Chapter 6 The Laplace Transform

5 hours

- the method in Chapter 3 is not convenient to use for discontinuous or impulse forcing terms

§6.1 Definition of the Laplace Transform

integral transform

$$\underbrace{F(s)}_{\text{transform of } f} = \int_\alpha^\beta \underbrace{K(s,t)}_{\text{kernel of the transform}} f(t)\,dt$$

$$\begin{array}{ccc} \text{problem on } f & \xrightarrow{\text{transform}} & \text{problem on } F \\ & & \Downarrow \\ \text{solution of } f\text{-problem} & \xleftarrow[\text{inverse transform}]{} & \text{solution of } F\text{-problem} \end{array}$$

Laplace transform

$$K(s,t) = e^{-st}, \quad \alpha = 0, \ \beta = \infty$$

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Review of Improper Integral

$$\int_a^\infty f(t)\,dt = \lim_{A\to\infty}\int_a^A f(t)\,dt \begin{cases} \text{exists} & \text{convergence} \\ \text{DNE} & \text{divergence} \end{cases}$$

examples

(1) $$\int_0^\infty e^{ct}\,dt = \lim_{A\to\infty}\frac{1}{c}\left(e^{cA}-1\right) \begin{cases} = -\frac{1}{c} & \text{if } c<0 \\ \text{diverges} & \text{if } c\geq 0 \end{cases}$$

(2) $$\int_1^\infty \frac{1}{t}\,dt = \lim_{A\to\infty}\ln A \quad \text{diverges}$$

(3) $$\int_1^\infty t^{-p}\,dt = \lim_{A\to\infty}\frac{1}{1-p}\left(A^{1-p}-1\right) \begin{cases} = \frac{1}{p-1} & \text{if } p>1 \\ \text{diverges} & \text{if } p\leq 1 \end{cases}$$

- $f(t)$ is piecewise continuous on $[\alpha,\beta] = \bigcup_{i=1}^{n}[t_{i-1},t_i]$

$\Longleftrightarrow$ (1) $f(t)\in C(t_{i-1},t_i)$ for $i=1,\cdots,n$

(2) $\lim_{t\to t_i^-} f(t)$ and $\lim_{t\to t_{i-1}^+} f(t)$ are finite.

$$\int_\alpha^\beta f(t)\,dt = \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} f(t)\,dt$$

example #1 (p315)

- **Convergence Test for Improper Integral**

(1) $f\in PC[\alpha,\infty)$, $|f(t)|\leq g(t)$ $\forall t\geq M$

$\Longrightarrow$ " $\int_M^\infty g(t)\,dt$ converges $\Longrightarrow \int_a^\infty f(t)\,dt$ converges."

(2) $f(t)\geq g(t)\geq 0$ $\forall t\geq M$

$\Longrightarrow$ " $\int_M^\infty g(t)\,dt$ diverges $\Longrightarrow \int_a^\infty f(t)\,dt$ diverges"

Assumptions (1) $f \in PC[0,\infty)$

(2) $|f(t)| \leq K e^{at} \quad \forall t \geq M$.

$\Longrightarrow \mathcal{L}\{f(t)\} = F(s)$ exists $\forall s > a$.

$$\int_0^\infty e^{-st} f(t)\,dt = \int_0^M e^{-st} f(t)\,dt + \underbrace{\int_M^\infty e^{-st} f(t)\,dt}_{\leq \int_M^\infty K e^{(a-s)t}\,dt}$$

examples (1) $\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = -\frac{1}{s} e^{-st}\Big|_0^\infty = \frac{1}{s}$

(2) $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{(a-s)t}\,dt = \frac{1}{a-s} e^{(a-s)t}\Big|_0^\infty = \frac{1}{s-a}$ if $s > a$

(3) $\mathcal{L}\{\sin(at)\} = \int_0^\infty \sin(at)\, e^{-st}\,dt = I$

integration by part $\left.\begin{array}{l} u = \sin(at),\ v' = e^{-st} \\ u' = a\cos(at),\ v = -\frac{1}{s} e^{-st} \end{array}\right\} \Rightarrow I = -\frac{1}{s}\sin(at) e^{-st}\Big|_0^\infty + \frac{a}{s}\int_0^\infty \cos(at) e^{-st}\,dt$

$= \frac{1}{s} + \left[\frac{1}{s} - \frac{a}{s} I\right]\frac{a}{s}$

$\Longrightarrow I = \frac{s+a}{s^2+a^2}$.

