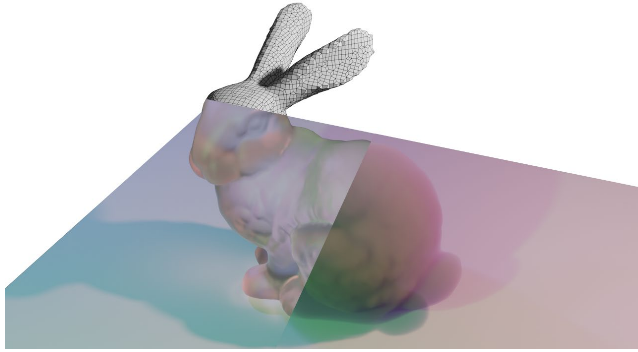
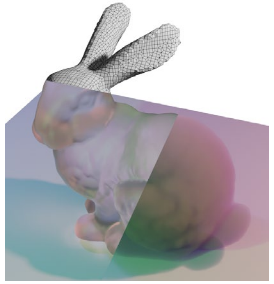


1. Radiometry and Light transport

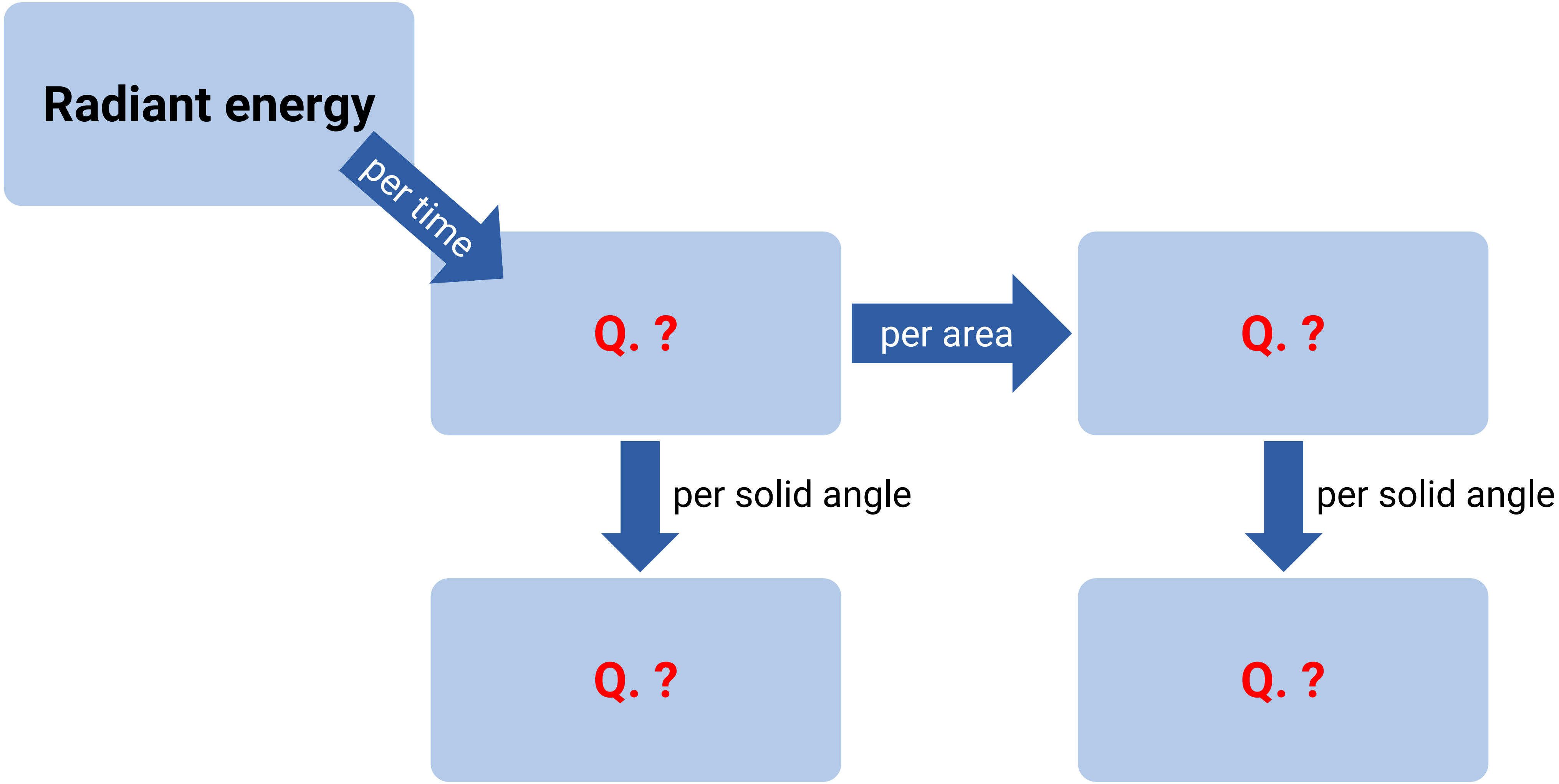
Physically Based Rendering

Shinyoung Yi (이신영)

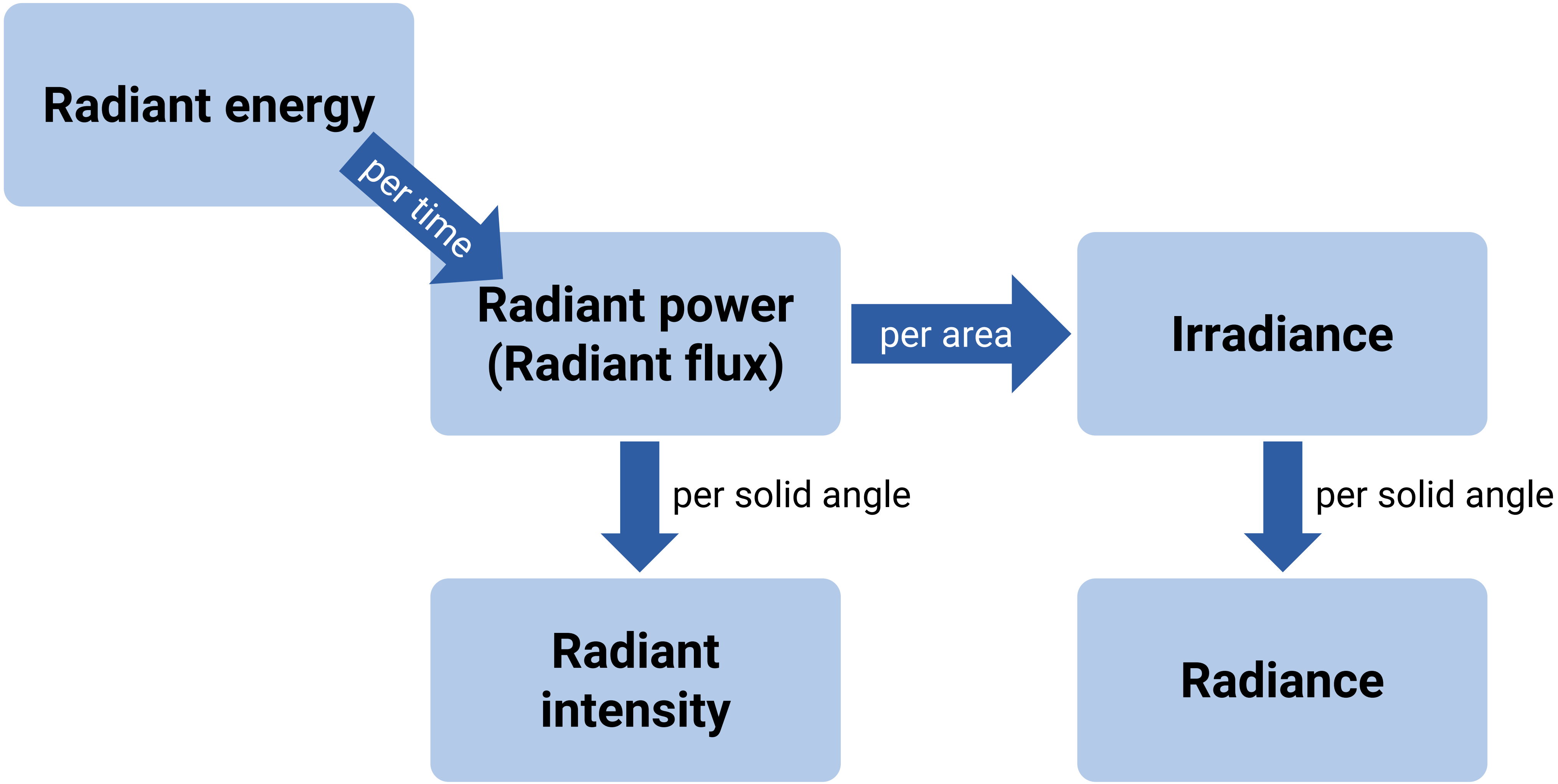


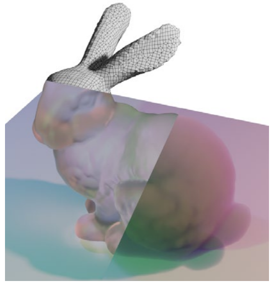
Quiz & Preview

Radiometric quantities



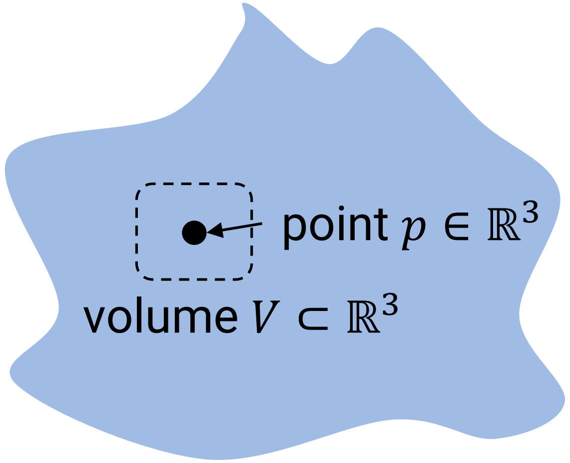
Radiometric quantities





Mass vs. Density

Concepts of mass vs. density

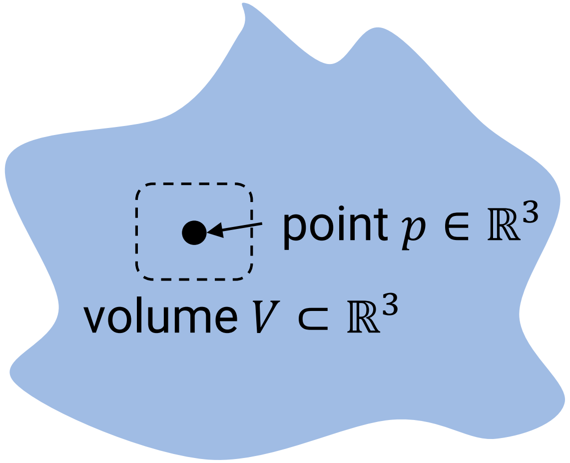


Object O

Mass → per volume → **Density**

Mass	Density
✓ Mass of the object O	✗ Density of the object O → illegal or "average density" of the object O
✓ Mass of some region (volume) V	✗ Density of some region (volume) V → illegal or "average density" of the volume V
✗ Mass at the point p → illegal or meaningless (always zero)	✓ Density at the point p

Concepts of mass vs. density

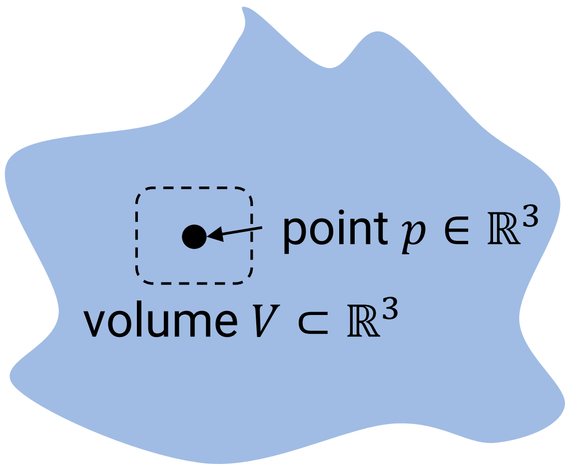


Mass of what? $V \subset \mathbb{R}^3$

- "mass of an object O"
  = "mass of the volume of O"

Density of what? $p \in \mathbb{R}^3$

Concepts of mass vs. density



Mass of $V \subset \mathbb{R}^3$

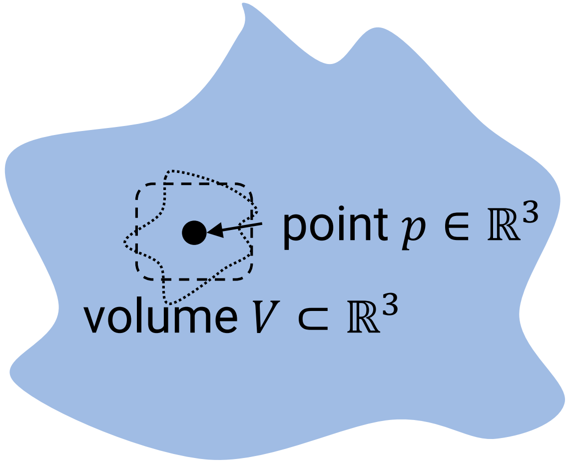
Density of $p \in \mathbb{R}^3$

$$\text{mass}(V)$$

per volume

$$\text{density}(p) = \lim_{\substack{\text{vol}(V)\to 0 \\ p \in V}} \frac{\text{mass}(V)}{\text{vol}(V)}$$

Concepts of mass vs. density



Mass of $V \subset \mathbb{R}^3$

Density of $p \in \mathbb{R}^3$

$$\text{mass}(V)$$

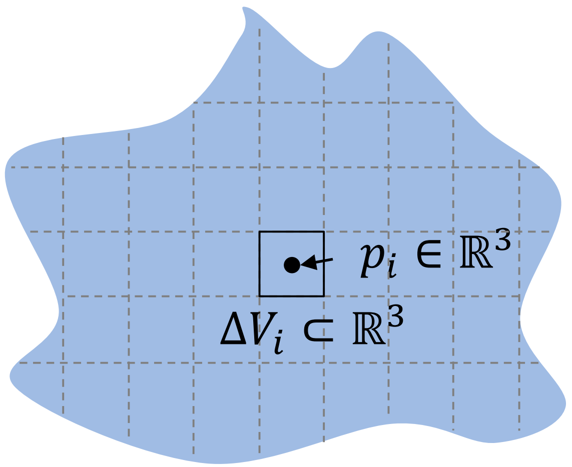
per volume

$$\text{density}(p) = \lim_{\substack{\text{vol}(V)\to 0 \\ p\in V}} \frac{\text{mass}(V)}{\text{vol}(V)}$$

$$= \lim_{\substack{\text{vol}(V)\to 0 \\ p\in V}} \frac{\text{mass}(V)}{\text{vol}(V)}$$

* The limit converges to the same value whenever $\text{vol}(V) \to 0$ and $p \in V$.

Concepts of mass vs. density



Mass of $V \subset \mathbb{R}^3$

$$\text{mass}(V) = \sum_i \text{mass}(\Delta V_i)$$

$$\approx \sum_i \text{density}(p_i)\,\text{vol}(\Delta V_i)$$

$\text{vol}(\Delta V_i) \to 0$

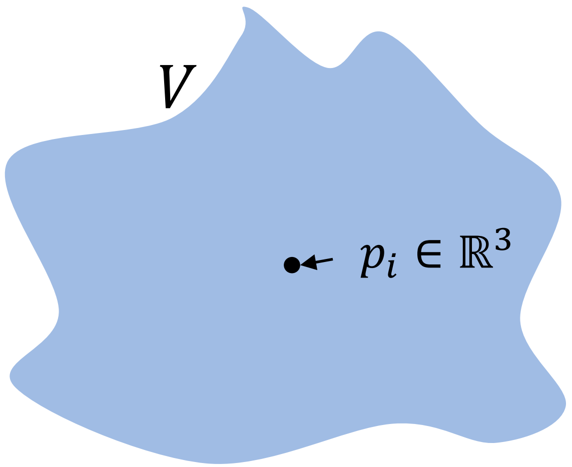
$$\int_V \text{density}(p)\,dp$$

over volume

Density of $p \in \mathbb{R}^3$

$$\boxed{\text{density}(p) = \lim_{\substack{\text{vol}(V)\to 0 \\ p\in V}} \frac{\text{mass}(V)}{\text{vol}(V)}}$$

Concepts of mass vs. density



Mass of $V \subset \mathbb{R}^3$ $[\,\mathrm{kg}\,]$

Density at $p \in \mathbb{R}^3$ $[\,\mathrm{kg/m^3}\,]$

Proposition

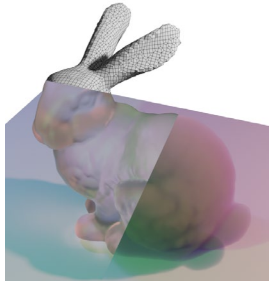
Relations between mass m of a volume $V \subset \mathbb{R}^3$ and density ρ at a point $p \in \mathbb{R}^3$ is that:

$$m(V) = \int_V \rho(p)\mathrm{d}p$$

per volume →

← over volume

$$\rho(p) = \lim_{\substack{|V|\to 0 \\ p\in V}} \frac{m(V)}{|V|}$$



Solid Angles

Solid angles

We roughly say…

"Solid angles" are
3D versions of "angles"

How strictly is this sentence correct?

Solid angles

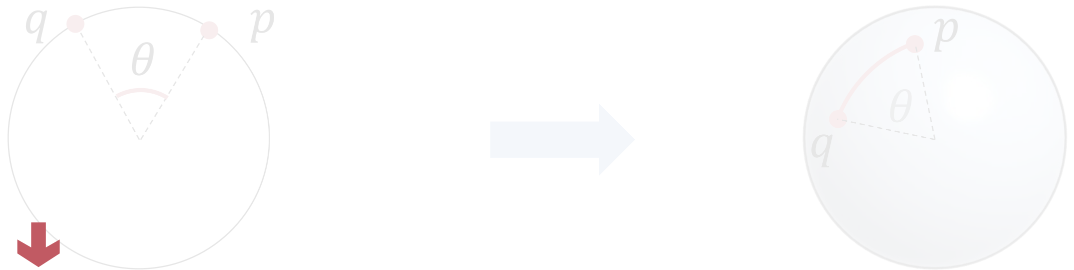
Can following statements be converted to sphere ($\mathbb{S}^2$) version using "solid angles"?

e.g. $p = (\theta, \phi)$

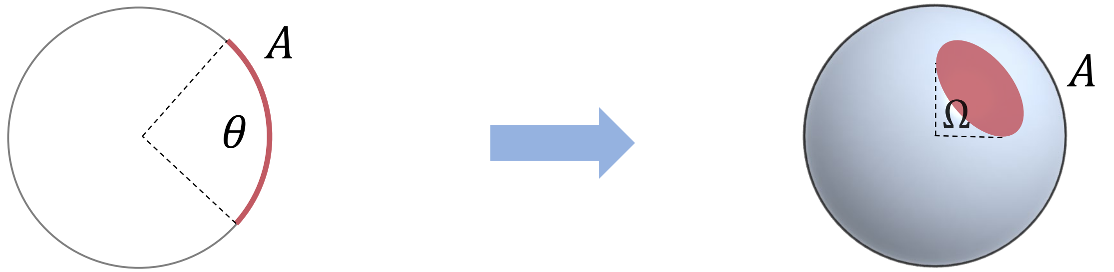
1. The position of a point p in the unit circle ($\mathbb{S}^1$) can be represented as the angle θ
   two coordinates such as (θ, ϕ)

"Solid angles" are only relevant to 3.

