Question

Given the following open loop transfer function of a marginally stable system, use frequency domain design techniques to design a controller which results in a system with a phase margin of approximately 60 degrees and a gain crossover frequency of approximately 1 rad/sec.

$$G(s) = \frac{1}{s^2}$$

Answer

From the problem statement, write out what you know so far:

- Plant transfer function: $G(s) = \frac{1}{s^2}$
- Desired phase margin: $PM^* = 60°$
  - $PM = \phi_{\omega_{gc}} + 180°$
  - Desired phase angle at ω_{gc}: $\phi^*_{\omega_{gc}} = 60° - 180° = -120°$
- Desired gain crossover frequency: $\omega^*_{gc} = 1\ rad/s$. Therefore $M_{\omega_{gc}} = 0\ dB$.

Now, draw the Bode plot of the uncompensated transfer function: $G(s) = \frac{1}{s^2}$ (see slides 137-141).

Table 1: Magnitude and phase angle data for uncompensated system

Magnitude Data (all in dB)					Phase Angle Data (all in Degrees)			
ω, rad/s	1/s **Integrator**	1/s **Integrator**	***Total***		**ω, rad/s**	1/s **Integrator**	1/s **Integrator**	***Total***
0.1	20	20	40		**0.1**	-90	-90	-180
1	0	0	0		**1**	-90	-90	-180
10	-20	-20	-40		**10**	-90	-90	-180
100	-40	-40	-80		**100**	-90	-90	-180
1000	-60	-60	-120		**1000**	-90	-90	-180

