List of Equations ABE 30800 – Spring 2018

Energy Balance

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_{gen} = \dot{E}_{st}$$

General Energy Transport Equation

$$\frac{\partial T}{\partial t} + \bar{u} \cdot \nabla T = \frac{k}{\rho c} \nabla^2 T + \frac{\dot{q}}{\rho c}$$

Heat Conduction (no convection)

- Cartesian Coordinates

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c}\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{\dot{q}}{\rho c}$$

- 1D – Heat Flow (x direction)

$$q_x = -kA\frac{dT(x)}{dx}$$

- 1D – Heat Flux (x direction)

$$q''_x = \frac{q_x}{A} = -k\frac{dT(x)}{dx}$$

- Cylindrical Geometry

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{\dot{q}}{\rho c}$$

- 1D – Heat Flow (r-direction – long cylinder)

$$q_r = -kA(r)\frac{dT(r)}{dr}$$

- 1D – Heat Flux (r-direction – long cylinder)

$$q''_r = \frac{q_r}{A(r)} = -k\frac{dT(r)}{dr}$$

- Spherical Geometry

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \varphi^2} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial T}{\partial \theta})\right] + \frac{\dot{q}}{\rho c}$$

- 1D – Heat Flow (r-direction)

$$q_r = -kA(r)\frac{dT(r)}{dr}$$

- 1D – Heat Flux (r-direction)

$$q_r'' = \frac{q_r}{A(r)} = -k\frac{dT(r)}{dr}$$

Heat Convection

- Heat Flow

$$q = hA(T_s - T_\infty)$$

- Heat Flux

$$q'' = \frac{q}{A} = h(T_s - T_\infty)$$

Radiation

$$\frac{q}{A} = q'' = \sigma T^4$$

$$\sigma = 5.676\ x10^{-8}\ \frac{W}{m^2 K^4}$$

$$\sigma = 0.174\ x10^{-8}\ \frac{BTU}{hr.ft^4.R^4}$$

Heat Transfer in a composite system (including conduction and convection)

- Convection in both sides and conduction through 3 layers – Cartesian Geometry – 1D

$$q_x = \frac{(T_h - T_c)}{\frac{1}{h_h A} + \frac{L_1}{k_1 A} + \frac{L_2}{k_2 A} + \frac{L_3}{k_3 A} + \frac{1}{h_c A}}$$

$$q_x = UA(T_h - T_c)$$

$$\frac{1}{U} = \frac{1}{h_h} + \frac{L_1}{k_1} + \frac{L_2}{k_2} + \frac{L_3}{k_3} + \frac{1}{h_c}$$

- Convection in both sides and conduction through 3 layers – Cylindrical Geometry – 1D

$$q_r = \frac{T_h - T_c}{\frac{1}{2\pi L r_i h_h} + \frac{ln(r_1 / r_i)}{2\pi k_1 L} + \frac{ln(r_2 / r_1)}{2\pi k_2 L} + \frac{ln(r_0 / r_2)}{2\pi k_3 L} + \frac{1}{2\pi L r_o h_c}}$$

$$q_r = UA(T_h - T_c)$$

$$U_i A_i = U_o A_o = U_1 A_1 = U_2 A_2$$

$$\frac{1}{U_i} = \frac{1}{h_h} + \frac{r_i}{k_1}\frac{ln(r_1 / r_i)}{} + \frac{r_i}{k_2}\frac{ln(r_2 / r_1)}{} + \frac{r_i}{k_3}\frac{ln(r_0 / r_2)}{} + \frac{r_i}{r_o}\frac{1}{h_c}$$

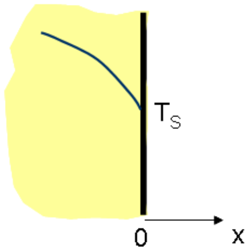
- Convection in both sides and conduction through 3 layers – Spherical Geometry – 1D

$$q_r = \frac{T_h - T_c}{\frac{1}{4\pi r_i^2 h_h} + \frac{r_1 - r_i}{4\pi k_1 r_1 r_i} + \frac{r_o - r_1}{4\pi k_2 r_o r_1} + \frac{1}{4\pi r_o^2 h_c}}$$

