# 7

# Design of Engineering Experiments

## CHAPTER OUTLINE

## LEARNING OBJECTIVES

After careful study of this chapter, you should be able to do the following:

1. Design and conduct engineering experiments involving several factors using the factorial design approach.
2. Know how to analyze and interpret main effects and interactions.
3. Understand how the ANOVA is used to analyze the data from these experiments.
4. Assess model adequacy with residual plots.
5. Know how to use the two-level series of factorial designs.
6. Understand the role of center points and how two-level factorial designs can be run in blocks.
7. Design and analyze two-level fractional factorial designs.

# 7-1 Why Design Experiments

1. Reduce design and development time
2. Improve design via a systematic analysis of variables and their effect on the yield
3. Reduce cost of operations

## Applications

1. Evaluate design configurations
2. Material selection
3. Determination of critical parameters that has the most effect on the process or gives the best process performance

# 7-1 The Strategy of Experimentation

Every experiment involves a sequence of activities:

1. Conjecture – the original hypothesis that motivates the experiment.

2. Experiment – the test performed to investigate the conjecture.

3. Analysis – the statistical analysis of the data from the experiment.

4. Conclusion – what has been learned about the original conjecture from the experiment. Often the experiment will lead to a revised conjecture, and a new experiment, and so forth.

# 7-2 Factorial Experiments

By a **factorial experiment** we mean that in each complete replicate of the experiment all possible combinations of the levels of the factors are investigated.

Table 7-1 A Factorial Experiment without Interaction

| Factor $A$ | Factor $B$ | |
|---|---|---|
| | $B_{low}$ | $B_{high}$ |
| $A_{low}$ | 10 | 20 |
| $A_{high}$ | 30 | 40 |

Table 7-2 A Factorial Experiment with Interaction

| Factor $A$ | Factor $B$ | |
|---|---|---|
| | $B_{low}$ | $B_{high}$ |
| $A_{low}$ | 10 | 20 |
| $A_{high}$ | 30 | 0 |

# 7-2 Factorial Experiments

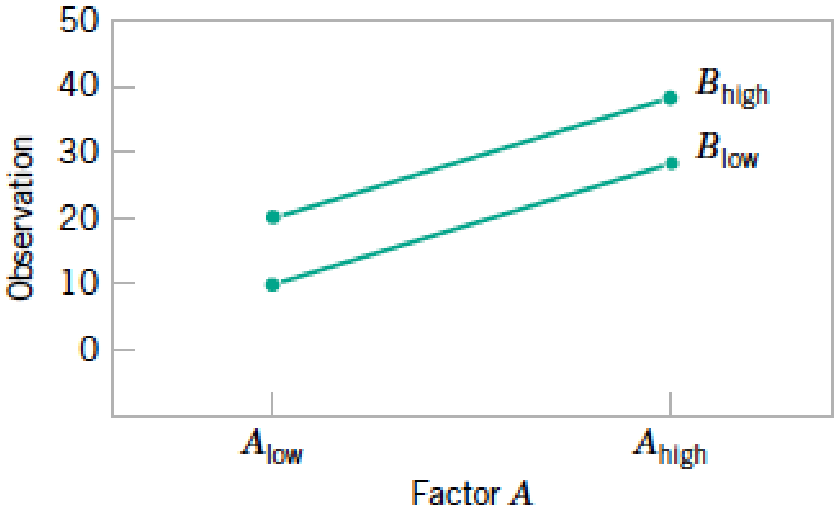



Figure 7-1 An interaction plot of a factorial experiment, no interaction.

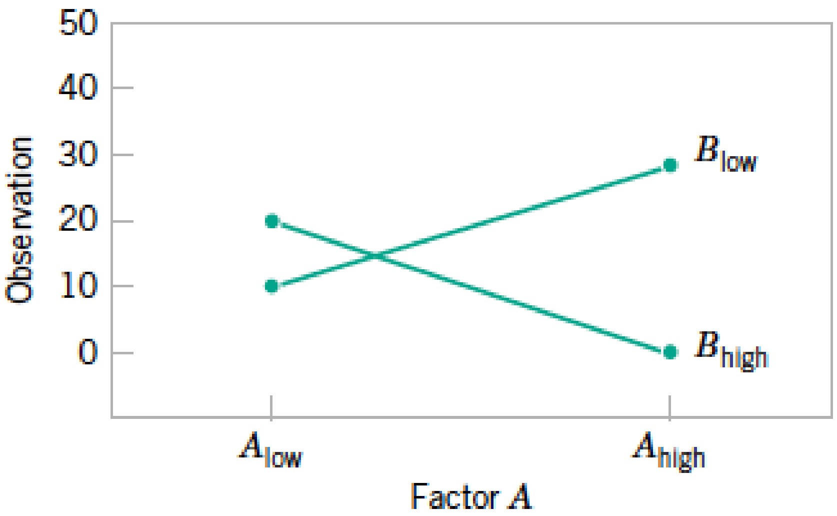



Figure 7-2 An interaction plot of a factorial experiment, with interaction.

