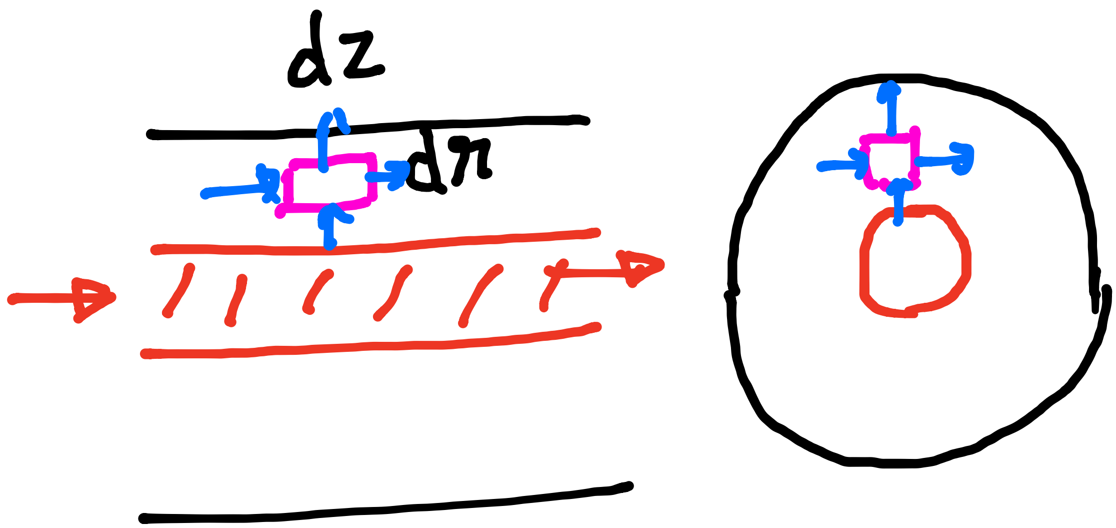
1. The schematic of capillary with surrounding tissue is shown below.



(a) A mass balance for the volume element gives,

Rate in at r = $-D\left.\frac{\partial c}{\partial r}\right|_{r} 2\pi r\, dz$

Rate out at r+dr = $-D\left.\frac{\partial c}{\partial r}\right|_{r+dr} 2\pi (r+dr)\, dz$

Rate in at z = $-D\left.\frac{\partial c}{\partial z}\right|_{z} 2\pi r\, dr$

Rate out at z+dz = $-D\left.\frac{\partial c}{\partial z}\right|_{z+dz} 2\pi r\, dr$

Rate of consumption = $2\pi r\, dr\, dz\, R_{O_2}$

At steady state, rate in - rate out = rate of consumption. Dividing this equation by

$2\pi r\, dr\, dz$ and taking the limit as dr and dz tending to zero, one obtains,

$$\frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right) + D\frac{\partial^2 c}{\partial z^2} = R_{O_2} \qquad (1)$$

(b) Boundary conditions,

$$r = R_c \quad c(R_c, z) = c_{blood}(z) \quad (2)$$

$$r = R_o, \quad -D\frac{\partial c}{\partial r} = 0 \quad \forall z \quad (3)$$

$$z = 0 \quad c(R_c, 0) = c_{blood,0} \quad (4)$$

$$z = L \quad \left.\frac{\partial c}{\partial z}\right|_{R_c, L} = 0 \quad (5)$$

(c) Neglecting axial diffusion, eq. (1) becomes

$$\frac{D}{r}\frac{d}{dr}\left(r\frac{dc}{dr}\right) = R_{O_2} \quad (6)$$

case (i)

$$R_{O_2} = \frac{R_{max}\,c}{K_M + c}$$

$$K_M \ll c \Rightarrow R_{O_2} = R_{max}$$

For a fixed axial distance of L1, we get,

$$\frac{D}{r}\frac{d}{dr}\left(r\frac{dc}{dr}\right) = R_{max} \quad (7)$$

with the boundary conditions

$$r = R_c \quad c(R_c, L_1) = c_{blood}(L_1) \quad (8)$$

$$r = R_o \quad -D\frac{dc(L_1, r)}{dr} = 0 \quad (9)$$

Following the class notes, the solution is given by

$$c(r, L_1) = c_{blood}(L_1) + \frac{R_{max} R_0^2}{2D}\left[\frac{1}{2}\left\{\left(\frac{r}{R_0}\right)^2 - \left(\frac{R_c}{R_0}\right)^2\right\} - \ln\left(\frac{r}{R_c}\right)\right] \quad (10)$$

case (ii) $K_M \gg c$

$$\therefore R_{O_2} = \frac{R_{max}\, c}{K_M}$$

For a fixed L1, we get,

$$\frac{D}{r}\frac{d}{dr}\left(r\frac{dc}{dr}\right) = \frac{R_{max}\, c}{K_M} \quad (11)$$

with the same boundary conditions (8) and (9)

