Chapter 10 PDEs and Fourier Series

- ODE eigenvalue problem (§10.1)
- Fourier Series (§10.2 – 10.4)
- seperation of variables (§10.5 – 10.8)

§10.1 Two-Pts Boundary Value Problems

initial-value problem $\begin{cases} y'' + p(t)\,y' + q(t)\,y = g(t) \\ y(t_0) = y_0, \quad y'(t_0) = y_0' \end{cases} \xrightarrow[\text{assumption}]{\text{mild}}$ unique solution

two-pts boundary value prob $\begin{cases} y'' + p(t)\,y' + q(t)\,y = g(t) \\ y(\alpha) = y_0, \quad y(\beta) = y_1 \end{cases} \Longrightarrow \begin{cases} \text{unique solution} \\ \text{no solution} \\ \text{infinite many solutions} \end{cases}$

homogeneous $y_0 = y_1 = 0, \quad g(t) = 0$

Linear Algebra

- $Ax = b$ has a unique solution $\Longleftrightarrow$ $Ax = 0$ has only trivial solution $x = 0$
- $Ax = b$ has either no solution or infinite many solutions $\Longleftrightarrow$ $Ax = 0$ has nonzero solutions.

Examples 1. $\begin{cases} y'' + 2y = 0 \\ y(0) = 1, \ y(\pi) = 0 \end{cases}$

$0 = r^2 + 2 \implies r = \pm\sqrt{2}\, i \implies y = c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x)$

$\begin{cases} 1 = y(0) = c_1 \\ 0 = y(\pi) = c_1 \cos(\sqrt{2}\pi) + c_2 \sin(\sqrt{2}\pi) \end{cases} \implies y = \cos(\sqrt{2}x) - \cot(\sqrt{2}\pi)\sin(\sqrt{2}x)$

unique solution

2. $\begin{cases} y'' + y = 0 \\ y(0) = 1, \ y(\pi) = a \end{cases}$

$0 = r^2 + 1 \implies r = \pm i \implies y = c_1 \cos x + c_2 \sin x$

$\begin{cases} 1 = y(0) = c_1 \\ a = y(\pi) = -c_1 \end{cases} \implies \begin{cases} a \neq -1 \implies \text{no solution} \\ a = -1 \implies \text{many solutions} \quad y = \cos x + c_2 \sin x \end{cases}$

3. $\begin{cases} y'' + 2y = 0 \\ y(0) = y(\pi) = 0 \end{cases}$ $\qquad y = c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x)$

$\begin{cases} 0 = y(0) = c_1 \\ 0 = y(\pi) = c_1 \cos(\sqrt{2}\pi) + c_2 \sin(\sqrt{2}\pi) \end{cases} \implies y(x) \equiv 0$

4. $\begin{cases} y'' + y = 0 \\ y(0) = y(\pi) = 0 \end{cases}$ $\qquad y = c_1 \cos x + c_2 \sin x$

$\begin{cases} 0 = y(0) = c_1 \\ 0 = y(\pi) = -c_1 \end{cases} \implies y = c_2 \sin x \qquad$ many solutions

Eigenvalue Problem

$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = y(\pi) = 0 \end{cases}$$

$\underline{\lambda = 2}$ $\quad y = 0$

$\underline{\lambda = 1}$ $\quad y = c_2 \sin x$ — eigenfunction, $\quad \lambda = 1$ — eigenvalue

$\underline{\text{Case } \lambda > 0}$ $\quad$ Let $\lambda = \mu^2$ with $\mu > 0$

$0 = r^2 + \mu^2 \Longrightarrow \lambda = \pm \mu i \Longrightarrow y = c_1 \cos(\mu x) + c_2 \sin(\mu x)$

$$\begin{cases} 0 = y(0) = c_1 \\ 0 = y(\pi) = c_2 \sin(\mu \pi) \Longrightarrow \sin(\mu\pi) = 0 \Longrightarrow \mu = 1, 2, 3, \cdots \end{cases}$$

$$\boxed{\begin{array}{ll} \text{eigenvalues} & \underline{\lambda = \mu^2}: \ \lambda_1 = 1, \ \lambda_2 = 2^2, \cdots, \lambda_n = n^2, \cdots \\ \text{eigenfunctions} & \underline{y = \sin(\mu x)}: \ y_1 = \sin x, \ y_2 = \sin(2x), \cdots, y_n = \sin(nx), \cdots \end{array}}$$

