7-4 Center Points and Blocking in 2^k Designs

7-4.1 Addition of Center Points

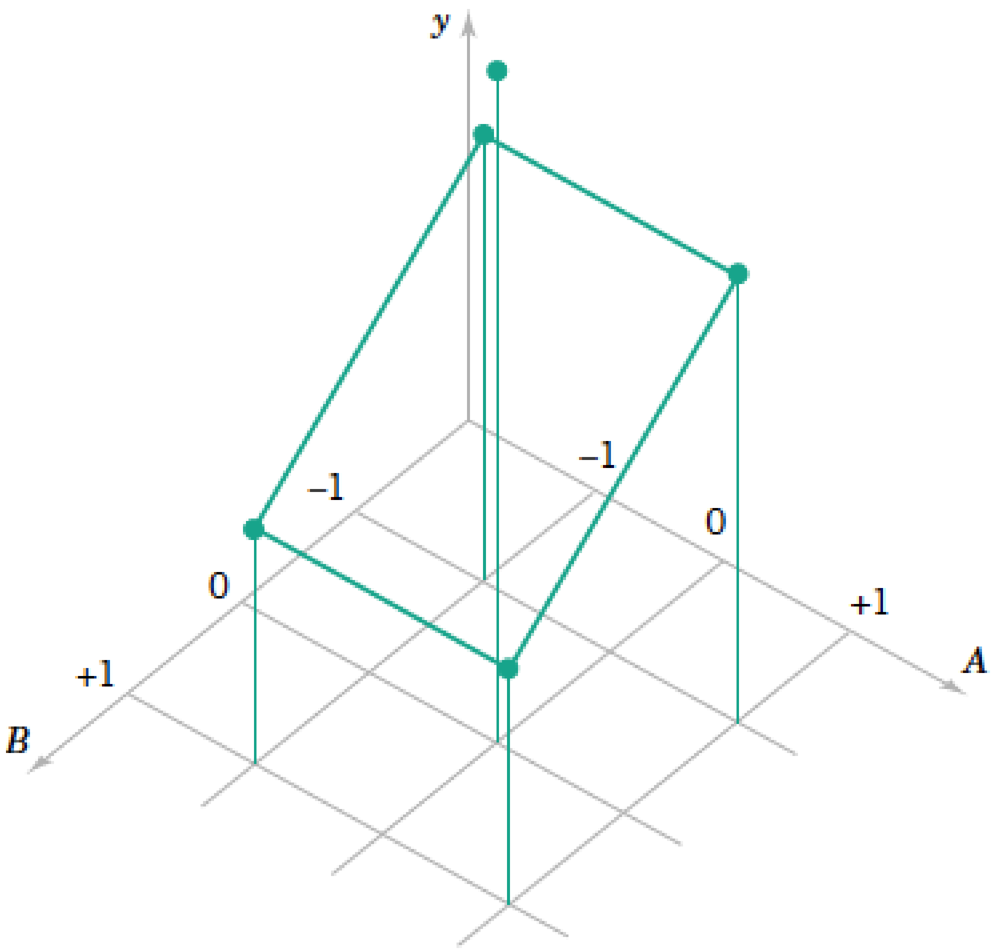

Figure 7-19 A 2^2 design with center points.

7-4 Center Points and Blocking in 2^k Designs

7-4.1 Addition of Center Points

A *t*-test statistic for curvature is given by

$$t_{\text{Curvature}} = \frac{\bar{y}_F - \bar{y}_C}{\sqrt{\hat{\sigma}^2\left(\frac{1}{n_F} + \frac{1}{n_C}\right)}} \tag{7-15}$$

where n_F is the number of factorial design points and n_C is the number of center points.

$$Y = \beta_0 + \sum_{j=1}^{k} \beta_j x_j + \sum_{i<j}\sum \beta_{ij} x_i x_j + \sum_{j=1}^{k} \beta_{jj} x_j^2 + \epsilon$$

7-4 Center Points and Blocking in 2^k Designs

7-4.1 Addition of Center Points

$$H_0\text{: } \sum_{j=1}^{k} \beta_{jj} = 0 \qquad H_1\text{: } \sum_{j=1}^{k} \beta_{jj} \neq 0$$

7-4 Center Points and Blocking in 2^k Designs

EXAMPLE 7-4

Process Yield

A chemical engineer is studying the percent conversion or yield of a process. There are two variables of interest, reaction time and reaction temperature. Because she is uncertain about the assumption of linearity over the region of exploration, the engineer decides to conduct a 2^2 design (with a single replicate of each factorial run) augmented with five center points. The design and the yield data are shown in Fig. 7-20.