# 7-2 Factorial Experiments

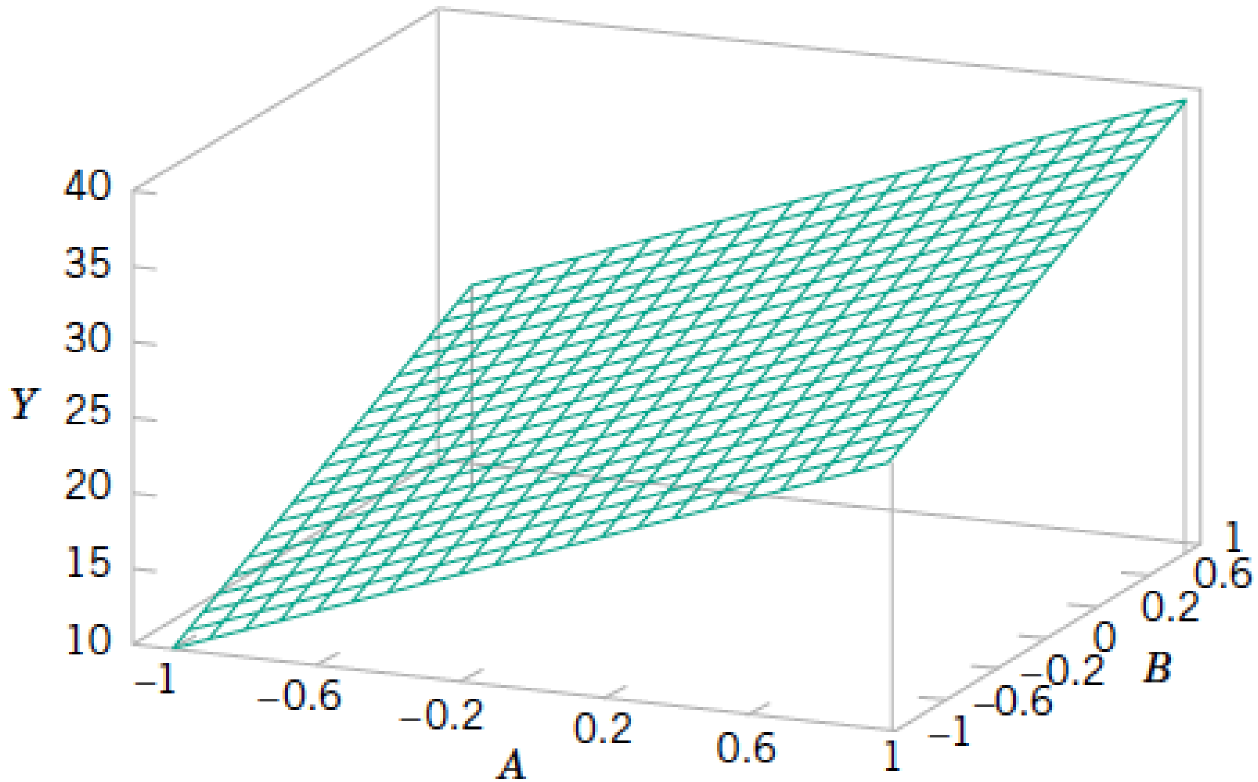



Figure 7-3 Three-dimensional surface plot for the data from Table 7-1, showing main effects of the two factors *A* and *B*.

