Chapter 5 Series Solutions of 2^{nd}-Order Linear Eqs

9 hours = 6×1.5

2^{nd}-Order Linear Eqs

- constant coefficients: systematical procedure for constructing fundamental solutions
- variable coefficients: $P(x)y'' + Q(x)y' + R(x)y = 0$

power series solution at x_0
- ordinary pt $P(x_0) \neq 0$
- singular pt $P(x_0) = 0$
  - regular singular pt $\lim_{x \to x_0} (x-x_0)\frac{Q(x)}{P(x)}$, $\lim (x-x_0)^2 \frac{R(x)}{P(x)}$ finite
  - irregular singular pt.

§5.1 Review of Power Series

Definition (1) power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$

(2) $\sum a_n (x-x_0)^n$ converges at $x \iff \lim_{m\to\infty} \sum_{n=0}^{m} a_n (x-x_0)^n$ exists for x

(3) $\sum a_n (x-x_0)^n$ converges absolutely at $x \iff \sum \left| a_n (x-x_0)^n \right|$ converges at x

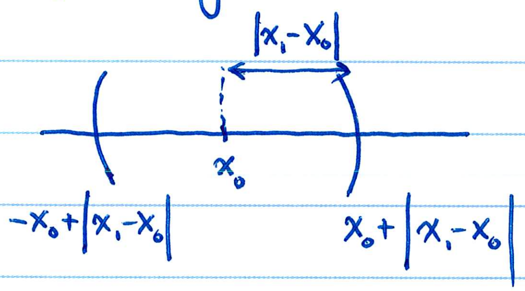
Ratio Test

$$\overline{\lim_{n\to\infty}} \left| \frac{a_{n+1}(x-x_0)^{n+1}}{a_n (x-x_0)^n} \right| = |x-x_0| \overline{\lim_{n\to\infty}} \left| \frac{a_{n+1}}{a_n} \right| \equiv L$$

$$L \begin{cases} <1 & \text{absolutely convergent at } x \\ >1 & \text{divergent at } x \\ =1 & \text{unknown} \end{cases}$$

Ex. 1 For which values of x does $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} (x-1)^n$ converge?

Property $\sum a_n (x-x_0)^n$ converges at $x = x_1 \Longrightarrow$ it converges abs. for all x s.t. $|x-x_0| < |x_1 - x_0|$

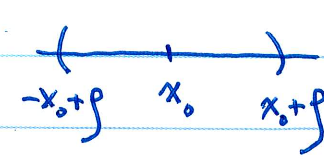


Radius of Convergence

$\rho > 0$ is called the radius of convergece of the power series $\sum a_n (x-x_0)^n$

$\Longleftrightarrow$ it converges abs. $\forall x$ s.t. $|x-x_0| < \rho$

and diverges $\forall x$ s.t. $|x - x_0| > \rho$



Ex. 2 Determine ρ of $\sum \frac{(x+1)^n}{n 2^n}$, $\sum (-1)^n \frac{n^2 (x+2)^n}{3^n}$

Manipulations $\sum a_n (x-x_0)^n \longrightarrow f(x)$, $\sum b_n (x-x_0)^n \longrightarrow g(x)$ with $\rho > 0$

(1) $f(x) \pm g(x) = \sum (a_n \pm b_n)(x-x_0)^n$ with at least $\rho > |x-x_0|$

(2) $f(x) g(x) = \sum c_n (x-x_0)^n$ converges at least for $|x-x_0| < \rho$

with $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}$

(3) assume that $g(x_0) \neq 0$

$$\frac{f(x)}{g(x)} = \sum d_n (x-x_0)^n$$ with the radius of convergence $\leq \rho$

(4) $f(x)$ is cont. and has derivatives of all order for $|x-x_0| < \rho$

$$f^{(k)}(x) = \sum \left[a_n (x-x_0)^n\right]^{(k)}, \quad \int f(x)\,dx = \sum a_n \int (x-x_0)^n\,dx.$$

Taylor Expansion $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$$ with $\rho > 0$

