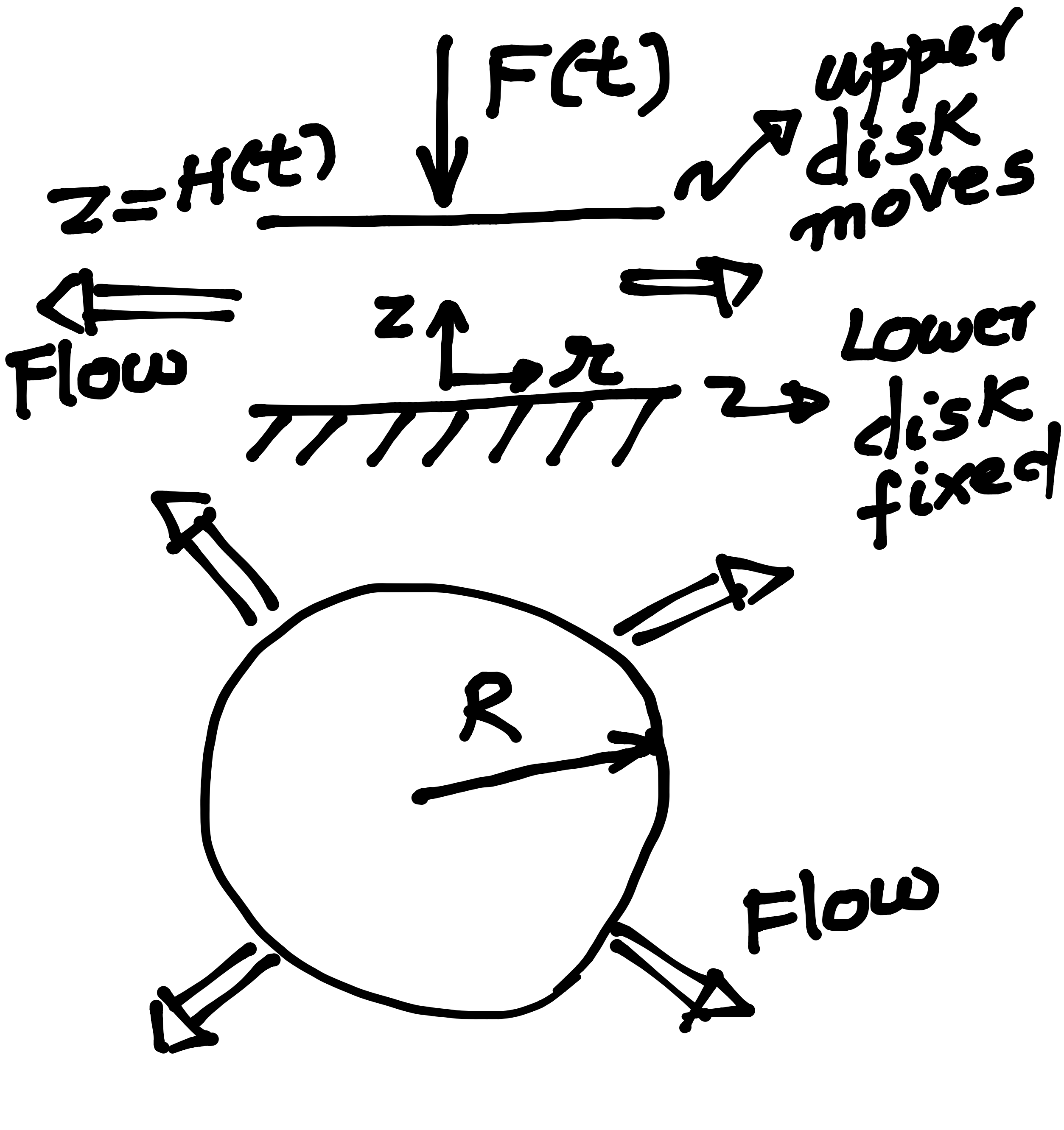
Squeezing flow between two parallel diskes:



Let the top disk be moving with a constant velocity v0. The initial gap between the two plates is much smaller than the radius, i.e. H0<<R. As the top plate is squeezed with a constant velocity, the liquid between the two plates is squeezed redially outward as indicted above. Eventhough the gap between the two plates is changing with time, the system can be assumed to be in quasi steady stete, i.e. the timescale of variation of gap between the two plates is much larger than the time

scale of liquid motion. Note that the flow is not one dimensional. The coordinate system is defined above. There are non zero velocity components along radial (r) and axial (z) directions.