2. How far apart two points p and $q \in \mathbb{S}^1$ can be represented as the angle θ between them.
   angle θ



3. The size of a region $A \subset \mathbb{S}^1$ can be represented as the angle θ
   solid angle Ω



Solid angles

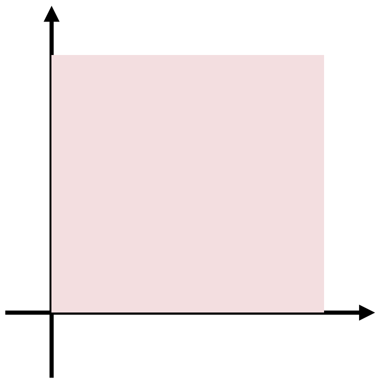
In many times,

several concepts can be treated
as a single concept in lower dimensions,

but they become different in higher dimensions

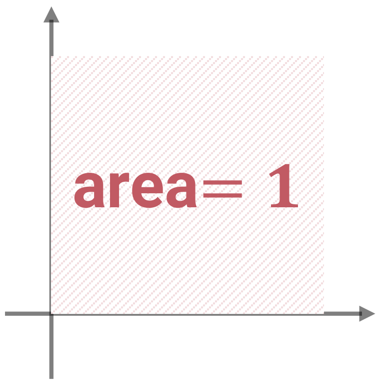
Solid angles

We call the both "area". (similarly for "volume")



a 2-dimensional *subset*

$A = \{(x, y) \in \mathbb{R}^2 | 0 \leq x, y \leq 1\}$

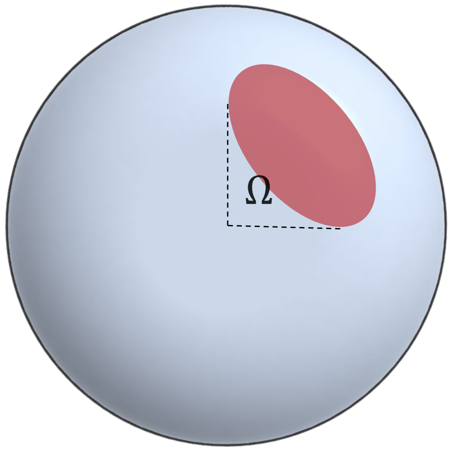


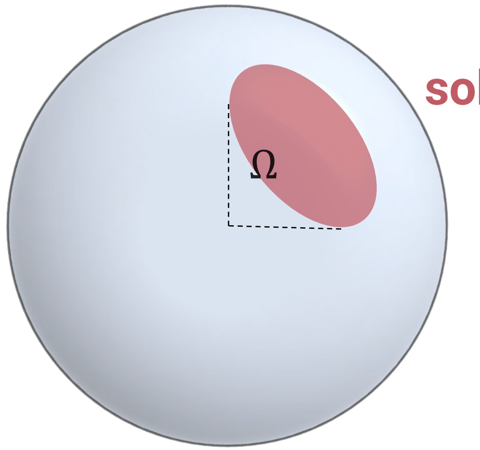
measure of a 2-dimensional subset

$|A| = 1$

Solid angles

Also for “solid angle”





a spherical *subset*

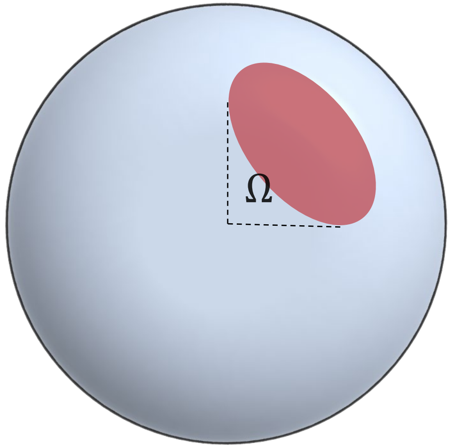
$\Omega = \{\widehat{\omega} \in \mathbb{S}^2 | \hat{u} \cdot \widehat{\omega} \geq 0.7\}$

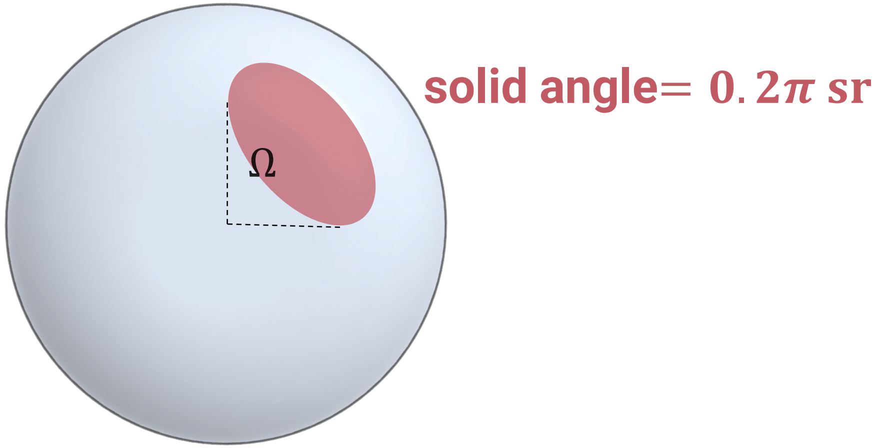
measure of a spherical subset

$|\Omega| = 1$

Solid angles

(not common) terminology in this seminar





a spherical *subset*

$\Omega = \{\hat{\omega} \in \mathbb{S}^2 | \hat{u} \cdot \hat{\omega} \geq 0.7\}$

solid angle *region*

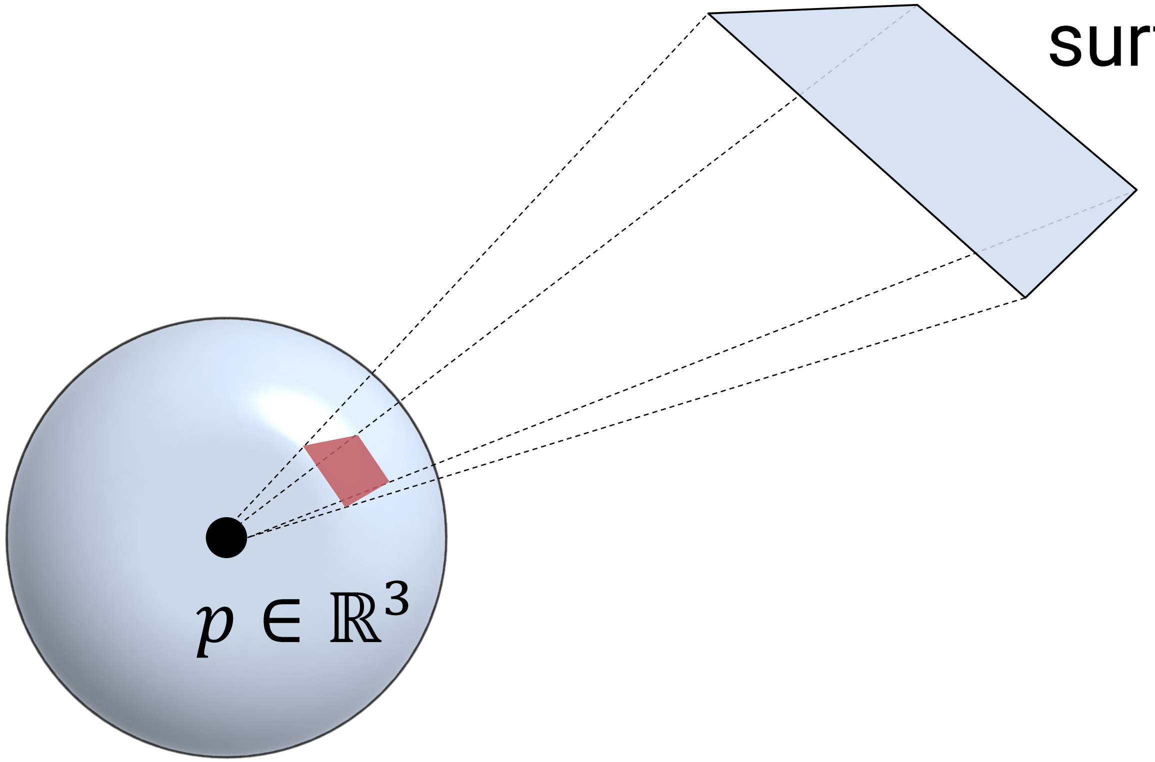
measure of a spherical subset

$|\Omega| = 1$

solid angle *measure*

Surface to solid angle

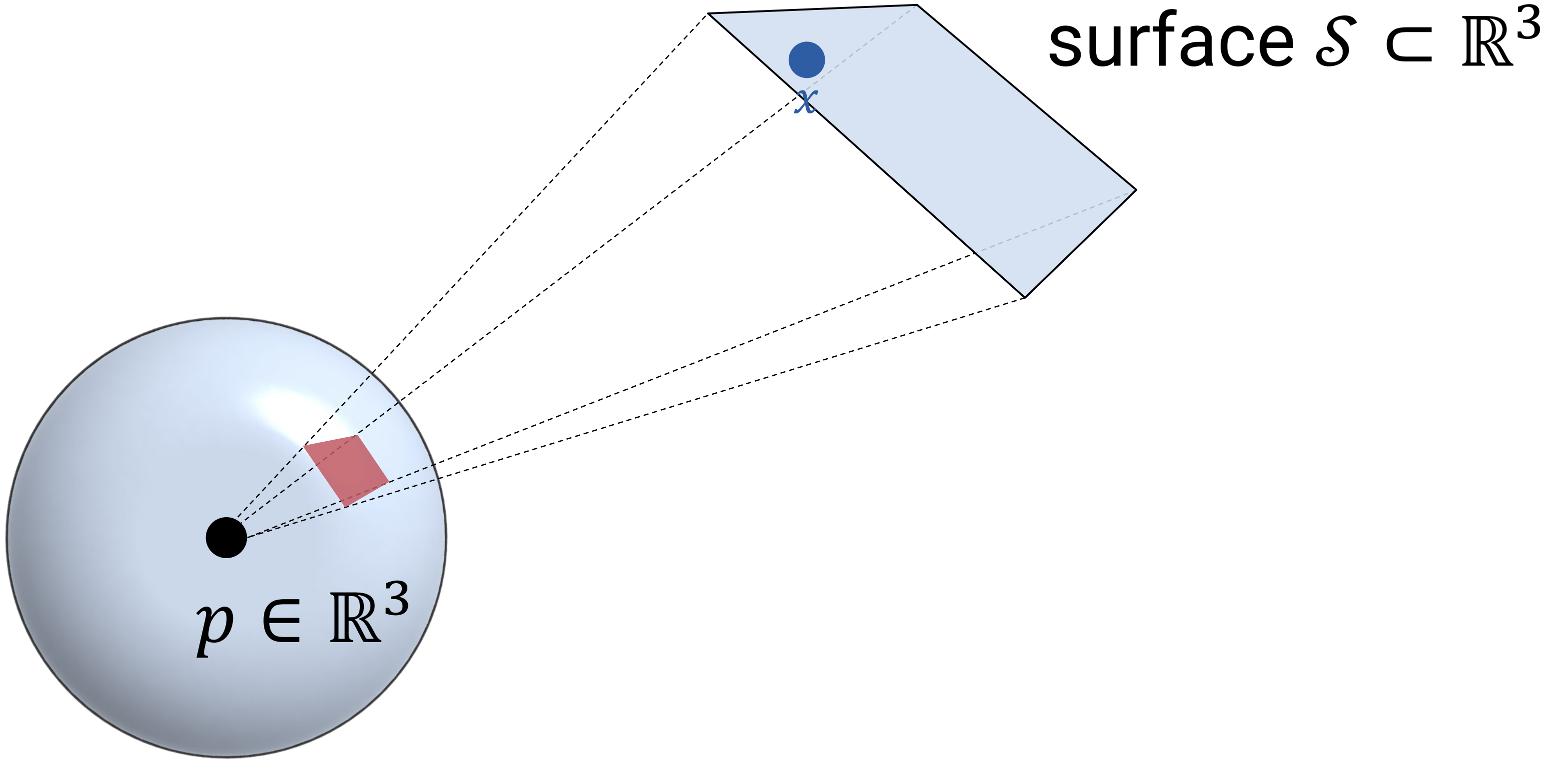
How large $\mathcal{S}$ appears to and observer at p?



Solid angle!

Surface to solid angle

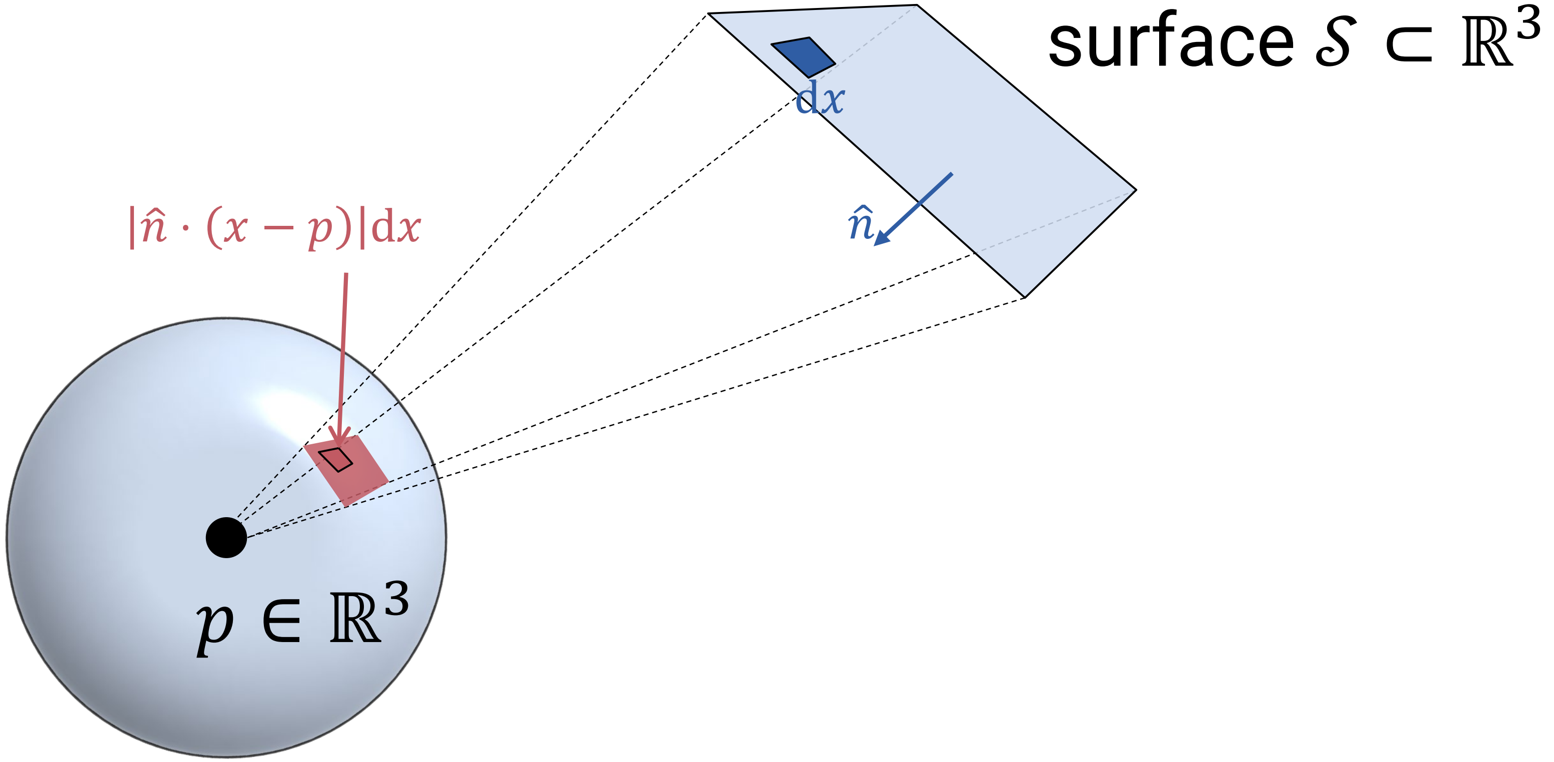
Try to write in an area integral on $\mathcal{S}$



