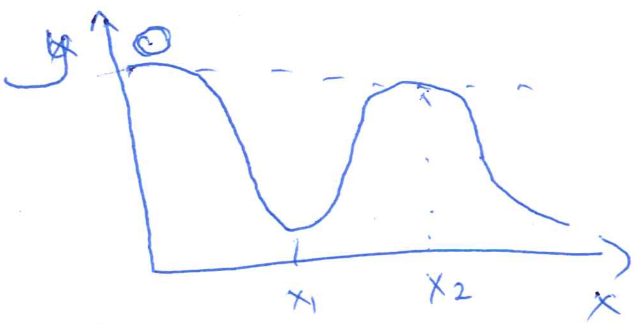
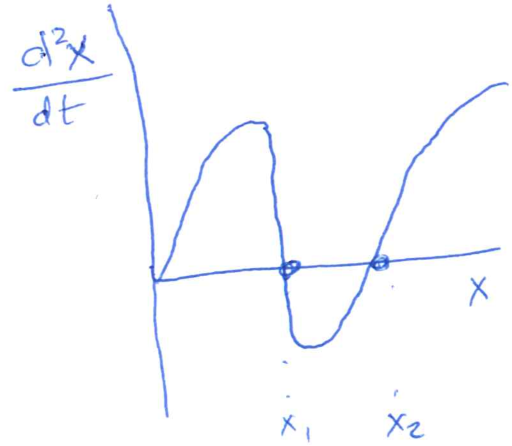
LINEAR STABILITY ANALYSIS ("REVIEW") 9/20/18

an example





steady ~~ts~~ state or stationary point in trough where velocity is zero $\left(\frac{dx}{dt} = \frac{dy}{dt} = 0\right)$

Steady state (stable stationary pt) is when $+/- \varepsilon$ from stationary point pull it back to stationary pt.

Systems are stable when the "acceleration" is *negative*, unstable when *positive

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More generally

$$\frac{\partial x}{\partial t} = f(x,y) \qquad \text{e.g. mass balance of protein 1}$$

$$\frac{\partial y}{\partial t} = g(x,y) \qquad \text{" " protein 2}$$

<u>Nullclines</u> are defined as

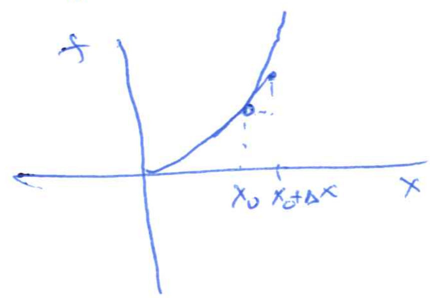
$$\frac{\partial x}{\partial t} = 0 \rightarrow f(x_0, y_0) = 0 \quad \text{stationary pts}$$

$$\frac{\partial y}{\partial t} = 0 \rightarrow g(x_0, y_0) = 0$$

↳ stationary pts as a fn of system parameters

Let's simplify f & g by linearizing w/ a Taylor Series expansions

$$f(x_0 + \Delta x) = f(x_0) + \frac{df}{dx}\Delta x$$



in multiple dimensions

$$f(x_0 + \Delta x,\ y_0 + \Delta y) \approx \overset{0}{\cancel{f(x_0, y_0)}} + \left.\frac{\partial f}{\partial x}\right|_{x_0, y_0} \Delta x + \left.\frac{\partial f}{\partial y}\right|_{x_0, y_0} \Delta y$$

$$g(x_0 + \Delta x,\ y_0 + \Delta y) \approx \underbrace{\cancel{g(x_0, y_0)}}_{0 \text{ stationary points}} + \left.\frac{\partial g}{\partial x}\right|_{x_0, y_0} \Delta x + \left.\frac{\partial g}{\partial y}\right|_{x_0, y_0} \Delta y$$

$$\Rightarrow \frac{\partial x}{\partial t} = f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$$

$$\Rightarrow \frac{\partial y}{\partial t} = g \approx \frac{\partial g}{\partial x}\Delta x + \frac{\partial g}{\partial y}\Delta y$$

Partial Derivatives

e.g. $f = xy^2$

$$\frac{\partial f}{\partial x} = y^2$$

$$\frac{\partial f}{\partial y} = 2xy.$$

$$\begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{bmatrix} = \vec{x} = \underbrace{\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

