6.2 Solution of Initial Value Problems

In this section we show how the Laplace transform can be used to solve initial value problems for linear differential equations with constant coefficients. The usefulness of the Laplace transform for this purpose rests primarily on the fact that the transform of f' is related in a simple way to the transform of f. The relationship is expressed in the following theorem.

Theorem 6.2.1 Suppose that f is continuous and f' is piecewise continuous on any interval $0 \le t \le A$. Suppose further that there exist constants K, a, and M such that $|f(t)| \le Ke^{at}$ for $t \ge M$. Then $\mathcal{L}\{f'(t)\}$ exists for $s > a$, and moreover,

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0). \tag{1}$$

To prove this theorem, we consider the integral

$$\int_0^A e^{-st} f'(t)\, dt,$$

whose limit as $A \to \infty$, if it exists, is the Laplace transform of f'. To calculate this limit we first need to write the integral in a suitable form. If f' has points of discontinuity in the interval $0 \le t \le A$, let them be denoted by $t_1, t_2, \dots, t_k$. Then we can write the integral as

$$\int_0^A e^{-st} f'(t)\, dt = \int_0^{t_1} e^{-st} f'(t)\, dt + \int_{t_1}^{t_2} e^{-st} f'(t)\, dt + \cdots + \int_{t_k}^{A} e^{-st} f'(t)\, dt.$$

Integrating each term on the right by parts yields

$$\int_0^A e^{-st} f'(t)\, dt = e^{-st} f(t)\Big|_0^{t_1} + e^{-st} f(t)\Big|_{t_1}^{t_2} + \cdots + e^{-st} f(t)\Big|_{t_k}^{A}$$
$$+ s\left[\int_0^{t_1} e^{-st} f(t)\, dt + \int_{t_1}^{t_2} e^{-st} f(t)\, dt + \cdots + \int_{t_k}^{A} e^{-st} f(t)\, dt\right].$$

Since f is continuous, the contributions of the integrated terms at $t_1, t_2, \dots, t_k$ cancel. Further, the integrals on the right side can be combined into a single integral, so that we obtain

$$\int_0^A e^{-st} f'(t)\, dt = e^{-sA} f(A) - f(0) + s\int_0^A e^{-st} f(t)\, dt. \tag{2}$$

Now we let $A \to \infty$ in Eq. (2). The integral on the right side of this equation approaches $\mathcal{L}\{f(t)\}$. Further, for $A \ge M$, we have $|f(A)| \le Ke^{aA}$; consequently, $|e^{-sA} f(A)| \le Ke^{-(s-a)A}$. Hence $e^{-sA} f(A) \to 0$ as $A \to \infty$ whenever $s > a$. Thus the right side of Eq. (2) has the limit $s\mathcal{L}\{f(t)\} - f(0)$. Consequently, the left side of Eq. (2) also has a limit, and as noted above, this limit is $\mathcal{L}\{f'(t)\}$. Therefore, for $s > a$, we conclude that

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0),$$

which establishes the theorem.

If f' and f'' satisfy the same conditions that are imposed on f and f', respectively, in Theorem 6.2.1, then it follows that the Laplace transform of f'' also exists for $s > a$ and is given by

$$\begin{aligned}\mathcal{L}\{f''(t)\} &= s\mathcal{L}\{f'(t)\} - f'(0)\\ &= s[s\mathcal{L}\{f(t)\} - f(0)] - f'(0)\\ &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).\end{aligned} \tag{3}$$

Indeed, provided the function f and its derivatives satisfy suitable conditions, an expression for the transform of the nth derivative $f^{(n)}$ can be derived by n successive applications of this theorem. The result is given in the following corollary.

Corollary 6.2.2 Suppose that the functions $f, f', \dots, f^{(n-1)}$ are continuous and that $f^{(n)}$ is piecewise continuous on any interval $0 \le t \le A$. Suppose further that there exist constants K, a, and M such that $|f(t)| \le Ke^{at}$, $|f'(t)| \le Ke^{at}, \dots, |f^{(n-1)}(t)| \le Ke^{at}$ for $t \ge M$. Then $\mathcal{L}\{f^{(n)}(t)\}$ exists for $s > a$ and is given by

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0). \tag{4}$$

We now show how the Laplace transform can be used to solve initial value problems. It is most useful for problems involving nonhomogeneous differential equations, as we will demonstrate in later sections of this chapter. However, we begin by looking at some homogeneous equations, which are a bit simpler.

EXAMPLE 1

Consider the differential equation

$$y'' - y' - 2y = 0 \tag{5}$$

and the initial conditions

$$y(0) = 1, \qquad y'(0) = 0. \tag{6}$$

This problem is easily solved by the methods of Section 3.1. The characteristic equation is

$$r^2 - r - 2 = (r - 2)(r + 1) = 0,$$

and consequently, the general solution of Eq. (5) is

$$y = c_1e^{-t} + c_2e^{2t}. \tag{7}$$

To satisfy the initial conditions (6), we must have $c_1 + c_2 = 1$ and $-c_1 + 2c_2 = 0$; hence $c_1 = \frac{2}{3}$ and $c_2 = \frac{1}{3}$, so the solution of the initial value problem (5) and (6) is

$$y = \phi(t) = \tfrac{2}{3}e^{-t} + \tfrac{1}{3}e^{2t}. \tag{8}$$

Now let us solve the same problem by using the Laplace transform. To do this, we must assume that the problem has a solution $y = \phi(t)$, which with its first two derivatives satisfies the conditions of Corollary 6.2.2. Then, taking the Laplace transform of the differential equation (5), we obtain

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0, \tag{9}$$

where we have used the linearity of the transform to write the transform of a sum as the sum of the separate transforms. Upon using the corollary to express $\mathcal{L}\{y''\}$ and $\mathcal{L}\{y'\}$ in terms of $\mathcal{L}\{y\}$, we find that Eq. (9) becomes

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - [s\mathcal{L}\{y\} - y(0)] - 2\mathcal{L}\{y\} = 0,$$

or

$$(s^2 - s - 2)Y(s) + (1 - s)y(0) - y'(0) = 0, \tag{10}$$

where $Y(s) = \mathcal{L}\{y\}$. Substituting for $y(0)$ and $y'(0)$ in Eq. (10) from the initial conditions (6), and then solving for $Y(s)$, we obtain

$$Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}. \tag{11}$$

We have thus obtained an expression for the Laplace transform $Y(s)$ of the solution $y = \phi(t)$ of the given initial value problem. To determine the function ϕ, we must find the function whose Laplace transform is $Y(s)$, as given by Eq. (11).

This can be done most easily by expanding the right side of Eq. (11) in partial fractions. Thus we write

$$Y(s) = \frac{s-1}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1) + b(s-2)}{(s-2)(s+1)}, \tag{12}$$

where the coefficients a and b are to be determined. By equating numerators of the second and fourth members of Eq. (12), we obtain

$$s - 1 = a(s+1) + b(s-2),$$

an equation that must hold for all s. In particular, if we set $s = 2$, then it follows that $a = \frac{1}{3}$. Similarly, if we set $s = -1$, then we find that $b = \frac{2}{3}$. By substituting these values for a and b, respectively, we have

$$Y(s) = \frac{1/3}{s-2} + \frac{2/3}{s+1}. \tag{13}$$

Finally, if we use the result of Example 5 of Section 6.1, it follows that $\frac{1}{3}e^{2t}$ has the transform $\frac{1}{3}(s-2)^{-1}$; similarly, $\frac{2}{3}e^{-t}$ has the transform $\frac{2}{3}(s+1)^{-1}$. Hence, by the linearity of the Laplace transform,

$$y = \phi(t) = \tfrac{1}{3}e^{2t} + \tfrac{2}{3}e^{-t}$$

has the transform (13) and is therefore the solution of the initial value problem (5), (6). Observe that it does satisfy the conditions of Corollary 6.2.2, as we assumed initially. Of course, this is the same solution that we obtained earlier.

The same procedure can be applied to the general second order linear equation with constant coefficients

$$ay'' + by' + cy = f(t). \tag{14}$$

Assuming that the solution $y = \phi(t)$ satisfies the conditions of Corollary 6.2.2 for $n = 2$, we can take the transform of Eq. (14) and thereby obtain

$$a[s^2Y(s) - sy(0) - y'(0)] + b[sY(s) - y(0)] + cY(s) = F(s), \tag{15}$$

where $F(s)$ is the transform of $f(t)$. By solving Eq. (15) for $Y(s)$, we find that

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}. \tag{16}$$

The problem is then solved, provided that we can find the function $y = \phi(t)$ whose transform is $Y(s)$.

Even at this early stage of our discussion we can point out some of the essential features of the transform method. In the first place, the transform $Y(s)$ of the unknown function $y = \phi(t)$ is found by solving an *algebraic equation* rather than a *differential equation*, Eq. (10) rather than Eq. (5) in Example 1, or in general Eq. (15) rather than Eq. (14). This is the key to the usefulness of Laplace transforms for solving linear, constant coefficient, ordinary differential equations—the problem is reduced from a differential equation to an algebraic one. Next, the solution satisfying given initial conditions is automatically found, so that the task of determining appropriate values for the arbitrary constants in the general solution does not arise. Further, as indicated in Eq. (15), nonhomogeneous equations are handled in exactly the same way as homogeneous ones; it is not necessary to solve the corresponding homogeneous equation first. Finally, the method can be applied in the same way to higher order equations, as long as we assume that the solution satisfies the conditions of Corollary 6.2.2 for the appropriate value of n.

Observe that the polynomial $as^2 + bs + c$ in the denominator on the right side of Eq. (16) is precisely the characteristic polynomial associated with Eq. (14). Since the use of a partial fraction expansion of $Y(s)$ to determine $\phi(t)$ requires us to factor this polynomial, the use of Laplace transforms does not avoid the necessity of finding roots of the characteristic equation. For equations of higher than second order, this may require a numerical approximation, particularly if the roots are irrational or complex.

The main difficulty that occurs in solving initial value problems by the transform method lies in the problem of determining the function $y = \phi(t)$ corresponding to the transform $Y(s)$. This problem is known as the inversion problem for the Laplace transform; $\phi(t)$ is called the inverse transform corresponding to $Y(s)$, and the process of finding $\phi(t)$ from $Y(s)$ is known as inverting the transform. We also use the notation $\mathcal{L}^{-1}\{Y(s)\}$ to denote the inverse transform of $Y(s)$. There is a general formula for the inverse Laplace transform, but its use requires a familiarity with functions of a complex variable, and we do not consider it in this book. However, it is still possible to develop many important properties of the Laplace transform, and to solve many interesting problems, without the use of complex variables.

In solving the initial value problem (5), (6), we did not consider the question of whether there may be functions other than the one given by Eq. (8) that also have the transform (13). By Theorem 3.2.1 we know that the initial value problem has no other solutions. We also know that the unique solution (8) of the initial value problem is continuous. Consistent with this fact, it can be shown that if f and g are continuous functions with the same Laplace transform, then f and g must be identical. On the other hand, if f and g are only piecewise continuous, then they may differ at one or more points of discontinuity and yet have the same Laplace transform; see Example 6 in Section 6.1. This lack of uniqueness of the inverse Laplace transform for piecewise continuous functions is of no practical significance in applications.

Thus there is essentially a one-to-one correspondence between functions and their Laplace transforms. This fact suggests the compilation of a table, such as Table 6.2.1, giving the transforms of functions frequently encountered, and vice versa. The entries in the second column of Table 6.2.1 are the transforms of those in the first column. Perhaps more important, the functions in the first column are the inverse transforms

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\dfrac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. $t^n, \quad n = \text{positive integer}$	$\dfrac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 31
4. $t^p, \quad p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 31
5. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 7
6. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \lvert a\rvert$	Sec. 6.1; Prob. 8
8. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \lvert a\rvert$	Sec. 6.1; Prob. 7
9. $e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}, \quad n = \text{positive integer}$	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct}f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 25
16. $\displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2; Cor. 6.2.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 29

of those in the second column. Thus, for example, if the transform of the solution of a differential equation is known, the solution itself can often be found merely by looking it up in the table. Some of the entries in Table 6.2.1 have been used as examples, or appear as problems in Section 6.1, while others will be developed later in the chapter. The third column of the table indicates where the derivation of the given transforms may be found. Although Table 6.2.1 is sufficient for the examples and problems in this book, much larger tables are also available (see the list of references at the end of the chapter). Transforms and inverse transforms can also be readily obtained electronically by using a computer algebra system.

Frequently, a Laplace transform $F(s)$ is expressible as a sum of several terms

$$F(s) = F_1(s) + F_2(s) + \cdots + F_n(s). \tag{17}$$

Suppose that $f_1(t) = \mathcal{L}^{-1}\{F_1(s)\}, \ldots, f_n(t) = \mathcal{L}^{-1}\{F_n(s)\}$. Then the function

$$f(t) = f_1(t) + \cdots + f_n(t)$$

has the Laplace transform $F(s)$. By the uniqueness property stated previously, no other continuous function f has the same transform. Thus

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F_1(s)\} + \cdots + \mathcal{L}^{-1}\{F_n(s)\}; \tag{18}$$

that is, the inverse Laplace transform is also a linear operator.

In many problems it is convenient to make use of this property by decomposing a given transform into a sum of functions whose inverse transforms are already known or can be found in the table. Partial fraction expansions are particularly useful for this purpose, and a general result covering many cases is given in Problem 39. Other useful properties of Laplace transforms are derived later in this chapter.

As further illustrations of the technique of solving initial value problems by means of the Laplace transform and partial fraction expansions, consider the following examples.

EXAMPLE 2

Find the solution of the differential equation

$$y'' + y = \sin 2t \tag{19}$$

satisfying the initial conditions

$$y(0) = 2, \qquad y'(0) = 1. \tag{20}$$

We assume that this initial value problem has a solution $y = \phi(t)$, which with its first two derivatives satisfies the conditions of Corollary 6.2.2. Then, taking the Laplace transform of the differential equation, we have

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 2/(s^2 + 4),$$

where the transform of $\sin 2t$ has been obtained from line 5 of Table 6.2.1. Substituting for $y(0)$ and $y'(0)$ from the initial conditions and solving for $Y(s)$, we obtain

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2 + 1)(s^2 + 4)}. \tag{21}$$

Using partial fractions, we can write $Y(s)$ in the form

$$Y(s) = \frac{as + b}{s^2 + 1} + \frac{cs + d}{s^2 + 4} = \frac{(as + b)(s^2 + 4) + (cs + d)(s^2 + 1)}{(s^2 + 1)(s^2 + 4)}. \tag{22}$$

By expanding the numerator on the right side of Eq. (22) and equating it to the numerator in Eq. (21), we find that

$$2s^3 + s^2 + 8s + 6 = (a + c)s^3 + (b + d)s^2 + (4a + c)s + (4b + d)$$

for all s. Then, comparing coefficients of like powers of s, we have

$$a + c = 2, \qquad b + d = 1,$$
$$4a + c = 8, \qquad 4b + d = 6.$$

Consequently, $a = 2$, $c = 0$, $b = \frac{5}{3}$, and $d = -\frac{2}{3}$, from which it follows that

$$Y(s) = \frac{2s}{s^2 + 1} + \frac{5/3}{s^2 + 1} - \frac{2/3}{s^2 + 4}. \tag{23}$$

From lines 5 and 6 of Table 6.2.1, the solution of the given initial value problem is

$$y = \phi(t) = 2\cos t + \tfrac{5}{3}\sin t - \tfrac{1}{3}\sin 2t. \tag{24}$$

EXAMPLE 3

Find the solution of the initial value problem

$$y^{(4)} - y = 0, \tag{25}$$

$$y(0) = 0, \qquad y'(0) = 1, \qquad y''(0) = 0, \qquad y'''(0) = 0. \tag{26}$$

In this problem we need to assume that the solution $y = \phi(t)$ satisfies the conditions of Corollary 6.2.2 for $n = 4$. The Laplace transform of the differential equation (25) is

$$s^4 Y(s) - s^3 y(0) - s^2 y'(0) - sy''(0) - y'''(0) - Y(s) = 0.$$

Then, using the initial conditions (26) and solving for $Y(s)$, we have

$$Y(s) = \frac{s^2}{s^4 - 1}. \tag{27}$$

A partial fraction expansion of $Y(s)$ is

$$Y(s) = \frac{as + b}{s^2 - 1} + \frac{cs + d}{s^2 + 1},$$

and it follows that

$$(as + b)(s^2 + 1) + (cs + d)(s^2 - 1) = s^2 \tag{28}$$

for all s. By setting $s = 1$ and $s = -1$, respectively, in Eq. (28), we obtain the pair of equations

$$2(a + b) = 1, \qquad 2(-a + b) = 1,$$

and therefore $a = 0$ and $b = \frac{1}{2}$. If we set $s = 0$ in Eq. (28), then $b - d = 0$, so $d = \frac{1}{2}$. Finally, equating the coefficients of the cubic terms on each side of Eq. (28), we find that $a + c = 0$, so $c = 0$. Thus

$$Y(s) = \frac{1/2}{s^2 - 1} + \frac{1/2}{s^2 + 1}, \tag{29}$$

and from lines 7 and 5 of Table 6.2.1, the solution of the initial value problem (25), (26) is

$$y = \phi(t) = \frac{\sinh t + \sin t}{2}. \tag{30}$$

The most important elementary applications of the Laplace transform are in the study of mechanical vibrations and in the analysis of electric circuits; the governing equations were derived in Section 3.7. A vibrating spring–mass system has the equation of motion

$$m\frac{d^2u}{dt^2} + \gamma\frac{du}{dt} + ku = F(t), \tag{31}$$

where m is the mass, γ the damping coefficient, k the spring constant, and $F(t)$ the applied external force. The equation that describes an electric circuit containing an inductance L, a resistance R, and a capacitance C (an LRC circuit) is

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t), \tag{32}$$

where $Q(t)$ is the charge on the capacitor and $E(t)$ is the applied voltage. In terms of the current $I(t) = dQ(t)/dt$, we can differentiate Eq. (32) and write

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}(t). \tag{33}$$

Suitable initial conditions on u, Q, or I must also be prescribed.

We have noted previously, in Section 3.7, that Eq. (31) for the spring–mass system and Eqs. (32) or (33) for the electric circuit are identical mathematically, differing only in the interpretation of the constants and variables appearing in them. There are other physical problems that also lead to the same differential equation. Thus, once the mathematical problem is solved, its solution can be interpreted in terms of whichever corresponding physical problem is of immediate interest.

In the problem lists following this and other sections in this chapter are numerous initial value problems for second order linear differential equations with constant coefficients. Many can be interpreted as models of particular physical systems, but usually we do not point this out explicitly.

PROBLEMS

In each of Problems 1 through 10, find the inverse Laplace transform of the given function.

1. $F(s) = \dfrac{3}{s^2 + 4}$
2. $F(s) = \dfrac{4}{(s-1)^3}$
3. $F(s) = \dfrac{2}{s^2 + 3s - 4}$
4. $F(s) = \dfrac{3s}{s^2 - s - 6}$
5. $F(s) = \dfrac{2s + 2}{s^2 + 2s + 5}$
6. $F(s) = \dfrac{2s - 3}{s^2 - 4}$
7. $F(s) = \dfrac{2s + 1}{s^2 - 2s + 2}$
8. $F(s) = \dfrac{8s^2 - 4s + 12}{s(s^2 + 4)}$
9. $F(s) = \dfrac{1 - 2s}{s^2 + 4s + 5}$
10. $F(s) = \dfrac{2s - 3}{s^2 + 2s + 10}$

In each of Problems 11 through 23, use the Laplace transform to solve the given initial value problem.

11. $y'' - y' - 6y = 0; \quad y(0) = 1, \quad y'(0) = -1$
12. $y'' + 3y' + 2y = 0; \quad y(0) = 1, \quad y'(0) = 0$

13. $y'' - 2y' + 2y = 0; \qquad y(0) = 0, \quad y'(0) = 1$
14. $y'' - 4y' + 4y = 0; \qquad y(0) = 1, \quad y'(0) = 1$
15. $y'' - 2y' + 4y = 0; \qquad y(0) = 2, \quad y'(0) = 0$
16. $y'' + 2y' + 5y = 0; \qquad y(0) = 2, \quad y'(0) = -1$
17. $y^{(4)} - 4y''' + 6y'' - 4y' + y = 0; \qquad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 1$
18. $y^{(4)} - y = 0; \qquad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1, \quad y'''(0) = 0$
19. $y^{(4)} - 4y = 0; \qquad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2, \quad y'''(0) = 0$
20. $y'' + \omega^2 y = \cos 2t, \quad \omega^2 \neq 4; \qquad y(0) = 1, \quad y'(0) = 0$
21. $y'' - 2y' + 2y = \cos t; \qquad y(0) = 1, \quad y'(0) = 0$
22. $y'' - 2y' + 2y = e^{-t}; \qquad y(0) = 0, \quad y'(0) = 1$
23. $y'' + 2y' + y = 4e^{-t}; \qquad y(0) = 2, \quad y'(0) = -1$

In each of Problems 24 through 27, find the Laplace transform $Y(s) = \mathcal{L}\{y\}$ of the solution of the given initial value problem. A method of determining the inverse transform is developed in Section 6.3. You may wish to refer to Problems 21 through 24 in Section 6.1.

24. $y'' + 4y = \begin{cases} 1, & 0 \le t < \pi, \\ 0, & \pi \le t < \infty; \end{cases} \qquad y(0) = 1, \quad y'(0) = 0$

25. $y'' + y = \begin{cases} t, & 0 \le t < 1, \\ 0, & 1 \le t < \infty; \end{cases} \qquad y(0) = 0, \quad y'(0) = 0$

26. $y'' + 4y = \begin{cases} t, & 0 \le t < 1, \\ 1, & 1 \le t < \infty; \end{cases} \qquad y(0) = 0, \quad y'(0) = 0$

27. $y'' + y = \begin{cases} t, & 0 \le t < 1, \\ 2 - t, & 1 \le t < 2, \\ 0, & 2 \le t < \infty; \end{cases} \qquad y(0) = 0, \quad y'(0) = 0$

28. The Laplace transforms of certain functions can be found conveniently from their Taylor series expansions.

    (a) Using the Taylor series for $\sin t$

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!},$$

and assuming that the Laplace transform of this series can be computed term by term, verify that

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2 + 1}, \qquad s > 1.$$

    (b) Let

$$f(t) = \begin{cases} (\sin t)/t, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

Find the Taylor series for f about $t = 0$. Assuming that the Laplace transform of this function can be computed term by term, verify that

$$\mathcal{L}\{f(t)\} = \arctan(1/s), \qquad s > 1.$$

(c) The Bessel function of the first kind of order zero, J_0, has the Taylor series (see Section 5.7)

$$J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}.$$

Assuming that the following Laplace transforms can be computed term by term, verify that

$$\mathcal{L}\{J_0(t)\} = (s^2+1)^{-1/2}, \qquad s > 1$$

and

$$\mathcal{L}\{J_0(\sqrt{t})\} = s^{-1}e^{-1/(4s)}, \qquad s > 0.$$

Problems 29 through 37 are concerned with differentiation of the Laplace transform.

29. Let

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

It is possible to show that as long as f satisfies the conditions of Theorem 6.1.2, it is legitimate to differentiate under the integral sign with respect to the parameter s when $s > a$.

(a) Show that $F'(s) = \mathcal{L}\{-tf(t)\}$.

(b) Show that $F^{(n)}(s) = \mathcal{L}\{(-t)^n f(t)\}$; hence differentiating the Laplace transform corresponds to multiplying the original function by $-t$.

In each of Problems 30 through 35, use the result of Problem 29 to find the Laplace transform of the given function; a and b are real numbers and n is a positive integer.

30. $f(t) = te^{at}$

31. $f(t) = t^2 \sin bt$

32. $f(t) = t^n$

33. $f(t) = t^n e^{at}$

34. $f(t) = te^{at} \sin bt$

35. $f(t) = te^{at} \cos bt$

36. Consider Bessel's equation of order zero

$$ty'' + y' + ty = 0.$$

Recall from Section 5.7 that $t = 0$ is a regular singular point for this equation, and therefore solutions may become unbounded as $t \to 0$. However, let us try to determine whether there are any solutions that remain finite at $t = 0$ and have finite derivatives there. Assuming that there is such a solution $y = \phi(t)$, let $Y(s) = \mathcal{L}\{\phi(t)\}$.

(a) Show that $Y(s)$ satisfies

$$(1+s^2)Y'(s) + sY(s) = 0.$$

(b) Show that $Y(s) = c(1+s^2)^{-1/2}$, where c is an arbitrary constant.

(c) Writing $(1+s^2)^{-1/2} = s^{-1}(1+s^{-2})^{-1/2}$, expanding in a binomial series valid for $s > 1$, and assuming that it is permissible to take the inverse transform term by term, show that

$$y = c\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2} = cJ_0(t),$$

where J_0 is the Bessel function of the first kind of order zero. Note that $J_0(0) = 1$ and that J_0 has finite derivatives of all orders at $t = 0$. It was shown in Section 5.7 that the second solution of this equation becomes unbounded as $t \to 0$.

37. For each of the following initial value problems, use the results of Problem 29 to find the differential equation satisfied by $Y(s) = \mathcal{L}\{\phi(t)\}$, where $y = \phi(t)$ is the solution of the given initial value problem.
(a) $y'' - ty = 0;\quad y(0) = 1,\quad y'(0) = 0$ (Airy's equation)
(b) $(1 - t^2)y'' - 2ty' + \alpha(\alpha + 1)y = 0;\quad y(0) = 0,\quad y'(0) = 1$ (Legendre's equation)
Note that the differential equation for $Y(s)$ is of first order in part (a), but of second order in part (b). This is due to the fact that t appears at most to the first power in the equation of part (a), whereas it appears to the second power in that of part (b). This illustrates that the Laplace transform is not often useful in solving differential equations with variable coefficients, unless all the coefficients are at most linear functions of the independent variable.

38. Suppose that
$$g(t) = \int_0^t f(\tau)\,d\tau.$$
If $G(s)$ and $F(s)$ are the Laplace transforms of $g(t)$ and $f(t)$, respectively, show that
$$G(s) = F(s)/s.$$

39. In this problem we show how a general partial fraction expansion can be used to calculate many inverse Laplace transforms. Suppose that
$$F(s) = P(s)/Q(s),$$
where $Q(s)$ is a polynomial of degree n with distinct zeros $r_1, \ldots, r_n$, and $P(s)$ is a polynomial of degree less than n. In this case it is possible to show that $P(s)/Q(s)$ has a partial fraction expansion of the form
$$\frac{P(s)}{Q(s)} = \frac{A_1}{s - r_1} + \cdots + \frac{A_n}{s - r_n}, \tag{i}$$
where the coefficients $A_1, \ldots, A_n$ must be determined.
(a) Show that
$$A_k = P(r_k)/Q'(r_k), \qquad k = 1, \ldots, n. \tag{ii}$$
Hint: One way to do this is to multiply Eq. (i) by $s - r_k$ and then to take the limit as $s \to r_k$.
(b) Show that
$$\mathcal{L}^{-1}\{F(s)\} = \sum_{k=1}^{n} \frac{P(r_k)}{Q'(r_k)} e^{r_k t}. \tag{iii}$$

6.3 Step Functions

In Section 6.2 we outlined the general procedure involved in solving initial value problems by means of the Laplace transform. Some of the most interesting elementary applications of the transform method occur in the solution of linear differential equations with discontinuous or impulsive forcing functions. Equations of this type frequently arise in the analysis of the flow of current in electric circuits or the vibrations of mechanical systems. In this section and the following ones, we develop some additional properties of the Laplace transform that are useful in the solution of such problems. Unless a specific statement is made to the contrary, all functions appearing below will be assumed to be piecewise continuous and of exponential order, so that their Laplace transforms exist, at least for s sufficiently large.

