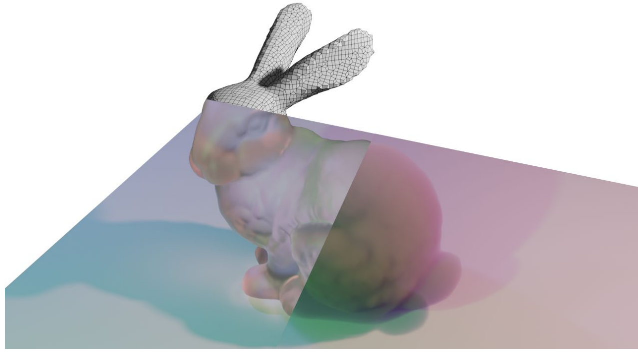
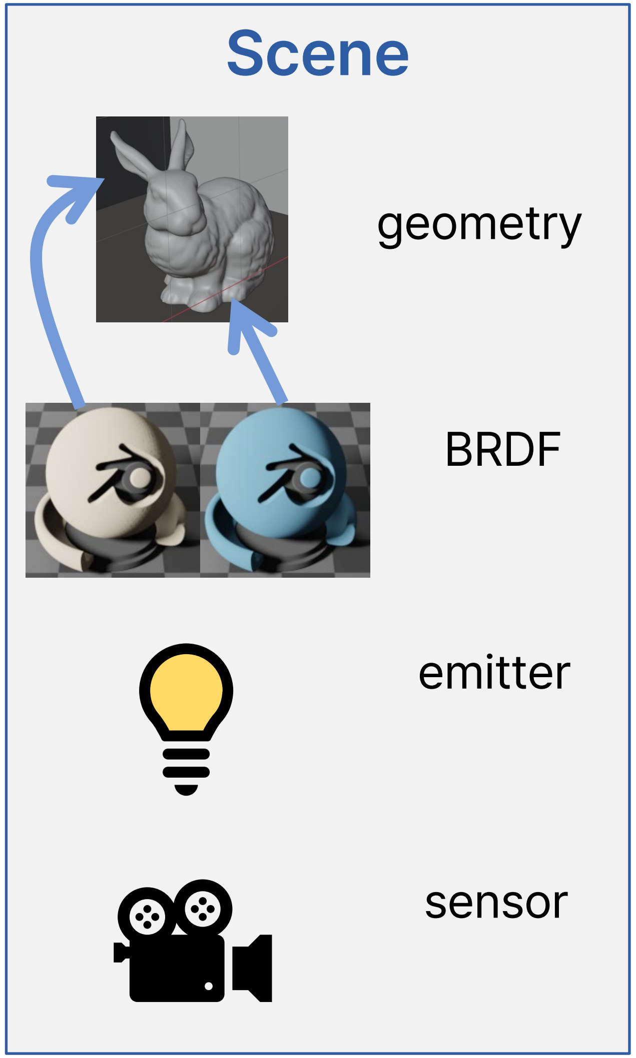


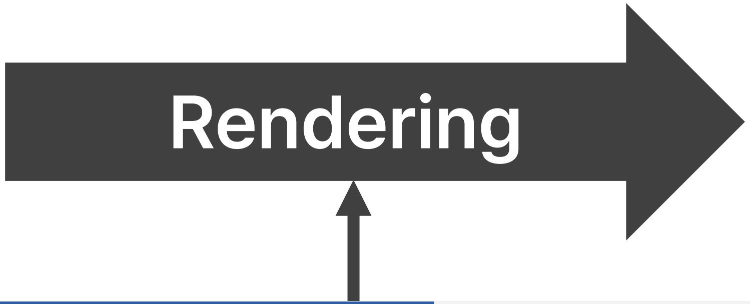
1. Radiometry and Light transport

Physically Based Rendering

Shinyoung Yi (이신영)

Rendering

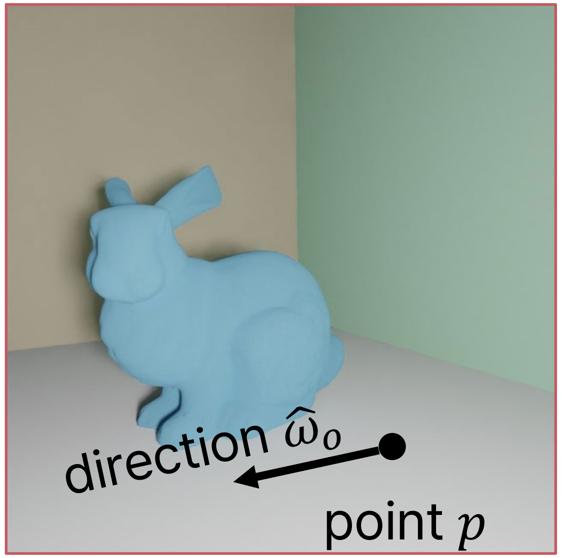




Rendering equation ?

Find scalar $\boxed{\text{radiance } L(p, \hat{\omega})}$
at any point p, along any direction $\hat{\omega}$

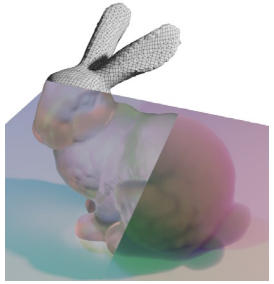
Image $\mathbf{I} \in \mathbb{R}^{H \times W}$



Q. With which physical quantities can we describe intensity of *light*?
→ radiometric quantities

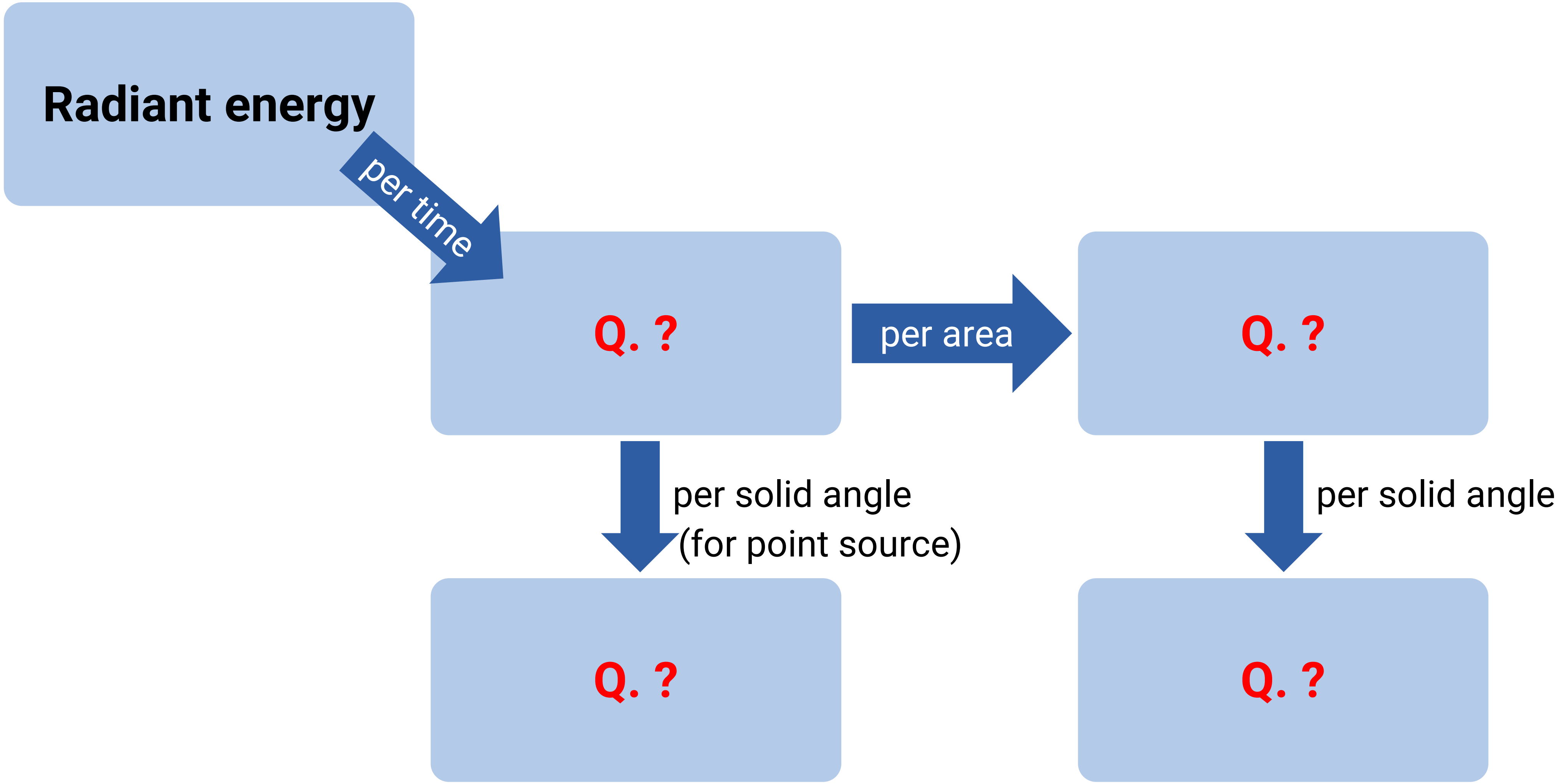
Q. How can we describe *material*?
→ BRDF

Q. How can we evaluate (simulate) *interactions* of light and material?
→ Rendering equation

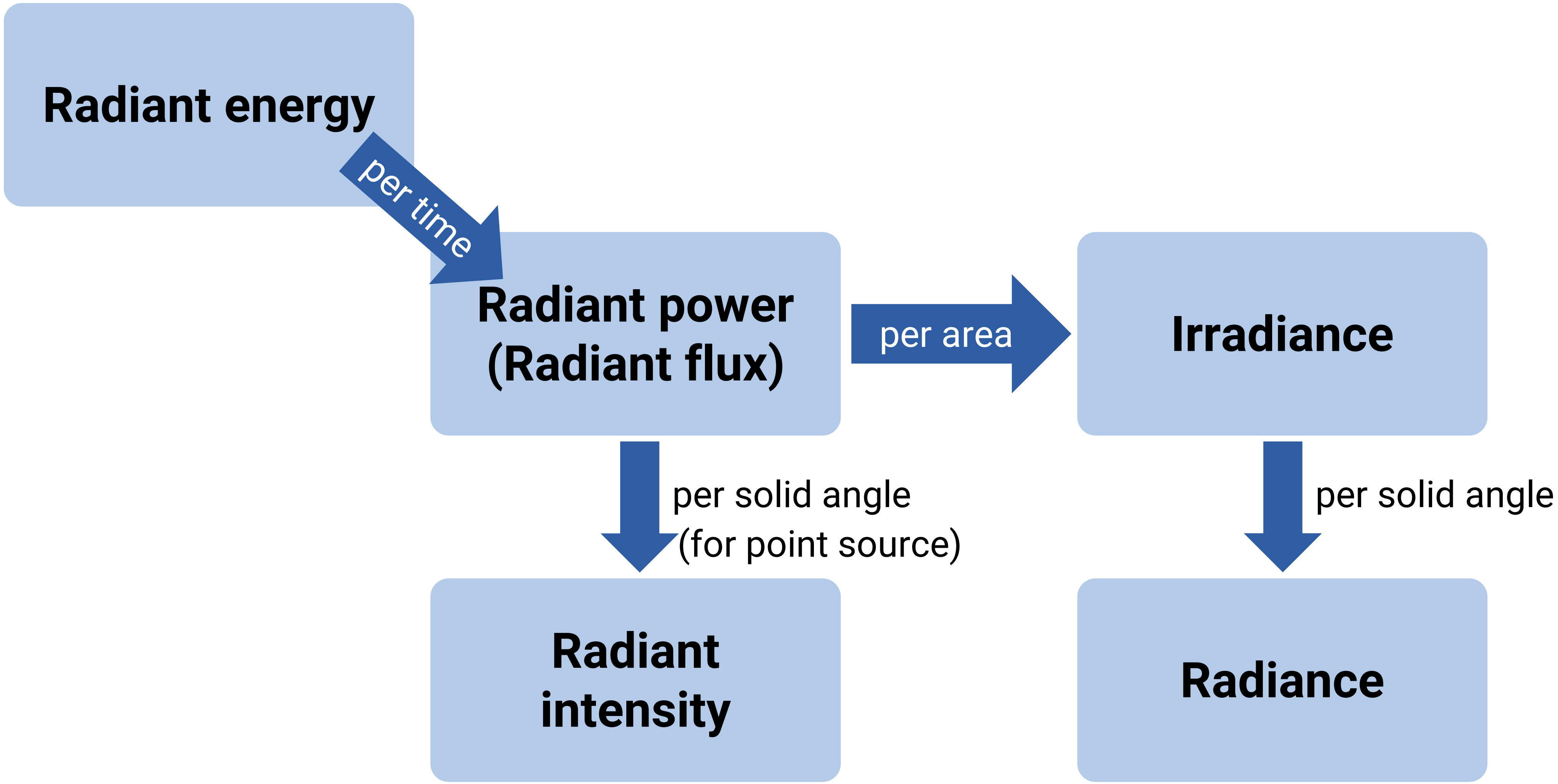


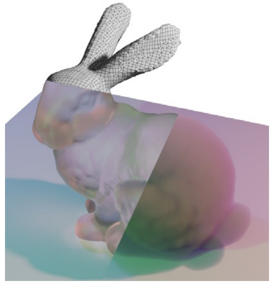
Quiz & Preview for Radiometric Quantities

Radiometric quantities



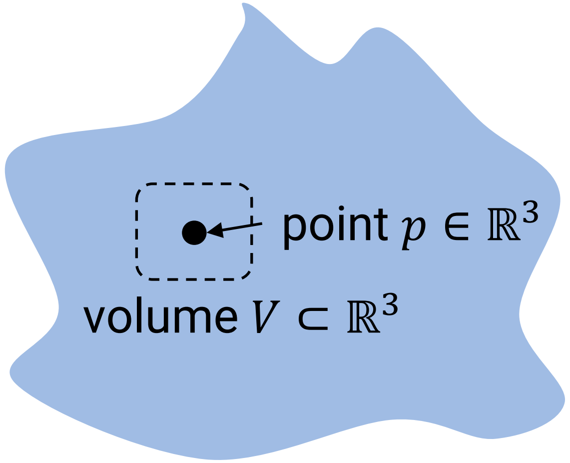
Radiometric quantities





Mass vs. Density

Concepts of mass vs. density



Object O

Mass | per volume → | **Density**

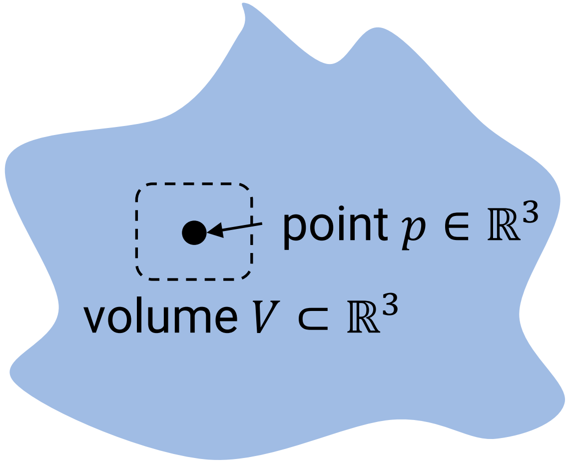
Mass

- ✓ Mass of the object O
- ✓ Mass of some region (volume) V
- ✗ Mass at the point p
  → illegal or meaningless (always zero)

Density

- ✗ Density of the object O
  → illegal or "average density" of the object O
- ✗ Density of some region (volume) V
  → illegal or "average density" of the volume V
- ✓ Density at the point p

Concepts of mass vs. density



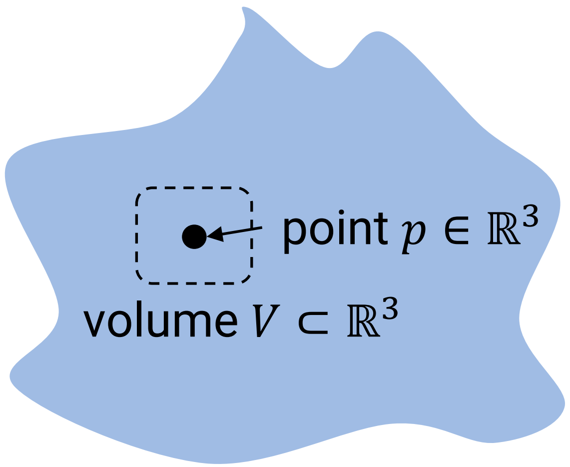
Mass of *what?* $V \subset \mathbb{R}^3$