$f(x)$ is analytic at $x = x_0 \Longleftrightarrow \rho > 0$

Shift of Index of Summation

$$\sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = \sum_{j=0}^{\infty} \frac{2^j x^j}{j!}$$

Ex. 3 $$\sum_{n=2}^{\infty} a_n x^n \xrightarrow{?} \sum_{m=0}^{\infty} a_{m+2}\, x^{m+2}$$

$$m \equiv n-2 = 0 \implies n = m+2$$

Ex. 4 $$\sum_{n=2}^{\infty} (n+2)(n+1)\, a_n \, (x-x_0)^{n-2} \xrightarrow{?} \sum_{m=0}^{\infty} (n+4)(n+3)\underline{(x-x_0)^m}\, a_{n+2}$$

$$n \to m+2$$

$$m = n-2 \implies n = m+2$$

§5.2 Series Solutions Near an Ordinary Pt, Part I

DEs with variable coefficients given $P(x)$, $Q(x)$, $R(x)$

$$P(x)\, y''(x) + Q(x)\, y'(x) + R(x)\, y(x) = 0$$

set $p(x) = \frac{Q(x)}{P(x)}$, $q = \frac{R}{P}$ $\Rightarrow y''(x) + p(x) y'(x) + q(x) y(x) = 0$

examples

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0 \quad \text{the Bessel eq.}$$

$$(1-x^2) y'' - 2x y' + \alpha(\alpha+1) y = 0 \quad \text{the Legendre eq.}$$

Assumption $P(x)$, $Q(x)$, $R(x)$ are polynomials with no common factor

Definition
- Ordinary pt $x_0 \iff P(x_0) \neq 0$ §5.2-5.3
- Singular pt $x_0 \iff P(x_0) = 0$ §5.4-5.6

$$\Rightarrow \lim_{x \to x_0} p(x) = \infty \text{ and } \lim_{x \to x_0} q(x) = \infty$$

Series Solution

$$y(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots$$

$$= \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

Questions $a_n = ?$ and $\rho = ?$

Example 1 $\quad y'' + y = 0$

solution 1 $\quad y = e^{rx} \Rightarrow (r^2+1)e^{rx} = 0 \Rightarrow r^2 = 1$

$$\Rightarrow r = \pm i \Rightarrow e^{ix} = \cos x + i \sin x$$

$$\Rightarrow y(x) = c_1 \cos x + c_2 \sin x$$

solution 2 $\quad P(x) = 1 \Rightarrow$ every x_0 is an ordinary pt

$x_0 = 0 \quad y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\Rightarrow 0 = y'' + y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$$

$$\Big\Downarrow \; \begin{matrix} m = n-2 \\ n = m+2 \end{matrix}$$

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m$$

$m = n$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n\right] x^n$$

$$\Rightarrow (n+2)(n+1) a_{n+2} + a_n = 0 \quad \text{for } n = 0, 1, \ldots$$

$$\Rightarrow \quad a_{n+2} = -\frac{1}{(n+1)(n+2)}\, a_n \qquad \text{recurrence relation}$$

$$a_2 = -\frac{a_0}{1\cdot 2}, \quad a_4 = -\frac{a_2}{3\cdot 4} = (-1)^2 \frac{a_0}{4!} \Longrightarrow a_{2k} = (-1)^k \frac{a_0}{(2k)!}$$

$$a_3 = -\frac{a_1}{2\cdot 3}, \quad a_5 = -\frac{a_3}{4\cdot 5} = (-1)^2 \frac{a_1}{5!} \Longrightarrow a_{2k+1} = (-1)^k \frac{a_1}{(2k+1)!}$$

$$\Rightarrow y(x) = a_0 \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} x^{2k} + a_1 \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} x^{2k+1}$$

$$= a_0 \cos x + a_1 \sin x$$

<u>ratio test</u> $\rho = \infty$

see Figures 5.2.1 and 5.2.2

<u>Ex. 2</u> Airy's eq. $y'' - xy = 0$

(1) $y(x) = \sum_{n=0}^{\infty} a_n x^n = ?$

(2) $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n = ?$

<u>Hand Graded HW</u> 14 (a, b)

