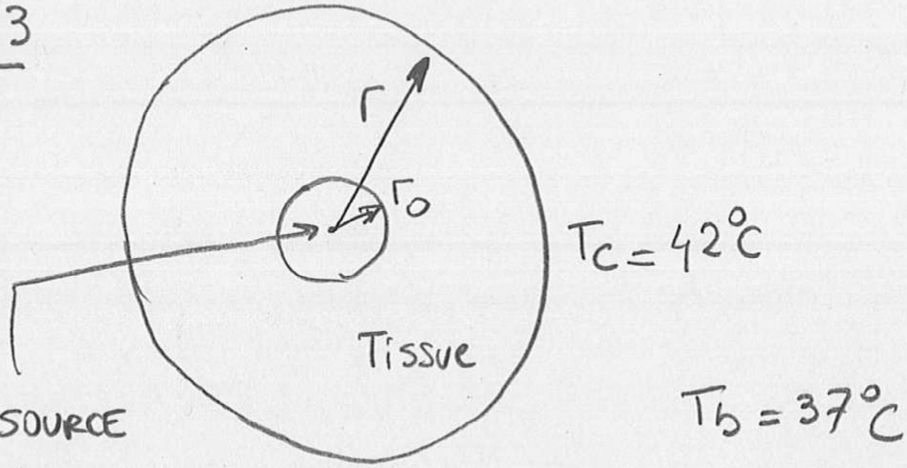
substituting values into Eq.(9)

$$200 \frac{W}{m^2} = \frac{T_o - 25}{\frac{1}{5}} + \frac{T_o - 50}{\frac{200\times10^{-3}}{4} + \frac{1}{20}}$$

$$200 = 5T_o - 125 + 10T_o - 500$$

$$15T_o = 200 + 125 + 500 = 825 \implies \underline{T_o = 55°C}$$

Problem 3



Let's consider spherical coordinates, radial flow, constant properties and steady state. It is important to note that there is no heat generation in the tissue so the conduction is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0 \qquad (1)$$

however the heat flow can be considered as a boundary condition

$$\text{at } r = r_0 \qquad q_0 = -KA\frac{dT}{dr} \qquad (1a)$$

$$\text{at } r \to \infty \qquad T = T_b \qquad (1b)$$

Integration of Eq.(1) yields

$$r^2\frac{dT}{dr} = C_1$$

$$\frac{dT}{dr} = \frac{C_1}{r^2} \qquad (2)$$

and by integrating again

$$T(r) = -\frac{C_1}{r} + C_2 \quad (3)$$

by using the boundary condition (1a)

$$-K \underbrace{4\pi \cancel{r_0^2}}_{A} \times \frac{C_1}{\cancel{r_0^2}} = q_0$$

$$C_1 = -\frac{q_0}{4\pi K} \quad (4)$$

substituting Eq. (4) into Eq. (3)

$$T(r) = \frac{q_0}{4\pi K}\frac{1}{r} + C_2 \quad (5)$$

Using Eq. (1b)

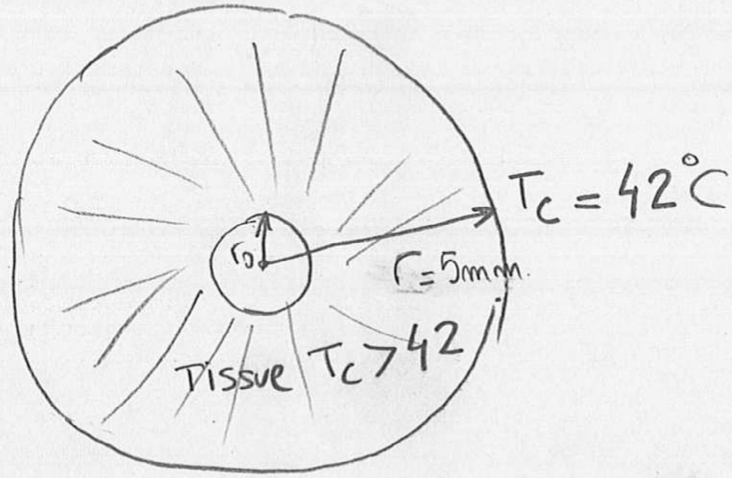
$$T(r \to \infty) = T_b = C_2 \quad (6)$$

substituting Eq. (6) into Eq. (5)

$$\boxed{T(r) = T_b + \frac{q_0}{4\pi K}\frac{1}{r}} \quad (7)$$

To calculate the flow q_0 we can obtain from Eq. (7)