- "mass of an object O"

  = "mass of the volume of O"

Density of *what?* $p \in \mathbb{R}^3$

Concepts of mass vs. density



Mass of $V \subset \mathbb{R}^3$

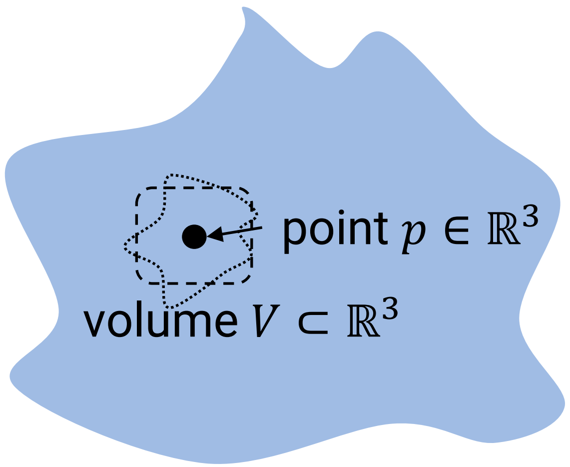
Density of $p \in \mathbb{R}^3$

$$\mathrm{mass}(V)$$

per volume

$$\mathrm{density}(p) = \lim_{\substack{\mathrm{vol}(V)\to 0 \\ p\in V}} \frac{\mathrm{mass}(V)}{\mathrm{vol}(V)}$$

Concepts of mass vs. density



Mass of $V \subset \mathbb{R}^3$

Density of $p \in \mathbb{R}^3$

$$\text{mass}(V)$$

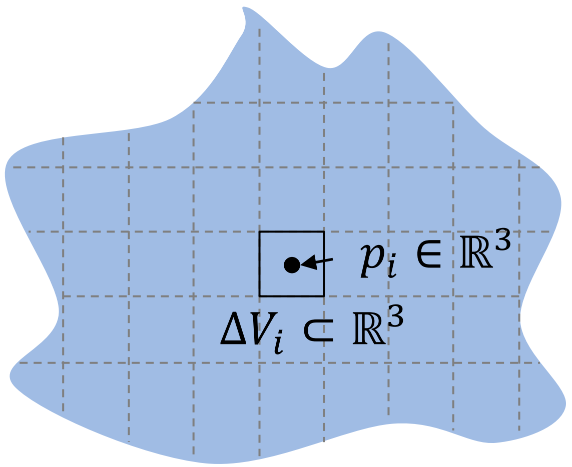
per volume

$$\text{density}(p) = \lim_{\substack{\text{vol}(V)\to 0 \\ p \in V}} \frac{\text{mass}(V)}{\text{vol}(V)}$$

$$= \lim_{\substack{\text{vol}(V)\to 0 \\ p \in V}} \frac{\text{mass}(V)}{\text{vol}(V)}$$

* The limit converges to the same value whenever $\text{vol}(V) \to 0$ and $p \in V$.

Concepts of mass vs. density



Mass of $V \subset \mathbb{R}^3$

Density of $p \in \mathbb{R}^3$

$$\text{mass}(V) = \sum_i \text{mass}(\Delta V_i)$$

$$\approx \sum_i \text{density}(p_i)\,\text{vol}(\Delta V_i)$$

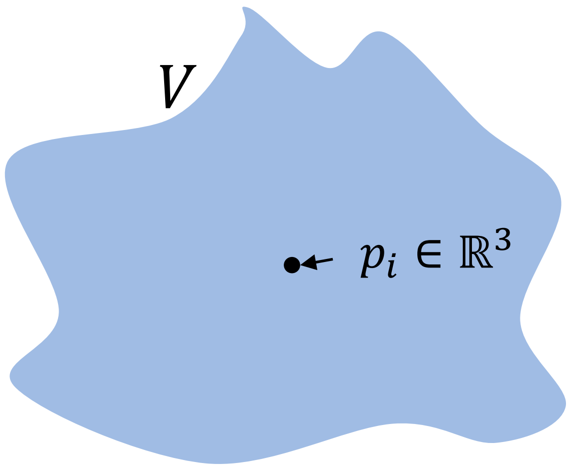
$$\int_V \text{density}(p)\,dp$$

$\text{vol}(\Delta V_i) \to 0$

over volume

$$\boxed{\text{density}(p) = \lim_{\substack{\text{vol}(V)\to 0 \\ p\in V}} \frac{\text{mass}(V)}{\text{vol}(V)}}$$

Concepts of mass vs. density



Mass of $V \subset \mathbb{R}^3$ [kg]

Density at $p \in \mathbb{R}^3$ [kg/m^3]

Proposition

Relations between mass m of a volume $V \subset \mathbb{R}^3$ and density ρ at a point $p \in \mathbb{R}^3$ is that:

$$m(V) = \int_V \rho(p) \mathrm{d}p$$

per volume →

← over volume

$$\rho(p) = \lim_{\substack{|V| \to 0 \\ p \in V}} \frac{m(V)}{|V|}$$

Concepts of mass vs. density

Notation comparison

Other text often write $\frac{\mathrm{d}m}{\mathrm{d}V}$ instead of $\lim_{|V|\to 0, p\in V} \frac{m(V)}{|V|}$ but the former notation may give a misunderstanding that m is a function of a real number (volume measure) rather than one of a subset of $\mathbb{R}^3$ (volume region). The formula $\frac{\mathrm{d}m}{\mathrm{d}V}$ can be correctly understood only if it denoted a Radon-Nikodym derivative, which is dealt in *measure theory* (4th grade in Math. major).
We do not assume measure theory as a prerequisite, so we use the latter notation $\lim_{|V|\to 0, p\in V} \frac{m(V)}{|V|}$ for explicitness.

Proposition

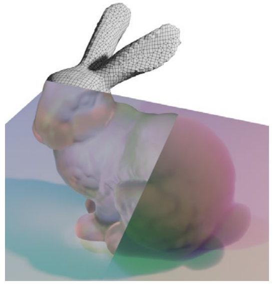
Relations between mass m of a volume $V \subset \mathbb{R}^3$ and density ρ at a point $p \in \mathbb{R}^3$ is that:

$$m(V) = \int_V \rho(p)\mathrm{d}p$$

per volume

over volume

$$\rho(p) = \lim_{\substack{|V|\to 0 \\ p\in V}} \frac{m(V)}{|V|}$$



Solid Angles

Solid angles

We roughly say…

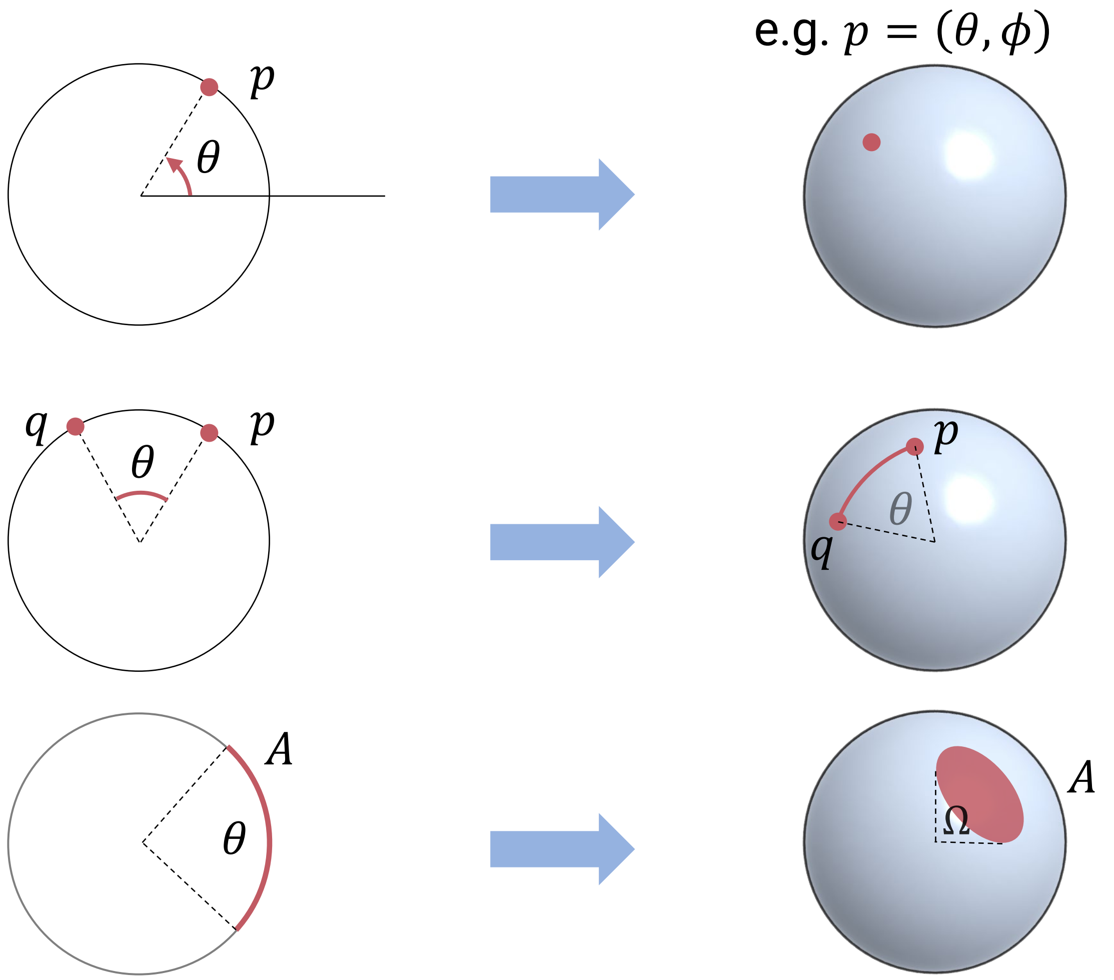
"Solid angles" are
3D versions of "angles"

How strictly is this sentence correct?

Solid angles

Can following statements be converted to sphere ($\mathbb{S}^2$) version using "solid angles"?

1. The position of a point p in the unit circle ($\mathbb{S}^1$) can be represented as the angle θ
   two coordinates such as (θ, ϕ)
2. How far apart two points p and $q \in \mathbb{S}^1$ can be represented as the angle θ between them.
   angle θ
3. The size of a region $A \subset \mathbb{S}^1$ can be represented as the angle θ
   solid angle Ω



Solid angles

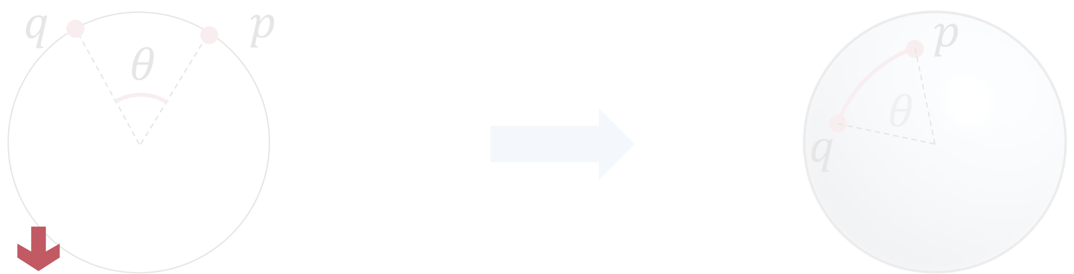
Can following statements be converted to sphere ($\mathbb{S}^2$) version using "solid angles"?

e.g. $p = (\theta, \phi)$

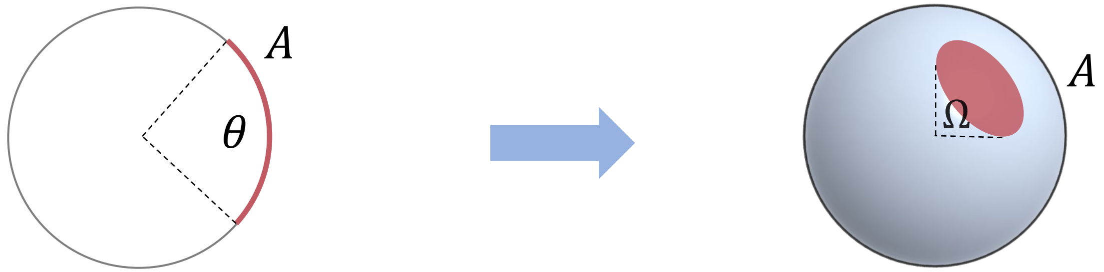
1. The position of a point p in the $\boxed{\text{unit circle } (\mathbb{S}^1)}$ can be represented as the $\boxed{\text{angle } \theta}$
   two coordinates such as (θ, ϕ)

"Solid angles" are only relevant to 3.

2. How far apart two points p and $q \in \boxed{\mathbb{S}^1}$ can be represented as the $\boxed{\text{angle } \theta}$ between them.
   angle θ



3. The size of a region $A \subset \boxed{\mathbb{S}^1}$ can be represented as the $\boxed{\text{angle } \theta}$
   solid angle Ω



Solid angles

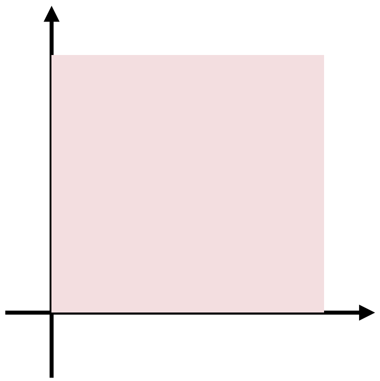
In many times,

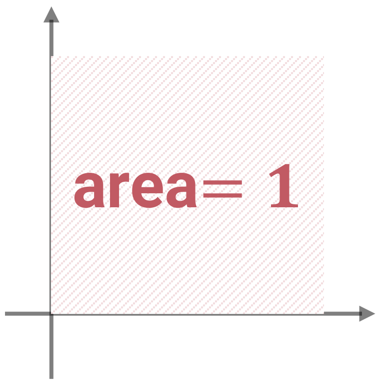
several concepts can be treated
as a single concept in lower dimensions,

but they become different in higher dimensions

Solid angles

We call the both "area". (similarly for "volume")





a 2-dimensional *subset*

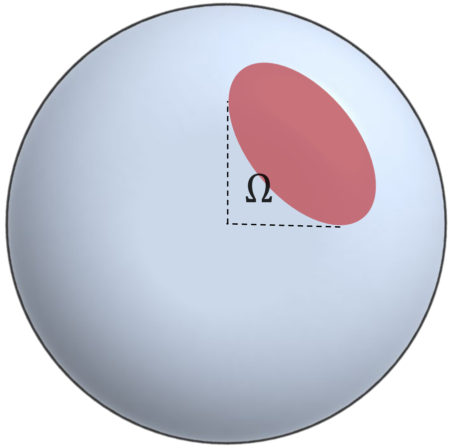
$A = \{(x, y) \in \mathbb{R}^2 | 0 \leq x, y \leq 1\}$

measure of a 2-dimensional subset

$|A| = 1$

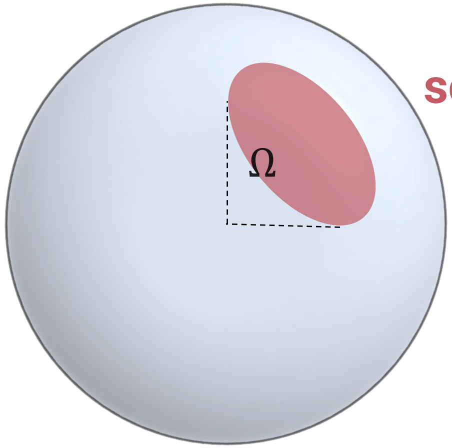
Solid angles

Also for “solid angle”



a spherical *subset*

$$\Omega = \{\hat{\omega} \in \mathbb{S}^2 | \hat{u} \cdot \hat{\omega} \geq 0.7\}$$