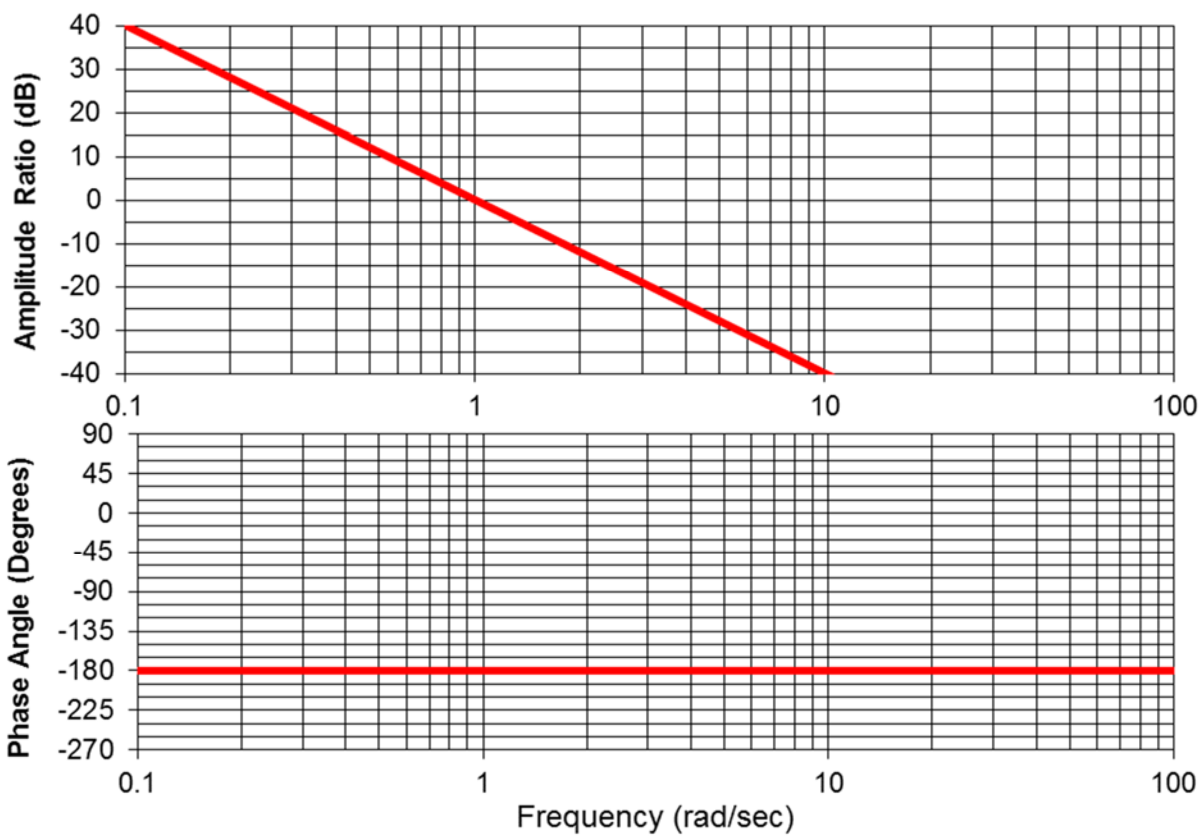

Figure 1: Bode plot of uncompensated system

Notice from our Bode plot of the uncompensated system, that our current phase margin is 0° and our current gain crossover frequency (ω_{gc}) is 1 rad/s (which also happens to be ω_{gc}^*).
In order to meet our requirements of $PM^* = 60°$ and $\omega_{gc}^* = 1\ rad/s$, we need to leave the magnitude plot relatively unchanged and we need to increase our phase plot by 60° at 1 rad/s.

Since we need to use a controller, look at how any particular controller affects the Bode plot. Refer to slides 244-245 and 280-282. The Bode plots shown on these slides will add to the uncompensated Bode plot drawn above. Since we want to add positive phase angle (+60°), don't use a phase-lag or PI controller since they contribute negative phase angle. Both phase-lead and PD contribute positive phase angle and the low-frequency magnitude data is relatively unchanged. Let's choose a PD controller in order to meet our requirements of $PM^* = 60°$ and $\omega_{gc}^* = 1\ rad/s$.

Our controller transfer function will take on the form:

$$G_c(s) = K_p\left(1 + \frac{K_d}{K_p}s\right)$$

Notice that a PD controller is made up of a gain (K_p) and a first order lead ($1 + \frac{K_d}{K_p}s$) that is in the form of ($\tau s + 1$).

Start with the controller design by focusing on the phase angle requirement first. We need to increase the phase angle by 60° at 1 rad/s (increase the angle from -180° to -120°). The angle contribution from a PD controller is defined by the equation:

$$\phi = \tan^{-1}\frac{\omega}{\left(\frac{K_p}{K_d}\right)}$$

We know that ϕ needs to be +60° and we know that this occurs at $\omega = 1\ rad/s$, we can rearrange the equation and find K_p/K_d.

$$\frac{K_p}{K_d} = \frac{\omega}{\tan(\phi)} = \frac{1\ rad/s}{\tan(60°)} = 0.577$$

This ratio of K_p/K_d is used to find the value of τ.

$$\tau = \frac{K_d}{K_p} = \frac{1}{0.577} = 1.732$$

Now our controller equation looks like: $G_c(s) = K_p(1 + 1.732s)$. All we have left to do is find the value of K_p and then we'll have designed our controller. Since K_p will only contribute to the magnitude portion of the Bode plot (see slide 137), let's add the first order lead $(1 + 1.732s)$ to the Bode plot that we drew above in Figure 1.