$$q_0 = \left(T(r) - T_b\right) \times 4\pi K r \quad (8)$$



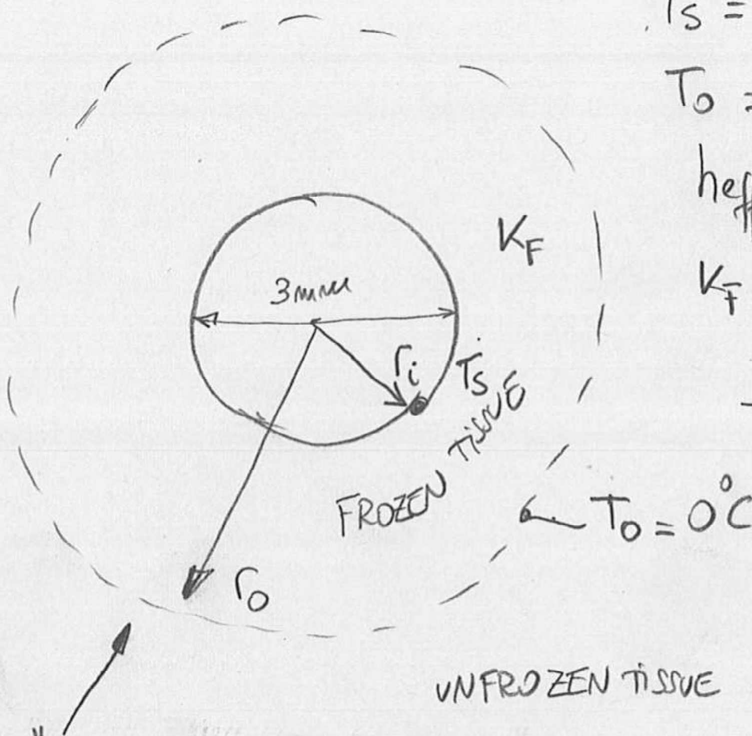
$$T(r = 5\,mm) = T_c = 42^\circ C$$

substituting values into Eq. (8)

$$q_0 = (42-37)^\circ \cancel{C} \times 4\pi \times 0.5 \frac{W}{\cancel{m}^\circ \cancel{C}} \times 5 \times 10^{-3}\, \cancel{m} = 0.157\, W$$

$$\boxed{q_0 = 0.157\, W}$$

Problem 4



$T_s = -30°C$

$T_0 = 0°C$

$h_{eff} = 50 \frac{W}{m^2K}$

$K_F = 1.5 \frac{W}{m.K}$

$T_t = 37°C$

$q''_r = h_{eff}(T_t - T_0)$

$$q''_r = \frac{q_r}{4\pi r_0^2} = h_{eff}(T_t - T_0) = 50 \frac{W}{m^2K}(37-0)K = 1850 \frac{W}{m^2}$$

$$q''_r = 1850 \frac{W}{m^2}$$

Let's consider now the heat transfer between any point in the unfrozen layer and the surface of the probe at $T = T_s$

$$q_r = \frac{T_t - T_s}{\Sigma Res} = \frac{T_t - T_s}{\frac{1}{4\pi r_0^2 h_{eff}} + \frac{r_0 - r_i}{4\pi K_F r_0 r_i}} \quad (1)$$

or

$$\frac{q}{4\pi r_0^2} = \frac{T_t - T_s}{\frac{1}{h_{eff}} + \frac{r_0 - r_i}{K_F}\frac{r_0}{r_i}} = q''_r$$

$$\frac{T_t - T_s}{q''_r} = \frac{1}{h_{eff}} + \frac{r_o - r_i}{k_F}\frac{r_o}{r_i}$$

$$\frac{r_o^2}{r_i k_F} - \frac{r_o}{k_F} + \frac{1}{h_{eff}} - \frac{T_t - T_s}{q''_r} = 0$$

$$\frac{r_o^2}{1.5\times10^{-3}\times 1.5} - \frac{r_o}{1.5} + \frac{1}{50} - \frac{37-(-30)}{1850}$$

$$444.4\, r_o^2 - 0.667\, r_o - 0.0162 = 0$$

quadratic equation to find r_o

$$r_o = \frac{0.667 \pm \sqrt{0.667^2 + 4\times 444.4\times 0.0162}}{2\times 444.4}$$

Let's consider the positive root

$$r_o = 0.00684\,m = 6.84\,mm$$

Thickness $= r_o - r_i = 6.84 - 1.50 = 5.34\,mm$

Thickness of the Frozen tissue $= 5.34\,mm$