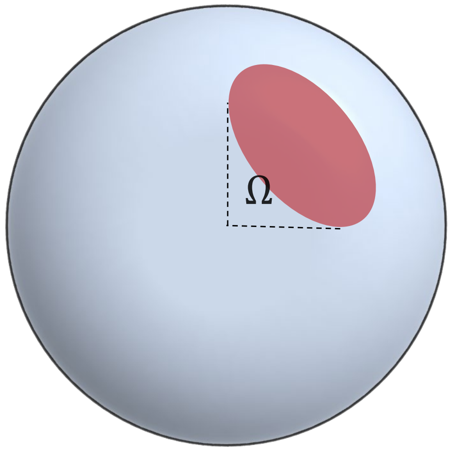


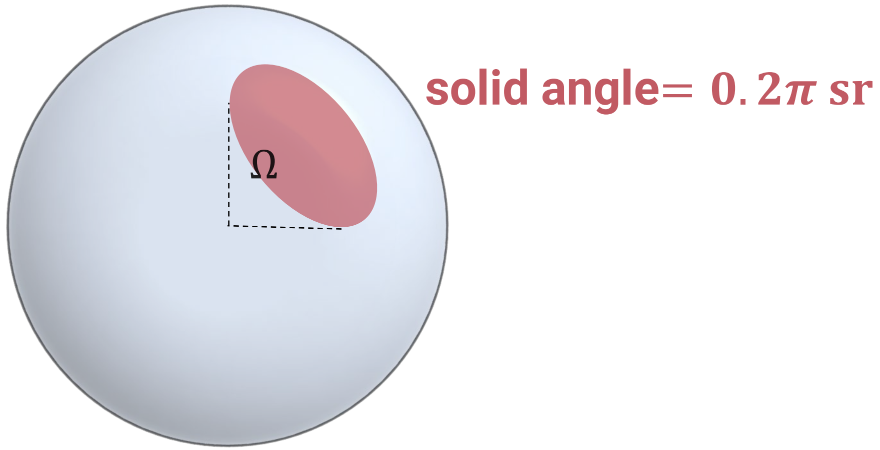
measure of a spherical subset

$$|\Omega| = 1$$

Solid angles

(not common) terminology in this seminar





a spherical *subset*

$\Omega = \{\hat{\omega} \in \mathbb{S}^2 | \hat{u} \cdot \hat{\omega} \geq 0.7\}$

solid angle *region*

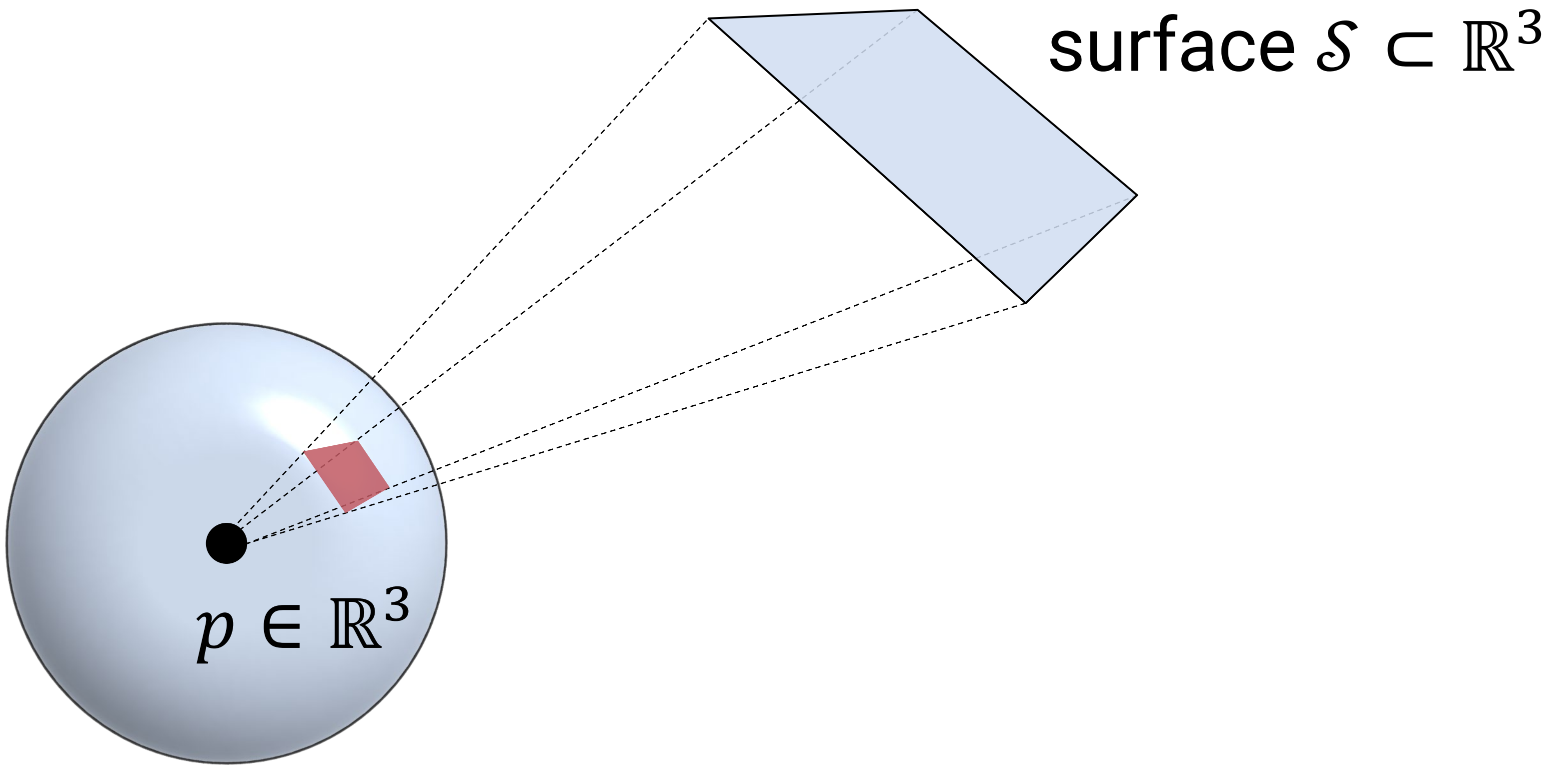
measure of a spherical subset

$|\Omega| = 1$

solid angle *measure*

Surface to solid angle

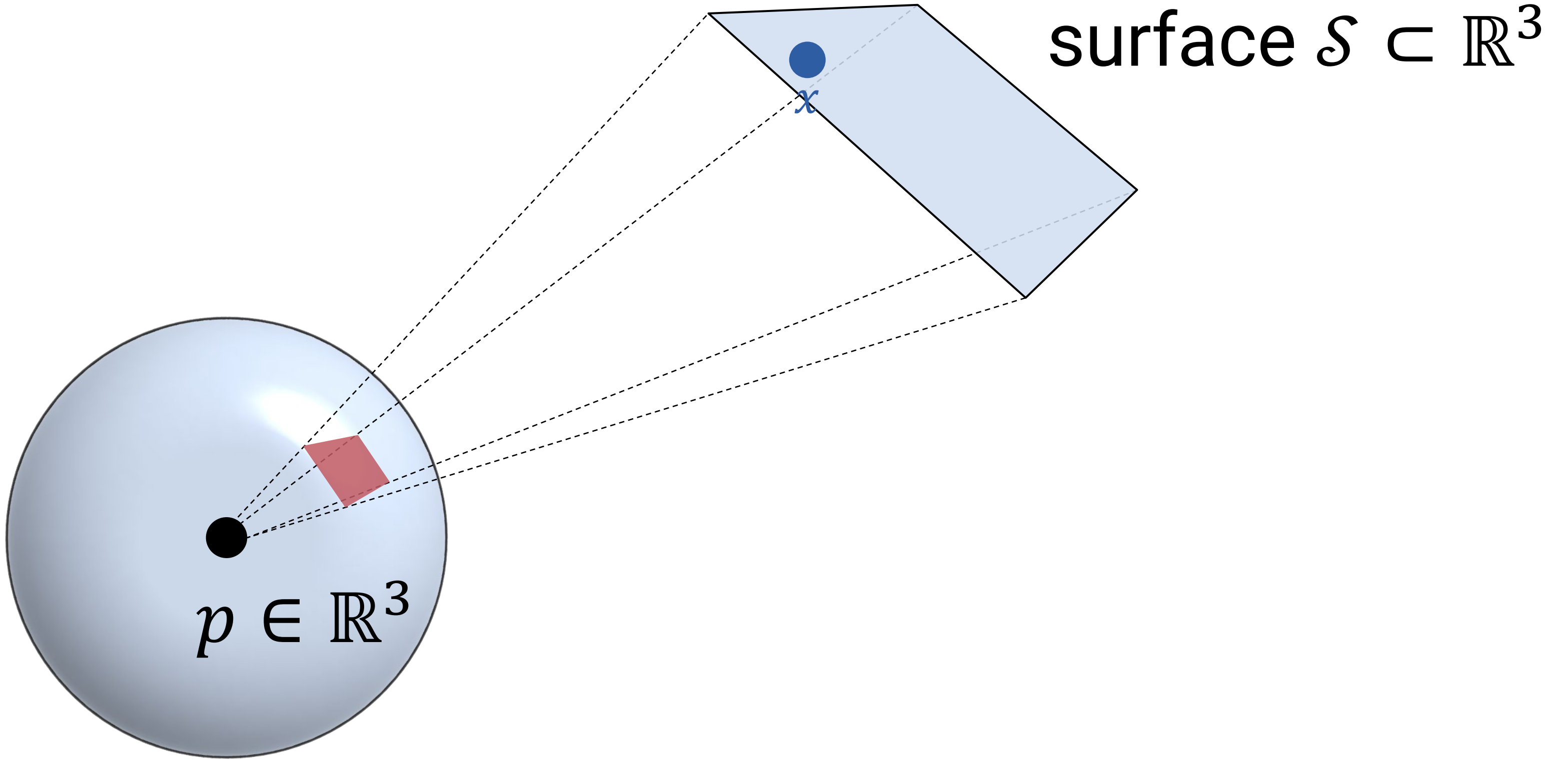
How large $\mathcal{S}$ appears to and observer at p?



Solid angle!

Surface to solid angle

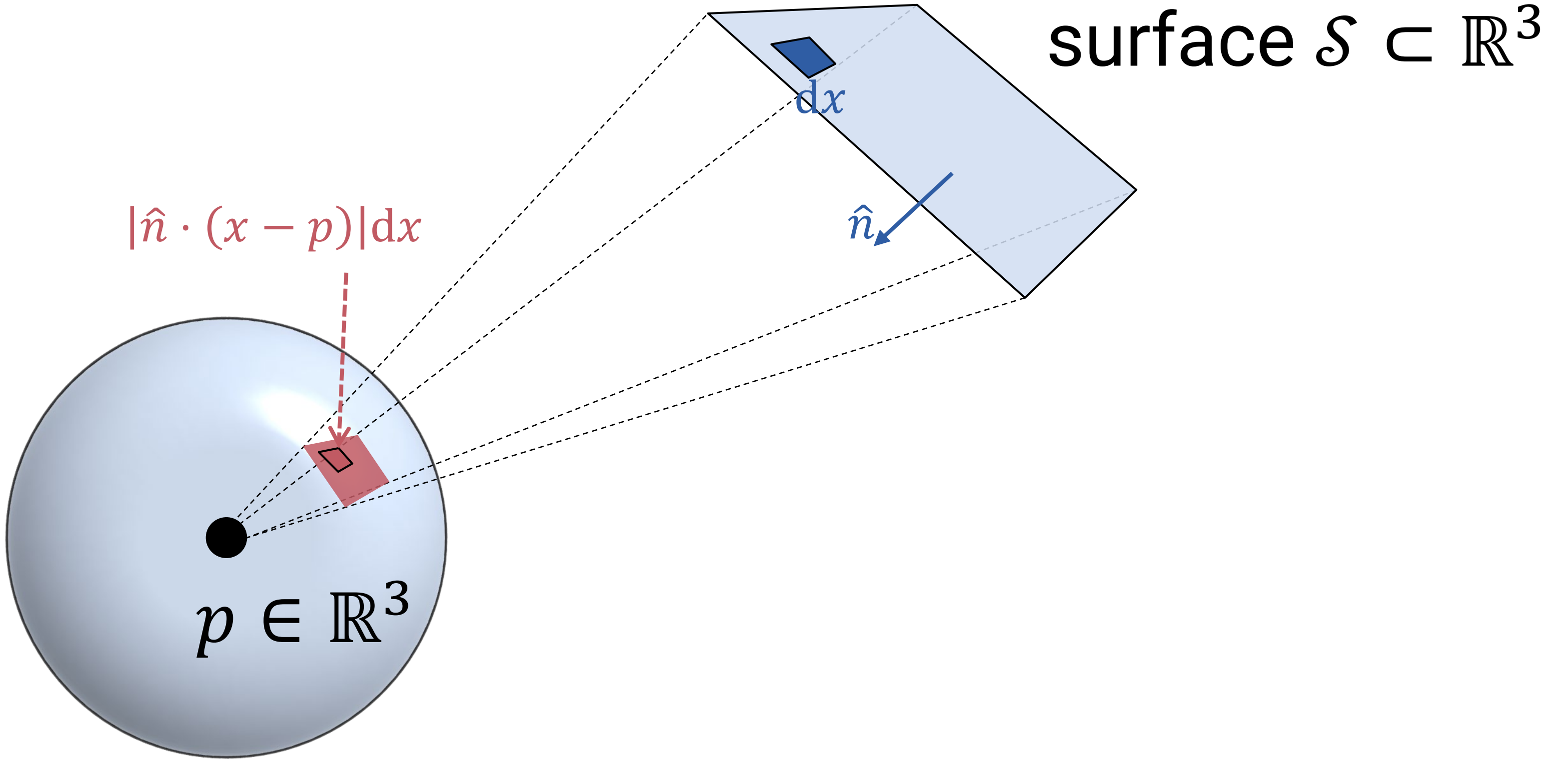
Try to write in an area integral on $\mathcal{S}$



Solid angle= $\int_{\mathcal{S}} \; ? \;\; \mathrm{d}x$

Surface to solid angle

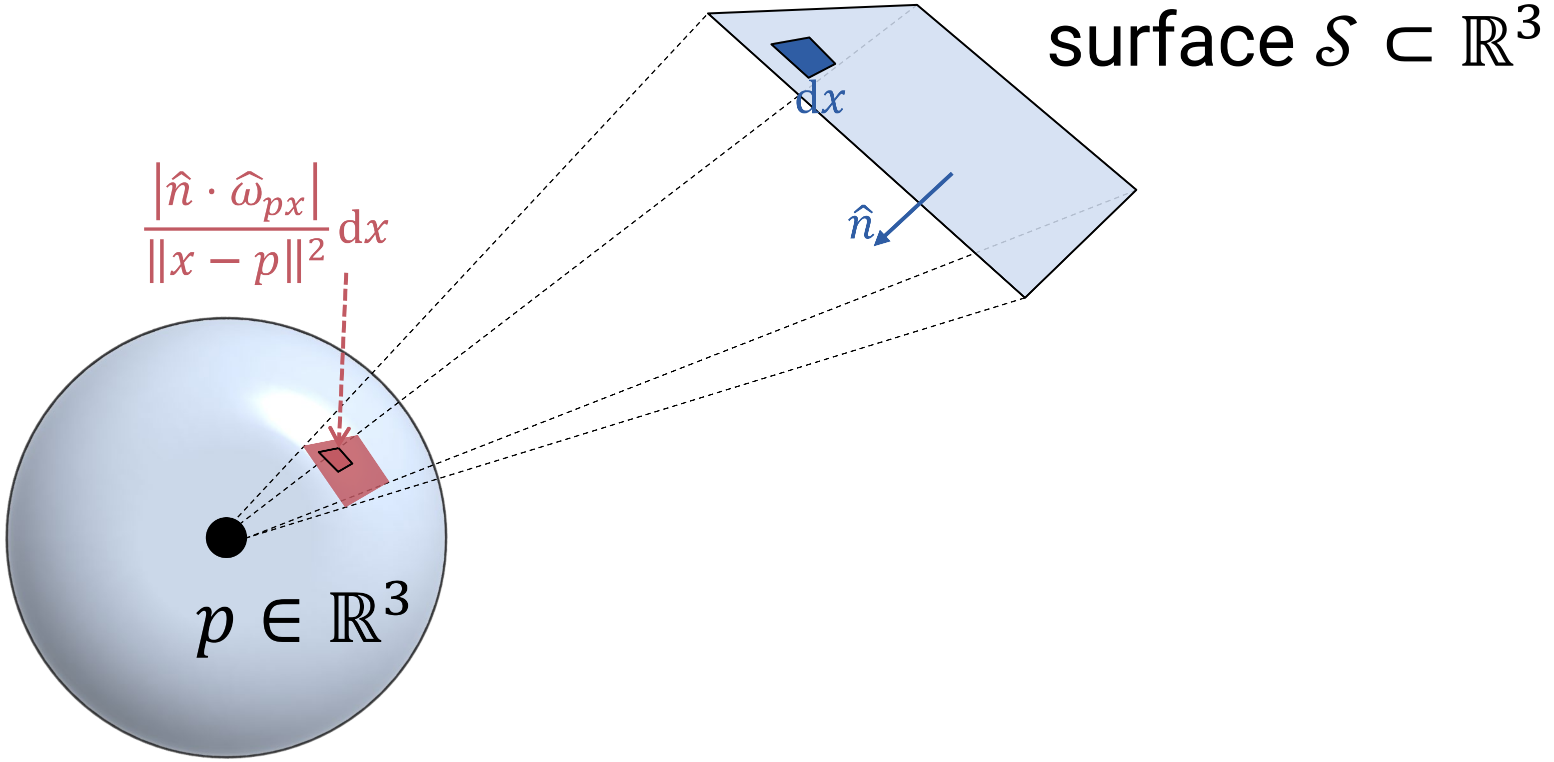
Try to write in an area integral on $\mathcal{S}$