# 7-2 Factorial Experiments

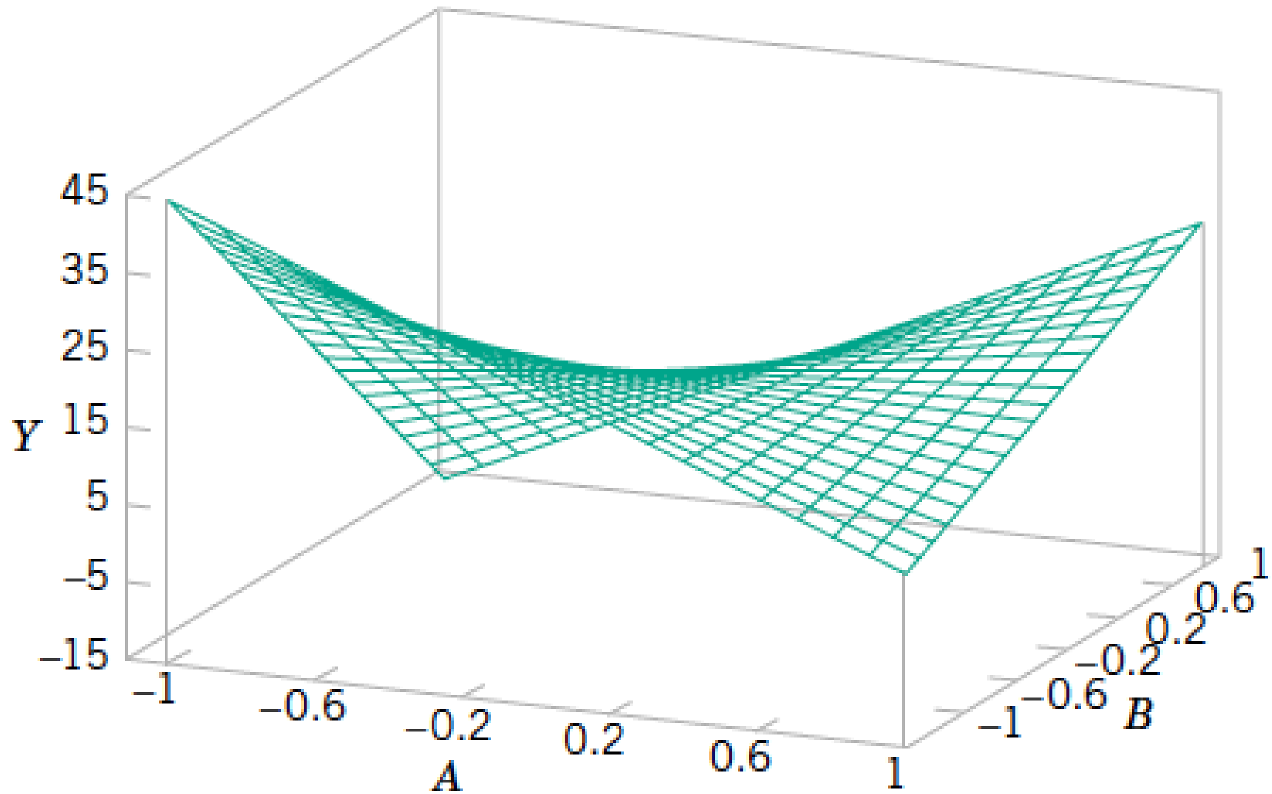



**Figure 7-4** Three-dimensional surface plot for the data from Table 7-2, showing the effect of the *A* and *B* interaction.

# 7-2 Factorial Experiments

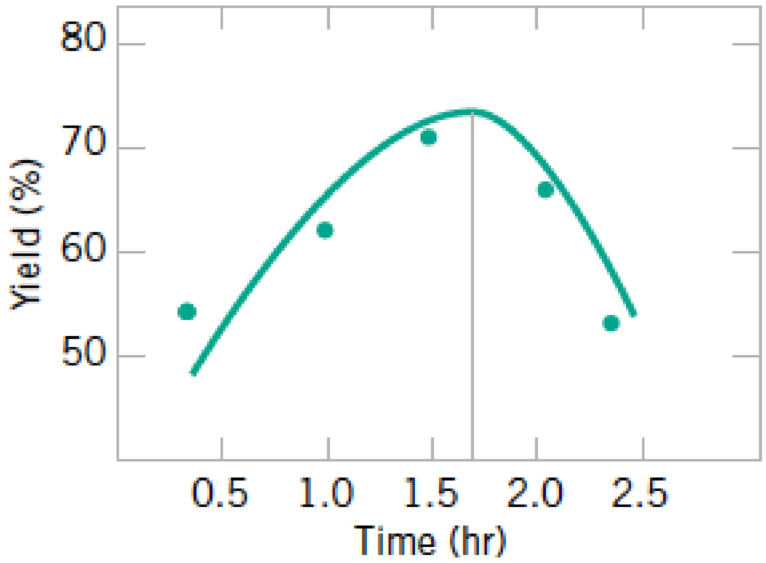



Figure 7-5 Yield versus reaction time with temperature constant at 155°F.

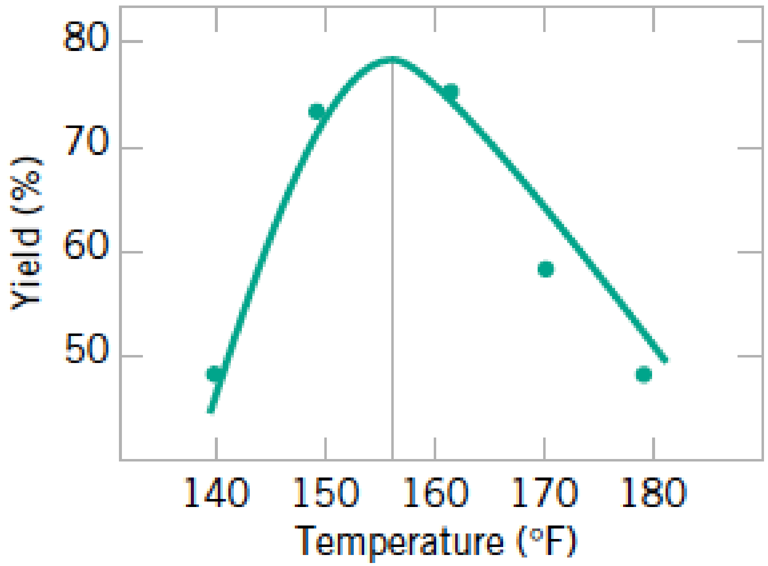



Figure 7-6 Yield versus temperature with reaction time constant at 1.7 hours.