Solid angle= $\int_{\mathcal{S}} \; ? \; \mathrm{d}x$

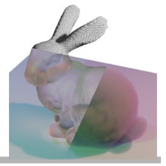
Surface to solid angle

Try to write in an area integral on $\mathcal{S}$

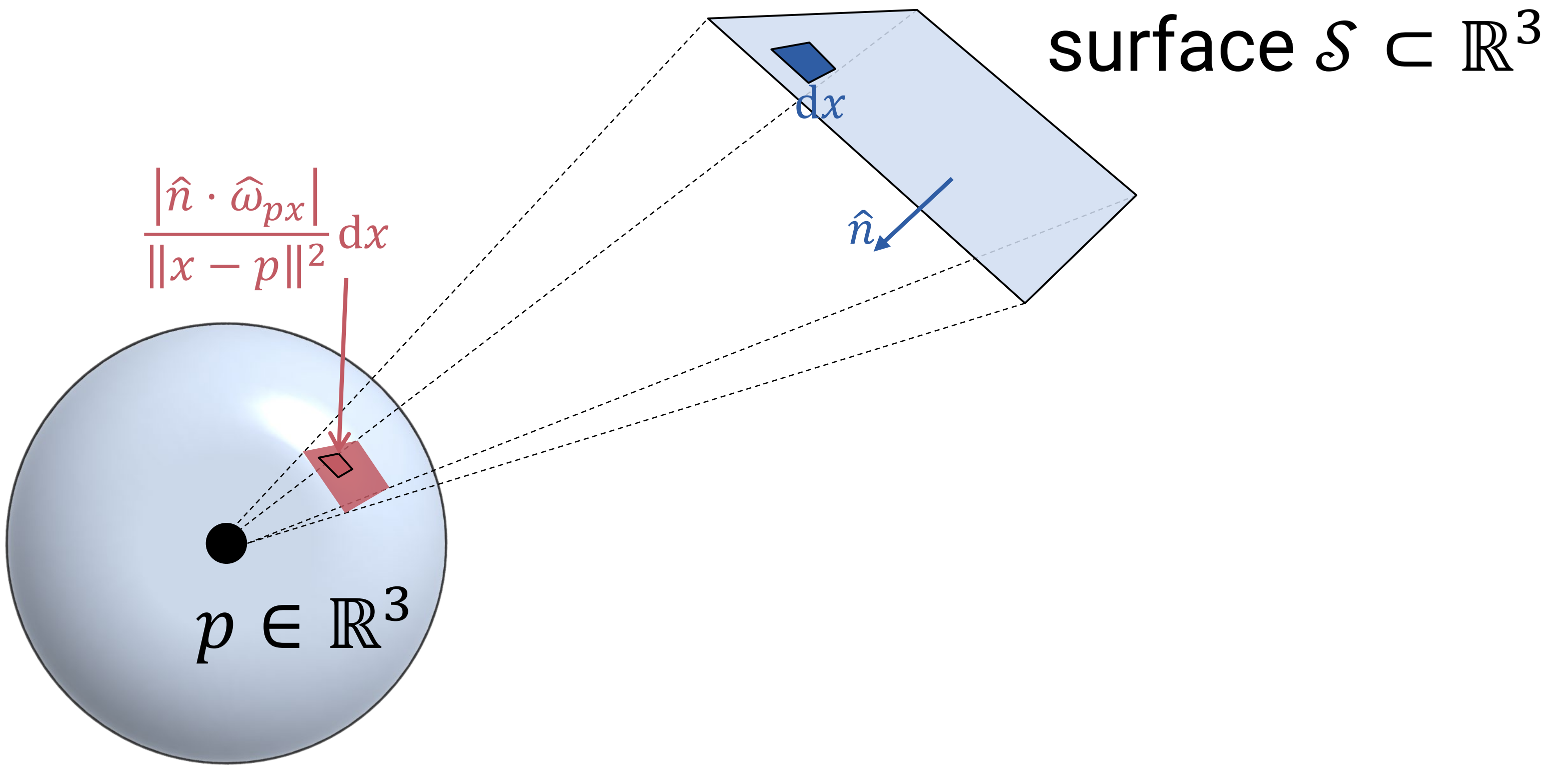


Solid angle= $\int_{\mathcal{S}} \; ? \;\; \mathrm{d}x$

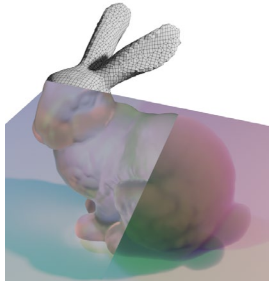
Surface to solid angle



Try to write in an area integral on $\mathcal{S}$

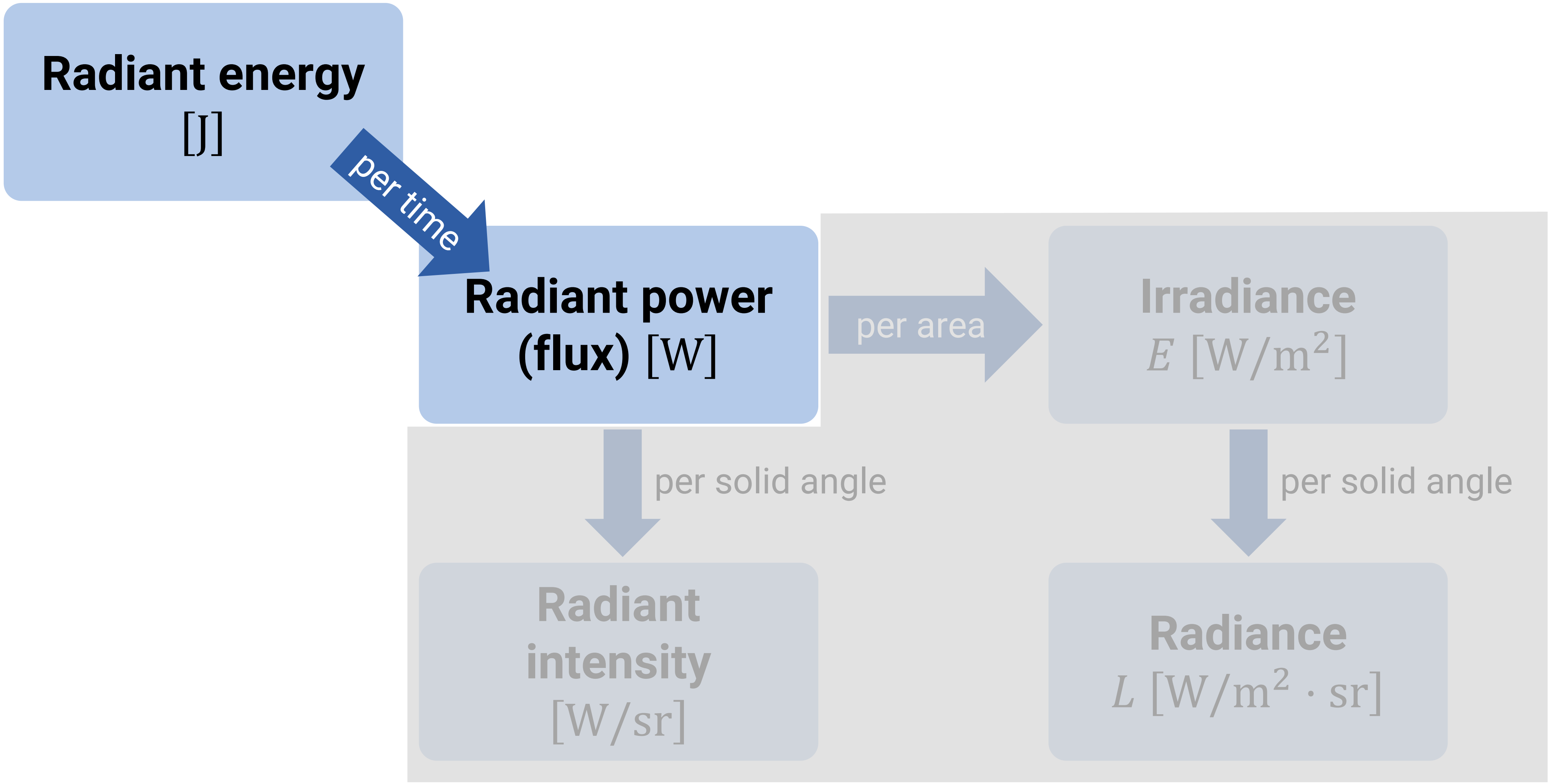


Solid angle= $\int_{\mathcal{S}} \frac{|\hat{n} \cdot \widehat{\omega}_{px}|}{\|x-p\|^2}\mathrm{d}x$

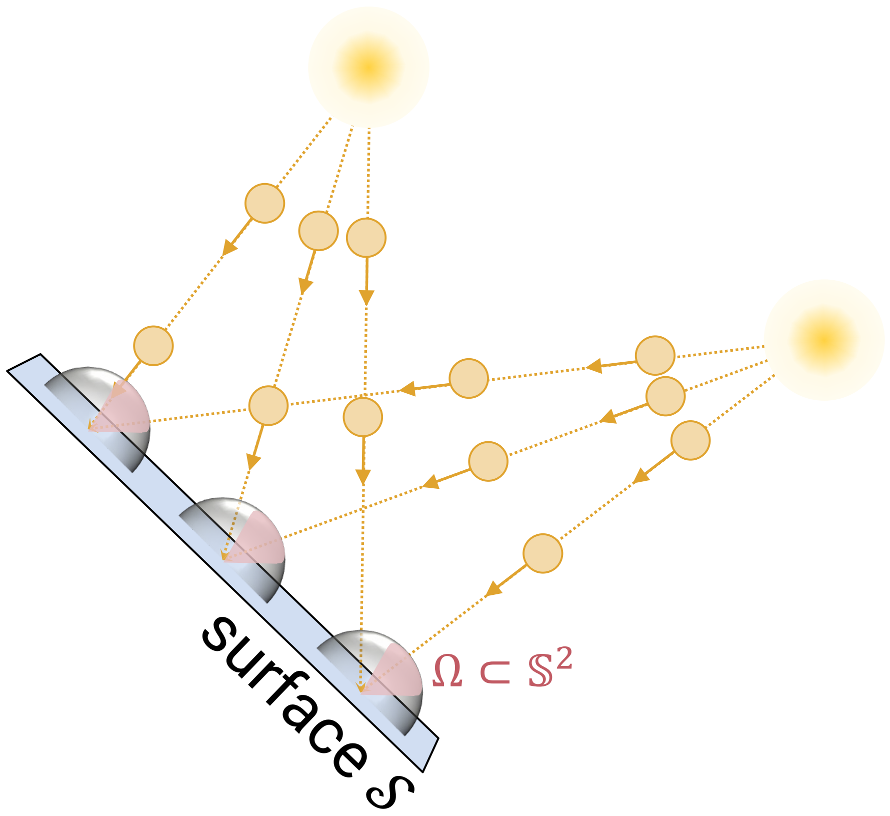


Radiometric Quantities

Radiometric quantities



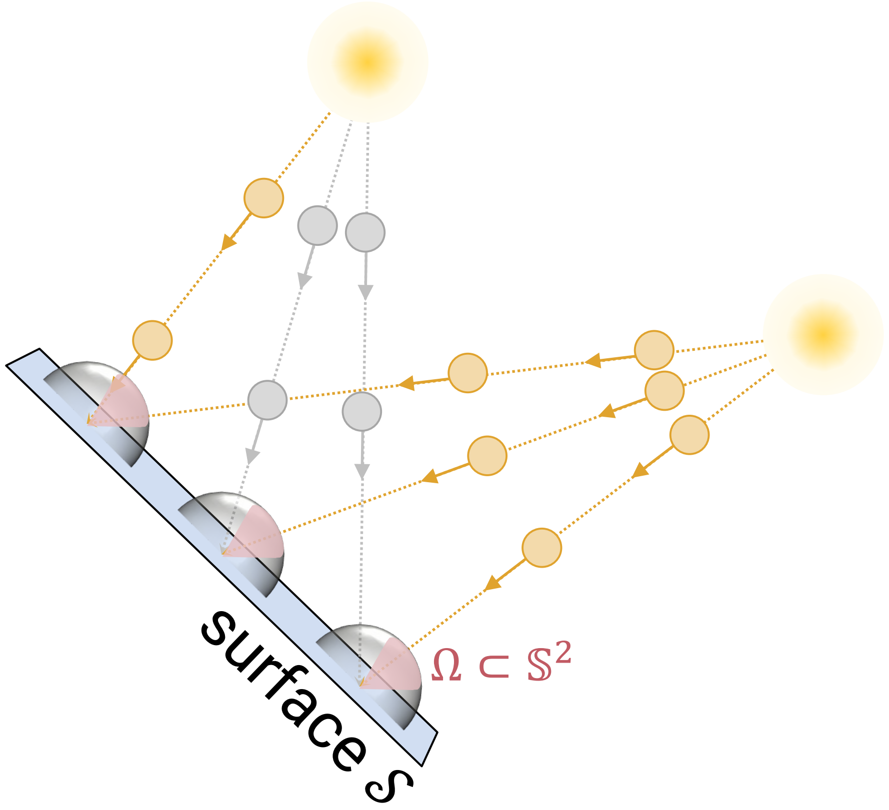
Radiant energy



Radiant energy **of** ***what?*** surface $S \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$,

- "Energy" is "energy"!
- *Number of "hits" of photons on the surface*

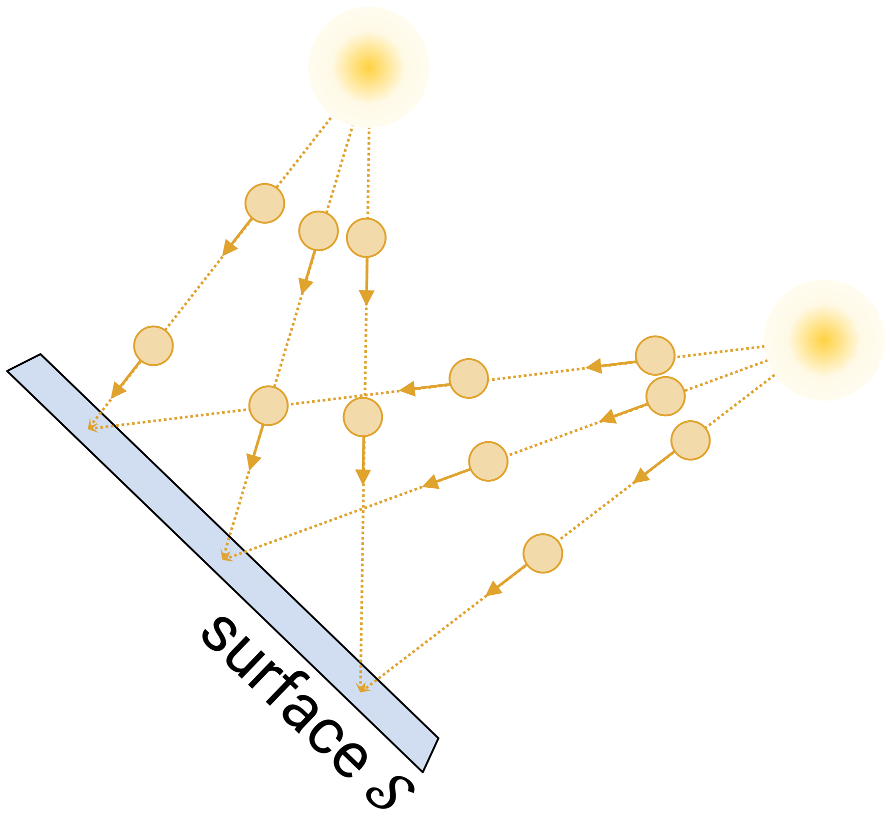
Radiant energy



Radiant energy of surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$,
time interval $[t_1, t_2] \subset \mathbb{R}$

- “Energy” is “energy”!
- *Number of “hits” of photons on the surface*

Radiant energy

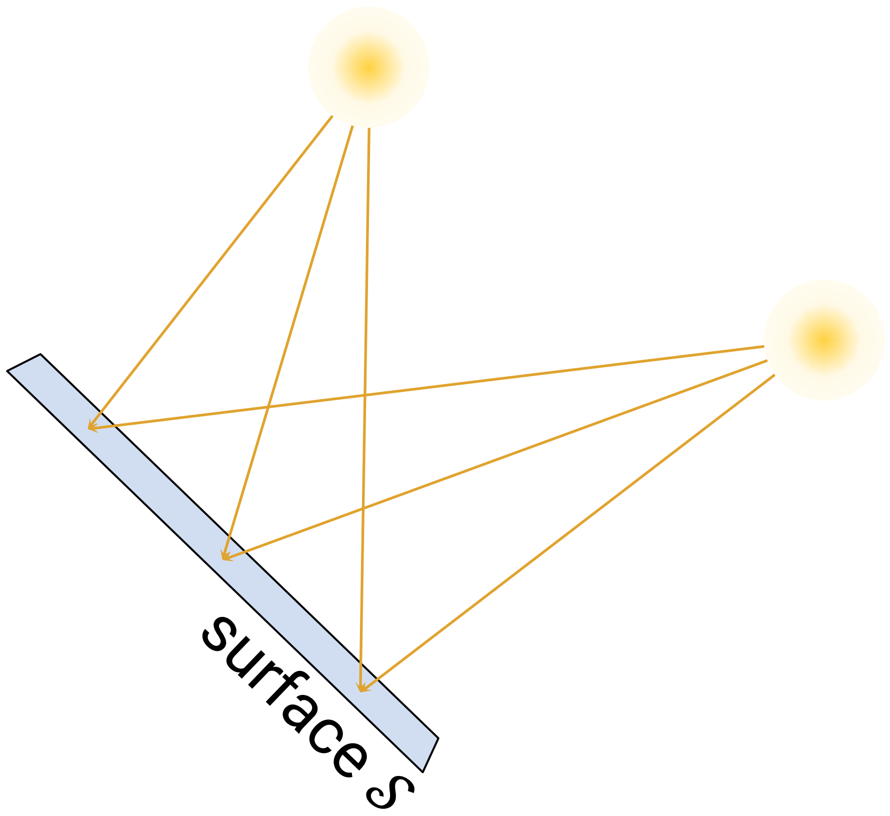


Radiant energy of surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \overset{\text{default}}{\in} \mathbb{S}^2$,
time interval $[t_1, t_2] \subset \mathbb{R}$

$$Q(\mathcal{S}, \Omega, [t_1, t_2]) \quad [\mathrm{J}]$$

- “Energy” is “energy”!
- *Number of “hits” of photons on the surface*

Radiant flux (radiant power)



Radiant flux **of what?** surface $\mathcal{S} \subset \mathbb{R}^3$,
(Radiant power)