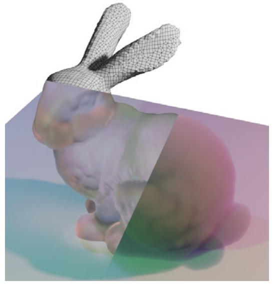
Solid angle= $\int_{\mathcal{S}} \; ? \;\; \mathrm{d}x$

Surface to solid angle

Try to write in an area integral on $\mathcal{S}$

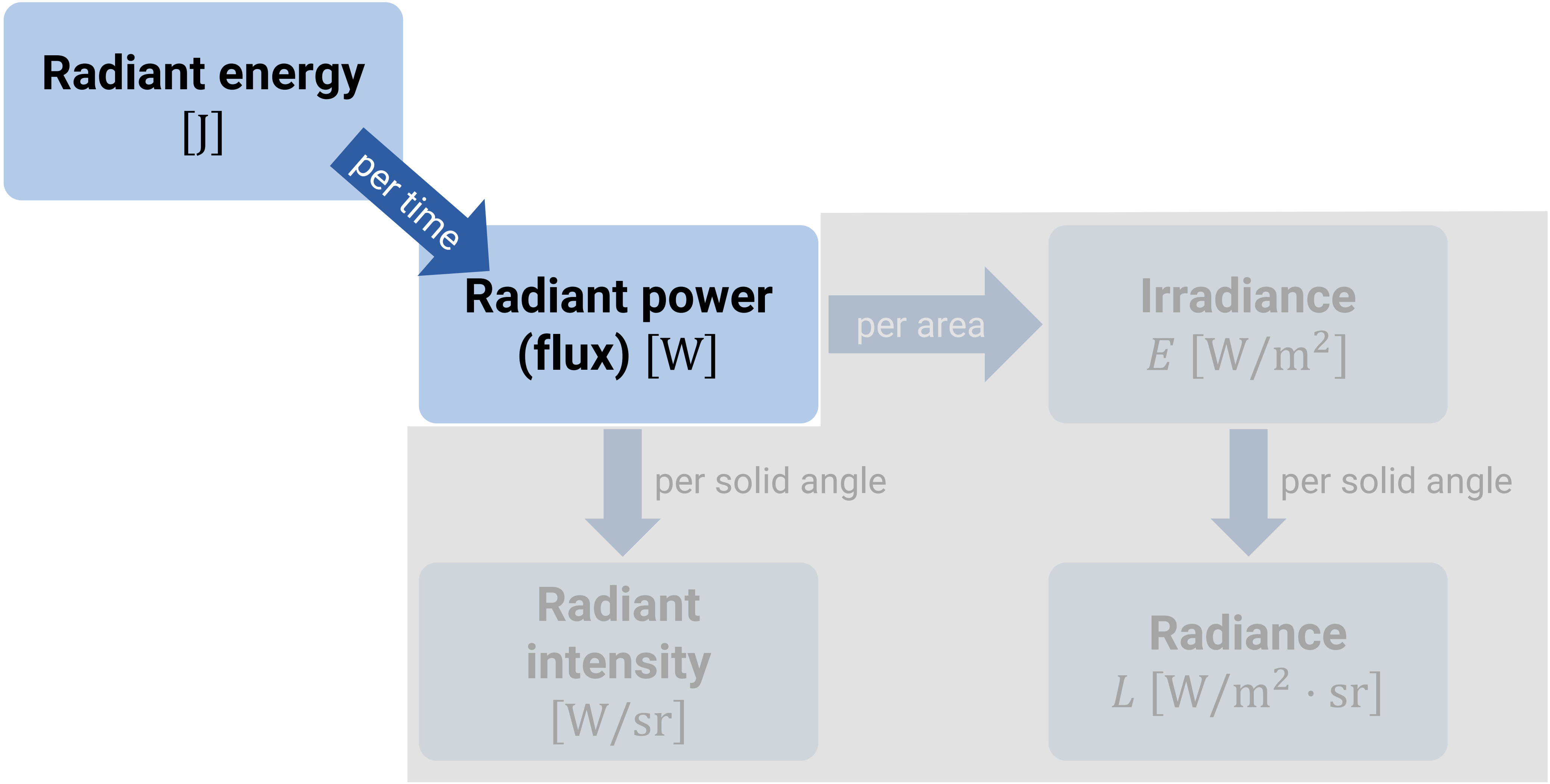


Solid angle= $\displaystyle\int_{\mathcal{S}} \frac{|\hat{n} \cdot \widehat{\omega}_{px}|}{\|x-p\|^2} \mathrm{d}x$



Radiometric Quantities

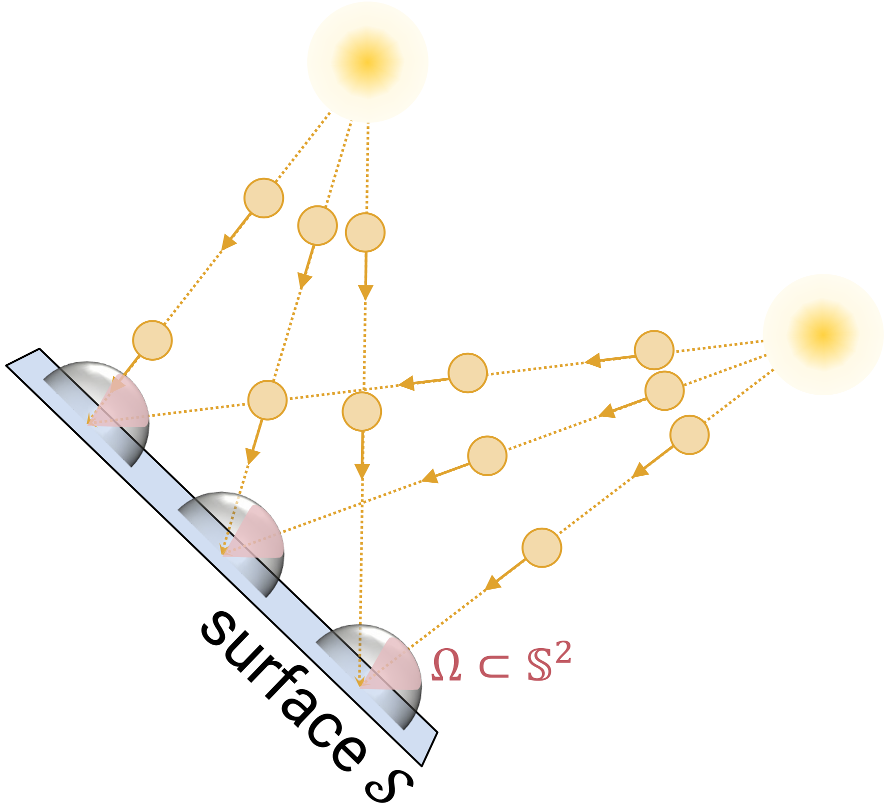
Radiometric quantities



Radiant energy

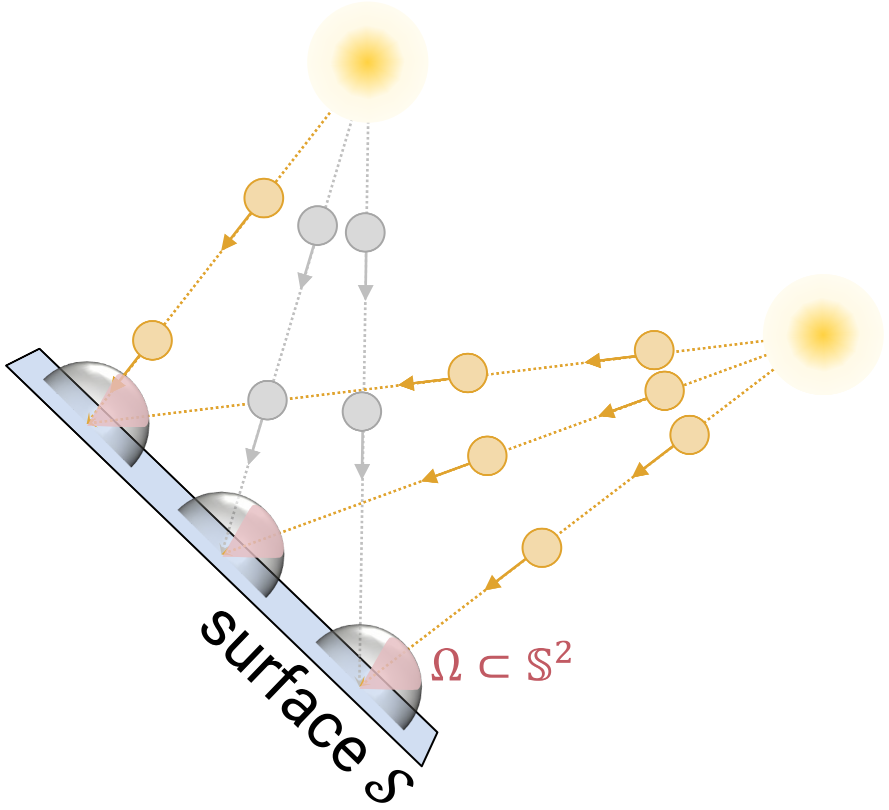
Radiant energy **of what?** surface $S \subset \mathbb{R}^3$,

solid angle $\Omega \subset \mathbb{S}^2$,



- "Energy" is "energy"!
- *Number of "hits" of photons on the surface*

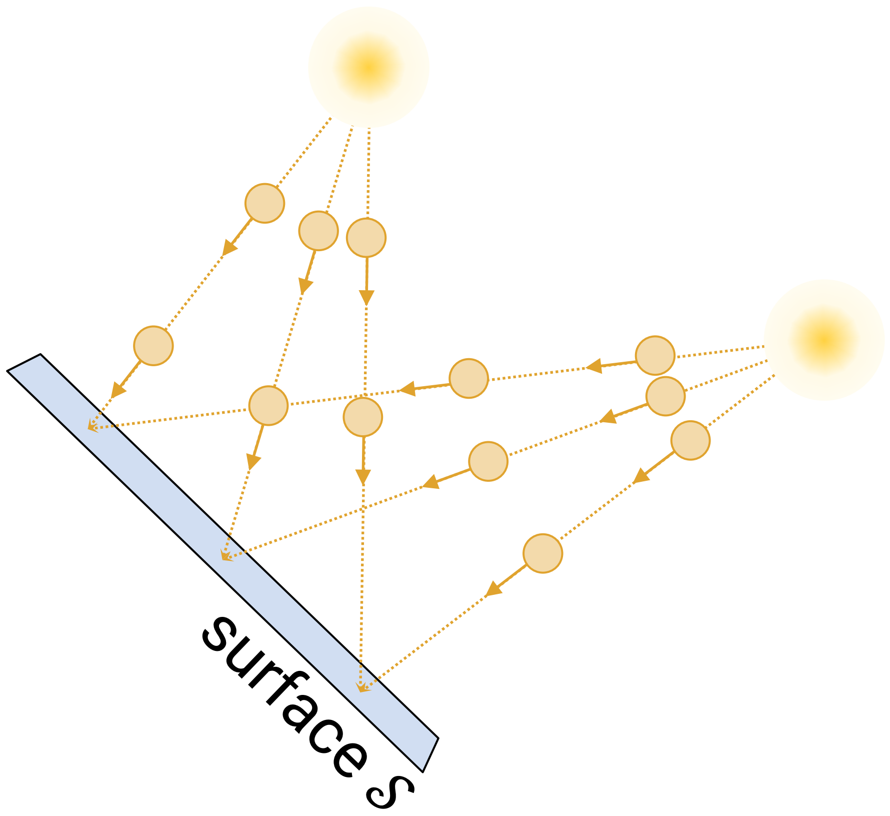
Radiant energy



Radiant energy of surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$,
time interval $[t_1, t_2] \subset \mathbb{R}$

- “Energy” is “energy”!
- *Number of “hits” of photons on the surface*

Radiant energy

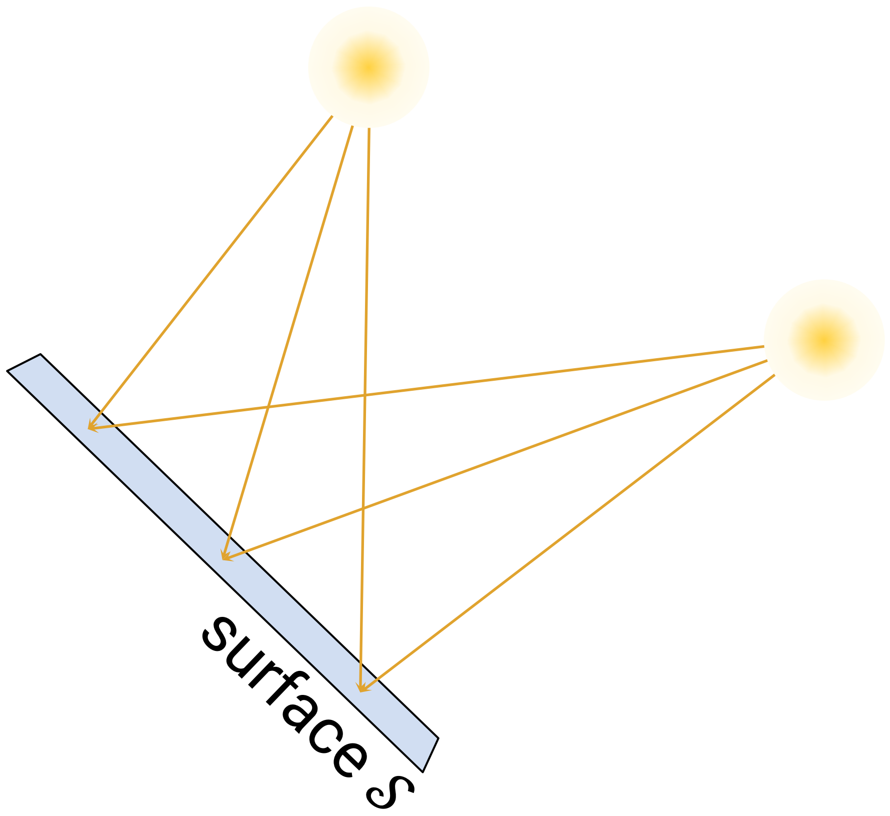


Radiant energy of surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \overset{\text{default}}{\in} \mathbb{S}^2$,
time interval $[t_1, t_2] \subset \mathbb{R}$

$$Q(\mathcal{S}, \Omega, [t_1, t_2]) \quad [\mathrm{J}]$$

- "Energy" is "energy"!
- *Number of "hits" of photons on the surface*

Radiant flux (radiant power)



