Chapter 8 Numerical Methods

1 lecture (§8.1 – 8.2)

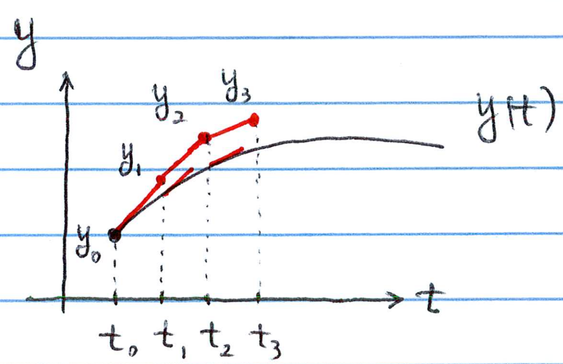
§8.1 The Euler or Tangent Line Method

IVP
$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

forward/explicit Euler

$$\left.\begin{aligned} y'(t_n) &= f(t_n, y(t_n)) \\ y'(t_n) &\approx \frac{y(t_n+h) - y(t_n)}{h} \end{aligned}\right\} \Longrightarrow y(t_n+h) \approx y(t_n) + h f(t_n, y(t_n))$$

$$\boxed{y_{n+1} = y_n + h f(t_n, y_n)} \quad \text{for } n = 0, 1, 2, \cdots$$



example 1
$$\begin{cases} y' = 1 - t + 4y \\ y(0) = 1 \end{cases}$$

the exact solution $y(t) = \frac{1}{4}t - \frac{3}{16} + \frac{19}{16}e^{4t}$

Euler method $y_{n+1} = y_n + h[1 - t_n + 4y_n]$

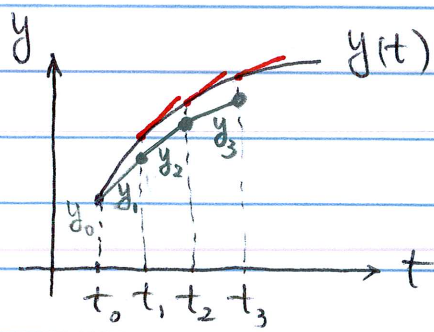
Table 8.1.1 $h = 0.05,\ 0.025,\ 0.01,\ 0.001$

backward/implicit Euler

$$y'(t_{n+1}) = f(t_{n+1}, y(t_{n+1}))$$

$$y'(t_{n+1}) \approx \frac{y(t_{n+1}) - y(t_n)}{h} \implies y(t_{n+1}) \approx y(t_n) + h f(t_{n+1}, y(t_{n+1}))$$

$$\boxed{y_{n+1} = y_n + h f(t_{n+1}, y_{n+1})} \quad \text{for } n = 0, 1, \ldots$$



example 2
$$\begin{cases} y' = 1 - t + 4y \\ y(0) = 1 \end{cases}$$

$$y_{n+1} = y_n + h\left[1 - t_{n+1} + 4y_{n+1}\right]$$

$$\implies y_{n+1} = \left\{ y_n + h(1 - t_{n+1}) \right\} \Big/ (1 - 4h)$$

Table 8.1.2 $\quad h = 0.05,\ 0.025,\ 0.01,\ 0.001$

§8.2 Improved Euler Method

trapezoid

$$y'(t)=f(t,y(t)) \Longrightarrow y(t_{n+1})=y(t_n)+\int_{t_n}^{t_{n+1}} f(t,y(t))\,dt$$

$$\approx y(t_n)+\frac{h}{2}\left[f(t_n,y(t_n))+f(t_{n+1},y(t_{n+1}))\right]$$

$$\Longrightarrow y_{n+1}=y_n+\frac{h}{2}\left[f(t_n,y_n)+f(t_{n+1},y_{n+1})\right]$$

$$=\frac{1}{2}\underbrace{\left[y_n+hf(t_n,y_n)\right]}_{\text{explicit Euler}}+\frac{1}{2}\underbrace{\left[y_n+hf(t_{n+1},y_{n+1})\right]}_{\text{implicit Euler}}$$

implicit $\Longleftrightarrow$ solving (possibly nonlinear) algebraic eq.

improved Euler

$$y_{n+1}=y_n+\frac{h}{2}\left[f(t_n,y_n)+f(t_n+h,\ y_n+hf(t_n,y_n))\right]$$

or

$$\begin{cases} \bar{y}_{n+1}=y_n+hf(t_n,y_n) & \text{predictor} \\ y_{n+1}=y_n+\frac{h}{2}\left[f(t_n,y_n)+f(t_n+h,\bar{y}_{n+1})\right] & \text{corrector} \end{cases}$$

example $\begin{cases} y'=1-t+4y \\ y(0)=1 \end{cases}$

Table 8.2.1