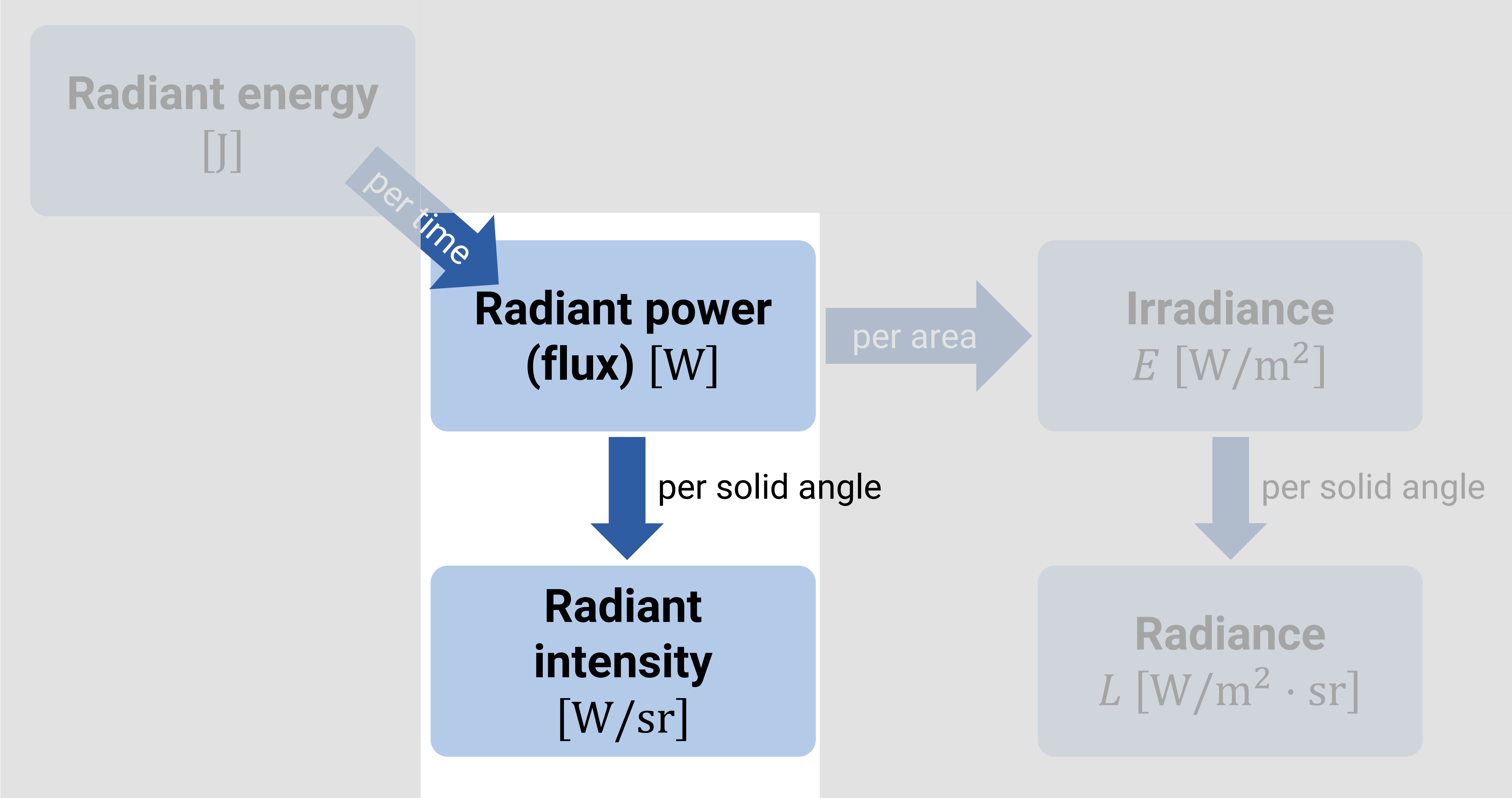
solid angle $\Omega \subset \mathbb{S}^2$

time $t \in \mathbb{R}$ (steady state)

$$\Phi(\mathcal{S}, \Omega, t) \quad [\mathrm{J/s} = \mathrm{W}]$$

- "Power" is "energy per time"!
- *Number of "intersecting rays" on the surface*

Radiant intensity

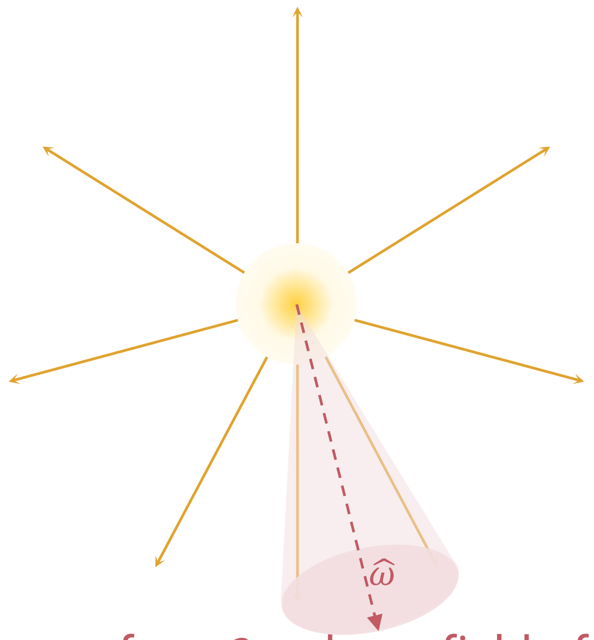


Radiant intensity

Radiant flux **of** surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$

per solid angle

Radiant intensity **of** point source $\mathbb{R}^3$, ?
direction $\hat{\omega}$



surface $\mathcal{S}_\Omega$ whose field of view from the point source is $\Omega \subset \mathbb{S}^2$

Proposition

Relations between radiant flux Φ of a surface $\mathcal{S} \subset \mathbb{R}^3$ and radiant intensity of a point source at a position $p \in \mathbb{R}^3$ is that:

$$\Phi(S_\Omega) = \int_\Omega I(\hat{\omega})\mathrm{d}\hat{\omega}$$

per solid angle →

← over solid angle

$$I(\hat{\omega}) = \lim_{\substack{|\Omega|\to 0\\ \hat{\omega}\in\Omega}} \frac{\Phi(\mathcal{S}_\Omega)}{|\Omega|},$$

solid angle measure

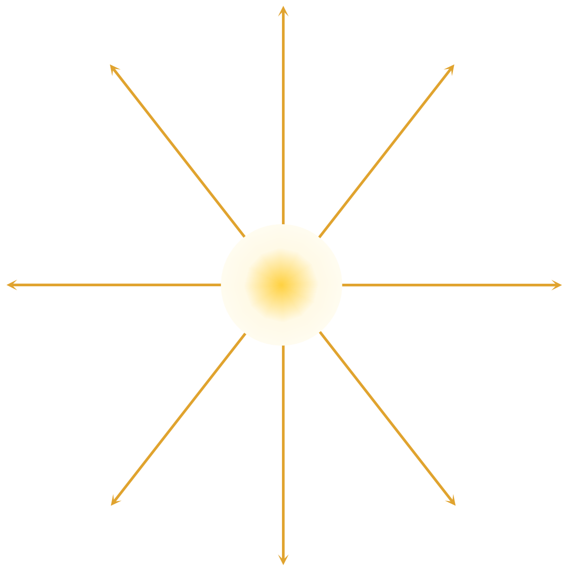
where Ω is the solid angle region

Radiant intensity

Radiant flux **of** surface $\mathcal{S} \subset \mathbb{R}^3$, solid angle $\Omega \subset \mathbb{S}^2$

per solid angle

Radiant intensity **of** point source, direction $\hat{\omega}$



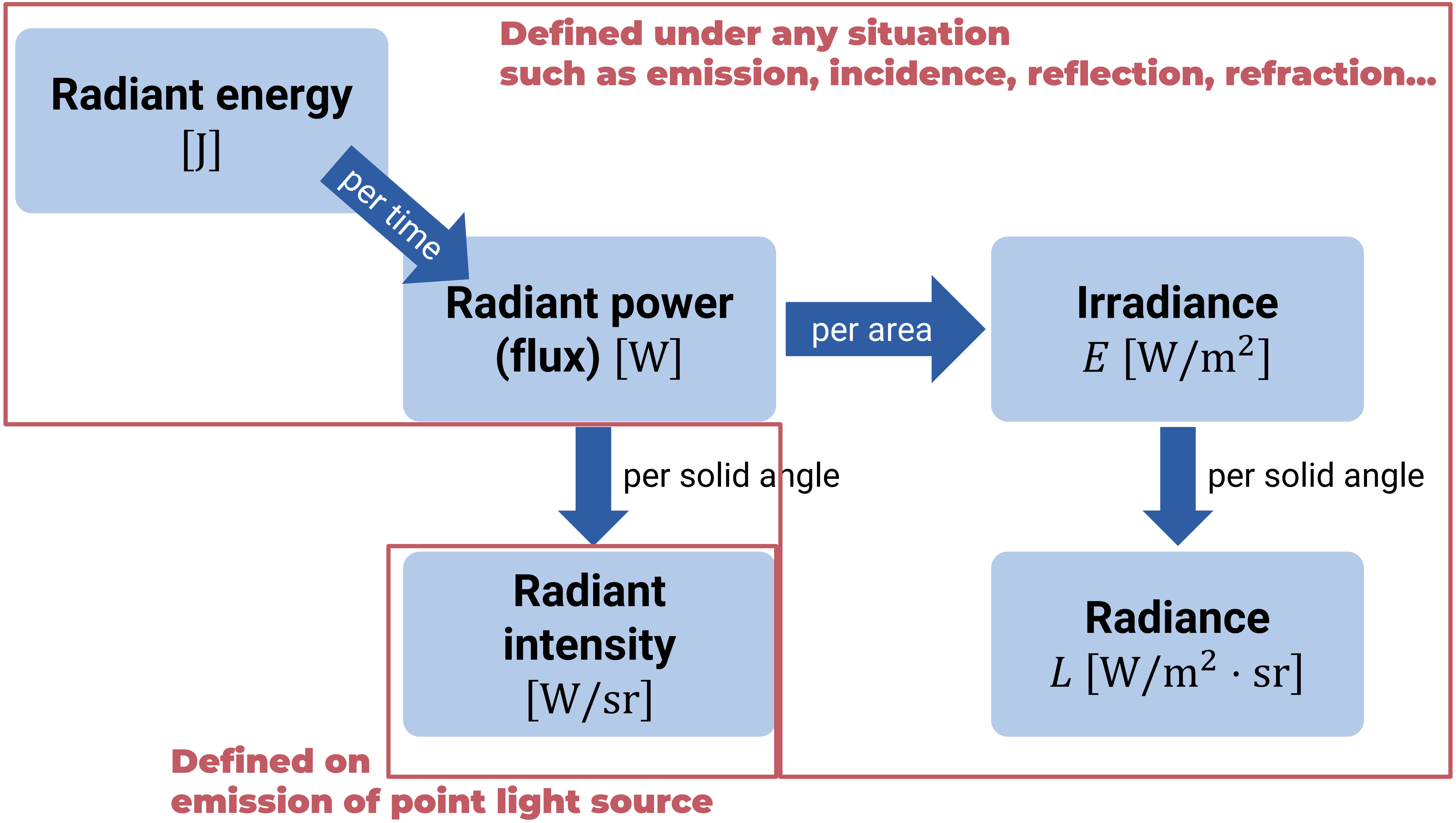
Practice

There is an isotropic point light source with radiant flux Φ.

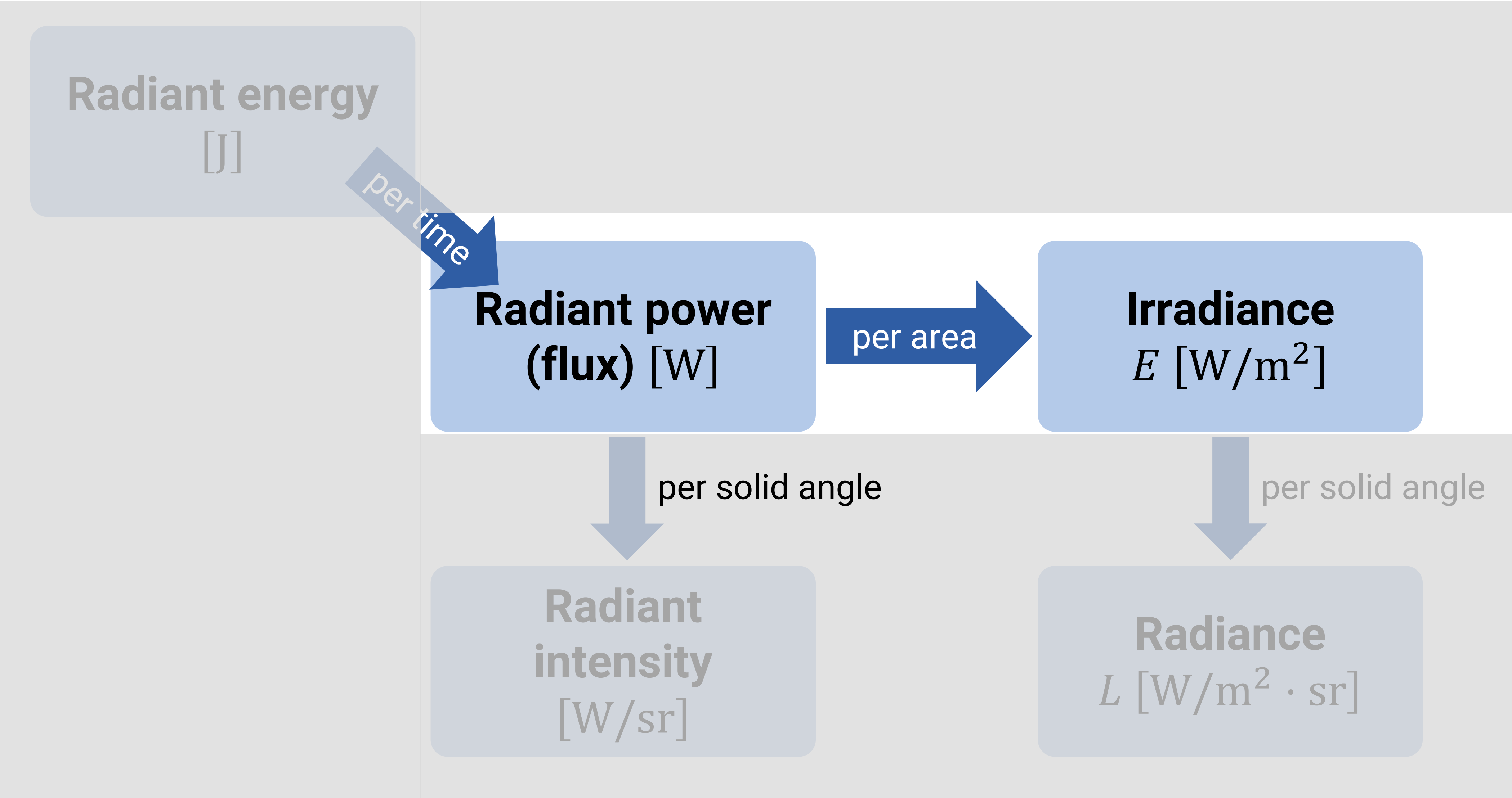
The radiant intensity of the source is?

$$I(\hat{\omega}) = \frac{\Phi}{4\pi}$$

Radiant intensity



Irradiance



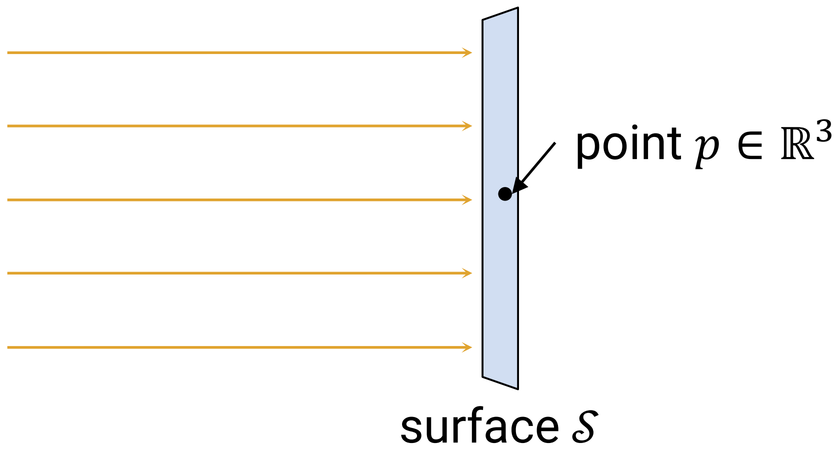
Radiant energy
[J]
per time
Radiant power
(flux) [W]
per area
Irradiance
E [W/m^2]
per solid angle
per solid angle
Radiant
intensity
[W/sr]
Radiance
L [W/m^2 · sr]

Irradiance

Radiant flux of surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$

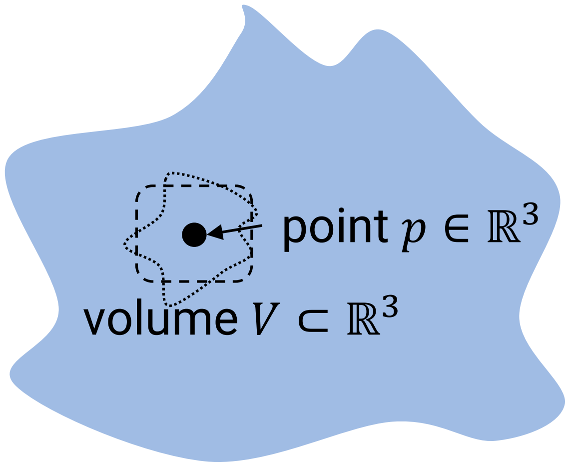
per area

irradiance of point $p \in \mathbb{R}^3$, ?
solid angle $\Omega \subset \mathbb{S}^2$



? $E(p, \Omega) = \lim_{\substack{\text{area}(\mathcal{S}) \to 0 \\ p \in \mathcal{S}}} \frac{\Phi(\mathcal{S}, \Omega)}{\text{area}(\mathcal{S})}$ **is it enough?**

Review: Concepts of mass vs. density



Mass of $V \subset \mathbb{R}^3$

Density of $p \in \mathbb{R}^3$

$\mathrm{mass}(V)$

per volume $\Rightarrow$

$$\mathrm{density}(p) = \lim_{\substack{\mathrm{vol}(V)\to 0 \\ p \in V}} \frac{\mathrm{mass}(V)}{\mathrm{vol}(V)}$$

$$= \lim_{\substack{\mathrm{vol}(V)\to 0 \\ p \in V}} \frac{\mathrm{mass}(V)}{\mathrm{vol}(V)}$$

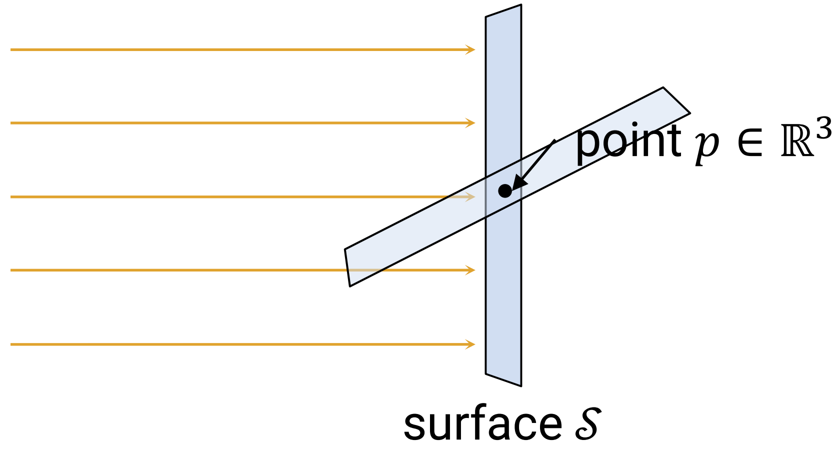
* **The limit converges to the same value whenever** $\mathrm{vol}(V) \to 0$ **and** $p \in V$.

