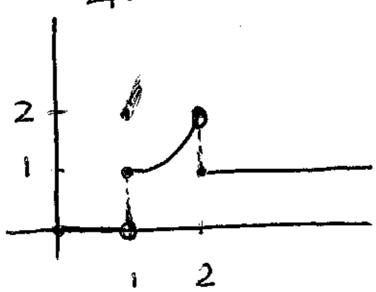
Name: ____________
PUID#: ____________

Midterm 2– Math 303 (03/30/17)
SHOW ALL RELEVANT WORK!!!

1. (16pts) Sketch the graph and find the Laplace transform (using step function) of

$$f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t^2 - 2t + 2, & 1 \le t < 2 \\ 1, & 2 \le t < \infty. \end{cases}$$

4



5

$$f(t) = \left(u_1(t) - u_2(t)\right)(t^2 - 2t + 2) + u_2(t)$$

$$= u_1(t)\left[(t-1)^2 + 1\right] + u_2(t)\left[-t^2 + 2t - 1\right]$$

$$= u_1(t)\left[(t-1)^2 + 1\right] - u_2(t)\left[t-1\right]^2$$

$$= u_1(t)\left[(t-1)^2 + 1\right] - u_2(t)\left[(t-2)^2 + 2(t-2) + 1\right]$$

$$(t-1)^2 = (t-2+1)^2$$
$$= (t-2)^2 + 2(t-2) + 1$$

$$\mathcal{L}\{f(t)\} = e^{-s}\mathcal{L}\{t^2+1\} - e^{-2s}\mathcal{L}\{t^2+2t+1\}$$

7

$$= e^{-s}\left[\frac{2}{s^3} + \frac{1}{s}\right] - e^{-2s}\left[\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right]$$

2. (16pts) (a) Find the Laplace transform of $f(t) = \int_0^t e^{-t+\tau}\cosh(\tau)d\tau$, where $\cosh t$ is given by

$$\cosh t = \left(e^t + e^{-t}\right)/2.$$

and (b) find the inverse Laplace transform of $F(s) = \frac{1}{(s+3)(s^2+3s)}$.

8 (a) $\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-t}\}\,\mathcal{L}\{\cosh t\}$

$$= \frac{1}{s+1} \cdot \frac{s}{s^2-1}$$

8 (b) $F(s) = \frac{1}{s(s+3)^2}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\} = e^{-3t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= e^{-3t} \cdot t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{(s+3)^2}\right\} = \int_0^t \tau e^{-3\tau}d\tau$$

$$\underset{u'=1,\ v=-\frac{1}{3}e^{-3\tau}}{\underline{\underline{u=\tau,\ v'=e^{-3\tau}}}} \ -\frac{1}{3}\tau e^{-3\tau}\Big|_0^t + \frac{1}{3}\int_0^t e^{-3\tau}d\tau$$

$$= -\frac{1}{3}te^{-3t} - \frac{1}{3}\cdot\frac{1}{3}e^{-3\tau}\Big|_0^t$$

$$= -\frac{1}{3}te^{-3t} - \frac{1}{9}e^{-3t} + \frac{1}{9}$$

3. (18pts) Use the Laplace transform to solve the initial value problem

$$y'' + 3y = u_\pi(t) + \delta(t-\pi), \quad y(0) = 0, \quad y'(0) = 0.$$

$$Y(s) = \mathcal{L}\{y(t)\} \qquad (s^2+3)Y(s) = \mathcal{L}\{u_\pi(t) + \delta(t-\pi)\}$$

$$= e^{-\pi s}\frac{1}{s} + e^{-\pi s}$$

$$Y(s) \overset{8}{=} e^{-\pi s}\left[\frac{1}{s^2+3} + \frac{1}{s(s^2+3)}\right]$$

$$\frac{1}{s(s^2+3)} = \frac{A}{s} + \frac{Bs+C}{s^2+3} = \frac{A(s^2+3)+(Bs+C)s}{s(s^2+3)}$$

$$= e^{-\pi s}\left[\frac{1}{s^2+3} + \frac{1}{3}\cdot\frac{1}{s} - \frac{1}{3}\frac{s}{s^2+3}\right]$$

