Chapter 7 Systems of 1^{st} Order Linear Equations

§7.1 Introduction

- higher order DEs $\Longrightarrow$ system of 1^{st} order DEs

$$u''+u'+2u=0 \xrightarrow[x_2=u']{x_1=u} \begin{cases} x_1'=x_2 \\ x_2'=u''=-2u-u' \\ \qquad =-2x_1-x_2 \end{cases} \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$u^{(4)}+u'''-u=0 \xrightarrow[\substack{x_3=u'' \\ x_4=u'''}]{\substack{x_1=u \\ x_2=u'}} \begin{cases} x_1'=x_2 \\ x_2'=x_3 \\ x_3'=x_4 \\ x_4'=u^{(4)}=u-u''' \\ \qquad =x_1-x_3 \end{cases} \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

- system of 1^{st} order DEs $\Longrightarrow$ a higher order DE

$$\begin{cases} x_1'=x_2 \\ x_2'=-2x_1-x_2 \end{cases} \xrightarrow{\text{diff. } 2^{nd} \text{ eq.}} x_2''=-2x_1'-x_2' \xlongequal{\text{use } 1^{st} \text{ eq}} -2x_2-x_2' \Longrightarrow x_2''+x_2'+2x_2=0$$

$$\begin{cases} x_1'=3x_1-2x_2 \\ x_2'=2x_1-2x_2 \end{cases} \qquad \begin{cases} x_1(0)=3 \\ x_2(0)=\frac{1}{2} \end{cases}$$

$$\xRightarrow{\text{diff } 2^{nd} \text{ eq.}} x_2''=2x_1'-2x_2' \xlongequal[\text{use } 1^{st} \text{ eq}]{} 2(3x_1-2x_2)-2x_2'=6x_1-4x_2-2x_2'$$

$$\xlongequal{\text{use } 2^{nd} \text{ eq.}} 6\left[\frac{1}{2}x_2'+x_2\right]-4x_2-2x_2' = x_2'+2x_2 \Longrightarrow x_2''-x_2'-2x_2=0$$

ICs $\quad x_2(0)=\frac{1}{2}, \quad x_2'(0)=2x_1(0)-2x_2(0)=6-1=5.$

1^{st}-Order System

$$(*)\quad \begin{cases} \vec{x}\,' = \vec{F}(t,\vec{x}) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}, \quad \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \vec{F} = \begin{pmatrix} F_1(t,\vec{x}) \\ \vdots \\ F_n(t,\vec{x}) \end{pmatrix}$$

initial condition

- linear system $\quad \vec{F}(t,\vec{x}) = P_{n\times n}(t)\,\vec{x} + \vec{g}(t)$

Thrm F_i, $\dfrac{\partial F_i}{\partial x_j}$ are continuous on $\left\{ \begin{matrix} \alpha < t < \beta \\ \alpha_i < x_i < \beta_i \end{matrix} \right\}$ containing $(t_0, \vec{x}_0)$

$\Longrightarrow$ $\exists$ an interval $|t - t_0| < h$ s.t. $(*)$ has a unique solution.

§7.2 Review of Matrices

$$A = (a_{ij})_{n\times n}, \quad A^t = (a_{ji}), \quad \bar{A} = (\bar{a}_{ij}), \quad A^* = \bar{A}^t$$

- $A_{n\times p} \pm B_{n\times p}$, αA, $A_{n\times r} B_{r\times m} = (c_{ij})_{n\times m}$, $c_{ij} = \sum_{k=1}^{r} a_{ik} b_{kj}$

- $x^t y = \sum_{i=1}^{n} x_i y_i$, $(x, y) = \sum_i \bar{x}_i y_i$

- Gaussian Elimination

- $\dfrac{d}{dt}(A + B) = \dfrac{d}{dt}A + \dfrac{d}{dt}B$, $\dfrac{d}{dt}(AB) = \left(\dfrac{d}{dt}A\right)B + A\dfrac{d}{dt}B$

examples (P376) #2, 10, 12

§7.3 Systems of Linear Algebraic Eq; Linear Indep., Eigenvalue, Eigenvector

System of Linear Equations

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \quad \cdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = b_n \end{cases} \qquad A_{n\times n}\vec{x}_{n\times 1} = \vec{b}_{n\times 1}$$

- $\vec{b} = \vec{0}$ homogeneous, $\vec{b} \neq \vec{0}$ non-homogeneous
- A is nonsingular : $\det(A) \neq 0 \iff A^{-1}$ exists.

