Sample from a given probability distribution

1. Know the probability distribution of x. Say it is $e^{-\lambda x}$. Say $\lambda = 1$ $\therefore$ it is e^{-x}.

2. Obtain its cumulative distribution:

$$F(x) = \int_0^x e^{-x'} dx' = 1 - e^{-x}.$$

3. Obtain a random number in the range [0 1]. Say it is ξ.

4. Equate $F(x) = \xi$ and solve for x.

$$\xi = 1 - e^{-x}$$

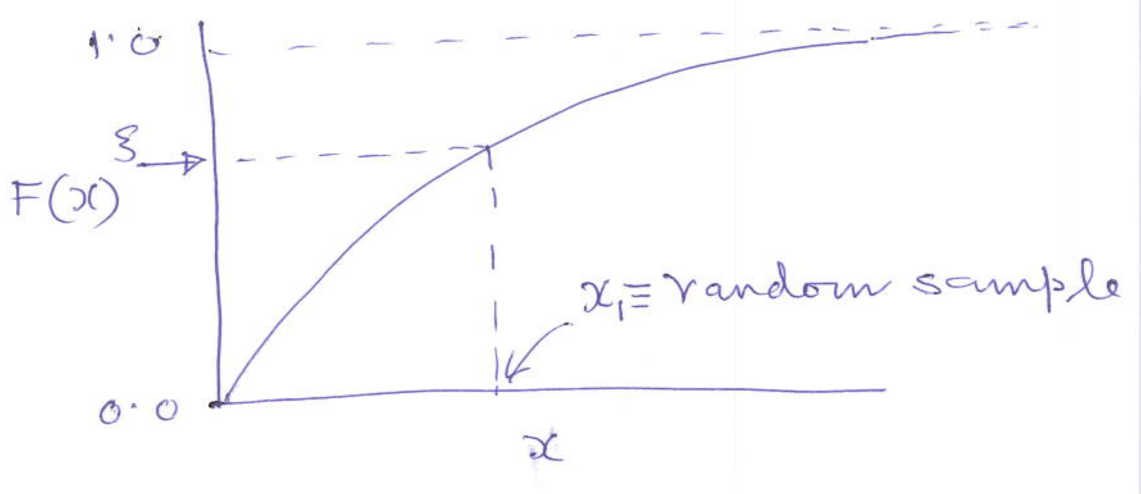
$$e^{-x} = (1 - \xi)$$

$$-x = \ln(1 - \xi)$$

$$x = -\ln(1 - \xi)$$

5. This value is your random sample.

geometrically:



(2). Central limit theorem: Example:

The distribution of $\bar{x}$ is normal with

$\mu = 20$ and $\sigma = \frac{0.5}{\sqrt{40}} = 0.079$

we are looking for $P(\bar{x} > 20.1)$

$$= P(20.1 \leq \bar{x} \leq \infty)$$

$$= P\left(\frac{20.1 - 20}{.079} \leq z \leq \infty\right)$$

$$= P(1.265 \leq z \leq \infty)$$

$$= \Phi(\infty) - \Phi(1.265)$$

$$= 1 - 0.896 = \underline{0.104}.$$

Linear function of dependent random variable:

$$Y = X_1 + X_2$$

$$E(X_1) = \mu_1 \quad E(X_2) = \mu_2 \quad V(X_1) = \sigma_1^2 \quad V(X_2) = \sigma_2^2$$

$$E(Y) = E(X_1) + E(X_2) = \mu_1 + \mu_2$$

$V(Y) = ?$

$$V(X) = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)\, dx$$

$$= \int_{-\infty}^{+\infty} (x^2 + \mu^2 - 2\mu x) f(x)\, dx$$

$$= \int_{-\infty}^{+\infty} x^2 f(x)\, dx + \mu^2 \int_{-\infty}^{+\infty} f(x)\, dx - 2\mu \int_{-\infty}^{+\infty} x f(x)\, dx$$