§5.3 Series Solutions Near an Ordinary Pt, Part II

(*) $P(x)y'' + Q(x)y' + R(x)y = 0$, P, Q, R are poly. with no common factor

$$y'' + py' + qy = 0 \quad \text{with } p = \frac{Q}{P} \text{ and } q = \frac{R}{P}.$$

solution $\quad y(x) = \varphi(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$

$$= \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(x_0)}{n!}(x-x_0)^n \implies a_n = \frac{\varphi^{(n)}(x_0)}{n!}, \quad n = 0, 1, 2, \cdots$$

$\varphi^{(n)}(x_0) = ?$ $\quad$ assume that $P(x) \neq 0$ in an interval containing x_0

$$\varphi(x_0) = a_0, \quad \varphi'(x_0) = a_1$$

$$\varphi''(x) = -p\varphi' - q\varphi \implies \varphi''(x_0) = -\left[p(x_0)a_1 + q(x_0)a_0\right]$$

$$\varphi'''(x_0) = -\left[p\varphi' + q\varphi\right]'_{x=x_0} = -\left[p\varphi'' + (p'+q)\varphi' + q'\varphi\right]_{x=x_0}$$

$$\varphi'''(x_0) = -\left[2!\,p(x_0)a_2 + \left(p'(x_0) + q(x_0)\right)a_1 + q'(x_0)a_0\right]$$

Assumption $p(x)$ and $q(x)$ are analytic at x_0.

$$\Longrightarrow \quad p(x) = \sum_{n=0}^{\infty} p_n (x-x_0)^2 \quad \text{and} \quad q(x) = \sum_{n=0}^{\infty} q_n (x-x_0)^2$$

Definition

- Ordinary pt $x_0 \iff p$ and q are analytic at x_0
- Singular pt $x_0 \iff$ either p or q is not analytic at x_0

Theorem Assume that x_0 is an ordinary pt of $(*)$, the general solution is

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 y_1(x) + a_1 y_2(x)$$

where a_0 and a_1 are arbitrary constants

$y_1(x)$ and $y_2(x)$ are linearly indep. series solutions that are analytic at x_0

$$\rho_{y_1}, \ \rho_{y_2} \geq \min \{\rho_p, \rho_q\}.$$

$\rho_p = ?$ $\rho_q = ?$ (1) power series of p and $q \Longrightarrow \rho_p, \rho_q$

(2) If P, Q, R are polynomials and $P(x_0) \neq 0$

$\Rightarrow p = \frac{Q}{P}$ has a convergent power series expansion at x_0

$$\rho_p = \text{dist}\{x_0, \text{the nearest zero of } P\}$$

Examples Taylor series and ρ

(1) $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$ at $x_0 = 0$

$\begin{cases} \text{ratio test} & \rho = 1 \\ \text{root} & x = \pm i \end{cases}$ $\quad \text{dist}\{0, \pm i\} = 1$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

(2) $\frac{1}{x^2 - 2x + 2}$ $\begin{cases} x_0 = 0 \\ x_0 = 1 \end{cases}$ $\quad x = 1 \pm i \quad$ $\begin{cases} \text{dist}\{0, 1 \pm i\} = \sqrt{2} \\ \text{dist}\{1, 1 \pm i\} = 1 \end{cases}$

Examples determine a lower bound for ρ

(3) Legendre eq. $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$ at $x_0 = 0$

$P(x) = 1 - x^2, \quad Q(x) = -2x, \quad R(x) = \alpha(\alpha+1)$

$P(x) = 0 \Downarrow x = \pm 1 \Longrightarrow \rho_p = \text{dist}\{0, \pm 1\} = 1 = \rho_q$

$\Longrightarrow \rho_y \geq 1$

(4) $(1+x^2)y'' + 2xy' + 4x^2 y$ at $x_0 = 0$ and $x_0 = -\frac{1}{2}$

$P(x) = 1 + x^2 = 0 \Longrightarrow x = \pm i \Longrightarrow \rho_p = \rho_q = \text{dist}\{0, \pm i\} = 1 \leq \rho_y$