  $\vec{x} = A^{-1}\vec{b} \quad (\vec{b} = \vec{0} \Rightarrow \vec{x} = \vec{0})$
- A is singular : $\det(A) = 0 \iff A^{-1}$ does not exists.

  (1) $A\vec{x} = \vec{0}$ has (infinite many) nonzero solutions.

  (2) $\vec{b} \perp \{\vec{y} \mid A^*\vec{y} = \vec{0}\}$ or $\vec{b} \in \text{Range}(A)$

  $\Longrightarrow A\vec{x} = \vec{b}$ has solutions of the form

  $$\vec{x} = \underbrace{\vec{\xi}}_{\text{general solution of } A\vec{x}=\vec{0}} + \underbrace{\vec{x}^{(0)}}_{\text{a particular solution}}$$

Examples (1) Find solution of $A\vec{x}=\vec{b}$

$$\left(\begin{array}{ccc:c} 1 & -2 & 3 & 7 \\ -1 & 1 & -2 & -5 \\ 2 & -1 & -1 & 4 \end{array}\right) \xrightarrow{GE} \left(\begin{array}{ccc:c} 1 & -2 & 3 & 7 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

(2) Discuss solutions of $A\vec{x}=\vec{b}$ for various of b_i, $A=\begin{pmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & 3 \end{pmatrix}$.

$$\left(\begin{array}{ccc:c} 1 & -2 & 3 & b_1 \\ -1 & 1 & -2 & b_2 \\ 2 & -1 & 3 & b_3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc:c} 1 & -2 & 3 & b_1 \\ 0 & 1 & -1 & -b_1-b_2 \\ 0 & 0 & 0 & b_1+3b_2+b_3 \end{array}\right)$$

$$\Longrightarrow b_1+3b_2+b_3=0 \Longleftrightarrow \vec{b} \perp \underbrace{\{\vec{y} \mid A^*\vec{y}=\vec{0}\}}_{= \operatorname{span}\left\{\begin{pmatrix}1\\3\\1\end{pmatrix}\right\}}$$

example $\vec{b}=(2,1,-5)^t$

$$\Longrightarrow \begin{cases} x_1-2x_2+3x_3=2 \\ x_2-x_3=-3 \end{cases} \Longrightarrow \begin{cases} x_1=-x_3-4 \\ x_2=x_3-3 \\ x_3=x_3 \end{cases} \Longrightarrow \vec{x}=x_3\begin{pmatrix}-1\\1\\1\end{pmatrix}+\begin{pmatrix}-4\\-3\\1\end{pmatrix}$$

Linear Dependence $\vec{x}^{(1)},\cdots,\vec{x}^{(k)}$ are linearly dep.

$$\Longleftrightarrow \exists\ c_1,\cdots,c_k \text{ not all zeros s.t. } \sum_{i=1}^{k} c_i\vec{x}^{(i)}=\vec{0}$$

Linear Indep. $\vec{x}^{(1)},\cdots,\vec{x}^{(k)}$ are linearly indep.

$$\Longleftrightarrow \text{"} \sum_{i=1}^{k} c_i\vec{x}^{(i)}=\vec{0} \Longrightarrow c_i=0 \text{ for } i=1,\cdots,k. \text{"}$$

$k=n$ and $\vec{x}^{(i)} \in R^n$

$$\sum_{i=1}^{n} c_i \vec{x}^{(i)} = \vec{0} \iff X\vec{c} = \vec{0} \quad \text{with } X = \left(\vec{x}^{(1)}, \cdots, \vec{x}^{(n)}\right)$$

$$\Longrightarrow \quad \text{l. dep.} \quad \det X = 0.$$
$$\text{l. indep.} \quad \det X \neq 0.$$

<u>example</u> $\vec{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \vec{x}^{(2)} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \vec{x}^{(3)} = \begin{pmatrix} -4 \\ 1 \\ -11 \end{pmatrix}$ l. indep or l. dep. ? if l. dep., find a linear relation.