$$U_i A_i = U_o A_o = U_1 A_1 = U_2 A_2$$

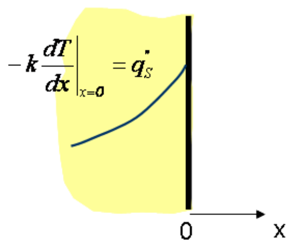
- Boundary Conditions

1. Surface Temperature is Specified



$$T\big|_{x=0} = T_s$$

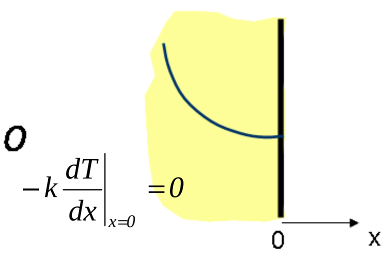
2. Surface Heat flow is specified



$$-k\frac{dT}{dx}\Big|_{x=0} = q_S''$$

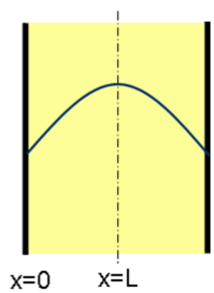
2a. Insulated Condition

$$-k\frac{dT}{dx}\Big|_{x=0} = 0$$



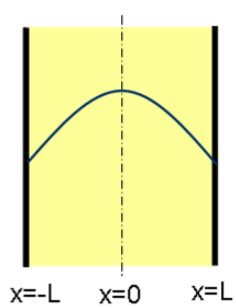
2b. Symmetry Condition

$$-k\frac{dT}{dx}\Big|_{x=L} = 0$$



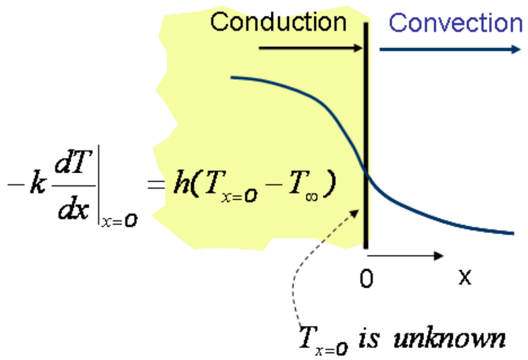
or

$$-k\frac{dT}{dx}\Big|_{x=0} = 0$$



This is the Boundary condition we have used so far, with x = 0 at the center of the infinite slab

2c. Convection at the surface



Steady State Heat Transfer with Heat Generation

- Cartesian Coordinate – Heat flow in only one direction (this is one of the multiple solution you can find out)

$$\frac{d^2T}{\partial x^2} = -\frac{\dot{q}}{k}$$
$$T = T_1 \quad at \quad x = -L$$
$$T = T_1 \quad at \quad x = L$$
$$\frac{dT}{dx} = 0 \quad at \quad x = 0$$

$$T = T_1 + \frac{\dot{q}L^2}{2k}\cdot\left(1 - \frac{x^2}{L^2}\right)$$

$$T_{\max} = T_1 + \frac{\dot{q}L^2}{2k}$$

- Cylindrical and Spherical Geometry

Equations are specific to the boundary conditions used (see examples given in class)

Steady State Heat Transfer for Extended Surfaces

$$T - T_{\infty} = (T_b - T_{\infty})\cdot e^{-mx} \qquad m^2 = \frac{hP}{kA_c}$$

$$\frac{q_{fin}}{q_{no-fin}} = \sqrt{\frac{k}{h}\cdot\frac{P}{A_c}}$$

Conduction Heat Transfer: Unsteady State

Lumped parameter Analysis

$$\frac{T - T_{\infty}}{T_i - T_{\infty}} = \exp\left(-\frac{UA}{mc}t\right)$$

$Bi = \frac{hL}{k}$ and $Bi > 0.1$, it can be "relaxed up to $Bi > 0.25$

<u>When the Lumped Analysis cannot be used, i.e. internal thermal resistance to heat transfer due to conduction is not negligible use Charts or Approximate Solutions</u>

Using Charts- x in slab is replaced by r and L by R in cylindrical and spherical geometries

$$Y=\frac{T(x,t)-T_\infty}{T_i-T_\infty}$$

$$F_o=\frac{\alpha t}{L^2}=\frac{k}{\rho c L^2}\cdot t \qquad m=\frac{1}{Bi}=\frac{k}{hL} \qquad n=\frac{x}{L}$$