Irradiance

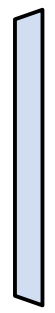
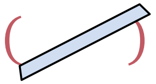
Radiant flux of surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$

per area

irradiance of point $p \in \mathbb{R}^3$, ?
solid angle $\Omega \subset \mathbb{S}^2$



? $$E(p, \Omega) = \lim_{\substack{\text{area}(\mathcal{S}) \to 0 \\ p \in \mathcal{S}}} \frac{\Phi(\mathcal{S}, \Omega)}{\text{area}(\mathcal{S})}$$ **is it enough?**

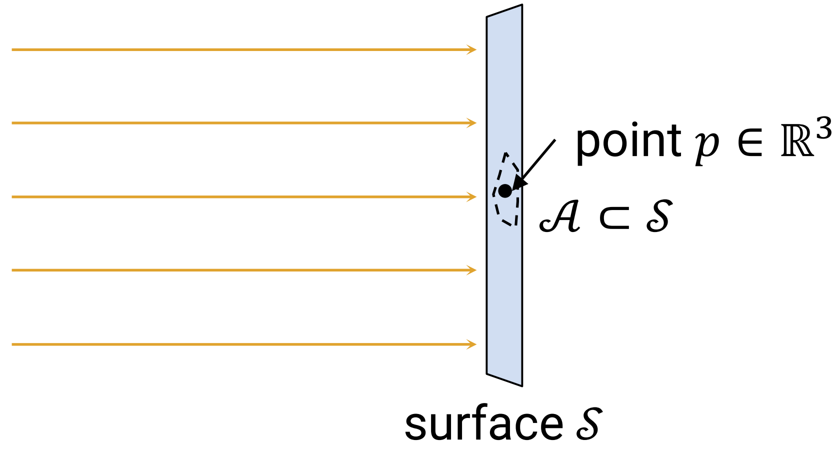
$\mathcal{S} = ($  $)$ and $\mathcal{S} = ($  $)$ yield different limits

Irradiance

Radiant flux **of** surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$

per area

irradiance **of** point $p \in \mathbb{R}^3$, ?
solid angle $\Omega \subset \mathbb{S}^2$



$$E_{\mathcal{S}}(p, \Omega) = \lim_{\substack{\mathrm{area}(\mathcal{A}) \to 0 \\ p \in \mathcal{A} \subset \mathcal{S}}} \frac{\Phi(\mathcal{A}, \Omega)}{\mathrm{area}(\mathcal{A})}$$

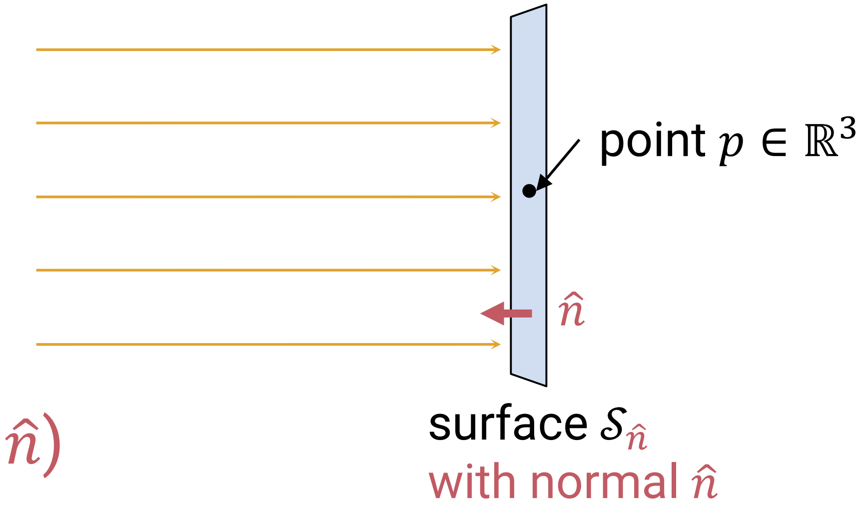
Irradiance defined as the limit about a subset of given fixed surface

Irradiance

Radiant flux **of** surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$

per area

irradiance **of** point $p \in \mathcal{S}$ (or $p \in \mathbb{R}^3$ and $\hat{n}$)
solid angle $\Omega \subset \mathbb{S}^2$



$$E_{\hat{n}}(p, \Omega) = \lim_{\substack{\text{area}(\mathcal{S}_{\hat{n}}) \to 0 \\ p \in \mathcal{S}_{\hat{n}}}} \frac{\Phi(\mathcal{S}_{\hat{n}}, \Omega)}{\text{area}(\mathcal{S}_{\hat{n}})}$$

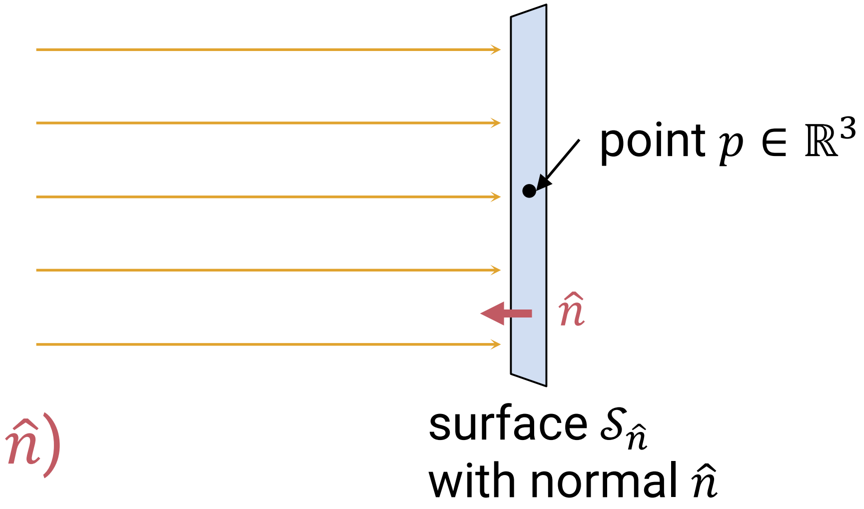
Irradiance defined as the limit about a subset of given fixed surface,
or a surface with given fixed normal

Irradiance

Radiant flux **of** surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$

per area

irradiance **of** point $p \in \mathcal{S}$ (or $p \in \mathbb{R}^3$ and $\hat{n}$)
solid angle $\Omega \subset \mathbb{S}^2$



$$E_{\hat{n}}(p, \Omega) = \lim_{\substack{\mathrm{area}(\mathcal{S}_{\hat{n}}) \to 0 \\ p \in \mathcal{S}_{\hat{n}}}} \frac{\Phi(\mathcal{S}_{\hat{n}}, \Omega)}{\mathrm{area}(\mathcal{S}_{\hat{n}})}$$

Irradiance defined as the limit about a subset of given fixed surface,
or a surface with given fixed normal

We don't say just "irradiance at p" when p is not on any surface

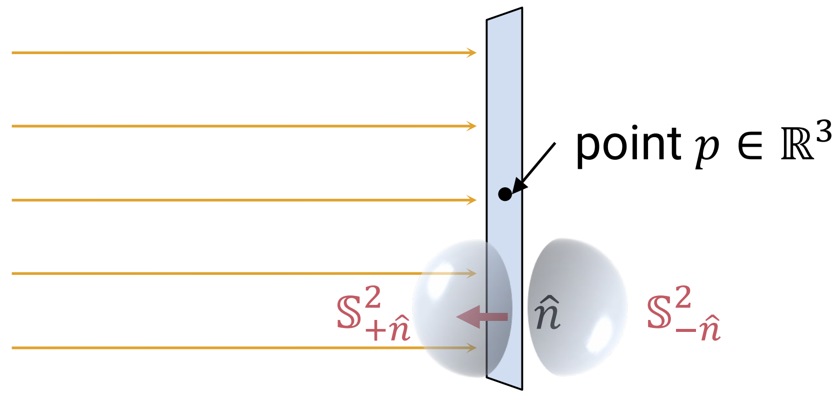
Irradiance

Radiant flux **of** surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$

per area

irradiance **of** point $p \in \mathcal{S}$ (or $p \in \mathbb{R}^3$ and $\hat{n}$)

default: solid angle $\Omega = \mathbb{S}^2, \mathbb{S}^2_{+\hat{n}},$ or $\mathbb{S}^2_{-\hat{n}}$



$$E_{\hat{n}}(p, \Omega) = \lim_{\substack{\text{area}(\mathcal{S}_{\hat{n}}) \to 0 \\ p \in \mathcal{S}_{\hat{n}}}} \frac{\Phi(\mathcal{S}_{\hat{n}}, \Omega)}{\text{area}(\mathcal{S}_{\hat{n}})}$$

The default solid angle changes depending on the context

How?

Irradiance

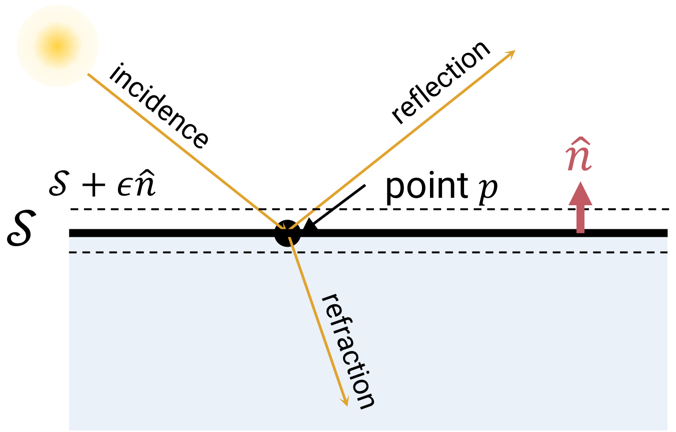
Radiant flux **of** surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$

per area

irradiance **of** point $p \in \mathcal{S}$ (or $p \in \mathbb{R}^3$ and $\hat{n}$)

default: solid angle $\Omega = \mathbb{S}^2, \mathbb{S}^2_{+\hat{n}}$, or $\mathbb{S}^2_{-\hat{n}}$

Some details depending on context…



- "incoming" irradiance: $E^{(\mathrm{in})}_{\mathcal{S}}(p) = E_{\mathcal{S}+\epsilon\hat{n}}(p, \mathbb{S}^2_{-\hat{n}})$
- "reflected" irradiance: $E^{(\mathrm{refl})}_{\mathcal{S}}(p) = E_{\mathcal{S}+\epsilon\hat{n}}(p, \mathbb{S}^2_{+\hat{n}})$
- "refracted" irradiance: $E^{(\mathrm{refr})}_{\mathcal{S}}(p) = E_{\mathcal{S}-\epsilon\hat{n}}(p, \mathbb{S}^2_{-\hat{n}})$
- "outgoing" irradiance: $E^{(\mathrm{out})}_{\mathcal{S}}(p) = E^{(\mathrm{refl})}_{\mathcal{S}}(p) + E^{(\mathrm{refr})}_{\mathcal{S}}(p)$

Our intuition easily can do this!

Irradiance

Radiant flux **of** surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$

per area

irradiance **of** point $p \in \mathcal{S}$ (or $p \in \mathbb{R}^3$ and $\hat{n}$)
solid angle $\Omega \subset \mathbb{S}^2$

$$\Phi(\mathcal{S}, \Omega) = \int_{\mathcal{S}} E_{\mathcal{S}}(p, \Omega) \mathrm{d}p$$

per area

over area

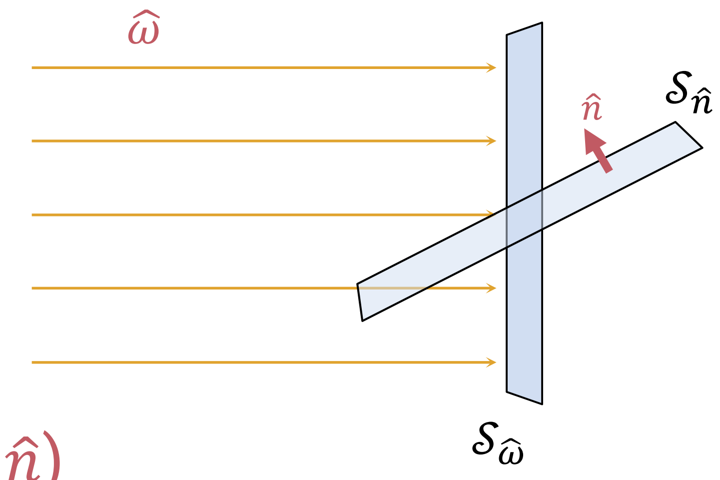
$$E_{\mathcal{S}}(p, \Omega) = \lim_{\substack{\text{area}(\mathcal{A}) \to 0 \\ p \in \mathcal{A} \subset \mathcal{S}}} \frac{\Phi(\mathcal{A}, \Omega)}{\text{area}(\mathcal{A})}$$

Irradiance

Radiant flux **of** surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$

per area

irradiance **of** point $p \in \mathcal{S}$ (or $p \in \mathbb{R}^3$ and $\hat{n}$)
solid angle $\Omega \subset \mathbb{S}^2$

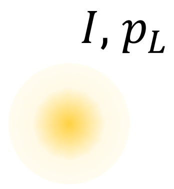


Practice

There is directional light with $\hat{\omega}$.
What is the relationship between $E_{\hat{\omega}}(p)$ and $E_{\hat{n}}(p)$?

$$E_{\hat{n}}(p) = E_{\hat{\omega}}(p)|\hat{n} \cdot \hat{\omega}|$$

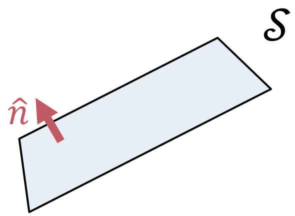
Irradiance



Radiant flux **of** surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$

↓ per area

irradiance **of** point $p \in \mathcal{S}$ (or $p \in \mathbb{R}^3$ and $\hat{n}$)
solid angle $\Omega \subset \mathbb{S}^2$



Practice

There is a point light source at p_L with the radiant intensity $I(\widehat{\omega})$.
What is the incident irradiance at p on a surface $\mathcal{S}$, $E_{\mathcal{S}}(p) = ?$

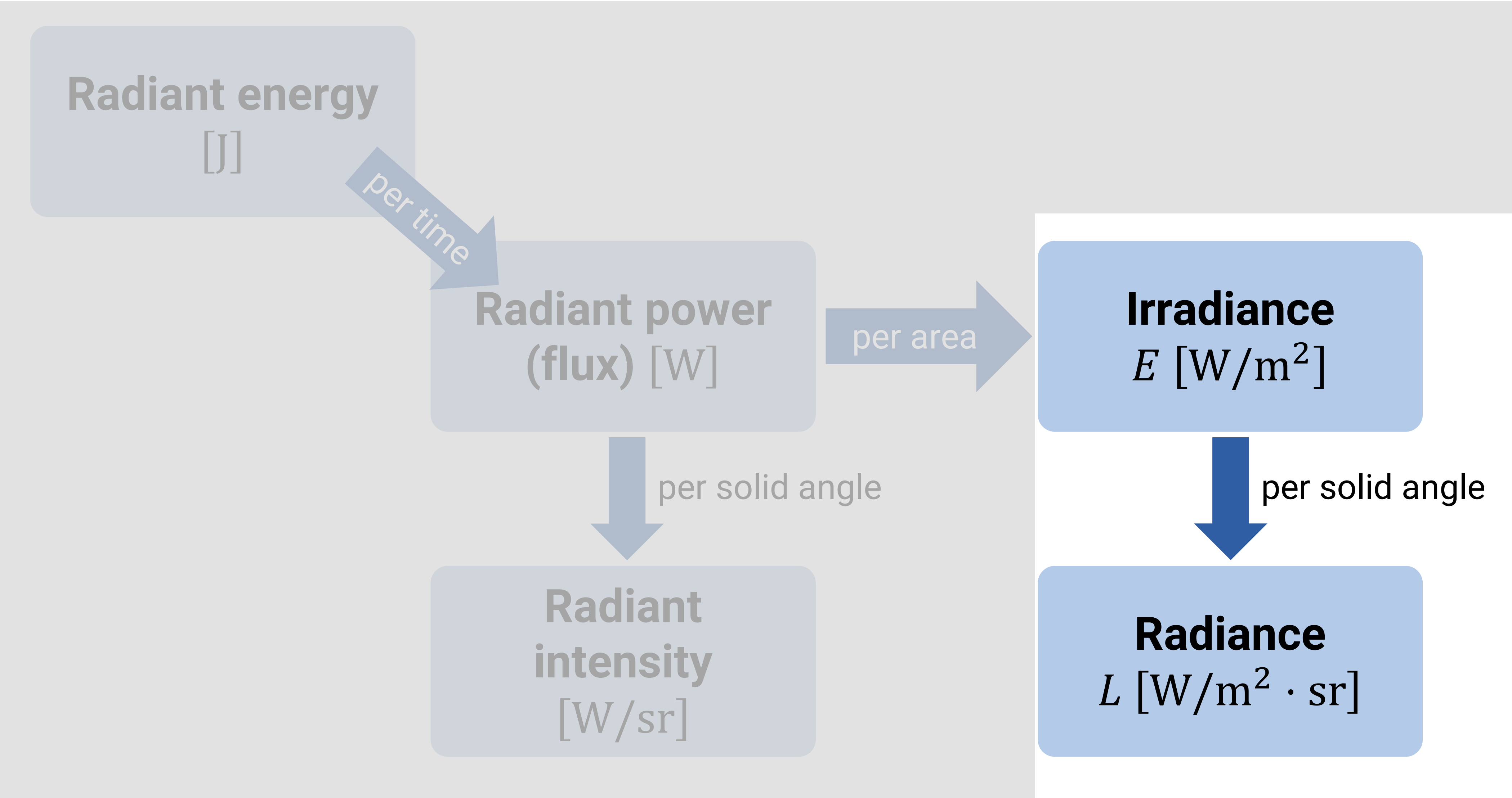
$$E_{\mathcal{S}}(p) = \lim_{\substack{\mathrm{area}(\mathcal{A})\to 0 \\ p\in\mathcal{A}\subset\mathcal{S}}} \frac{\Phi(\mathcal{A})}{\mathrm{area}(\mathcal{A})} = \lim_{\substack{\mathrm{area}(\mathcal{A})\to 0 \\ p\in\mathcal{A}\subset\mathcal{S}}} \frac{I(\widehat{\omega}_{p_L p})\,\mathrm{sol.ang.}(\mathcal{A}, p_L)}{\mathrm{area}(\mathcal{A})} = \frac{I(\widehat{\omega}_{p_L p})\,|\hat{n} \cdot \widehat{\omega}_{p p_L}|}{\|p - p_L\|^2}$$