$\det X = 0 \Rightarrow$ l. dep.

$$X\vec{c} = \vec{0} \Longrightarrow \vec{c} = c_3 \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \Longrightarrow -2\vec{x}^{(1)} + 3\vec{x}^{(2)} + \vec{x}^{(3)} = \vec{0}.$$

<u>Remark</u> $\det(A) \neq 0 \iff$ column (or row) vectors are l. indep.

<u>Definition</u> (1) $\vec{x}^{(1)}(t), \cdots, \vec{x}^{(k)}(t)$ for $t \in (\alpha, \beta)$ are l. dep.

$\iff \exists\, c_1, \cdots, c_k$ not all zero s.t. $c_1\vec{x}^{(1)}(t) + \cdots + c_k\vec{x}^{(k)}(t) = \vec{0} \ \forall t \in (\alpha, \beta)$

(2) otherwise, they are l. indep.

Eigenvalue and Eigenvector

characteristic equation

$$A\vec{x} = \lambda\vec{x} \iff (A-\lambda I)\vec{x} = \vec{0} \implies \underbrace{\det(A-\lambda I)}_{=\,P_n(\lambda)} = 0$$

- $\exists\ \lambda_1, \cdots, \lambda_n$ (may repeat) eigenvalues
- λ appears m times (algebraic multiplicity m)

$$\implies \exists\ 1 \le g \le m \text{ l. indep. eigenvectors}$$

(geometric multiplicity g).

examples

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Self-Adjoint or Hermitian $\quad A^* = A \iff a_{ij} = \bar{a}_{ji}$

(1) $\lambda_1, \cdots, \lambda_n$ are real;

(2) $\exists$ n l. indep. eigenvectors; $\vec{x}^{(1)}, \cdots, \vec{x}^{(n)}$

(3) $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ are eigenvectors corresponding to $\lambda_1 \ne \lambda_2 \implies \vec{x}^{(1)} \perp \vec{x}^{(2)}$

$\implies$ "all eigenvalues are simple $\implies \{\vec{x}^{(1)}, \cdots, \vec{x}^{(n)}\}$ is orthogonal set"

(4) $\lambda_1 < \lambda_2 < \cdots < \lambda_k$ with $m_1 + m_2 + \cdots + m_k = n$

$\implies \exists\ m_i$ eigenvectors corresponding to λ_i.

§7.3 #25

§7.5 Homogeneous Linear Systems with Constant Coefficients

$$\vec{x}'(t) = A_{n\times n}\,\vec{x}(t) \qquad A \text{ — const. matrix}$$

$n=1$ $\quad x'(t) = a\,x(t)$

$$\Rightarrow \quad x(t) = c\,e^{at} \begin{cases} = 0 & \text{the only equilibrium solution} \\ \to 0 & \text{approaching} \quad " \quad " \quad \text{for } a<0 \\ \to \infty & \text{departing from} \quad " \quad " \quad \text{for } a>0 \end{cases}$$

equilibrium solution $\vec{x}$ $\qquad A\vec{x} = \vec{0}$

$$\det(A) \neq 0 \Longrightarrow \vec{x} = \vec{0} \text{ the only equilibrium solution}$$

Question: As $t \to \infty$, do other solutions $\left\langle \begin{matrix}\text{approach}\\ \text{depart from}\end{matrix} \right\rangle$ the equilibrium solution?

$n=2$ $\quad$ phase plane $\quad$ a direction field of tangent vectors to solutions

(Fig. 7.5.1) $\quad \Longleftrightarrow$ vector field of $A\vec{x}$

phase portrait $\quad$ a representative sample of trajectories

Fig. 7.5.2

$\Longrightarrow$ a qualitative understanding

Construction of General Solution

$$\vec{x} = \vec{\xi} e^{rt} \qquad \vec{\xi} - \text{const vector}, \quad r - \text{exponent}$$

$$\vec{x}' = A\vec{x} \implies r\vec{\xi}e^{rt} = A\vec{\xi}e^{rt}$$

$$\implies \boxed{A\vec{\xi} = r\vec{\xi}}$$

examples (1) $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x}$ (2) $\vec{x}' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \vec{x}$

- use phase plane to plot direction field and some trajectories

(1) $r_1 = -1, \quad r_2 = 3$

$\vec{\xi}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \vec{\xi}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\vec{x}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}, \quad \vec{x}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$