$\underline{\text{Case } \lambda < 0}$ $\quad$ Let $\lambda = -\mu^2$

$\Longrightarrow y = c_1 \cosh(\mu x) + c_2 \sinh(\mu x) \Longrightarrow y(x) \equiv 0$ $\quad$ no nagative eigenvalues

$\underline{\text{Case } \lambda = 0}$ $\Longrightarrow y = c_1 x + c_2 \Longrightarrow y = 0 \Longrightarrow \lambda = 0$ is not an eigenvalue

$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = y(L) = 0 \end{cases}$$

$$\boxed{\begin{array}{lll} \text{eigenvalues} & \lambda_n = \dfrac{n^2 \pi^2}{L^2} & \text{for } n = 1, 2, \cdots \\ \text{eigenfunctions} & y_n(x) = \sin\left(\dfrac{n\pi x}{L}\right) & \end{array}}$$

§10.2 Fourier Series

F-series $\qquad \dfrac{a_0}{2} + \displaystyle\sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right)$

periodicity of the Sine and Cosine function

- $f(x)$ is periodic with period $T>0 \iff \begin{cases} x \in \mathrm{dom}(f) \Rightarrow x+T \in \mathrm{dom}(f) \\ f(x+T) = f(x) \end{cases}$

- fundamental period $= \min \{ T \mid T \text{ is period of } f \}$

properties (1) $f(x+T) = f(x) \Longrightarrow f(x+kT) = f(x) \quad \forall k$ integer

(2) f and g have the common period $T \Longrightarrow$ so is $c_1 f(x) + c_2 g(x)$

example $\quad \sin\left(\dfrac{m\pi x}{L}\right), \cos\left(\dfrac{m\pi x}{L}\right)$ for $m = 1, 2, \cdots$

$$\sin\left(\frac{m\pi x}{L}\right) = \sin\left(\frac{m\pi x}{L} + 2\pi\right) = \sin \frac{m\pi}{L}\left(x + \frac{2L}{m}\right) \Longrightarrow T = \frac{2L}{m}$$

orthogonality of the Sine and Cosine functions

- inner product on $[\alpha, \beta]$: $\quad (u, v) = \displaystyle\int_\alpha^\beta u(x)\, v(x)\, dx$

- $u \perp v \iff (u, v) = 0$

- A set of functions are mutually orthogonal $\iff$ each distinct pair is orthogonal

example $\sin\left(\frac{m\pi x}{L}\right)$, $\cos\left(\frac{m\pi x}{L}\right)$, $m=1,2,\cdots$ are mutually orth. on $[-L, L]$

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \begin{cases} 0 & m\neq n \\ L & m=n \end{cases} = \int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx$$

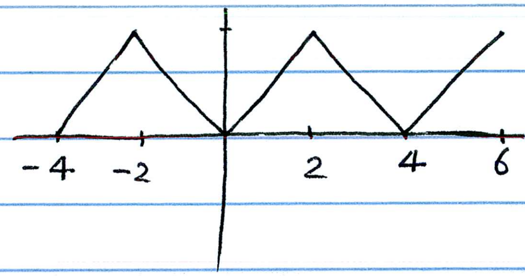
$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = 0$$

Euler-Fourier Fourmula $$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m\cos\frac{m\pi x}{L} + b_m\sin\frac{m\pi x}{L}\right)$$