(4) $\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}$

$\Longrightarrow \mathcal{L}$ is a linear operator

Table 6.2.1 (P321) Elementary Laplace Transforms

§6.2 Solution of Initial Value Problem (IVP)

Thrm $f \in C[0,\infty)$, ~~and~~ $f' \in PC[0,\infty)$, and $|f(t)| \le Ke^{at}$ $\forall t \ge M$

$$\Longrightarrow \mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Example $\begin{cases} y'' - y' - 2y = 0 \\ y(0) = 1,\ y'(0) = 0 \end{cases}$ $\Bigg|$ $r^2 - r - 2 = (r-2)(r+1) = 0 \Rightarrow r_1 = -1,\ r_2 = 2$

$y(t) = c_1 e^{-t} + c_2 e^{2t} = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}$

$$0 = \mathcal{L}\{y'' - y' - 2y\} = \mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\}$$

$$= \left[s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right] - \left[s\mathcal{L}\{y\} - y(0)\right] - 2\mathcal{L}\{y\}$$

$\underline{Y(s) = \mathcal{L}\{y\}}$ $\quad = (s^2 - s - 2)Y(s) + (1-s)y(0) - y'(0)$

$$\Longrightarrow Y(s) = \frac{s-1}{(s-2)(s+1)} = \frac{1}{3}\frac{1}{s-2} + \frac{2}{3}\frac{1}{s+1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

example #1 (p324)

$\underline{ay'' + by' + cy = f}$

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = ?$$

<u>examples</u> (1) $\begin{cases} y'' + y = \sin 2t \\ y(0) = 2,\ y'(0) = 1 \end{cases}$

$$(s^2+1)\,Y(s) = s\,y(0) + y'(0) + \frac{2}{s^2+4} = 2s + 1 + \frac{2}{s^2+4}$$

$$Y(s) = \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)}$$

$$\frac{2}{(s^2+1)(s^2+4)} = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4} = \frac{2}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right]$$

$$= \frac{2s}{s^2+1} + \frac{5}{3}\cdot\frac{1}{s^2+1} - \frac{2}{3}\cdot\frac{1}{s^2+4}$$

$$\Rightarrow y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$$

(2) $\begin{cases} y^{(4)} - y = 0 \\ y(0) = 0,\ y'(0) = 1,\ y''(0) = 0,\ y'''(0) = 0 \end{cases}$

$$Y(s) = \frac{s^2}{s^4-1} = \frac{s^2}{(s^2-1)(s^2+1)} = \frac{as+b}{s^2-1} + \frac{cs+d}{s^2+1} = \frac{1}{2}\left[\frac{1}{s^2-1} + \frac{1}{s^2+1}\right]$$

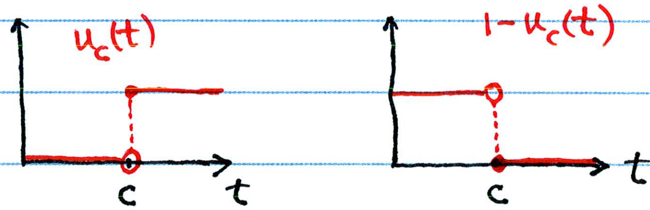
$$\Rightarrow y(t) = \frac{1}{2}\left[\sinh t + \sin t\right].$$

§6.3 Step Functions

unit step (Heaviside) function

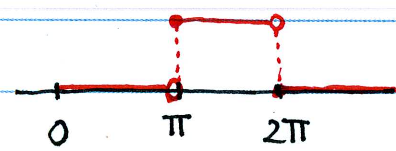
$$u_c(t) = \begin{cases} 0, & 0 \le t < c \\ 1, & c \le t \end{cases}$$

$$1 - u_c(t) = \begin{cases} 1, & 0 \le t < c \\ 0, & c \le t \end{cases}$$



rectangular pulse

example (1) sketch the graph of $h(t) = u_\pi(t) - u_{2\pi}(t)$ for $t \ge 0$



(2) #7 (P333) express $f(t)$ in terms of $u_c(t)$

$$f(t) = \begin{cases} 0, & 0 \le t < 3, \\ -2, & 3 \le t < 5, \\ 2, & 5 \le t < 7 \\ 1, & 7 \le t. \end{cases}$$