Definition of irradiance

Definition of radiant intensity, small areal $\mathcal{A}$

Relation between an area and a solid angle, small areal $\mathcal{A}$

Radiance



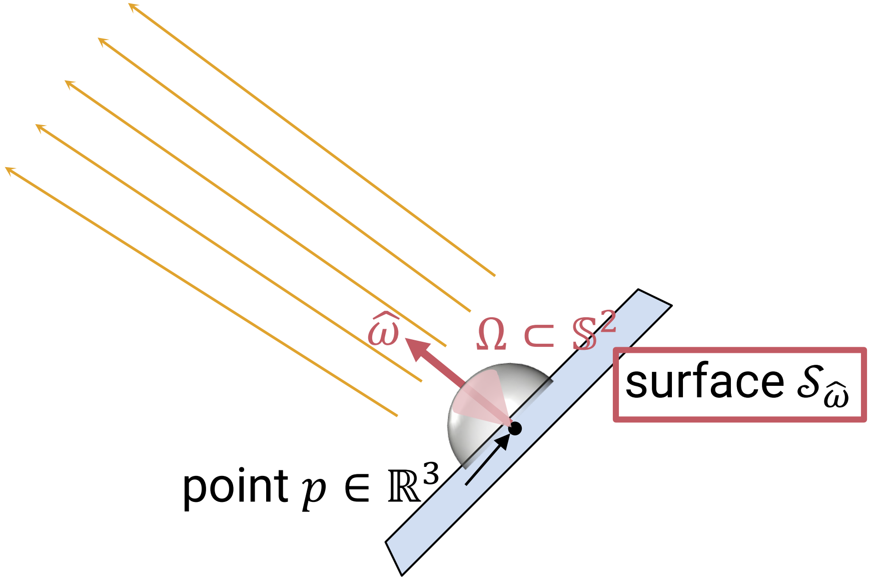
Radiant energy
[J]
per time
Radiant power (flux) [W]
per area
Irradiance
E [W/m^2]
per solid angle
Radiant intensity
[W/sr]
per solid angle
Radiance
L [W/m^2 · sr]

Radiance

Irradiance **of** point $p \in \mathcal{S}$,
solid angle $\Omega \subset \mathbb{S}^2$

per solid angle

Radiance **of** point $p \in \mathbb{R}^3$,
direction $\hat{\omega} \in \mathbb{S}^2$



$$L(p, \hat{\omega}) = \lim_{\substack{\text{sol.ang.}(\Omega) \to 0 \\ \hat{\omega} \in \Omega}} \frac{E_{\boxed{\mathcal{S}_{\hat{\omega}}}}(p, \Omega)}{\text{sol. ang.}(\Omega)}$$

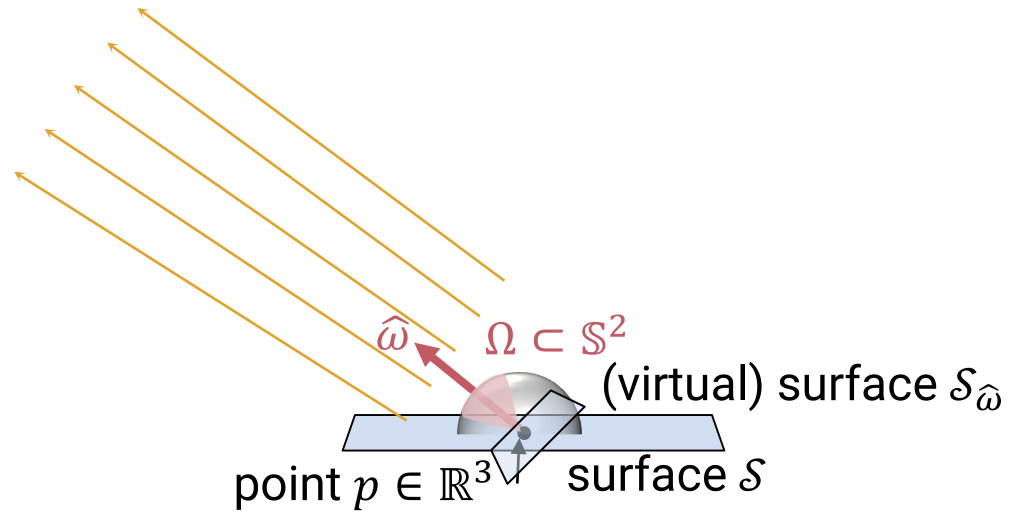
For a point $p \in \mathbb{R}^3$ (on or not on a surface),
the radiance $L(p, \hat{\omega})$ is defined as the limit about a virtual surface facing $\hat{\omega}$

Radiance

Irradiance **of** point $p \in \mathcal{S}$,
solid angle $\Omega \subset \mathbb{S}^2$

per solid angle

Radiance **of** point $p \in \mathbb{R}^3$,
direction $\hat{\omega} \in \mathbb{S}^2$



small solid angle around $\hat{\omega}$!

$$L(p, \hat{\omega}) = \lim_{\substack{\text{sol.ang.}(\Omega)\to 0 \\ \hat{\omega}\in\Omega}} \frac{\boxed{E_{\mathcal{S}_{\hat{\omega}}}(p, \Omega)}}{\text{sol. ang. }(\Omega)} = \lim_{\substack{\text{sol.ang.}(\Omega)\to 0 \\ \hat{\omega}\in\Omega}} \boxed{\frac{1}{|\hat{n}\cdot\hat{\omega}|}} \frac{\boxed{E_{\mathcal{S}}(p, \Omega)}}{\text{sol. ang. }(\Omega)}$$

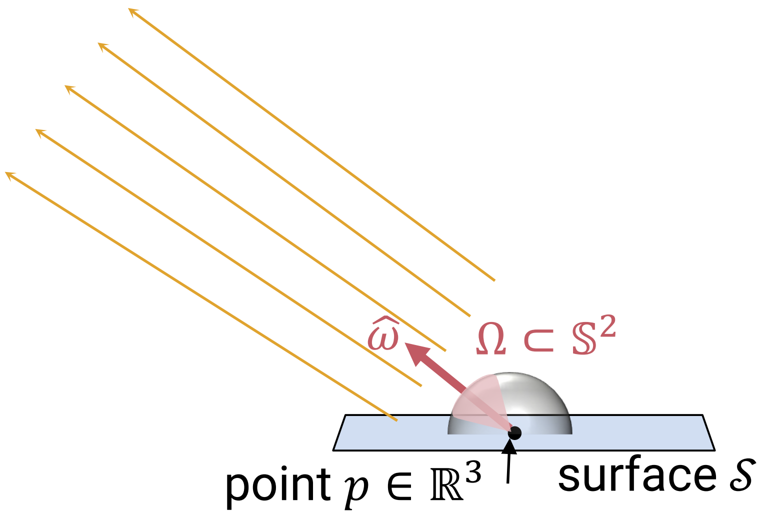
For a point p on given surface $\mathcal{S}$,
the radiance $L(p, \hat{\omega})$ can also be written as the limit about the irradiance on $\mathcal{S}$

Radiance

Irradiance **of** point $p \in \mathcal{S}$,
solid angle $\Omega \subset \mathbb{S}^2$

per solid angle

Radiance **of** point $p \in \mathbb{R}^3$,
direction $\hat{\omega} \in \mathbb{S}^2$



$$E_{\mathcal{S}}(p, \Omega) = \int_{\Omega} L(p, \hat{\omega}) |\hat{n} \cdot \hat{\omega}| \mathrm{d}\hat{\omega}$$

per solid angle →

← over solid angle

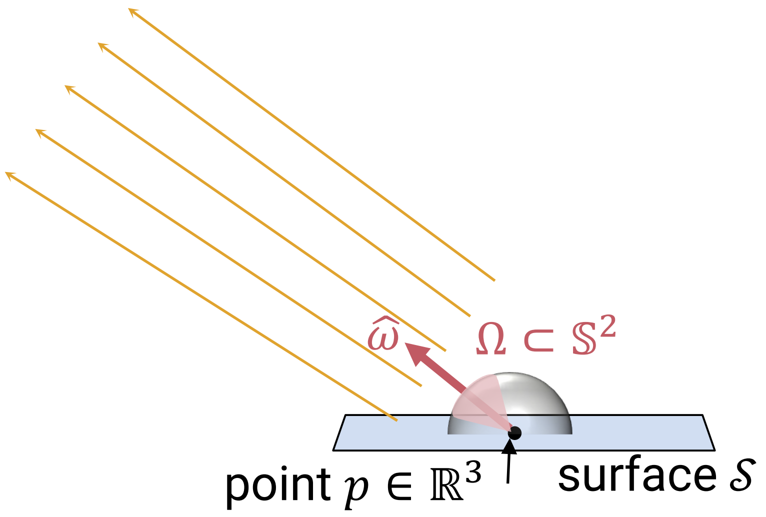
$$L(p, \hat{\omega}) = \lim_{\substack{\text{sol.ang.}(\Omega) \to 0 \\ \hat{\omega} \in \Omega}} \frac{1}{|\hat{n} \cdot \hat{\omega}|} \frac{E_{\mathcal{S}}(p, \Omega)}{\text{sol. ang. }(\Omega)}$$

Radiance

Irradiance **of** point $p \in \mathcal{S}$,
solid angle $\Omega \subset \mathbb{S}^2$

per solid angle

Radiance **of** point $p \in \mathbb{R}^3$,
direction $\hat{\omega} \in \mathbb{S}^2$



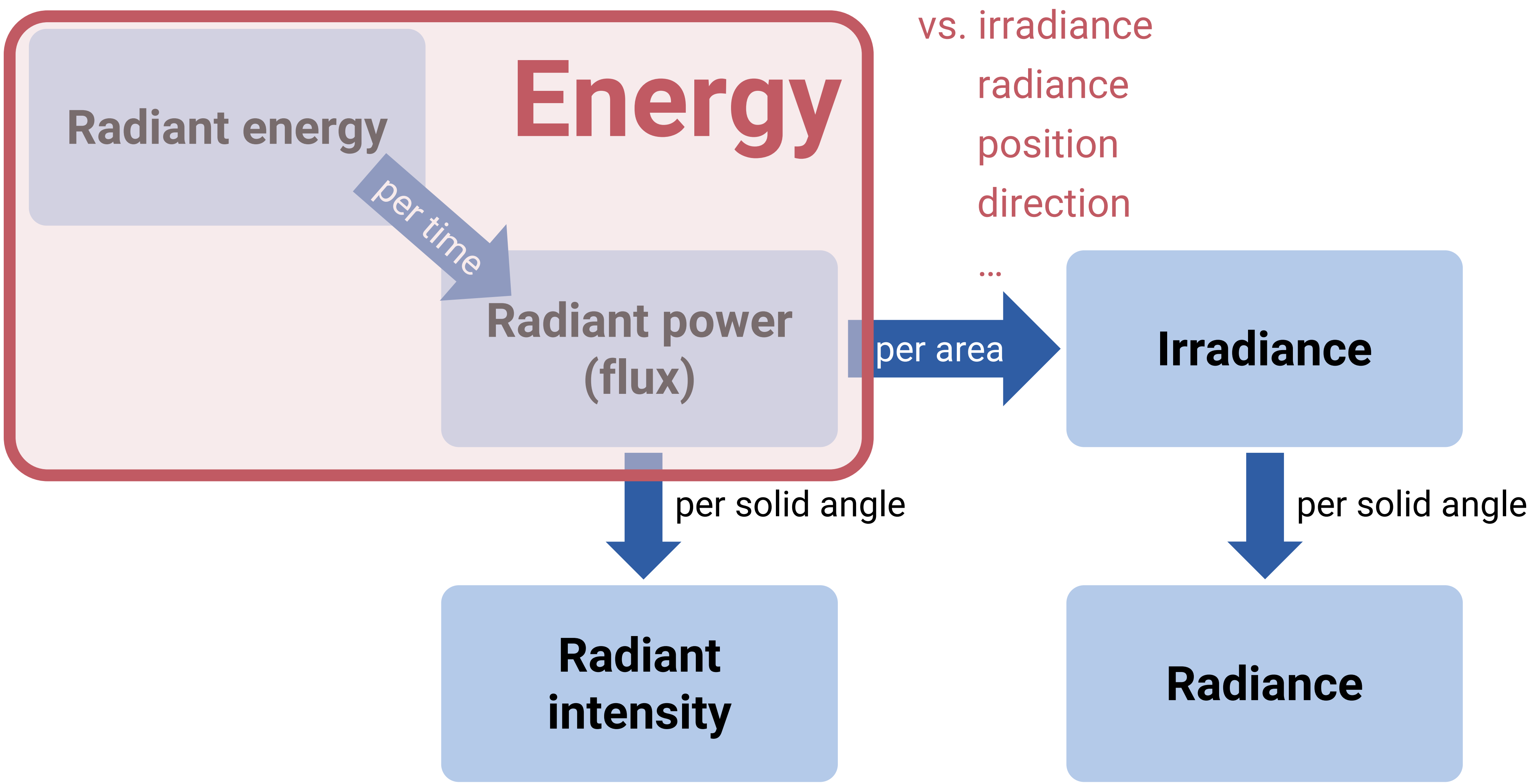
Practice

Radiance is invariant along ray:

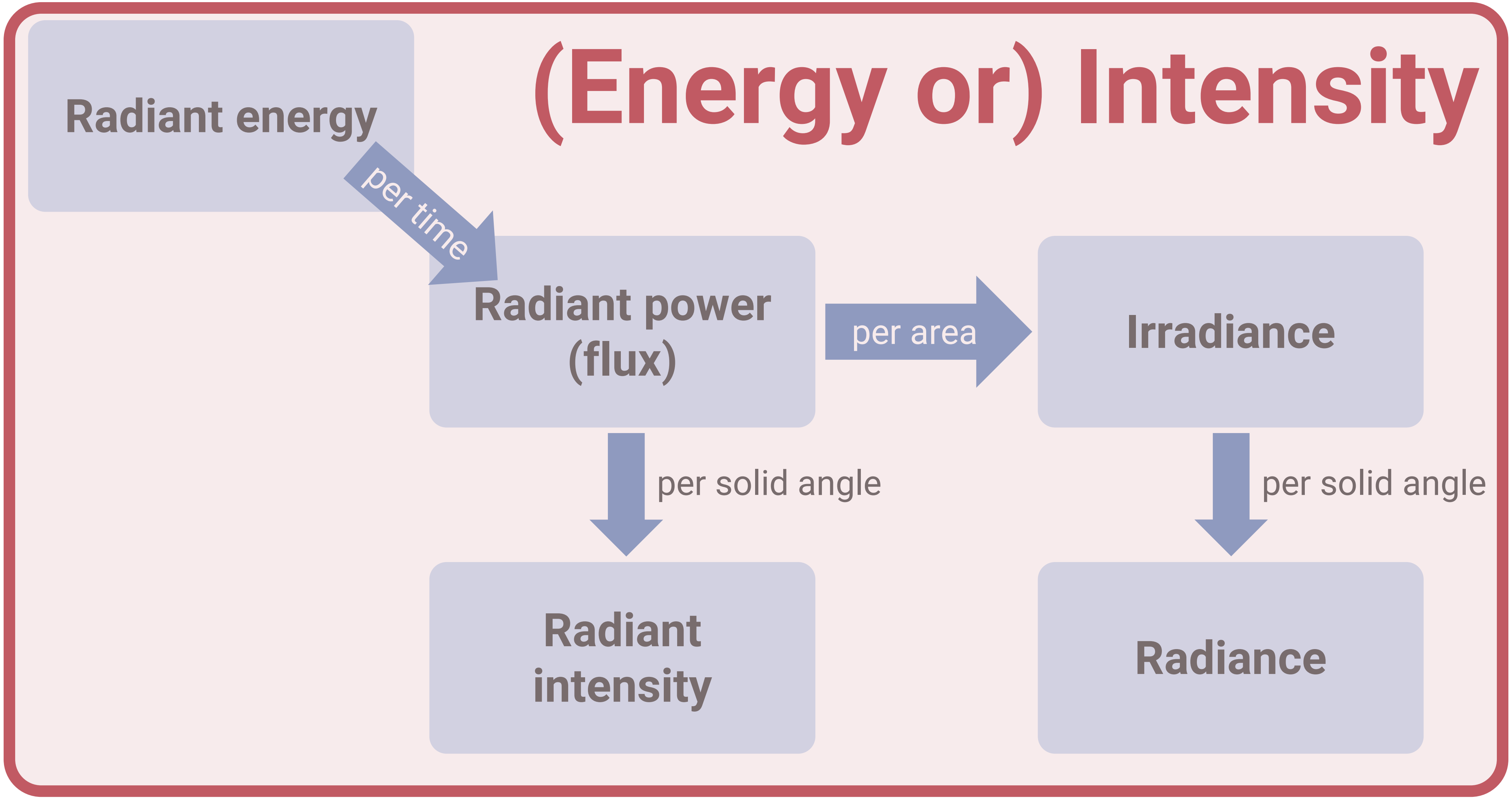
$$L(p, \hat{\omega}) = L(p + t\hat{\omega}, \hat{\omega}) \; \forall t \in \mathbb{R}$$