# 7-2 Factorial Experiments

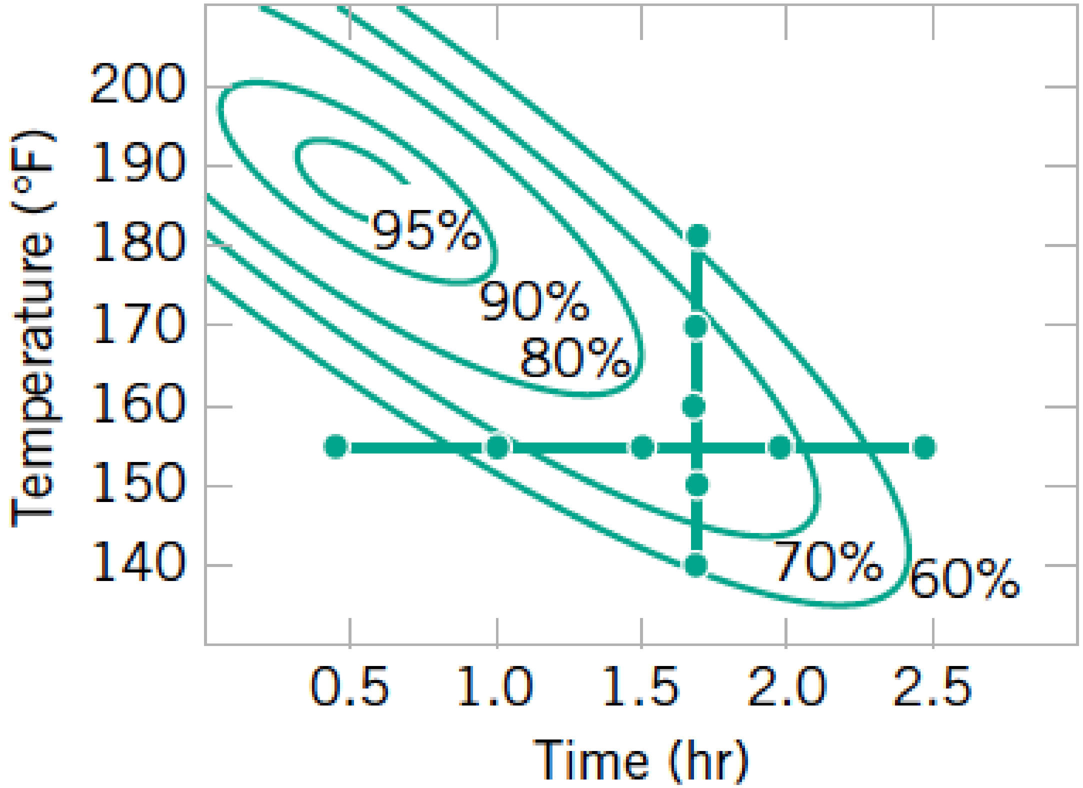



Figure 7-7 Contour plot of a yield function and an optimization experiment using the one-factor-at-a-time method.



***One factor at a time approach ignores interaction between variables***

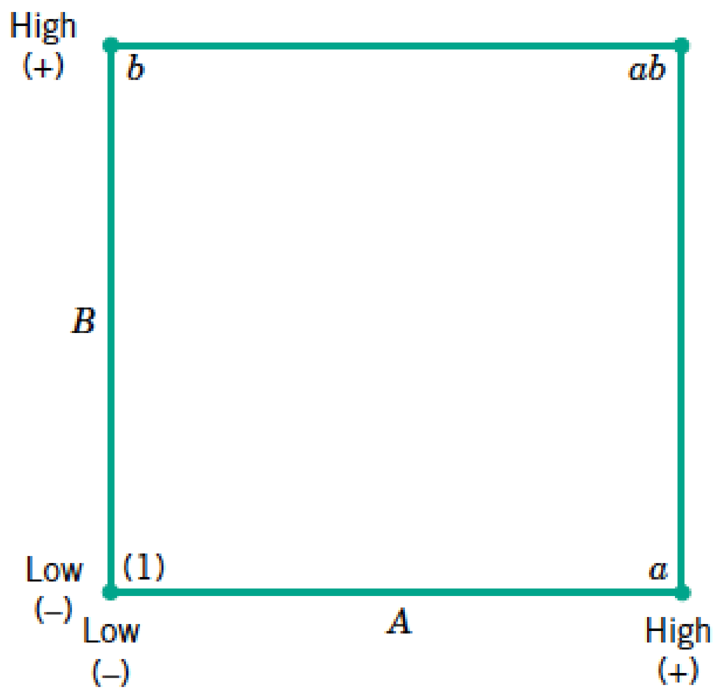
# 7-3 $2^k$ Factorial Design

**Two** Levels and '**k**' variables (or factors) ⇒ $\mathbf{2^k}$

- Basic building block
- Allows to test a range of variables
- Assumes a linear response

# 7-3 $2^k$ Factorial Design

## 7-3.1 $2^2$ Example : 2 factors (variables) at 2 levels




(*a*) Geometric view

| Run | Factor $A$ | Factor $B$ | Label |
|---|---|---|---|
| 1 | – | – | (1) |
| 2 | + | – | $a$ |
| 3 | – | + | $b$ |
| 4 | + | + | $ab$ |

(*b*) Design or test matrix for the $2^2$ factorial design

Figure 7-8 The $2^2$ factorial design.