$\rho_p = \rho_q = \text{dist}\{-\frac{1}{2}, \pm i\} = \frac{\sqrt{5}}{2} \leq \rho_y$

(5) $y'' + (\sin x)y' + (1+x^2)y = 0$ at $x_0 = 0$

§5.4 Euler Eqs; Regular Singular Pts

$$L[y] = x^2 y'' + \alpha x y' + \beta y = 0, \quad \alpha, \beta - \text{constants}$$

$P(x) = x^2 = 0 \Rightarrow x = 0$ is the only singular pt

$x \cdot \frac{Q}{P} = \alpha, \quad x^2 \frac{R}{P} = \beta \Longrightarrow x = 0$ regular singular pt

$\underline{x > 0}$ $\quad y = x^r \Longrightarrow L[x^r] = x^r \{ \underbrace{r(r-1) + \alpha r + \beta}_{F(r) = r^2 + (\alpha-1) r + \beta} \}$

$L[x^r] = 0 \Longrightarrow F(r) = 0 \Longrightarrow r_1, r_2 = \frac{1}{2} \left\{ -(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\beta} \right\}$

2^{nd}-order DE with constant coefficient $\qquad$ Euler Eq

$y = e^{r\underline{x}}$ $\qquad$ $y = x^r = e^{r \underline{\ln x}}$

<u>roots</u> (1) distinct real, (2) equal, (3) complex

(1) $r_1 \neq r_2$ (real) $\quad y_1 = x^{r_1}, \; y_2 = x^{r_2}, \quad W(y_1, y_2) \neq 0$

$$y = c_1 x^{r_1} + c_2 x^{r_2}$$

Example $\quad 2x^2 y'' + 3x y' - y = 0$ for $x > 0$

$$\Rightarrow r_1 = \frac{1}{2}, \; r_2 = -1$$

$$y = c_1 x^{\frac{1}{2}} + c_2 x^{-1}$$

(2) $r_1 = r_2$ $\quad y_1 = x^{r_1}$ — one solution

$y_2(x) = ?$ $\quad$ (method of reduction of order $y = v(x)\, y_1(x)$)

$$r_1 = r_2 \Longrightarrow F(r) = (r - r_1)^2$$

$$\Longrightarrow F(r_1) = 0 \text{ and } F'(r_1) = 0$$

$$L[x^r] = x^r F(r)$$

$$\Longrightarrow \frac{\partial}{\partial r} L[x^r] = \frac{\partial}{\partial r}\left[x^r F(r)\right] = \frac{\partial x^r}{\partial r} F(r) + x^r F'(r)$$

$$\| \qquad\qquad = x^r \ln x\, F(r) + x^r F'(r)$$

$$L\left[\frac{\partial}{\partial r} x^r\right] = L[x^r \ln x] \qquad = x^r \ln x\, (r - r_1)^2 + x^r\, 2(r - r_1)$$

$$= 0 \quad \text{for } r = r_1$$

$$\Longrightarrow y_2(x) = x^{r_1} \underline{\ln x}$$

$$y(x) = c_1 x^{r_1} + c_2 x^{r_1} \ln x$$

<u>Example</u> $x^2 y'' + 5 x y' + 4 y = 0$, $\quad r_1 = r_2 = -2$

$$y = c_1 x^{-2} + c_2 x^{-2} \ln x$$

(3) $r_{1,2} = \lambda \pm i\mu \quad (\mu \neq 0)$ $\qquad x^r = e^{r \ln x}$

$$x^{\lambda + i\mu} = e^{(\lambda + i\mu)\ln x} = e^{\lambda \ln x}\left(\cos(\mu \ln x) + i \sin(\mu \ln x)\right)$$