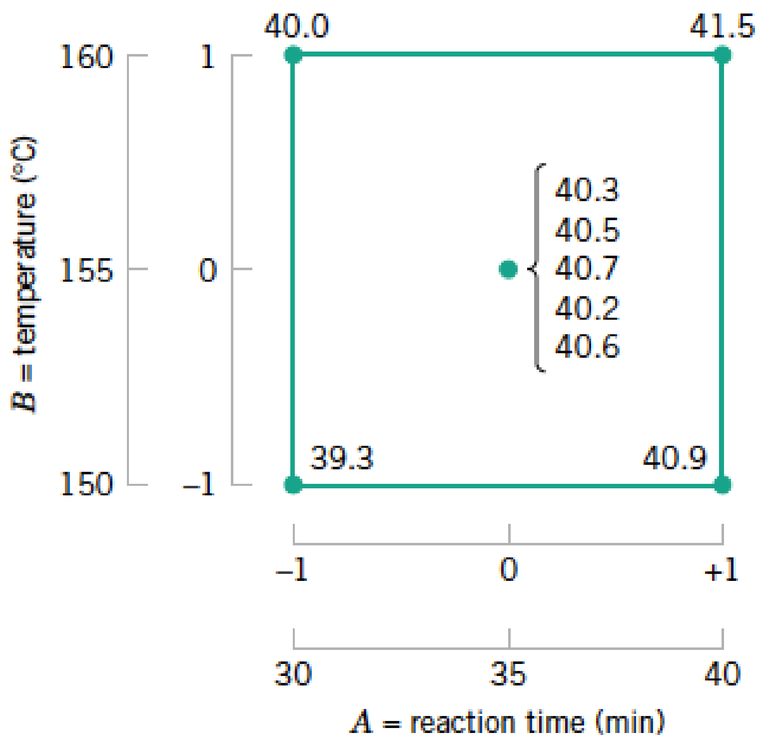

Figure 7-20 The 2^2 design with five center points for Example 7-4.

7-4 Center Points and Blocking in 2^k Designs

EXAMPLE 7-4

Solution. Table 7-14 summarizes the analysis for this experiment. The estimate of pure error is calculated from the center points as follows:

$$\hat{\sigma}^2 = \frac{\sum_{\text{center points}} (y_i - \bar{y}_C)^2}{n_C - 1} = \frac{\sum_{i=1}^{5} (y_i - 40.46)^2}{4} = \frac{0.1720}{4} = 0.0430$$

The average of the points in the factorial portion of the design is $\bar{y}_F = 40.425$, and the average of the points at the center is $\bar{y}_C = 40.46$. The difference $\bar{y}_F - \bar{y}_C = 40.425 - 40.46 = -0.035$ appears to be small. The curvature t-ratio is computed from equation 7-15 as follows:

$$t_{\text{Curvature}} = \frac{\bar{y}_F - \bar{y}_C}{\sqrt{\hat{\sigma}^2\left(\frac{1}{n_F} + \frac{1}{n_C}\right)}} = \frac{-0.035}{\sqrt{0.0430\left(\frac{1}{4} + \frac{1}{5}\right)}} = -0.252$$

The analysis indicates that there is no evidence of curvature in the response over the region of exploration; that is, the null hypothesis H_0: $\Sigma_{J=1}^{2} \beta_{jj} = 0$ cannot be rejected.

7-4 Center Points and Blocking in 2^k Designs

Table 7-14 Analysis for Example 7-4 Process Yield from Minitab

Factorial Design

Full Factorial Design

Factors:	2	Base Design:	2, 4
Runs:	9	Replicates:	1
Blocks:	none	Center pts (total):	5

All terms are free from aliasing

Fractional Factorial Fit

Estimated Effects and Coefficients for y

Term	Effect	Coef	StDev Coef	T	P
Constant		40.4444	0.06231	649.07	0.000
A	1.5500	0.7750	0.09347	8.29	0.000
B	0.6500	0.3250	0.09347	3.48	0.018
A*B	−0.0500	−0.0250	0.09347	−0.27	0.800