To deal effectively with functions having jump discontinuities, it is very helpful to introduce a function known as the **unit step function** or **Heaviside function**. This function will be denoted by u_c and is defined by

$$u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c. \end{cases} \tag{1}$$

Since the Laplace transform involves values of t in the interval $[0, \infty)$, we are also interested only in nonnegative values of c. The graph of $y = u_c(t)$ is shown in Figure 6.3.1. We have somewhat arbitrarily assigned the value one to u_c at $t = c$. However, for a piecewise continuous function such as u_c, the value at a discontinuity point is usually irrelevant. The step can also be negative. For instance, Figure 6.3.2 shows the graph of $y = 1 - u_c(t)$.

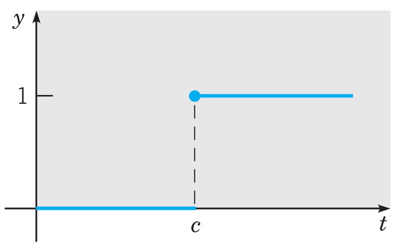

FIGURE 6.3.1 Graph of $y = u_c(t)$.

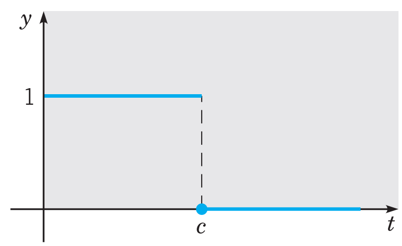

FIGURE 6.3.2 Graph of $y = 1 - u_c(t)$.

EXAMPLE 1

Sketch the graph of $y = h(t)$, where

$$h(t) = u_\pi(t) - u_{2\pi}(t), \qquad t \geq 0.$$

From the definition of $u_c(t)$ in Eq. (1), we have

$$h(t) = \begin{cases} 0 - 0 = 0, & 0 \leq t < \pi, \\ 1 - 0 = 1, & \pi \leq t < 2\pi, \\ 1 - 1 = 0, & 2\pi \leq t < \infty. \end{cases}$$

Thus the equation $y = h(t)$ has the graph shown in Figure 6.3.3. This function can be thought of as a rectangular pulse.

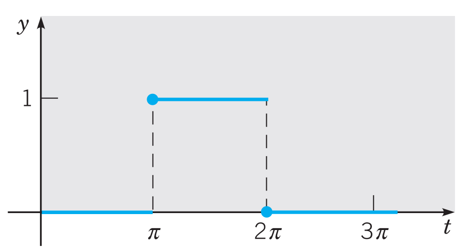

FIGURE 6.3.3 Graph of $y = u_\pi(t) - u_{2\pi}(t)$.

EXAMPLE 2

Consider the function

$$f(t) = \begin{cases} 2, & 0 \le t < 4, \\ 5, & 4 \le t < 7, \\ -1, & 7 \le t < 9, \\ 1, & t \ge 9, \end{cases} \tag{2}$$

whose graph is shown in Figure 6.3.4. Express $f(t)$ in terms of $u_c(t)$.

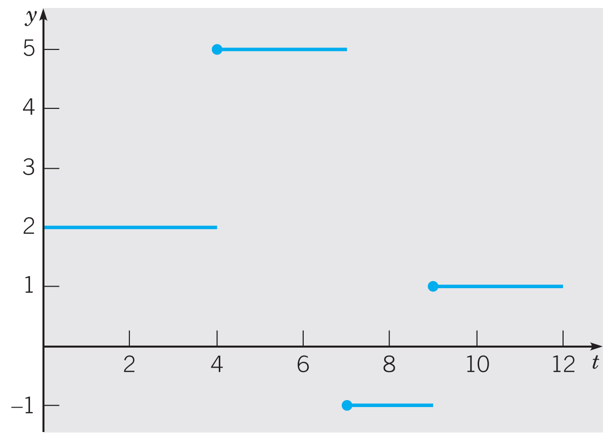

FIGURE 6.3.4 Graph of the function in Eq. (2).

We start with the function $f_1(t) = 2$, which agrees with $f(t)$ on $[0, 4)$. To produce the jump of three units at $t = 4$, we add $3u_4(t)$ to $f_1(t)$, obtaining

$$f_2(t) = 2 + 3u_4(t),$$

which agrees with $f(t)$ on $[0, 7)$. The negative jump of six units at $t = 7$ corresponds to adding $-6u_7(t)$, which gives

$$f_3(t) = 2 + 3u_4(t) - 6u_7(t).$$

Finally, we must add $2u_9(t)$ to match the jump of two units at $t = 9$. Thus we obtain

$$f(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t). \tag{3}$$

The Laplace transform of u_c for $c \ge 0$ is easily determined:

$$\begin{aligned} \mathcal{L}\{u_c(t)\} &= \int_0^\infty e^{-st}u_c(t)\,dt = \int_c^\infty e^{-st}\,dt \\ &= \frac{e^{-cs}}{s}, \qquad s > 0. \end{aligned} \tag{4}$$

For a given function f defined for $t \ge 0$, we will often want to consider the related function g defined by

$$y = g(t) = \begin{cases} 0, & t < c, \\ f(t-c), & t \ge c, \end{cases}$$

which represents a translation of f a distance c in the positive t direction; see Figure 6.3.5. In terms of the unit step function we can write $g(t)$ in the convenient form

$$g(t) = u_c(t)f(t-c).$$

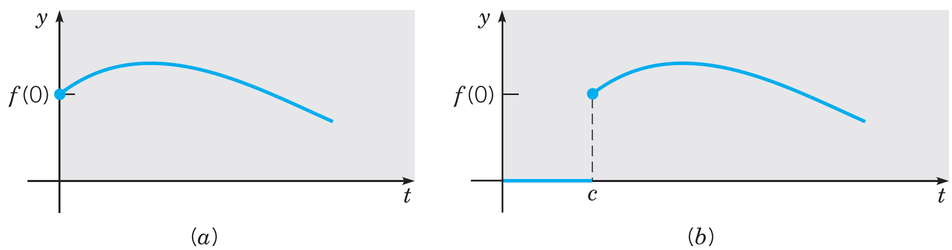

FIGURE 6.3.5 A translation of the given function. (a) $y = f(t)$; (b) $y = u_c(t)f(t-c)$.

The unit step function is particularly important in transform use because of the following relation between the transform of $f(t)$ and that of its translation $u_c(t)f(t-c)$.

Theorem 6.3.1

If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and if c is a positive constant, then

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s), \qquad s > a. \tag{5}$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$u_c(t)f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}. \tag{6}$$

Theorem 6.3.1 simply states that the translation of $f(t)$ a distance c in the positive t direction corresponds to the multiplication of $F(s)$ by e^{-cs}. To prove Theorem 6.3.1, it is sufficient to compute the transform of $u_c(t)f(t-c)$:

$$\begin{aligned}\mathcal{L}\{u_c(t)f(t-c)\} &= \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt\\ &= \int_c^\infty e^{-st}f(t-c)\,dt.\end{aligned}$$

Introducing a new integration variable $\xi = t - c$, we have

$$\begin{aligned}\mathcal{L}\{u_c(t)f(t-c)\} &= \int_0^\infty e^{-(\xi+c)s}f(\xi)\,d\xi = e^{-cs}\int_0^\infty e^{-s\xi}f(\xi)\,d\xi\\ &= e^{-cs}F(s).\end{aligned}$$

Thus Eq. (5) is established; Eq. (6) follows by taking the inverse transform of both sides of Eq. (5).

A simple example of this theorem occurs if we take $f(t) = 1$. Recalling that $\mathcal{L}\{1\} = 1/s$, we immediately have from Eq. (5) that $\mathcal{L}\{u_c(t)\} = e^{-cs}/s$. This result agrees with that of Eq. (4). Examples 3 and 4 illustrate further how Theorem 6.3.1 can be used in the calculation of transforms and inverse transforms.

EXAMPLE 3

If the function f is defined by

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi/4, \\ \sin t + \cos(t - \pi/4), & t \ge \pi/4, \end{cases}$$

find $\mathcal{L}\{f(t)\}$. The graph of $y = f(t)$ is shown in Figure 6.3.6.

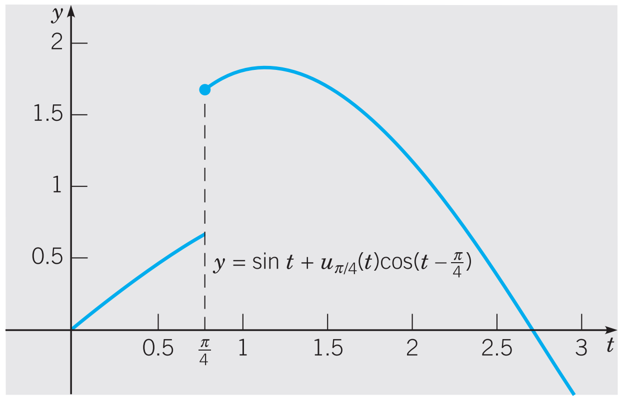

FIGURE 6.3.6 Graph of the function in Example 3.

Note that $f(t) = \sin t + g(t)$, where

$$g(t) = \begin{cases} 0, & t < \pi/4, \\ \cos(t - \pi/4), & t \ge \pi/4. \end{cases}$$

Thus

$$g(t) = u_{\pi/4}(t)\cos(t - \pi/4)$$

and

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{\sin t\} + \mathcal{L}\{u_{\pi/4}(t)\cos(t - \pi/4)\} \\ &= \mathcal{L}\{\sin t\} + e^{-\pi s/4}\mathcal{L}\{\cos t\}. \end{aligned}$$

Introducing the transforms of $\sin t$ and $\cos t$, we obtain

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2+1} + e^{-\pi s/4}\frac{s}{s^2+1} = \frac{1 + se^{-\pi s/4}}{s^2+1}.$$

You should compare this method with the calculation of $\mathcal{L}\{f(t)\}$ directly from the definition.

EXAMPLE 4

Find the inverse transform of

$$F(s) = \frac{1 - e^{-2s}}{s^2}.$$

From the linearity of the inverse transform, we have

$$\begin{aligned} f(t) = \mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} \\ &= t - u_2(t)(t - 2). \end{aligned}$$

The function f may also be written as

$$f(t) = \begin{cases} t, & 0 \le t < 2, \\ 2, & t \ge 2. \end{cases}$$

The following theorem contains another very useful property of Laplace transforms that is somewhat analogous to that given in Theorem 6.3.1.

Theorem 6.3.2 If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \ge 0$, and if c is a constant, then

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c), \qquad s > a + c. \tag{7}$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$e^{ct}f(t) = \mathcal{L}^{-1}\{F(s-c)\}. \tag{8}$$

According to Theorem 6.3.2, multiplication of $f(t)$ by e^{ct} results in a translation of the transform $F(s)$ a distance c in the positive s direction, and conversely. To prove this theorem, we evaluate $\mathcal{L}\{e^{ct}f(t)\}$. Thus

$$\begin{aligned} \mathcal{L}\{e^{ct}f(t)\} &= \int_0^\infty e^{-st}e^{ct}f(t)\,dt = \int_0^\infty e^{-(s-c)t}f(t)\,dt \\ &= F(s-c), \end{aligned}$$

which is Eq. (7). The restriction $s > a + c$ follows from the observation that, according to hypothesis (ii) of Theorem 6.1.2, $|f(t)| \le Ke^{at}$; hence $|e^{ct}f(t)| \le Ke^{(a+c)t}$. Equation (8) is obtained by taking the inverse transform of Eq. (7), and the proof is complete.

The principal application of Theorem 6.3.2 is in the evaluation of certain inverse transforms, as illustrated by Example 5.

EXAMPLE 5

Find the inverse transform of

$$G(s) = \frac{1}{s^2 - 4s + 5}.$$

By completing the square in the denominator, we can write

$$G(s) = \frac{1}{(s-2)^2 + 1} = F(s-2),$$

where $F(s) = (s^2+1)^{-1}$. Since $\mathcal{L}^{-1}\{F(s)\} = \sin t$, it follows from Theorem 6.3.2 that

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = e^{2t}\sin t.$$

The results of this section are often useful in solving differential equations, particularly those that have discontinuous forcing functions. The next section is devoted to examples illustrating this fact.

PROBLEMS

In each of Problems 1 through 6, sketch the graph of the given function on the interval $t \geq 0$.

1. $g(t) = u_1(t) + 2u_3(t) - 6u_4(t)$
2. $g(t) = (t-3)u_2(t) - (t-2)u_3(t)$
3. $g(t) = f(t-\pi)u_\pi(t), \quad$ where $f(t) = t^2$
4. $g(t) = f(t-3)u_3(t), \quad$ where $f(t) = \sin t$
5. $g(t) = f(t-1)u_2(t), \quad$ where $f(t) = 2t$
6. $g(t) = (t-1)u_1(t) - 2(t-2)u_2(t) + (t-3)u_3(t)$

In each of Problems 7 through 12:

(a) Sketch the graph of the given function.

(b) Express $f(t)$ in terms of the unit step function $u_c(t)$.

7. $f(t) = \begin{cases} 0, & 0 \leq t < 3, \\ -2, & 3 \leq t < 5, \\ 2, & 5 \leq t < 7, \\ 1, & t \geq 7. \end{cases}$
8. $f(t) = \begin{cases} 1, & 0 \leq t < 1, \\ -1, & 1 \leq t < 2, \\ 1, & 2 \leq t < 3, \\ -1, & 3 \leq t < 4, \\ 0, & t \geq 4. \end{cases}$
9. $f(t) = \begin{cases} 1, & 0 \leq t < 2, \\ e^{-(t-2)}, & t \geq 2. \end{cases}$
10. $f(t) = \begin{cases} t^2, & 0 \leq t < 2, \\ 1, & t \geq 2. \end{cases}$
11. $f(t) = \begin{cases} t, & 0 \leq t < 1, \\ t-1, & 1 \leq t < 2, \\ t-2, & 2 \leq t < 3, \\ 0, & t \geq 3. \end{cases}$
12. $f(t) = \begin{cases} t, & 0 \leq t < 2, \\ 2, & 2 \leq t < 5, \\ 7-t, & 5 \leq t < 7, \\ 0, & t \geq 7. \end{cases}$

In each of Problems 13 through 18, find the Laplace transform of the given function.

13. $f(t) = \begin{cases} 0, & t < 2 \\ (t-2)^2, & t \geq 2 \end{cases}$
14. $f(t) = \begin{cases} 0, & t < 1 \\ t^2 - 2t + 2, & t \geq 1 \end{cases}$
15. $f(t) = \begin{cases} 0, & t < \pi \\ t-\pi, & \pi \leq t < 2\pi \\ 0, & t \geq 2\pi \end{cases}$
16. $f(t) = u_1(t) + 2u_3(t) - 6u_4(t)$
17. $f(t) = (t-3)u_2(t) - (t-2)u_3(t)$
18. $f(t) = t - u_1(t)(t-1), \qquad t \geq 0$

In each of Problems 19 through 24, find the inverse Laplace transform of the given function.

19. $F(s) = \dfrac{3!}{(s-2)^4}$
20. $F(s) = \dfrac{e^{-2s}}{s^2 + s - 2}$
21. $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2 - 2s + 2}$
22. $F(s) = \dfrac{2e^{-2s}}{s^2 - 4}$
23. $F(s) = \dfrac{(s-2)e^{-s}}{s^2 - 4s + 3}$
24. $F(s) = \dfrac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}$
25. Suppose that $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$.
    (a) Show that if c is a positive constant, then

$$\mathcal{L}\{f(ct)\} = \frac{1}{c}F\left(\frac{s}{c}\right), \qquad s > ca.$$

(b) Show that if k is a positive constant, then

$$\mathcal{L}^{-1}\{F(ks)\} = \frac{1}{k} f\left(\frac{t}{k}\right).$$

(c) Show that if a and b are constants with $a > 0$, then

$$\mathcal{L}^{-1}\{F(as+b)\} = \frac{1}{a} e^{-bt/a} f\left(\frac{t}{a}\right).$$

In each of Problems 26 through 29, use the results of Problem 25 to find the inverse Laplace transform of the given function.

26. $F(s) = \dfrac{2^{n+1} n!}{s^{n+1}}$

27. $F(s) = \dfrac{2s+1}{4s^2+4s+5}$

28. $F(s) = \dfrac{1}{9s^2-12s+3}$

29. $F(s) = \dfrac{e^2 e^{-4s}}{2s-1}$

In each of Problems 30 through 33, find the Laplace transform of the given function. In Problem 33, assume that term-by-term integration of the infinite series is permissible.

30. $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$

31. $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \\ 1, & 2 \le t < 3 \\ 0, & t \ge 3 \end{cases}$

32. $f(t) = 1 - u_1(t) + \cdots + u_{2n}(t) - u_{2n+1}(t) = 1 + \displaystyle\sum_{k=1}^{2n+1} (-1)^k u_k(t)$

33. $f(t) = 1 + \displaystyle\sum_{k=1}^{\infty} (-1)^k u_k(t)$. See Figure 6.3.7.

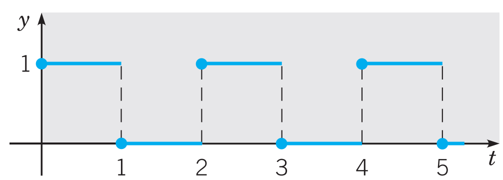

FIGURE 6.3.7 The function $f(t)$ in Problem 33; a square wave.

34. Let f satisfy $f(t+T) = f(t)$ for all $t \ge 0$ and for some fixed positive number T; f is said to be periodic with period T on $0 \le t < \infty$. Show that

$$\mathcal{L}\{f(t)\} = \frac{\displaystyle\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}.$$

In each of Problems 35 through 38, use the result of Problem 34 to find the Laplace transform of the given function.

35. $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & 1 \le t < 2; \end{cases}$

$f(t+2) = f(t)$.

Compare with Problem 33.

36. $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ -1, & 1 \le t < 2; \end{cases}$

$f(t+2) = f(t)$.

See Figure 6.3.8.

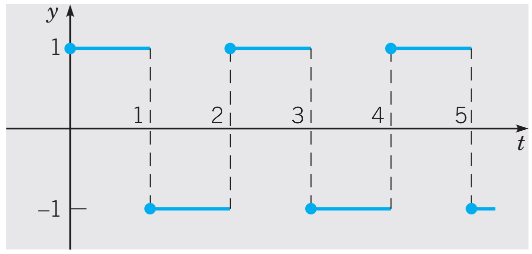

FIGURE 6.3.8 The function $f(t)$ in Problem 36; a square wave.

37. $f(t) = t, \quad 0 \le t < 1;$
$f(t+1) = f(t).$
See Figure 6.3.9.

38. $f(t) = \sin t, \quad 0 \le t < \pi;$
$f(t+\pi) = f(t).$
See Figure 6.3.10.

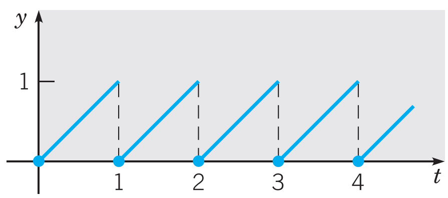

FIGURE 6.3.9 The function $f(t)$ in Problem 37; a sawtooth wave.

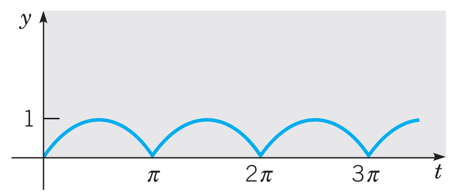

FIGURE 6.3.10 The function $f(t)$ in Problem 38; a rectified sine wave.

39. (a) If $f(t) = 1 - u_1(t)$, find $\mathcal{L}\{f(t)\}$; compare with Problem 30. Sketch the graph of $y = f(t)$.

(b) Let $g(t) = \int_0^t f(\xi)\, d\xi$, where the function f is defined in part (a). Sketch the graph of $y = g(t)$ and find $\mathcal{L}\{g(t)\}$.

(c) Let $h(t) = g(t) - u_1(t)g(t-1)$, where g is defined in part (b). Sketch the graph of $y = h(t)$ and find $\mathcal{L}\{h(t)\}$.

40. Consider the function p defined by

$$p(t) = \begin{cases} t, & 0 \le t < 1, \\ 2 - t, & 1 \le t < 2; \end{cases} \qquad p(t+2) = p(t).$$

(a) Sketch the graph of $y = p(t)$.

(b) Find $\mathcal{L}\{p(t)\}$ by noting that p is the periodic extension of the function h in Problem 39(c) and then using the result of Problem 34.

(c) Find $\mathcal{L}\{p(t)\}$ by noting that

$$p(t) = \int_0^t f(t)\, dt,$$

where f is the function in Problem 36, and then using Theorem 6.2.1.

6.4 Differential Equations with Discontinuous Forcing Functions

In this section we turn our attention to some examples in which the nonhomogeneous term, or forcing function, is discontinuous.

EXAMPLE 1

Find the solution of the differential equation

$$2y'' + y' + 2y = g(t), \tag{1}$$

where

$$g(t) = u_5(t) - u_{20}(t) = \begin{cases} 1, & 5 \le t < 20, \\ 0, & 0 \le t < 5 \quad \text{and} \quad t \ge 20. \end{cases} \tag{2}$$

Assume that the initial conditions are

$$y(0) = 0, \qquad y'(0) = 0. \tag{3}$$

This problem governs the charge on the capacitor in a simple electric circuit with a unit voltage pulse for $5 \le t < 20$. Alternatively, y may represent the response of a damped oscillator subject to the applied force $g(t)$.

The Laplace transform of Eq. (1) is

$$\begin{aligned} 2s^2Y(s) - 2sy(0) - 2y'(0) + sY(s) - y(0) + 2Y(s) &= \mathcal{L}\{u_5(t)\} - \mathcal{L}\{u_{20}(t)\} \\ &= (e^{-5s} - e^{-20s})/s. \end{aligned}$$

Introducing the initial values (3) and solving for $Y(s)$, we obtain

$$Y(s) = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)}. \tag{4}$$

To find $y = \phi(t)$, it is convenient to write $Y(s)$ as

$$Y(s) = (e^{-5s} - e^{-20s})H(s), \tag{5}$$

where

$$H(s) = \frac{1}{s(2s^2 + s + 2)}. \tag{6}$$

Then, if $h(t) = \mathcal{L}^{-1}\{H(s)\}$, we have

$$y = \phi(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20). \tag{7}$$

Observe that we have used Theorem 6.3.1 to write the inverse transforms of $e^{-5s}H(s)$ and $e^{-20s}H(s)$, respectively. Finally, to determine $h(t)$, we use the partial fraction expansion of $H(s)$:

$$H(s) = \frac{a}{s} + \frac{bs + c}{2s^2 + s + 2}. \tag{8}$$

Upon determining the coefficients, we find that $a = \frac{1}{2}$, $b = -1$, and $c = -\frac{1}{2}$. Thus

$$\begin{aligned} H(s) &= \frac{1/2}{s} - \frac{s + \frac{1}{2}}{2s^2 + s + 2} = \frac{1/2}{s} - \left(\frac{1}{2}\right)\frac{\left(s + \frac{1}{4}\right) + \frac{1}{4}}{\left(s + \frac{1}{4}\right)^2 + \frac{15}{16}} \\ &= \frac{1/2}{s} - \left(\frac{1}{2}\right)\left[\frac{s + \frac{1}{4}}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2} + \frac{1}{\sqrt{15}}\frac{\sqrt{15}/4}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2}\right]. \end{aligned} \tag{9}$$

Then, by referring to lines 9 and 10 of Table 6.2.1, we obtain

$$h(t) = \tfrac{1}{2} - \tfrac{1}{2}\left[e^{-t/4}\cos(\sqrt{15}\,t/4) + (\sqrt{15}/15)e^{-t/4}\sin(\sqrt{15}\,t/4)\right]. \tag{10}$$

In Figure 6.4.1 the graph of $y = \phi(t)$ from Eqs. (7) and (10) shows that the solution consists of three distinct parts. For $0 < t < 5$, the differential equation is

$$2y'' + y' + 2y = 0, \tag{11}$$

and the initial conditions are given by Eq. (3). Since the initial conditions impart no energy to the system, and since there is no external forcing, the system remains at rest; that is, $y = 0$ for $0 < t < 5$. This can be confirmed by solving Eq. (11) subject to the initial conditions (3). In particular, evaluating the solution and its derivative at $t = 5$, or, more precisely, as t approaches 5 from below, we have

$$y(5) = 0, \qquad y'(5) = 0. \tag{12}$$

Once $t > 5$, the differential equation becomes

$$2y'' + y' + 2y = 1, \tag{13}$$

whose solution is the sum of a constant (the response to the constant forcing function) and a damped oscillation (the solution of the corresponding homogeneous equation). The plot in Figure 6.4.1 shows this behavior clearly for the interval $5 \le t \le 20$. An expression for this portion of the solution can be found by solving the differential equation (13) subject to the initial conditions (12). Finally, for $t > 20$ the differential equation becomes Eq. (11) again, and the initial conditions are obtained by evaluating the solution of Eqs. (13), (12) and its derivative at $t = 20$. These values are

$$y(20) \cong 0.50162, \qquad y'(20) \cong 0.01125. \tag{14}$$

The initial value problem (11), (14) contains no external forcing, so its solution is a damped oscillation about $y = 0$, as can be seen in Figure 6.4.1.

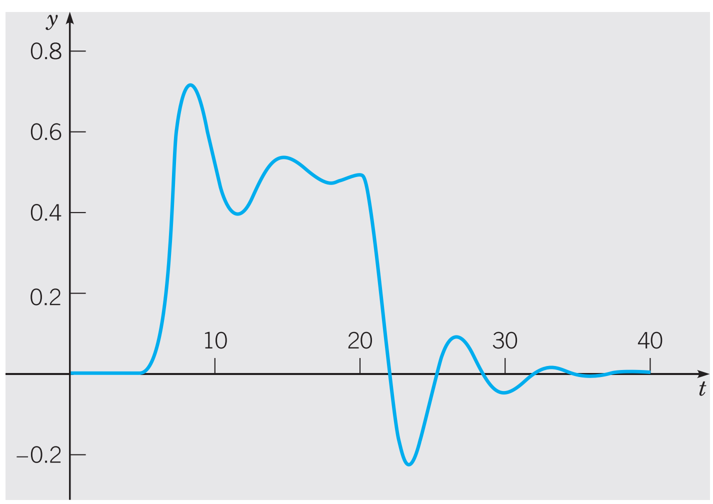

FIGURE 6.4.1 Solution of the initial value problem (1), (2), (3): $2y'' + y' + 2y = u_5(t) - u_{20}(t)$, $y(0) = 0$, $y'(0) = 0$.