$$= x^{\lambda} \cos(\mu \ln x) + i\, x^{\lambda} \sin(\mu \ln x)$$

$$y(x) = c_1 x^\lambda \cos(\mu \ln x) + c_2 x^\lambda \sin(\mu \ln x)$$

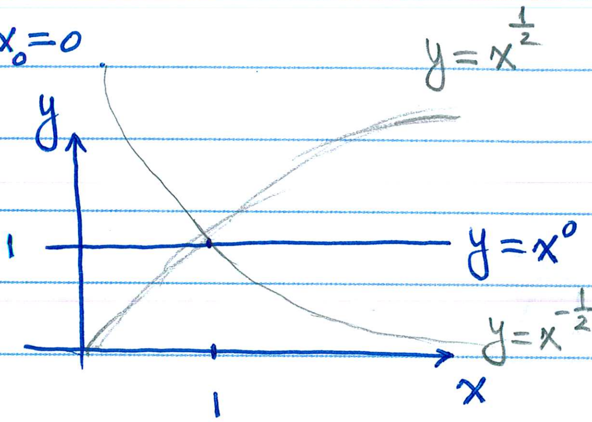
<u>Example</u> $\quad x^2 y'' + x y' + y = 0, \qquad r = \pm i$

$$y(x) = c_1 \cos(\ln x) + c_2 \sin(\ln x)$$

<u>Qualitative Behavior</u> near the singular pt $x_0 = 0$

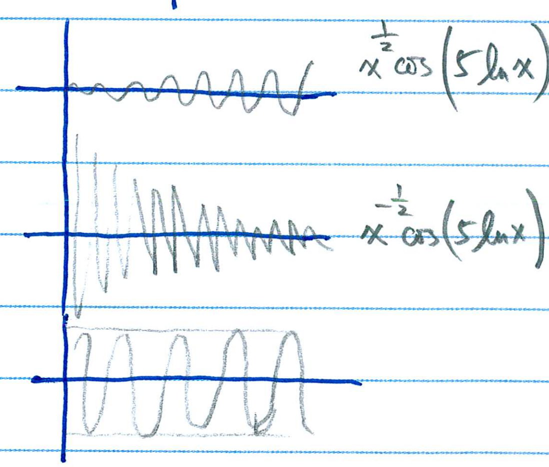
<u>r-real</u>

$$x^r \begin{cases} \to 0 & r > 0 \\ \text{unbounded} & r < 0 \\ 1 & r = 0 \end{cases}$$



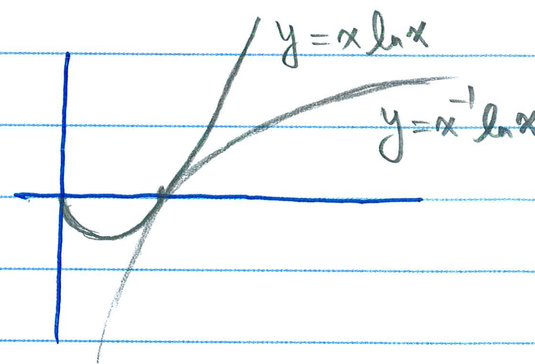
<u>r-complex</u>

$$x^\lambda \cos(\mu \ln x) \begin{cases} \to 0 & \lambda > 0 \\ \text{unbounded} & \lambda < 0 \\ \text{const. amplitude} & \lambda = 0 \end{cases}$$



<u>r-repeated</u>

$$x^r \ln x \begin{cases} \to 0 & r > 0 \\ \text{unbounded} & r < 0 \end{cases}$$



Regular Singular Pts

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0, \quad P, Q, R \text{ — poly. having no common factor}$$

singular pts: $S = \{ x \mid P(x) = 0 \}$

regular singular pts:

(1) P, Q, R — poly. $\quad RS = \left\{ x_0 \in S \;\middle|\; \lim\limits_{x \to x_0} \dfrac{(x-x_0)Q}{P} \;\&\; \lim\limits_{x \to x_0} \dfrac{(x-x_0)^2 R}{P} \text{ finite} \right\}$

(2) non-poly. $\quad RS = \left\{ x_0 \in S \;\middle|\; \dfrac{(x-x_0)Q}{P} \;\&\; \dfrac{(x-x_0)^2 R}{P} \text{ analytic at } x_0 \right\}$

Irregular singular pts: $\quad S \setminus RS$

Determine the singular pts: $S = ?\quad RS = ?$

Examples

(4) $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$ the Legendre eq.