Variables (ie. factors): A, B Levels: High (+1) ; Low (-1)

# 7-3 $2^k$ Factorial Design

## 7-3.1 $2^2$ Example

$$A = \bar{y}_{A+} - \bar{y}_{A-} = \frac{a + ab}{2n} - \frac{b + (1)}{2n} = \frac{1}{2n}[a + ab - b - (1)] \tag{7-1}$$

$$B = \bar{y}_{B+} - \bar{y}_{B-} = \frac{b + ab}{2n} - \frac{a + (1)}{2n} = \frac{1}{2n}[b + ab - a - (1)] \tag{7-2}$$

$$AB = \frac{ab + (1)}{2n} - \frac{a + b}{2n} = \frac{1}{2n}[ab + (1) - a - b] \tag{7-3}$$

# 7-3 $2^k$ Factorial Design

## 7-3.1 $2^2$ Example

$$A = \bar{y}_{A+} - \bar{y}_{A-} = \frac{a + ab}{2n} - \frac{b + (1)}{2n} = \frac{1}{2n}[a + ab - b - (1)] \qquad (7\text{-}1)$$

$$\text{Contrast}_A = a + ab - b - (1)$$

# 7-3 $2^k$ Factorial Design

## 7-3.1 $2^2$ Example

Table 7-3 Signs for Effects in the $2^2$ Design

| Treatment Combination | Factorial Effect | | | |
|---|---|---|---|---|
| | $I$ | $A$ | $B$ | $AB$ |
| (1) | + | − | − | + |
| $a$ | + | + | − | − |
| $b$ | + | − | + | − |
| $ab$ | + | + | + | + |

# 7-3 $2^k$ Factorial Design

## EXAMPLE 7-1
### Epitaxial Process

An article in the *AT&T Technical Journal* (Vol. 65, March/April 1986, pp. 39–50) describes the application of two-level factorial designs to integrated circuit manufacturing. A basic processing step in this industry is to grow an epitaxial layer on polished silicon wafers. The wafers are mounted on a susceptor and positioned inside a bell jar. Chemical vapors are introduced through nozzles near the top of the jar. The susceptor is rotated, and heat is applied. These conditions are maintained until the epitaxial layer is thick enough.

Table 7-4 presents the results of a $2^2$ factorial design with $n = 4$ replicates using the factors $A$ = deposition time and $B$ = arsenic flow rate. The two levels of deposition time are $-$ = short and $+$ = long, and the two levels of arsenic flow rate are $-$ = 55% and $+$ = 59%. The response variable is epitaxial layer thickness (μm). Find the estimate of the effects and assess the importance of the effects.

**Table 7-4** The $2^2$ Design for the Epitaxial Process Experiment

| Treatment Combination | Factorial Effect | | | Thickness (μm) | | | | Thickness (mm) | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $A$ | $B$ | $AB$ | | | | | Total | Average | Variance |
| (1) | − | − | + | 14.037 | 14.165 | 13.972 | 13.907 | 56.081 | 14.020 | 0.0121 |
| *a* | + | − | − | 14.821 | 14.757 | 14.843 | 14.878 | 59.299 | 14.825 | 0.0026 |
| *b* | − | + | − | 13.880 | 13.860 | 14.032 | 13.914 | 55.686 | 13.922 | 0.0059 |
| *ab* | + | + | + | 14.888 | 14.921 | 14.415 | 14.932 | 59.156 | 14.789 | 0.0625 |

# 7-3 $2^k$ Factorial Design

## EXAMPLE 7-1

**Solution.** We find the estimates of the effects using equations 7-1, 7-2, and 7-3 as follows:

$$A = \frac{1}{2n}[a + ab - b - (1)]$$

$$= \frac{1}{2(4)}[59.299 + 59.156 - 55.686 - 56.081] = \boxed{0.836}$$

$$B = \frac{1}{2n}[b + ab - a - (1)]$$

$$= \frac{1}{2(4)}[55.686 + 59.156 - 59.299 - 56.081] = -0.067$$

$$AB = \frac{1}{2n}[ab + (1) - a - b]$$

$$= \frac{1}{2(4)}[59.156 + 56.081 - 59.299 - 55.686] = 0.032$$

**Practical interpretation:** The numerical estimates of the effects indicate that the effect of deposition time is large and has a positive direction (increasing deposition time increases thickness), because changing deposition time from low to high changes the mean epitaxial layer thickness by 0.836 μm. The effects of arsenic flow rate ($B$) and the $AB$ interaction appear small.

# Effect of variables

1. Standard Error of effects

1. Regression analysis of effects

1. Normal probability plot analysis of effects

# 7-3 $2^k$ Factorial Design

## 7-3.2 Statistical Analysis

### Standard Error of the Effects

$$V(\text{Effect}) = \frac{\sigma^2}{N/2} + \frac{\sigma^2}{N/2} = \frac{2\sigma^2}{N/2} = \frac{\sigma^2}{n2^{k-2}}$$

$$\hat{\sigma}_i^2 = \frac{\sum_{j=1}^{n} (y_{ij} - \bar{y}_{i.})^2}{(n-1)} \qquad i = 1, 2, \ldots, 2^k$$

$$\hat{\sigma}^2 = \sum_{i=1}^{2^k} \frac{\hat{\sigma}_i^2}{2^k}$$

# 7-3 $2^k$ Factorial Design

## 7-3.2 Statistical Analysis

### Standard Error of the Effects

To illustrate this approach for the epitaxial process experiment, we find that

$$\hat{\sigma}^2 = \frac{0.0121 + 0.0026 + 0.0059 + 0.0625}{4} = 0.0208$$

and the estimated standard error of each effect is

$$se(\text{Effect}) = \sqrt{[\hat{\sigma}^2/(n2^{k-2})]} = \sqrt{[0.0208/(4 \cdot 2^{2-2})]} = 0.072$$