volumetric flow rate = $\pi R^2 V_0 = 2\pi R\, H(t)\, V_r\big|_{r=R}$

$$\therefore\ V_r\big|_{r=R} = \frac{R V_0}{2H(t)}, \quad \text{Since } H(t) << R \Rightarrow V_r\big|_{r=R} \gg V_0$$

$$V_r \sim V_r\big|_{r=R}$$

$$V_z \sim V_0$$

Therefore, $V_r \gg V_z$

Now, $\dfrac{\partial V_r}{\partial r} \sim \left(\dfrac{R}{H}\right)\dfrac{V_0}{R} = \dfrac{V_0}{H}$

$$\frac{\partial V_z}{\partial z} \sim -\frac{V_0}{H}$$

Equation of continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(r V_r) + \frac{\partial V_z}{\partial z} = 0 \quad (1)$$

Equation of motion:

r component

$$0 = -\frac{\partial p}{\partial r} + \mu\left(\frac{\partial^2 V_r}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}(r V_r)\right) \quad (2)$$

z component

$$0 = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 V_z}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}(r V_z)\right) \quad (3)$$

Terms of smaller order of magnitude are neglected in the above equations. z component of equation of motion gives the following,

$$\boxed{\frac{\partial p}{\partial z}=0} \qquad (4)$$

i.e. p is not a function of z. This is known as **LUBRICATION APPROXIMATION** that is valid for thin films of liquid.

Equation (2) gives,

$$\frac{\partial^2 v_r}{\partial z^2} = \frac{1}{\mu}\left(\frac{\partial p}{\partial r}\right)$$

$$\frac{\partial v_r}{\partial z} = \frac{1}{\mu}\left(\frac{\partial p}{\partial r}\right)z + C_1$$

$$v_r = \frac{1}{2\mu}\left(\frac{\partial p}{\partial r}\right)z^2 + C_1 z + C_2$$

$$z=0 \quad v_r=0 \qquad \text{NO SLIP} \Rightarrow C_2=0$$

$$z=H(t) \quad v_r=0$$

$$0=\frac{1}{2\mu}\left(\frac{\partial p}{\partial r}\right)H^2 + C_1 H$$

$$C_1 = -\frac{1}{2\mu}\left(\frac{dp}{dr}\right)H$$

$$v_r = \frac{1}{2\mu}\left(\frac{dp}{dr}\right)z(z-H) \qquad (5)$$

Substituting eq. (5) in equation of continuity, we get,

$$\frac{1}{2\mu}z(z-H)\frac{1}{r}\frac{d}{dr}\left(r\frac{dp}{dr}\right) + \frac{dv_z}{dz} = 0$$

Integrating, one obtains,

$$v_z = -\frac{1}{2\mu}\frac{1}{r}\frac{d}{dr}\left(r\frac{dp}{dr}\right)\left[\frac{z^3}{3}-\frac{Hz^2}{2}\right] + C_3$$

$z=0 \quad V_z=0 \quad \text{Symmetry} \Rightarrow C_3=0$

$z=H(t) \quad V_z=-V_0$

$$\therefore \; -V_0 = -\frac{1}{2\mu}\frac{1}{r}\frac{d}{dr}\left(r\frac{dp}{dr}\right)\left[\frac{H^3}{3}-\frac{H^3}{2}\right]$$

$$V_0 = -\frac{1}{12\mu}\frac{1}{r}\frac{d}{dr}\left(r\frac{dp}{dr}\right)H^3$$

Let us calculate the force F(t) that is applied on the top plate.

$$F = \int_0^R 2\pi r\,(p-p_{atm})\,dr$$

$$= 2\pi\frac{r^2}{2}(p-p_{atm})\Big|_0^R - \int_0^R 2\pi\frac{r^2}{2}\frac{dp}{dr}\,dr$$

Now, $\dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{dp}{dr}\right) = -\dfrac{12\mu V_0}{H^3}$

Integrating,

$$r\frac{dp}{dr} = -\frac{12\mu V_0}{H^3}\frac{r^2}{2}$$

$$\therefore \; -\int_0^R \pi r^2\frac{dp}{dr}\,dr = \frac{6\pi\mu V_0}{H^3(t)}\int_0^R r^3\,dr$$

Therefore,

$$F(t) = \frac{3}{2}\frac{\pi\mu V_0 R^4}{H^3(t)}$$

If constant force F0 is applied, we have,

$$F_0 = \frac{3}{2}\frac{\pi\mu R^4}{H^3(t)}\frac{dH}{dt}$$

$$\therefore \frac{2F_0}{3\pi\mu R^4}\int_0^t dt' = \int_{H_0}^{H(t)} \frac{dH}{H^3} = -\frac{1}{2}\left(\frac{1}{H^2} - \frac{1}{H_0^2}\right)$$

$$\frac{1}{H^2} = \frac{1}{H_0^2} + \frac{4F_0 t}{3\pi\mu R^4}$$