(2) $r_1 = -1, \quad r_2 = -4$

$\vec{\xi}_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \quad \vec{\xi}_2 = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$

$\vec{x}_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}, \quad \vec{x}_2 = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$

Determination of r and $\vec{\xi}$ $\qquad A\vec{\xi} = r\vec{\xi}$

eigenvalues: $\det(A - rI) = 0$

eigenvectors: $(A - r_i I)\vec{\xi}_i = \vec{0}$ for $i = 1, \cdots, n$

example $\vec{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \vec{x}$

§7.6 Complex Eigenvalues

$$\vec{x}\,' = A\vec{x}, \qquad \underset{n\times n}{A} \text{ — real} \qquad A\vec{\xi} = r\vec{\xi}$$

Thrm (1) $r_1 = \lambda + i\mu$ is an eigenvalue $\Longrightarrow$ so is $r_2 = \lambda - i\mu$

(2) $\vec{\xi}^{(1)}$ is an eigenvector corresponding to r_1 $\Longrightarrow$ $\vec{\xi}^{(2)} = \overline{\vec{\xi}^{(1)}}$ is an eigenvector corresponding to r_2.

Proof $(A - r_1 I)\vec{\xi}^{(1)} = 0 \Longrightarrow 0 = \overline{(A - r_1 I)\vec{\xi}^{(1)}} = (A - r_2 I)\overline{\vec{\xi}^{(1)}}$.

solutions $\vec{x}^{(1)} = \vec{\xi}^{(1)} e^{r_1 t}$ and $\vec{x}^{(2)} = \overline{\vec{\xi}^{(1)}} e^{\overline{r_1} t}$

$$\vec{\xi} = \vec{a} + i\vec{b}$$

$$\vec{x}^{(1)} = (\vec{a} + i\vec{b})\, e^{(\lambda + i\mu)t} = \vec{u}(t) + i\,\vec{v}(t)$$

$$\vec{u}(t) = e^{\lambda t}\left(\vec{a}\cos(\mu t) - \vec{b}\sin(\mu t)\right) \text{ and } \vec{v}(t) = e^{\lambda t}\left(\vec{a}\sin(\mu t) + \vec{b}\cos(\mu t)\right)$$

examples (1) $\vec{x}\,' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}\vec{x}$, (2) $\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}$, (3) $\vec{x}\,' = \begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix}\vec{x}$

#27 (P419) (a) §7.3 $r_1 \neq \overline{r_1} \Longrightarrow \vec{\xi}^{(1)} = \vec{a} + i\vec{b}$ and $\overline{\vec{\xi}^{(1)}} = \vec{a} - i\vec{b}$ are l. indep.

$$0 \neq W\left[\vec{\xi}^{(1)}, \overline{\vec{\xi}^{(1)}}\right] = \left|\vec{\xi}^{(1)}\ \overline{\vec{\xi}^{(1)}}\right| = -i\,2\left|\vec{a}\ \vec{b}\right| \Longrightarrow \vec{a}, \vec{b} \text{ — l. indep.}$$

(b) $W[\vec{u}(0), \vec{v}(0)] = \left|\vec{a}\ \vec{b}\right| \neq 0 \Longrightarrow \vec{u}(t), \vec{v}(t)$ — l. indep.

§7.7 Fundamental Matrices

$$\vec{x}'(t) = \underset{n\times n}{P(t)}\,\vec{x}(t) \quad \text{on } (\alpha, \beta)$$

fundamental matrix $\quad \underset{n\times n}{\vec{\Psi}}(t) = \left(\vec{x}^{(1)}(t), \cdots, \vec{x}^{(n)}(t)\right)$

$$\Longleftrightarrow \left\{\vec{x}^{(1)}(t), \cdots, \vec{x}^{(n)}(t)\right\} \text{ is a fundamental set of solutions.}$$

example find $\vec{\Psi}(t)$ for $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x}$

$$\vec{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \quad \vec{x}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} \Longrightarrow \Psi(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$$

general solution $\quad \vec{x}(t) = c_1 \vec{x}^{(1)}(t) + \cdots + c_n \vec{x}^{(n)}(t) = \Psi(t)\vec{c}$ with $\vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$