Jacobian matrix
↳ matrix of first order partial derivatives
$= A$

How does the system change w/ respect to time

Matrices can be represented by a characteristic root or eigenvalue,

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = Ax = \lambda x \quad \Rightarrow \quad \begin{array}{l} \Delta x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ \Delta y = c_3 e^{\lambda_1 t} + c_4 e^{\lambda_4 t} \end{array}$$

eigenvalues represent how the variables change over time as we move from a stationary pt

Allow easy solution of

if $\lambda_i > 0$, system diverge $\Rightarrow$ unstable
$\lambda_i < 0$, system converges to zero
↳ system is stable

λ complex $\Rightarrow \lambda_i = \lambda_{Re} + i\lambda_I$

$$\Delta x = c_1 e^{\lambda_{Re} t} \underbrace{e^{i\lambda_I t}} + c_2 e^{\lambda_2 t}$$

$$= c_1 e^{\lambda_{Re} t} (\cos \lambda_I t + i \sin \lambda_I t) + c_2 e^{\lambda_2 t}$$

$\Rightarrow$ periodic fn!
oscillations

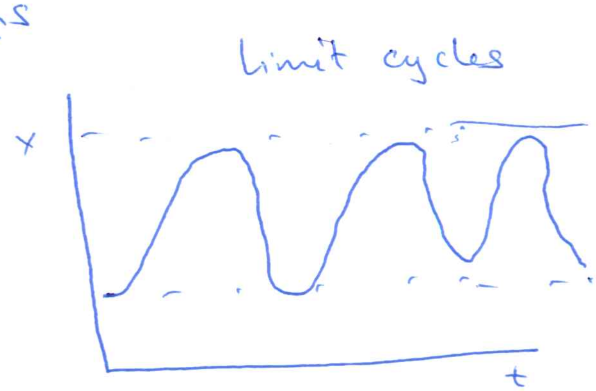
More generally

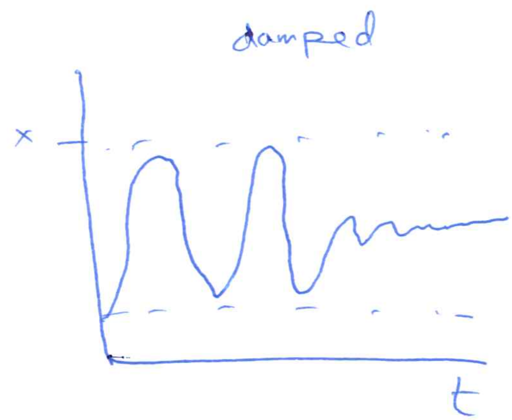
1) if ALL the real components of the Jacobian are negative, the stationary pt is stable

2) if ANY of the real components of the Jacobian are positive, the stationary pt. is unstable

3) if ANY of the eigenvalues are complex, the system will oscillate. Stable oscillations (limit cycles) may arise from unstable systems ($\lambda_{Re} > 0$)

Oscillations





How do we calculate eigenvalues?

Another property

Matrices have a determinant defined by:

$$A = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \qquad A = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}$$

$$= c_{11}c_{22} - c_{21}c_{12}$$

$$\det(A) = c_{11}\begin{vmatrix} c_{22} & c_{23} \\ c_{32} & c_{33} \end{vmatrix} - c_{12}\begin{vmatrix} \cancel{c_{22}}\,c_{21} & c_{23} \\ c_{31} & c_{33} \end{vmatrix} + c_{13}\begin{vmatrix} c_{21} & c_{22} \\ c_{31} & c_{32} \end{vmatrix}$$

eigenvalue given by a characteristic eq$^{\underline{n}}$

$$\det(A - \lambda \underbrace{I}_{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}) = 0$$

for 2×2:

$$0 = \begin{vmatrix} c_{11}-\lambda & c_{12} \\ c_{21} & c_{22}-\lambda \end{vmatrix}$$

$$= (c_{11}-\lambda)(c_{22}-\lambda) - c_{12}c_{21}$$

$$0 = \lambda^2 - c_{11}\lambda - c_{22}\lambda + c_{11}c_{22} - c_{12}c_{21}$$

Quadratic eq^n

$$\lambda = \frac{c_{11}+c_{22} \sqrt{[-1(c_{11}+c_{22})]^2 - 4(c_{11}c_{22}-c_{12}c_{21})}}{2}$$