(5) $2x(x-2)^2 y'' + 3xy' + (x-2)y = 0$

(6) $\left(x - \frac{\pi}{2}\right)^2 y'' + (\cos x)\,y' + (\sin x)\,y = 0$

§5.5 Series Solutions Near a Regular Singular Pt, Part I

(*) $P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0$

$x_0 = 0$ regular singular pt

$x\,p(x) = x\,\dfrac{Q}{P}$ and $x^2 q(x) = x^2\,\dfrac{R}{P}$ have finite limit as $x \to 0$
and are analytic at $x = 0$

$\Longrightarrow$ $x\,p(x) = \sum p_n x^n$ and $x^2 q(x) = \sum q_n x^n$ for all $|x| < \rho$

$\dfrac{x^2}{P} \times (*)$ $\Longrightarrow$ $x^2 y'' + x\left(x\,p(x)\right) y' + \left(x^2 q(x)\right) y = 0$ (**)

Assume that $p_0 = \lim\limits_{x\to 0} \dfrac{xQ}{P}$ and $q_0 = \lim \dfrac{x^2 R}{P}$, $p_n = q_n = 0$ for $n = 1, 2, \cdots$

$\overset{(**)}{\Longrightarrow}$ $x^2 y'' + p_0\,x\,y' + q_0\,y = 0$ Euler's equation

$p_n \neq 0$ & $q_n \neq 0$ the solution of (**) has essential character of the solution of Euler eq.

$\Longrightarrow$ find $y(x) = x^r \sum\limits_{n=0}^{\infty} a_n x^n = \sum\limits_{n=0} a_n x^{n+r}$ for $x > 0$

Questions $r = ?$, $a_n = ?$, $\rho = ?$

Example 1 $\quad 2x^2y'' - xy' + (1+x)y = 0$

$x_0 = 0$ — regular singular pt

$xp(x) = -\frac{1}{2} \Rightarrow p_0 = -\frac{1}{2}$; $\quad x^2q(x) = \frac{1+x}{2} \Rightarrow q_0 = \frac{1}{2} = q_1$

$2x^2y'' - xy' + y = 0$ — the corresponding Euler eq.

Series Solution $\quad y = \sum_{n=0}^{\infty} a_n x^{n+r}$, $\quad y' = \sum (n+r) a_n x^{n+r-1}$, $\quad y'' = \sum (n+r)(n+r-1) a_n x^{n+r-2}$

$$0 = 2x^2y'' - xy' + (1+x)y$$

$$= \sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} + \underbrace{\sum_{n=0}^{\infty} a_n x^{n+r+1}}_{\substack{m=n+1 \\ \sum_{m=1}^{\infty} a_{m-1} x^{m+r}}}$$

$$= a_0\left[2r(r-1) - r + 1\right]x^r$$

$$+ \sum_{n=1}^{\infty}\left\{\left[2(n+r)(n+r-1) - (n+r) + 1\right]a_n + a_{n-1}\right\}x^{n+r}$$

$$2(n+r)^2 - 3(n+r) + 1 = \left[2(n+r)-1\right]\left[(n+r)-1\right]$$

$$\Longrightarrow \begin{cases} 0 = 2r(r-1) - r + 1 = (r-1)(2r-1) \Rightarrow r_1 = 1,\ r_2 = \frac{1}{2} \\ \text{the indicial eq.} \\ 0 = \left[2(n+r)-1\right](n+r-1)a_n + a_{n-1} \Rightarrow a_n = \dfrac{-a_{n-1}}{\left[2(n+r)-1\right](n+r-1)} \text{ for } n \geq 1 \end{cases}$$