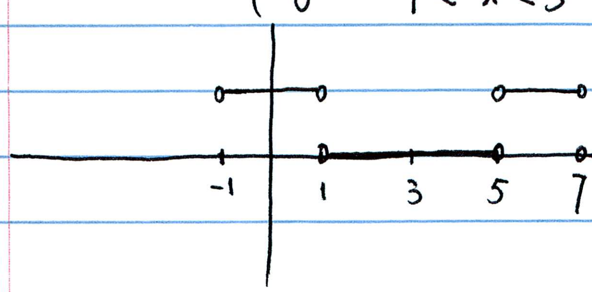
$$\Longrightarrow\quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \quad \text{for } n=0,1,2,\cdots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \quad \text{for } n=1,2,\cdots$$

examples 1. $f(x) = \begin{cases} -x & -2\leq x<0 \\ x & 0\leq x<2 \end{cases}$, $f(x+4)=f(x)$



2. $f(x) = \begin{cases} 0 & -3<x<-1 \\ 1 & -1<x<1 \\ 0 & 1<x<3 \end{cases}$ $f(x+6)=f(x)$



§10.3 The Fourier Convergence Thrm

- $f(x)$ is piecewise continuous on $[a,b] = \bigcup_{i=1}^{n} [x_{i-1}, x_i]$ $\quad f \in PC[a,b]$

$\iff$ $\begin{cases} (1)\ f \text{ is continuous on } (x_{i-1}, x_i) \quad f \in C(x_{i-1}, x_i) \\ (2)\ \lim_{x \to x_{i-1}^+} f(x) \text{ and } \lim_{x \to x_i^-} f(x) \text{ are finite} \end{cases}$

Thrm Assume that (1) $f, f' \in PC[-L, L]$

(2) $f(x+2L) = f(x)$

$\Longrightarrow$ (1) $f(x)$ has a F-series: $\dfrac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \dfrac{m\pi x}{L} + b_m \sin \dfrac{m\pi x}{L} \right)$

with a_m, b_m given by Euler-Fourier formulas

(2) F-series $\longrightarrow \begin{cases} f(x) & \text{at continuous pts} \\ \dfrac{f(x^-)+f(x^+)}{2} & \text{at discontinuous pts} \end{cases}$

Remark functions are not included

(1) infinite discont., e.g., $\dfrac{1}{x^2} \to \infty$ as $x \to 0$; $\ln|x-L| \to \infty$ as $x \to L$

(2) infinite number of jump discont.

examples 1. $f(x) = \begin{cases} 0 & -L < x < 0 \\ L & 0 < x < L \end{cases}$ $\quad f(x+2L) = f(x)$

- compute F-series
- graph the function
- no #3 of webassign HW

F-series to solve IVP with periodic forcing terms

#13 (p612)

$$\begin{cases} y'' + w^2 y = \sin nt \\ y(0) = 0 = y'(0) \end{cases}$$

$$0 = r^2 + w^2 \Rightarrow r = \pm w i \Rightarrow y_c(t) = c_1 \cos wt + c_2 \sin wt$$

Case $w^2 \neq n^2$ $\quad Y(t) = A \sin nt + B \cos nt, \quad Y'' = -n^2 Y$

$$\Rightarrow \sin(nt) = (w^2 - n^2) Y = (w^2 - n^2)(A \sin nt + B \cos nt)$$

$c_1 = 0$

$c_2 = -\frac{n}{w(w^2-n^2)}$

$$\Rightarrow B = 0 \text{ and } A = \frac{1}{w^2 - n^2} \Rightarrow Y(t) = \frac{1}{w^2 - n^2} \sin nt$$

Case $w^2 = n^2$ $\quad Y(t) = t(A \sin nt + B \cos nt)$

$$\Rightarrow A = 0 \text{ and } B = -\frac{1}{2n} \Rightarrow Y(t) = -\frac{1}{2n} t \cos nt$$

#14 (p612)

$$\begin{cases} y'' + w^2 y = \sum_{n=1}^{\infty} b_n \sin nt \\ y(0) = y'(0) = 0 \end{cases}$$