Analysis of Variance for y

Source	DF	Seq SS	Adj SS	Adj MS	F	P
Main Effects	2	2.82500	2.82500	1.41250	40.42	0.001
2-Way Interactions	1	0.00250	0.00250	0.00250	0.07	0.800
Residual Error	5	0.17472	0.17472	0.03494		
Curvature	1	0.00272	0.00272	0.00272	0.06	0.814
Pure Error	4	0.17200	0.17200	0.04300		
Total	8	3.00222				

7-4 Center Points and Blocking in 2^k Designs

EXAMPLE 7-4

Table 7-14 displays output from Minitab for this example. The effect of A is $(41.5 + 40.9 - 40.0 - 39.3)/2 = 1.55$, and the other effects are obtained similarly. The pure-error estimate (0.043) agrees with our previous result. Recall from regression modeling that the square of a t-ratio is an F-ratio. Consequently, Minitab uses $0.252^2 = 0.06$ as an F-ratio to obtain an identical test for curvature. The sum of squares for curvature is an intermediate step in the calculation of the F-ratio that equals the square of the t-ratio when the estimate of σ^2 is omitted. That is,

$$SS_{\text{Curvature}} = \frac{(\bar{y}_F - \bar{y}_C)^2}{\dfrac{1}{n_F} + \dfrac{1}{n_C}} \tag{7-18}$$

Furthermore, Minitab adds the sum of squares for curvature and for pure error to obtain the residual sum of squares (0.17472) with 5 degrees of freedom. The residual mean square (0.03494) is a pooled estimate of σ^2, and it is used in the calculation of the t-ratio for the A, B and AB effects. The pooled estimate is close to the pure-error estimate in this example because curvature is negligible. If curvature were significant, the pooling would not be appropriate. The estimate of the intercept β_0 (40.444) is the mean of all nine measurements. ■

7-6 Response Surface Methods and Designs

Response surface methodology, or RSM, is a collection of mathematical and statistical techniques that are useful for modeling and analysis in applications where a response of interest is influenced by several variables and the objective is to **optimize** this response. For example, suppose that a chemical engineer wishes to find the levels of temperature (x_1) and feed concentration (x_2) that maximize the yield (y) of a process. The process yield is a function of the levels of temperature and feed concentration—say,

$$Y = f(x_1, x_2) + \epsilon \tag{7-19}$$

where ϵ represents the noise or error observed in the response Y. If we denote the expected response by $E(Y) = f(x_1, x_2)$, the surface represented by $f(x_1, x_2)$ is called a **response surface.**