$\underline{s=0}$ $\quad 1 = 3A \Rightarrow A = \frac{1}{3}$

$\underline{s=\sqrt{3}i}$

$$1 = \sqrt{3}i(B\sqrt{3}i + C)$$

$$= -3B + \sqrt{3}Ci$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \mathcal{L}^{-1}\left\{e^{-\pi s}\left(\frac{1}{s^2+3} + \frac{1}{3}\cdot\frac{1}{s} - \frac{1}{3}\cdot\frac{s}{s^2+3}\right)\right\} \Rightarrow B = -\frac{1}{3}, \; C = 0$$

$$\overset{10}{=} u_\pi(t)\left[\frac{1}{\sqrt{3}}\sin\left(\sqrt{3}(t-\pi)\right) + \frac{1}{3} - \frac{1}{3}\cos\left(\sqrt{3}(t-\pi)\right)\right]$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+3}\right\} = \frac{1}{\sqrt{3}}\sin(\sqrt{3}t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+3}\right\} = \cos(\sqrt{3}t)$$

4. (20pts) For the system $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -6 & -4 \end{pmatrix} \mathbf{x}$,

(a) find the general solution; (b) describe the behavior of the solutions as $t \to \infty$; sketch a few trajectories of the system on the pplane (make sure you use arrows to indicate the direction of increasing t); and (c) find the soultion satisfying the initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

(a) $0 = \det\begin{pmatrix} 1-r & 1 \\ -6 & -4-r \end{pmatrix} = (r-1)(r+4)+6 = r^2+3r+2 = (r+1)(r+2)$

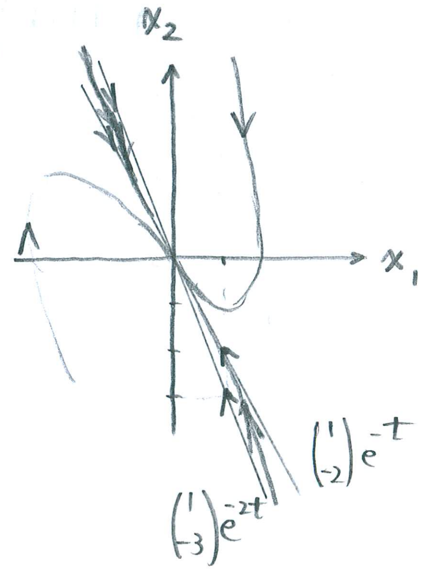
$\Rightarrow r_1 = -2, \; r_2 = -1$ 4

$\underline{r_1=-2}$ $\quad 3\xi_1 + \xi_2 = 0 \Rightarrow \vec{\xi}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \Rightarrow \vec{x}^{(1)} = e^{-2t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ 4

$\underline{r_2=-1}$ $\quad 2\xi_1 + \xi_2 = 0 \Rightarrow \vec{\xi}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow \vec{x}^{(2)} = e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

4 $\quad \vec{x}(t) = c_1 e^{-2t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

4 (b)



4 (c) $\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\Rightarrow c_1 = 0$ and $c_2 = 1$

$\Rightarrow \vec{x}(t) = e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

5. (15pts) (a) Find the general solution of the system $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\mathbf{x}$; and (b) Describe the behavior of the solutions as $t \to \infty$ and sketch a few trajectories on the pplane (make sure you use arrows to indicate the direction of increasing t).

