Name: ____________

PUID#: ____________

Midterm 1– Math 303 (02/13/17)

SHOW ALL RELEVANT WORK!!!

1. (20pts) Determine the radius of convergence of the given power series.

-5 $\rho = \frac{1}{3}$

(a) $\sum_{n=0}^{\infty} \frac{(x-1)^n}{n3^n}$

$$L = \overline{\lim_{n\to\infty}} \left| \frac{\frac{(x-1)^{n+1}}{(n+1)3^{n+1}}}{\frac{(x-1)^n}{n\,3^n}} \right| = \overline{\lim_{n\to\infty}} \frac{n}{3(n+1)} |x-1|$$

$$= \frac{1}{3}|x-1| < 1 \implies |x-1| < 3 = \rho$$

(b) $1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

$$L = \overline{\lim_{n\to\infty}} \left| \frac{\frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2}}{\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}} \right| = \overline{\lim_{n\to\infty}} \frac{1}{4(n+1)^2} x^2 = 0 < 1$$

$$\implies \rho = \infty$$

4 $L = \frac{1}{4}x^2 < 1 \Rightarrow \rho = 2$

5 $\rho = \infty$ but not enough evidence

2. (20pts) For the differential equation $y'' - xy' - y = 0$ at at $x_0 = 0$, **(a)** find the recurrence relation of the series solution, **(b)** find three terms in each of two linearly independent series solutions, and **(c)** if possible, find the general term in each solution.

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$0 = y'' - xy' - y = \underbrace{\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}}_{= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n} - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - (n+1) a_n \right] x^n \quad 4$$

(a) $\quad a_{n+2} = \frac{1}{n+2} a_n \quad \text{for } n = 0, 1, 2, \cdots \quad 2$

(c) & (b) $\quad a_2 = \frac{1}{2} a_0 \qquad\qquad a_3 = \frac{1}{3} a_1 \quad 2$

$a_4 = \frac{1}{4} a_2 = \frac{1}{2\cdot 4} a_0 \quad 2 \qquad a_5 = \frac{1}{5} a_3 = \frac{1}{3\cdot 5} a_1$

$$a_{2k} = \frac{1}{2\cdot 4 \cdots (2k)} a_0 = \frac{1}{2^k k!} a_0 \qquad a_{2k+1} = \frac{1}{3\cdot 5 \cdots (2k+1)} a_1 = \frac{2^k k!}{(2k+1)!} a_1$$

$$y(x) = \left[a_0 + a_2 x^2 + \cdots + a_{2k} x^{2k} + \cdots \right] + \left[a_1 x + a_3 x^3 + \cdots + a_{2k+1} x^{2k+1} + \cdots \right]$$

$$= a_0 \left[1 + \frac{1}{2} x^2 + \frac{1}{2\cdot 4} x^4 + \cdots + \frac{1}{2^k k!} x^{2k} + \cdots \right] \quad 2 \quad 3$$

$$+ a_1 \left[x + \frac{1}{3} x^3 + \frac{1}{3\cdot 5} x^5 + \cdots + \frac{2^k k!}{(2k+1)!} x^{2k+1} + \cdots \right] \quad 2 \quad 3$$

3. (10pts) Determine the lower bounds for the radiuses of convergence of series solutions at $x_0 = 4$ and $x_0 = 0$ for

$$(x^2 - 2x - 3)y'' + xy' + 4y = 0.$$

$$0 = P(x) = x^2 - 2x - 3 = (x+1)(x-3)$$

$$\Rightarrow \quad x = -1, \quad x = 3 \qquad 4$$

$$\begin{cases} \text{dist}(-1, 4) = 5 \\ \text{dist}(3, 4) = 1 \end{cases} \Longrightarrow \rho = 1 \quad \text{for } x_0 = 4$$

3

$$\begin{cases} \text{dist}(-1, 0) = 1 \\ \text{dist}(3, 0) = 3 \end{cases} \Longrightarrow \rho = 1 \quad \text{for } x_0 = 0.$$

3

4. (10pts) Consider the differential equation

$$2x^2y'' - xy' + (1+x)y = 0.$$

(a) Show that $x = 0$ is a regular singular point.
(b) Find the indicial equation and its roots.

(a) $P(x) = 2x^2 = 0 \Rightarrow x = 0$ is a singular pt

$$\lim_{x\to 0} x\frac{Q}{P} = \lim_{x\to 0} x \cdot \frac{-x}{2x^2} = -\frac{1}{2} = p_0$$

$$\lim_{x\to 0} x^2\frac{R}{P} = \lim_{x\to 0} x^2 \cdot \frac{1+x}{2x^2} = \frac{1}{2} = q_0$$

$\Rightarrow x = 0$ is a regular singular point

(b) the corresponding Euler eq. $y'' - \frac{1}{2}xy' + \frac{1}{2}y = 0$

$$\Rightarrow r(r-1) - \frac{1}{2}r + \frac{1}{2} = 0 = r^2 - \frac{3}{2}r + \frac{1}{2} = \frac{1}{2}(2r-1)(r-1)$$

$\Rightarrow r = \frac{1}{2}, 1$

5. (10pts) Find all singular points of the differential equation

$$(1-x^2)^2y'' + x(1-x)y' + (1+x)y = 0$$

and determine whether each one is regular or irregular.

$$0 = P(x) = (1-x^2)^2 \Rightarrow x^2 = 1 \Rightarrow x = -1, \ x = 1$$

singular pts 4

$\underline{x=-1}$ $$\lim_{x\to -1} (x+1)\frac{Q}{P} = \lim_{x\to -1} (x+1)\frac{x(1-x)}{(1-x)^2(1+x)^2} = \infty$$

$x = -1$ is irregular singular pt 3

$\underline{x=1}$ $$\lim_{x\to 1} (x-1)\frac{Q}{P} = \lim_{x\to 1} (x-1)\frac{x(1-x)}{(1-x)^2(1+x)^2} = -\frac{1}{4}$$

$$\lim_{x\to 1} (x-1)^2\frac{R}{P} = \lim_{x\to 1} (x-1)^2\frac{1+x}{(1-x)^2(1+x)^2} = \frac{1}{2}$$

$x = 1$ is regular singular pt. 3

6. (15pts) Determine the general solution of $x^2y'' - xy' + y = 0$ in any interval not including the singular point.

$$0 = r(r-1) - r + 1 = r^2 - 2r + 1 = (r-1)^2 \Rightarrow r = 1$$

$$y(x) = c_1 x + c_2 x \ln x$$

7. (15pts) Determine the general solution of $x^2y'' + 3xy' + 5y = 0$ in any interval not including the singular point.

$$0 = r(r-1) + 3r + 5 = r^2 + 2r + 5 = (r+1)^2 + 4$$

$$\Rightarrow r = -1 \pm 2i$$

$$y(x) = c_1 x^{-1} \cos(2 \ln x) + c_2 x^{-1} \sin(2 \ln x)$$

-4 if r is wrong

-8 solution form is partially wrong.

10:30

99	89	77	66
98	89	77	59
97	86	73	57
97	86	72	50
96	84	72	47
96	82	72	
94	82	72	
93	81	70	
92	81		
92			
91			
90			
12	9	8	5 = 34

12

100	99	92	89	77	62
100	98	91	86	74	61
	96	91	85	74	58
	96	91	85	70	
	95	90	83		
	95		83		
	94				
	93				
	93				
	93				
2	15		6	4	3 = 30