7-6 Response Surface Methods and Designs

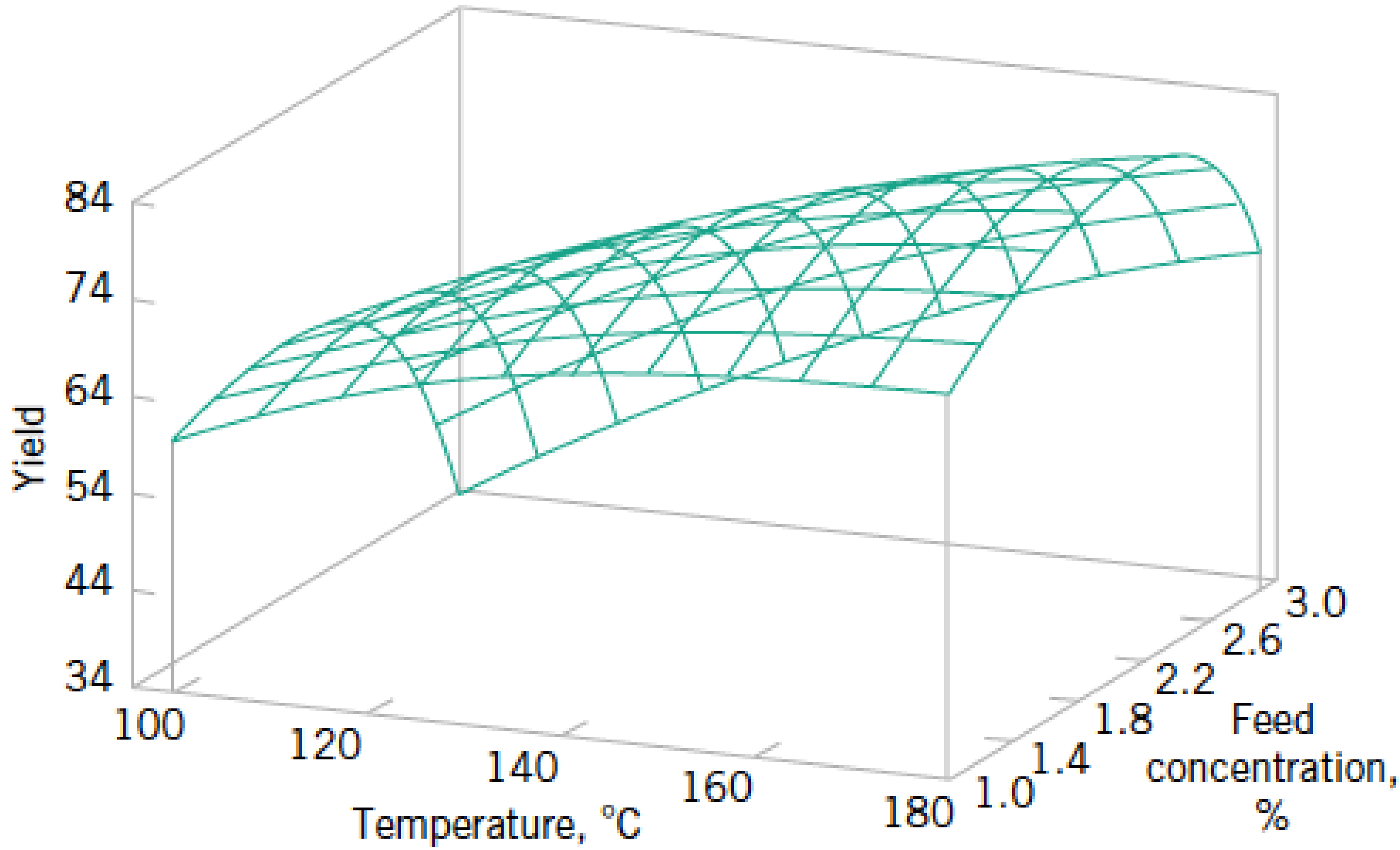

Figure 7-34 A three-dimensional response surface showing the expected yield as a function of temperature and feed concentration.