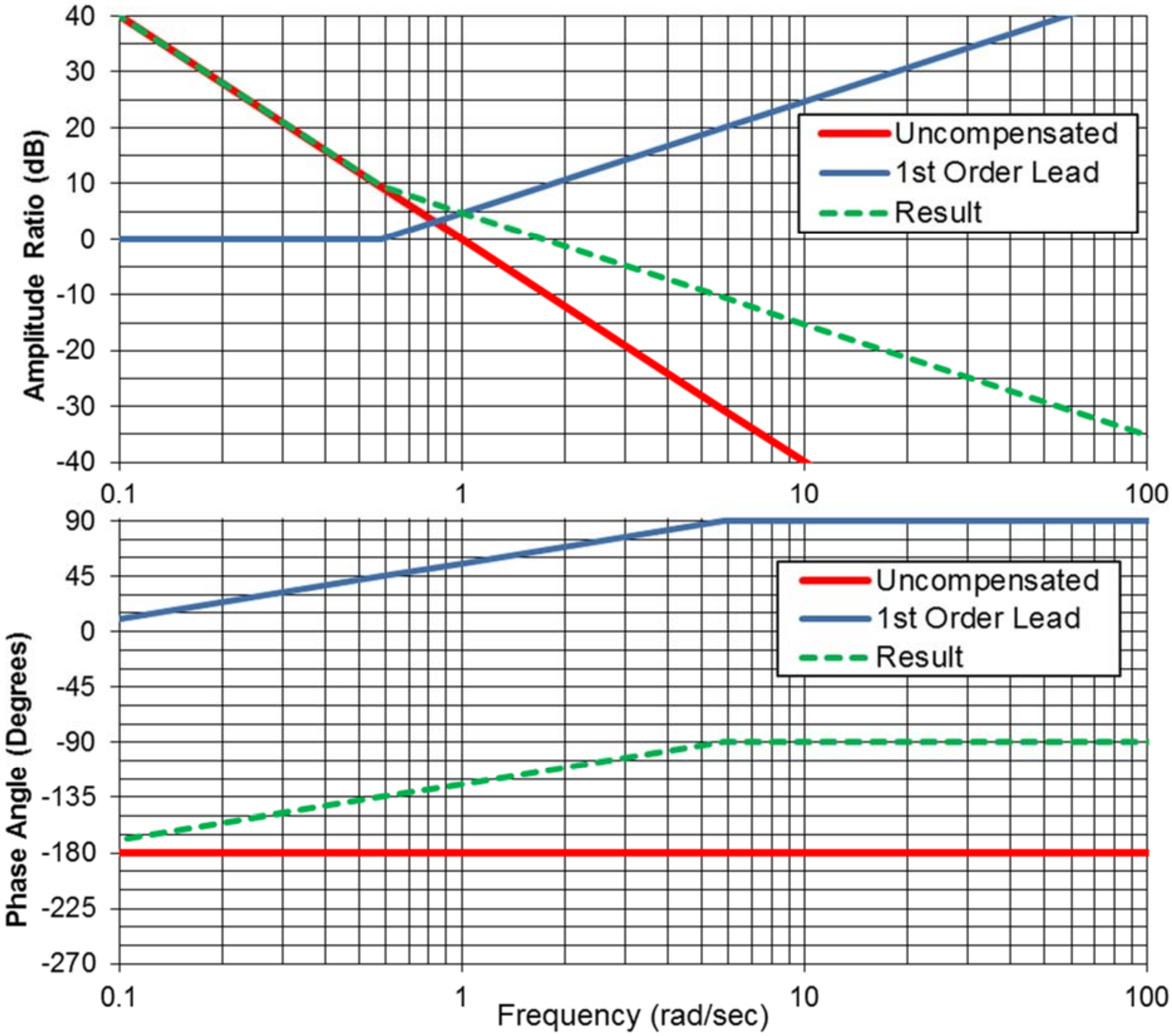

Figure 2: Bode plot with uncompensated system and 1st order lead portion of PD controller

Above in Figure 2 you can see how the 1st order lead portion of the PD controller is drawn on a Bode plot. The values from the blue and red lines will add together and result in the green dotted line. Note that we have only applied the 1st order lead portion of the PD controller. We still need to specify a gain K_p in order to fully describe the PD controller. Remember that the gain K_p will only affect the magnitude plot (it will shift it up or down); therefore, the phase plot will not be affected by any value of K_p.

Looking at the phase plot, we can see that the resulting green dotted line phase angle at 1 rad/s is about -125°, which is close enough to our desired value of 120°. We are drawing our Bode plots using straight line approximation, so it will not show the exact/actual Bode plot paths. Also, when you draw this plots by hand, we introduce even more error. This is why our angle of -125° is good enough.

Looking at the magnitude plot, the green dotted line has a value of about 5 dB at 1 rad/s. Recall that our gain crossover frequency (ω_{gc}) is supposed to be at 1 rad/s. Therefore, the value of the green dotted line needs to be 0 dB at 1 rad/s. Using K_p we can shift the green dotted line down 5 dB so that the magnitude plot has a value of 0 dB at 1 rad/s, thus making the gain crossover frequency 1 rad/s. In order to shift the green dotted line down 5 dB we follow the equations shown below (see slide 137).

$$dB = 20\log(K_p)$$
$$-5\ dB = 20\log(K_p)$$
$$K_p = 10^{\frac{-5}{20}} = 0.562$$

Below in Figure 3 we see the final result of the system when using the full PD controller. The green dotted line in the magnitude plot has been shifted down to become the yellow line. The yellow line now crossed 0 dB at 1 rad/s, thus making $\omega_{gc} = 1\ rad/s$. Notice that the phase plot is unchanged; the green dotted line and the yellow line are on top of each other.