Although it may be helpful to visualize the solution shown in Figure 6.4.1 as composed of solutions of three separate initial value problems in three separate intervals, it is somewhat tedious to find the solution by solving these separate problems. Laplace transform methods provide a much more convenient and elegant approach to this problem and to others that have discontinuous forcing functions.

The effect of the discontinuity in the forcing function can be seen if we examine the solution $\phi(t)$ of Example 1 more closely. According to the existence and uniqueness Theorem 3.2.1, the solution ϕ and its first two derivatives are continuous except possibly at the points $t = 5$ and $t = 20$, where g is discontinuous. This can also be seen at once from Eq. (7). One can also show by direct computation from Eq. (7) that ϕ and ϕ' are continuous even at $t = 5$ and $t = 20$. However, if we calculate ϕ'', we find that

$$\lim_{t\to 5-} \phi''(t) = 0, \qquad \lim_{t\to 5+} \phi''(t) = 1/2.$$

Consequently, $\phi''(t)$ has a jump of $1/2$ at $t = 5$. In a similar way, we can show that $\phi''(t)$ has a jump of $-1/2$ at $t = 20$. Thus the jump in the forcing term $g(t)$ at these points is balanced by a corresponding jump in the highest order term $2y''$ on the left side of the equation.

Consider now the general second order linear equation

$$y'' + p(t)y' + q(t)y = g(t), \tag{15}$$

where p and q are continuous on some interval $\alpha < t < \beta$, but g is only piecewise continuous there. If $y = \psi(t)$ is a solution of Eq. (15), then ψ and ψ' are continuous on $\alpha < t < \beta$, but ψ'' has jump discontinuities at the same points as g. Similar remarks apply to higher order equations; the highest derivative of the solution appearing in the differential equation has jump discontinuities at the same points as the forcing function, but the solution itself and its lower derivatives are continuous even at those points.

EXAMPLE 2

Describe the qualitative nature of the solution of the initial value problem

$$y'' + 4y = g(t), \tag{16}$$

$$y(0) = 0, \qquad y'(0) = 0, \tag{17}$$

where

$$g(t) = \begin{cases} 0, & 0 \le t < 5, \\ (t-5)/5, & 5 \le t < 10, \\ 1, & t \ge 10, \end{cases} \tag{18}$$

and then find the solution.

In this example the forcing function has the graph shown in Figure 6.4.2 and is known as ramp loading. It is relatively easy to identify the general form of the solution. For $t < 5$ the solution is simply $y = 0$. On the other hand, for $t > 10$ the solution has the form

$$y = c_1 \cos 2t + c_2 \sin 2t + 1/4. \tag{19}$$

The constant $1/4$ is a particular solution of the nonhomogeneous equation, while the other two terms are the general solution of the corresponding homogeneous equation. Thus the solution (19) is a simple harmonic oscillation about $y = 1/4$. Similarly, in the intermediate range $5 < t < 10$, the solution is an oscillation about a certain linear function. In an engineering context, for example, we might be interested in knowing the amplitude of the eventual steady oscillation.

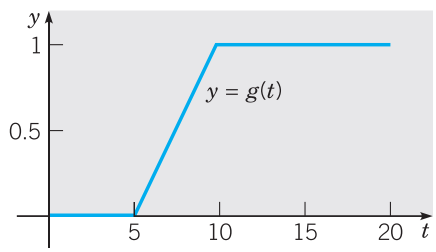

FIGURE 6.4.2 Ramp loading; $y = g(t)$ from Eq. (18) or Eq. (20).

To solve the problem, it is convenient to write

$$g(t) = \left[u_5(t)(t-5) - u_{10}(t)(t-10)\right]/5, \tag{20}$$

as you may verify. Then we take the Laplace transform of the differential equation and use the initial conditions, thereby obtaining

$$(s^2+4)Y(s) = (e^{-5s} - e^{-10s})/5s^2$$

or

$$Y(s) = (e^{-5s} - e^{-10s})H(s)/5, \tag{21}$$

where

$$H(s) = \frac{1}{s^2(s^2+4)}. \tag{22}$$

Thus the solution of the initial value problem (16), (17), (18) is

$$y = \phi(t) = \left[u_5(t)h(t-5) - u_{10}(t)h(t-10)\right]/5, \tag{23}$$

where $h(t)$ is the inverse transform of $H(s)$. The partial fraction expansion of $H(s)$ is

$$H(s) = \frac{1/4}{s^2} - \frac{1/4}{s^2+4}, \tag{24}$$

and it then follows from lines 3 and 5 of Table 6.2.1 that

$$h(t) = \tfrac{1}{4}t - \tfrac{1}{8}\sin 2t. \tag{25}$$

The graph of $y = \phi(t)$ is shown in Figure 6.4.3. Observe that it has the qualitative form that we indicated earlier. To find the amplitude of the eventual steady oscillation, it is sufficient to locate one of the maximum or minimum points for $t > 10$. Setting the derivative of the solution (23) equal to zero, we find that the first maximum is located approximately at (10.642, 0.2979), so the amplitude of the oscillation is approximately 0.0479.

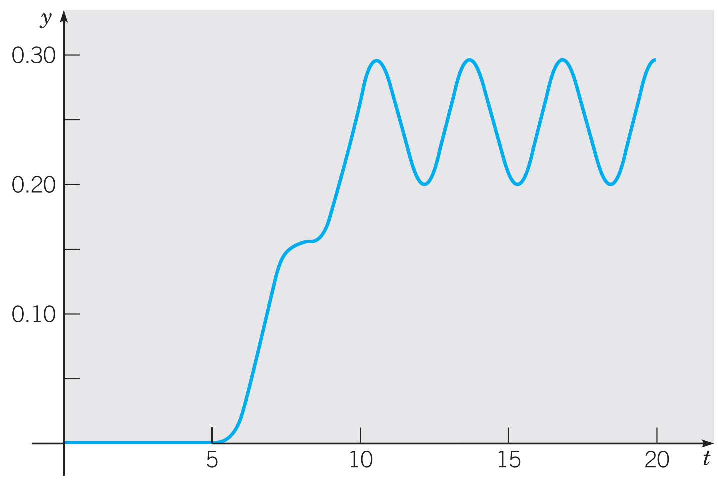

FIGURE 6.4.3 Solution of the initial value problem (16), (17), (18).

Note that in this example, the forcing function g is continuous but g' is discontinuous at $t = 5$ and $t = 10$. It follows that the solution ϕ and its first two derivatives are continuous everywhere, but ϕ''' has discontinuities at $t = 5$ and at $t = 10$ that match the discontinuities in g' at those points.

PROBLEMS

In each of Problems 1 through 13:
(a) Find the solution of the given initial value problem.
(b) Draw the graphs of the solution and of the forcing function; explain how they are related.

1. $y'' + y = f(t); \quad y(0) = 0, \quad y'(0) = 1; \quad f(t) = \begin{cases} 1, & 0 \le t < 3\pi \\ 0, & 3\pi \le t < \infty \end{cases}$
2. $y'' + 2y' + 2y = h(t); \quad y(0) = 0, \quad y'(0) = 1; \quad h(t) = \begin{cases} 1, & \pi \le t < 2\pi \\ 0, & 0 \le t < \pi \quad \text{and} \quad t \ge 2\pi \end{cases}$
3. $y'' + 4y = \sin t - u_{2\pi}(t)\sin(t - 2\pi); \quad y(0) = 0, \quad y'(0) = 0$
4. $y'' + 4y = \sin t + u_{\pi}(t)\sin(t - \pi); \quad y(0) = 0, \quad y'(0) = 0$
5. $y'' + 3y' + 2y = f(t); \quad y(0) = 0, \quad y'(0) = 0; \quad f(t) = \begin{cases} 1, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$
6. $y'' + 3y' + 2y = u_2(t); \quad y(0) = 0, \quad y'(0) = 1$
7. $y'' + y = u_{3\pi}(t); \quad y(0) = 1, \quad y'(0) = 0$
8. $y'' + y' + \frac{5}{4}y = t - u_{\pi/2}(t)(t - \pi/2); \quad y(0) = 0, \quad y'(0) = 0$
9. $y'' + y = g(t); \quad y(0) = 0, \quad y'(0) = 1; \quad g(t) = \begin{cases} t/2, & 0 \le t < 6 \\ 3, & t \ge 6 \end{cases}$
10. $y'' + y' + \frac{5}{4}y = g(t); \quad y(0) = 0, \quad y'(0) = 0; \quad g(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$
11. $y'' + 4y = u_{\pi}(t) - u_{3\pi}(t); \quad y(0) = 0, \quad y'(0) = 0$

12. $y^{(4)} - y = u_1(t) - u_2(t); \qquad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$
13. $y^{(4)} + 5y'' + 4y = 1 - u_\pi(t); \qquad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$
14. Find an expression involving $u_c(t)$ for a function f that ramps up from zero at $t = t_0$ to the value h at $t = t_0 + k$.
15. Find an expression involving $u_c(t)$ for a function g that ramps up from zero at $t = t_0$ to the value h at $t = t_0 + k$ and then ramps back down to zero at $t = t_0 + 2k$.
16. A certain spring–mass system satisfies the initial value problem

$$u'' + \tfrac{1}{4}u' + u = kg(t), \qquad u(0) = 0, \quad u'(0) = 0,$$

where $g(t) = u_{3/2}(t) - u_{5/2}(t)$ and $k > 0$ is a parameter.

(a) Sketch the graph of $g(t)$. Observe that it is a pulse of unit magnitude extending over one time unit.

(b) Solve the initial value problem.

(c) Plot the solution for $k = 1/2, k = 1$, and $k = 2$. Describe the principal features of the solution and how they depend on k.

(d) Find, to two decimal places, the smallest value of k for which the solution $u(t)$ reaches the value 2.

(e) Suppose $k = 2$. Find the time τ after which $|u(t)| < 0.1$ for all $t > \tau$.

17. Modify the problem in Example 2 of this section by replacing the given forcing function $g(t)$ by

$$f(t) = \left[u_5(t)(t-5) - u_{5+k}(t)(t-5-k)\right]/k.$$

(a) Sketch the graph of $f(t)$ and describe how it depends on k. For what value of k is $f(t)$ identical to $g(t)$ in the example?

(b) Solve the initial value problem

$$y'' + 4y = f(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

(c) The solution in part (b) depends on k, but for sufficiently large t the solution is always a simple harmonic oscillation about $y = 1/4$. Try to decide how the amplitude of this eventual oscillation depends on k. Then confirm your conclusion by plotting the solution for a few different values of k.

18. Consider the initial value problem

$$y'' + \tfrac{1}{3}y' + 4y = f_k(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where

$$f_k(t) = \begin{cases} 1/2k, & 4-k \le t < 4+k \\ 0, & 0 \le t < 4-k \quad \text{and} \quad t \ge 4+k \end{cases}$$

and $0 < k < 4$.

(a) Sketch the graph of $f_k(t)$. Observe that the area under the graph is independent of k. If $f_k(t)$ represents a force, this means that the product of the magnitude of the force and the time interval during which it acts does not depend on k.

(b) Write $f_k(t)$ in terms of the unit step function and then solve the given initial value problem.

(c) Plot the solution for $k = 2, k = 1$, and $k = \frac{1}{2}$. Describe how the solution depends on k.

Resonance and Beats. In Section 3.8 we observed that an undamped harmonic oscillator (such as a spring–mass system) with a sinusoidal forcing term experiences resonance if the frequency of the forcing term is the same as the natural frequency. If the forcing frequency is slightly different from the natural frequency, then the system exhibits a beat. In Problems 19 through 23 we explore the effect of some nonsinusoidal periodic forcing functions.

19. Consider the initial value problem

$$y'' + y = f(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where

$$f(t) = u_0(t) + 2\sum_{k=1}^{n}(-1)^k u_{k\pi}(t).$$

(a) Draw the graph of $f(t)$ on an interval such as $0 \le t \le 6\pi$.

(b) Find the solution of the initial value problem.

(c) Let $n = 15$ and plot the graph of the solution for $0 \le t \le 60$. Describe the solution and explain why it behaves as it does.

(d) Investigate how the solution changes as n increases. What happens as $n \to \infty$?

20. Consider the initial value problem

$$y'' + 0.1y' + y = f(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where $f(t)$ is the same as in Problem 19.

(a) Plot the graph of the solution. Use a large enough value of n and a long enough t-interval so that the transient part of the solution has become negligible and the steady state is clearly shown.

(b) Estimate the amplitude and frequency of the steady state part of the solution.

(c) Compare the results of part (b) with those from Section 3.8 for a sinusoidally forced oscillator.

21. Consider the initial value problem

$$y'' + y = g(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where

$$g(t) = u_0(t) + \sum_{k=1}^{n}(-1)^k u_{k\pi}(t).$$

(a) Draw the graph of $g(t)$ on an interval such as $0 \le t \le 6\pi$. Compare the graph with that of $f(t)$ in Problem 19(a).

(b) Find the solution of the initial value problem.

(c) Let $n = 15$ and plot the graph of the solution for $0 \le t \le 60$. Describe the solution and explain why it behaves as it does. Compare it with the solution of Problem 19.

(d) Investigate how the solution changes as n increases. What happens as $n \to \infty$?

22. Consider the initial value problem

$$y'' + 0.1y' + y = g(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where $g(t)$ is the same as in Problem 21.

(a) Plot the graph of the solution. Use a large enough value of n and a long enough t-interval so that the transient part of the solution has become negligible and the steady state is clearly shown.

(b) Estimate the amplitude and frequency of the steady state part of the solution.

(c) Compare the results of part (b) with those from Problem 20 and from Section 3.8 for a sinusoidally forced oscillator.

23. Consider the initial value problem

$$y'' + y = h(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where

$$f(t) = u_0(t) + 2\sum_{k=1}^{n}(-1)^k u_{11k/4}(t).$$

Observe that this problem is identical to Problem 19 except that the frequency of the forcing term has been increased somewhat.

(a) Find the solution of this initial value problem.

(b) Let $n \geq 33$ and plot the solution for $0 \leq t \leq 90$ or longer. Your plot should show a clearly recognizable beat.

(c) From the graph in part (b), estimate the "slow period" and the "fast period" for this oscillator.

(d) For a sinusoidally forced oscillator, it was shown in Section 3.8 that the "slow frequency" is given by $|\omega - \omega_0|/2$, where ω_0 is the natural frequency of the system and ω is the forcing frequency. Similarly, the "fast frequency" is $(\omega + \omega_0)/2$. Use these expressions to calculate the "fast period" and the "slow period" for the oscillator in this problem. How well do the results compare with your estimates from part (c)?

6.5 Impulse Functions

In some applications it is necessary to deal with phenomena of an impulsive nature—for example, voltages or forces of large magnitude that act over very short time intervals. Such problems often lead to differential equations of the form

$$ay'' + by' + cy = g(t), \tag{1}$$

where $g(t)$ is large during a short interval $t_0 - \tau < t < t_0 + \tau$ for some $\tau > 0$, and is otherwise zero.

The integral $I(\tau)$, defined by

$$I(\tau) = \int_{t_0-\tau}^{t_0+\tau} g(t)\,dt, \tag{2}$$

or, since $g(t) = 0$ outside of the interval $(t_0 - \tau, t_0 + \tau)$, by

$$I(\tau) = \int_{-\infty}^{\infty} g(t)\,dt, \tag{3}$$

is a measure of the strength of the forcing function. In a mechanical system, where $g(t)$ is a force, $I(\tau)$ is the total **impulse** of the force $g(t)$ over the time interval $(t_0 - \tau, t_0 + \tau)$. Similarly, if y is the current in an electric circuit and $g(t)$ is the time derivative of the voltage, then $I(\tau)$ represents the total voltage impressed on the circuit during the interval $(t_0 - \tau, t_0 + \tau)$.

In particular, let us suppose that t_0 is zero and that $g(t)$ is given by

$$g(t) = d_\tau(t) = \begin{cases} 1/(2\tau), & -\tau < t < \tau, \\ 0, & t \le -\tau \quad \text{or} \quad t \ge \tau, \end{cases} \tag{4}$$

where τ is a small positive constant (see Figure 6.5.1). According to Eq. (2) or (3), it follows immediately that in this case, $I(\tau) = 1$ independent of the value of τ, as long as $\tau \neq 0$. Now let us idealize the forcing function d_τ by prescribing it to act over shorter and shorter time intervals; that is, we require that $\tau \to 0^+$, as indicated in Figure 6.5.2. As a result of this limiting operation, we obtain

$$\lim_{\tau \to 0^+} d_\tau(t) = 0, \qquad t \neq 0. \tag{5}$$

Further, since $I(\tau) = 1$ for each $\tau \neq 0$, it follows that

$$\lim_{\tau \to 0^+} I(\tau) = 1. \tag{6}$$

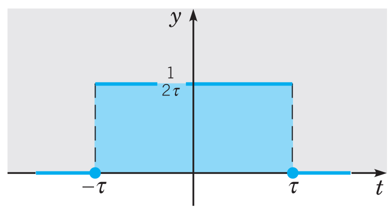

FIGURE 6.5.1 Graph of $y = d_\tau(t)$.

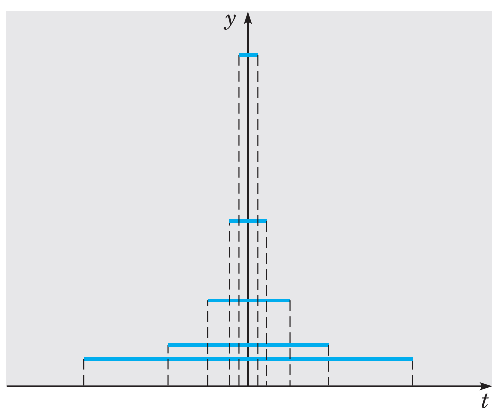

FIGURE 6.5.2 Graphs of $y = d_\tau(t)$ as $\tau \to 0^+$.

Equations (5) and (6) can be used to define an idealized **unit impulse function** δ, which imparts an impulse of magnitude one at $t = 0$ but is zero for all values of t other than zero. That is, the "function" δ is defined to have the properties

$$\delta(t) = 0, \qquad t \neq 0; \tag{7}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1. \tag{8}$$

There is no ordinary function of the kind studied in elementary calculus that satisfies both Eqs. (7) and (8). The "function" δ, defined by those equations, is an example of what are known as generalized functions; it is usually called the Dirac[3] **delta function**. Since $\delta(t)$ corresponds to a unit impulse at $t = 0$, a unit impulse at an arbitrary point $t = t_0$ is given by $\delta(t - t_0)$. From Eqs. (7) and (8), it follows that

$$\delta(t - t_0) = 0, \qquad t \neq t_0; \tag{9}$$

$$\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1. \tag{10}$$

The delta function does not satisfy the conditions of Theorem 6.1.2, but its Laplace transform can nevertheless be formally defined. Since $\delta(t)$ is defined as the limit of $d_\tau(t)$ as $\tau \to 0^+$, it is natural to define the Laplace transform of δ as a similar limit of the transform of d_τ. In particular, we will assume that $t_0 > 0$ and will define $\mathcal{L}\{\delta(t - t_0)\}$ by the equation

$$\mathcal{L}\{\delta(t - t_0)\} = \lim_{\tau \to 0^+} \mathcal{L}\{d_\tau(t - t_0)\}. \tag{11}$$

To evaluate the limit in Eq. (11), we first observe that if $\tau < t_0$, which must eventually be the case as $\tau \to 0^+$, then $t_0 - \tau > 0$. Since $d_\tau(t - t_0)$ is nonzero only in the interval from $t_0 - \tau$ to $t_0 + \tau$, we have

$$\begin{aligned} \mathcal{L}\{d_\tau(t - t_0)\} &= \int_0^{\infty} e^{-st} d_\tau(t - t_0)\, dt \\ &= \int_{t_0 - \tau}^{t_0 + \tau} e^{-st} d_\tau(t - t_0)\, dt. \end{aligned}$$

Substituting for $d_\tau(t - t_0)$ from Eq. (4), we obtain

$$\begin{aligned} \mathcal{L}\{d_\tau(t - t_0)\} &= \frac{1}{2\tau} \int_{t_0 - \tau}^{t_0 + \tau} e^{-st}\, dt = -\frac{1}{2s\tau} e^{-st} \Big|_{t = t_0 - \tau}^{t = t_0 + \tau} \\ &= \frac{1}{2s\tau} e^{-st_0} (e^{s\tau} - e^{-s\tau}) \end{aligned}$$

or

$$\mathcal{L}\{d_\tau(t - t_0)\} = \frac{\sinh s\tau}{s\tau} e^{-st_0}. \tag{12}$$

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[3]Paul A. M. Dirac (1902–1984), English mathematical physicist, received his Ph.D. from Cambridge in 1926 and was professor of mathematics there until 1969. He was awarded the Nobel Prize for Physics in 1933 (with Erwin Schrödinger) for fundamental work in quantum mechanics. His most celebrated result was the relativistic equation for the electron, published in 1928. From this equation he predicted the existence of an "anti-electron," or positron, which was first observed in 1932. Following his retirement from Cambridge, Dirac moved to the United States and held a research professorship at Florida State University.

The quotient $(\sinh s\tau)/s\tau$ is indeterminate as $\tau \to 0^+$, but its limit can be evaluated by L'Hôpital's[4] rule. We obtain

$$\lim_{\tau\to 0^+} \frac{\sinh s\tau}{s\tau} = \lim_{\tau\to 0^+} \frac{s\cosh s\tau}{s} = 1.$$

Then from Eq. (11) it follows that

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}. \tag{13}$$

Equation (13) defines $\mathcal{L}\{\delta(t-t_0)\}$ for any $t_0 > 0$. We extend this result, to allow t_0 to be zero, by letting $t_0 \to 0^+$ on the right side of Eq. (13); thus

$$\mathcal{L}\{\delta(t)\} = \lim_{t_0\to 0^+} e^{-st_0} = 1. \tag{14}$$

In a similar way, it is possible to define the integral of the product of the delta function and any continuous function f. We have

$$\int_{-\infty}^{\infty} \delta(t-t_0)f(t)\,dt = \lim_{\tau\to 0^+} \int_{-\infty}^{\infty} d_\tau(t-t_0)f(t)\,dt. \tag{15}$$

Using the definition (4) of $d_\tau(t)$ and the mean value theorem for integrals, we find that

$$\begin{aligned}\int_{-\infty}^{\infty} d_\tau(t-t_0)f(t)\,dt &= \frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau} f(t)\,dt \\ &= \frac{1}{2\tau}\cdot 2\tau\cdot f(t^*) = f(t^*),\end{aligned}$$

where $t_0 - \tau < t^* < t_0 + \tau$. Hence $t^* \to t_0$ as $\tau \to 0^+$, and it follows from Eq. (15) that

$$\int_{-\infty}^{\infty} \delta(t-t_0)f(t)\,dt = f(t_0). \tag{16}$$

The following example illustrates the use of the delta function in solving an initial value problem with an impulsive forcing function.

EXAMPLE 1

Find the solution of the initial value problem

$$2y'' + y' + 2y = \delta(t-5), \tag{17}$$

$$y(0) = 0, \qquad y'(0) = 0. \tag{18}$$

This initial value problem arises from the study of the same electric circuit or mechanical oscillator as in Example 1 of Section 6.4. The only difference is in the forcing term.

To solve the given problem, we take the Laplace transform of the differential equation and use the initial conditions, obtaining

$$(2s^2 + s + 2)Y(s) = e^{-5s}.$$

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[4]Marquis Guillaume de L'Hôpital (1661–1704) was a French nobleman with deep interest in mathematics. For a time he employed Johann Bernoulli as his private tutor in calculus. L'Hôpital published the first textbook on differential calculus in 1696; in it appears the differentiation rule that is named for him.

Thus

$$Y(s) = \frac{e^{-5s}}{2s^2 + s + 2} = \frac{e^{-5s}}{2} \frac{1}{\left(s + \frac{1}{4}\right)^2 + \frac{15}{16}}. \tag{19}$$

By Theorem 6.3.2 or from line 9 of Table 6.2.1,

$$\mathcal{L}^{-1}\left\{\frac{1}{\left(s + \frac{1}{4}\right)^2 + \frac{15}{16}}\right\} = \frac{4}{\sqrt{15}} e^{-t/4} \sin \frac{\sqrt{15}}{4} t. \tag{20}$$

Hence, by Theorem 6.3.1, we have

$$y = \mathcal{L}^{-1}\{Y(s)\} = \frac{2}{\sqrt{15}} u_5(t) e^{-(t-5)/4} \sin \frac{\sqrt{15}}{4}(t-5), \tag{21}$$

which is the formal solution of the given problem. It is also possible to write y in the form

$$y = \begin{cases} 0, & t < 5, \\ \dfrac{2}{\sqrt{15}} e^{-(t-5)/4} \sin \dfrac{\sqrt{15}}{4}(t-5), & t \geq 5. \end{cases} \tag{22}$$

The graph of Eq. (22) is shown in Figure 6.5.3. Since the initial conditions at $t = 0$ are homogeneous and there is no external excitation until $t = 5$, there is no response in the interval $0 < t < 5$. The impulse at $t = 5$ produces a decaying oscillation that persists indefinitely. The response is continuous at $t = 5$ despite the singularity in the forcing function at that point. However, the first derivative of the solution has a jump discontinuity at $t = 5$, and the second derivative has an infinite discontinuity there. This is required by the differential equation (17), since a singularity on one side of the equation must be balanced by a corresponding singularity on the other side.

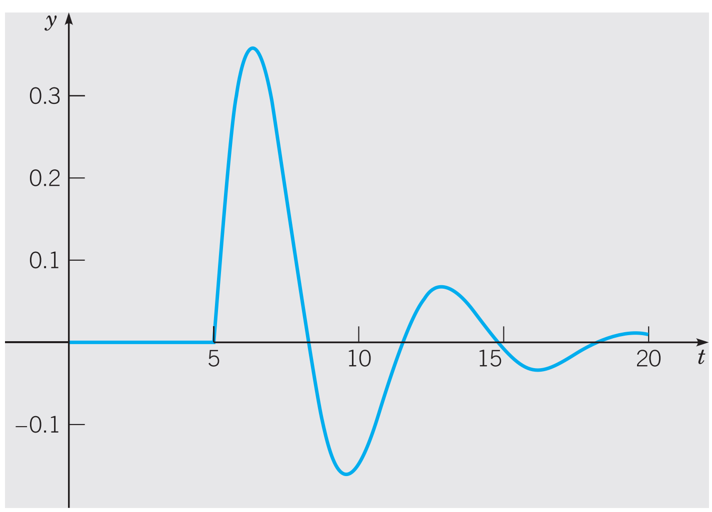


FIGURE 6.5.3 Solution of the initial value problem (17), (18): $2y'' + y' + 2y = \delta(t-5)$, $y(0) = 0$, $y'(0) = 0$.

In dealing with problems that involve impulsive forcing, the use of the delta function usually simplifies the mathematical calculations, often quite significantly.

However, if the actual excitation extends over a short, but nonzero, time interval, then an error will be introduced by modeling the excitation as taking place instantaneously. This error may be negligible, but in a practical problem it should not be dismissed without consideration. In Problem 16 you are asked to investigate this issue for a simple harmonic oscillator.

PROBLEMS

In each of Problems 1 through 12:

(a) Find the solution of the given initial value problem.