Case $w > 0$ not integer

$$Y(t) = \sum_{n=1}^{\infty} (B_n \sin nt + A_n \cos nt)$$

$$Y''(t) = -\sum_{n=1}^{\infty} n^2 (B_n \sin nt + A_n \cos nt)$$

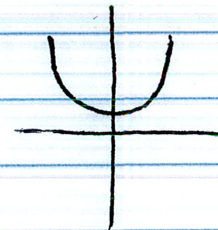
$$\Rightarrow \sum_{n=1}^{\infty} b_n \sin nt = Y'' + w^2 Y = \sum_{n=1}^{\infty} (w^2 - n^2) B_n \sin nt + \sum_{n=1}^{\infty} (1 - n^2) A_n \cos nt$$

$$\Rightarrow A_n = 0 \text{ and } B_n = \frac{b_n}{w^2 - n^2}.$$

§10.4 Even and Odd Functions

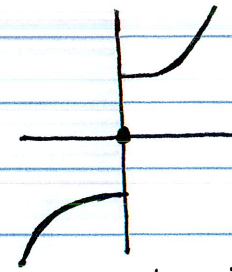
even function $f(-x) = f(x)$

odd function $f(-x) = -f(x)$



sym w.r.t. y-axis

e.g., $1, x^2, x^4, |x|, \cos x$



sym w.r.t. origion

e.g., $x, x^3, \sin x$

properties (1) even function $\pm, \times, /$ even function $=$ even function

(2) odd function $\pm$ odd function $=$ odd function

odd function $\times, /$ odd function $=$ even function

(3) odd function $\pm$ even function $=$ neither even nor odd

odd function $\times, /$ even function $=$ odd function

(4) f is even $\Longrightarrow \int_{-L}^{L} f(x)\,dx = 2\int_0^L f(x)\,dx$

(5) f is odd $\Longrightarrow \int_{-L}^{L} f(x)\,dx = 0$

Cosine series $f, f' \in PC[-L, L)$, $f(-x) = f(x)$, $f(x+2L) = f(x)$

$$\Longrightarrow b_n = 0$$

$$\Longrightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \quad \text{for } n = 0, 1, 2, \cdots$$

<u>Sine series</u> $f, f' \in PC[-L, L)$, $f(-x) = -f(x)$, $f(x+2L) = f(x)$

$$\Rightarrow a_n = 0, \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

<u>examples</u> 1. $f(x) = \begin{cases} x & -L < x < L \\ 0 & x = -L, L \end{cases}$ $\quad f(x+2L) = f(x)$, $\quad f(-x) = -f(x)$

2. $f(x) = \begin{cases} -x & -2 \le x < 0 \\ x & 0 \le x < 2 \end{cases}$ $\quad f(x+4) = f(x)$, $\quad f(-x) = f(x)$

<u>extension of $f(x)$ on $[0, L]$ to $[-L, L]$</u> with $f(x+2L) = f(x)$

- even extension $\quad g(x) = \begin{cases} f(x) & 0 \le x \le L \\ f(-x) & -L \le x \le 0 \end{cases}$

- odd extension $\quad h(x) = \begin{cases} f(x) & 0 < x < L \\ 0 & x = 0, L \\ -f(-x) & -L < x < 0 \end{cases}$

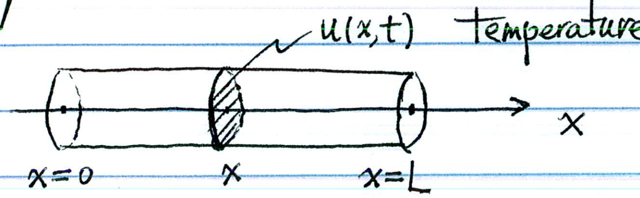
- other extension

<u>Criterion on extensions</u> based on the rapidity of convergence

<u>example</u> $f(x) = \begin{cases} 1-x & 0 < x < 1 \\ 0 & 1 < x \le 2 \end{cases}$ $\quad$ even/odd extensions $\Rightarrow$ cosine/sine series