ICs $\vec{x}(t_0) = \vec{x}^0 \Longrightarrow \vec{x}^0 = \Psi(t_0)\vec{c} \Longrightarrow \vec{c} = \Psi^{-1}(t_0)\vec{x}^0$

$$\Longrightarrow \boxed{\vec{x}(t) = \Psi(t)\Psi^{-1}(t_0)\vec{x}^0}$$

the special fundamental matrix $\quad \Phi(t) = \left(\vec{x}^{(1)}, \cdots, \vec{x}^{(n)}\right)$ and $\vec{x}^{(j)}(t_0) = \vec{e}^{(j)}$

$$\Longleftrightarrow \Phi(t_0) = I$$

$$\Longrightarrow \boxed{\vec{x}(t) = \Phi(t)\vec{x}^0}$$

example find $\Phi(t)$ for $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\vec{x}$, $\quad \vec{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$

find $\vec{x}^{(1)}(t)$ by setting $\vec{x}(0) = c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$

$$\Longrightarrow \vec{x}^{(1)}(t) = \begin{pmatrix}\frac{1}{2}(e^{3t}+e^{-t})\\ e^{3t}-e^{-t}\end{pmatrix}$$

$$\vec{x}^{(2)}(0) = \begin{pmatrix}0\\1\end{pmatrix} \Longrightarrow \vec{x}^{(2)}(t) = \begin{pmatrix}\frac{1}{4}(e^{3t}-e^{-t})\\ \frac{1}{2}(e^{3t}+e^{-t})\end{pmatrix}$$

matrix $\exp(At)$

$$\begin{cases} x' = ax \\ x(0) = x_0 \end{cases} \quad a\text{-const} \Longrightarrow x(t) = x_0 e^{at} = x_0\left(1 + \sum_{j=1}^{\infty} \frac{a^j}{j!} t^j\right)$$

$$\begin{cases} \vec{x}'(t) = A\vec{x}(t) \\ \vec{x}(0) = \vec{x}^0 \end{cases} \quad A_{n\times n}\text{-const} \Longrightarrow \vec{x}(t) = \Phi(t)\vec{x}^0 \text{ and } \Phi(0) = I$$
$$= \exp(At)\vec{x}^0$$

$$\exp(At) = I + \sum_{j=1}^{\infty} \frac{t^j}{j!} A^j$$

diagonalizable matrices

$$A_{n\times n} \text{ --- } \left\langle \begin{array}{ll} \text{eigenvalues} & r_1, \cdots, r_n \\ \text{eigenvectors} & \vec{\xi}^{(1)}, \cdots, \vec{\xi}^{(n)} \end{array} \right\rangle \quad T = \left(\vec{\xi}^{(1)}, \cdots, \vec{\xi}^{(n)}\right)$$

$$\Longrightarrow AT = TD \quad \text{with } D = \begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_n \end{pmatrix}$$

transformation $\vec{x}(t) = T\vec{y}(t) \Longrightarrow T\vec{y}'(t) = AT\vec{y}(t)$

$$\Longrightarrow \vec{y}'(t) = D\vec{y}(t) \Longrightarrow \text{fundamental matrix } Q = \begin{pmatrix} e^{r_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{r_n t} \end{pmatrix}$$

$$\Longrightarrow \text{fundamental matrix for } \vec{x}' = A\vec{x} \quad \Psi(t) = TQ = \left(e^{r_1 t}\vec{\xi}^{(1)}, \cdots, e^{r_n t}\vec{\xi}^{(n)}\right).$$

examples 3 and 4

§7.8 Repeated Eigenvalues

$\vec{x}' = A\vec{x}$ where A has a repeated eigenvalues (complex or real)

example $A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$, $\underbrace{r_1 = r_2 = 2}_{\text{eigenvalue}}$, $\vec{\xi} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\underbrace{\text{only one l. indep.}}_{\text{eigenvector}}$

$r = \rho$ with multiplicity k

eigenvectors $\vec{\xi}^{(1)}, \cdots, \vec{\xi}^{(m)}$ are l. indep. $\begin{cases} m = k & A\text{-Hermitian} \\ m < k \end{cases}$