Table 7-5 *t*-Tests of the Effects for Example 7-1 Epitaxial Process Experiment

| Effect | Effect Estimate | Estimated Standard Error | *t* Ratio | *P*-Value | Effect ± Two Estimated Standard Errors |
|---|---|---|---|---|---|
| *A* | 0.836 | 0.072 | 11.61 | 0.00 | 0.836 ± 0.144 |
| *B* | −0.067 | 0.072 | −0.93 | 0.38 | −0.067 ± 0.144 |
| *AB* | 0.032 | 0.072 | 0.44 | 0.67 | 0.032 ± 0.144 |

# 7-3 $2^k$ Factorial Design

## Regression Analysis of effects

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \epsilon$$

For Example 7-1, the least squares fitted model is

$$\hat{y} = 14.389 + \left(\frac{0.836}{2}\right) x_1 + \left(\frac{-0.067}{2}\right) x_2 + \left(\frac{0.032}{2}\right) x_1 x_2$$

Deposition time ($x_1 \Rightarrow \pm$ 1); Arsenic flow ($x_2 \Rightarrow \pm$ 1 ); Interaction ($x_1 x_2$)

# 7-3 $2^k$ Factorial Design

**Formulas for Two-Level Factorial Experiments with $k$ Factors Each at Two Levels and $N$ Total Trials**

$$\text{Coefficient} = \frac{\text{effect}}{2}$$

$$se(\text{Effect}) = \sqrt{\frac{2\hat{\sigma}^2}{N/2}} = \sqrt{\frac{\hat{\sigma}^2}{n2^{k-2}}}$$

$$se(\text{Coefficient}) = \frac{1}{2}\sqrt{\frac{2\hat{\sigma}^2}{N/2}} = \frac{1}{2}\sqrt{\frac{\hat{\sigma}^2}{n2^{k-2}}} \tag{7-7}$$

$$t \text{ ratio} = \frac{\text{effect}}{se(\text{effect})} = \frac{\text{coefficient}}{se(\text{coefficient})}$$

$$\hat{\sigma}^2 = \text{mean square error}$$

$$2^k(n-1) = \text{residual degrees of freedom}$$

# 7-3 $2^k$ Factorial Design

Table 7-6 Regression Analysis for Example 7-1. The regression equation is
Thickness = $14.4 + 0.418x_1 - 0.0336x_2 + 0.0158x_1x_2$

| Independent Variable | Coefficient Estimate | Standard Error of Coefficient | $t$ for $H_0$ Coefficient = 0 | $P$-Value |
|---|---|---|---|---|
| Intercept | 14.3889 | 0.0360 | 399.17 | 0.000 |
| $A$ or $X_1$ | 0.41800 | 0.03605 | 11.60 | 0.000 |
| $B$ or $X_2$ | −0.03363 | 0.03605 | −0.93 | 0.369 |
| $AB$ or $X_1X_2$ | 0.01575 | 0.03605 | 0.44 | 0.670 |

Analysis of Variance

| Source | Sum of Squares | Degrees of Freedom | Mean Square | $f_0$ | $P$-Value |
|---|---|---|---|---|---|
| Model | 2.81764 | 3 | 0.93921 | 45.18 | 0.000 |
| Error | 0.24948 | 12 | 0.02079 | | |
| Total | 3.06712 | 15 | | | |



t-test shows that the effect of ‘B’ and ‘AB’ on yield is not significant

# Table 7-7

ANOVA (Table 7-7 contd.)

| Source of Variation | Sum of Squares | Degree of Freedom | Mean Square | $f_0$ | $P$-Value |
|---|---|---|---|---|---|
| $A$ (deposition time) | 2.7956 | 1 | 2.7956 | 134.40 | 7.07 E-8 |
| $B$ (arsenic flow) | 0.0181 | 1 | 0.0181 | 0.87 | 0.38 |
| $AB$ | 0.0040 | 1 | 0.0040 | 0.19 | 0.67 |
| Error | 0.2495 | 12 | 0.0208 | | |
| Total | 3.0672 | 15 | | | |

# 7-3 $2^k$ Factorial Design

## 7-3.3 Residual Analysis and Model Checking

From Example 7-1,

$$\hat{y} = 14.389 + \left(\frac{0.836}{2}\right)x_1$$

when $x_1 = -1$:

$$\hat{y} = 14.389 + \left(\frac{0.836}{2}\right)(-1) = 13.971 \ \mu\text{m}$$

The resulting residuals are:

$$e_1 = 14.037 - 13.971 = 0.066 \qquad e_3 = 13.972 - 13.971 = 0.001$$

$$e_2 = 14.165 - 13.971 = 0.194 \qquad e_4 = 13.907 - 13.971 = -0.064$$

# Error Analysis

Low Deposition Time ($x_1$) = -1 and Low Arsenic flow rate -> row (1)

Low Deposition Time ($x_1$) = -1 and High Arsenic flow rate -> row (1)

High Deposition Time ($x_1$) = +1 and Low Arsenic flow rate -> row (1)

High Deposition Time ($x_1$) = +1 and Low Arsenic flow rate -> row (1)