Defining $R^* = \dfrac{R_{max}}{K_M D}$, the above equation can be written as

$$r^2 c'' + r c' + R^* r^2 c = 0 \quad (12)$$

Defining $y = R^{*1/2} R_0 \dfrac{r}{R_0}$, the above equation can be recast as

$$y^2 \frac{d^2c}{dy^2} + y\frac{dc}{dy} + y^2 c = 0 \quad (13)$$

The solution is given by

$$c(L_1, y) = J_0(y) \quad (14)$$

The boundary conditions are

$$y = R^{*1/2} R_c \quad c(L_1, y) = c_{blood}(L_1) \quad (15)$$

$$y = R^{*1/2} \quad \frac{dc}{dy}(L_1, y) = 0 \quad (16)$$

Since $J_0'(y) = J_1(y)$ from boundary condition (16), the solution can be written as,

$$y = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\lambda_n}{R^{*1/2} R_o} y\right)$$

where are zeroes of J1(y) from boundary condition (16). From boundary condition (15), one obtains

$$c_{blood}(L_1) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\lambda_n}{R^{*1/2} R_o} y\right) \quad (17)$$

The coefficients An can be determined from orthogonality of Bessel function.

(d) The rate of consumption of oxygen at axial distance L1 is given by

$$R(L_1) = \int_{R_c}^{R_o} 2\pi r\, R_{O_2}\{c(r, L_1)\}\, dr \quad (18)$$

one needs to substitute the solution of oxygen concentration $c(r, L_1)$ for the two cases in the above equation to determine the rate of oxygen consumption.

(e) A mass balance for oxygen in the capillary between z and z+dz gives,

Rate in at z $= \pi R_c^2 v\, c_{blood}(z)$

Rate out at z+dz $= \pi R_c^2 v\, c_{blood}(z+dz)$

Rate of consumption $= 2\pi R_c\, dz\, R(z)$, where R(z) is given by eq. (18) evaluated at z (instead of L1).

Dividing the mass balance equation by dz and simplkfying, one obtains

$$v R_c \frac{dc_{blood}}{dz} = -2R(z) \quad (19)$$

which can be solved to give the following

$$c_{blood}(z) = c_{blood}(0) - \frac{2}{vR_c} \int_0^z R(z')\, dz' \quad (20)$$

3. a. For steady state one dimensional transport gives

$$\frac{dN_{iz}}{dz} = 0 , \quad i = 1, 2$$

$$\frac{dN_{Na^+}}{dz} = 0 \; ; \quad \frac{dN_{Cl^-}}{dz} = 0 .$$

b. Since there is potential difference across the membrane, the expressions for flux for cation and anion are given by

$$N_+ = -D_+ \left[\frac{dc_+}{dz} + \frac{z_+ c_+ F}{RT} \frac{d\varphi}{dz} \right]$$

$$N_- = -D_- \left[\frac{dc_-}{dz} + \frac{z_- c_-}{RT} \frac{d\varphi}{dz} \right]$$

c. Electrical neutrality implies

$$z_+ N_+ + z_- N_- = 0$$

$$\text{or,} \quad z_+ N_+ = - z_- N_-$$

d. Applying electrical neutrality to the flux equation, one obtains

$$z_+ D_+ \left[\frac{dc_+}{dz} + \frac{z_+ c_+ F}{RT} \frac{d\varphi}{dz} \right] = - z_- D_- \left[\frac{dc_-}{dz} + \frac{z_- c_-}{RT} \frac{d\varphi}{dz} \right]$$

simplifying the above equation, we get

$$\frac{d\psi}{dz} = -\left(\frac{D_+ - D_-}{z_+ D_+ - z_- D_-}\right) \frac{RT}{F} \frac{1}{c_+} \frac{dc_+}{dz}$$

For NaCl, $z_+ = 1$; $z_- = -1$

Integrating the above equation, one obtains

$$\psi(L) - \psi(0) = \left(\frac{D_+ - D_-}{D_+ + D_-}\right) \frac{RT}{F} \ln\left(\frac{c_0}{c_L}\right)$$

Substitute for D+, D-, co and cL to calculate the potential difference