(b) Draw a graph of the solution.

1. $y'' + 2y' + 2y = \delta(t - \pi); \quad y(0) = 1, \quad y'(0) = 0$
2. $y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi); \quad y(0) = 0, \quad y'(0) = 0$
3. $y'' + 3y' + 2y = \delta(t - 5) + u_{10}(t); \quad y(0) = 0, \quad y'(0) = 1/2$
4. $y'' - y = -20\delta(t - 3); \quad y(0) = 1, \quad y'(0) = 0$
5. $y'' + 2y' + 3y = \sin t + \delta(t - 3\pi); \quad y(0) = 0, \quad y'(0) = 0$
6. $y'' + 4y = \delta(t - 4\pi); \quad y(0) = 1/2, \quad y'(0) = 0$
7. $y'' + y = \delta(t - 2\pi)\cos t; \quad y(0) = 0, \quad y'(0) = 1$
8. $y'' + 4y = 2\delta(t - \pi/4); \quad y(0) = 0, \quad y'(0) = 0$
9. $y'' + y = u_{\pi/2}(t) + 3\delta(t - 3\pi/2) - u_{2\pi}(t); \quad y(0) = 0, \quad y'(0) = 0$
10. $2y'' + y' + 4y = \delta(t - \pi/6)\sin t; \quad y(0) = 0, \quad y'(0) = 0$
11. $y'' + 2y' + 2y = \cos t + \delta(t - \pi/2); \quad y(0) = 0, \quad y'(0) = 0$
12. $y^{(4)} - y = \delta(t - 1); \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$
13. Consider again the system in Example 1 of this section, in which an oscillation is excited by a unit impulse at $t = 5$. Suppose that it is desired to bring the system to rest again after exactly one cycle—that is, when the response first returns to equilibrium moving in the positive direction.

    (a) Determine the impulse $k\delta(t - t_0)$ that should be applied to the system in order to accomplish this objective. Note that k is the magnitude of the impulse and t_0 is the time of its application.

    (b) Solve the resulting initial value problem, and plot its solution to confirm that it behaves in the specified manner.
14. Consider the initial value problem

    $$y'' + \gamma y' + y = \delta(t - 1), \qquad y(0) = 0, \quad y'(0) = 0,$$

    where γ is the damping coefficient (or resistance).

    (a) Let $\gamma = \frac{1}{2}$. Find the solution of the initial value problem and plot its graph.

    (b) Find the time t_1 at which the solution attains its maximum value. Also find the maximum value y_1 of the solution.

    (c) Let $\gamma = \frac{1}{4}$ and repeat parts (a) and (b).

    (d) Determine how t_1 and y_1 vary as γ decreases. What are the values of t_1 and y_1 when $\gamma = 0$?
15. Consider the initial value problem

    $$y'' + \gamma y' + y = k\delta(t - 1), \qquad y(0) = 0, \quad y'(0) = 0,$$

where k is the magnitude of an impulse at $t = 1$, and γ is the damping coefficient (or resistance).

(a) Let $\gamma = \frac{1}{2}$. Find the value of k for which the response has a peak value of 2; call this value k_1.

(b) Repeat part (a) for $\gamma = \frac{1}{4}$.

(c) Determine how k_1 varies as γ decreases. What is the value of k_1 when $\gamma = 0$?

16. Consider the initial value problem

$$y'' + y = f_k(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where $f_k(t) = [u_{4-k}(t) - u_{4+k}(t)]/2k$ with $0 < k \leq 1$.

(a) Find the solution $y = \phi(t, k)$ of the initial value problem.

(b) Calculate $\lim\limits_{k\to 0^+} \phi(t, k)$ from the solution found in part (a).

(c) Observe that $\lim\limits_{k\to 0^+} f_k(t) = \delta(t - 4)$. Find the solution $\phi_0(t)$ of the given initial value problem with $f_k(t)$ replaced by $\delta(t - 4)$. Is it true that $\phi_0(t) = \lim\limits_{k\to 0^+} \phi(t, k)$?

(d) Plot $\phi(t, 1/2)$, $\phi(t, 1/4)$, and $\phi_0(t)$ on the same axes. Describe the relation between $\phi(t, k)$ and $\phi_0(t)$.

Problems 17 through 22 deal with the effect of a sequence of impulses on an undamped oscillator. Suppose that

$$y'' + y = f(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

For each of the following choices for $f(t)$:

(a) Try to predict the nature of the solution without solving the problem.

(b) Test your prediction by finding the solution and drawing its graph.

(c) Determine what happens after the sequence of impulses ends.

17. $f(t) = \sum\limits_{k=1}^{20} \delta(t - k\pi)$

18. $f(t) = \sum\limits_{k=1}^{20} (-1)^{k+1}\delta(t - k\pi)$

19. $f(t) = \sum\limits_{k=1}^{20} \delta(t - k\pi/2)$

20. $f(t) = \sum\limits_{k=1}^{20} (-1)^{k+1}\delta(t - k\pi/2)$

21. $f(t) = \sum\limits_{k=1}^{15} \delta[t - (2k - 1)\pi]$

22. $f(t) = \sum\limits_{k=1}^{40} (-1)^{k+1}\delta(t - 11k/4)$

23. The position of a certain lightly damped oscillator satisfies the initial value problem

$$y'' + 0.1y' + y = \sum_{k=1}^{20} (-1)^{k+1}\delta(t - k\pi), \qquad y(0) = 0, \quad y'(0) = 0.$$

Observe that, except for the damping term, this problem is the same as Problem 18.

(a) Try to predict the nature of the solution without solving the problem.

(b) Test your prediction by finding the solution and drawing its graph.

(c) Determine what happens after the sequence of impulses ends.

24. Proceed as in Problem 23 for the oscillator satisfying

$$y'' + 0.1y' + y = \sum_{k=1}^{15} \delta[t - (2k - 1)\pi], \qquad y(0) = 0, \quad y'(0) = 0.$$

Observe that, except for the damping term, this problem is the same as Problem 21.

25. (a) By the method of variation of parameters, show that the solution of the initial value problem

$$y'' + 2y' + 2y = f(t); \qquad y(0) = 0, \quad y'(0) = 0$$

is

$$y = \int_0^t e^{-(t-\tau)} f(\tau) \sin(t-\tau)\, d\tau.$$

(b) Show that if $f(t) = \delta(t-\pi)$, then the solution of part (a) reduces to

$$y = u_\pi(t) e^{-(t-\pi)} \sin(t-\pi).$$

(c) Use a Laplace transform to solve the given initial value problem with $f(t) = \delta(t-\pi)$, and confirm that the solution agrees with the result of part (b).

6.6 The Convolution Integral

Sometimes it is possible to identify a Laplace transform $H(s)$ as the product of two other transforms $F(s)$ and $G(s)$, the latter transforms corresponding to known functions f and g, respectively. In this event, we might anticipate that $H(s)$ would be the transform of the product of f and g. However, this is not the case; in other words, the Laplace transform cannot be commuted with ordinary multiplication. On the other hand, if an appropriately defined "generalized product" is introduced, then the situation changes, as stated in the following theorem.

Theorem 6.6.1 If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ both exist for $s > a \geq 0$, then

$$H(s) = F(s)G(s) = \mathcal{L}\{h(t)\}, \qquad s > a, \tag{1}$$

where

$$h(t) = \int_0^t f(t-\tau)g(\tau)\, d\tau = \int_0^t f(\tau)g(t-\tau)\, d\tau. \tag{2}$$

The function h is known as the convolution of f and g; the integrals in Eq. (2) are called convolution integrals.

The equality of the two integrals in Eq. (2) follows by making the change of variable $t - \tau = \xi$ in the first integral. Before giving the proof of this theorem, let us make some observations about the convolution integral. According to this theorem, the transform of the convolution of two functions, rather than the transform of their ordinary product, is given by the product of the separate transforms. It is conventional to emphasize that the convolution integral can be thought of as a "generalized product" by writing

$$h(t) = (f * g)(t). \tag{3}$$

In particular, the notation $(f * g)(t)$ serves to indicate the first integral appearing in Eq. (2).

The convolution $f * g$ has many of the properties of ordinary multiplication. For example, it is relatively simple to show that

$$f * g = g * f \qquad \text{(commutative law)} \tag{4}$$

$$f * (g_1 + g_2) = f * g_1 + f * g_2 \qquad \text{(distributive law)} \tag{5}$$

$$(f * g) * h = f * (g * h) \qquad \text{(associative law)} \tag{6}$$

$$f * 0 = 0 * f = 0. \tag{7}$$

In Eq. (7) the zeros denote not the number 0 but the function that has the value 0 for each value of t. The proofs of these properties are left to you as exercises.

However, there are other properties of ordinary multiplication that the convolution integral does not have. For example, it is not true in general that $f * 1$ is equal to f. To see this, note that

$$(f * 1)(t) = \int_0^t f(t-\tau) \cdot 1 \, d\tau = \int_0^t f(t-\tau)\, d\tau.$$

If, for example, $f(t) = \cos t$, then

$$\begin{aligned} (f * 1)(t) &= \int_0^t \cos(t-\tau)\, d\tau = -\sin(t-\tau)\Big|_{\tau=0}^{\tau=t} \\ &= -\sin 0 + \sin t \\ &= \sin t. \end{aligned}$$

Clearly, $(f * 1)(t) \neq f(t)$ in this case. Similarly, it may not be true that $f * f$ is nonnegative. See Problem 3 for an example.

Convolution integrals arise in various applications in which the behavior of the system at time t depends not only on its state at time t but also on its past history. Systems of this kind are sometimes called hereditary systems and occur in such diverse fields as neutron transport, viscoelasticity, and population dynamics, among others.

Turning now to the proof of Theorem 6.6.1, we note first that if

$$F(s) = \int_0^\infty e^{-s\xi} f(\xi)\, d\xi$$

and

$$G(s) = \int_0^\infty e^{-s\tau} g(\tau)\, d\tau,$$

then

$$F(s)G(s) = \int_0^\infty e^{-s\xi} f(\xi)\, d\xi \int_0^\infty e^{-s\tau} g(\tau)\, d\tau. \tag{8}$$

Since the integrand of the first integral does not depend on the integration variable of the second, we can write $F(s)G(s)$ as an iterated integral

$$\begin{aligned} F(s)G(s) &= \int_0^\infty e^{-s\tau} g(\tau) \left[\int_0^\infty e^{-s\xi} f(\xi)\, d\xi \right] d\tau \\ &= \int_0^\infty g(\tau) \left[\int_0^\infty e^{-s(\xi+\tau)} f(\xi)\, d\xi \right] d\tau. \end{aligned} \tag{9}$$

The latter integral can be put into a more convenient form by introducing a change of variable. Let $\xi = t - \tau$, for fixed τ, so that $d\xi = dt$. Further, $\xi = 0$ corresponds to $t = \tau$, and $\xi = \infty$ corresponds to $t = \infty$; then the integral with respect to ξ in Eq. (9) is transformed into one with respect to t:

$$F(s)G(s) = \int_0^\infty g(\tau)\left[\int_\tau^\infty e^{-st} f(t-\tau)\,dt\right] d\tau. \tag{10}$$

The iterated integral on the right side of Eq. (10) is carried out over the shaded wedge-shaped region extending to infinity in the $t\tau$-plane shown in Figure 6.6.1. Assuming that the order of integration can be reversed, we rewrite Eq. (10) so that the integration with respect to τ is executed first. In this way we obtain

$$F(s)G(s) = \int_0^\infty e^{-st}\left[\int_0^t f(t-\tau)g(\tau)\,d\tau\right] dt \tag{11}$$

or

$$F(s)G(s) = \int_0^\infty e^{-st} h(t)\,dt = \mathcal{L}\{h(t)\}, \tag{12}$$

where $h(t)$ is defined by Eq. (2). This completes the proof of Theorem 6.6.1.

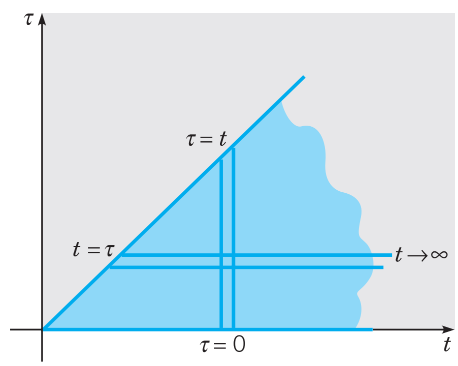


FIGURE 6.6.1 Region of integration in $F(s)G(s)$.

EXAMPLE 1

Find the inverse transform of

$$H(s) = \frac{a}{s^2(s^2+a^2)}. \tag{13}$$

It is convenient to think of $H(s)$ as the product of s^{-2} and $a/(s^2+a^2)$, which, according to lines 3 and 5 of Table 6.2.1, are the transforms of t and $\sin at$, respectively. Hence, by Theorem 6.6.1, the inverse transform of $H(s)$ is

$$h(t) = \int_0^t (t-\tau)\sin a\tau\,d\tau = \frac{at - \sin at}{a^2}. \tag{14}$$

You can verify that the same result is obtained if $h(t)$ is written in the alternative form

$$h(t) = \int_0^t \tau \sin a(t-\tau)\,d\tau,$$

which confirms Eq. (2) in this case. Of course, $h(t)$ can also be found by expanding $H(s)$ in partial fractions.

EXAMPLE 2

Find the solution of the initial value problem

$$y'' + 4y = g(t), \tag{15}$$

$$y(0) = 3, \qquad y'(0) = -1. \tag{16}$$

By taking the Laplace transform of the differential equation and using the initial conditions, we obtain

$$s^2Y(s) - 3s + 1 + 4Y(s) = G(s)$$

or

$$Y(s) = \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4}. \tag{17}$$

Observe that the first and second terms on the right side of Eq. (17) contain the dependence of $Y(s)$ on the initial conditions and forcing function, respectively. It is convenient to write $Y(s)$ in the form

$$Y(s) = 3\frac{s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}G(s). \tag{18}$$

Then, using lines 5 and 6 of Table 6.2.1 and Theorem 6.6.1, we obtain

$$y = 3\cos 2t - \tfrac{1}{2}\sin 2t + \tfrac{1}{2}\int_0^t \sin 2(t-\tau)g(\tau)\,d\tau. \tag{19}$$

If a specific forcing function g is given, then the integral in Eq. (19) can be evaluated (by numerical means, if necessary).

Example 2 illustrates the power of the convolution integral as a tool for writing the solution of an initial value problem in terms of an integral. In fact, it is possible to proceed in much the same way in more general problems. Consider the problem consisting of the differential equation

$$ay'' + by' + cy = g(t), \tag{20}$$

where $a, b,$ and c are real constants and g is a given function, together with the initial conditions

$$y(0) = y_0, \qquad y'(0) = y_0'. \tag{21}$$

The transform approach yields some important insights concerning the structure of the solution of any problem of this type.

The initial value problem (20), (21) is often referred to as an input–output problem. The coefficients $a, b,$ and c describe the properties of some physical system, and $g(t)$ is the input to the system. The values y_0 and y_0' describe the initial state, and the solution y is the output at time t.

By taking the Laplace transform of Eq. (20) and using the initial conditions (21), we obtain

$$(as^2 + bs + c)Y(s) - (as + b)y_0 - ay_0' = G(s).$$

If we let

$$\Phi(s) = \frac{(as+b)y_0 + ay_0'}{as^2+bs+c}, \qquad \Psi(s) = \frac{G(s)}{as^2+bs+c}, \tag{22}$$

then we can write

$$Y(s) = \Phi(s) + \Psi(s). \tag{23}$$

Consequently,

$$y = \phi(t) + \psi(t), \tag{24}$$

where $\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\}$ and $\psi(t) = \mathcal{L}^{-1}\{\Psi(s)\}$. Observe that $\phi(t)$ is the solution of the initial value problem

$$ay'' + by' + cy = 0, \qquad y(0) = y_0, \quad y'(0) = y_0', \tag{25}$$

obtained from Eqs. (20) and (21) by setting $g(t)$ equal to zero. Similarly, $\psi(t)$ is the solution of

$$ay'' + by' + cy = g(t), \qquad y(0) = 0, \quad y'(0) = 0, \tag{26}$$

in which the initial values y_0 and y_0' are each replaced by zero.

Once specific values of $a, b,$ and c are given, we can find $\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\}$ by using Table 6.2.1, possibly in conjunction with a translation or a partial fraction expansion. To find $\psi(t) = \mathcal{L}^{-1}\{\Psi(s)\}$, it is convenient to write $\Psi(s)$ as

$$\Psi(s) = H(s)G(s), \tag{27}$$

where $H(s) = (as^2 + bs + c)^{-1}$. The function H is known as the **transfer function**[5] and depends only on the properties of the system under consideration; that is, $H(s)$ is determined entirely by the coefficients $a, b,$ and c. On the other hand, $G(s)$ depends only on the external excitation $g(t)$ that is applied to the system. By the convolution theorem we can write

$$\psi(t) = \mathcal{L}^{-1}\{H(s)G(s)\} = \int_0^t h(t-\tau)g(\tau)\,d\tau, \tag{28}$$

where $h(t) = \mathcal{L}^{-1}\{H(s)\}$, and $g(t)$ is the given forcing function.

To obtain a better understanding of the significance of $h(t)$, we consider the case in which $G(s) = 1$; consequently, $g(t) = \delta(t)$ and $\Psi(s) = H(s)$. This means that $y = h(t)$ is the solution of the initial value problem

$$ay'' + by' + cy = \delta(t), \qquad y(0) = 0, \quad y'(0) = 0, \tag{29}$$

obtained from Eq. (26) by replacing $g(t)$ by $\delta(t)$. Thus $h(t)$ is the response of the system to a unit impulse applied at $t = 0$, and it is natural to call $h(t)$ the **impulse response** of the system. Equation (28) then says that $\psi(t)$ is the convolution of the impulse response and the forcing function.

Referring to Example 2, we note that in that case, the transfer function is $H(s) = 1/(s^2 + 4)$ and the impulse response is $h(t) = (\sin 2t)/2$. Also, the first two terms on the right side of Eq. (19) constitute the function $\phi(t)$, the solution of the corresponding homogeneous equation that satisfies the given initial conditions.

PROBLEMS

1. Establish the commutative, distributive, and associative properties of the convolution integral.
   (a) $f * g = g * f$
   (b) $f * (g_1 + g_2) = f * g_1 + f * g_2$
   (c) $f * (g * h) = (f * g) * h$

---

[5] This terminology arises from the fact that $H(s)$ is the ratio of the transforms of the output and the input of the problem (26).

2. Find an example different from the one in the text showing that $(f * 1)(t)$ need not be equal to $f(t)$.
3. Show, by means of the example $f(t) = \sin t$, that $f * f$ is not necessarily nonnegative.

In each of Problems 4 through 7, find the Laplace transform of the given function.

4. $f(t) = \int_0^t (t-\tau)^2 \cos 2\tau \, d\tau$
5. $f(t) = \int_0^t e^{-(t-\tau)} \sin \tau \, d\tau$
6. $f(t) = \int_0^t (t-\tau) e^{\tau} \, d\tau$
7. $f(t) = \int_0^t \sin(t-\tau) \cos \tau \, d\tau$

In each of Problems 8 through 11, find the inverse Laplace transform of the given function by using the convolution theorem.

8. $F(s) = \dfrac{1}{s^4(s^2+1)}$
9. $F(s) = \dfrac{s}{(s+1)(s^2+4)}$
10. $F(s) = \dfrac{1}{(s+1)^2(s^2+4)}$
11. $F(s) = \dfrac{G(s)}{s^2+1}$
12. (a) If $f(t) = t^m$ and $g(t) = t^n$, where m and n are positive integers, show that

$$f * g = t^{m+n+1} \int_0^1 u^m (1-u)^n \, du.$$

(b) Use the convolution theorem to show that

$$\int_0^1 u^m (1-u)^n \, du = \frac{m!\, n!}{(m+n+1)!}.$$

(c) Extend the result of part (b) to the case where m and n are positive numbers but not necessarily integers.

In each of Problems 13 through 20, express the solution of the given initial value problem in terms of a convolution integral.

13. $y'' + \omega^2 y = g(t); \quad y(0) = 0, \quad y'(0) = 1$
14. $y'' + 2y' + 2y = \sin \alpha t; \quad y(0) = 0, \quad y'(0) = 0$
15. $4y'' + 4y' + 17y = g(t); \quad y(0) = 0, \quad y'(0) = 0$
16. $y'' + y' + \frac{5}{4}y = 1 - u_\pi(t); \quad y(0) = 1, \quad y'(0) = -1$
17. $y'' + 4y' + 4y = g(t); \quad y(0) = 2, \quad y'(0) = -3$
18. $y'' + 3y' + 2y = \cos \alpha t; \quad y(0) = 1, \quad y'(0) = 0$
19. $y^{(4)} - y = g(t); \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$
20. $y^{(4)} + 5y'' + 4y = g(t); \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$
21. Consider the equation

$$\phi(t) + \int_0^t k(t-\xi)\phi(\xi) \, d\xi = f(t),$$

in which f and k are known functions, and ϕ is to be determined. Since the unknown function ϕ appears under an integral sign, the given equation is called an **integral equation**; in particular, it belongs to a class of integral equations known as Volterra integral equations. Take the Laplace transform of the given integral equation and obtain an expression for $\mathcal{L}\{\phi(t)\}$ in terms of the transforms $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{k(t)\}$ of the given functions f and k. The inverse transform of $\mathcal{L}\{\phi(t)\}$ is the solution of the original integral equation.

22. Consider the Volterra integral equation (see Problem 21)

$$\phi(t) + \int_0^t (t-\xi)\phi(\xi)\,d\xi = \sin 2t. \tag{i}$$

(a) Solve the integral equation (i) by using the Laplace transform.

(b) By differentiating Eq. (i) twice, show that $\phi(t)$ satisfies the differential equation

$$\phi''(t) + \phi(t) = -4\sin 2t.$$

Show also that the initial conditions are

$$\phi(0) = 0, \qquad \phi'(0) = 2.$$

(c) Solve the initial value problem in part (b), and verify that the solution is the same as the one in part (a).

In each of Problems 23 through 25:

(a) Solve the given Volterra integral equation by using the Laplace transform.

(b) Convert the integral equation into an initial value problem, as in Problem 22(b).

(c) Solve the initial value problem in part (b), and verify that the solution is the same as the one in part (a).

23. $\phi(t) + \int_0^t (t-\xi)\phi(\xi)\,d\xi = 1$

24. $\phi(t) - \int_0^t (t-\xi)\phi(\xi)\,d\xi = 1$

25. $\phi(t) + 2\int_0^t \cos(t-\xi)\phi(\xi)\,d\xi = e^{-t}$

There are also equations, known as **integro-differential equations**, in which both derivatives and integrals of the unknown function appear. In each of Problems 26 through 28:

(a) Solve the given integro-differential equation by using the Laplace transform.

(b) By differentiating the integro-differential equation a sufficient number of times, convert it into an initial value problem.

(c) Solve the initial value problem in part (b), and verify that the solution is the same as the one in part (a).

26. $\phi'(t) + \int_0^t (t-\xi)\phi(\xi)\,d\xi = t, \qquad \phi(0) = 0$

27. $\phi'(t) - \frac{1}{2}\int_0^t (t-\xi)^2\phi(\xi)\,d\xi = -t, \qquad \phi(0) = 1$

28. $\phi'(t) + \phi(t) = \int_0^t \sin(t-\xi)\phi(\xi)\,d\xi, \qquad \phi(0) = 1$

29. **The Tautochrone.** A problem of interest in the history of mathematics is that of finding the tautochrone[6]—the curve down which a particle will slide freely under gravity alone, reaching the bottom in the same time regardless of its starting point on the curve. This problem arose in the construction of a clock pendulum whose period is independent of the amplitude of its motion. The tautochrone was found by Christian Huygens (1629–1695) in 1673 by geometrical methods, and later by Leibniz and Jakob Bernoulli using analytical arguments. Bernoulli's solution (in 1690) was one of the first occasions on which

[6]The word "tautochrone" comes from the Greek words *tauto*, which means "same," and *chronos*, which means "time."

a differential equation was explicitly solved. The geometric configuration is shown in Figure 6.6.2. The starting point $P(a, b)$ is joined to the terminal point $(0, 0)$ by the arc C. Arc length s is measured from the origin, and $f(y)$ denotes the rate of change of s with respect to y:

$$f(y) = \frac{ds}{dy} = \left[1 + \left(\frac{dx}{dy}\right)^2\right]^{1/2}. \tag{i}$$

Then it follows from the principle of conservation of energy that the time $T(b)$ required for a particle to slide from P to the origin is

$$T(b) = \frac{1}{\sqrt{2g}} \int_0^b \frac{f(y)}{\sqrt{b-y}}\, dy. \tag{ii}$$

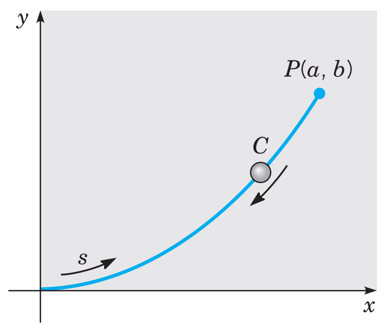


FIGURE 6.6.2 The tautochrone.

(a) Assume that $T(b) = T_0$, a constant, for each b. By taking the Laplace transform of Eq. (ii) in this case, and using the convolution theorem, show that

$$F(s) = \sqrt{\frac{2g}{\pi}}\, \frac{T_0}{\sqrt{s}}; \tag{iii}$$

then show that

$$f(y) = \frac{\sqrt{2g}}{\pi}\, \frac{T_0}{\sqrt{y}}. \tag{iv}$$

Hint: See Problem 31 of Section 6.1.

(b) Combining Eqs. (i) and (iv), show that

$$\frac{dx}{dy} = \sqrt{\frac{2\alpha - y}{y}}, \tag{v}$$

where $\alpha = gT_0^2/\pi^2$.

(c) Use the substitution $y = 2\alpha \sin^2(\theta/2)$ to solve Eq. (v), and show that

$$x = \alpha(\theta + \sin\theta), \qquad y = \alpha(1 - \cos\theta). \tag{vi}$$

Equations (vi) can be identified as parametric equations of a cycloid. Thus the tautochrone is an arc of a cycloid.

(a) Show that any solution $\mathbf{x} = \mathbf{z}(t)$ can be written in the form

$$\mathbf{z}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$$

for suitable constants $c_1, \ldots, c_n$.
Hint: Use the result of Problem 12 of Section 7.3, and also Problem 8 above.

(b) Show that the expression for the solution $\mathbf{z}(t)$ in part (a) is unique; that is, if $\mathbf{z}(t) = k_1\mathbf{x}^{(1)}(t) + \cdots + k_n\mathbf{x}^{(n)}(t)$, then $k_1 = c_1, \ldots, k_n = c_n$.
Hint: Show that $(k_1 - c_1)\mathbf{x}^{(1)}(t) + \cdots + (k_n - c_n)\mathbf{x}^{(n)}(t) = \mathbf{0}$ for each t in $\alpha < t < \beta$, and use the linear independence of $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$.

