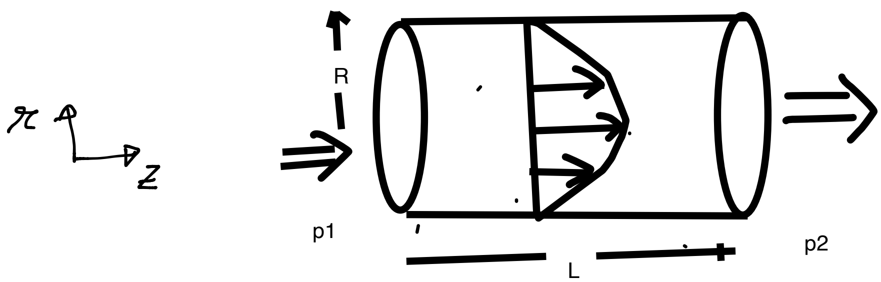
Infer power law rheological parameters from the experimental measurements of pressure drop vs flow rate in a capillary. Let R and L be the radius and length of the capillary.

Assumptions:

1. steady incompressible flow.
2. Laminar flow
3. Fully developed flow.
4. Axial symmetry $V_\theta = 0$
5. Flow is one dimensional
6. Creeping flow.



$V_z \neq 0 \quad V_r = V_\theta = 0$

Equation of continuity

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\cancel{r V_r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\cancel{V_\theta} + \frac{\partial}{\partial z} V_z = 0 \qquad (1)$$

Equation of motion along z direction

$$\rho\left(\cancel{\frac{\partial V_z}{\partial t}} + \cancel{V_r}\frac{\partial V_z}{\partial r} + \cancel{\frac{V_\theta}{r}}\frac{\partial V_z}{\partial \theta} + V_z \cancel{\frac{\partial V_z}{\partial z}}\right) = -\frac{\partial p}{\partial z}$$

$$-\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\frac{\partial \cancelto{0}{\tau_{\theta z}}}{\partial \theta} + \frac{\partial \cancelto{0}{\tau_{zz}}}{\partial z}\right) + \rho \cancelto{0}{g_z}$$

$$0 = -\frac{dp}{dz} - \frac{1}{r}\frac{d}{dr}(r\tau_{rz}) \qquad (2)$$

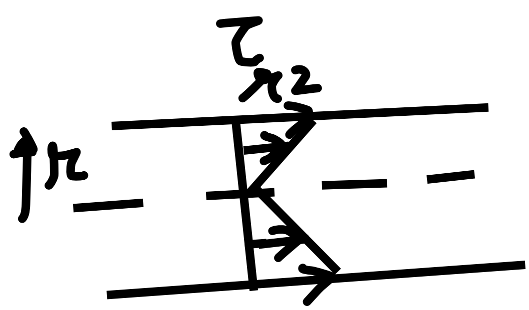
$$\frac{1}{r}\frac{d}{dr}(r\tau_{rz}) = -\frac{dp}{dz} = \frac{p_1 - p_2}{L} = \frac{\Delta P}{L}$$

$$\frac{d}{dr}(r\tau_{rz}) = \frac{\Delta P r}{L}$$

$$r\tau_{rz} = \frac{\Delta P r^2}{2L} + C_1$$

$$\tau_{rz} = \frac{\Delta P r}{2L} + \cancelto{0}{\frac{C_1}{r}} \quad \text{since } \tau_{rz} \text{ is finite.}$$

$$\tau_w = \text{Wall shear stress} = \frac{\Delta P R}{2L} \qquad (3)$$



For Newtonian

$$\tau_{rz} = -\mu\left(\frac{dV_z}{dr}\right) \quad (4a)$$

For Power law

$$\tau_{rz} = -\mu_{app}\left(\frac{dV_z}{dr}\right) \quad (4b)$$

$$\mu_{app} = K\left|\frac{dV_z}{dr}\right|^{n-1} \quad (5)$$

$$\tau_{rz} = -K\left|\frac{dV_z}{dr}\right|^{n-1}\left(\frac{dV_z}{dr}\right)$$

Here, $\left(\frac{dV_z}{dr}\right) < 0 \qquad \left|\frac{dV_z}{dr}\right| = -\left(\frac{dV_z}{dr}\right)$

$$\tau_{rz} = K\left(-\frac{dV_z}{dr}\right)^n$$ Constitutive equation

$$K\left(-\frac{dV_z}{dr}\right)^n = \frac{\Delta P r}{2L}$$

$$\frac{dV_z}{dr} = -\left(\frac{\Delta P r}{2KL}\right)^{1/n} \qquad \dot{\gamma}_W = -\left(\frac{dV_z}{dr}\right)_{r=R} = \left(\frac{\Delta P R}{2KL}\right)^{1/n}$$