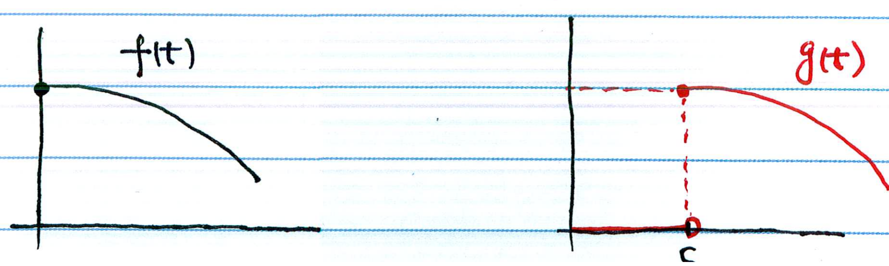
$$= -2u_3(t) + 4u_5(t) - u_7(t)$$

Laplace transform

$$\mathcal{L}\{u_c(t)\} = \frac{1}{s} e^{-cs}, \quad s > 0$$

translation of $f(t)$ for $t \ge 0$

$$g(t) = u_c(t) f(t-c) = \begin{cases} 0, & 0 \le t < c \\ f(t-c), & c \le t \end{cases}$$



Thrm (1) $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s)$

(2) $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)\,f(t-c)$

Proof $\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt = \int_c^\infty e^{-st}f(t-c)\,dt$

example (1) $f(t) = \begin{cases} \sin t, & 0 \le t < \frac{\pi}{4} \\ \sin t + \cos\left(t-\frac{\pi}{4}\right), & \frac{\pi}{4} \le t \end{cases}$

$= \sin t + u_{\frac{\pi}{4}}(t)\cos\left(t-\frac{\pi}{4}\right)$

$? = \mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} + \mathcal{L}\{u_{\frac{\pi}{4}}(t)\cos(t-\frac{\pi}{4})\}$

$= \frac{1}{s^2+1} + e^{-\pi s/4}\frac{s}{s^2+1}$

(2) $F(s) = \frac{1}{s}\left(1-e^{-2s}\right)$, $? = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{\frac{1}{s}\} - \mathcal{L}^{-1}\{e^{-2s}\frac{1}{s}\} = t - u_2(t)\,(t-2)$

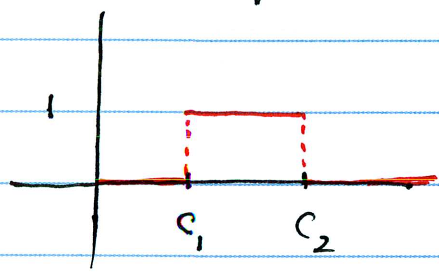
Thrm (1) $\mathcal{L}\{e^{ct}f(t)\} = F(s-c)$ where $F(s) = \mathcal{L}\{f(t)\}$

(2) $\mathcal{L}^{-1}\{F(s-c)\} = e^{ct}f(t)$

example $? = \mathcal{L}^{-1}\left\{\frac{1}{s^2-4s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+1}\right\} = \mathcal{L}^{-1}\{F(s-2)\} = e^{2t}\sin t$

with $F(s) = \frac{1}{s^2+1}$ and $\mathcal{L}^{-1}\{F(s)\} = \sin t$

How to rewrite a piecewise defined function in a formula?