7.5 Homogeneous Linear Systems with Constant Coefficients

We will concentrate most of our attention on systems of homogeneous linear equations with constant coefficients—that is, systems of the form

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{1}$$

where $\mathbf{A}$ is a constant $n \times n$ matrix. Unless stated otherwise, we will assume further that all the elements of $\mathbf{A}$ are real (rather than complex) numbers.

If $n = 1$, then the system reduces to a single first order equation

$$\frac{dx}{dt} = ax, \tag{2}$$

whose solution is $x = ce^{at}$. Note that $x = 0$ is the only equilibrium solution if $a \neq 0$. If $a < 0$, then other solutions approach $x = 0$ as t increases, and in this case we say that $x = 0$ is an asymptotically stable equilibrium solution. On the other hand, if $a > 0$, then $x = 0$ is unstable, since other solutions depart from it with increasing t. For systems of n equations, the situation is similar but more complicated. Equilibrium solutions are found by solving $\mathbf{A}\mathbf{x} = \mathbf{0}$. We usually assume that $\det \mathbf{A} \neq 0$, so $\mathbf{x} = \mathbf{0}$ is the only equilibrium solution. An important question is whether other solutions approach this equilibrium solution or depart from it as t increases; in other words, is $\mathbf{x} = \mathbf{0}$ asymptotically stable or unstable? Or are there still other possibilities?

The case $n = 2$ is particularly important and lends itself to visualization in the x_1x_2-plane, called the **phase plane**. By evaluating $\mathbf{A}\mathbf{x}$ at a large number of points and plotting the resulting vectors, we obtain a direction field of tangent vectors to solutions of the system of differential equations. A qualitative understanding of the behavior of solutions can usually be gained from a direction field. More precise information results from including in the plot some solution curves, or trajectories. A plot that shows a representative sample of trajectories for a given system is called a **phase portrait**. A well-constructed phase portrait provides easily understood information about all solutions of a two-dimensional system in a single graphical display. Although creating quantitatively accurate phase portraits requires computer assistance, it is usually possible to sketch qualitatively accurate phase portraits by hand, as we demonstrate in Examples 2 and 3 below.

Our first task, however, is to show how to find solutions of systems such as Eq. (1). We start with a particularly simple example.

EXAMPLE 1

Find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \mathbf{x}. \tag{3}$$

The most important feature of this system is that the coefficient matrix is a diagonal matrix. Thus, by writing the system in scalar form, we obtain

$$x_1' = 2x_1, \qquad x_2' = -3x_2.$$

Each of these equations involves only one of the unknown variables, so we can solve the two equations separately. In this way we find that

$$x_1 = c_1 e^{2t}, \qquad x_2 = c_2 e^{-3t},$$

where c_1 and c_2 are arbitrary constants. Then, by writing the solution in vector form, we have

$$\mathbf{x} = \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{-3t} \end{pmatrix} = c_1 \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^{-3t} \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t}. \tag{4}$$

Now we define the two solutions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ so that

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t}, \tag{5}$$

The Wronskian of these solutions is

$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](t) = \begin{vmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{vmatrix} = e^{-t}, \tag{6}$$

which is never zero. Therefore, $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ form a fundamental set of solutions, and the general solution of Eq. (3) is given by Eq. (4).

In Example 1 we found two independent solutions of the given system (3) in the form of an exponential function multiplied by a vector. This was perhaps to be expected since we have found other linear equations with constant coefficients to have exponential solutions, and the unknown $\mathbf{x}$ in the system (3) is a vector. So let us try to extend this idea to the general system (1) by seeking solutions of the form

$$\mathbf{x} = \boldsymbol{\xi} e^{rt}, \tag{7}$$

where the exponent r and the vector $\boldsymbol{\xi}$ are to be determined. Substituting from Eq. (7) for $\mathbf{x}$ in the system (1) gives

$$r\boldsymbol{\xi} e^{rt} = \mathbf{A}\boldsymbol{\xi} e^{rt}.$$

Upon canceling the nonzero scalar factor e^{rt}, we obtain $\mathbf{A}\boldsymbol{\xi} = r\boldsymbol{\xi}$, or

$$(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \tag{8}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. Thus, to solve the system of differential equations (1), we must solve the system of algebraic equations (8). This latter problem is precisely the one that determines the eigenvalues and eigenvectors of the matrix $\mathbf{A}$.

Therefore, the vector $\mathbf{x}$ given by Eq. (7) is a solution of Eq. (1), provided that r is an eigenvalue and $\boldsymbol{\xi}$ an associated eigenvector of the coefficient matrix $\mathbf{A}$.

The following two examples are typical of 2×2 systems with eigenvalues that are real and different. In each example we will solve the system and construct a corresponding phase portrait. We will see that solutions have very distinct geometrical patterns depending on whether the eigenvalues have the same sign or different signs. Later in the section we return to a further discussion of the general $n \times n$ system.

EXAMPLE 2

Consider the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}. \tag{9}$$

Plot a direction field and determine the qualitative behavior of solutions. Then find the general solution and draw a phase portrait showing several trajectories.

A direction field for this system is shown in Figure 7.5.1. By following the arrows in this figure, you can see that a typical solution in the second quadrant eventually moves into the first or third quadrant, and likewise for a typical solution in the fourth quadrant. On the other hand, no solution leaves either the first or the third quadrant. Further, it appears that a typical solution departs from the neighborhood of the origin and ultimately has a slope of approximately 2.

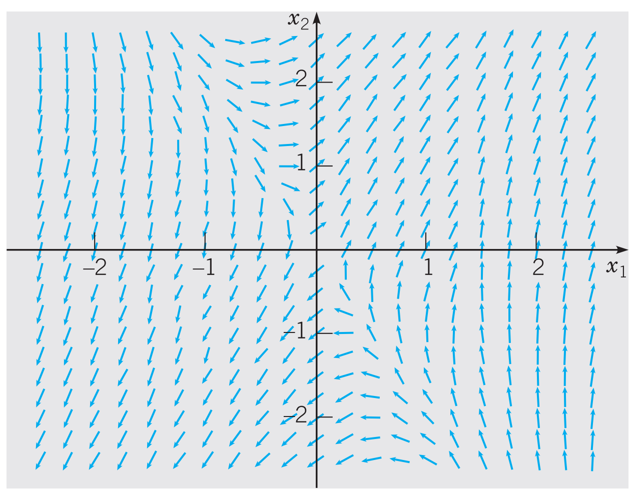


FIGURE 7.5.1 Direction field for the system (9).

To find solutions explicitly, we assume that $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ and substitute for $\mathbf{x}$ in Eq. (9). We are led to the system of algebraic equations

$$\begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{10}$$

Equations (10) have a nontrivial solution if and only if the determinant of coefficients is zero. Thus, allowable values of r are found from the equation

$$\begin{aligned}\begin{vmatrix}1-r & 1\\ 4 & 1-r\end{vmatrix} &= (1-r)^2 - 4\\ &= r^2 - 2r - 3 = (r-3)(r+1) = 0.\end{aligned} \tag{11}$$

Equation (11) has the roots $r_1 = 3$ and $r_2 = -1$; these are the eigenvalues of the coefficient matrix in Eq. (9). If $r = 3$, then the system (10) reduces to the single equation

$$-2\xi_1 + \xi_2 = 0. \tag{12}$$

Thus $\xi_2 = 2\xi_1$, and the eigenvector corresponding to $r_1 = 3$ can be taken as

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix}1\\2\end{pmatrix}. \tag{13}$$

Similarly, corresponding to $r_2 = -1$, we find that $\xi_2 = -2\xi_1$, so the eigenvector is

$$\boldsymbol{\xi}^{(2)} = \begin{pmatrix}1\\-2\end{pmatrix}. \tag{14}$$

The corresponding solutions of the differential equation are

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix}1\\2\end{pmatrix}e^{3t}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix}1\\-2\end{pmatrix}e^{-t}. \tag{15}$$

The Wronskian of these solutions is

$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](t) = \begin{vmatrix}e^{3t} & e^{-t}\\ 2e^{3t} & -2e^{-t}\end{vmatrix} = -4e^{2t}, \tag{16}$$

which is never zero. Hence the solutions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ form a fundamental set, and the general solution of the system (9) is

$$\begin{aligned}\mathbf{x} &= c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t)\\ &= c_1\begin{pmatrix}1\\2\end{pmatrix}e^{3t} + c_2\begin{pmatrix}1\\-2\end{pmatrix}e^{-t},\end{aligned} \tag{17}$$

where c_1 and c_2 are arbitrary constants.

To visualize the solution (17), it is helpful to consider its graph in the x_1x_2-plane for various values of the constants c_1 and c_2. We start with $\mathbf{x} = c_1\mathbf{x}^{(1)}(t)$ or, in scalar form,

$$x_1 = c_1e^{3t}, \qquad x_2 = 2c_1e^{3t}.$$

By eliminating t between these two equations, we see that this solution lies on the straight line $x_2 = 2x_1$; see Figure 7.5.2*a*. This is the line through the origin in the direction of the eigenvector $\boldsymbol{\xi}^{(1)}$. If we look on the solution as the trajectory of a moving particle, then the particle is in the first quadrant when $c_1 > 0$ and in the third quadrant when $c_1 < 0$. In either case the particle departs from the origin as t increases. Next consider $\mathbf{x} = c_2\mathbf{x}^{(2)}(t)$, or

$$x_1 = c_2e^{-t}, \qquad x_2 = -2c_2e^{-t}.$$

This solution lies on the line $x_2 = -2x_1$, whose direction is determined by the eigenvector $\boldsymbol{\xi}^{(2)}$. The solution is in the fourth quadrant when $c_2 > 0$ and in the second quadrant when $c_2 < 0$, as shown in Figure 7.5.2*a*. In both cases the particle moves toward the origin as t increases. The solution (17) is a combination of $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$. For large t the term $c_1\mathbf{x}^{(1)}(t)$ is dominant and the term $c_2\mathbf{x}^{(2)}(t)$ becomes negligible. Thus all solutions for which $c_1 \neq 0$ are asymptotic to the line $x_2 = 2x_1$ as $t \to \infty$. Similarly, all solutions for which $c_2 \neq 0$ are asymptotic to the line $x_2 = -2x_1$ as $t \to -\infty$. A phase portrait for the system including the graphs of several solutions is shown in Figure 7.5.2*a*. The pattern of trajectories in this figure is typical of all 2×2 systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for which the eigenvalues are real and of opposite signs. The origin is called a **saddle point** in this case. Saddle points are always unstable because almost all trajectories depart from them as t increases.

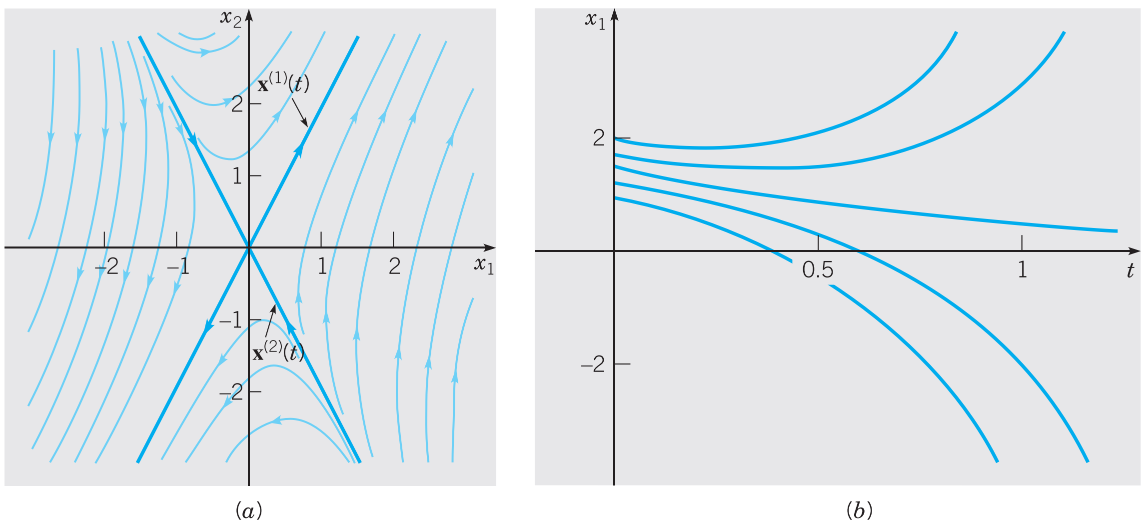


FIGURE 7.5.2 (*a*) A phase portrait for the system (9); the origin is a saddle point. (*b*) Typical plots of x_1 versus t for the system (9).

In the preceding paragraph, we have described how to draw by hand a qualitatively correct sketch of the trajectories of a system such as Eq. (9), once the eigenvalues and eigenvectors have been determined. However, to produce a detailed and accurate drawing, such as Figure 7.5.2*a* and other figures that appear later in this chapter, a computer is extremely helpful, if not indispensable.

As an alternative to Figure 7.5.2*a*, you can also plot x_1 or x_2 as a function of t; some typical plots of x_1 versus t are shown in Figure 7.5.2*b*, and those of x_2 versus t are similar. For certain initial conditions it follows that $c_1 = 0$ in Eq. (17), so that $x_1 = c_2e^{-t}$ and $x_1 \to 0$ as $t \to \infty$. One such graph is shown in Figure 7.5.2*b*, corresponding to a trajectory that approaches the origin in Figure 7.5.2*a*. For most initial conditions, however, $c_1 \neq 0$ and x_1 is given by $x_1 = c_1e^{3t} + c_2e^{-t}$. Then the presence of the positive exponential term causes x_1 to grow exponentially in magnitude as t increases. Several graphs of this type are shown in Figure 7.5.2*b*, corresponding to trajectories that depart from the neighborhood of the origin in Figure 7.5.2*a*. It is important to understand the relation between parts (*a*) and (*b*) of Figure 7.5.2 and other similar figures that appear later, since you may want to visualize solutions either in the x_1x_2-plane or as functions of the independent variable t.

EXAMPLE 3

Consider the system

$$\mathbf{x}' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \mathbf{x}. \tag{18}$$

Draw a direction field for this system and find its general solution. Then plot a phase portrait showing several typical trajectories in the phase plane.

The direction field for the system (18) in Figure 7.5.3 shows clearly that all solutions approach the origin. To find the solutions, we assume that $\mathbf{x} = \boldsymbol{\xi} e^{rt}$; then we obtain the algebraic system

$$\begin{pmatrix} -3 - r & \sqrt{2} \\ \sqrt{2} & -2 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{19}$$

The eigenvalues satisfy

$$\begin{aligned} (-3 - r)(-2 - r) - 2 &= r^2 + 5r + 4 \\ &= (r + 1)(r + 4) = 0, \end{aligned} \tag{20}$$

so $r_1 = -1$ and $r_2 = -4$. For $r = -1$, Eq. (19) becomes

$$\begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{21}$$

Hence $\xi_2 = \sqrt{2}\,\xi_1$, and the eigenvector $\boldsymbol{\xi}^{(1)}$ corresponding to the eigenvalue $r_1 = -1$ can be taken as

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}. \tag{22}$$

Similarly, corresponding to the eigenvalue $r_2 = -4$ we have $\xi_1 = -\sqrt{2}\,\xi_2$, so the eigenvector is

$$\boldsymbol{\xi}^{(2)} = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}. \tag{23}$$

Thus a fundamental set of solutions of the system (18) is

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}, \tag{24}$$

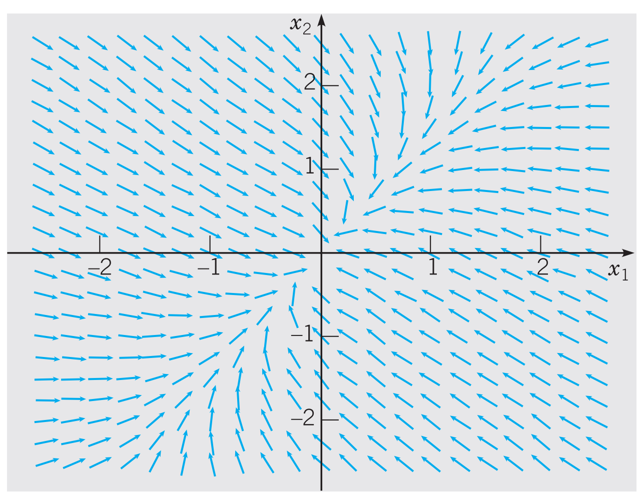


FIGURE 7.5.3 Direction field for the system (18).

and the general solution is

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)} = c_1\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + c_2\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}. \tag{25}$$

A phase portrait for the system (18) is constructed by drawing graphs of the solution (25) for several values of c_1 and c_2, as shown in Figure 7.5.4*a*. The solution $\mathbf{x}^{(1)}(t)$ approaches the origin along the line $x_2 = \sqrt{2}x_1$, and the solution $\mathbf{x}^{(2)}(t)$ approaches the origin along the line $x_1 = -\sqrt{2}x_2$. The directions of these lines are determined by the eigenvectors $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$, respectively. In general, we have a combination of these two fundamental solutions. As $t \to \infty$, the solution $\mathbf{x}^{(2)}(t)$ is negligible compared to $\mathbf{x}^{(1)}(t)$. Thus, unless $c_1 = 0$, the solution (25) approaches the origin tangent to the line $x_2 = \sqrt{2}x_1$. The pattern of trajectories shown in Figure 7.5.4*a* is typical of all 2×2 systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for which the eigenvalues are real, different, and of the same sign. The origin is called a **node** for such a system. If the eigenvalues were positive rather than negative, then the trajectories would be similar but traversed in the outward direction. Nodes are asymptotically stable if the eigenvalues are negative and unstable if the eigenvalues are positive.

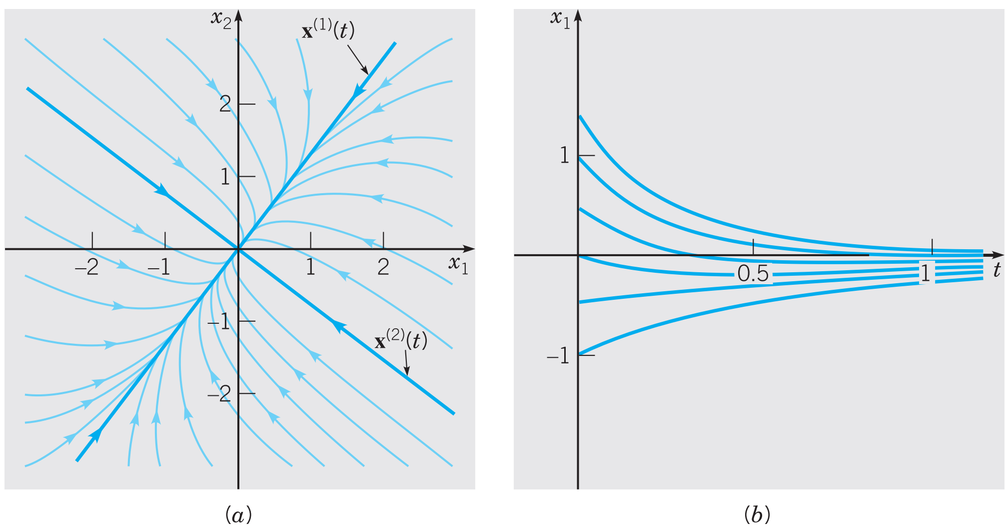


FIGURE 7.5.4 (*a*) A phase portrait for the system (18); the origin is an asynptotically stable node. (*b*) Typical plots of x_1 versus t for the system (18).

Although Figure 7.5.4*a* was computer-generated, a qualitatively correct sketch of the trajectories can be drawn quickly by hand on the basis of a knowledge of the eigenvalues and eigenvectors.

Some typical plots of x_1 versus t are shown in Figure 7.5.4*b*. Observe that each of the graphs approaches the t-axis asymptotically as t increases, corresponding to a trajectory that approaches the origin in Figure 7.5.2*a*. The behavior of x_2 as a function of t is similar.

Examples 2 and 3 illustrate the two main cases for 2×2 systems having eigenvalues that are real and different. The eigenvalues have either opposite signs (Example 2) or the same sign (Example 3). The other possibility is that zero is an eigenvalue, but in this case it follows that $\det \mathbf{A} = 0$, which violates the assumption made at the beginning of this section. However, see Problems 7 and 8.

Returning to the general system (1), we proceed as in the examples. To find solutions of the differential equation (1), we must find the eigenvalues and eigenvectors of $\mathbf{A}$ from the associated algebraic system (8). The eigenvalues $r_1, \ldots, r_n$ (which need not all be different) are roots of the nth degree polynomial equation

$$\det(\mathbf{A} - r\mathbf{I}) = 0. \tag{26}$$

The nature of the eigenvalues and the corresponding eigenvectors determines the nature of the general solution of the system (1). If we assume that $\mathbf{A}$ is a real-valued matrix, then we must consider the following possibilities for the eigenvalues of $\mathbf{A}$:

1. All eigenvalues are real and different from each other.
2. Some eigenvalues occur in complex conjugate pairs.
3. Some eigenvalues, either real or complex, are repeated.

If the n eigenvalues are all real and different, as in the three preceding examples, then associated with each eigenvalue r_i is a real eigenvector $\boldsymbol{\xi}^{(i)}$, and the n eigenvectors $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$ are linearly independent. The corresponding solutions of the differential system (1) are

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t}, \quad \ldots, \quad \mathbf{x}^{(n)}(t) = \boldsymbol{\xi}^{(n)} e^{r_n t}. \tag{27}$$

To show that these solutions form a fundamental set, we evaluate their Wronskian:

$$W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}](t) = \begin{vmatrix} \xi_1^{(1)} e^{r_1 t} & \cdots & \xi_1^{(n)} e^{r_n t} \\ \vdots & & \vdots \\ \xi_n^{(1)} e^{r_1 t} & \cdots & \xi_n^{(n)} e^{r_n t} \end{vmatrix}$$

$$= e^{(r_1 + \cdots + r_n)t} \begin{vmatrix} \xi_1^{(1)} & \cdots & \xi_1^{(n)} \\ \vdots & & \vdots \\ \xi_n^{(1)} & \cdots & \xi_n^{(n)} \end{vmatrix}. \tag{28}$$

First, we observe that the exponential function is never zero. Next, since the eigenvectors $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$ are linearly independent, the determinant in the last term of Eq. (28) is nonzero. As a consequence, the Wronskian $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}](t)$ is never zero; hence $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ form a fundamental set of solutions. Thus the general solution of Eq. (1) is

$$\mathbf{x} = c_1 \boldsymbol{\xi}^{(1)} e^{r_1 t} + \cdots + c_n \boldsymbol{\xi}^{(n)} e^{r_n t}. \tag{29}$$

If $\mathbf{A}$ is real and symmetric (a special case of Hermitian matrices), recall from Section 7.3 that all the eigenvalues $r_1, \ldots, r_n$ must be real. Further, even if some of the eigenvalues are repeated, there is always a full set of n eigenvectors $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$ that are linearly independent (in fact, orthogonal). Hence the corresponding solutions of the differential system (1) given by Eq. (27) again form a fundamental set of solutions, and the general solution is again given by Eq. (29). The following example illustrates this case.

EXAMPLE 4

Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{x}. \tag{30}$$

Observe that the coefficient matrix is real and symmetric. The eigenvalues and eigenvectors of this matrix were found in Example 5 of Section 7.3:

$$r_1 = 2, \qquad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \tag{31}$$

$$r_2 = -1, \qquad r_3 = -1; \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \boldsymbol{\xi}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \tag{32}$$

Hence a fundamental set of solutions of Eq. (30) is

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t}, \qquad \mathbf{x}^{(3)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}, \tag{33}$$

and the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}. \tag{34}$$

This example illustrates the fact that even though an eigenvalue ($r = -1$) has algebraic multiplicity 2, it may still be possible to find two linearly independent eigenvectors $\boldsymbol{\xi}^{(2)}$ and $\boldsymbol{\xi}^{(3)}$ and, as a consequence, to construct the general solution (34).

The behavior of the solution (34) depends critically on the initial conditions. For large t the first term on the right side of Eq. (34) is the dominant one; therefore, if $c_1 \neq 0$, all components of $\mathbf{x}$ become unbounded as $t \to \infty$. On the other hand, for certain initial points c_1 will be zero. In this case, the solution involves only the negative exponential terms, and $\mathbf{x} \to \mathbf{0}$ as $t \to \infty$. The initial points that cause c_1 to be zero are precisely those that lie in the plane determined by the eigenvectors $\boldsymbol{\xi}^{(2)}$ and $\boldsymbol{\xi}^{(3)}$ corresponding to the two negative eigenvalues. Thus solutions that start in this plane approach the origin as $t \to \infty$, while all other solutions become unbounded.

If some of the eigenvalues occur in complex conjugate pairs, then there are still n linearly independent solutions of the form (27), provided that all the eigenvalues are different. Of course, the solutions arising from complex eigenvalues are complex-valued. However, as in Section 3.3, it is possible to obtain a full set of real-valued solutions. This is discussed in Section 7.6.

More serious difficulties can occur if an eigenvalue is repeated. In this event the number of corresponding linearly independent eigenvectors may be smaller than the algebraic multiplicity of the eigenvalue. If so, the number of linearly independent solutions of the form $\boldsymbol{\xi}e^{rt}$ will be smaller than n. To construct a fundamental set of solutions, it is then necessary to seek additional solutions of another form. The situation is somewhat analogous to that for an nth order linear equation with constant coefficients; a repeated root of the characteristic equation gave rise to solutions of the form $e^{rt}, te^{rt}, t^2e^{rt}, \ldots$. The case of repeated eigenvalues is treated in Section 7.8.

Finally, if $\mathbf{A}$ is complex, then complex eigenvalues need not occur in conjugate pairs, and the eigenvectors are normally complex-valued even though the associated eigenvalue may be real. The solutions of the differential equation (1) are still of the form (27), provided that there are n linearly independent eigenvectors, but in general all the solutions are complex-valued.

PROBLEMS

In each of Problems 1 through 6:

(a) Find the general solution of the given system of equations and describe the behavior of the solution as $t \to \infty$.

(b) Draw a direction field and plot a few trajectories of the system.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\mathbf{x}$
2. $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}\mathbf{x}$
3. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x}$
4. $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{x}$
5. $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{x}$
6. $\mathbf{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix}\mathbf{x}$

In each of Problems 7 and 8:

(a) Find the general solution of the given system of equations.

(b) Draw a direction field and a few of the trajectories. In each of these problems, the coefficient matrix has a zero eigenvalue. As a result, the pattern of trajectories is different from those in the examples in the text.

7. $\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}\mathbf{x}$
8. $\mathbf{x}' = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}\mathbf{x}$

In each of Problems 9 through 14, find the general solution of the given system of equations.

9. $\mathbf{x}' = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\mathbf{x}$
10. $\mathbf{x}' = \begin{pmatrix} 2 & 2+i \\ -1 & -1-i \end{pmatrix}\mathbf{x}$
11. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}\mathbf{x}$
12. $\mathbf{x}' = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}\mathbf{x}$
13. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix}\mathbf{x}$
14. $\mathbf{x}' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}\mathbf{x}$

In each of Problems 15 through 18, solve the given initial value problem. Describe the behavior of the solution as $t \to \infty$.