L: characteristic length, it depends on the geometry

Multi-dimensional Problems

$$\frac{T(x,y,z,t)-T_\infty}{T_i-T_\infty}=\left(\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right)_{\text{infinite slab}}\cdot\left(\frac{T(y,t)-T_\infty}{T_i-T_\infty}\right)_{\text{infinite slab}}\cdot\left(\frac{T(z,t)-T_\infty}{T_i-T_\infty}\right)_{\text{infinite slab}}$$

$$\frac{T(r,z,t)-T_\infty}{T_i-T_\infty}=\left(\frac{T(r,t)-T_\infty}{T_i-T_\infty}\right)_{\text{infinite cylinder}}\cdot\left(\frac{T(z,t)-T_\infty}{T_i-T_\infty}\right)_{\text{infinite slab}}$$

Analytical Solution (complete) - Slab only

$$Y=\frac{T(x,t)-T_\infty}{T_i-T_\infty}=\sum_{n=0}^{\infty}\frac{4(-1)^n}{(2n+1)\pi}\cos\frac{(2n+1)\pi x}{2L}\cdot e^{-\alpha\left(\frac{(2n+1)\pi}{2L}\right)^2 t}$$

Analytical Solution (approximated for long times) - Slab only

$$Y=\frac{T(x,t)-T_\infty}{T_i-T_\infty}=\frac{4}{\pi}\cdot\cos\frac{\pi x}{2L}\cdot e^{-\alpha\left(\frac{\pi}{2L}\right)^2 t}$$

$$\ln\frac{T_{av}-T_\infty}{T_i-T_\infty}=\ln\frac{8}{\pi^2}-\alpha\left(\frac{\pi}{2L}\right)^2 t$$

Transient heat Transfer in a Semi-infinite region

<u>Constant temperature boundary condition</u>

$$\frac{T(x,t)-T_i}{T_s-T_i}=1-erf\left[\frac{x}{2\sqrt{\alpha t}}\right]$$

C<u>onvection boundary condition</u>

$$\frac{T(x,t)-T_i}{T_\infty-T_i}=1-erf\left[\frac{x}{2\sqrt{\alpha t}}\right]-\exp\left(\frac{hx}{k}+\frac{h^2\alpha t}{k^2}\right)\cdot\left(1-erf\left(\frac{x}{2\sqrt{\alpha t}}+\frac{h\sqrt{\alpha t}}{k}\right)\right)$$

<u>Approximated Solution to the Unsteady State Equation (also used for Mass Transfer) for Three Different Geometries without Using Charts.</u>

These equations can be used for Unsteady State Heat Transfer. In the case of heat transfer problems, concentrations are replaced by Temperatures.

For relative long times, i.e. for $F_o > 0.2$ the infinite series solution can be approximated by the first term of the series. During lectures, we discussed that equation for slab geometries which is:

$$\frac{T(x,t)-T_\infty}{T_i - T_\infty} = \frac{4}{\pi} \cdot \cos\frac{\pi x}{2L} \cdot e^{-\alpha \cdot \left(\frac{\pi}{2L}\right)^2 t}$$

However, no equations were given to estimate the temperatures with position and time for **<u>cylindrical and spherical surfaces</u>**. The approach to estimate these temperatures (or concentrations in the second part of this class) was to use specific charts for these geometries. The use of charts sometimes may result cumbersome so a more general approach is used to estimate these temperatures (and concentrations) for Fo > 0.2. <u>For Fo < 0.2 the complete solution (infinite series) should be used.</u>

<u>The general approach</u> is summarized in the following equations, e.g. applied to heat transfer:

- For Slab

$$\frac{T(x,t)-T_\infty}{T_i - T_\infty} = C_1 \cdot e^{-\gamma_1^2 Fo} \cdot \cos(\gamma_1 \cdot \frac{x}{L})$$

$$and \ \ for \ x \to 0 \qquad \frac{T(x,t)-T_\infty}{T_i - T_\infty} = C_1 \cdot e^{-\gamma_1^2 Fo}$$

- For Cylinder

$$\frac{T(r,t)-T_\infty}{T_i - T_\infty} = C_1 \cdot e^{-\gamma_1^2 Fo} \cdot J_o\left(\gamma_1 \frac{r}{R}\right)$$

$$and \ \ for \ r \to 0 \qquad \frac{T(r,t)-T_\infty}{T_i - T_\infty} = C_1 \cdot e^{-\gamma_1^2 Fo}$$