Integrating,

$$V_z = -\left(\frac{\Delta P}{2KL}\right)^{1/n}\frac{r^{\frac{1}{n}+1}}{\frac{1}{n}+1} + C_2$$

B.C. $r = R \quad V_z = 0$

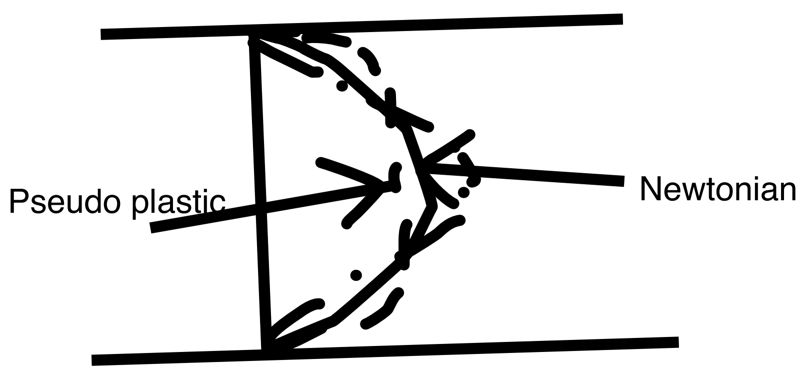
$$0 = -\left(\frac{\Delta P}{2KL}\right)^{1/n}\frac{R^{\frac{1}{n}+1}}{\frac{1}{n}+1} + C_2$$

$$C_2 = \left(\frac{\Delta P}{2KL}\right)^{\frac{1}{n}} \frac{R^{\frac{1}{n}+1}}{\frac{1}{n}+1}$$

Velocity Profile

$$V_z = \left(\frac{\Delta P}{2KL}\right)^{\frac{1}{n}} \frac{n}{n+1} R^{\frac{n+1}{n}} \left[1 - \left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right] \quad (6)$$

When n=1, the above profile reduces to parabolic profile for Newtonian fluid.



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The velocity profile is not parabolic for Non Newtonian.

$$Q = 2\pi \int_0^R V_z(r)\, r\, dr$$

$$Q = 2\pi \int_0^R \left(\frac{\Delta P}{2KL}\right)^{\frac{1}{n}} \frac{n}{n+1} R^{\frac{n+1}{n}} \left[1 - \left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right] r\, dr$$

$$Q = 2\pi \left(\frac{\Delta P}{2KL}\right)^{\frac{1}{n}} \frac{n}{n+1} R^{\frac{n+1}{n}} R^2 \int_0^1 \left(1 - x^{\frac{n+1}{n}}\right) x\, dx$$

$$Q = 2\pi \left(\frac{\Delta P}{2KL}\right)^{\frac{1}{n}} R^{\frac{3n+1}{n}} \frac{n}{n+1} \underbrace{\left[\frac{x^2}{2} - \frac{x^{\frac{3n+1}{n}}}{3n+1/n}\right]_0^1}$$

$$Q = \pi \left(\frac{\Delta P}{2KL}\right)^{\frac{1}{n}} R^{\frac{3n+1}{n}} \frac{n}{n+1} 2\left(\frac{1}{2} - \frac{n}{3n+1}\right)$$

$$Q = \pi \left(\frac{\Delta P}{2KL}\right)^{\frac{1}{n}} R^{\frac{3n+1}{n}} \frac{n}{n+1} \frac{n+1}{3n+1}$$

$$Q = \pi \left(\frac{\Delta P}{2KL}\right)^{\frac{1}{n}} R^{\frac{3n+1}{n}} \frac{n}{3n+1} \qquad (7)$$

For Newtonian fluid, we have,

$$Q = \left(\frac{\pi \Delta P}{8\mu L}\right) R^4 \qquad (8)$$

$$\frac{4Q}{\pi R^3} = \frac{\Delta P R}{2\mu L} = -\left(\frac{dV_z}{dr}\right)_{r=R} = \dot{\gamma}_{ow} \qquad (9)$$

For power law fluid

$$\frac{4Q}{\pi R^3} = \left(\frac{\Delta P}{2KL}\right)^{\frac{1}{n}} R^{\frac{1}{n}} \frac{4n}{3n+1} \qquad (10)$$

$$\dot{\gamma}_{OW} = \dot{\gamma}_W \frac{4n}{(3n+1)} \qquad (11)$$

Therefore,

$$\dot{\gamma}_W = \dot{\gamma}_{OW} \left(\frac{3n+1}{4n}\right) \longleftarrow \text{CORRECTION FACTOR} \qquad (12)$$