whenever there is no material between p and $p + t\hat{\omega}$

Slight abuse of terminology



Slight abuse of terminology

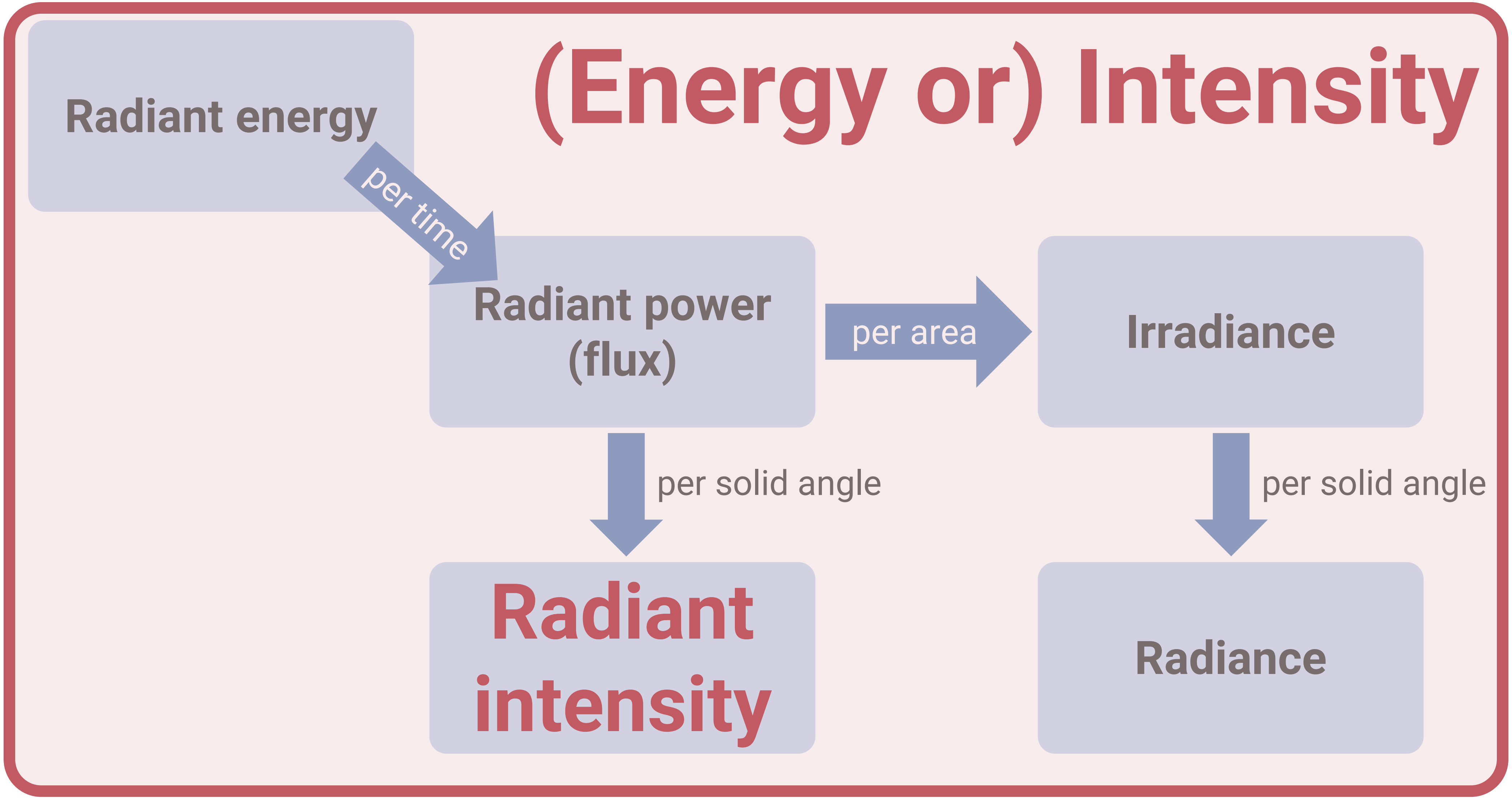


vs. position

direction

…

Unfortunate ambiguity

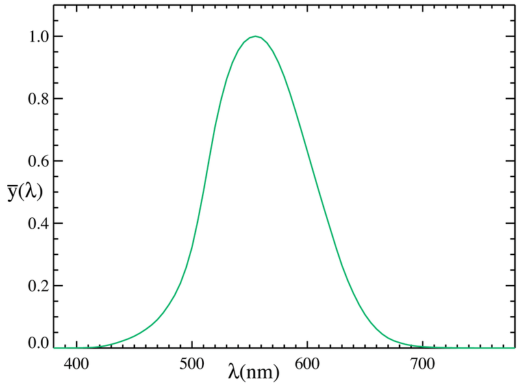


Photometry?

Radiometry: physical energy

Photometry: how bright human perceive

$$\int_{380\mathrm{nm}}^{700\mathrm{nm}} (\text{radiometric quantity per wavelength})\,(\text{luminous efficiency function})\,d\lambda$$



Photometry?

radiant → luminous

radiance → luminance

Radiant energy

per time

Radiant power (flux)

per area

Irradiance

per solid angle

Radiant intensity

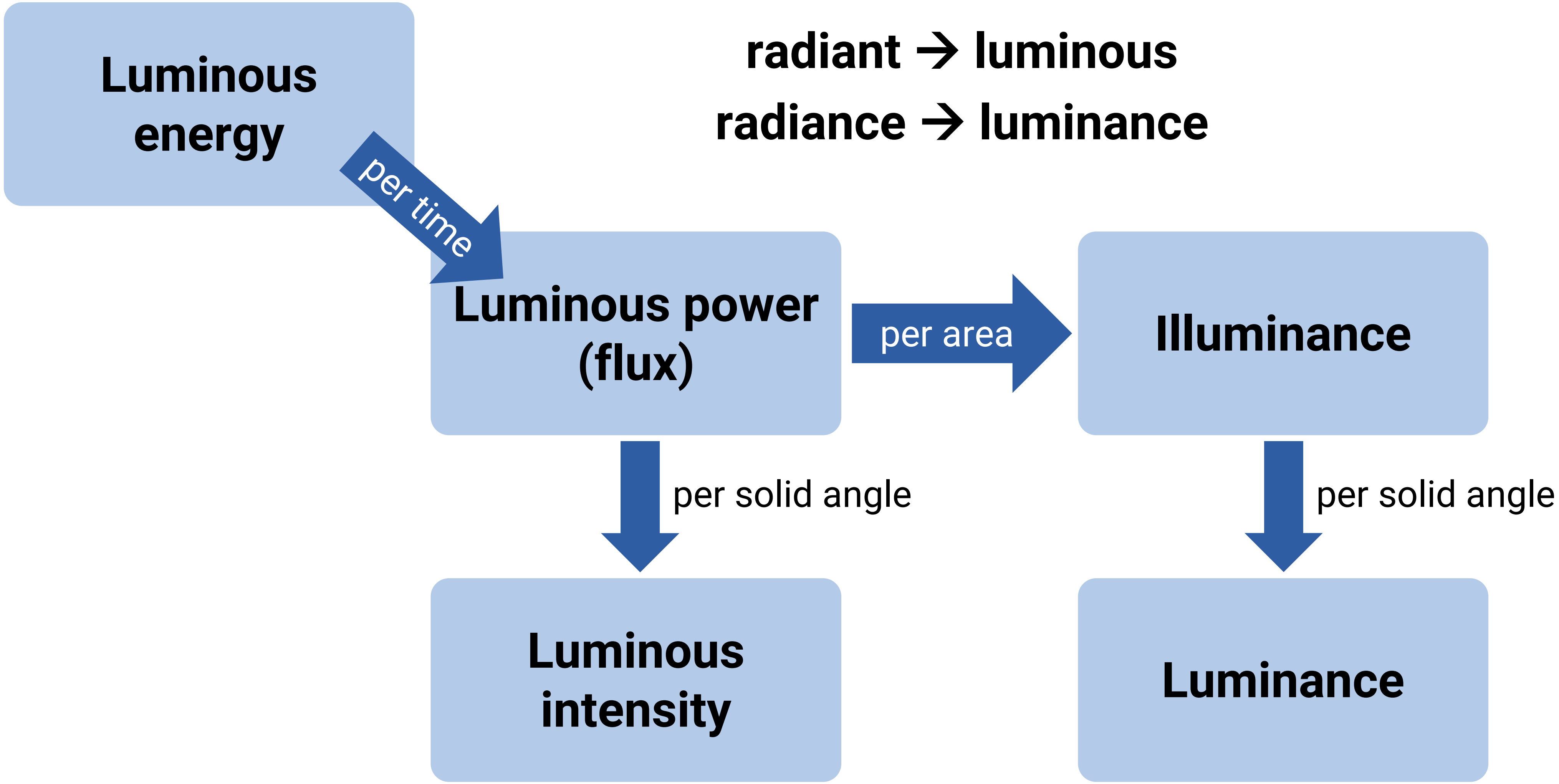
per solid angle

Radiance

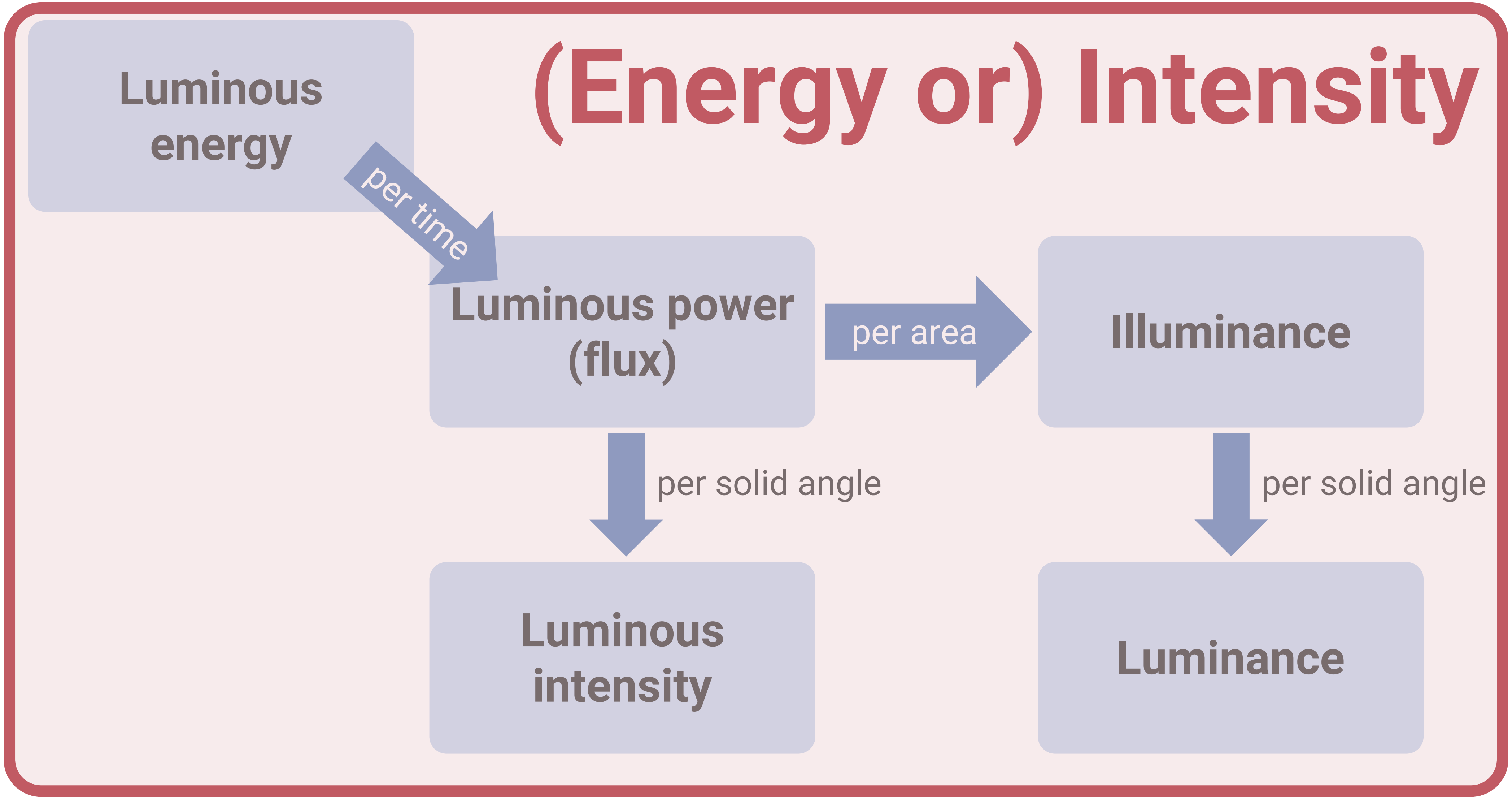
Photometry

radiant → luminous

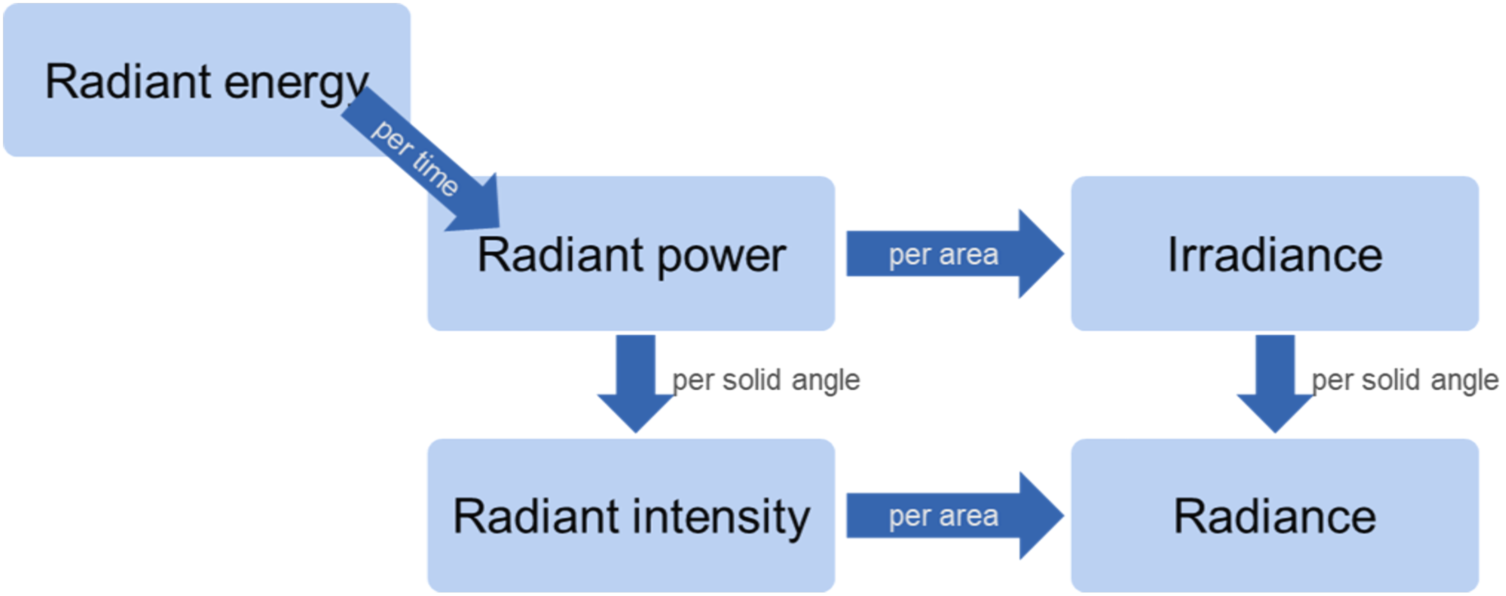
radiance → luminance



Slight abuse of terminology



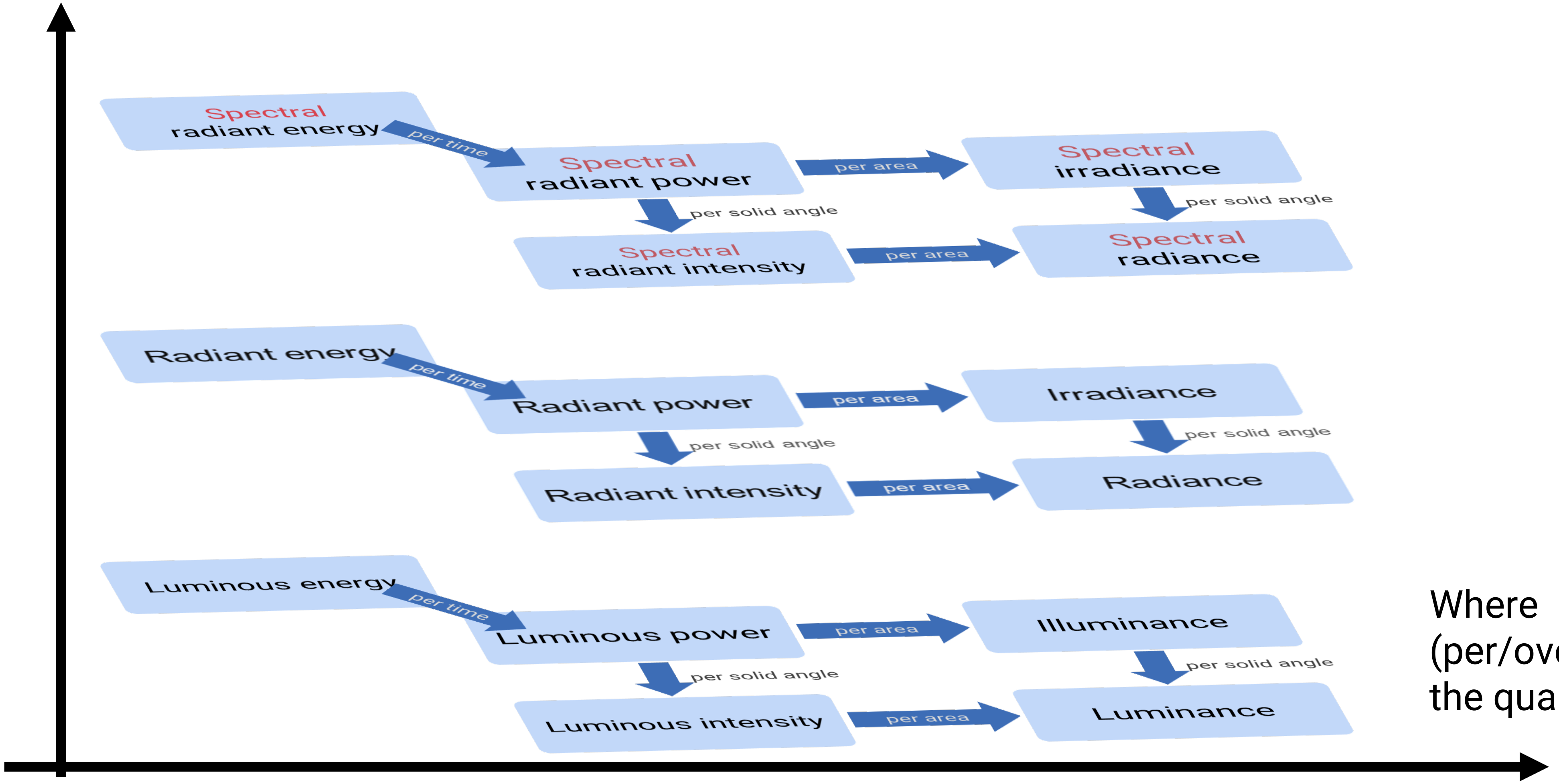
Radiometric quantities



Radiometric quantities

Physical energy? (per wavelength?)

Visible brightness for human?



Where
(per/over position? direction?)
the quantity is defined

Bidirectional reflectance distribution function

We roughly say....