$$= E(X^2) + \mu^2 - 2\mu \cdot \mu = \underline{E(X^2) - \mu^2}$$

$$V(Y) = E\left[(X_1 + X_2)^2\right] - [\mu_1 + \mu_2]^2$$

$$= E\left[X_1^2 + X_2^2 + 2X_1X_2\right] - (\mu_1^2 + \mu_2^2 + 2\mu_1\mu_2)$$

$$= E(X_1^2) + E(X_2^2) + 2E(X_1X_2) - \mu_1^2 - \mu_2^2 - 2\mu_1\mu_2$$

$$= V(X_1) + V(X_2) + 2\left[E(X_1X_2) - \mu_1\mu_2\right]$$

If x_1 & X_2 are independent, the corresponding formulae will apply to the above eqn. In that case $E(X_1 X_2)$ is $\mu_1 \mu_2$ and the formulae reduces to that of independent random variables.

Hence, $[E(X_1 X_2) - \mu_1 \mu_2]$ is a measure of dependance. This is called "covariance"

... is called covariance.

Remember the expression

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$$S_{xy} = \sum x_i \cdot y_i - \frac{\sum x_i \sum y_i}{n}$$

$$= n\left[\frac{\sum x_i y_i}{n} - \frac{\sum x_i}{n} \cdot \frac{\sum y_i}{n}\right]$$

$$= \overline{x_i y_i} - \bar{x}_i \cdot \bar{y}_i$$

May be viewed as: $E(x_i y_i) - E(x_i)E(y_i)$
(not exactly).

Similar way, the correlation coefficient (previously it was sample correlation coefficient because it was based on a finite sample) is defined as:

$$\rho_{x_1,x_2} = \frac{E(x_1 x_2) - E(x_1)E(x_2)}{\sqrt{\sigma_1^2 \sigma_2^2}}$$

$$-1 \leq \rho_{x_1,x_2} \leq +1$$

$$E\left[\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}\right] = \frac{1}{n-1}\,E\left[\sum_{i=1}^{n}(x_i-\bar{x})^2\right]$$

$$= \frac{1}{n-1}\,E\left[\sum_{i=1}^{n}\left(x_i^2+\bar{x}^2-2x_i\bar{x}\right)\right]$$

$$= \frac{1}{n-1}\,E\left[\sum x_i^2+\bar{x}^2\sum-2\bar{x}\sum x_i\right]$$

$$= \frac{1}{n-1}\,E\left[\sum x_i^2+n\bar{x}^2-2\bar{x}\cdot n\bar{x}\right]$$

$$= \frac{1}{n-1}\,E\left[\sum x_i^2-n\bar{x}^2\right]$$

$$= \frac{1}{n-1}\left[\sum E(x_i^2)-n\,E(\bar{x}^2)\right]$$

$$E\left[(\bar{x}-\mu)^2\right]$$

$$= E\left[\bar{x}^2+\mu^2-2\bar{x}\mu\right]$$

$$= E(\bar{x}^2)+E(\mu^2)-2\mu E(\bar{x})$$

$$= E(\bar{x}^2)+\mu^2-2\mu E(\bar{x})+\left[E(\bar{x})\right]^2-\left[E(\bar{x})\right]^2$$

$$= \left[E(\bar{x}^2)-\{E(\bar{x})\}^2\right]+\underbrace{\left[\mu-E(\bar{x})\right]^2}_{\text{bias.}}$$

$$= \underbrace{E(\bar{x}^2)-\{E(\bar{x})\}^2}_{\text{Variance.}}$$

10 Samples:

12·8, 9·4, 8·7, 11·6, 13·1, 9·8, 14·1, 8·5
12·1, 10·3

Arrange in order:

8·5, 8·7, 9·4, 9·8, 10·3, 11·6, 12·1, 12·8,
13·1, 14·1.

a) Sample mean: $\bar{x}$: $E(\bar{x}) = \mu$.

b) Median: $\frac{1}{2}(5^{th} + 6^{th}$ value).

$$E(\text{Median}) = \frac{1}{2}E(5^{th}) + \frac{1}{2}E(6^{th}) = \mu.$$

c) Randomly selected value

E(") = μ.