15. $\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$
16. $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$
17. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$
18. $\mathbf{x}' = \begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 7 \\ 5 \\ 5 \end{pmatrix}$
19. The system $t\mathbf{x}' = \mathbf{A}\mathbf{x}$ is analogous to the second order Euler equation (Section 5.4). Assuming that $\mathbf{x} = \boldsymbol{\xi} t^r$, where $\boldsymbol{\xi}$ is a constant vector, show that $\boldsymbol{\xi}$ and r must satisfy $(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$ in order to obtain nontrivial solutions of the given differential equation.

Referring to Problem 19, solve the given system of equations in each of Problems 20 through 23. Assume that $t > 0$.

20. $t\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x}$

21. $t\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\mathbf{x}$

22. $t\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}\mathbf{x}$

23. $t\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\mathbf{x}$

In each of Problems 24 through 27, the eigenvalues and eigenvectors of a matrix $\mathbf{A}$ are given. Consider the corresponding system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

(a) Sketch a phase portrait of the system.

(b) Sketch the trajectory passing through the initial point $(2, 3)$.

(c) For the trajectory in part (b), sketch the graphs of x_1 versus t and of x_2 versus t on the same set of axes.

24. $r_1 = -1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \qquad r_2 = -2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

25. $r_1 = 1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \qquad r_2 = -2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

26. $r_1 = -1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \qquad r_2 = 2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

27. $r_1 = 1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \qquad r_2 = 2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

28. Consider a 2×2 system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. If we assume that $r_1 \neq r_2$, the general solution is $\mathbf{x} = c_1\boldsymbol{\xi}^{(1)}e^{r_1 t} + c_2\boldsymbol{\xi}^{(2)}e^{r_2 t}$, provided that $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly independent. In this problem we establish the linear independence of $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ by assuming that they are linearly dependent and then showing that this leads to a contradiction.

    (a) Note that $\boldsymbol{\xi}^{(1)}$ satisfies the matrix equation $(\mathbf{A} - r_1\mathbf{I})\boldsymbol{\xi}^{(1)} = \mathbf{0}$; similarly, note that $(\mathbf{A} - r_2\mathbf{I})\boldsymbol{\xi}^{(2)} = \mathbf{0}$.

    (b) Show that $(\mathbf{A} - r_2\mathbf{I})\boldsymbol{\xi}^{(1)} = (r_1 - r_2)\boldsymbol{\xi}^{(1)}$.

    (c) Suppose that $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly dependent. Then $c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)} = \mathbf{0}$ and at least one of c_1 and c_2 (say c_1) is not zero. Show that $(\mathbf{A} - r_2\mathbf{I})(c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}) = \mathbf{0}$, and also show that $(\mathbf{A} - r_2\mathbf{I})(c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}) = c_1(r_1 - r_2)\boldsymbol{\xi}^{(1)}$. Hence $c_1 = 0$, which is a contradiction. Therefore, $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly independent.

    (d) Modify the argument of part (c) if we assume that $c_2 \neq 0$.

    (e) Carry out a similar argument for the case in which the order n is equal to 3; note that the procedure can be extended to an arbitrary value of n.

29. Consider the equation

$$ay'' + by' + cy = 0, \tag{i}$$

    where a, b, and c are constants with $a \neq 0$. In Chapter 3 it was shown that the general solution depended on the roots of the characteristic equation

$$ar^2 + br + c = 0. \tag{ii}$$

    (a) Transform Eq. (i) into a system of first order equations by letting $x_1 = y, x_2 = y'$. Find the system of equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$ satisfied by $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

(b) Find the equation that determines the eigenvalues of the coefficient matrix **A** in part (a). Note that this equation is just the characteristic equation (ii) of Eq. (i).

30. The two-tank system of Problem 22 in Section 7.1 leads to the initial value problem

$$\mathbf{x}' = \begin{pmatrix} -\frac{1}{10} & \frac{3}{40} \\ \frac{1}{10} & -\frac{1}{5} \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} -17 \\ -21 \end{pmatrix},$$

where x_1 and x_2 are the deviations of the salt levels Q_1 and Q_2 from their respective equilibria.

(a) Find the solution of the given initial value problem.

(b) Plot x_1 versus t and x_2 versus t on the same set of axes.

(c) Find the smallest time T such that $|x_1(t)| \le 0.5$ and $|x_2(t)| \le 0.5$ for all $t \ge T$.

31. Consider the system

$$\mathbf{x}' = \begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix} \mathbf{x}.$$

(a) Solve the system for $\alpha = 0.5$. What are the eigenvalues of the coefficient matrix? Classify the equilibrium point at the origin as to type.

(b) Solve the system for $\alpha = 2$. What are the eigenvalues of the coefficient matrix? Classify the equilibrium point at the origin as to type.

(c) In parts (a) and (b), solutions of the system exhibit two quite different types of behavior. Find the eigenvalues of the coefficient matrix in terms of α, and determine the value of α between 0.5 and 2 where the transition from one type of behavior to the other occurs.

Electric Circuits. Problems 32 and 33 are concerned with the electric circuit described by the system of differential equations in Problem 21 of Section 7.1:

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -\dfrac{R_1}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C} & -\dfrac{1}{CR_2} \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}. \tag{i}$$

32. (a) Find the general solution of Eq. (i) if $R_1 = 1\ \Omega$, $R_2 = \frac{3}{5}\ \Omega$, $L = 2$ H, and $C = \frac{2}{3}$ F.

(b) Show that $I(t) \to 0$ and $V(t) \to 0$ as $t \to \infty$, regardless of the initial values $I(0)$ and $V(0)$.

33. Consider the preceding system of differential equations (i).

(a) Find a condition on R_1, R_2, C, and L that must be satisfied if the eigenvalues of the coefficient matrix are to be real and different.

(b) If the condition found in part (a) is satisfied, show that both eigenvalues are negative. Then show that $I(t) \to 0$ and $V(t) \to 0$ as $t \to \infty$, regardless of the initial conditions.

(c) If the condition found in part (a) is not satisfied, then the eigenvalues are either complex or repeated. Do you think that $I(t) \to 0$ and $V(t) \to 0$ as $t \to \infty$ in these cases as well?

Hint: In part (c), one approach is to change the system (i) into a single second order equation. We also discuss complex and repeated eigenvalues in Sections 7.6 and 7.8.

7.6 Complex Eigenvalues

In this section we consider again a system of n linear homogeneous equations with constant coefficients

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{1}$$

where the coefficient matrix $\mathbf{A}$ is real-valued. If we seek solutions of the form $\mathbf{x} = \boldsymbol{\xi} e^{rt}$, then it follows, as in Section 7.5, that r must be an eigenvalue and $\boldsymbol{\xi}$ a corresponding eigenvector of the coefficient matrix $\mathbf{A}$. Recall that the eigenvalues $r_1, \dots, r_n$ of $\mathbf{A}$ are the roots of the characteristic equation

$$\det(\mathbf{A} - r\mathbf{I}) = 0 \tag{2}$$

and that the corresponding eigenvectors satisfy

$$(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}. \tag{3}$$

If $\mathbf{A}$ is real, then the coefficients in the polynomial equation (2) for r are real, and any complex eigenvalues must occur in conjugate pairs. For example, if $r_1 = \lambda + i\mu$, where λ and μ are real, is an eigenvalue of $\mathbf{A}$, then so is $r_2 = \lambda - i\mu$. To explore the effect of complex eigenvalues, we begin with an example.

EXAMPLE 1

Find a fundamental set of real-valued solutions of the system

$$\mathbf{x}' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \mathbf{x}. \tag{4}$$

Plot a phase portrait and graphs of components of typical solutions.

A direction field for the system (4) is shown in Figure 7.6.1. This plot suggests that the trajectories in the phase plane spiral clockwise toward the origin.

To find a fundamental set of solutions, we assume that

$$\mathbf{x} = \boldsymbol{\xi} e^{rt} \tag{5}$$

and obtain the set of linear algebraic equations

$$\begin{pmatrix} -\frac{1}{2} - r & 1 \\ -1 & -\frac{1}{2} - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6}$$

for the eigenvalues and eigenvectors of $\mathbf{A}$. The characteristic equation is

$$\begin{vmatrix} -\frac{1}{2} - r & 1 \\ -1 & -\frac{1}{2} - r \end{vmatrix} = r^2 + r + \tfrac{5}{4} = 0; \tag{7}$$

therefore the eigenvalues are $r_1 = -\frac{1}{2} + i$ and $r_2 = -\frac{1}{2} - i$. From Eq. (6) a straightforward calculation shows that the corresponding eigenvectors are

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}. \tag{8}$$

Observe that the eigenvectors $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are also complex conjugates. Hence a fundamental set of solutions of the system (4) is

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1/2+i)t}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(-1/2-i)t}. \tag{9}$$

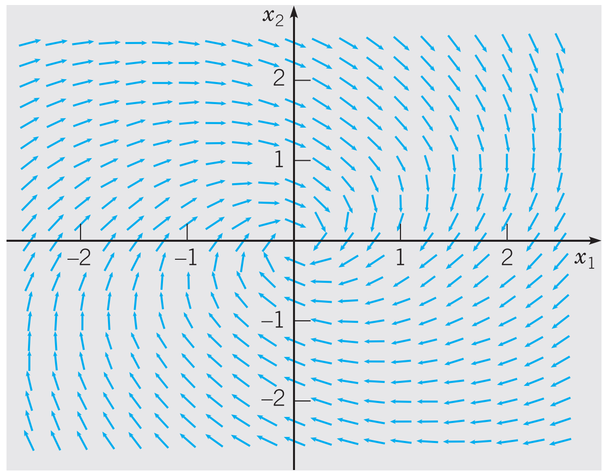


FIGURE 7.6.1 A direction field for the system (4).

To obtain a set of real-valued solutions, we can (by Theorem 7.4.5) choose the real and imaginary parts of either $\mathbf{x}^{(1)}$ or $\mathbf{x}^{(2)}$. In fact,

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-t/2}(\cos t + i \sin t) = \begin{pmatrix} e^{-t/2}\cos t \\ -e^{-t/2}\sin t \end{pmatrix} + i \begin{pmatrix} e^{-t/2}\sin t \\ e^{-t/2}\cos t \end{pmatrix}. \tag{10}$$

Hence a set of real-valued solutions of (Eq. 4) is

$$\mathbf{u}(t) = e^{-t/2}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \qquad \mathbf{v}(t) = e^{-t/2}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}. \tag{11}$$

To verify that $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are linearly independent, we compute their Wronskian:

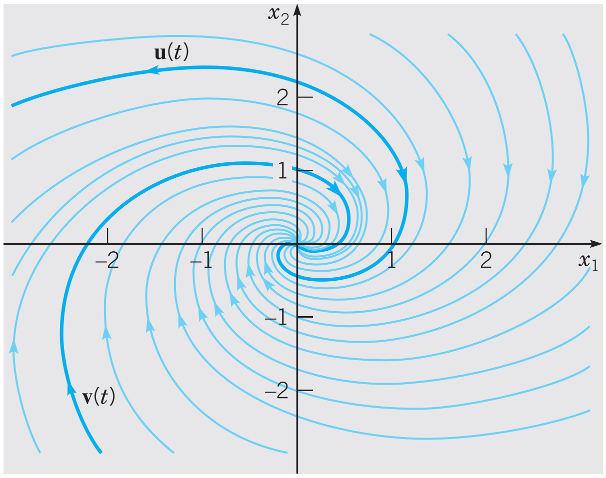
$$\begin{aligned} W(\mathbf{u},\mathbf{v})(t) &= \begin{vmatrix} e^{-t/2}\cos t & e^{-t/2}\sin t \\ -e^{-t/2}\sin t & e^{-t/2}\cos t \end{vmatrix} \\ &= e^{-t}(\cos^2 t + \sin^2 t) = e^{-t}. \end{aligned}$$

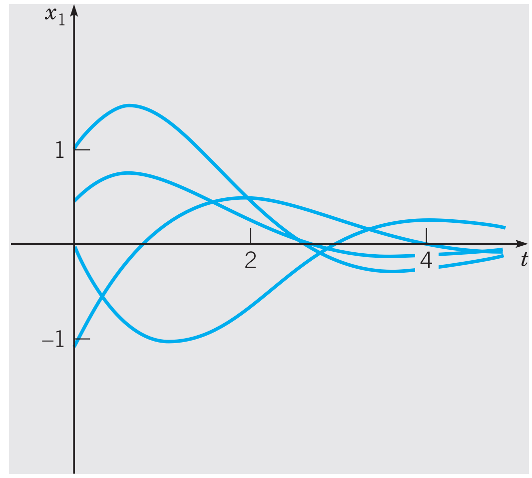
The Wronskian $W(\mathbf{u},\mathbf{v})(t)$ is never zero, so it follows that $\mathbf{u}(t)$ and $\mathbf{v}(t)$ constitute a fundamental set of (real-valued) solutions of the system (4).

The graphs of the solutions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are shown in Figure 7.6.2*a*. Since

$$\mathbf{u}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

the graphs of $\mathbf{u}(t)$ and $\mathbf{v}(t)$ pass through the points $(1, 0)$ and $(0, 1)$, respectively. Other solutions of the system (4) are linear combinations of $\mathbf{u}(t)$ and $\mathbf{v}(t)$, and graphs of a few of these solutions are also shown in Figure 7.6.2*a*; this figure is a phase portrait for the system (4). Each trajectory approaches the origin along a spiral path as $t \to \infty$, making infinitely many circuits about the origin; this is due to the fact that the solutions (11) are products of decaying exponential and sine or cosine factors. Some typical graphs of x_1 versus t are shown in Figure 7.6.2*b*; each one represents a decaying oscillation in time.





(*a*) (*b*)

FIGURE 7.6.2 (*a*) A phase portrait for the system (4); the origin is a spiral point. (*b*) Plots of x_1 versus t for the system (4); graphs of x_2 versus t are similar.

Figure 7.6.2*a* is typical of all 2×2 systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ whose eigenvalues are complex with negative real part. The origin is called a **spiral point** and is asymptotically stable because all trajectories approach it as t increases. For a system whose eigenvalues have a positive real part, the trajectories are similar to those in Figure 7.6.2*a*, but the direction of motion is away from the origin, and the trajectories become unbounded. In this case, the origin is unstable. If the real part of the eigenvalues is zero, then the trajectories neither approach the origin nor become unbounded but instead repeatedly traverse a closed curve about the origin. Examples of this behavior can be seen in Figures 7.6.3*b* and 7.6.4*b* below. In this case the origin is called a **center** and is said to be stable, but not asymptotically stable. In all three cases, the direction of motion may be either clockwise, as in this example, or counterclockwise, depending on the elements of the coefficient matrix **A**.

The phase portrait in Figure 7.6.2*a* was drawn by a computer, but it is possible to produce a useful sketch of the phase portrait by hand. We have noted that when the eigenvalues $\lambda \pm i\mu$ are complex, then the trajectories either spiral in ($\lambda < 0$), spiral out ($\lambda > 0$), or repeatedly traverse a closed curve ($\lambda = 0$). To determine whether the direction of motion is clockwise or counterclockwise, we only need to determine the direction of motion at a single convenient point. For instance, in the system (4) we might choose $\mathbf{x} = (0, 1)^T$. Then $\mathbf{A}\mathbf{x} = (1, -\frac{1}{2})^T$. Thus at the point $(0, 1)$ in the phase plane the tangent vector $\mathbf{x}'$ to the trajectory at that point has a positive x_1-component and therefore is directed from the second quadrant into the first. The direction of motion is therefore clockwise for the trajectories of this system.

Returning to the general equation (1)

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

we can proceed just as in the example. Suppose that there is a pair of complex conjugate eigenvalues, $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$. Then the corresponding eigenvectors $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are also complex conjugates. To see that this is so, recall that

r_1 and $\boldsymbol{\xi}^{(1)}$ satisfy

$$(\mathbf{A} - r_1\mathbf{I})\boldsymbol{\xi}^{(1)} = \mathbf{0}. \tag{12}$$

On taking the complex conjugate of this equation and noting that $\mathbf{A}$ and $\mathbf{I}$ are real-valued, we obtain

$$(\mathbf{A} - \bar{r}_1\mathbf{I})\bar{\boldsymbol{\xi}}^{(1)} = \mathbf{0}, \tag{13}$$

where $\bar{r}_1$ and $\bar{\boldsymbol{\xi}}^{(1)}$ are the complex conjugates of r_1 and $\boldsymbol{\xi}^{(1)}$, respectively. In other words, $r_2 = \bar{r}_1$ is also an eigenvalue, and $\boldsymbol{\xi}^{(2)} = \bar{\boldsymbol{\xi}}^{(1)}$ is a corresponding eigenvector. The corresponding solutions

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)}e^{r_1 t}, \qquad \mathbf{x}^{(2)}(t) = \bar{\boldsymbol{\xi}}^{(1)}e^{\bar{r}_1 t} \tag{14}$$

of the differential equation (1) are then complex conjugates of each other. Therefore, as in Example 1, we can find two real-valued solutions of Eq. (1) corresponding to the eigenvalues r_1 and r_2 by taking the real and imaginary parts of $\mathbf{x}^{(1)}(t)$ or $\mathbf{x}^{(2)}(t)$ given by Eq. (14).

Let us write $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are real; then we have

$$\begin{aligned} \mathbf{x}^{(1)}(t) &= (\mathbf{a} + i\mathbf{b})e^{(\lambda + i\mu)t} \\ &= (\mathbf{a} + i\mathbf{b})e^{\lambda t}(\cos \mu t + i \sin \mu t). \end{aligned} \tag{15}$$

Upon separating $\mathbf{x}^{(1)}(t)$ into its real and imaginary parts, we obtain

$$\mathbf{x}^{(1)}(t) = e^{\lambda t}(\mathbf{a}\cos \mu t - \mathbf{b}\sin \mu t) + ie^{\lambda t}(\mathbf{a}\sin \mu t + \mathbf{b}\cos \mu t). \tag{16}$$

If we write $\mathbf{x}^{(1)}(t) = \mathbf{u}(t) + i\mathbf{v}(t)$, then the vectors

$$\begin{aligned} \mathbf{u}(t) &= e^{\lambda t}(\mathbf{a}\cos \mu t - \mathbf{b}\sin \mu t), \\ \mathbf{v}(t) &= e^{\lambda t}(\mathbf{a}\sin \mu t + \mathbf{b}\cos \mu t) \end{aligned} \tag{17}$$

are real-valued solutions of Eq. (1). It is possible to show that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent solutions (see Problem 27).

For example, suppose that the matrix $\mathbf{A}$ has two complex eigenvalues $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$, and that $r_3, \dots, r_n$ are all real and distinct. Let the corresponding eigenvectors be $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$, $\boldsymbol{\xi}^{(2)} = \mathbf{a} - i\mathbf{b}$, $\boldsymbol{\xi}^{(3)}, \dots, \boldsymbol{\xi}^{(n)}$. Then the general solution of Eq. (1) is

$$\mathbf{x} = c_1\mathbf{u}(t) + c_2\mathbf{v}(t) + c_3\boldsymbol{\xi}^{(3)}e^{r_3 t} + \cdots + c_n\boldsymbol{\xi}^{(n)}e^{r_n t}, \tag{18}$$

where $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are given by Eqs. (17). We emphasize that this analysis applies only if the coefficient matrix $\mathbf{A}$ in Eq. (1) is real, for it is only then that complex eigenvalues and eigenvectors must occur in conjugate pairs.

For 2×2 systems with real coefficients, we have now completed our description of the three main cases that can occur.

1. Eigenvalues are real and have opposite signs; $\mathbf{x} = \mathbf{0}$ is a saddle point.
2. Eigenvalues are real and have the same sign but are unequal; $\mathbf{x} = \mathbf{0}$ is a node.
3. Eigenvalues are complex with nonzero real part; $\mathbf{x} = \mathbf{0}$ is a spiral point.

Other possibilities are of less importance and occur as transitions between two of the cases just listed. For example, a zero eigenvalue occurs during the transition between a saddle point and a node. Purely imaginary eigenvalues occur during a transition between asymptotically stable and unstable spiral points. Finally, real and equal eigenvalues appear during the transition between nodes and spiral points.

EXAMPLE 2

The system

$$\mathbf{x}' = \begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix} \mathbf{x} \tag{19}$$

contains a parameter α. Describe how the solutions depend qualitatively on α; in particular, find the critical values of α at which the qualitative behavior of the trajectories in the phase plane changes markedly.

The behavior of the trajectories is controlled by the eigenvalues of the coefficient matrix. The characteristic equation is

$$r^2 - \alpha r + 4 = 0, \tag{20}$$

so the eigenvalues are

$$r = \frac{\alpha \pm \sqrt{\alpha^2 - 16}}{2}. \tag{21}$$

From Eq. (21) it follows that the eigenvalues are complex conjugates for $-4 < \alpha < 4$ and are real otherwise. Thus two critical values are $\alpha = -4$ and $\alpha = 4$, where the eigenvalues change from real to complex, or vice versa. For $\alpha < -4$ both eigenvalues are negative, so all trajectories approach the origin, which is an asymptotically stable node. For $\alpha > 4$ both eigenvalues are positive, so the origin is again a node, this time unstable; all trajectories (except $\mathbf{x} = \mathbf{0}$) become unbounded. In the intermediate range, $-4 < \alpha < 4$, the eigenvalues are complex and the trajectories are spirals. However, for $-4 < \alpha < 0$ the real part of the eigenvalues is negative, the spirals are directed inward, and the origin is asymptotically stable, whereas for $0 < \alpha < 4$ the real part of the eigenvalues is positive and the origin is unstable. Thus $\alpha = 0$ is also a critical value where the direction of the spirals changes from inward to outward. For this value of α, the origin is a center and the trajectories are closed curves about the origin, corresponding to solutions that are periodic in time. The other critical values, $\alpha = \pm 4$, yield eigenvalues that are real and equal. In this case the origin is again a node, but the phase portrait differs somewhat from those in Section 7.5. We take up this case in Section 7.8.

A Multiple Spring–Mass System. Consider the system of two masses and three springs shown in Figure 7.1.1, whose equations of motion are given by Eqs. (1) in Section 7.1. If we assume that there are no external forces, then $F_1(t) = 0$, $F_2(t) = 0$, and the resulting equations are

$$\begin{aligned} m_1 \frac{d^2 x_1}{dt^2} &= -(k_1 + k_2)x_1 + k_2 x_2, \\ m_2 \frac{d^2 x_2}{dt^2} &= k_2 x_1 - (k_2 + k_3)x_2. \end{aligned} \tag{22}$$

These equations can be solved as a system of two second order equations (see Problem 29), but, as is consistent with our approach in this chapter, we will transform them into a system of four first order equations. Let $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1'$, and $y_4 = x_2'$. Then

$$y_1' = y_3, \qquad y_2' = y_4, \tag{23}$$

and, from Eqs. (22),

$$m_1 y_3' = -(k_1 + k_2)y_1 + k_2 y_2, \qquad m_2 y_4' = k_2 y_1 - (k_2 + k_3)y_2. \tag{24}$$

The following example deals with a particular case of this two-mass, three-spring system.

EXAMPLE 3

Suppose that $m_1 = 2$, $m_2 = 9/4$, $k_1 = 1$, $k_2 = 3$, and $k_3 = 15/4$ in Eqs. (23) and (24) so that these equations become

$$y_1' = y_3, \qquad y_2' = y_4, \qquad y_3' = -2y_1 + \tfrac{3}{2}y_2, \qquad y_4' = \tfrac{4}{3}y_1 - 3y_2. \tag{25}$$

Analyze the possible motions described by Eqs. (25), and draw graphs showing typical behavior.

We can write the system (25) in matrix form as

$$\mathbf{y}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & \frac{3}{2} & 0 & 0 \\ \frac{4}{3} & -3 & 0 & 0 \end{pmatrix} \mathbf{y} = \mathbf{A}\mathbf{y}. \tag{26}$$

Keep in mind that y_1 and y_2 are the positions of the two masses, relative to their equilibrium positions, and that y_3 and y_4 are their velocities. We assume, as usual, that $\mathbf{y} = \boldsymbol{\xi} e^{rt}$, where r must be an eigenvalue of the matrix $\mathbf{A}$ and $\boldsymbol{\xi}$ a corresponding eigenvector. It is possible, though a bit tedious, to find the eigenvalues and eigenvectors of $\mathbf{A}$ by hand, but it is easy with appropriate computer software. The characteristic polynomial of $\mathbf{A}$ is

$$r^4 + 5r^2 + 4 = (r^2 + 1)(r^2 + 4), \tag{27}$$

so the eigenvalues are $r_1 = i$, $r_2 = -i$, $r_3 = 2i$, and $r_4 = -2i$. The corresponding eigenvectors are

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 3 \\ 2 \\ 3i \\ 2i \end{pmatrix}, \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 3 \\ 2 \\ -3i \\ -2i \end{pmatrix}, \qquad \boldsymbol{\xi}^{(3)} = \begin{pmatrix} 3 \\ -4 \\ 6i \\ -8i \end{pmatrix}, \qquad \boldsymbol{\xi}^{(4)} = \begin{pmatrix} 3 \\ -4 \\ -6i \\ 8i \end{pmatrix}. \tag{28}$$

The complex-valued solutions $\boldsymbol{\xi}^{(1)} e^{it}$ and $\boldsymbol{\xi}^{(2)} e^{-it}$ are complex conjugates, so two real-valued solutions can be found by finding the real and imaginary parts of either of them. For instance, we have

$$\begin{aligned} \boldsymbol{\xi}^{(1)} e^{it} &= \begin{pmatrix} 3 \\ 2 \\ 3i \\ 2i \end{pmatrix} (\cos t + i \sin t) \\ &= \begin{pmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{pmatrix} + i \begin{pmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{pmatrix} = \mathbf{u}^{(1)}(t) + i\mathbf{v}^{(1)}(t). \end{aligned} \tag{29}$$

In a similar way, we obtain

$$\boldsymbol{\xi}^{(3)}e^{2it} = \begin{pmatrix} 3 \\ -4 \\ 6i \\ -8i \end{pmatrix}(\cos 2t + i\sin 2t)$$

$$= \begin{pmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{pmatrix} + i\begin{pmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{pmatrix} = \mathbf{u}^{(2)}(t) + i\mathbf{v}^{(2)}(t). \tag{30}$$

We leave it to you to verify that $\mathbf{u}^{(1)}$, $\mathbf{v}^{(1)}$, $\mathbf{u}^{(2)}$, and $\mathbf{v}^{(2)}$ are linearly independent and therefore form a fundamental set of solutions. Thus the general solution of Eq. (26) is

$$\mathbf{y} = c_1\begin{pmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{pmatrix} + c_2\begin{pmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{pmatrix} + c_3\begin{pmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{pmatrix} + c_4\begin{pmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{pmatrix}, \tag{31}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants.

The phase space for this system is four-dimensional, and each solution, obtained by a particular set of values for $c_1, \dots, c_4$ in Eq. (31), corresponds to a trajectory in this space. Since each solution, given by Eq. (31), is periodic with period 2π, each trajectory is a closed curve. No matter where the trajectory starts at $t = 0$, it returns to that point at $t = 2\pi$, $t = 4\pi$, and so forth, repeatedly traversing the same curve in each time interval of length 2π. We do not attempt to show any of these four-dimensional trajectories here. Instead, in the figures below we show projections of certain trajectories in the y_1y_3- or y_2y_4-plane, thereby showing the motion of each mass separately.