7-6 Response Surface Methods and Designs

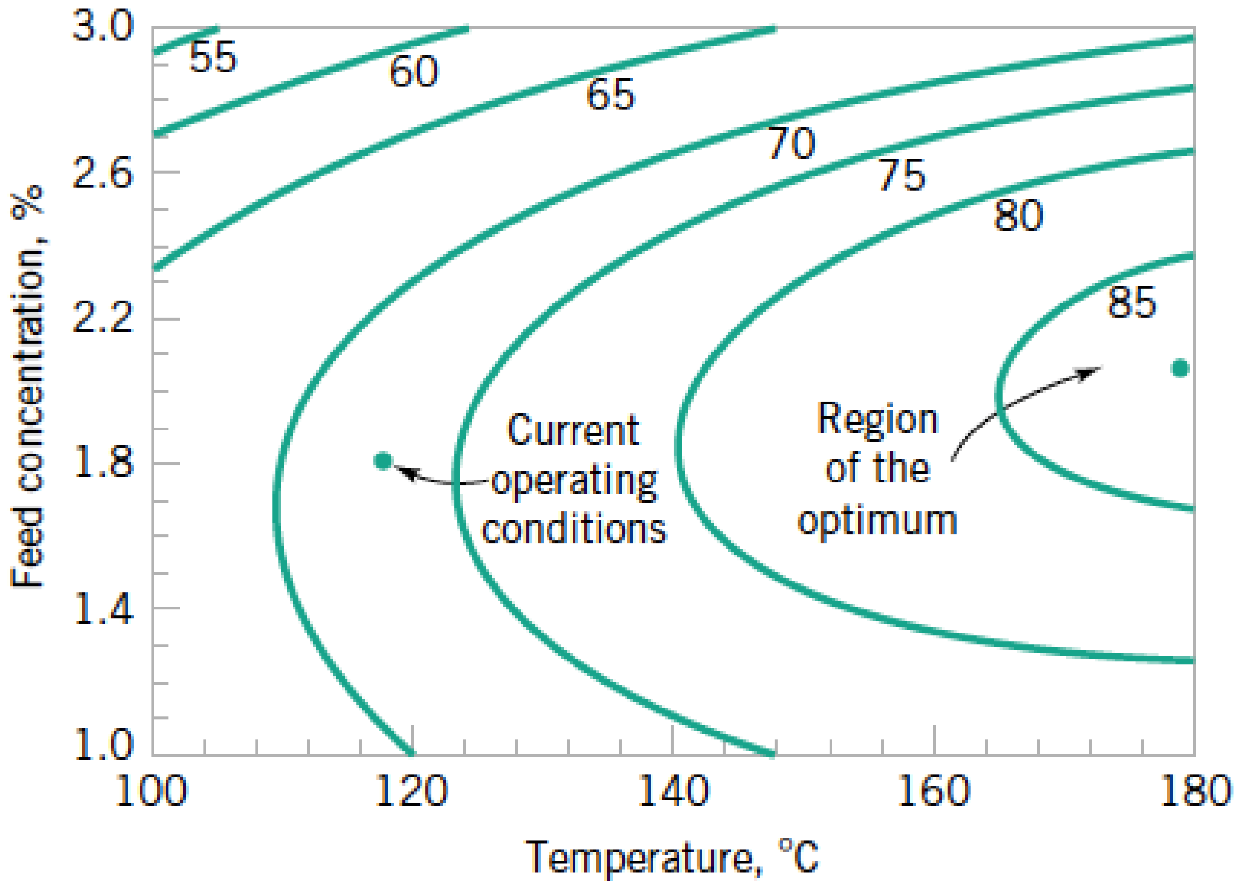

Figure 7-35 A contour plot of the yield response surface in Figure 7-34.

7-6 Response Surface Methods and Designs

First-order model:

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k + \epsilon$$

Second-order model:

$$Y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \sum_{i<j}\sum \beta_{ij} x_i x_j + \epsilon$$

7-6 Response Surface Methods and Designs

7-6.1 Method of Steepest Ascent

The **method of steepest ascent** is a procedure for moving sequentially along the path of steepest ascent—that is, in the direction of the maximum increase in the response. Of course, if **minimization** is desired, we are talking about the **method of steepest descent.** The fitted first-order model is

$$\hat{y} = \hat{\beta}_0 + \sum_{i=1}^{k} \hat{\beta}_i x_i \qquad (7\text{-}21)$$