2

(a) $0 = \det\begin{pmatrix} 1-r & -1 \\ 1 & 3-r \end{pmatrix} = (r-1)(r-3)+1 = r^2-4r+4 = (r-2)^2 \Rightarrow r=2$

2

$\underline{r=2}$ $\quad -\xi_1 - \xi_2 = 0 \Rightarrow \vec{\xi} \overset{2}{=} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \vec{x}^{(1)}(t) = e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$(A-2I)\vec{\eta} = \vec{\xi} \Rightarrow \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

3

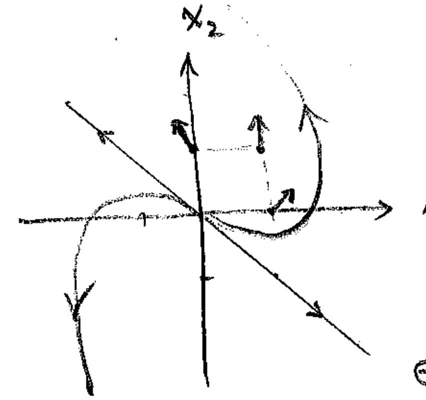
$\Rightarrow -\eta_1 - \eta_2 = 1 \Rightarrow \vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ -\eta_1 - 1 \end{pmatrix} = \eta_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \overset{\eta_1 = 0}{=} \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$\vec{x}^{(2)}(t) = te^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t}\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ $\quad$ a different $\vec{\eta}$ also okay if it is a solution of $(A-2I)\vec{\eta} = \vec{\xi}$

4

$\vec{x}(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 te^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

4 (b)



$e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$

6. (15pts) Find three linearly independent solutions in terms of real-valued functions of the system

$$\mathbf{x}' = \begin{pmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} \end{pmatrix} \mathbf{x}.$$

each solution is 5pts $3\times5=15$

or

eigenvalues	3
eigenvectors	3
$\vec{x}^{(1)}(t)$	3
$\vec{x}^{(2)}(t)$	3
$\vec{x}^{(3)}(t)$	3
	15

$$0 = \det\begin{pmatrix} -\frac{1}{4}-r & 1 & 0 \\ -1 & -\frac{1}{4}-r & 0 \\ 0 & 0 & -\frac{1}{4}-r \end{pmatrix} = -\left(r+\frac{1}{4}\right)\det\begin{pmatrix} -\frac{1}{4}-r & 1 \\ -1 & -\frac{1}{4}-r \end{pmatrix}$$

$$= -\left(r+\frac{1}{4}\right)\left[\left(r+\frac{1}{4}\right)^2+1\right]$$

$$\Rightarrow r_1 = -\frac{1}{4}, \quad r_{2,3} = -\frac{1}{4} \pm i$$

$\underline{r_1 = -\frac{1}{4}}$

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \vec{0} \Rightarrow \begin{matrix} \xi_2 = 0 \\ -\xi_1 = 0 \end{matrix} \Rightarrow \vec{\xi} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \vec{x}^{(1)} = e^{-\frac{1}{4}t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\underline{r_2 = -\frac{1}{4}+i}$

$$\begin{pmatrix} -i & 1 & 0 \\ -1 & -i & 0 \\ 0 & 0 & -i \end{pmatrix}\vec{\xi} = \vec{0} \rightarrow i\begin{pmatrix} -i & 1 & 0 \\ -1 & -i & 0 \\ 0 & 0 & -i \end{pmatrix} \rightarrow \begin{pmatrix} -i & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix}$$

$$\Rightarrow \begin{cases} -i\xi_1 + \xi_2 = 0 \\ -i\xi_3 = 0 \end{cases} \Rightarrow \vec{\xi} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{x}^{(2)}(t) = e^{-\frac{1}{4}t}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\cos t - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\sin t\right] = e^{-\frac{1}{4}t}\begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix}$$

$$\vec{x}^{(3)}(t) = e^{-\frac{1}{4}t}\begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix}$$

~~12 Noon~~ 10:30 am

100 96 88 77 68 59
100 96 88 76 67 55
100 95 86 76 62 54
94 84 76
94 82 76

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92 74
74
74
73
73

73
72
71
71

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3 + 6 + 5 + 13 + 3 + 3 = 33

12 noon

100 99 89 79 68
100 98 89 79 65
97 88 79 61
97 87 78 61
95 86 76

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91 85 74
85 74
84
84
84

84

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2 + 6 + 11 + 7 + 4 = 30