The first two terms on the right side of Eq. (31) describe motions with frequency 1 and period 2π. Note that $y_2 = (2/3)y_1$ in these terms and that $y_4 = (2/3)y_3$. This means that the two masses move back and forth together, always going in the same direction, but with the second mass moving only two-thirds as far as the first mass. If we focus on the solution $\mathbf{u}^{(1)}(t)$ and plot y_1 versus t and y_2 versus t on the same axes, we obtain the cosine graphs of amplitude 3 and 2, respectively, shown in Figure 7.6.3*a*. The trajectory of the first mass in the y_1y_3-plane lies on the circle of radius 3 shown in Figure 7.6.3*b*, traversed clockwise starting at the point $(3, 0)$ and completing a circuit in time 2π. Also shown in this figure is the trajectory of the second mass in the y_2y_4-plane, which lies on the circle of radius 2, also traversed clockwise starting at $(2, 0)$ and also completing a circuit in time 2π. The origin is a center in the respective y_1y_3- and y_2y_4-planes. Similar graphs (with an appropriate shift in time) are obtained from $\mathbf{v}^{(1)}$ or from a linear combination of $\mathbf{u}^{(1)}$ and $\mathbf{v}^{(1)}$.

The remaining terms on the right side of Eq. (31) describe motions with frequency 2 and period π. Observe that in this case, $y_2 = -(4/3)y_1$ and $y_4 = -(4/3)y_3$. This means that the two masses are always moving in opposite directions and that the second mass moves four-thirds as far as the first mass. If we look only at $\mathbf{u}^{(2)}(t)$ and plot y_1 versus t and y_2 versus t on the same axes, we obtain Figure 7.6.4*a*. There is a phase difference of π, and the amplitude of y_2 is four-thirds that of y_1, confirming the preceding statements about the motions of the masses. Figure 7.6.4*b* shows a superposition of the trajectories for the two masses in their respective phase planes. Both graphs are ellipses, the inner one corresponding to the first mass and the outer one to the second. The trajectory on the inner ellipse starts at $(3, 0)$, and the trajectory on the outer ellipse starts at $(-4, 0)$. Both are traversed clockwise, and a circuit is completed in time π. The origin is a center in the respective y_1y_3- and y_2y_4-planes. Once again, similar graphs are obtained from $\mathbf{v}^{(2)}$ or from a linear combination of $\mathbf{u}^{(2)}$ and $\mathbf{v}^{(2)}$.

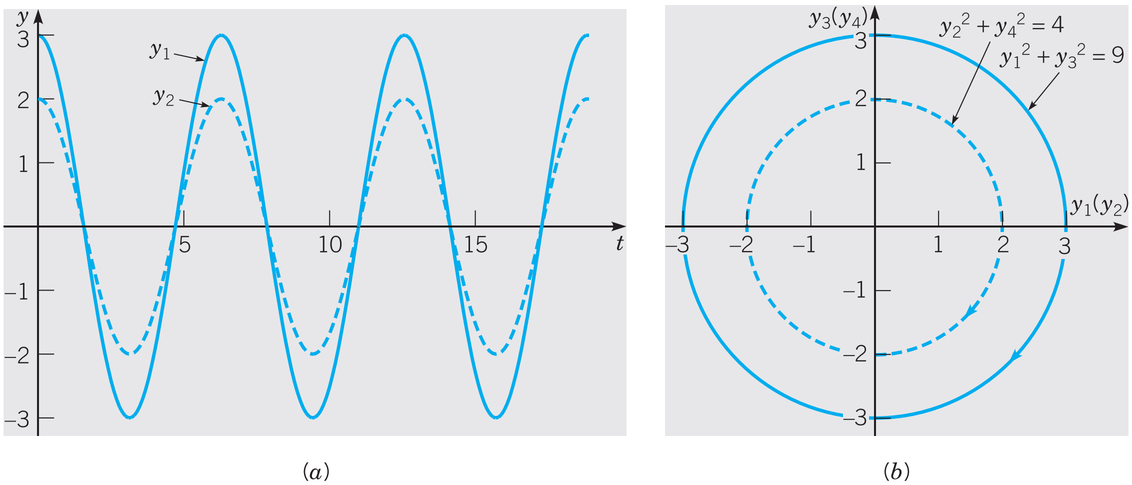


FIGURE 7.6.3 (a) A plot of y_1 versus t and y_2 versus t for the solution $\mathbf{u}^{(1)}(t)$. (b) Superposition of projections of trajectories in the y_1y_3- and y_2y_4-planes for the solution $\mathbf{u}^{(1)}(t)$.

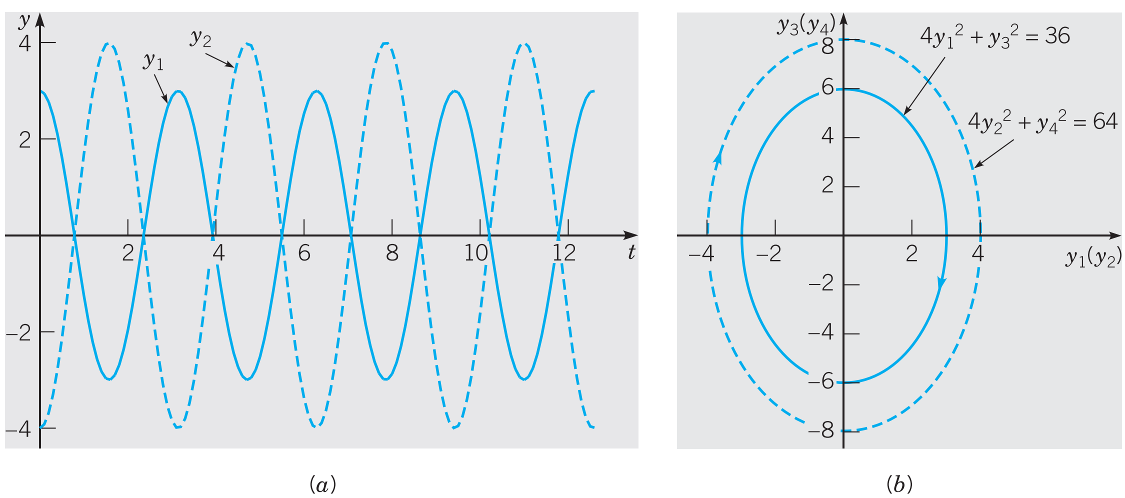
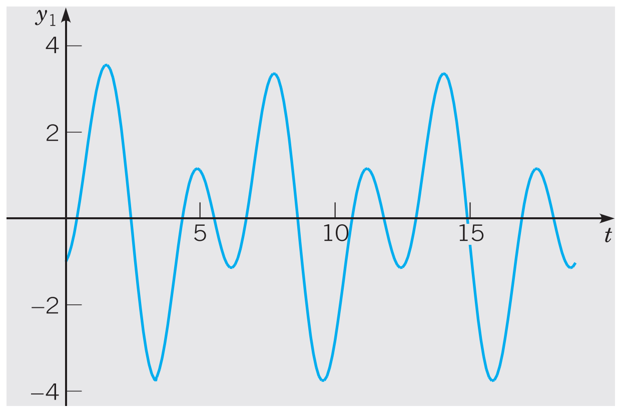


FIGURE 7.6.4 (a) A plot of y_1 versus t and y_2 versus t for the solution $\mathbf{u}^{(2)}(t)$. (b) Superposition of projections of trajectories in the y_1y_3- and y_2y_4-planes for the solution $\mathbf{u}^{(2)}(t)$.

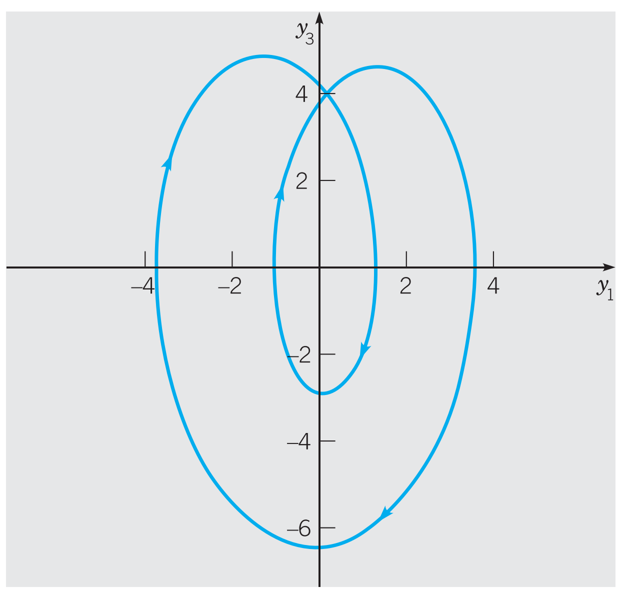
The types of motion described in the two preceding paragraphs are called **fundamental modes** of vibration for the two-mass system. Each of them results from fairly special initial conditions. For example, to obtain the fundamental mode of frequency 1, both of the constants c_3 and c_4 in Eq. (31) must be zero. This occurs only for initial conditions in which

$3y_2(0) = 2y_1(0)$ and $3y_4(0) = 2y_3(0)$. Similarly, the mode of frequency 2 is obtained only when both of the constants c_1 and c_2 in Eq. (31) are zero—that is, when the initial conditions are such that $3y_2(0) = -4y_1(0)$ and $3y_4(0) = -4y_3(0)$.

For more general initial conditions the solution is a combination of the two fundamental modes. A plot of y_1 versus t for a typical case is shown in Figure 7.6.5*a*, and the projection of the corresponding trajectory in the y_1y_3-plane is shown in Figure 7.6.5*b*. Observe that this latter figure may be a bit misleading in that it shows the projection of the trajectory crossing itself. This cannot be the case for the actual trajectory in four dimensions, because it would violate the general uniqueness theorem: there cannot be two different solutions issuing from the same initial point.



(*a*)



(*b*)

FIGURE 7.6.5 A solution of the system (25) satisfying the initial condition $\mathbf{y}(0) = (-1, 4, 1, 1)^T$. (*a*) A plot of y_1 versus t. (*b*) The projection of the trajectory in the y_1y_3-plane. As stated in the text, the actual trajectory in four dimensions does not intersect itself.

PROBLEMS

In each of Problems 1 through 6:

(a) Express the general solution of the given system of equations in terms of real-valued functions.

(b) Also draw a direction field, sketch a few of the trajectories, and describe the behavior of the solutions as $t \to \infty$.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

3. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}$

4. $\mathbf{x}' = \begin{pmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{pmatrix} \mathbf{x}$

5. $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \mathbf{x}$

6. $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \mathbf{x}$

In each of Problems 7 and 8, express the general solution of the given system of equations in terms of real-valued functions.

7. $\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{x}$

8. $\mathbf{x}' = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} \mathbf{x}$

In each of Problems 9 and 10, find the solution of the given initial value problem. Describe the behavior of the solution as $t \to \infty$.

9. $\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

10. $\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

In each of Problems 11 and 12:

(a) Find the eigenvalues of the given system.

(b) Choose an initial point (other than the origin) and draw the corresponding trajectory in the x_1x_2-plane.

(c) For your trajectory in part (b), draw the graphs of x_1 versus t and of x_2 versus t.

(d) For your trajectory in part (b), draw the corresponding graph in three-dimensional tx_1x_2-space.

11. $\mathbf{x}' = \begin{pmatrix} \frac{3}{4} & -2 \\ 1 & -\frac{5}{4} \end{pmatrix} \mathbf{x}$

12. $\mathbf{x}' = \begin{pmatrix} -\frac{4}{5} & 2 \\ -1 & \frac{6}{5} \end{pmatrix} \mathbf{x}$

In each of Problems 13 through 20, the coefficient matrix contains a parameter α. In each of these problems:

(a) Determine the eigenvalues in terms of α.

(b) Find the critical value or values of α where the qualitative nature of the phase portrait for the system changes.

(c) Draw a phase portrait for a value of α slightly below, and for another value slightly above, each critical value.

13. $\mathbf{x}' = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix} \mathbf{x}$

14. $\mathbf{x}' = \begin{pmatrix} 0 & -5 \\ 1 & \alpha \end{pmatrix} \mathbf{x}$

15. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ \alpha & -2 \end{pmatrix} \mathbf{x}$

16. $\mathbf{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \alpha & \frac{5}{4} \end{pmatrix} \mathbf{x}$

17. $\mathbf{x}' = \begin{pmatrix} -1 & \alpha \\ -1 & -1 \end{pmatrix} \mathbf{x}$

18. $\mathbf{x}' = \begin{pmatrix} 3 & \alpha \\ -6 & -4 \end{pmatrix} \mathbf{x}$

19. $\mathbf{x}' = \begin{pmatrix} \alpha & 10 \\ -1 & -4 \end{pmatrix} \mathbf{x}$

20. $\mathbf{x}' = \begin{pmatrix} 4 & \alpha \\ 8 & -6 \end{pmatrix} \mathbf{x}$

In each of Problems 21 and 22, solve the given system of equations by the method of Problem 19 of Section 7.5. Assume that $t > 0$.

21. $t\mathbf{x}' = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} \mathbf{x}$

22. $t\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}$

In each of Problems 23 and 24:

(a) Find the eigenvalues of the given system.

(b) Choose an initial point (other than the origin) and draw the corresponding trajectory in the x_1x_2-plane. Also draw the trajectories in the x_1x_3- and x_2x_3-planes.

(c) For the initial point in part (b), draw the corresponding trajectory in $x_1x_2x_3$-space.

23. $\mathbf{x}' = \begin{pmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} \end{pmatrix} \mathbf{x}$

24. $\mathbf{x}' = \begin{pmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{10} \end{pmatrix} \mathbf{x}$

25. Consider the electric circuit shown in Figure 7.6.6. Suppose that $R_1 = R_2 = 4\ \Omega$, $C = \frac{1}{2}$ F, and $L = 8$ H.

(a) Show that this circuit is described by the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}, \tag{i}$$

where I is the current through the inductor and V is the voltage drop across the capacitor. *Hint:* See Problem 20 of Section 7.1.

(b) Find the general solution of Eqs. (i) in terms of real-valued functions.

(c) Find $I(t)$ and $V(t)$ if $I(0) = 2$ A and $V(0) = 3$ V.

(d) Determine the limiting values of $I(t)$ and $V(t)$ as $t \to \infty$. Do these limiting values depend on the initial conditions?

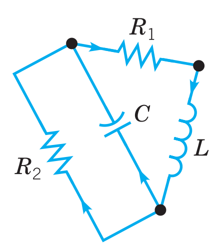


FIGURE 7.6.6 The circuit in Problem 25.

26. The electric circuit shown in Figure 7.6.7 is described by the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{1}{L} \\ -\dfrac{1}{C} & -\dfrac{1}{RC} \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}, \tag{i}$$

where I is the current through the inductor and V is the voltage drop across the capacitor. These differential equations were derived in Problem 19 of Section 7.1.

(a) Show that the eigenvalues of the coefficient matrix are real and different if $L > 4R^2C$; show that they are complex conjugates if $L < 4R^2C$.

(b) Suppose that $R = 1\ \Omega$, $C = \frac{1}{2}$ F, and $L = 1$ H. Find the general solution of the system (i) in this case.

(c) Find $I(t)$ and $V(t)$ if $I(0) = 2$ A and $V(0) = 1$ V.

(d) For the circuit of part (b) determine the limiting values of $I(t)$ and $V(t)$ as $t \to \infty$. Do these limiting values depend on the initial conditions?

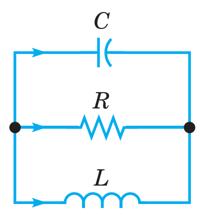


FIGURE 7.6.7 The circuit in Problem 26.

27. In this problem we indicate how to show that $\mathbf{u}(t)$ and $\mathbf{v}(t)$, as given by Eqs. (17), are linearly independent. Let $r_1 = \lambda + i\mu$ and $\overline{r}_1 = \lambda - i\mu$ be a pair of conjugate eigenvalues of the coefficient matrix $\mathbf{A}$ of Eq. (1); let $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$ and $\overline{\boldsymbol{\xi}}^{(1)} = \mathbf{a} - i\mathbf{b}$ be the corresponding eigenvectors. Recall that it was stated in Section 7.3 that two different eigenvalues have linearly independent eigenvectors, so if $r_1 \neq \overline{r}_1$, then $\boldsymbol{\xi}^{(1)}$ and $\overline{\boldsymbol{\xi}}^{(1)}$ are linearly independent.

(a) First we show that $\mathbf{a}$ and $\mathbf{b}$ are linearly independent. Consider the equation $c_1\mathbf{a} + c_2\mathbf{b} = \mathbf{0}$. Express $\mathbf{a}$ and $\mathbf{b}$ in terms of $\boldsymbol{\xi}^{(1)}$ and $\overline{\boldsymbol{\xi}}^{(1)}$, and then show that $(c_1 - ic_2)\boldsymbol{\xi}^{(1)} + (c_1 + ic_2)\overline{\boldsymbol{\xi}}^{(1)} = \mathbf{0}$.

(b) Show that $c_1 - ic_2 = 0$ and $c_1 + ic_2 = 0$ and then that $c_1 = 0$ and $c_2 = 0$. Consequently, $\mathbf{a}$ and $\mathbf{b}$ are linearly independent.

(c) To show that $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are linearly independent, consider the equation $c_1\mathbf{u}(t_0) + c_2\mathbf{v}(t_0) = \mathbf{0}$, where t_0 is an arbitrary point. Rewrite this equation in terms of $\mathbf{a}$ and $\mathbf{b}$, and then proceed as in part (b) to show that $c_1 = 0$ and $c_2 = 0$. Hence $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are linearly independent at the arbitrary point t_0. Therefore, they are linearly independent at every point and on every interval.

28. A mass m on a spring with constant k satisfies the differential equation (see Section 3.7)

$$mu'' + ku = 0,$$

where $u(t)$ is the displacement at time t of the mass from its equilibrium position.

(a) Let $x_1 = u, x_2 = u'$, and show that the resulting system is

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -k/m & 0 \end{pmatrix} \mathbf{x}.$$

(b) Find the eigenvalues of the matrix for the system in part (a).

(c) Sketch several trajectories of the system. Choose one of your trajectories, and sketch the corresponding graphs of x_1 versus t and x_2 versus t. Sketch both graphs on one set of axes.

(d) What is the relation between the eigenvalues of the coefficient matrix and the natural frequency of the spring–mass system?

29. Consider the two-mass, three-spring system of Example 3 in the text. Instead of converting the problem into a system of four first order equations, we indicate here how to proceed directly from Eqs. (22).

(a) Show that Eqs. (22) can be written in the form

$$\mathbf{x}'' = \begin{pmatrix} -2 & \frac{3}{2} \\ \frac{4}{3} & -3 \end{pmatrix} \mathbf{x} = \mathbf{A}\mathbf{x}. \tag{i}$$

(b) Assume that $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ and show that

$$(\mathbf{A} - r^2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}.$$

Note that r^2 (rather than r) is an eigenvalue of $\mathbf{A}$ corresponding to an eigenvector $\boldsymbol{\xi}$.

(c) Find the eigenvalues and eigenvectors of $\mathbf{A}$.

(d) Write down expressions for x_1 and x_2. There should be four arbitrary constants in these expressions.

(e) By differentiating the results from part (d), write down expressions for x_1' and x_2'. Your results from parts (d) and (e) should agree with Eq. (31) in the text.

30. Consider the two-mass, three-spring system whose equations of motion are Eqs. (22) in the text. Let $m_1 = 1, m_2 = 4/3, k_1 = 1, k_2 = 3$, and $k_3 = 4/3$.

(a) As in the text, convert the system to four first order equations of the form $\mathbf{y}' = \mathbf{A}\mathbf{y}$. Determine the coefficient matrix $\mathbf{A}$.

(b) Find the eigenvalues and eigenvectors of $\mathbf{A}$.

(c) Write down the general solution of the system.

(d) Describe the fundamental modes of vibration. For each fundamental mode draw graphs of y_1 versus t and y_2 versus t. Also draw the corresponding trajectories in the y_1y_3- and y_2y_4-planes.

(e) Consider the initial conditions $\mathbf{y}(0) = (2, 1, 0, 0)^T$. Evaluate the arbitrary constants in the general solution in part (c). What is the period of the motion in this case? Plot graphs of y_1 versus t and y_2 versus t. Also plot the corresponding trajectories in the y_1y_3- and y_2y_4-planes. Be sure you understand how the trajectories are traversed for a full period.

(f) Consider other initial conditions of your own choice, and plot graphs similar to those requested in part (e).

31. Consider the two-mass, three-spring system whose equations of motion are Eqs. (22) in the text. Let $m_1 = m_2 = 1$ and $k_1 = k_2 = k_3 = 1$.

(a) As in the text, convert the system to four first order equations of the form $\mathbf{y}' = \mathbf{A}\mathbf{y}$. Determine the coefficient matrix $\mathbf{A}$.

(b) Find the eigenvalues and eigenvectors of $\mathbf{A}$.

(c) Write down the general solution of the system.

(d) Describe the fundamental modes of vibration. For each fundamental mode draw graphs of y_1 versus t and y_2 versus t. Also draw the corresponding trajectories in the y_1y_3- and y_2y_4-planes.

(e) Consider the initial conditions $\mathbf{y}(0) = (-1, 3, 0, 0)^T$. Evaluate the arbitrary constants in the general solution in part (c). Plot y_1 versus t and y_2 versus t. Do you think the solution is periodic? Also draw the trajectories in the y_1y_3- and y_2y_4-planes.

(f) Consider other initial conditions of your own choice, and plot graphs similar to those requested in part (e).

(a) Assuming that a solution $\mathbf{x} = \boldsymbol{\phi}(t)$ exists, show that it must satisfy the integral equation

$$\boldsymbol{\phi}(t) = \mathbf{x}^0 + \int_0^t \mathbf{A}\boldsymbol{\phi}(s)\,ds. \tag{ii}$$

(b) Start with the initial approximation $\boldsymbol{\phi}^{(0)}(t) = \mathbf{x}^0$. Substitute this expression for $\boldsymbol{\phi}(s)$ in the right side of Eq. (ii) and obtain a new approximation $\boldsymbol{\phi}^{(1)}(t)$. Show that

$$\boldsymbol{\phi}^{(1)}(t) = (\mathbf{I} + \mathbf{A}t)\mathbf{x}^0. \tag{iii}$$

(c) Repeat this process and thereby obtain a sequence of approximations $\boldsymbol{\phi}^{(0)}$, $\boldsymbol{\phi}^{(1)}$, $\boldsymbol{\phi}^{(2)}, \dots, \boldsymbol{\phi}^{(n)}, \dots$. Use an inductive argument to show that

$$\boldsymbol{\phi}^{(n)}(t) = \left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \cdots + \mathbf{A}^n\frac{t^n}{n!}\right)\mathbf{x}^0. \tag{iv}$$

(d) Let $n \to \infty$ and show that the solution of the initial value problem (i) is

$$\boldsymbol{\phi}(t) = \exp(\mathbf{A}t)\mathbf{x}^0. \tag{v}$$

7.8 Repeated Eigenvalues

We conclude our consideration of the linear homogeneous system with constant coefficients

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \tag{1}$$

with a discussion of the case in which the matrix $\mathbf{A}$ has a repeated eigenvalue. Recall that in Section 7.3 we stated that a repeated eigenvalue with algebraic multiplicity $m \geq 2$ may have a geometric multiplicity less than m. In other words, there may be fewer than m linearly independent eigenvectors associated with this eigenvalue. The following example illustrates this possibility.

EXAMPLE 1

Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}. \tag{2}$$

The eigenvalues r and eigenvectors $\boldsymbol{\xi}$ satisfy the equation $(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$, or

$$\begin{pmatrix} 1-r & -1 \\ 1 & 3-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3}$$

The eigenvalues are the roots of the equation

$$\det(\mathbf{A} - r\mathbf{I}) = \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = r^2 - 4r + 4 = (r-2)^2 = 0. \tag{4}$$

Thus the two eigenvalues are $r_1 = 2$ and $r_2 = 2$; that is, the eigenvalue 2 has algebraic multiplicity 2.

To determine the eigenvectors, we must return to Eq. (3) and use for r the value 2. This gives

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5}$$

Hence we obtain the single condition $\xi_1 + \xi_2 = 0$, which determines ξ_2 in terms of ξ_1, or vice versa. Thus the eigenvector corresponding to the eigenvalue $r = 2$ is

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{6}$$

or any nonzero multiple of this vector. Observe that there is only one linearly independent eigenvector associated with the double eigenvalue.

Returning to the system (1), suppose that $r = \rho$ is an m-fold root of the characteristic equation

$$\det(\mathbf{A} - r\mathbf{I}) = 0. \tag{7}$$

Then ρ is an eigenvalue of algebraic multiplicity m of the matrix $\mathbf{A}$. In this event, there are two possibilities: either there are m linearly independent eigenvectors corresponding to the eigenvalue ρ, or else there are fewer than m such eigenvectors.

In the first case, let $\boldsymbol{\xi}^{(1)}, \dots, \boldsymbol{\xi}^{(m)}$ be m linearly independent eigenvectors associated with the eigenvalue ρ of algebraic multiplicity m. Then there are m linearly independent solutions $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)}e^{\rho t}, \dots, \mathbf{x}^{(m)}(t) = \boldsymbol{\xi}^{(m)}e^{\rho t}$ of Eq. (1). Thus in this case it makes no difference that the eigenvalue $r = \rho$ is repeated; there is still a fundamental set of solutions of Eq. (1) of the form $\boldsymbol{\xi}e^{rt}$. This case always occurs if the coefficient matrix $\mathbf{A}$ is Hermitian (or real and symmetric).

However, if the coefficient matrix is not Hermitian, then there may be fewer than m independent eigenvectors corresponding to an eigenvalue ρ of algebraic multiplicity m, and if so, there will be fewer than m solutions of Eq. (1) of the form $\boldsymbol{\xi}e^{\rho t}$ associated with this eigenvalue. Therefore, to construct the general solution of Eq. (1), it is necessary to find other solutions of a different form. Recall that a similar situation occurred in Section 3.4 for the linear equation $ay'' + by' + cy = 0$ when the characteristic equation has a double root r. In that case we found one exponential solution $y_1(t) = e^{rt}$, but a second independent solution had the form $y_2(t) = te^{rt}$. With that result in mind, consider the following example.

EXAMPLE 2

Find a fundamental set of solutions of

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x} \tag{8}$$

and draw a phase portrait for this system.

A direction field for the system (8) is shown in Figure 7.8.1. From this figure it appears that all nonzero solutions depart from the origin.

To solve the system, observe that the coefficient matrix $\mathbf{A}$ is the same as the matrix in Example 1. Thus we know that $r = 2$ is a double eigenvalue and that it has only a single corresponding eigenvector, which we may take as $\boldsymbol{\xi}^T = (1, -1)$. Thus one solution of the system (8) is

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}, \tag{9}$$

but there is no second solution of the form $\mathbf{x} = \boldsymbol{\xi}e^{rt}$.