§10.5 Separation of Variables; Heat Conduction in a Rod

physical phenomena: diffusive, oscillatory, and steady processes
mathematical equations: heat conduction, wave, and potential equations



heat conduction eq. $\quad u_t = \alpha^2 u_{xx}$ for $x \in (0, L)$ and $t > 0$

$$\alpha^2 = \frac{K}{\rho s}$$

K — thermal conductivity

α^2 — the thermal diffusivity

s — specific heat of the material

ρ — density

initial temperature distribution $\quad u(x,0) = f(x)$ for $x \in [0, L]$

boundary conditions $\quad u(0,t) = 0, \; u(L,t) = 0$ for $t > 0$

method of separation of variables

$$u(x,t) = X(x)\,T(t)$$

$$u_t = \alpha u_{xx} \Rightarrow X T' = \alpha^2 X'' T \Longrightarrow \frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = -\lambda \quad \text{constant}$$

$$\Longrightarrow \begin{cases} X'' + \lambda X = 0 \\ T' + \alpha^2 \lambda T = 0 \end{cases}$$

$$\left.\begin{aligned} 0 = u(0,t) = X(0)\,T(t) \\ 0 = u(L,t) = X(L)\,T(t) \end{aligned}\right. \Longrightarrow \left.\begin{aligned} X(0) = 0,\; X(L) = 0 \\ X'' + \lambda X = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} X_n(x) &= \sin\frac{n\pi x}{L} \\ \lambda_n &= \frac{n^2\pi^2}{L^2} \end{aligned}$$

for $n = 1, 2, \cdots$

$$T' + \left(\frac{\alpha n\pi}{L}\right)^2 T = 0 \implies T(t) = c\, e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}$$

$$\implies u_n(x,t) = e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \sin\frac{n\pi x}{L} \quad \text{for } n=1,2,3,\cdots$$

$$\implies \boxed{u(x,t) = \sum_{n=1}^{\infty} c_n\, e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \sin\frac{n\pi x}{L}}$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} \implies \boxed{c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx}$$

example 1 $\quad L = 50\text{ cm}, \quad u(x,0) = f(x) = 20^\circ C, \quad u(0,t) = u(50,t) = 0$

$u(x,t) = ?$

example 2 $\quad$ #3, 4 (p630)

§10.6 Other Heat Conduction Problems

PDE $\quad u_t = \alpha^2 u_{xx}, \quad x \in (0,L), \quad t > 0$

BCs $\quad u(0,t) = 0$ and $u(L,t) = 0$

IC $\quad u(x,0) = f(x)$

$$u(x,t) = \sum_{n=1}^{\infty} c_n\, e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

non-homogeneous BCs $u(0,t) = T_1, \quad u(L,t) = T_2$

as $t \to \infty$, expect $u(x,t) \longrightarrow v(x)$ steady temperature distribution

$$0 = v_{xx}, \quad \cancel{v(0,t)} \; v(0) = T_1 \text{ and } V(L) = T_2$$

$$\Longrightarrow \quad v(x) = T_1 + (T_2 - T_1)\frac{x}{L}.$$

$$u(x,t) = \underbrace{v(x)}_{\text{steady state solution}} + w(x,t) \sim \text{transient part}$$

$$\Longrightarrow \begin{cases} w_t = \alpha w_{xx} \\ w(0,t) = 0, \; w(L,t) = 0 \\ w(x,0) = f(x) - v(x) \end{cases}$$

$$\Longrightarrow \quad u(x,t) = T_1 + (T_2 - T_1)\frac{x}{L} + \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \sin\frac{n\pi x}{L}$$

$$c_n = \frac{2}{L}\int_0^L \left[f(x) - T_1 - (T_2 - T_1)\frac{x}{L}\right] \sin\frac{n\pi x}{L}\,dx$$

example 1 $T_1 = 20, \; T_2 = 50, \; L = 30, \; f(x) = 60 - 2x$

$u(x,t) = ?$

bar with insulated ends

no heat flow $u_x(0,t) = 0, \; u_x(L,t) = 0$ for $t > 0$

$$\begin{cases} X'' + \lambda X = 0 \\ T' + \alpha^2 \lambda T = 0 \end{cases}$$

$$\left.\begin{array}{l} 0 = u_x(0,t) = X'(0)\,T(t) \\ 0 = u_x(L,t) = X'(L)\,T(t) \end{array}\right\} \Rightarrow \quad X'(0) = X'(L) = 0$$