All are unbiased estimators.

Variance:

a) Sample mean: $\frac{\sigma^2}{n} = \frac{\sigma^2}{10}$

b) Median.

$$V(\text{median}) = V\left\{\frac{1}{2}(5^{th}\text{ value}) + \frac{1}{2}X_6\right\}$$

$$= \frac{1}{4}\sigma_5^2 + \frac{1}{4}\sigma_6^2$$

$$= \frac{2\sigma^2}{4} = \frac{\sigma^2}{2} = \frac{\sigma^2}{2}$$

c) Randomly selected value: σ^2.

$$E(\hat{\theta}_1) = \mu$$

$$E(\hat{\theta}_2) = \frac{3}{2}\mu - \frac{1}{2}\mu + \mu = 2\mu.$$

$$V(\hat{\theta}_1) = \sigma^2/9$$

$$V(\hat{\theta}_2) = \left(\frac{9}{4} + \frac{1}{4} + \frac{4}{4}\right)\sigma^2 = \frac{7}{2}\sigma^2 = 3.5\sigma^2$$

$$MSE(\hat{\theta}_1) = \frac{\sigma^2}{9} + 0 = \sigma^2/9$$

$$MSE(\hat{\theta}_2) = 3.5\sigma^2 + 4\mu^2$$

$$E(x_i^2) = \int_{-\infty}^{+\infty} x_i^2 \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[\frac{-(x_i-\mu)^2}{2\sigma^2}\right] dx_i$$

Substitute: $\frac{x_i-\mu}{\sigma} = z \qquad x_i = \sigma z + \mu.$

$$dx_i = \sigma\, dz$$

$$= \int_{-\infty}^{+\infty} (\sigma z+\mu)^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz$$

$$= \int_{-\infty}^{+\infty} (\sigma^2 z^2 + \mu^2 + 2\mu\sigma z) \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)\, dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} z^2 \exp(-z^2/2)\, dz + \frac{\mu^2}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-z^2/2)\, dz$$

$$+ \frac{2\mu\sigma}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} z \exp(-z^2/2)\, dz$$

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-z^2/2)\, dz = 1.0$ (Integration of std. gaussian)

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} z \exp(-z^2/2)\, dz = 0.0$$

$$\int_{-\infty}^{+\infty} z^2 \exp(-z^2/2)\, dz = \sqrt{2\pi}$$

Substituting:

$$E(x_i^2) = \frac{\sigma^2}{\sqrt{2\pi}} \cdot \sqrt{2\pi} + \mu^2 = \sigma^2 + \mu^2$$

similarly:

$$E(\bar{x}^2) = \int_{-\infty}^{+\infty} (\bar{x})^2 \frac{1}{\sqrt{2\pi}\,(\sigma/\sqrt{n})} \exp\left(-\frac{(\bar{x}-\mu)^2}{2\,[\sigma^2/n]}\right) d\bar{x}$$

Remember: if x_i has normal distribution with μ & σ; $\bar{x}$ is normal with μ & σ^2/n

Substitute:

$$\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = z \qquad \frac{d\bar{x}}{\sigma/\sqrt{n}} = dz$$

$$\bar{x} = \left[z\frac{\sigma}{\sqrt{n}} + \mu\right]$$

$$\therefore E(\bar{x}^2) = \int_{-\infty}^{+\infty} \left(\frac{z^2\sigma^2}{n} + \mu^2 + \frac{2z\sigma\mu}{\sqrt{n}}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-z^2/2\right) dz$$

$$= \frac{\sigma^2}{n} \cdot \left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{+\infty} z^2 \exp\left(-z^2/2\right) dz + \mu^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-z^2/2\right) dz$$

$$+ \frac{2\mu\sigma}{\sqrt{n}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} z \exp\left(-z^2/2\right) dz$$

$$= \frac{\sigma^2}{n} + \mu^2$$