$\underline{r = r_1 = 1}$ $\quad a_n = -\dfrac{a_{n-1}}{n(2n+1)}$ for $n \geq 1$

$$a_1 = -\frac{1}{1\cdot 3}a_0, \quad a_2 = -\frac{a_1}{2\cdot 5} = (-1)^2 \frac{a_0}{(1\cdot 2)(3\cdot 5)} = (-1)^2 \frac{a_0\, 2\cdot 4}{2!\ 5!} = (-1)^2 \frac{2^2}{5!} a_0$$

$$a_3 = -\frac{a_2}{3\cdot 7} = (-1)^3 \frac{a_0}{(1\cdot 2\cdot 3)(3\cdot 5\cdot 7)} = (-1)^3 \frac{a_0 (2\cdot 4\cdot 6)}{3!\ 7!} = (-1)^3 \frac{2^3}{7!} a_0$$

$$\Longrightarrow a_n = (-1)^n \frac{2^n}{(2n+1)!} a_0 \quad \text{for } n \geq 1$$

$$\Longrightarrow y_1(x) = \frac{1}{a_0}\sum_{n=0}^{\infty} a_n x^{n+1} = x \left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{2^n}{(2n+1)!} x^n\right]$$

$$L(x) = \lim \left|\frac{a_{n+1} x^{n+1}}{a_n x^n}\right| = \lim \frac{2|x|}{(2n+2)(2n+3)} = 0 \Longrightarrow \rho = +\infty.$$

$\underline{r = r_2 = \frac{1}{2}}$ $\quad a_n = -\dfrac{a_{n-1}}{n(2n-1)}$ for $n \geq 1 \Longrightarrow a_n = (-1)^n \dfrac{2^n}{(2n)!} a_0$

$$\Longrightarrow y_2(x) = x^{\frac{1}{2}} \left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{2^n}{(2n)!} x^n\right] \quad \text{with } \rho = +\infty$$

$$\Longrightarrow \quad y(x) = c_1 y_1(x) + c_2 y_2(x) \quad \text{for } x > 0$$

<u>regular singular pt at $x_0 \neq 0$</u>

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r}$$

§5.6 Series Solutions Near a Regular Singular Pt, Part II

$$L[y] = x^2 y'' + x\left(x p(x)\right) y' + \left(x^2 q(x)\right) y = 0$$

Assumptions (1) $x p(x) = \sum_{n=0}^{\infty} p_n x^n$, $x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n$ converge for $|x| < \rho$

(2) $x_0 = 0$ is a regular singular pt

Euler's eq. $x^2 y'' + p_0 x y' + q_0 y = 0$

the indicial eq. $\boxed{0 \doteq F(r)} \equiv r(r-1) + p_0 r + q_0$

Question $? = y = \varphi(r, x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$

$$\Longrightarrow 0 = L[y] = L[\varphi](r, x) = a_0 F(r) x^r + \sum_{n=1}^{\infty} \left\{ F(r+n) a_n + \sum_{k=0}^{n-1} \left[(r+k) p_{n-k} + q_{n-k}\right] a_k \right\} x^{n+r}$$

$$\Longrightarrow \begin{cases} F(r) = 0 \Longrightarrow \text{roots } \underline{r_1 \geq r_2} \text{ (real)} & \text{exponents at the singularity} \\ F(r+n) a_n + \sum_{k=0}^{n-1} \left[(r+k) p_{n-k} + q_{n-k}\right] a_k = 0 \text{ for } n \geq 1 & \text{recurrence relation} \end{cases}$$

$$\xrightarrow{F(r+n) \neq 0} \quad a_n = -\frac{1}{F(r+n)} \sum_{k=0}^{n-1} \left[(r+k) p_{n-k} + q_{n-k}\right] a_k$$

one solution $r_1 \geq r_2 \Longrightarrow r_1 + n \neq r_1$ or r_2 for $n \geq 1 \Longrightarrow F(r_1+n) \neq 0$

$$\Longrightarrow y_1(x) = x^{r_1}\left[1 + \sum_{n=1}^{\infty} a_n(r_1)\, x^n\right] \quad \text{for } x>0$$

2nd solution (1) $r_2 \neq r_1$ and $r_1 - r_2 \neq$ positive integer

$$\Longrightarrow r_2 + n \neq r_1 \Longrightarrow F(r_2+n) \neq 0$$

$$\Longrightarrow y_2(x) = x^{r_2}\left[1 + \sum_{n=1}^{\infty} a_n(r_2)\, x^n\right] \quad \text{for } x>0$$

Remarks (a) $y_1(x)$ and $y_2(x)$ converge at least for all $|x| < \rho$;

(b) $1 + \sum a_n(r_i)\, x^n$ define functions that are analytic at $x_0 = 0$;

(c) the singular behavior is due to the factor x^{r_1} and x^{r_2};

(d) when $x<0$, replacing x^{r_i} by $|x|^{r_i}$;

(e) $r_1 = \bar{r}_2$, two real-valued series solutions: real and imaginary parts

(f) $r_1 = r_2$ or $r_1 - r_2 = N$ to be discussed later.