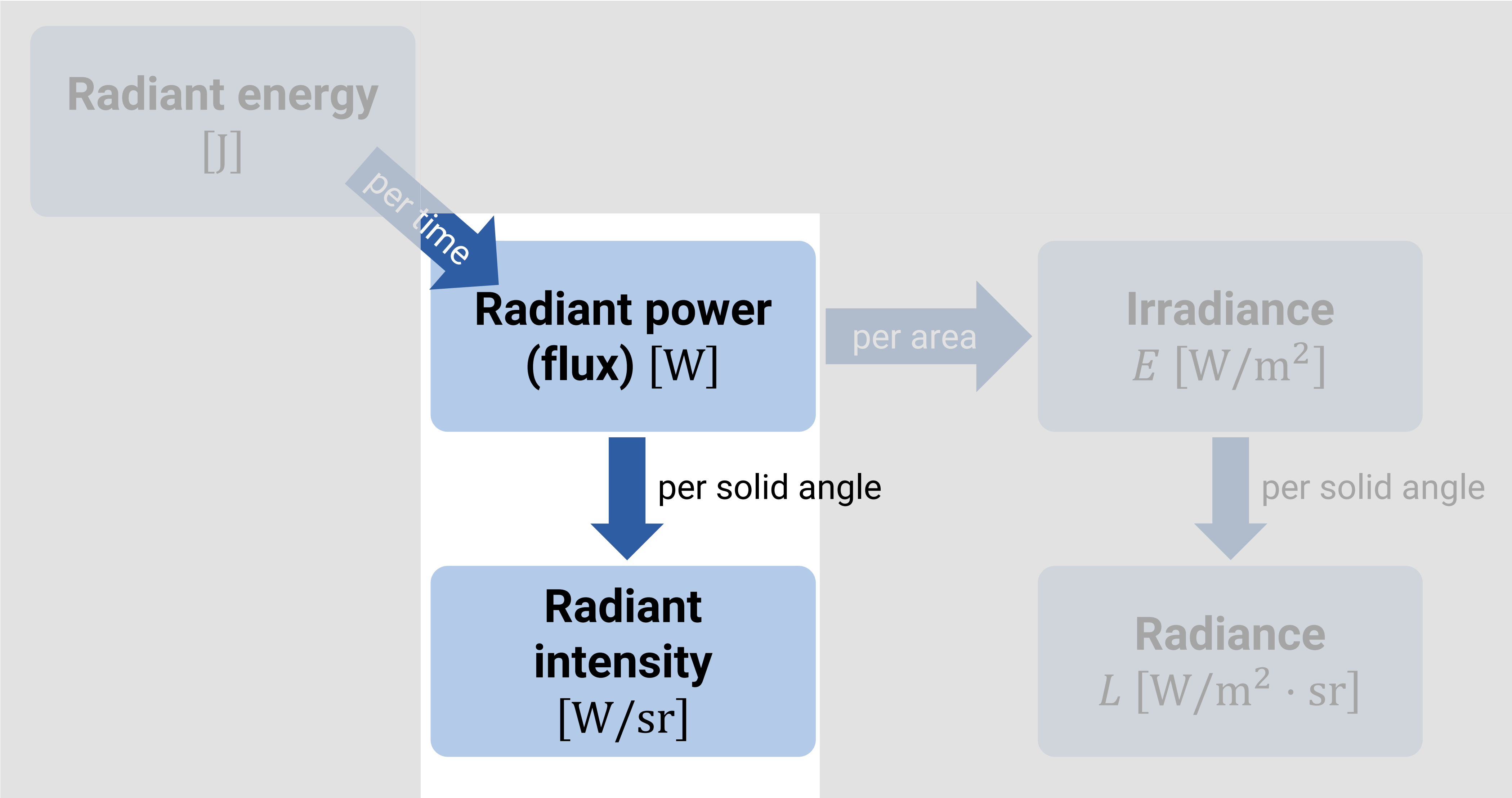
Radiant flux **of what?**
(Radiant power)

surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$
time $t \in \mathbb{R}$ (steady state)

$$\Phi(\mathcal{S}, \Omega, t) \quad [\mathrm{J/s} = \mathrm{W}]$$

- "Power" is "energy per time"!
- *Number of "intersecting rays" on the surface*

Radiant intensity

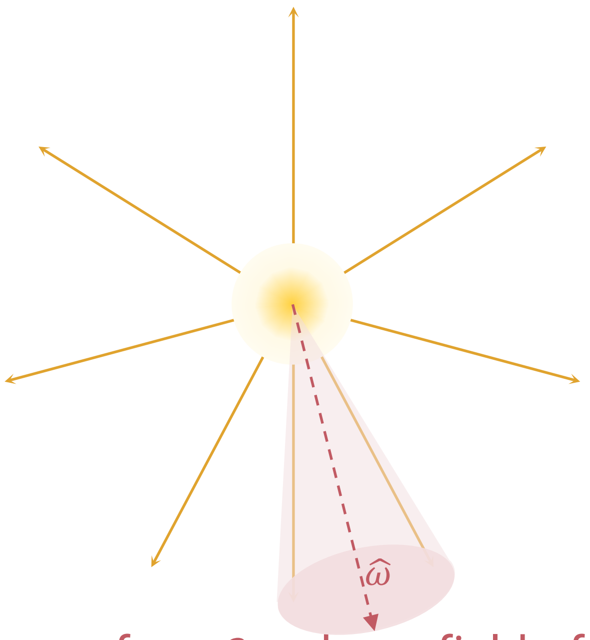


Radiant intensity

Radiant flux of surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$

per solid angle

Radiant intensity of point source, $\mathbb{R}^3$, ?
direction $\hat{\omega}$



surface $\mathcal{S}_\Omega$ whose field of view from the point source is $\Omega \subset \mathbb{S}^2$

Proposition

Relations between radiant flux Φ of a surface $\mathcal{S} \subset \mathbb{R}^3$ and radiant intensity of a point source at a position $p \in \mathbb{R}^3$ is that:

$$\Phi(\mathcal{S}_\Omega) = \int_\Omega I(\hat{\omega})\mathrm{d}\hat{\omega}$$

per solid angle →

← over solid angle

$$I(\hat{\omega}) = \lim_{\substack{|\Omega| \to 0 \\ \hat{\omega} \in \Omega}} \frac{\Phi(\mathcal{S}_\Omega)}{|\Omega|},$$

solid angle measure

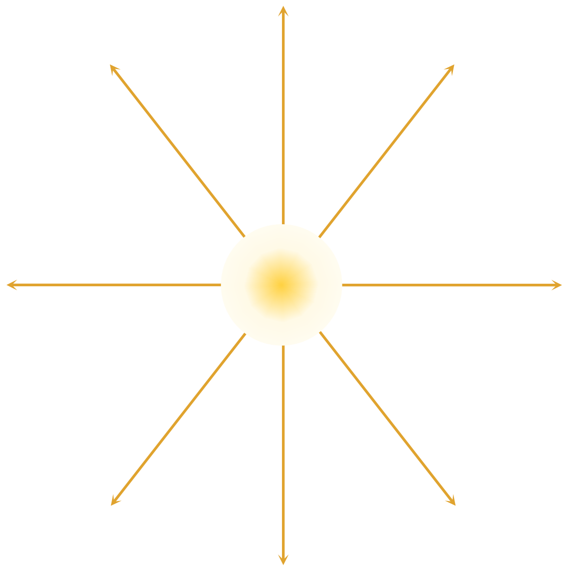
where Ω is the solid angle region

Radiant intensity

Radiant flux **of** surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$

per solid angle

Radiant intensity **of** point source,
direction $\hat{\omega}$



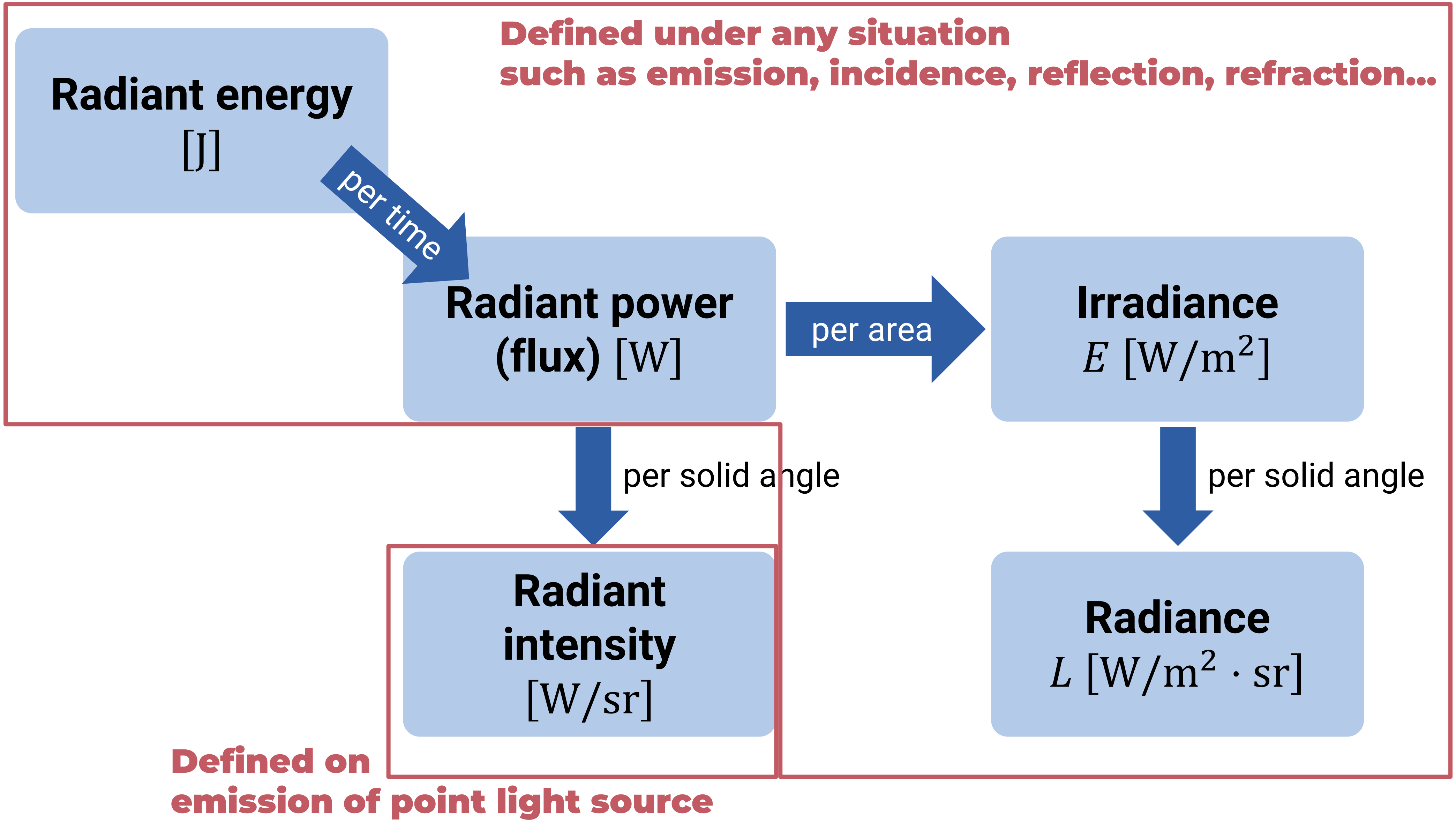
Practice

There is an isotropic point light source with radiant flux Φ.

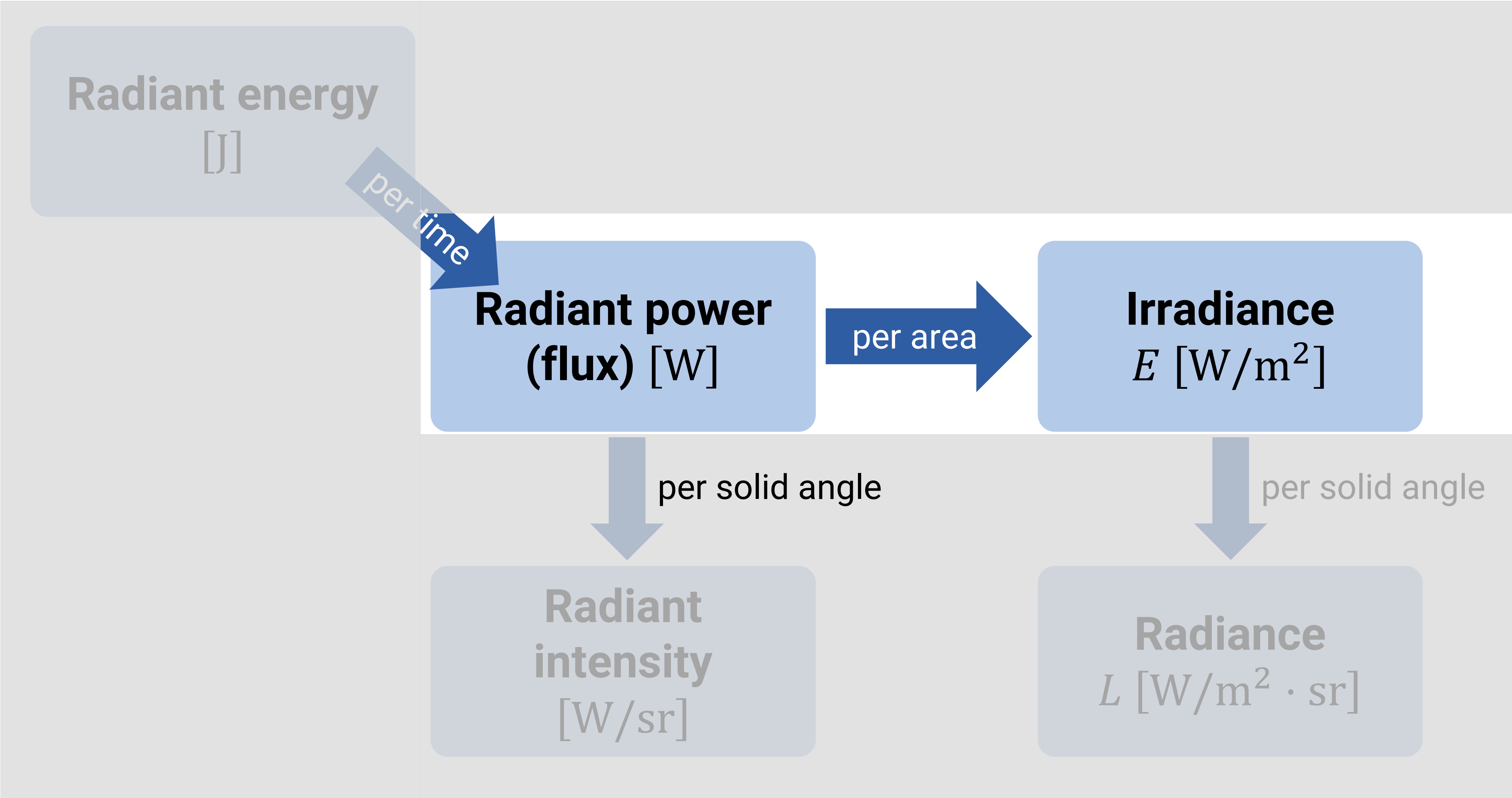
The radiant intensity of the source is?

$$I(\hat{\omega}) = \frac{\Phi}{4\pi}$$

Radiant intensity



Irradiance

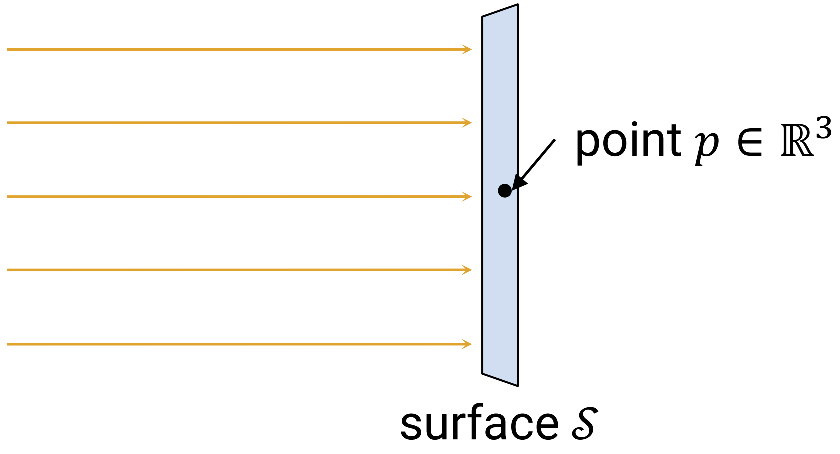

Radiant energy
[J]
per time
Radiant power
(flux) [W]
per area
Irradiance
E [W/m^2]
per solid angle
per solid angle
Radiant
intensity
[W/sr]
Radiance
L [W/m$^2 \cdot$ sr]