- For Sphere

$$\frac{T(r,t)-T_\infty}{T_i - T_\infty} = C_1 \cdot e^{-\gamma_1^2 Fo} \cdot \frac{\sin(\gamma_1 \frac{r}{R})}{\gamma_1 \frac{r}{R}}$$

$$and \ \ for \ r \to 0 \qquad \frac{T(r,t)-T_\infty}{T_i - T_\infty} = C_1 \cdot e^{-\gamma_1^2 Fo}$$

Where C_1 and γ_1 are two parameters that depend on the geometry and are tabulated in the table given in the table below as a function of the Biot number. J_0 is the zero Bessel function of the first kind:

	Plane wall		**Infinite Cylinder**		**Sphere**	
Bi	$\gamma_1(rad)$	C_1	$\gamma_1(rad)$	C_1	$\gamma_1(rad)$	C_1
0.01	0.098	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.03	0.1732	1.0049	0.2439	1.0075	0.2989	1.0090
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.05	0.2217	1.0082	0.3142	1.0124	0.3582	1.0149
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.07	0.2615	1.0114	0.3708	1.0173	0.4550	1.0209
0.08	0.2791	1.01130	0.3960	1.0197	0.4860	1.0239
0.09	0.2956	1.0145	0.4195	1.0222	0.5150	1.0268
0.10	0.3111	1.0160	0.4417	1.0246	0.5423	1.0298
0.15	0.3779	1.0327	0.5376	1.0365	0.6608	1.0445
0.20	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.25	0.4801	1.0382	0.6856	1.0598	0.8448	1.0737
0.30	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.40	0.5932	1.0580	0.8516	1.0932	1.0528	1.1164
0.50	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.60	0.7051	1.0814	1.0185	1.1346	1.2644	1.1713
0.70	0.7506	1.0919	1.0873	1.1539	1.3525	1.1978
0.80	0.7910	1.1016	1.1490	1.1725	1.4320	1.2236
0.90	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2	1.0769	1.1795	1.5995	1.3384	2.0288	1.4793
3	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4	1.2646	1.2287	1.9081	1.4698	2.4556	1.7201
5	1.3138	1.2402	1.9898	1.5029	2.5704	1.7870
6	1.3494	1.2479	2.0490	1.5253	2.6537	1.8338
7	1.3766	1.2532	2.0937	1.5411	2.7165	1.8674
8	1.3978	1.2570	2.1286	1.5526	2.7654	1.8921
9	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20	1.4961	1.2699	2.2881	1.5919	2.9857	1.9781

	Plane wall		Infinite Cylinder		Sphere	
Bi	$\gamma_1(rad)$	C_1	$\gamma_1(rad)$	C_1	$\gamma_1(rad)$	C_1
30	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5707	1.2733	2.4050	1.6018	3.1415	2.000

$Bi = \frac{hL}{k}$, L is the characteristic length and the Biot number could be the heat or the mass Biot number, x is used for a slab and r for cylindrical and spherical geometries.

The Bessel Functions of the First Kind arise from the solution of the partial differential equation describing the unsteady state heat conduction model in cylindrical coordinates. The subject has been probably covered in MA 30300. If you need to know more about it, please see http://mathworld.wolfram.com/BesselFunctionoftheFirstKind.html. Also tabulated on the right

The zeroth- and first-order Bessel functions of the first kind

η	$J_0(\eta)$	$J_1(\eta)$
0.0	1.0000	0.0000
0.1	0.9975	0.0499
0.2	0.9900	0.0995
0.3	0.9776	0.1483
0.4	0.9604	0.1960
0.5	0.9385	0.2423
0.6	0.9120	0.2867
0.7	0.8812	0.3290
0.8	0.8463	0.3688
0.9	0.8075	0.4059
1.0	0.7652	0.4400
1.1	0.7196	0.4709
1.2	0.6711	0.4983
1.3	0.6201	0.5220
1.4	0.5669	0.5419
1.5	0.5118	0.5579
1.6	0.4554	0.5699
1.7	0.3980	0.5778
1.8	0.3400	0.5815
1.9	0.2818	0.5812
2.0	0.2239	0.5767
2.1	0.1666	0.5683
2.2	0.1104	0.5560
2.3	0.0555	0.5399
2.4	0.0025	0.5202
2.6	−0.0968	−0.4708
2.8	−0.1850	−0.4097
3.0	−0.2601	−0.3391
3.2	−0.3202	−0.2613