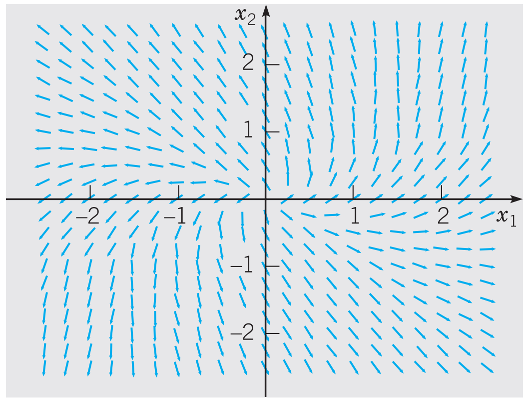


FIGURE 7.8.1 A direction field for the system (8).

Based on the procedure used for second order linear equations in Section 3.4, it may be natural to attempt to find a second independent solution of the system (8) of the form

$$\mathbf{x} = \boldsymbol{\xi} t e^{2t}, \tag{10}$$

where $\boldsymbol{\xi}$ is a constant vector to be determined. Substituting for $\mathbf{x}$ in Eq. (8), we obtain

$$2\boldsymbol{\xi} t e^{2t} + \boldsymbol{\xi} e^{2t} - \mathbf{A}\boldsymbol{\xi} t e^{2t} = \mathbf{0}. \tag{11}$$

For Eq. (11) to be satisfied for all t, it is necessary for the coefficients of te^{2t} and e^{2t} both to be zero. From the term in e^{2t} we find that

$$\boldsymbol{\xi} = \mathbf{0}. \tag{12}$$

Hence there is no nonzero solution of the system (8) of the form (10).

Since Eq. (11) contains terms in both te^{2t} and e^{2t}, it appears that in addition to $\boldsymbol{\xi} te^{2t}$, the second solution must contain a term of the form $\boldsymbol{\eta} e^{2t}$; in other words, we need to assume that

$$\mathbf{x} = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}, \tag{13}$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are constant vectors to be determined. Upon substituting this expression for $\mathbf{x}$ in Eq. (8), we obtain

$$2\boldsymbol{\xi} t e^{2t} + (\boldsymbol{\xi} + 2\boldsymbol{\eta}) e^{2t} = \mathbf{A}(\boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}). \tag{14}$$

Equating coefficients of te^{2t} and e^{2t} on each side of Eq. (14) gives the two conditions

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = \mathbf{0} \tag{15}$$

and

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi} \tag{16}$$

for the determination of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. Equation (15) is satisfied if $\boldsymbol{\xi}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue $r = 2$, such as $\boldsymbol{\xi}^T = (1, -1)$. Since $\det(\mathbf{A} - 2\mathbf{I})$ is zero, Eq. (16) is solvable only if the right side $\boldsymbol{\xi}$ satisfies a certain condition. Fortunately, $\boldsymbol{\xi}$ and its multiples are exactly the vectors that allow Eq. (16) to be solved. The augmented matrix for Eq. (16) is

$$\left(\begin{array}{cc|c} -1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right).$$

The second row of this matrix is proportional to the first, so the system is solvable. We have

$$-\eta_1 - \eta_2 = 1,$$

so if $\eta_1 = k$, where k is arbitrary, then $\eta_2 = -k - 1$. If we write

$$\boldsymbol{\eta} = \begin{pmatrix} k \\ -1-k \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + k\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{17}$$

then by substituting for $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ in Eq. (13), we obtain

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} + k\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}. \tag{18}$$

The last term in Eq. (18) is merely a multiple of the first solution $\mathbf{x}^{(1)}(t)$ and may be ignored, but the first two terms constitute a new solution:

$$\mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t}. \tag{19}$$

An elementary calculation shows that $W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](t) = -e^{4t} \neq 0$, and therefore $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ form a fundamental set of solutions of the system (8). The general solution is

$$\begin{aligned} \mathbf{x} &= c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) \\ &= c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t}\right]. \end{aligned} \tag{20}$$

The main features of a phase portrait for the solution (20) follow from the presence of the exponential factor e^{2t} in every term. Therefore $\mathbf{x} \to \mathbf{0}$ as $t \to -\infty$ and, unless both c_1 and c_2 are zero, $\mathbf{x}$ becomes unbounded as $t \to \infty$. If c_1 and c_2 are not both zero, then along any trajectory we have

$$\lim_{t\to-\infty} \frac{x_2(t)}{x_1(t)} = \lim_{t\to-\infty} \frac{-c_1 - c_2 t - c_2}{c_1 + c_2 t} = -1.$$

Therefore, as $t \to -\infty$, every trajectory approaches the origin tangent to the line $x_2 = -x_1$ determined by the eigenvector; this behavior is clearly evident in Figure 7.8.2*a*. Further, as $t \to \infty$, the slope of each trajectory also approaches -1. However, it is possible to show that trajectories do not approach asymptotes as $t \to \infty$. Several trajectories of the system (8) are shown in Figure 7.8.2*a*, and some typical plots of x_1 versus t are shown in Figure 7.8.2*b*. The pattern of trajectories in this figure is typical of 2×2 systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with equal eigenvalues and only one independent eigenvector. The origin is called an **improper node** in this case. If the eigenvalues are negative, then the trajectories are similar but are traversed in the inward direction. An improper node is asymptotically stable or unstable, depending on whether the eigenvalues are negative or positive.

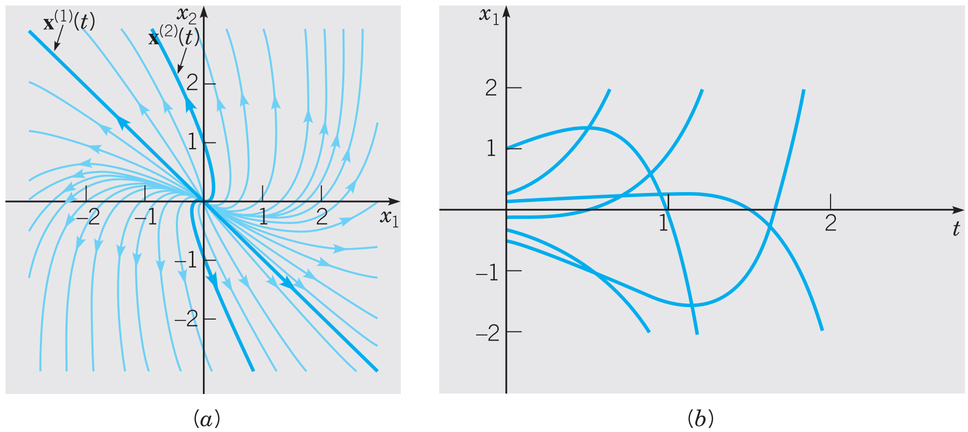


FIGURE 7.8.2 (*a*) Phase portrait of the system (8); the origin is an improper node. (*b*) Plots of x_1 versus t for the system (8).

One difference between a system of two first order equations and a single second order equation is evident from the preceding example. For a second order linear equation with a repeated root r_1 of the characteristic equation, a term $ce^{r_1 t}$ in the second solution is not required since it is a multiple of the first solution. On the other hand, for a system of two first order equations, the term $\boldsymbol{\eta}e^{r_1 t}$ of Eq. (13) with $r_1 = 2$ is not, in general, a multiple of the first solution $\boldsymbol{\xi}e^{r_1 t}$, so the term $\boldsymbol{\eta}e^{r_1 t}$ must be retained.

Example 2 is entirely typical of the general case when there is a double eigenvalue and a single associated eigenvector. Consider again the system (1), and suppose that $r = \rho$ is a double eigenvalue of $\mathbf{A}$, but that there is only one corresponding eigenvector $\boldsymbol{\xi}$. Then one solution [similar to Eq. (9)] is

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}e^{\rho t}, \tag{21}$$

where $\boldsymbol{\xi}$ satisfies

$$(\mathbf{A} - \rho\mathbf{I})\boldsymbol{\xi} = \mathbf{0}. \tag{22}$$

By proceeding as in Example 2, we find that a second solution [similar to Eq. (19)] is

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi}te^{\rho t} + \boldsymbol{\eta}e^{\rho t}, \tag{23}$$

where $\boldsymbol{\xi}$ satisfies Eq. (22) and $\boldsymbol{\eta}$ is determined from

$$(\mathbf{A} - \rho\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}. \tag{24}$$

Even though $\det(\mathbf{A} - \rho\mathbf{I}) = 0$, it can be shown that it is always possible to solve Eq. (24) for $\boldsymbol{\eta}$. Note that if we multiply Eq. (24) by $\mathbf{A} - \rho\mathbf{I}$ and use Eq. (22), then we obtain

$$(\mathbf{A} - \rho\mathbf{I})^2\boldsymbol{\eta} = \mathbf{0}.$$

The vector $\boldsymbol{\eta}$ is called a **generalized eigenvector** of the matrix $\mathbf{A}$ corresponding to the eigenvalue ρ.

Fundamental Matrices. As explained in Section 7.7, fundamental matrices are formed by arranging linearly independent solutions in columns. Thus, for example, a fundamental matrix for the system (8) can be formed from the solutions $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ from Eqs. (9) and (19), respectively:

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{pmatrix} = e^{2t}\begin{pmatrix} 1 & t \\ -1 & -1-t \end{pmatrix}. \tag{25}$$

The particular fundamental matrix $\boldsymbol{\Phi}$ that satisfies $\boldsymbol{\Phi}(0) = \mathbf{I}$ can also be readily found from the relation $\boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0)$. For Eq. (8) we have

$$\boldsymbol{\Psi}(0) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \qquad \boldsymbol{\Psi}^{-1}(0) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \tag{26}$$

and then

$$\begin{aligned} \boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0) &= e^{2t}\begin{pmatrix} 1 & t \\ -1 & -1-t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \\ &= e^{2t}\begin{pmatrix} 1-t & -t \\ t & 1+t \end{pmatrix}. \end{aligned} \tag{27}$$

The latter matrix is also the exponential matrix $\exp(\mathbf{A}t)$.

Jordan Forms. An $n \times n$ matrix $\mathbf{A}$ can be diagonalized as discussed in Section 7.7 only if it has a full complement of n linearly independent eigenvectors. If there is a shortage of eigenvectors (because of repeated eigenvalues), then $\mathbf{A}$ can always be transformed into a nearly diagonal matrix called its Jordan[6] form, which has the eigenvalues of $\mathbf{A}$ on the main diagonal, ones in certain positions on the diagonal above the main diagonal, and zeros elsewhere.

Consider again the matrix $\mathbf{A}$ given by Eq. (2). To transform $\mathbf{A}$ into its Jordan form, we construct the transformation matrix $\mathbf{T}$ with the single eigenvector $\boldsymbol{\xi}$ from Eq. (6) in its first column and the generalized eigenvector $\boldsymbol{\eta}$ from Eq. (17) with $k = 0$ in the second column. Then $\mathbf{T}$ and its inverse are given by

$$\mathbf{T} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \qquad \mathbf{T}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}. \tag{28}$$

As you can verify, it follows that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \mathbf{J}. \tag{29}$$

The matrix $\mathbf{J}$ in Eq. (29) is the Jordan form of $\mathbf{A}$. It is typical of all Jordan forms in that it has a 1 above the main diagonal in the column corresponding to the eigenvector that is lacking (and is replaced in $\mathbf{T}$ by the generalized eigenvector).

If we start again from Eq. (1)

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

---

[6]Marie Ennemond Camille Jordan (1838–1922) was professor at the École Polytechnique and the Collège de France. He is known for his important contributions to analysis and to topology (the Jordan curve theorem) and especially for his foundational work in group theory. The Jordan form of a matrix appeared in his influential book *Traité des substitutions et des équations algébriques*, published in 1870.

the transformation $\mathbf{x} = \mathbf{T}\mathbf{y}$, where $\mathbf{T}$ is given by Eq. (28), produces the system

$$\mathbf{y}' = \mathbf{J}\mathbf{y}, \tag{30}$$

where $\mathbf{J}$ is given by Eq. (29). In scalar form the system (30) is

$$y_1' = 2y_1 + y_2, \qquad y_2' = 2y_2. \tag{31}$$

These equations can be solved readily in reverse order—that is, by starting with the equation for y_2. In this way we obtain

$$y_2 = c_1 e^{2t}, \qquad y_1 = c_1 t e^{2t} + c_2 e^{2t}. \tag{32}$$

Thus two independent solutions of the system (30) are

$$\mathbf{y}^{(1)}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}, \qquad \mathbf{y}^{(2)}(t) = \begin{pmatrix} t \\ 1 \end{pmatrix} e^{2t}, \tag{33}$$

and the corresponding fundamental matrix is

$$\hat{\mathbf{\Psi}}(t) = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}. \tag{34}$$

Since $\hat{\mathbf{\Psi}}(0) = \mathbf{I}$, we can also identify the matrix in Eq. (34) as $\exp(\mathbf{J}t)$. The same result can be reached by calculating powers of $\mathbf{J}$ and substituting them into the exponential series (see Problems 20 through 22). To obtain a fundamental matrix for the original system, we now form the product

$$\mathbf{\Psi}(t) = \mathbf{T}\exp(\mathbf{J}t) = \begin{pmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -e^{2t} - te^{2t} \end{pmatrix}, \tag{35}$$

which is the same as the fundamental matrix given in Eq. (25).

We will not discuss $n \times n$ systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ in more detail here. For large n it is possible that there may be eigenvalues of high algebraic multiplicity m, perhaps with much lower geometric multiplicity q, thus giving rise to $m - q$ generalized eigenvectors. For $n \geq 4$ there may also be repeated complex eigenvalues. A full discussion[7] of the Jordan form of a general $n \times n$ matrix requires a greater background in linear algebra than we assume for most readers of this book. Problems 18 through 22 ask you to explore the use of Jordan forms for systems of three equations.

The amount of arithmetic required in the analysis of a general $n \times n$ system may be prohibitive to do by hand even if n is no greater than 3 or 4. Consequently, suitable computer software should be used routinely in most cases. This does not overcome all difficulties by any means, but it does make many problems much more tractable. Finally, for a set of equations arising from modeling a physical system, it is likely that some of the elements in the coefficient matrix $\mathbf{A}$ result from measurements of some physical quantity. The inevitable uncertainties in such measurements lead to uncertainties in the values of the eigenvalues of $\mathbf{A}$. For example, in such a case it may not be clear whether two eigenvalues are actually equal or are merely close together.

[7]For example, see the books listed in the References at the end of this chapter.

PROBLEMS

In each of Problems 1 through 4:

(a) Draw a direction field and sketch a few trajectories.

(b) Describe how the solutions behave as $t \to \infty$.

(c) Find the general solution of the system of equations.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$
2. $\mathbf{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \mathbf{x}$
3. $\mathbf{x}' = \begin{pmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix} \mathbf{x}$
4. $\mathbf{x}' = \begin{pmatrix} -3 & \frac{5}{2} \\ -\frac{5}{2} & 2 \end{pmatrix} \mathbf{x}$

In each of Problems 5 and 6, find the general solution of the given system of equations.

5. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \mathbf{x}$
6. $\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{x}$

In each of Problems 7 through 10:

(a) Find the solution of the given initial value problem.

(b) Draw the trajectory of the solution in the x_1x_2-plane, and also draw the graph of x_1 versus t.

7. $\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$
8. $\mathbf{x}' = \begin{pmatrix} -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$
9. $\mathbf{x}' = \begin{pmatrix} 2 & \frac{3}{2} \\ -\frac{3}{2} & -1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$
10. $\mathbf{x}' = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

In each of Problems 11 and 12:

(a) Find the solution of the given initial value problem.

(b) Draw the corresponding trajectory in $x_1x_2x_3$-space, and also draw the graph of x_1 versus t.

11. $\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix}$
12. $\mathbf{x}' = \begin{pmatrix} -\frac{5}{2} & 1 & 1 \\ 1 & -\frac{5}{2} & 1 \\ 1 & 1 & -\frac{5}{2} \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

In each of Problems 13 and 14, solve the given system of equations by the method of Problem 19 of Section 7.5. Assume that $t > 0$.

13. $t\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$
14. $t\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}$

15. Show that all solutions of the system

$$\mathbf{x}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{x}$$

approach zero as $t \to \infty$ if and only if $a + d < 0$ and $ad - bc > 0$. Compare this result with that of Problem 37 in Section 3.4.

16. Consider again the electric circuit in Problem 26 of Section 7.6. This circuit is described by the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{1}{L} \\ -\dfrac{1}{C} & -\dfrac{1}{RC} \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}.$$

(a) Show that the eigenvalues are real and equal if $L = 4R^2C$.

(b) Suppose that $R = 1\ \Omega$, $C = 1$ F, and $L = 4$ H. Suppose also that $I(0) = 1$ A and $V(0) = 2$ V. Find $I(t)$ and $V(t)$.

17. Consider again the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x} \tag{i}$$

that we discussed in Example 2. We found there that $\mathbf{A}$ has a double eigenvalue $r_1 = r_2 = 2$ with a single independent eigenvector $\boldsymbol{\xi}^{(1)} = (1, -1)^T$, or any nonzero multiple thereof. Thus one solution of the system (i) is $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{2t}$ and a second independent solution has the form

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t},$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ satisfy

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \qquad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}. \tag{ii}$$

In the text we solved the first equation for $\boldsymbol{\xi}$ and then the second equation for $\boldsymbol{\eta}$. Here we ask you to proceed in the reverse order.

(a) Show that $\boldsymbol{\eta}$ satisfies $(\mathbf{A} - 2\mathbf{I})^2\boldsymbol{\eta} = \mathbf{0}$.

(b) Show that $(\mathbf{A} - 2\mathbf{I})^2 = \mathbf{0}$. Thus the generalized eigenvector $\boldsymbol{\eta}$ can be chosen arbitrarily, except that it must be independent of $\boldsymbol{\xi}^{(1)}$.

(c) Let $\boldsymbol{\eta} = (0, -1)^T$. Then determine $\boldsymbol{\xi}$ from the second of Eqs. (ii) and observe that $\boldsymbol{\xi} = (1, -1)^T = \boldsymbol{\xi}^{(1)}$. This choice of $\boldsymbol{\eta}$ reproduces the solution found in Example 2.

(d) Let $\boldsymbol{\eta} = (1, 0)^T$ and determine the corresponding eigenvector $\boldsymbol{\xi}$.

(e) Let $\boldsymbol{\eta} = (k_1, k_2)^T$, where k_1 and k_2 are arbitrary numbers. Then determine $\boldsymbol{\xi}$. How is it related to the eigenvector $\boldsymbol{\xi}^{(1)}$?

Eigenvalues of Multiplicity 3. If the matrix $\mathbf{A}$ has an eigenvalue of algebraic multiplicity 3, then there may be either one, two, or three corresponding linearly independent eigenvectors. The general solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is different, depending on the number of eigenvectors associated with the triple eigenvalue. As noted in the text, there is no difficulty if there are three eigenvectors, since then there are three independent solutions of the form $\mathbf{x} = \boldsymbol{\xi} e^{rt}$. The following two problems illustrate the solution procedure for a triple eigenvalue with one or two eigenvectors, respectively.

18. Consider the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} \mathbf{x}. \qquad \text{(i)}$$

(a) Show that $r = 2$ is an eigenvalue of algebraic multiplicity 3 of the coefficient matrix $\mathbf{A}$ and that there is only one corresponding eigenvector, namely,

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

(b) Using the information in part (a), write down one solution $\mathbf{x}^{(1)}(t)$ of the system (i). There is no other solution of the purely exponential form $\mathbf{x} = \boldsymbol{\xi} e^{rt}$.

(c) To find a second solution, assume that $\mathbf{x} = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}$. Show that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ satisfy the equations

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \qquad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}.$$

Since $\boldsymbol{\xi}$ has already been found in part (a), solve the second equation for $\boldsymbol{\eta}$. Neglect the multiple of $\boldsymbol{\xi}^{(1)}$ that appears in $\boldsymbol{\eta}$, since it leads only to a multiple of the first solution $\mathbf{x}^{(1)}$. Then write down a second solution $\mathbf{x}^{(2)}(t)$ of the system (i).

(d) To find a third solution, assume that $\mathbf{x} = \boldsymbol{\xi}(t^2/2)e^{2t} + \boldsymbol{\eta} t e^{2t} + \boldsymbol{\zeta} e^{2t}$. Show that $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, and $\boldsymbol{\zeta}$ satisfy the equations

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \qquad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}, \qquad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\zeta} = \boldsymbol{\eta}.$$

The first two equations are the same as in part (c), so solve the third equation for $\boldsymbol{\zeta}$, again neglecting the multiple of $\boldsymbol{\xi}^{(1)}$ that appears. Then write down a third solution $\mathbf{x}^{(3)}(t)$ of the system (i).

(e) Write down a fundamental matrix $\boldsymbol{\Psi}(t)$ for the system (i).

(f) Form a matrix $\mathbf{T}$ with the eigenvector $\boldsymbol{\xi}^{(1)}$ in the first column and the generalized eigenvectors $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ in the second and third columns. Then find $\mathbf{T}^{-1}$ and form the product $\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$. The matrix $\mathbf{J}$ is the Jordan form of $\mathbf{A}$.

19. Consider the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix} \mathbf{x}. \qquad \text{(i)}$$

(a) Show that $r = 1$ is a triple eigenvalue of the coefficient matrix $\mathbf{A}$ and that there are only two linearly independent eigenvectors, which we may take as

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}. \qquad \text{(ii)}$$

Write down two linearly independent solutions $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ of Eq. (i).

(b) To find a third solution, assume that $\mathbf{x} = \boldsymbol{\xi} t e^{t} + \boldsymbol{\eta} e^{t}$; then show that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ must satisfy

$$(\mathbf{A} - \mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \qquad \text{(iii)}$$

$$(\mathbf{A} - \mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}. \qquad \text{(iv)}$$

(c) Equation (iii) is satisfied if $\boldsymbol{\xi}$ is an eigenvector, so one way to proceed is to choose $\boldsymbol{\xi}$ to be a suitable linear combination of $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ so that Eq. (iv) is solvable, and then to solve that equation for $\boldsymbol{\eta}$. However, let us proceed in a different way and follow the pattern of Problem 17. First, show that $\boldsymbol{\eta}$ satisfies

$$(\mathbf{A} - \mathbf{I})^2\boldsymbol{\eta} = \mathbf{0}.$$

Further, show that $(\mathbf{A} - \mathbf{I})^2 = \mathbf{0}$. Thus $\boldsymbol{\eta}$ can be chosen arbitrarily, except that it must be independent of $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$.

(d) A convenient choice for $\boldsymbol{\eta}$ is $\boldsymbol{\eta} = (0, 0, 1)^T$. Find the corresponding $\boldsymbol{\xi}$ from Eq. (iv). Verify that $\boldsymbol{\xi}$ is an eigenvector.

(e) Write down a fundamental matrix $\boldsymbol{\Psi}(t)$ for the system (i).

(f) Form a matrix $\mathbf{T}$ with the eigenvector $\boldsymbol{\xi}^{(1)}$ in the first column and with the eigenvector $\boldsymbol{\xi}$ from part (d) and the generalized eigenvector $\boldsymbol{\eta}$ in the other two columns. Find $\mathbf{T}^{-1}$ and form the product $\mathbf{J} = \mathbf{T}^{-1}\mathbf{AT}$. The matrix $\mathbf{J}$ is the Jordan form of $\mathbf{A}$.

20. Let $\mathbf{J} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, where λ is an arbitrary real number.

(a) Find $\mathbf{J}^2$, $\mathbf{J}^3$, and $\mathbf{J}^4$.

(b) Use an inductive argument to show that $\mathbf{J}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$.

(c) Determine $\exp(\mathbf{J}t)$.

(d) Use $\exp(\mathbf{J}t)$ to solve the initial value problem $\mathbf{x}' = \mathbf{Jx}$, $\mathbf{x}(0) = \mathbf{x}^0$.

21. Let

$$\mathbf{J} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

where λ is an arbitrary real number.

(a) Find $\mathbf{J}^2$, $\mathbf{J}^3$, and $\mathbf{J}^4$.

(b) Use an inductive argument to show that

$$\mathbf{J}^n = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

(c) Determine $\exp(\mathbf{J}t)$.

(d) Observe that if you choose $\lambda = 1$, then the matrix $\mathbf{J}$ in this problem is the same as the matrix $\mathbf{J}$ in Problem 19(f). Using the matrix $\mathbf{T}$ from Problem 19(f), form the product $\mathbf{T}\exp(\mathbf{J}t)$ with $\lambda = 1$. Is the resulting matrix the same as the fundamental matrix $\boldsymbol{\Psi}(t)$ in Problem 19(e)? If not, explain the discrepancy.

22. Let

$$\mathbf{J} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

where λ is an arbitrary real number.

(a) Find $\mathbf{J}^2$, $\mathbf{J}^3$, and $\mathbf{J}^4$.

(b) Use an inductive argument to show that

$$\mathbf{J}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & [n(n-1)/2]\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

(c) Determine $\exp(\mathbf{J}t)$.

(d) Note that if you choose $\lambda = 2$, then the matrix $\mathbf{J}$ in this problem is the same as the matrix $\mathbf{J}$ in Problem 18(f). Using the matrix $\mathbf{T}$ from Problem 18(f), form the product $\mathbf{T}\exp(\mathbf{J}t)$ with $\lambda = 2$. The resulting matrix is the same as the fundamental matrix $\mathbf{\Psi}(t)$ in Problem 18(e).

7.9 Nonhomogeneous Linear Systems

In this section we turn to the nonhomogeneous system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t), \tag{1}$$

where the $n \times n$ matrix $\mathbf{P}(t)$ and $n \times 1$ vector $\mathbf{g}(t)$ are continuous for $\alpha < t < \beta$. By the same argument as in Section 3.5 (see also Problem 16 in this section), the general solution of Eq. (1) can be expressed as

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) + \mathbf{v}(t), \tag{2}$$

where $c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$ is the general solution of the homogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, and $\mathbf{v}(t)$ is a particular solution of the nonhomogeneous system (1). We will briefly describe several methods for determining $\mathbf{v}(t)$.

Diagonalization. We begin with systems of the form

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t), \tag{3}$$

where $\mathbf{A}$ is an $n \times n$ diagonalizable constant matrix. By diagonalizing the coefficient matrix $\mathbf{A}$, as indicated in Section 7.7, we can transform Eq. (3) into a system of equations that is readily solvable.

Let $\mathbf{T}$ be the matrix whose columns are the eigenvectors $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$ of $\mathbf{A}$, and define a new dependent variable $\mathbf{y}$ by

$$\mathbf{x} = \mathbf{T}\mathbf{y}. \tag{4}$$

Then, substituting for $\mathbf{x}$ in Eq. (3), we obtain

$$\mathbf{T}\mathbf{y}' = \mathbf{A}\mathbf{T}\mathbf{y} + \mathbf{g}(t).$$

When we multiply by $\mathbf{T}^{-1}$, it follows that

$$\mathbf{y}' = (\mathbf{T}^{-1}\mathbf{A}\mathbf{T})\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}(t) = \mathbf{D}\mathbf{y} + \mathbf{h}(t), \tag{5}$$

where $\mathbf{h}(t) = \mathbf{T}^{-1}\mathbf{g}(t)$ and where $\mathbf{D}$ is the diagonal matrix whose diagonal entries are the eigenvalues $r_1, \ldots, r_n$ of $\mathbf{A}$, arranged in the same order as the corresponding eigenvectors $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$ that appear as columns of $\mathbf{T}$. Equation (5) is a system of