Irradiance

Radiant flux **of** surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$

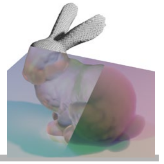
per area

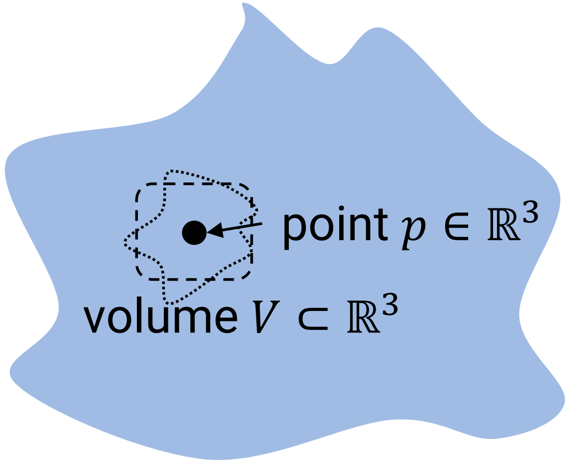
irradiance **of** point $p \in \mathbb{R}^3$, **?**
solid angle $\Omega \subset \mathbb{S}^2$



? $$E(p, \Omega) = \lim_{\substack{\text{area}(\mathcal{S}) \to 0 \\ p \in \mathcal{S}}} \frac{\Phi(\mathcal{S}, \Omega)}{\text{area}(\mathcal{S})}$$ **is it enough?**

Review: Concepts of mass vs. density





Mass of $V \subset \mathbb{R}^3$

Density of $p \in \mathbb{R}^3$

$\mathrm{mass}(V)$

per volume

$$\mathrm{density}(p) = \lim_{\substack{\mathrm{vol}(V)\to 0\\ p\in V}} \frac{\mathrm{mass}(V)}{\mathrm{vol}(V)}$$

$$= \lim_{\substack{\mathrm{vol}(V)\to 0\\ p\in V}} \frac{\mathrm{mass}(V)}{\mathrm{vol}(V)}$$

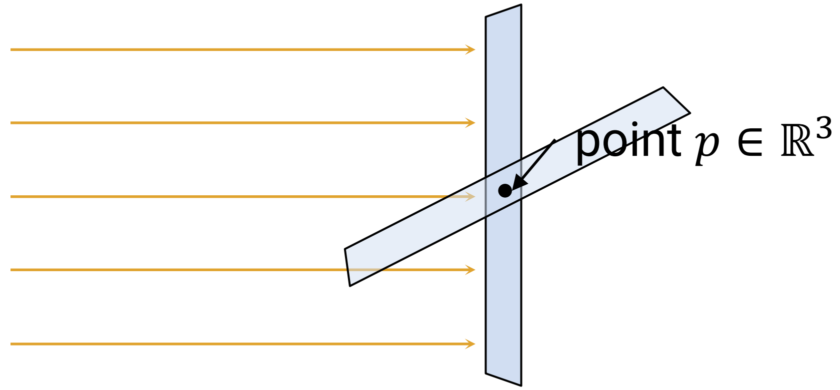
* **The limit converges to the same value whenever** $\mathrm{vol}(V) \to 0$ **and** $p \in V$.

Irradiance

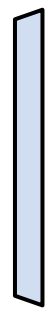
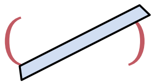
Radiant flux **of** surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$

per area

irradiance **of** point $p \in \mathbb{R}^3$, **?**
solid angle $\Omega \subset \mathbb{S}^2$



? $$E(p, \Omega) = \lim_{\substack{\text{area}(\mathcal{S}) \to 0 \\ p \in \mathcal{S}}} \frac{\Phi(\mathcal{S}, \Omega)}{\text{area}(\mathcal{S})}$$ **is it enough?**

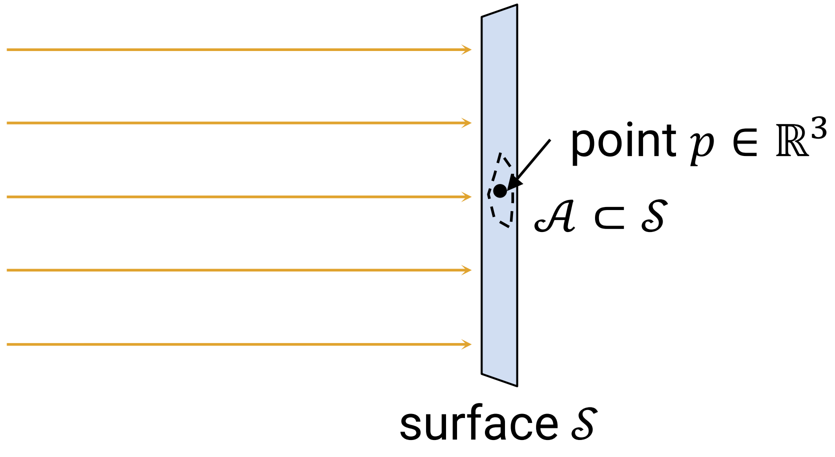
$\mathcal{S} = ($  $)$ and $\mathcal{S} = ($  $)$ yield different limits

Irradiance

Radiant flux **of** surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$

per area

irradiance **of** point $p \in \mathbb{R}^3$, **?**
solid angle $\Omega \subset \mathbb{S}^2$



$$E(p, \Omega) = \lim_{\substack{\text{area}(\mathcal{S}) \to 0 \\ p \in \mathcal{S}}} \frac{\Phi(\mathcal{S}, \Omega)}{\text{area}(\mathcal{S})}$$

Irradiance

Radiant flux **of** surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$

per area

irradiance **of** point $p \in \mathbb{R}^3$, ?
solid angle $\Omega \subset \mathbb{S}^2$

$$E_{\mathcal{S}}(p, \Omega) = \lim_{\substack{\text{area}(\mathcal{A}) \to 0 \\ p \in \mathcal{A} \subset \mathcal{S}}} \frac{\Phi(\mathcal{A}, \Omega)}{\text{area}(\mathcal{A})}$$

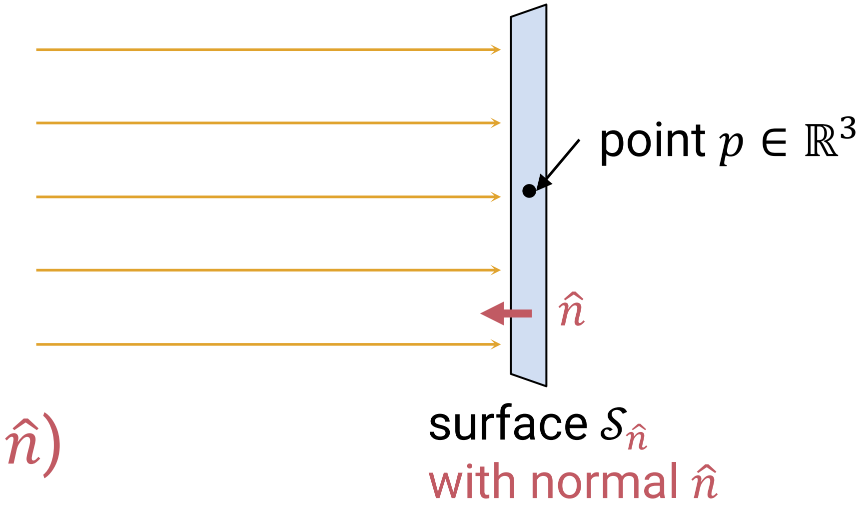
Irradiance defined as the limit about a subset of given fixed surface

Irradiance

Radiant flux **of** surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$

per area

irradiance **of** point $p \in \mathcal{S}$ (or $p \in \mathbb{R}^3$ and $\hat{n}$)
solid angle $\Omega \subset \mathbb{S}^2$



$$E_{\hat{n}}(p, \Omega) = \lim_{\substack{\mathrm{area}(\mathcal{S}_{\hat{n}}) \to 0 \\ p \in \mathcal{S}_{\hat{n}}}} \frac{\Phi(\mathcal{S}_{\hat{n}}, \Omega)}{\mathrm{area}(\mathcal{S}_{\hat{n}})}$$

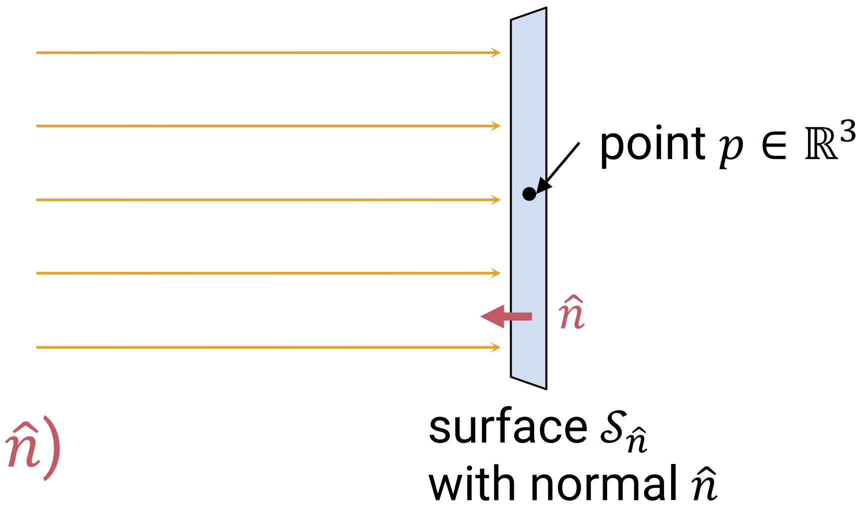
Irradiance defined as the limit about a subset of given fixed surface,
or a surface with given fixed normal

Irradiance

Radiant flux **of** surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$

per area

irradiance **of** point $p \in \mathcal{S}$ (or $p \in \mathbb{R}^3$ and $\hat{n}$)
solid angle $\Omega \subset \mathbb{S}^2$



$$E_{\hat{n}}(p, \Omega) = \lim_{\substack{\text{area}(\mathcal{S}_{\hat{n}}) \to 0 \\ p \in \mathcal{S}_{\hat{n}}}} \frac{\Phi(\mathcal{S}_{\hat{n}}, \Omega)}{\text{area}(\mathcal{S}_{\hat{n}})}$$

Irradiance defined as the limit about a subset of given fixed surface,
or a surface with given fixed normal

We don't say just "irradiance at p" when p is not on any surface

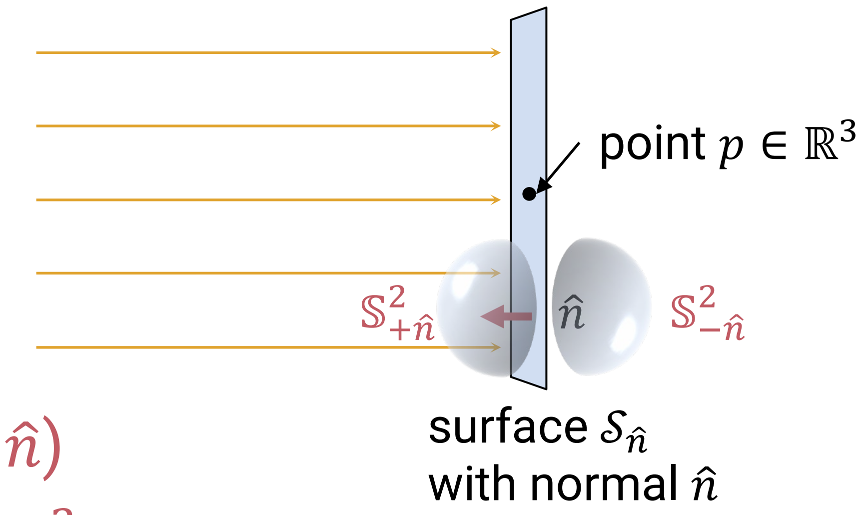
Irradiance

Radiant flux **of** surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$

$\Big\downarrow$ per area

irradiance **of** point $p \in \mathcal{S}$ (or $p \in \mathbb{R}^3$ and $\hat{n}$)

default: solid angle $\Omega = \mathbb{S}^2, \mathbb{S}^2_{+\hat{n}},$ or $\mathbb{S}^2_{-\hat{n}}$



$$E_{\hat{n}}(p, \Omega) = \lim_{\substack{\text{area}(\mathcal{S}_{\hat{n}}) \to 0 \\ p \in \mathcal{S}_{\hat{n}}}} \frac{\Phi(\mathcal{S}_{\hat{n}}, \Omega)}{\text{area}(\mathcal{S}_{\hat{n}})}$$

The default solid angle changes depending on the context

How?

Irradiance

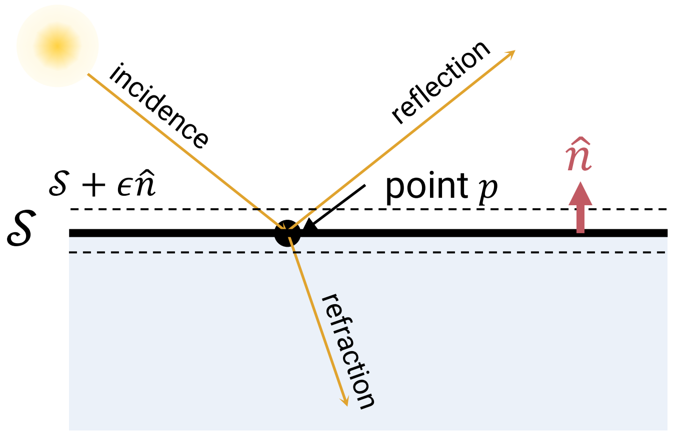
Radiant flux **of** surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$

per area

irradiance **of** point $p \in \mathcal{S}$ (or $p \in \mathbb{R}^3$ and $\hat{n}$)

default: solid angle $\Omega = \mathbb{S}^2, \mathbb{S}^2_{+\hat{n}}$, or $\mathbb{S}^2_{-\hat{n}}$

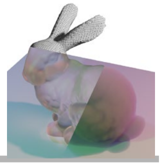
Some details depending on context…



- "incoming" irradiance: $E_{\mathcal{S}}^{(\mathrm{in})}(p) = E_{\mathcal{S}+\epsilon\hat{n}}\left(p, \mathbb{S}^2_{-\hat{n}}\right)$
- "reflected" irradiance: $E_{\mathcal{S}}^{(\mathrm{refl})}(p) = E_{\mathcal{S}+\epsilon\hat{n}}\left(p, \mathbb{S}^2_{+\hat{n}}\right)$
- "refracted" irradiance: $E_{\mathcal{S}}^{(\mathrm{refr})}(p) = E_{\mathcal{S}-\epsilon\hat{n}}\left(p, \mathbb{S}^2_{-\hat{n}}\right)$
- "outgoing" irradiance: $E_{\mathcal{S}}^{(\mathrm{out})}(p) = E_{\mathcal{S}}^{(\mathrm{refl})}(p) + E_{\mathcal{S}}^{(\mathrm{refr})}(p)$

Our intuition easily can do this!

Irradiance



Radiant flux **of** surface $\mathcal{S} \subset \mathbb{R}^3$, solid angle $\Omega \subset \mathbb{S}^2$

per area

irradiance **of** point $p \in \mathcal{S}$ (or $p \in \mathbb{R}^3$ and $\hat{n}$) solid angle $\Omega \subset \mathbb{S}^2$

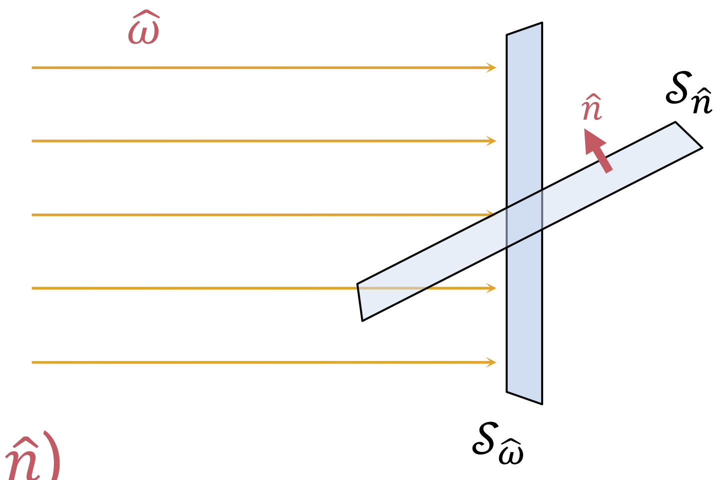
$$\Phi(\mathcal{S}, \Omega) = \int_{\mathcal{S}} E_{\mathcal{S}}(p, \Omega) \mathrm{d}p \quad \underset{\text{over area}}{\overset{\text{per area}}{\rightleftarrows}} \quad E_{\mathcal{S}}(p, \Omega) = \lim_{\substack{\operatorname{area}(\mathcal{A}) \to 0 \\ p \in \mathcal{A} \subset \mathcal{S}}} \frac{\Phi(\mathcal{A}, \Omega)}{\operatorname{area}(\mathcal{A})}$$

Irradiance

Radiant flux **of** surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$

per area

irradiance **of** point $p \in \mathcal{S}$ (or $p \in \mathbb{R}^3$ and $\hat{n}$)
solid angle $\Omega \subset \mathbb{S}^2$



Practice

There is directional light with $\hat{\omega}$.
What is the relationship between $E_{\hat{\omega}}(p)$ and $E_{\hat{n}}(p)$?