7-6 Response Surface Methods and Designs

7-6.1 Method of Steepest Ascent

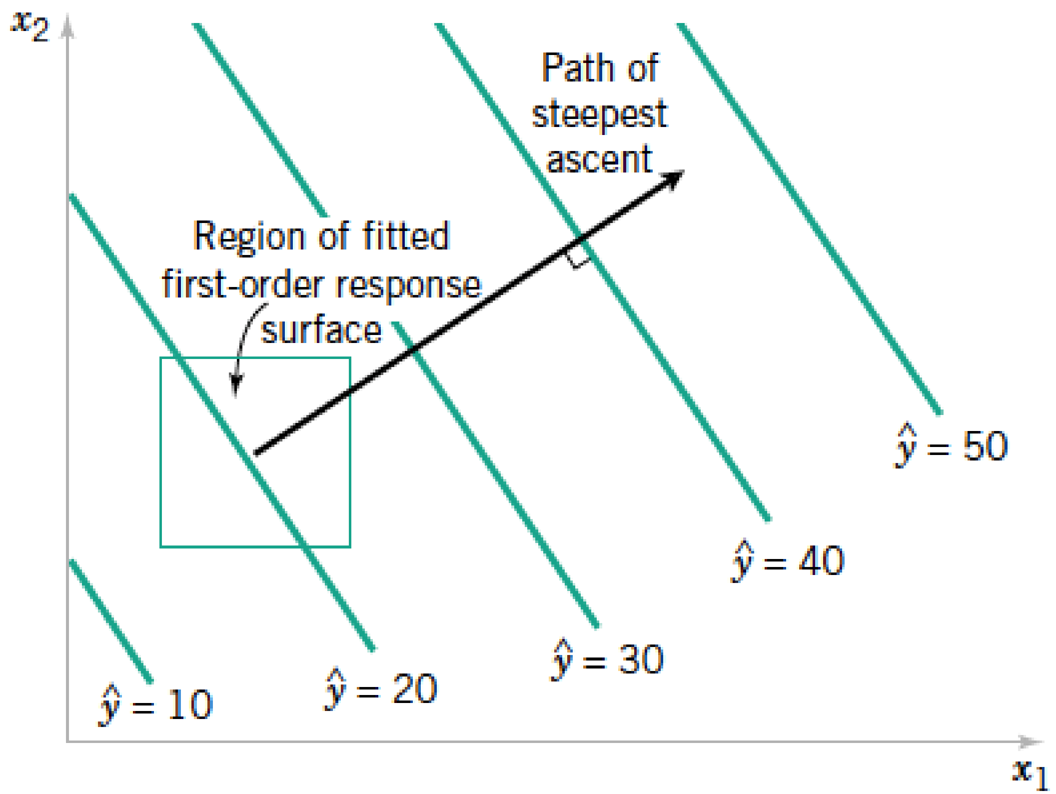

Figure 7-36 First-order response surface and path of steepest ascent.

7-6 Response Surface Methods and Designs

7-6.1 Method of Steepest Ascent

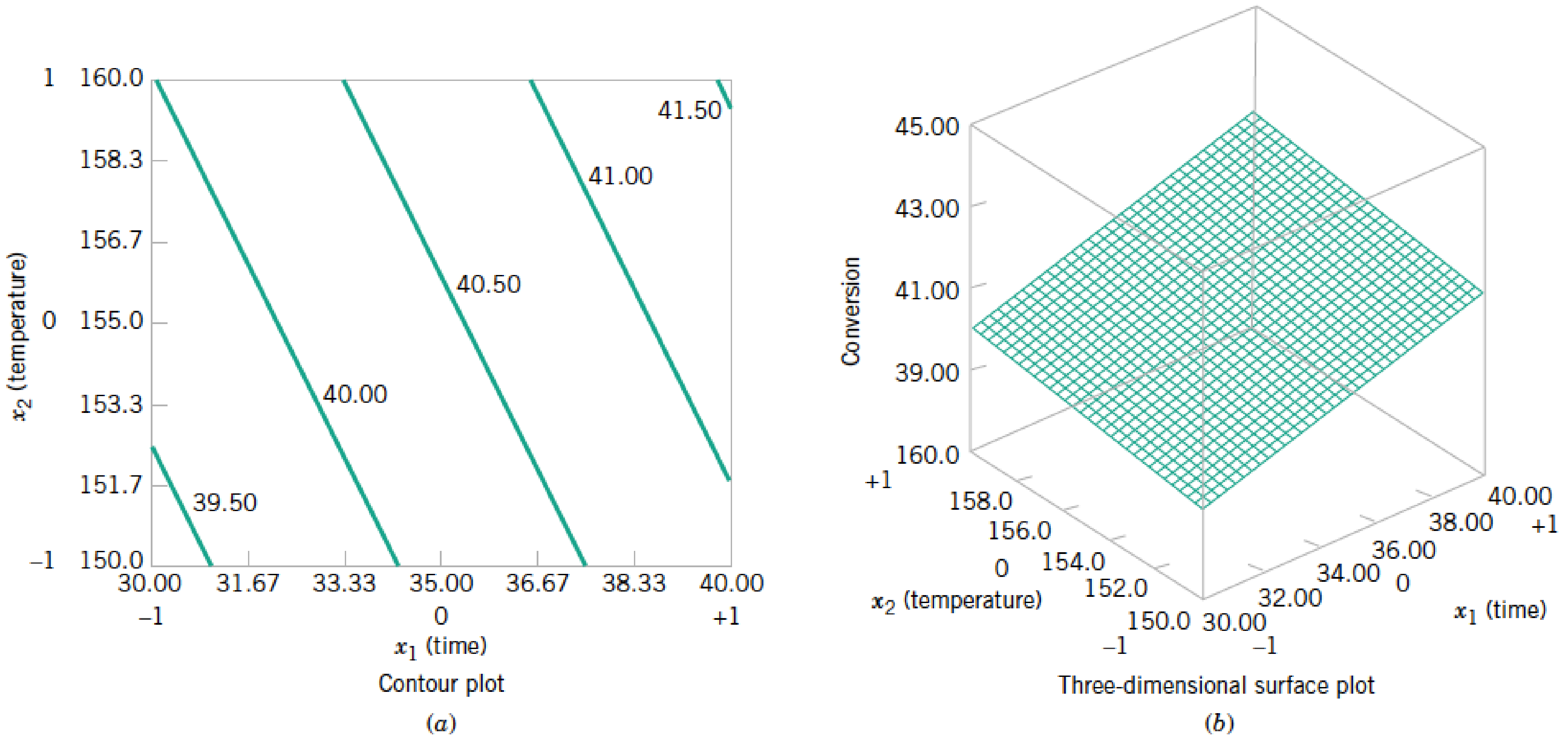

Figure 7-37 Response surface plots for the first-order model of reaction time and temperature. (a) Contour plot. (b) Three-dimensional surface plot.