$u_{c_1}(t) - u_{c_2}(t)$

$$f(t) = \begin{cases} g_1(t) & 0 \le t < c_1 \\ g_2(t) & c_1 \le t < c_2 \\ g_3(t) & c_2 \le t \end{cases}$$

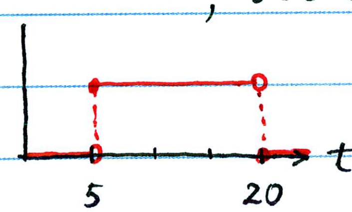
$$= g_1(t)\left(1 - u_{c_1}(t)\right) + g_2(t)\left(u_{c_1}(t) - u_{c_2}(t)\right) + g_3(t)\, u_{c_2}(t)$$

$$= g_1(t) + \left(g_2(t) - g_1(t)\right) u_{c_1}(t) + \left(g_3(t) - g_2(t)\right) u_{c_2}(t)$$

§6.4 DEs with Discontinuous Forcing Functions

<u>example 1</u>

$$\begin{cases} 2y'' + y' + 2y = g(t), \\ y(0)=0, \quad y'(0)=0, \end{cases} \qquad g(t) = u_5(t) - u_{20}(t) = \begin{cases} 1, & 5 \le t < 20, \\ 0, & 0 \le t < 5,\ t \ge 20, \end{cases}$$



$$\overset{\mathcal{L}}{\Longrightarrow} \left(2s^2+s+2\right) Y(s) = \frac{1}{s}\left(e^{-5s} - e^{-20s}\right)$$

$$\Rightarrow Y(s) = \left(e^{-5s} - e^{-20s}\right) H(s)$$

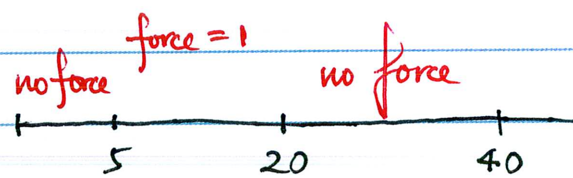
with
$$H(s) = \frac{1}{s(2s^2+s+2)} = \frac{a}{s} + \frac{bs+c}{2s^2+s+2} = \frac{1}{2}\cdot\frac{1}{s} - \frac{s+\frac{1}{2}}{2s^2+s+2}$$

$$= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{\left(s+\frac{1}{4}\right)+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}} = \frac{1}{2}\frac{1}{s} - \frac{1}{2}\left[\frac{s+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\left(\frac{\sqrt{15}}{4}\right)^2} + \frac{1}{\sqrt{15}}\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2+\left(\frac{\sqrt{15}}{4}\right)^2}\right]$$

$$\Longrightarrow h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{1}{2} - \frac{1}{2}\left[e^{-\frac{t}{4}}\cos\left(\frac{\sqrt{15}\,t}{4}\right) + \frac{1}{\sqrt{15}} e^{-\frac{t}{4}} \sin\left(\frac{\sqrt{15}\,t}{4}\right)\right]$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = u_5(t)\,h(t-5) - u_{20}(t)\,h(t-20)$$

<u>see Figure 6.4.1</u>



<u>effect of discontinuity in forcing function</u>

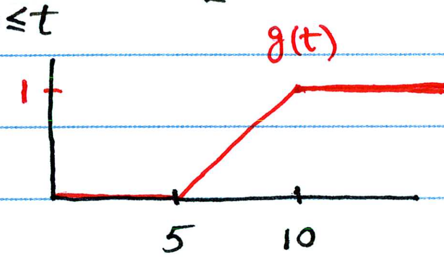
(1) y, y', and y'' are continuous except possibly at $t=5, 20$;

(2) y and y' are continuous at $t=5, 20$;

(3) $\left|[\![y'']\!]_{t=5}\right| = \left|\lim_{t\to5^+} y''(t) - \lim_{t\to5^-} y''(t)\right| = \left|\frac{1}{2}\right| = \frac{1}{2}, \quad [\![y'']\!]_{t=20} = -\frac{1}{2}$

example 2 $\begin{cases} y'' + 4y = g(t) \\ y(0) = y'(0) = 0 \end{cases}$

$$g(t) = \begin{cases} 0, & 0 \le t < 5 \\ \frac{t-5}{5}, & 5 \le t < 10 \\ 1, & 10 \le t \end{cases} = \frac{1}{5}\left[u_5(t)(t-5) - u_{10}(t)(t-10)\right]$$