$\Longrightarrow$ m l. indep. solutions $\vec{x}^{(1)}(t) = \vec{\xi}^{(1)} e^{\rho t}, \cdots, \vec{x}^{(m)}(t) = \vec{\xi}^{(m)} e^{\rho t}$

$m < k$ need to construct $k-m$ l. indep. solutions

example $\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \vec{x}$, $\vec{x}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} = \vec{\xi}^{(1)} e^{2t}$

$\vec{x}^{(2)}(t) = ?$

1st try $\vec{x} = \vec{\xi}\, t e^{2t}$, $\vec{\xi} = ?$

$\vec{x}' = A\vec{x} \Longrightarrow \vec{0} = \vec{\xi} e^{2t} + \left(2\vec{\xi} - A\vec{\xi}\right) t e^{2t} \Longrightarrow \vec{\xi} = \vec{0}$

2nd try $\vec{x} = \vec{\xi}\, t e^{2t} + \vec{\eta} e^{2t}$, $\vec{\xi} = ?$, $\vec{\eta} = ?$

$\vec{x}' = A\vec{x} \Longrightarrow \begin{cases} (A - 2I)\vec{\xi} = 0 \Longrightarrow \vec{\xi} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ (A - 2I)\vec{\eta} = \vec{\xi} \end{cases}$

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ -1-\eta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \eta_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{with } \eta_1 \text{ arbitrary}$$

$$\Longrightarrow \vec{x}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} + \eta_1 \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}}_{\vec{x}^{(1)}(t)}$$

choose $\vec{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t}$

$$W\left(\vec{x}^{(1)}(t), \vec{x}^{(2)}(t)\right) = \left| \left(\vec{x}^{(1)}(t) \;\; \vec{x}^{(2)}(t) \right) \right| = -e^{4t} \neq 0$$

general solution $\quad \vec{x}(t) = c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t)$

General Case $\quad r = \rho$ double eigenvalue with only one eigenvector $\vec{\xi}$

$$\vec{x}^{(1)}(t) = \vec{\xi} e^{\rho t} \qquad (A - \rho I)\vec{\xi} = \vec{0}$$

$$\vec{x}^{(2)}(t) = \vec{\xi} t e^{\rho t} + \vec{\eta} e^{\rho t} \qquad (A - \rho I)\vec{\eta} = \vec{\xi}$$

Comparison $\quad r = \rho$ with multiplicity 2

- 2^{nd}-order single eq : $\quad x^{(1)} = e^{\rho t}, \quad x^{(2)} = t e^{\rho t}$
- 1^{st}-order 2x2 system : $\quad \vec{x}^{(1)} = \vec{\xi} e^{\rho t}, \quad \vec{x}^{(2)} = \vec{\xi} t e^{\rho t} + \vec{\eta} e^{\rho t}$

§7.9 Nonhomogeneous Linear Systems

$$\vec{x}'(t) = P_{n\times n}(t)\,\vec{x}(t) + \vec{g}_{n\times 1}(t)$$

general solution $\quad \vec{x}(t) = \underbrace{\sum_{i=1}^{n} c_i \vec{x}^{(i)}(t)}_{\text{general solution of homog. system}} + \underbrace{\vec{v}(t)}_{\text{particular solution of non-homog. system}}$

methods of determine $\vec{v}(t)$

(1) diagonalization $\quad \vec{x}' = A\vec{x} + \vec{g} \quad$ A–diagonalizable const matrix

(i) $r_1, \cdots, r_n$ — eigenvalues

$\vec{\xi}^{(1)}, \cdots, \vec{\xi}^{(n)}$ — eigenvectors

$$T = \left(\vec{\xi}^{(1)}, \cdots, \vec{\xi}^{(n)}\right) \text{ and } D = \begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_n \end{pmatrix}$$

(ii) $\vec{x} = T\vec{y} \implies \vec{y}' = D\vec{y} + T^{-1}\vec{g}(t) = D\vec{y} + \vec{h}(t)$

$$\Rightarrow \begin{cases} y_1' = r_1 y_1 + h_1 \\ \quad\vdots \\ y_n' = r_n y_n + h_n \end{cases} \implies y_j = e^{r_j t}\int_{t_0}^{t} e^{-r_j s} h_j(s)\,ds + c_j e^{r_j t}$$

example $\quad A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \quad \vec{g} = \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$

$$D = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad T^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{cases} y_1' = -3y_1 + \frac{1}{\sqrt{2}}\left(2e^{-t} - 3t\right) \\ y_2' = -y_2 + \frac{1}{\sqrt{2}}\left(2e^{-t} + t\right) \end{cases}$$