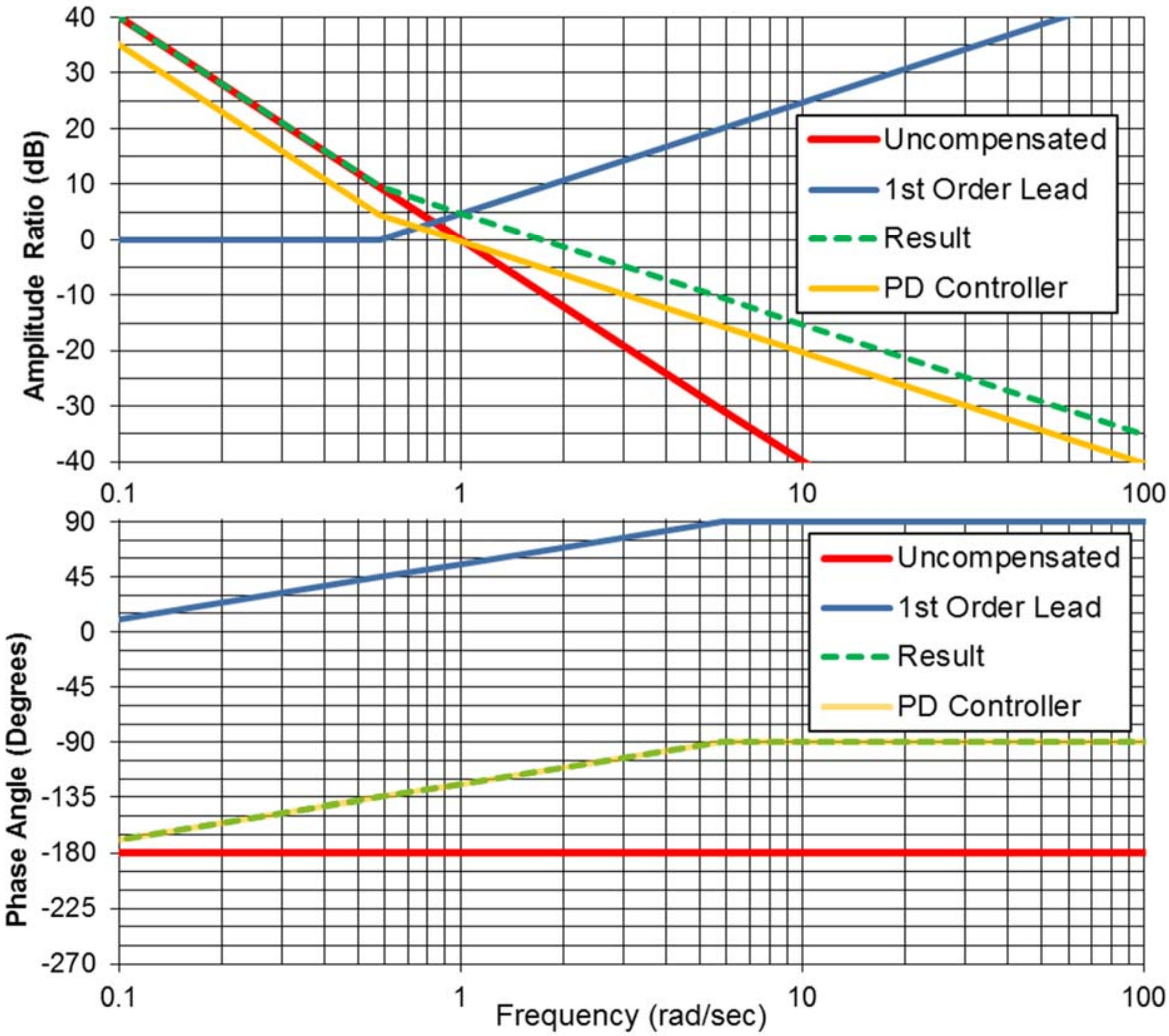

Figure 3: Final system frequency response with full PD controller

To finally answer the question, recall that our PD controller transfer function will take on the form:

$$G_c(s) = K_p\left(1 + \frac{K_d}{K_p}s\right)$$

We know that $K_p = 0.562$ and that $\frac{K_d}{K_p} = \tau = 1.732$. Therefore, $K_d = 0.973$.