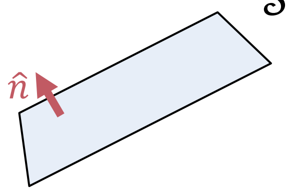
$$E_{\hat{n}}(p) = E_{\hat{\omega}}(p)|\hat{n} \cdot \hat{\omega}|$$

Irradiance

Radiant flux **of** surface $\mathcal{S} \subset \mathbb{R}^3$,
solid angle $\Omega \subset \mathbb{S}^2$

per area

irradiance **of** point $p \in \mathcal{S}$ (or $p \in \mathbb{R}^3$ and $\hat{n}$)
solid angle $\Omega \subset \mathbb{S}^2$



Practice

There is a point light source at p_L with the radiant intensity $I(\widehat{\omega})$.
What is the incident irradiance at p on a surface $\mathcal{S}$, $E_{\mathcal{S}}(p) = ?$

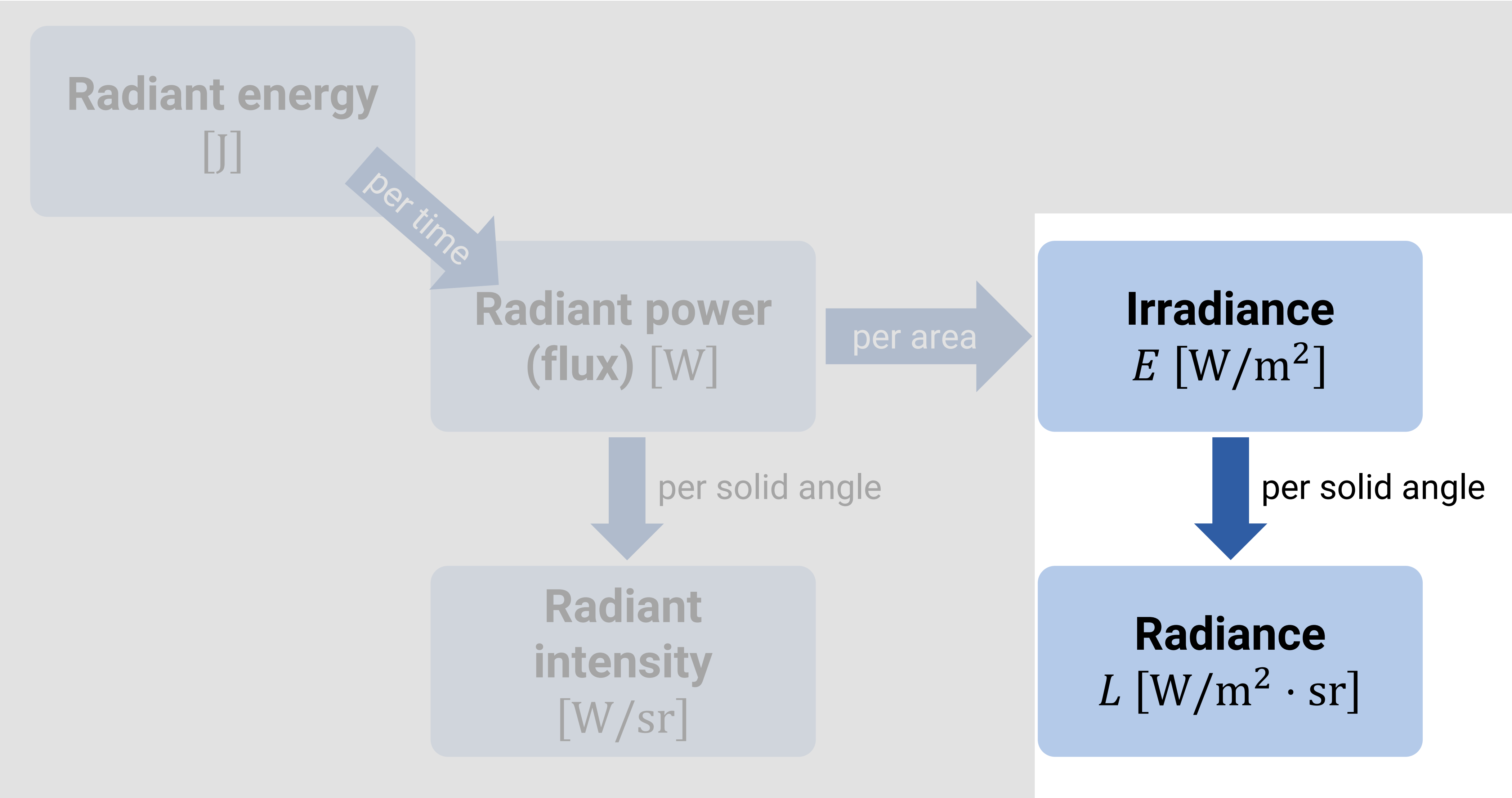
$$E_{\mathcal{S}}(p) = \lim_{\substack{\mathrm{area}(\mathcal{A})\to 0 \\ p\in\mathcal{A}\subset\mathcal{S}}} \frac{\Phi(\mathcal{A})}{\mathrm{area}(\mathcal{A})} = \lim_{\substack{\mathrm{area}(\mathcal{A})\to 0 \\ p\in\mathcal{A}\subset\mathcal{S}}} \frac{I\left(\widehat{\omega}_{p_L p}\right)\mathrm{sol.ang.}(\mathcal{A}, p_L)}{\mathrm{area}(\mathcal{A})} = \frac{I\left(\widehat{\omega}_{p_L p}\right)\left|\hat{n}\cdot\widehat{\omega}_{p p_L}\right|}{\|p - p_L\|^2}$$

Definition of irradiance

Definition of radiant intensity, small areal $\mathcal{A}$

Relation between an area and a solid angle, small areal $\mathcal{A}$

Radiance



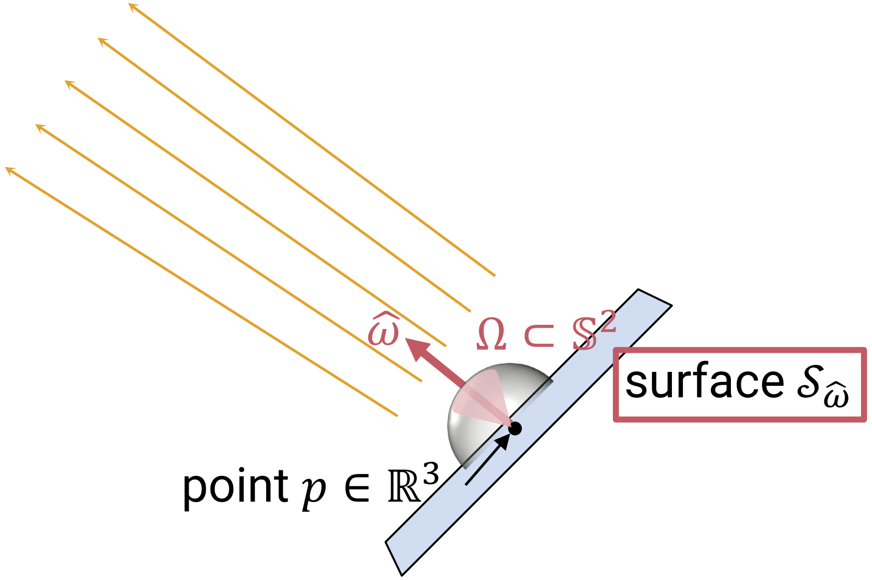
Radiant energy
[J]
per time
Radiant power
(flux) [W]
per area
Irradiance
E [W/m^2]
per solid angle
Radiant
intensity
[W/sr]
per solid angle
Radiance
L [W/m^2 · sr]

Radiance

Irradiance **of** point $p \in \mathcal{S}$,
solid angle $\Omega \subset \mathbb{S}^2$

per solid angle

Radiance **of** point $p \in \mathbb{R}^3$,
direction $\hat{\omega} \in \mathbb{S}^2$



$$L(p, \hat{\omega}) = \lim_{\substack{\text{sol.ang.}(\Omega)\to 0 \\ \hat{\omega}\in\Omega}} \frac{E_{\boxed{\mathcal{S}_{\hat{\omega}}}}(p, \Omega)}{\text{sol. ang. }(\Omega)}$$

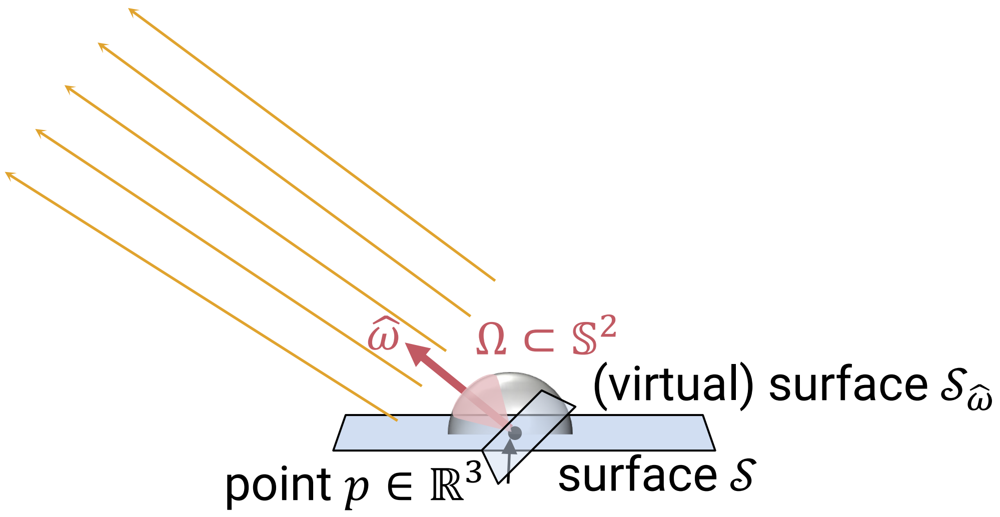
For a point $p \in \mathbb{R}^3$ (on or not on a surface),
the radiance $L(p, \hat{\omega})$ is defined as the limit about a virtual surface facing $\hat{\omega}$

Radiance

Irradiance **of** point $p \in \mathcal{S}$,
solid angle $\Omega \subset \mathbb{S}^2$

per solid angle

Radiance **of** point $p \in \mathbb{R}^3$,
direction $\hat{\omega} \in \mathbb{S}^2$



small solid angle around $\hat{\omega}$!

$$L(p, \hat{\omega}) = \lim_{\substack{\text{sol.ang.}(\Omega) \to 0 \\ \hat{\omega} \in \Omega}} \frac{\boxed{E_{\mathcal{S}_{\hat{\omega}}}(p, \Omega)}}{\text{sol. ang. } (\Omega)} = \lim_{\substack{\text{sol.ang.}(\Omega) \to 0 \\ \hat{\omega} \in \Omega}} \boxed{\frac{1}{|\hat{n} \cdot \hat{\omega}|}} \frac{\boxed{E_{\mathcal{S}}(p, \Omega)}}{\text{sol. ang. } (\Omega)}$$

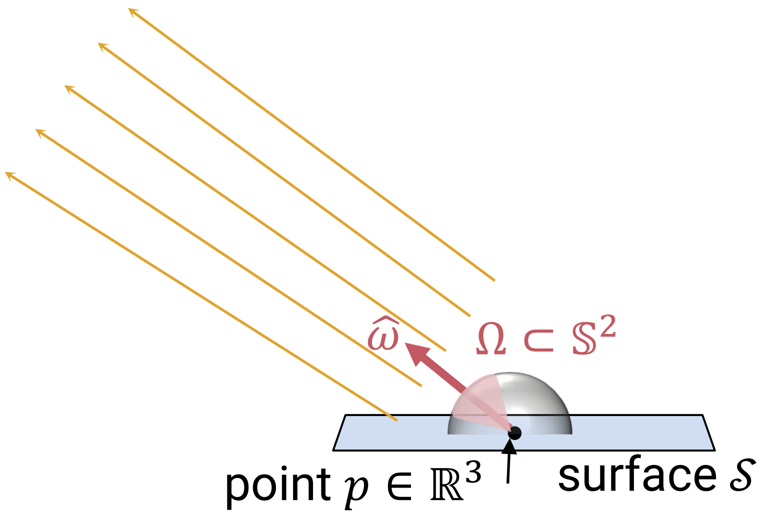
For a point p on given surface $\mathcal{S}$,
the radiance $L(p, \hat{\omega})$ can also be written as the limit about the irradiance on $\mathcal{S}$

Radiance

Irradiance **of** point $p \in \mathcal{S}$,
solid angle $\Omega \subset \mathbb{S}^2$

per solid angle

Radiance **of** point $p \in \mathbb{R}^3$,
direction $\hat{\omega} \in \mathbb{S}^2$



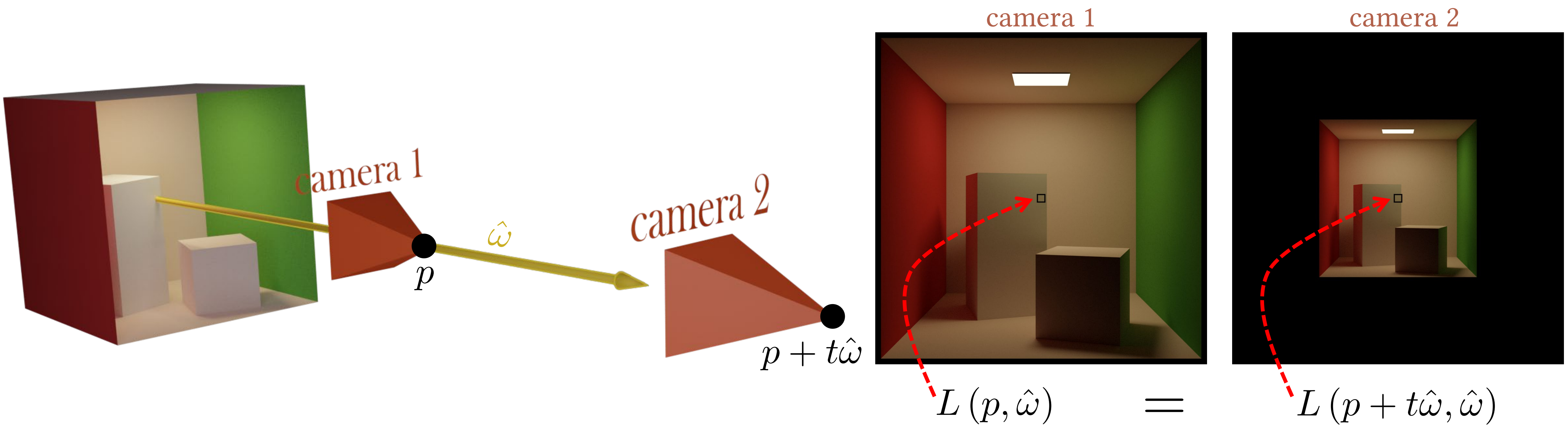
$$E_{\mathcal{S}}(p,\Omega) = \int_{\Omega} L(p,\hat{\omega})|\hat{n}\cdot\hat{\omega}|\mathrm{d}\hat{\omega}$$

per solid angle →

← over solid angle

$$L(p,\hat{\omega}) = \lim_{\substack{\text{sol.ang.}(\Omega)\to 0 \\ \hat{\omega}\in\Omega}} \frac{1}{|\hat{n}\cdot\hat{\omega}|}\frac{E_{\mathcal{S}}(p,\Omega)}{\text{sol. ang. }(\Omega)}$$