(2) undetermined coefficients (§3.5)

applicable to const matrix P, special forms of $\vec{g}$

review of §3.5 $\quad L[y] \equiv y'' + p(t)\,y' + q(t)\,y = g(t)$

$$y = \underbrace{c_1 y_1(t) + c_2 y_2(t)}_{\text{general solution of } L[y]=0} + \underbrace{Y(t)}_{\text{particular solution}}$$

$$g(t) = e^{\alpha t} \Longrightarrow Y(t) = \begin{cases} A e^{\alpha t} & L[e^{\alpha t}] \neq 0 \\ A t e^{\alpha t} & L[e^{\alpha t}] = 0 \end{cases}$$

$$\vec{g} = \vec{u} e^{\lambda t}, \ \lambda\text{-simple eigenvalue} \Longrightarrow \vec{v} = \vec{b} e^{\lambda t} + \vec{a} t e^{\lambda t}$$

the previous example $\quad \vec{g}(t) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} t, \quad D = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}$

$$\vec{v} = \left(\vec{b} e^{-t} + \vec{a} t e^{-t} \right) + \left(\vec{c} t + \vec{d} \right)$$

$$\vec{v}' = A\vec{v} + \vec{g} \Longrightarrow A\vec{a} = -\vec{a}, \quad A\vec{b} = \vec{a} - \vec{b} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad A\vec{c} = -\begin{pmatrix} 0 \\ 3 \end{pmatrix}, \quad A\vec{d} = \vec{c}$$

$\vec{a} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad$ can be solved only if $\alpha = 1$ $\quad \vec{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \vec{d} = -\frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix}$

$\alpha \neq 0$ arbitrary $\quad \vec{b} = k \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\forall k$

$k = 0 \Rightarrow \vec{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

(3) <u>variation of parameters</u> $\vec{x}' = P(t)\vec{x} + \vec{g}(t)$

$$\begin{cases} \Psi_{n\times n}(t) & \text{— fundamental matrix} \\ \vec{x}(t) = \Psi(t)\vec{c} & \text{— general solution} \end{cases} \quad \text{for } \vec{x}' = P(t)\vec{x}$$

<u>seek $\vec{x}(t) = \Psi_{n\times n}(t)\,\vec{u}(t)$ for $\vec{x}' = P\vec{x} + \vec{g}$</u>

$$\underset{\Psi'\vec{u} + \Psi\vec{u}'}{\underset{\parallel}{(\Psi\vec{u})'}} = P\Psi\vec{u} + \vec{g}$$

$$\xrightarrow{\Psi' = P\Psi} \Psi\vec{u}' = \vec{g} \Rightarrow \vec{u}' = \Psi^{-1}\vec{g} \Rightarrow \vec{u}(t) = \int \Psi^{-1}\vec{g}\,dt + \vec{c}$$

$$\Rightarrow \vec{x}(t) = \Psi\vec{c} + \Psi\int \Psi^{-1}\vec{g}\,dt$$

<u>ICs</u> $\vec{x}(t_0) = \vec{x}^{(0)}$,

$$\vec{x}(t) = \Phi(t)\vec{x}^{(0)} + \Phi\int_{t_0}^{t} \Phi^{-1}(s)\,\vec{g}(s)\,ds$$

<u>the previous example</u>

$$\vec{u}' = \Psi^{-1}\vec{g} = \begin{pmatrix} e^{2t} - \frac{3}{2}te^{3t} \\ 1 + \frac{3}{2}te^{t} \end{pmatrix}$$

$$\vec{u}(t) = \vec{c} + \begin{pmatrix} \frac{1}{2}e^{2t} - \frac{1}{2}te^{3t} + \frac{1}{6}e^{3t} \\ t + \frac{3}{2}te^{t} - \frac{3}{2}e^{t} \end{pmatrix}$$

$$\vec{x}(t) = \Psi\,\vec{u}(t)$$