Qualitative Nature

(1) On $[0, 5)$ $\quad g(t) = 0$

$\Longrightarrow y(t) = 0$

(2) On $(10, \infty)$ $\quad g(t) = 1$

$\Longrightarrow y(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{4}$ harmonic oscillation about $y = \frac{1}{4}$

(3) On $[5, 10]$, $\quad g(t) = \frac{t-5}{5}$ linear function — oscillation about a linear function

see Fig. 6.4.3

$$\xrightarrow{\mathcal{L}} (s^2+4)Y(s) = \left(e^{-5s} - e^{-10s}\right)\frac{1}{5s^2}$$

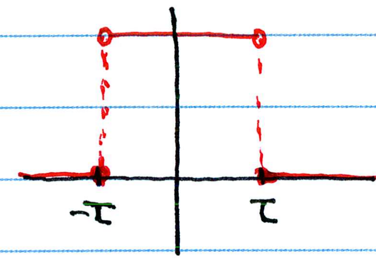
$$\Rightarrow Y(s) = \left(e^{-5s} - e^{-10s}\right)\frac{1}{5}H(s)$$

$$H(s) = \frac{1}{s^2(s^2+4)} = \frac{1}{4}\left[\frac{1}{s^2} - \frac{1}{s^2+4}\right] \Longrightarrow h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{1}{4}t - \frac{1}{8}\sin 2t$$

$$y(t) = \frac{1}{5}\left[u_5(t)h(t-5) - u_{10}(t)h(t-10)\right]$$

§6.5 Impulse Functions

$$d_\tau(t) = \begin{cases} \frac{1}{2\tau}, & -\tau < t < \tau \\ 0, & \text{otherwise} \end{cases}$$



$$I(\tau) = \int_{-\infty}^{\infty} d_\tau(t)\,dt = 1 \quad \forall\, \tau \neq 0$$

$$\lim_{\tau\to 0} d_\tau(t) = 0 \quad \forall\, t \neq 0 \qquad \text{and} \qquad \lim_{\tau \to 0} I(\tau) = 1$$

unit impulse function at 0 $\quad \delta(t) = \lim_{\tau\to 0^+} d_\tau(t)$ Dirac delta function

$$\delta(t) = 0 \quad \forall\, t \neq 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

at t_0
$$\begin{cases} \delta(t-t_0) = 0 \quad \forall\, t \neq t_0 \\ \int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1 \end{cases}$$

Laplace transform $\quad \mathcal{L}\{\delta(t-t_0)\} = e^{-t_0 s}$

identity $\quad \int_{-\infty}^{\infty} \delta(t-t_0) f(t)\,dt = f(t_0)$

<u>example</u>

$$\begin{cases} 2y'' + y' + 2y = \delta(t-5) \\ y(0) = y'(0) = 0 \end{cases}$$

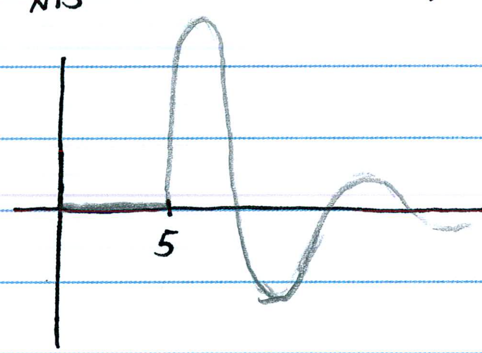
$$(2s^2 + s + 2)\, Y(s) = e^{-5s}$$

$$\Rightarrow Y(s) = \frac{1}{2} e^{-5s} \frac{1}{(s+\frac{1}{4})^2 + \frac{15}{16}} = \frac{1}{2} e^{-5s} H(s)$$