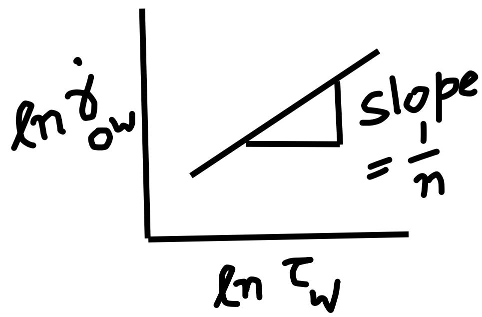
$$\dot{\gamma}_W = -\left(\frac{dV_z}{dr}\right)_{r-R} = \left(\frac{\Delta P R}{2KL}\right)^{1/n}$$

$$\dot{\gamma}_{OW} = \left(\frac{\Delta P R}{2KL}\right)^{1/n} \frac{4n}{(3n+1)} \qquad (13)$$

$$\tau_W = \frac{\Delta P R}{2L} \qquad (3)$$

$$\dot{\gamma}_{OW} = \tau_W^{1/n} \frac{4n}{(3n+1)} \frac{1}{K^{1/n}}$$

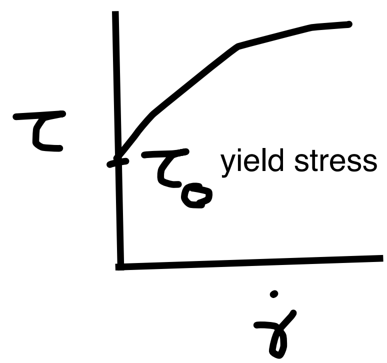
$$\ln \dot{\gamma}_{OW} = \frac{1}{n} \ln \tau_W + \ln \frac{4n}{(3n+1)} + \frac{1}{n} \ln K$$



Experimental data of Q vs deltaP.

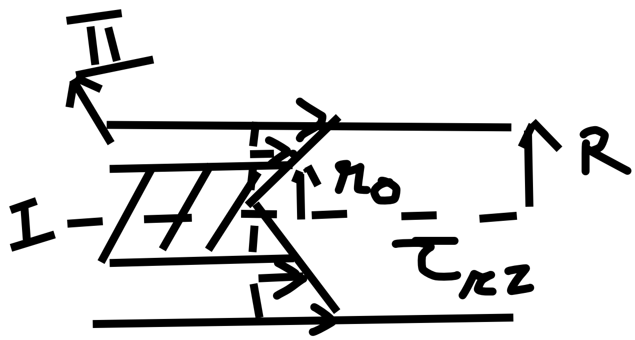
1. Convert ΔP to τ_w
2. Convert Q to $\dot{\gamma}_{ow}$
3. Plot $\ln \dot{\gamma}_{ow}$ vs $\ln \tau_w$
4. slope $= \frac{1}{n}$
5. Calculate $CF = \frac{3n+1}{4n}$
6. $\dot{\gamma}_w = \dot{\gamma}_{ow}\, CF$

Plastic fluid with an yield stress. Recall the plot of shear stress vs shear rate for a plastic fluid.



When $\tau \leq \tau_0$ $\dot{\gamma} = 0$

The shear stress distribution for flow through a capillary is shown below.



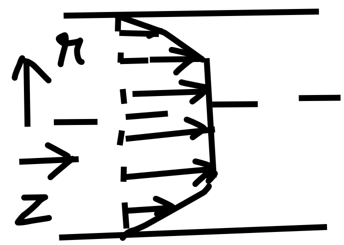
region I ($r \leq r_0$)

$$-\tau_{rz} < \tau_0$$

$$\Rightarrow \dot{\gamma} = 0 \quad r \leq r_0$$

$$\frac{dV_z}{dr} = 0 \quad r \leq r_0 \Rightarrow V_z \text{ is constant}$$

Since velocity is maximum at the center of the tube, the above condition implies that velocity is constant (zero gradient) in the inner core. The velocity profile will be as shown below.



The shear stress profile is given by,

$$\tau_{rz} = \frac{\Delta P r}{2L}$$

$$r_o = \frac{2L\tau_o}{\Delta P}$$

$r_o \downarrow$ as $\Delta P \uparrow$

When $r_o = R \Rightarrow V_z = 0$ everywhere

$$\Delta P = \frac{2L\tau_o}{R} = \Delta P_{min}$$

ΔP_{min} can be estimated by extrapolating ΔP vs Q to zero flow rate.

From ΔP_{min} you can evaluate τ_0. Using the previous analysis, you can convert ΔP to τ_w and Q to $\dot{\gamma}_w$. Fit the data at high flow rates to the following equation

to determine K and n.

$$(\tau_w - \tau_0) = K \dot{\gamma}_w^{\,n}$$