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0, \; X'(L) = 0 \end{cases}$$

<u>case $\lambda < 0$</u> $\quad X(x) = k_1 \sinh(\mu x) + k_2 \cosh(\mu x)$

$$X'(0) = X'(L) = 0 \Longrightarrow k_1 = k_2 = 0 \Longrightarrow X(x) \equiv 0$$

<u>case $\lambda = 0$</u> $\quad X(x) = k_1 x + k_2 \xrightarrow{X'(0)=0} k_1 = 0 \Longrightarrow \begin{cases} \lambda = 0 & \text{eigenvalue} \\ X(x) = 1 & \text{eigenfunction} \end{cases}$

<u>case $\lambda > 0$</u> $\quad$ let $\lambda = \mu^2$ with $\mu > 0$

$$X(x) = k_1 \sin \mu x + k_2 \cos \mu x \xrightarrow[\cancel{X'(L)=0}]{X'(0)=0} k_1 = 0$$

$$\xrightarrow{X'(L)=0} \quad 0 = -k_2 \mu \sin(\mu L) \Longrightarrow \mu L = n\pi \Longrightarrow \mu = \frac{n\pi}{L} \text{ for } n = 1, 2, \cdots$$

$$\begin{cases} \lambda_n = \left(\dfrac{n\pi}{L}\right)^2 & \text{for } n = 1, 2, \cdots & \text{eigenvalue} \\ X_n(x) = \cos\left(\dfrac{n\pi x}{L}\right) & & \text{eigenfunction} \end{cases}$$

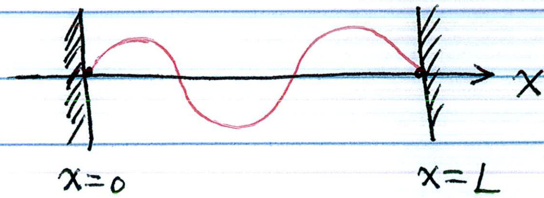
$$\Longrightarrow \quad \begin{aligned} &u_0(x,t) = 1 \\ &u_n(x,t) = e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \cos \frac{n\pi x}{L} \quad \text{for } n = 1, 2, \cdots \end{aligned}$$

$$\Longrightarrow \quad u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right)$$

$$c_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx$$

<u>example 2</u> $\quad L = 25\text{ cm}$, ~~metal ⇒ α~~ $u_x(0,t) = u_x(L,t) = 0$, $f(x) = x$.

§10.7 The Wave Equation: Vibration of an Elastic String



a vibrating string

PDE
BCs
ICs

$$\begin{cases} u_{tt} = a^2 u_{xx} \\ u(0,t)=0,\ u(L,t)=0 \\ u(x,0)=f(x),\ u_t(x,0)=g(x) \end{cases}$$

$a^2 = \dfrac{T}{\rho}$ ~ tension, ~ mass per unit length

compatibility condition $f(0)=f(L)=0,\ g(0)=g(L)=0$

elastic string with nonzero intial displacement $g(x)=0$ and $f(x)\neq 0$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0)=0,\ X(L)=0 \end{cases} \Longrightarrow \begin{array}{l} \lambda_n = \left(\dfrac{n\pi}{L}\right)^2 \\ X_n(x) = \sin\dfrac{n\pi x}{L} \end{array} \quad \text{for } n=1,2,\dots$$

$$\begin{cases} T'' + a^2\lambda T = 0 \\ T'(0)=0 \end{cases} \Longrightarrow T_n(t) = \cos\frac{n\pi a t}{L}$$