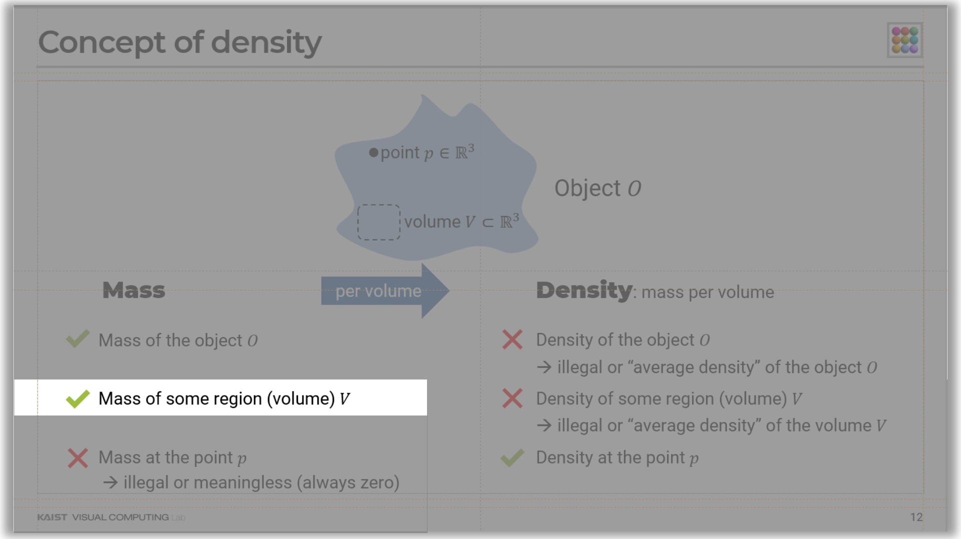
$$\text{BRDF } f_s\text{:} \frac{\text{outgoing radiance along } \omega_o}{\text{incident irradiance at } \omega_i}$$

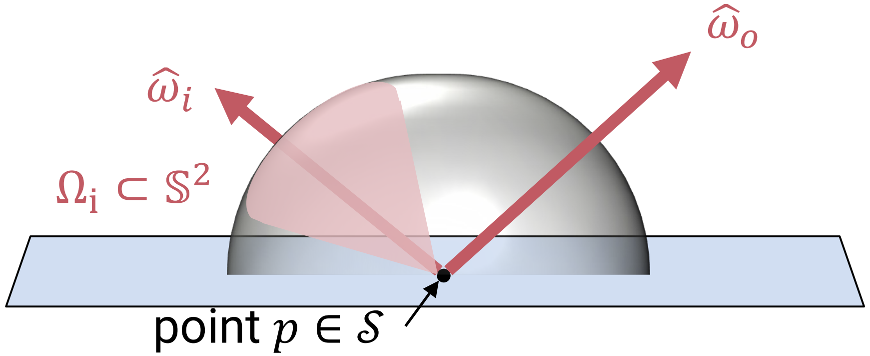
Previous slide

Irradiance ***at a point on a surface (dosen't depend on direction)***

????

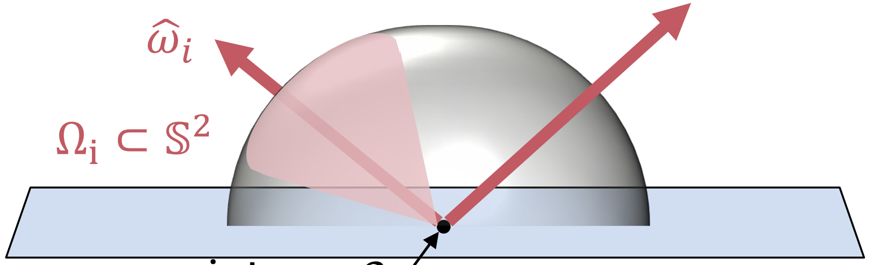
Bidirectional reflectance distribution function





$$\text{BRDF } f_s(p, \hat{\omega}_i, \hat{\omega}_o) = \lim_{\substack{\text{sol.ang.}(\Omega_i)\to 0 \\ \omega_i \in \Omega_i}} \frac{L(p, \hat{\omega}_o)}{E_{\mathcal{S}}^{(\text{in})}(p, \Omega_i)}$$

Bidirectional reflectance distribution function



$$\text{BRDF } f_s(p, \widehat{\omega}_i, \widehat{\omega}_o) = \lim_{\substack{\text{sol.ang.}(\Omega_i)\to 0 \\ \omega_i \in \Omega_i}} \frac{L(p, \widehat{\omega}_o)}{E_{\mathcal{S}}^{(\text{in})}(p, \Omega_i)}$$

$$= \lim_{\substack{\text{sol.ang.}(\Omega_i)\to 0 \\ \omega_i \in \Omega_i}} \frac{L(p, \widehat{\omega}_o)}{\int_{\Omega_i} L^{(\text{in})}(p, \widehat{\omega}_i)\, |\hat{n} \cdot \widehat{\omega}_i| \mathrm{d}\widehat{\omega}_i} = \lim_{\substack{\text{sol.ang.}(\Omega_i)\to 0 \\ \omega_i \in \Omega_i}} \frac{L(p, \widehat{\omega}_o)}{L^{(\text{in})}(p, \widehat{\omega}_i) |\hat{n} \cdot \widehat{\omega}_i| \text{sol. ang. } (\Omega_i)}$$

Rendering equation

$+L_e(p, \widehat{\omega}_o) \Rightarrow$**, then we get the rendering equation**

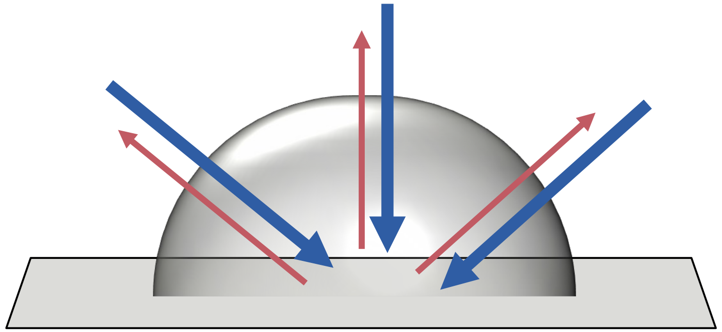
$$L^{(\mathrm{out})}(p, \widehat{\omega}_o) = \int_{\mathbb{S}^2} L^{(\mathrm{in})}(p, \widehat{\omega}_i) f_s(p, \widehat{\omega}_i, \widehat{\omega}_o) |\hat{n} \cdot \widehat{\omega}_i| \mathrm{d}\widehat{\omega}_i$$

per solid angle

over solid angle

$$f_s(p, \widehat{\omega}_i, \widehat{\omega}_o) = \lim_{\substack{\text{sol.ang.}(\Omega_i) \to 0 \\ \omega_i \in \Omega_i}} \frac{L(p, \widehat{\omega}_o)}{L^{(\mathrm{in})}(p, \widehat{\omega}_i) |\hat{n} \cdot \widehat{\omega}_i| \, \mathrm{sol.ang.}(\Omega_i)}$$

Properties of BRDF: energy conservation



$$\frac{\text{outgoing irradiance}}{\text{incident irradiance}} \leq 1, \text{ for any illumination condition}$$

$$\frac{\int_{\mathbb{S}^2} L^{(\text{out})}(p, \hat{\omega})|\hat{n} \cdot \hat{\omega}| \mathrm{d}\hat{\omega}}{\int_{\mathbb{S}^2} L^{(\text{in})}(p, \hat{\omega})|\hat{n} \cdot \hat{\omega}| \mathrm{d}\hat{\omega}} =$$

rendering equation

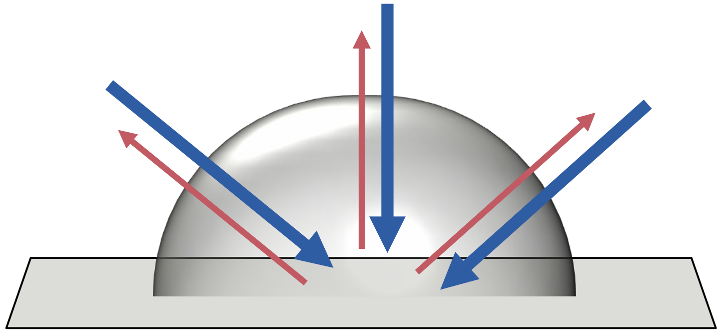
$$\frac{\int_{\mathbb{S}^2} \int_{\mathbb{S}^2} L^{(\text{in})}(p, \hat{\omega}_i) f_s(p, \hat{\omega}_i, \hat{\omega}_o)|\hat{n} \cdot \hat{\omega}_i| \mathrm{d}\hat{\omega}\, |\hat{n} \cdot \hat{\omega}_o| \mathrm{d}\hat{\omega}_o}{\int_{\mathbb{S}^2} L^{(\text{in})}(p, \hat{\omega})|\hat{n} \cdot \hat{\omega}| \mathrm{d}\hat{\omega}} \leq 1,$$

for **any positive function** $L^{(\text{in})}(p, \cdot)$.

Taking $L^{(\text{in})}(p, \cdot)$ as a Dirac delta function centered at $\hat{\omega}_i$,

$$\therefore \int_{\mathbb{S}^2} f_s(p, \hat{\omega}_i, \hat{\omega}_o)|\hat{n} \cdot \omega_o| \mathrm{d}\hat{\omega}_o \leq 1, \forall \hat{\omega}_i$$

Properties of BRDF: energy conservation



$$\frac{\text{outgoing irradiance}}{\text{incident irradiance}} \leq 1, \text{ for any illumination condition}$$

< 1: energy losses

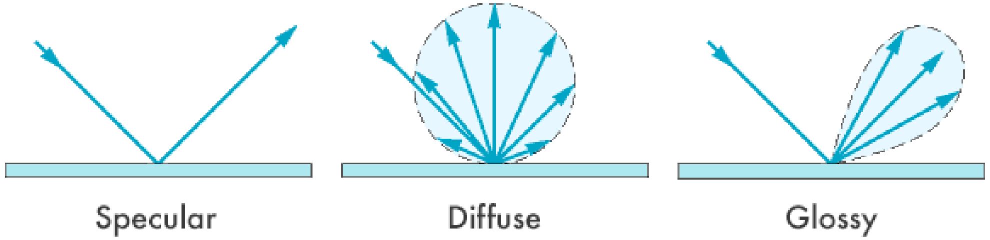
$= 1$: energy conserves

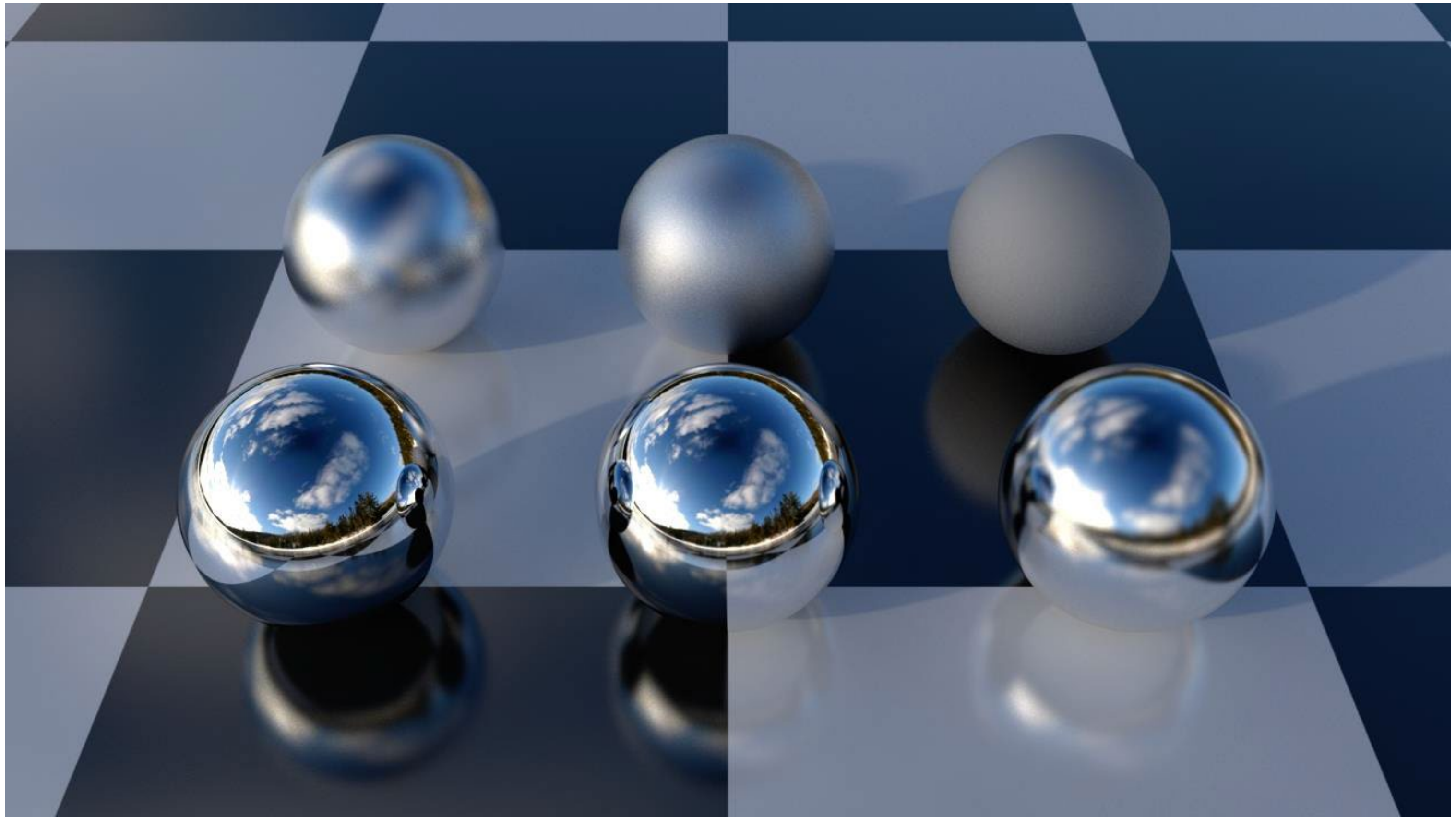
> 1: impossible!

Energy Conservation

$$\int_{\mathbb{S}^2} f_s(p, \hat{\omega}_i, \hat{\omega}_o) |\hat{n} \cdot \omega_o| \mathrm{d}\hat{\omega}_o \leq 1, \forall \hat{\omega}_i$$

Example BRDFs





source: Keenan Crane []

Example BRDFs

- Pure diffuse (Lambertian reflection)
  - Albedo ρ_d: ratio of energy conservation
  - f_s is a constant function on $\mathbb{S}^2_{\hat{z}}$

$$\int_{\mathbb{S}^2_{\hat{z}}} f_s |\hat{n} \cdot \widehat{\omega}_o| \mathrm{d}\omega_o = \pi f_s = \rho_d$$

$$\therefore f_s = \frac{\rho_d}{\pi}$$

Example BRDFs

- Pure specular
  - A Dirac delta function centered at $\mathrm{refl}_{\hat{n}}(\hat{\omega}_i)$....
  - Be careful when you treat Dirac delta functions

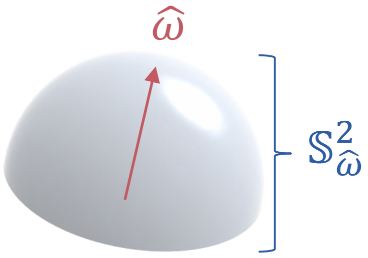
$$f_s(\widehat{\omega}_i, \widehat{\omega}_o) = a \cdot \delta_{\mathbb{S}^2}\big(\widehat{\omega}_o, \mathrm{refl}_{\hat{n}}(\widehat{\omega}_i)\big), \int_{\mathbb{S}^2_{\hat{n}}} f_s(\widehat{\omega}_i, \widehat{\omega}_o) |\hat{n} \cdot \widehat{\omega}_o| \mathrm{d}\widehat{\omega}_o = 1$$

$$\therefore\ f_s(\widehat{\omega}_i, \widehat{\omega}_o) = \frac{\delta_{\mathbb{S}^2}\big(\widehat{\omega}_o, \mathrm{refl}_{\hat{n}}(\widehat{\omega}_i)\big)}{\hat{n} \cdot \widehat{\omega}_o}$$

Notation table

Sets

$\mathbb{R}^n$	Euclidean space
$\mathbb{S}^2$	the unit sphere (the set of all unit vectors)
$\mathbb{S}^2_{\hat{\omega}}$	the hemisphere facing a direction $\hat{\omega} \in \mathbb{S}^2$



Convention of variables

$p \in \mathbb{R}^3$	point in the space (or a surface)
$\hat{\omega} \in \mathbb{S}^2$	direction (unit vector)
• $\hat{\omega}_{p_1 p_2} := \frac{p_2 - p_1}{\lVert p_2 - p_1 \rVert}$ for any $p_1, p_2 \in \mathbb{R}^3$	
$\hat{n} \in \mathbb{S}^2$	surface normal, where a point and a surface are given in context
$\mathcal{S} \subset \mathbb{R}^3$	surface
$\mathcal{V} \subset \mathbb{R}^3$	volume
$\Omega \subset \mathbb{S}^2$	solid angle (region on the unit sphere $\mathbb{S}^2$)

Radiometric quantities

* time dependency is omitted for simplicity
* (· ", Ω") is usually omitted and assumed as an entire $\mathbb{S}^2$ or hemisphere

$\Phi(\mathcal{S}, \Omega)$ [W]	radiant power (flux) at a surface $\mathcal{S} \subset \mathbb{R}^3$ and a solid angle $\Omega \subset \mathbb{S}^2$
$I(\hat{\omega})$ [W/sr]	radiant intensity at a direction $\hat{\omega} \in \mathbb{S}^2$, where a point source is given in context
$E(p, \Omega)$ [W/m^2]	irradiance at $p \in \mathcal{S}$ and $\Omega \subset \mathbb{S}^2$, where the surface $\mathcal{S} \subset \mathbb{R}^3$ is given in context
$L(p, \omega)$ [W/m^{2}sr]	radiance at $p \in \mathbb{R}^3$ and $\hat{\omega} \in \mathbb{S}^2$
$f_s(p, \omega_i, \omega_o)$ [sr^{-1}]	BSDF at $p \in \mathcal{S}$ from $\hat{\omega}_i \in \mathbb{S}^2$ to $\hat{\omega}_o \in \mathbb{S}^2$, where the surface $\mathcal{S}$ is given in context