1. Applying Darcy's law, one obtains

$$\underset{\sim}{v} = -K \nabla p \quad (1)$$

Equation of continuity for incompressible liquid gives

$$\nabla \cdot \underset{\sim}{v} = 0 \quad (2)$$

Substituting eq. (1) into eq. (2), one obtains,

$$\nabla \cdot (-K \nabla p) = 0$$

$$\text{or,} \quad \nabla^2 p = 0 \quad (3)$$

For flow through a spherical tumor, the above equation becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dp}{dr}\right) = 0 \quad (3a)$$

Integrating eq. (3a), one obtains,

$$\frac{dp}{dr} = \frac{C_1}{r^2}$$

$$p = -\frac{C_1}{r} + C_2 \quad (4)$$

The boundary conditions are

$$r = \delta \quad p = p_a \quad (5)$$

$$r = a \quad p = 0 \quad (6)$$

From boundary condition (6), we get,

$$0 = -\frac{C_1}{a} + C_2 \Rightarrow C_2 = \frac{C_1}{a} \quad (7)$$

From boundary condition (5), we get,

$$p_a = -\frac{C_1}{\delta} + C_2 = -\frac{C_1}{\delta} + \frac{C_1}{a} = C_1\left(\frac{1}{a} - \frac{1}{\delta}\right)$$

$$C_1 = \frac{-p_a}{\left(\frac{1}{\delta} - \frac{1}{a}\right)} \quad (8)$$

The flux is given by

$$\dot{j}_r = -K\frac{dp}{dr} = -\frac{KC_1}{r^2} = \frac{K p_a}{\left(\frac{1}{\delta} - \frac{1}{a}\right) r^2} \quad (9)$$

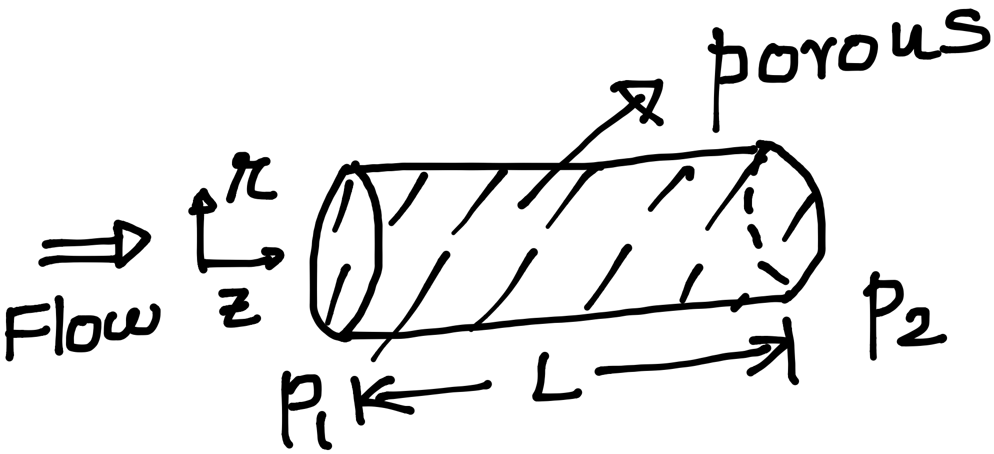
The volumetric flow rate Q is given by,

$$Q = 4\pi r^2 j_r = \frac{4\pi K p_a}{\left(\frac{1}{\delta} - \frac{1}{a}\right)} \quad (10)$$

From (4), the pressue profile is given by

$$p(r) = p_a \frac{\left(\frac{1}{r} - \frac{1}{a}\right)}{\left(\frac{1}{\delta} - \frac{1}{a}\right)} = \frac{Q}{4\pi K}\left(\frac{1}{r} - \frac{1}{a}\right) \quad (11)$$

2.



Assumptions: one dimensional flow, lubrication approximation, i.e.

$$v_z \neq 0, \; v_r = v_\theta = 0 \quad \frac{\partial p}{\partial r} = 0$$

Equation of continuity gives

$$\frac{\partial v_z}{\partial z} = 0 \quad (1)$$

Steady state equation of motion along z direction gives

$$\mu \nabla^2 \underset{\sim}{v} - \frac{\mu}{k} \underset{\sim}{v} - \nabla p = 0 \quad (2)$$

In the above equation, k is the permeability. For the current problem, the above equation becomes

$$\frac{\mu}{r} \frac{d}{dr}\left(r \frac{dv_z}{dr}\right) - \frac{\mu}{k} v_z - \frac{dp}{dz} = 0 \quad (3)$$

$$\text{or,}\quad \underbrace{\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)-\frac{\mu}{R}v_z}_{\text{Function of } r} = \underbrace{\frac{dp}{dz}}_{\text{function of } z} \qquad (3a)$$