$$\Longrightarrow u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$$

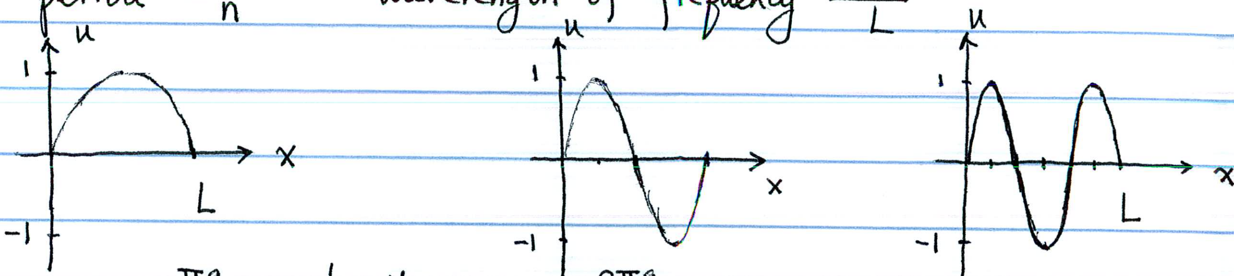
$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

fixed n $\quad \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi a t}{L}\right)$

time t (a vibratory motion)

period $\frac{2L}{na}$, natural frequency $\frac{na}{L}\pi$

space x period $\frac{2L}{n}$ — wavelength of frequency $\frac{na\pi}{L}$



Frequency $= \frac{\pi a}{L}$, wavelength $= 2L$ $\qquad \frac{2\pi a}{L}$, L $\qquad \frac{3\pi a}{L}$, $\frac{2L}{3}$

example 1

$$
\begin{cases}
u_{tt} = 4u_{xx} & (x,t)\in(0,30)\times(0,+\infty) \\
u(0,t) = u(30,t) = 0 \\
u(x,0) = f(x) = \begin{cases} \frac{x}{10} & 0 \le x \le 10 \\ \frac{30-x}{20} & 10 < x \le 30 \end{cases} \\
u_t(x,0) = 0
\end{cases}
$$

elastic string with nonzero initial velocity

$$
\begin{cases}
u_{tt} = a^2 u_{xx} & (x,t)\in(0,L)\times(0,+\infty) \\
u(0,t) = u(L,t) = 0, & t > 0 \\
u(x,0) = 0, \quad u_t(x,0) = g(x), & x\in[0,L]
\end{cases}
$$

$$
\begin{cases}
T'' + \left(\frac{n\pi a}{L}\right)^2 T = 0 \\
T(0) = 0
\end{cases}
\Longrightarrow T_n(t) = \sin\frac{n\pi a t}{L} \Longrightarrow u(x,t) = \sum_{n=1}^{\infty} k_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi a t}{L}\right)
$$

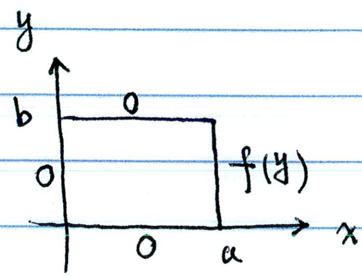
$$
\frac{n\pi a}{L} k_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx
$$

§10.8 Laplace's Equation

$$u_{xx} + u_{yy} = 0$$

Dirichlet problem on a rectangle

$$\begin{cases} u_{xx} + u_{yy} = 0 \text{ in } (0,a)\times(0,b) \\ u(x,0)=0, \ u(x,b)=0 \\ u(0,y)=0, \ u(a,y)=f(y) \end{cases}$$



$$u(x,y) = X(x)\,Y(y) \xRightarrow{u_{xx}+u_{yy}=0} \frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

$$\Longrightarrow \begin{cases} X'' - \lambda X = 0, & X(0)=0 \\ Y'' + \lambda Y = 0, & Y(0)=Y(b)=0 \end{cases} \Longrightarrow \begin{array}{l} \lambda_n = \left(\frac{n\pi}{b}\right)^2 \\ Y_n(y) = \sin\left(\frac{n\pi y}{b}\right) \end{array} \quad n=1,2,\cdots$$