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{4}{\sqrt{15}} e^{-\frac{t}{4}} \sin \frac{\sqrt{15}}{4} t$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{2}{\sqrt{15}} u_5(t)\, e^{-\frac{t-5}{4}} \sin \frac{\sqrt{15}}{4}(t-5)$$

see Fig. 6.5.3



§6.6 The Convolution Integral

Thrm $F(s)=\mathcal{L}\{f(t)\}$ and $G(s)=\mathcal{L}\{g(t)\}$

$$\Longrightarrow \quad H(s)=F(s)G(s)=\mathcal{L}\{h(t)\} \Longleftrightarrow h(t)=\mathcal{L}^{-1}\{F(s)G(s)\}$$

where $$h(t)=\int_0^t f(t-\tau)\,g(\tau)\,d\tau=\int_0^t f(\tau)\,g(t-\tau)\,d\tau$$
$$=(f*g)(t)$$

properties

$$f*g=g*f$$
$$f*(g_1+g_2)=f*g_1+f*g_2$$
$$f*(g*h)=(f*g)*h$$
$$f*0=0*f=0$$
$$f*1\neq f$$

proof of Thrm

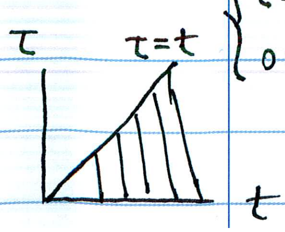
$$F(s)G(s)=\int_0^\infty e^{-s\xi}f(\xi)\,d\xi\int_0^\infty e^{-s\eta}g(\eta)\,d\eta$$
$$=\int_0^\infty\left(\int_0^\infty e^{-s(\xi+\eta)}f(\xi)\,d\xi\right)g(\eta)\,d\eta$$

$t=\xi+\eta$
$$=\int_0^\infty\left(\int_\eta^\infty e^{-st}f(t-\eta)\,dt\right)g(\eta)\,d\eta=\int_0^\infty\left(\int_\tau^\infty e^{-st}f(t-\tau)\,dt\right)g(\tau)\,d\tau$$

$$\begin{cases}\tau\le t<\infty\\ 0\le\tau<\infty\end{cases}\Longrightarrow\begin{cases}0\le\tau\le t\\ 0\le t<\infty\end{cases}$$

$$=\int_0^\infty e^{-st}\left(\int_0^t f(t-\tau)\,g(\tau)\,d\tau\right)dt$$

$$=\mathcal{L}\{h(t)\}.$$



<u>example</u> (1) $? = \mathcal{L}^{-1}\left\{\frac{a}{s^2(s^2+a^2)}\right\} = \mathcal{L}^{-1}\left\{\underset{F(s)}{\underset{\parallel}{\frac{1}{s^2}}} \cdot \underset{G(s)}{\underset{\parallel}{\frac{a}{s^2+a^2}}}\right\}$

$f(t) = t, \; g(t) = \sin(at)$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{a}{s^2(s^2+a^2)}\right\} = (f*g)(t) = \int_0^t (t-\tau)\sin(a\tau)\,d\tau$$

$$= \frac{1}{a^2}\left[at - \sin(at)\right]$$

(2) $\begin{cases} y'' + 4y = g(t) \\ y(0) = 3, \; y'(0) = -1 \end{cases}$

$$Y(s) = \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4} = 3\frac{s}{s^2+4} - \frac{1}{2}\cdot\frac{2}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}\cdot G(s)$$

$$y(t) = 3\cos(2t) - \frac{1}{2}\sin(2t) + \frac{1}{2}\int_0^t \sin(2(t-\tau))\,g(\tau)\,d\tau$$

(3) $\begin{cases} ay'' + by' + cy = g(t) \\ y(0) = y_0, \; y'(0) = y_0' \end{cases} \Longrightarrow \begin{cases} a\varphi'' + b\varphi' + c\varphi = 0 \\ \varphi(0) = y_0, \; \varphi'(0) = y_0' \end{cases}$ and $\begin{cases} a\psi'' + b\psi' + c\psi = g \\ \psi(0) = 0, \; \psi'(0) = 0 \end{cases}$.

$$\Rightarrow Y(s) = \Phi(s) + \Psi(s) \equiv \frac{y_0(as+b) - ay_0'}{as^2+bs+c} + \frac{1}{as^2+bs+c}\cdot G(s)$$

$$\Rightarrow y(t) = \varphi(t) + \psi(t) \equiv \mathcal{L}^{-1}\{\Phi(s)\} + \underset{(f*g)(t)}{\underset{\parallel}{\mathcal{L}^{-1}\{\Psi(s)\}}}$$ where $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{as^2+bs+c}\right\}$