Steps (a) $p_0 = \lim_{x\to 0} x\, p(x)$ and $q_0 = \lim_{x \to 0} x^2 q(x)$

(b) solving $0 = F(r) = r(r-1) + p_0 r + q_0$ for r_1 and r_2

(c) determine ρ_p and ρ_q for $x\,p(x) = \sum p_n x^n$ and $x^2 q(x) = \sum q_n x^n$, resp.

Example 1 $2x(1+x)y'' + (3+x)y' - xy = 0$

(a) regular singular pts

$$P(x) = 2x(1+x) = 0 \Longrightarrow \text{singular pts}: \ x=0, \ x=-1$$

$x=0$ $\quad p_0 = \lim_{x\to 0} x\frac{Q}{P} = \lim_{x\to 0}\frac{3+x}{2(1+x)} = \frac{3}{2}, \quad q_0 = \lim_{x\to 0} x^2\frac{R}{P} = 0 \Longrightarrow$ Regular singular pt.

$x=-1$ $\quad p_0 = \lim_{x\to -1}(x+1)\frac{Q}{P} = -1, \quad q_0 = \lim_{x\to -1}(x+1)^2\frac{R}{P} = 0 \Longrightarrow$ Regular singular pt.

(b) the indicial eqs and exponents

$x=0$ $\quad 0 = r(r-1) + \frac{3}{2}r = r\left(r+\frac{1}{2}\right) \Longrightarrow r_1 = 0, \ r_2 = -\frac{1}{2}$

$x=-1$ $\quad 0 = r(r-1) - r = r(r-2) \Longrightarrow r_1 = 2, \ r_2 = 0$

(c) series solutions and their behaviors

$x=0$ $\quad r_1 = 0 > r_2 = -\frac{1}{2}$ and $r_1 - r_2 = \frac{1}{2} \neq$ integer

$$y_1(x) = 1 + \sum_{n=1}^{\infty} a_n(0)\, x^n \qquad \text{bounded as } x\to 0$$

$$y_2(x) = |x|^{-\frac{1}{2}}\left[1 + \sum_{n=1}^{\infty} a_n\left(-\frac{1}{2}\right)x^n\right] \qquad \text{unbounded as } x\to 0$$

converge for $|x| < \rho$ with $\rho \geq \min\{\rho_p, \rho_q\} = \text{dist}\{0, \text{the nearest zero of } P\}$
$= 1$

$\underline{x=-1}$ $r_1 = 2 > r_2 = 0$, but $r_1 - r_2 = 2$ — integer

$$y_1(x) = (x+1)^2 \left[1 + \sum_{n=1}^{\infty} a_n(2)\, x^n \right] \quad \text{bounded as } x \to -1$$

converges for $|x+1| < \rho$ with $\rho \geq 1$.

$y_2(x) = ?$

$\underline{2^{nd} \text{ solution}}$ $\underline{(2)\ r_1 = r_2}$

$$y_2(x) = y_1(x) \ln|x| + |x|^{r_1} \sum_{n=1}^{\infty} b_n(r_1)\, x^n \quad \text{with } b_n(r_1) = \left.\frac{d a_n(r)}{dr}\right|_{r=r_1}$$

(3) $\underline{r_1 - r_2 = N}$

$$y_2(x) = a\, y_1(x) \ln|x| + |x|^{r_2} \left[1 + \sum_{n=1}^{\infty} c_n(r_2)\, x^n \right]$$

where $c_n(r_2) = \frac{d}{dr}\left[(r - r_2)\, a_n(r) \right]_{r=r_2}$ and $a = \lim_{r \to r_2} (r - r_2)\, a(r)$.