Purdue University Mar 23 2017
School of Chemical Engineering Time: 1 hr
Exam-2, Spring 2017 Full Marks: 100
Subject: Statistical Modeling and Quality Enhancement (CHE320)

Instructions:
Read the questions carefully and plan accordingly before answering. Use the backside of the pages if extra space is required. Do not discuss the content of this test until next week.

Name: ______________________________

Signature: ______________________________

1. (a) (10 marks) Consider the two estimators of mean:

$$\bar{X} = \frac{X_1 + X_2 + X_3 + X_4}{4}$$

$$X^* = \frac{X_1 + 2X_2 + 2X_3 + X_4}{6}$$

Evaluate the mean and variance of both the estimators to show that $\bar{X}$ is the Minimum Variance Unbiased Estimator (MVUE).

(b) (15 marks) If the second estimator is replaced by

$$X^* = \frac{X_1 + 2X_2 + 2X_3 + X_4}{8}$$

obtain an expression for the relative efficiency of the estimators as a function of population mean (μ) and variance (σ). Use $MSE(\hat{\Theta}) = V(\hat{\Theta}) + (\text{bias}^2)$.

(c) (5 marks) If a population has $\mu = 2.45$ and $\sigma = 2$, which estimator from (b) is a better estimator?

Problem 1

a) $E(\bar{X}) = \frac{4\mu}{4} = \mu \qquad E(X^*) = \frac{6\mu}{6} = \mu$

$V(\bar{X}) = \frac{4\sigma^2}{16} = \frac{1}{4}\sigma^2 \qquad V(X^*) = \frac{10}{36}\sigma^2 = \frac{5}{18}\sigma^2$

$\bar{X}$ is MVUE since $\frac{1}{4}\sigma^2 < \frac{5}{18}\sigma^2$.

b) $E(X^*) = \frac{6\mu}{8} = \frac{3}{4}\mu \qquad bias = \frac{3}{4}\mu - \mu = -\frac{1}{4}\mu$

$bias^2 = \frac{1}{16}\mu^2$

$V(X^*) = \frac{10}{64}\sigma^2 = \frac{5}{32}\sigma^2$

$MSE(X^*) = \frac{5}{32}\sigma^2 + \frac{1}{16}\mu^2$

$MSE(\bar{X}) = \frac{1}{4}\sigma^2$

Relative efficiency $= \dfrac{\frac{1}{4}\sigma^2}{\frac{5}{32}\sigma^2 + \frac{1}{16}\mu^2} = \dfrac{0.25\sigma^2}{0.156\sigma^2 + 0.0625\mu^2}$

c) If $\mu = 2.45$, $\sigma = 2$, then

Relative efficiency $= \dfrac{0.25(4)}{0.156(4) + 0.0625(2.45^2)} = 1.00084 > 1$

meaning X^* is a better estimator

(if rounding $\Rightarrow$ same estimators)

2. (40 marks) Motor from three different brands are tested for vibrations. The data and partial calculations are presented below: **Please note that Table 2 provides the value of y_{ij}^2.**

Table 1: Measured vibration of motors(micron)

Brand	Observations					
	1	2	3	4	Totals	Averages
Brand 1	13	15	14	14	56	14
Brand 2	16	16	17	15	64	16
Brand 3	14	14	12	14	54	13.5
					174	14.5

Table 2: Measured vibration^2 (micron^2)

Brand	Observations				
	1	2	3	4	Totals
Brand 1	169	225	196	196	____
Brand 2	256	256	289	225	____
Brand 3	196	196	144	196	____
					____

(a) (40 marks) We want to discard the brand with significantly higher vibration. Complete the ANOVA table to decide whether any significant difference exists among brands. Put bounds on P value. Significance level of 0.05 may be used.

Table 3: One way analysis of variance

Source	DF	SS	MS	F	P
Brand	____	____	____	____	____
Error	____	____	____		
Total	____	____			

(b) (5 marks (bonus)) Determine which one has significantly higher vibration than others.

Problem 2

a)

	DF	SS	MS	F	P
Brand	2	14	7	9	<0.025
Error	9	7	7/9		
Total	11	21			

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{n} y_{ij}^2 - \frac{y_{..}^2}{N}$$

$$= 2544 - \frac{174^2}{12} = 21$$

$$SS_{Treatments} = \sum_{i=1}^{a} \frac{y_{i.}^2}{n} - \frac{y_{..}^2}{N}$$

$$= \frac{56^2 + 64^2 + 54^2}{4} - \frac{174^2}{12} = 14$$

P-val: $f_{0.05,2,9} = 4.26$

$f_{0.025,2,9} = 5.71 \longrightarrow$ P-val < 0.025

b) $6\sqrt{MS_E/n} = 6\sqrt{7/9\,/4} = 0.441 \times 6 = 2.646$

~~Mean width covers Brands 1 & 3, but not Brand 2, which is also significantly higher.~~

Brand 2 is higher than Brands 1 & 3, and mean width does not necessarily cover it, so it is qualitatively significantly higher.

3. (30 marks) An experimenter constructs a 95% two sided CI for μ using a sample size of 16. Normality of the population may be assumed and the population σ is unknown. The upper and lower limit of the CI are 14 and 24 respectively.

If instead of constructing CI, the experimenter had tested

$$H_0 : \mu = 10$$

$$H_1 : \mu \neq 10$$

what is the numerical value of the test statistics?

Problem 3

$\boxed{t\text{-test}}$ $t_0 = \dfrac{\bar{x} - \mu}{s/\sqrt{n}}$ Find $\bar{x}$, s

$$\underbrace{\bar{x} - t_{\frac{\alpha}{2}, n-1}\frac{s}{\sqrt{n}}}_{=14} \le \mu \le \underbrace{\bar{x} + t_{\frac{\alpha}{2}, n-1}\frac{s}{\sqrt{n}}}_{=24}$$

$t_{0.025,15} = \boxed{2.131}$ 4 pts

plug in

$\bar{x} = 14 + 2.131\left(\dfrac{s}{\sqrt{16}}\right)$ ~~8~~ 6 pts

$\bar{x} = 24 - 2.131\left(\dfrac{s}{\sqrt{16}}\right)$ ~~8~~ 6 pts

$$14 + 2.131\left(\frac{s}{\sqrt{16}}\right) = 24 - 2.131\left(\frac{s}{\sqrt{16}}\right)$$

$$2\left[2.131\left(\frac{s}{\sqrt{16}}\right)\right] = 10$$

$\boxed{s = 9.385}$ 2 pts

$\bar{x} = 14 + 2.131\left(\dfrac{9.385}{\sqrt{16}}\right) = \boxed{19}$ 2 pts

$\Rightarrow t_0 = \dfrac{19 - 10}{9.385/\sqrt{16}} = \cancel{0.2387}$ $\boxed{3.8359}$ 5 pts