7-6 Response Surface Methods and Designs

7-6.1 Method of Steepest Ascent

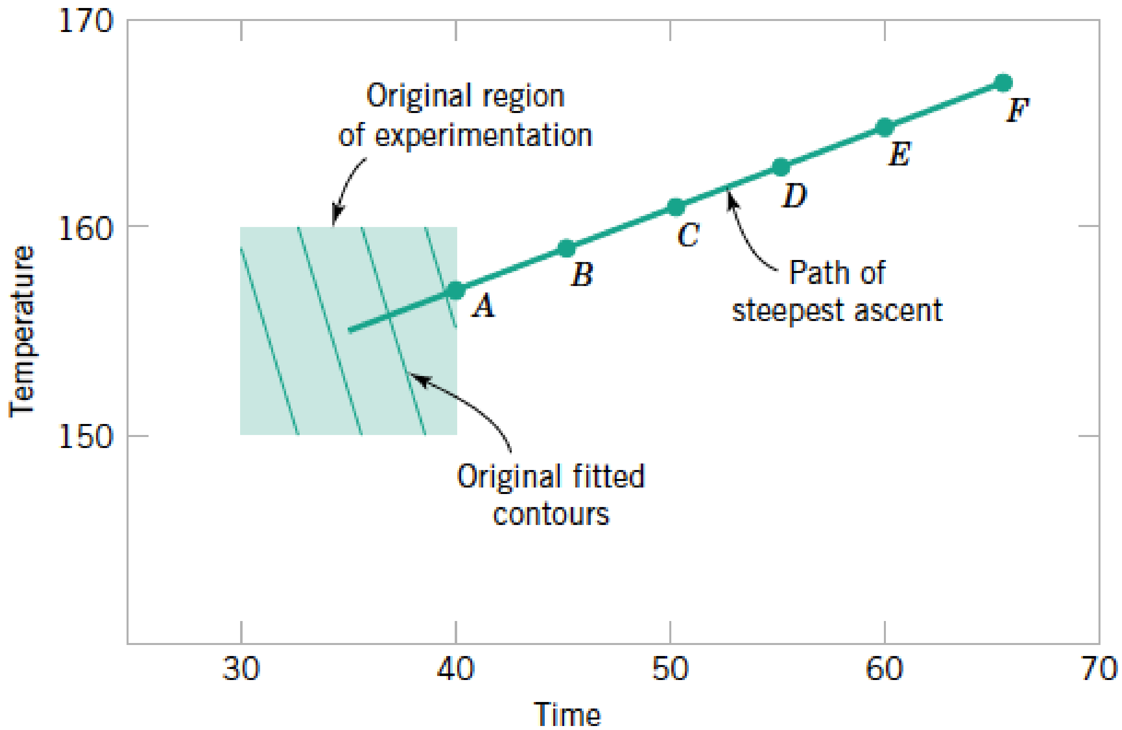

Figure 7-38 Steepest ascent experiment for the first-order model of reaction time and temperature.

Point A: 40 minutes, 157°F, $y = 40.5$
Point B: 45 minutes, 159°F, $y = 51.3$
Point C: 50 minutes, 161°F, $y = 59.6$
Point D: 55 minutes, 163°F, $y = 67.1$
Point E: 60 minutes, 165°F, $y = 63.6$
Point F: 65 minutes, 167°F, $y = 60.7$

IMPORTANT TERMS AND CONCEPTS

- 2^k factorial design
- 2^{k-p} fractional factorial design
- Aliases
- Blocking and confounding
- Center points in a 2^k factorial design
- Central composite design
- Contour plot
- Factorial design
- First-order model
- Interaction
- Interaction plot
- Main effect of a factor
- Method of steepest ascent
- Normal probability plot of effects
- Optimization experiments
- Regression model
- Residual analysis
- Residuals
- Response surface
- Screening experiment
- Second-order model
- Sequential experimentation