$$u(x,0)=0 \Longrightarrow X(x)\,Y(0)=0 \Longrightarrow Y(0)=0$$

$$u(x,b)=0 \Longrightarrow Y(b)=0, \quad u(0,y)=0 \Longrightarrow X(0)=0$$

$$\begin{cases} X'' - \lambda X = 0 \\ X(0)=0 \end{cases} \Longrightarrow X_n = \sinh\left(\frac{n\pi x}{b}\right)$$

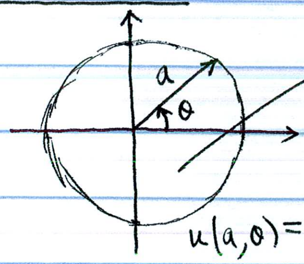
$$u(x,t) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$f(y) = u(a,y) \Longrightarrow c_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b f(y)\sin\left(\frac{n\pi y}{b}\right) dy$$

<u>example 1</u> $a=3, \ b=2, \ f(y) = \begin{cases} y & 0 \le y \le 1 \\ 2-y & 1 \le y \le 2 \end{cases}$

$$\Longrightarrow c_n = \frac{8\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2 \sinh\left(\frac{3n\pi}{2}\right)}$$

Dirichel problem on a circle



$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$

$(r,\theta) \in (0,a) \times [0, 2\pi)$

$u(a,\theta) = f(\theta)$

$$u(r,\theta) = R(r)\,\Theta(\theta)$$

$$\Longrightarrow \quad R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \quad \Longrightarrow \quad r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

$$\begin{cases} r^2R'' + rR' - \lambda R = 0 \\ \Theta'' + \lambda\Theta = 0 \end{cases}$$

(Problem 9) $\xrightarrow{\Theta \text{ is periodic}}$ λ is real

<u>case $\lambda < 0$</u> let $\lambda = -\mu^2$ with $\mu > 0$

$$\Longrightarrow \Theta(\theta) = c_1 e^{\mu\theta} + c_2 e^{-\mu\theta} \xrightarrow{\Theta(\theta) \text{ is periodic}} c_1 = c_2 = 0$$

<u>case $\lambda = 0$</u> $\quad \Theta(\theta) = c_1 + c_2\theta \xrightarrow{\Theta(\theta) \text{ is periodic}} c_2 = 0 \Longrightarrow \Theta(\theta) = c$

$$\xrightarrow{\lambda=0} r^2R'' + rR' = 0 \Longrightarrow R(r) = k_1 + k_2 \ln r$$

$$\xrightarrow{u(r,\theta) \text{ is bounded}} k_2 = 0 \quad \Longrightarrow \quad \begin{cases} \lambda = 0 \\ u_0(r,\theta) = 1 \end{cases}$$

<u>case $\lambda > 0$</u> let $\lambda = \mu^2$ with $\mu > 0$

$$\begin{cases} r^2R'' + rR' - \mu^2 R = 0 \\ \Theta'' + \mu^2\Theta = 0 \end{cases} \Longrightarrow \begin{array}{l} R(r) = k_1 r^{\mu} + k_2 r^{-\mu} \\ \Theta(\theta) = c_1 \sin(\mu\theta) + c_2\cos(\mu\theta) \end{array}$$

$\Theta(\theta + 2\pi) = \Theta(\theta) \Longrightarrow \mu$ is a positive integer n

$R(r)$ is bounded $\Longrightarrow k_2 = 0$

$$f(\theta) = u(a,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} a^n (c_n\cos n\theta + k_n \sin n\theta)$$

$$\Longrightarrow u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n (c_n \cos n\theta + k_n \sin n\theta) \quad \Longrightarrow \begin{cases} a^n c_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta \, d\theta \\ a^n k_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta \, d\theta \end{cases}$$