# 7-3 $2^k$ Factorial Design

## 7-3.3 Residual Analysis and Model Checking

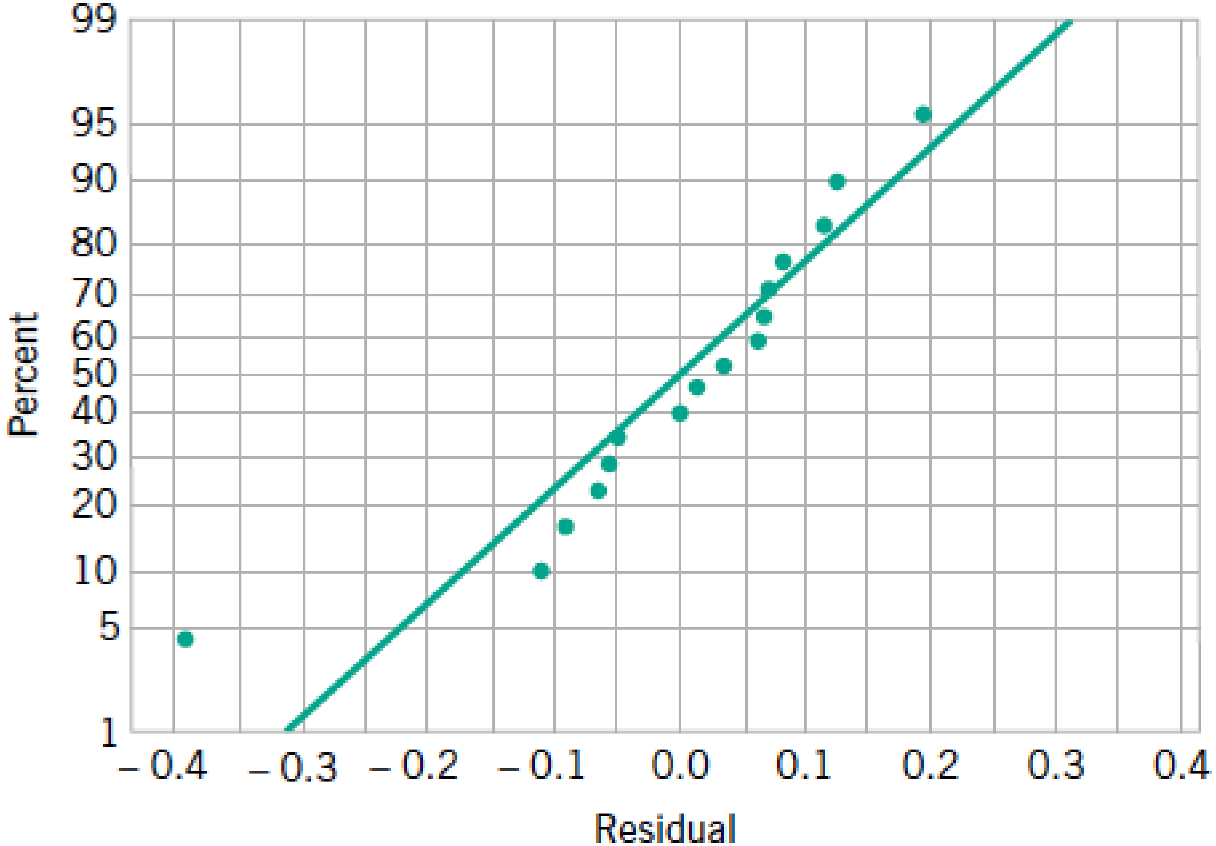



Figure 7-9 Normal probability plot of residuals for the epitaxial process experiment.



**Recall: Sec 3-6.1** **100[1- (j-0.5)/n]**

# 7-3 $2^k$ Factorial Design

## 7-3.3 Residual Analysis and Model Checking

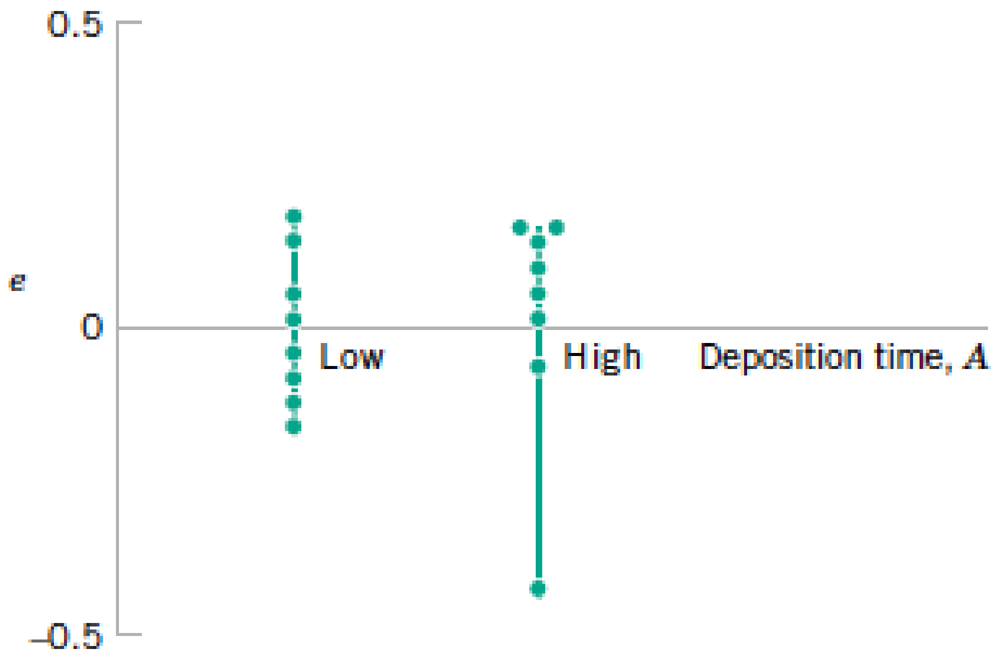



Figure 7-10 Plot of residuals versus deposition time.