Therefore,

$$\frac{dp}{dz} = \text{constant} = -\frac{(p_1-p_2)}{L} = -B \qquad (4)$$

$$\therefore \frac{d}{dr}\left(r\frac{dv_z}{dr}\right) - \frac{v_z}{R}r = -\frac{B}{\mu}r$$

$$r^2\frac{d^2v_z}{dr^2} + r\frac{dv_z}{dr} - \frac{r^2}{R}v_z + \frac{B}{\mu}r^2 = 0 \qquad (5)$$

$$v_z' = v_z - \frac{RB}{\mu}$$

$$\therefore r^2\frac{d^2v_z'}{dr^2} + r\frac{dv_z'}{dr} - \frac{r^2}{R}v_z' = 0$$

$$r = i\sqrt{R}\,y$$

$$\therefore y^2\frac{d^2v_z'}{dy^2} + y\frac{dv_z'}{dy} + v_z' = 0 \qquad (6)$$

Solution of eq. (6) is given by

$$v_z' = A\,J_0(y) \qquad (7)$$

Boundary conditions are

$$r=0 \quad \frac{dv_z}{dr} = 0 \quad \text{or} \quad \frac{dv_z'}{dr} = 0 \qquad (8)$$

$$r=R \quad v_z = 0 \quad \text{or} \quad v_z' = -\frac{RB}{\mu} \qquad (9)$$

From (7), $V_z'(r) = A\, J_0\left(-\frac{ir}{\sqrt{k}}\right)$ (10)

Applying (9), we get,

$$-\frac{k}{\mu}B = A\, J_0\left(-\frac{iR}{\sqrt{k}}\right)$$

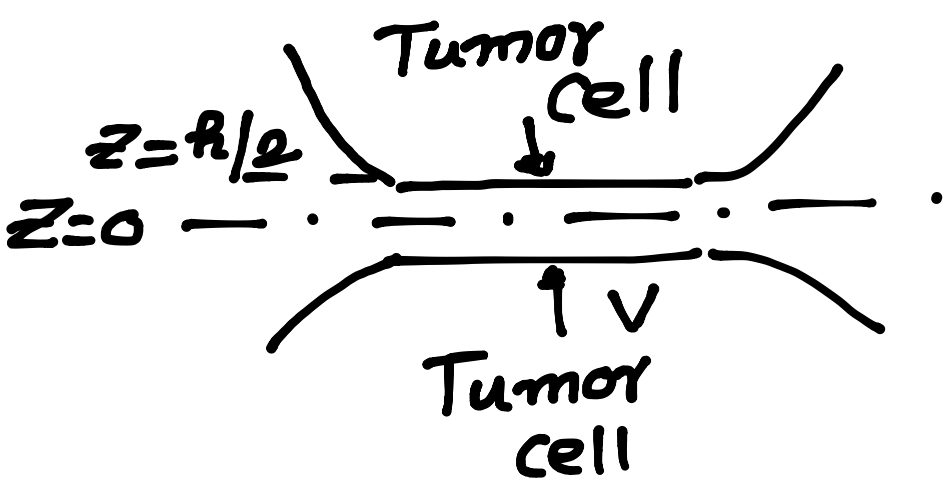
$$A = \frac{-kB/\mu}{J_0\left(-\frac{iR}{\sqrt{k}}\right)} \quad (11)$$

$$V_z(r) = \frac{kB}{\mu}\left[1 - \frac{J_0\left(-\frac{ir}{\sqrt{k}}\right)}{J_0\left(-\frac{iR}{\sqrt{k}}\right)}\right] \quad (11)$$

$$\therefore \frac{V_z(r)}{(kB/\mu)} = 1 - \frac{J_0\left\{-\frac{iR}{\sqrt{k}}\left(\frac{r}{R}\right)\right\}}{J_0\left\{-\frac{iR}{\sqrt{k}}\right\}}$$

The plots of velocity profile for different tube radii are shown in separate handout. Please see the supplemental solution to homework. The profile becomes flatter for larger tube radii.
Also, the plots of velocity profile for different values of permeability as shown in the supplemental indicate that the profile becomes flatter for smaller values of permeability Therefore, Darcy's law is valid for very small values of permeability.

3. See the handout on squeezing flow wherein the flow between two circular disks as a result of force applied on them is discussed. In this example, the gap between two neighboring tumor cells can be modelled as gap between two disks moving with a certain velocity V that is equal to the rate of growth of the tumor cells.



The gap is shown se\chematically above. The coordinate system is defined slightly differently here. z=0 refers to the plane of symmetry between the two disks. z=h/2 refers to the top disk (tumor cell). Because of symmetry one needs to consider only the top half. In this case, the boundary conditions become the same as discussed in the handout, i.e.
z = 0 vz = 0

z=h/2 vz = -V

Alternatively, one could define z = 0 as the bottome disk (tumor cell) and z = h as the top disk (tumor cell). In this case, the boundary conditions become
z = 0 vz = V

z=h vz = -V

Note that the above two formalisms are exactly the same.

For the first formalism, one can use the equation given in the handout by replacing "h" by "h/2". for the applied force for constant velocity, i.e.

$$F(R,t) = \frac{3}{2} \frac{\mu V R^4}{H(t)^3} \quad (1)$$

The stress N(R,t) is force per unit area of the disk and is therefore given by

$$N(R,t) = \frac{F(R,t)}{\pi R^2} = \frac{3}{2} \frac{\mu V R^2}{\pi H^3(t)} \quad (2)$$

Now, $\frac{dH}{dt} = -V$

Integrating, $H(t) = \frac{H_0}{2} - Vt \quad (3)$

The plots of normal stress for different radii and growth rates are given in supplemental. Normal stress is higher for larger radii and larger growth rates.