# 7-3 $2^k$ Factorial Design

## 7-3.3 Residual Analysis and Model Checking

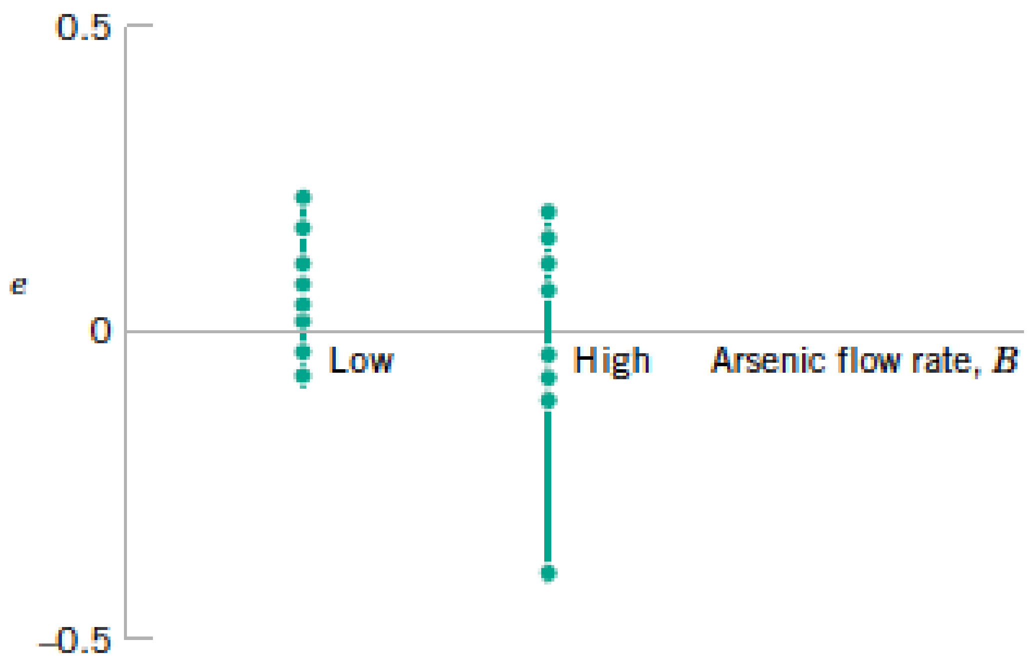



Figure 7-11 Plot of residuals versus arsenic flow rate.

# 7-3 $2^k$ Factorial Design

## 7-3.3 Residual Analysis and Model Checking

Ref. Table 7-4

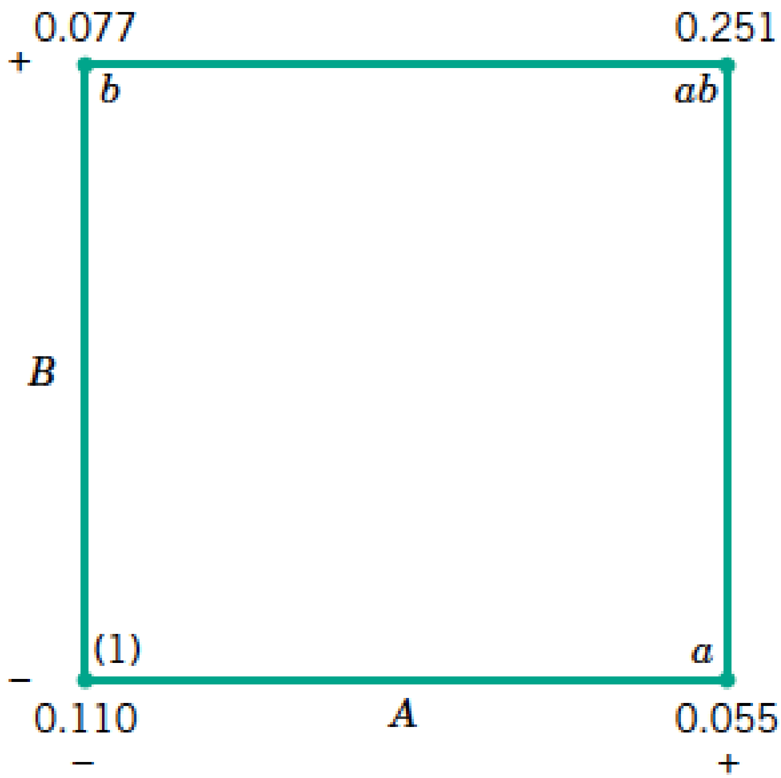



Figure 7-12 The standard deviation of epitaxial layer thickness at the four runs in the $2^2$ design.