Radiance



* if the pixel footprints are sufficiently small

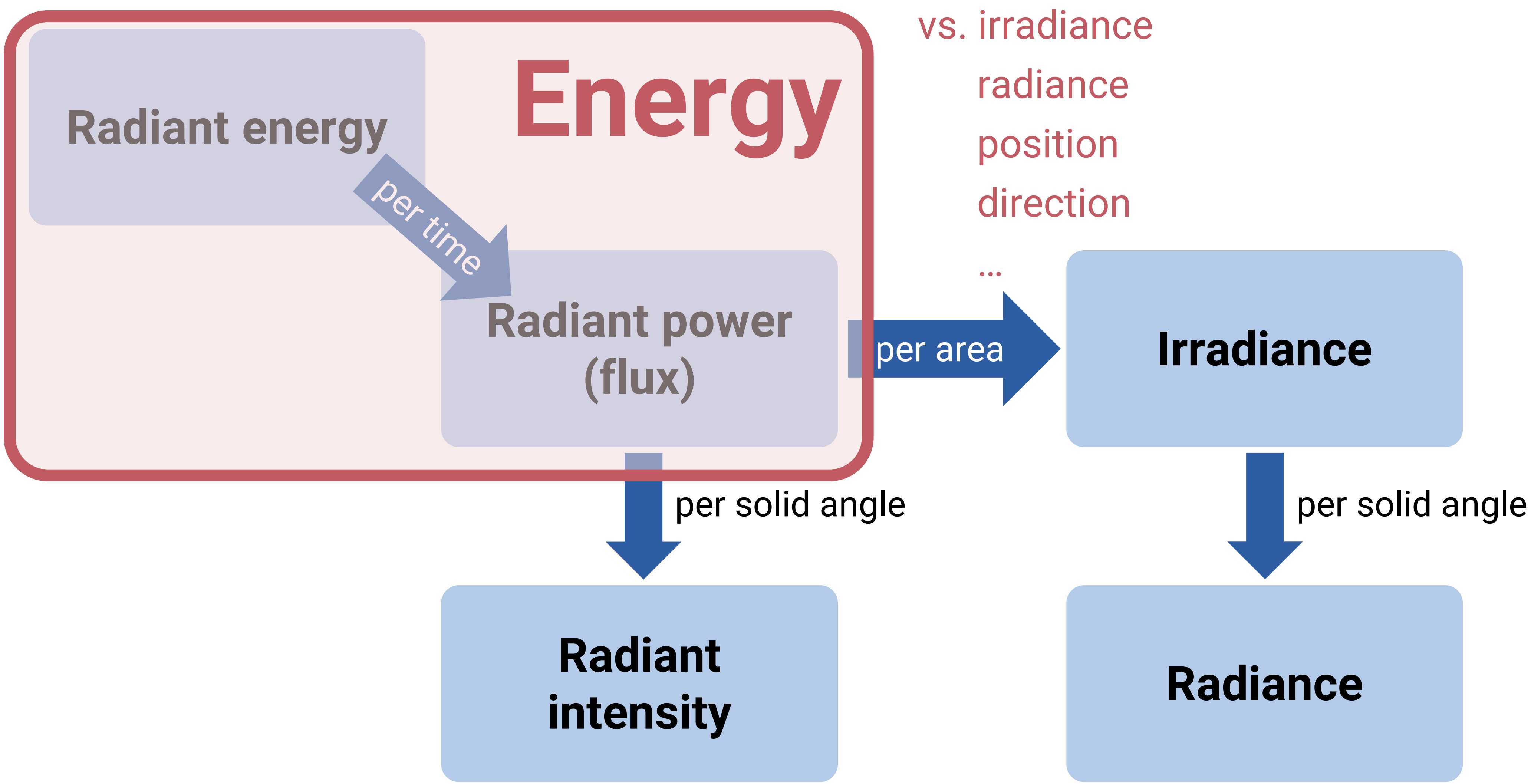
Practice

Radiance is invariant along ray:

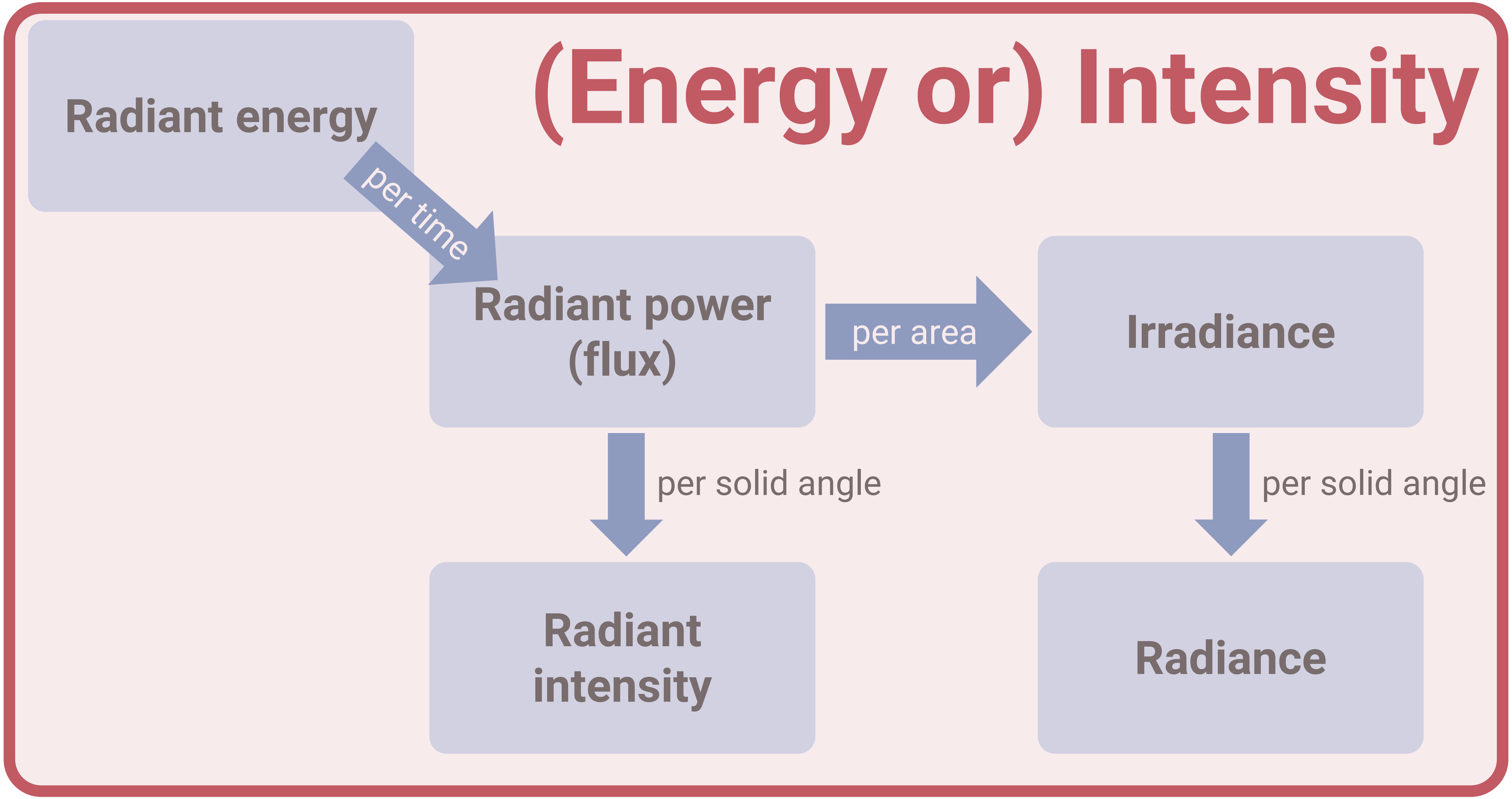
$$L(p, \widehat{\omega}) = L(p + t\widehat{\omega}, \widehat{\omega}) \ \forall t \in \mathbb{R}$$

whenever there is no material between p and $p + t\widehat{\omega}$

Slight abuse of terminology



Slight abuse of terminology

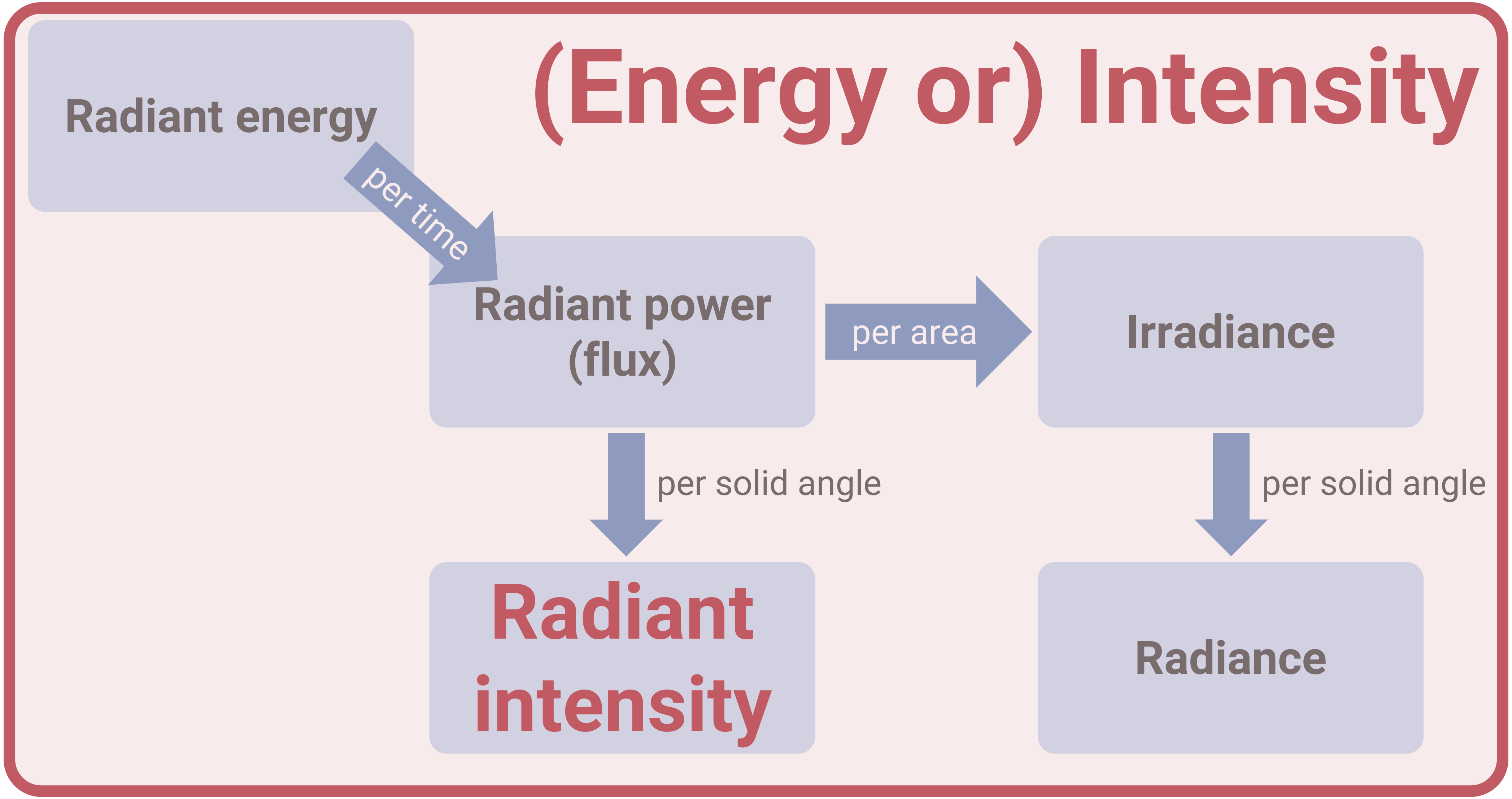


vs. position

direction

…

Unfortunate ambiguity

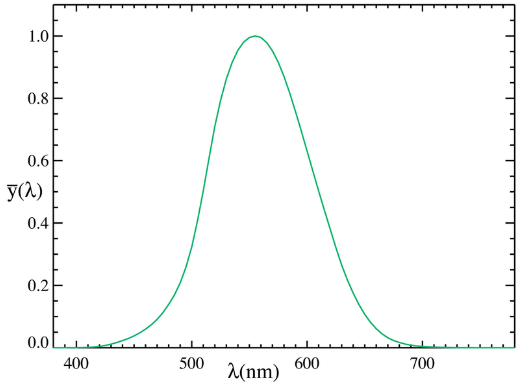


Photometry?

Radiometry: physical energy

Photometry: how bright human perceive

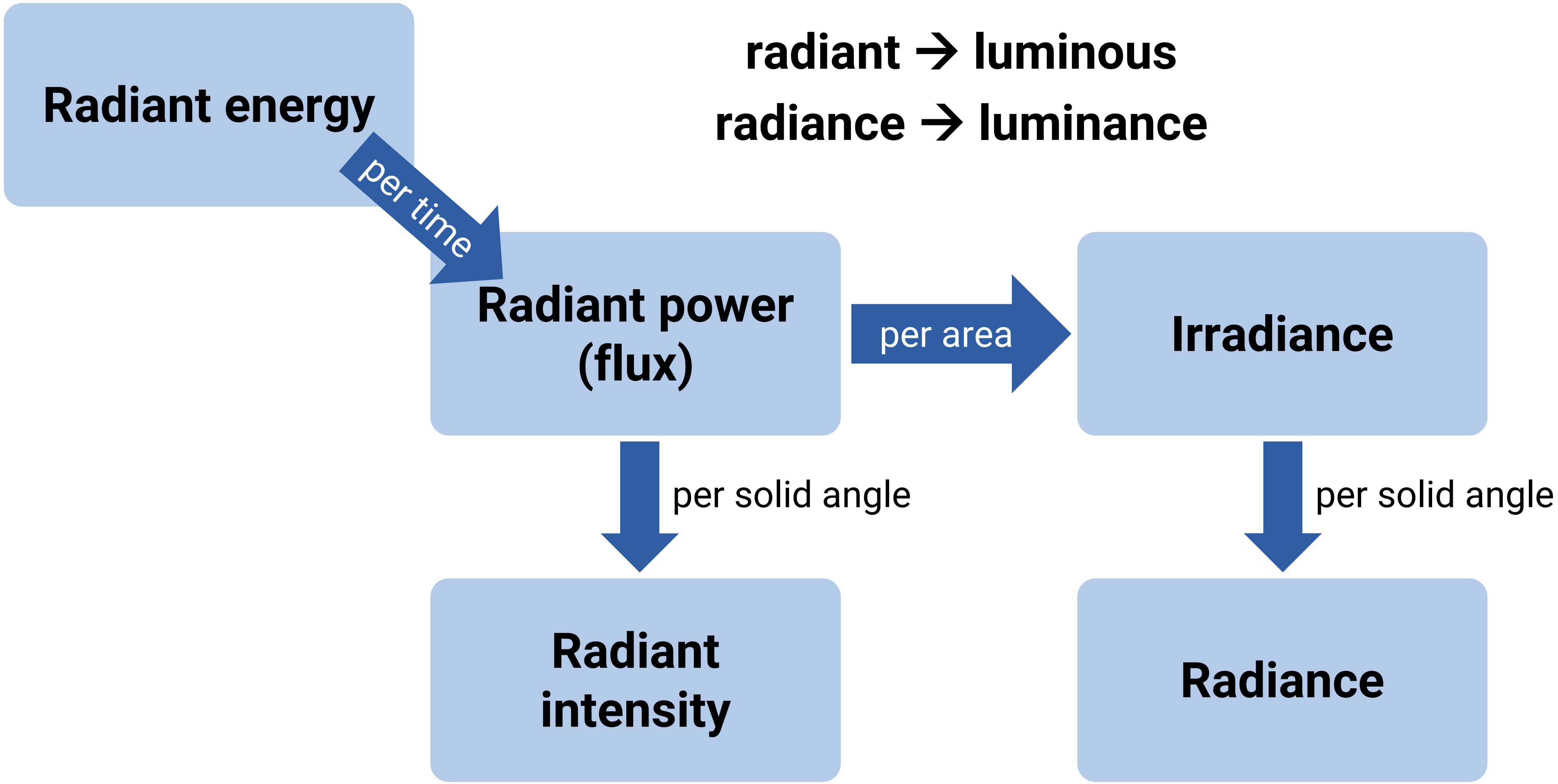
$$\int_{380\mathrm{nm}}^{700\mathrm{nm}} (\text{radiometric quantity per wavelength})\,(\text{luminous efficiency function})\,d\lambda$$



Photometry?

radiant → luminous

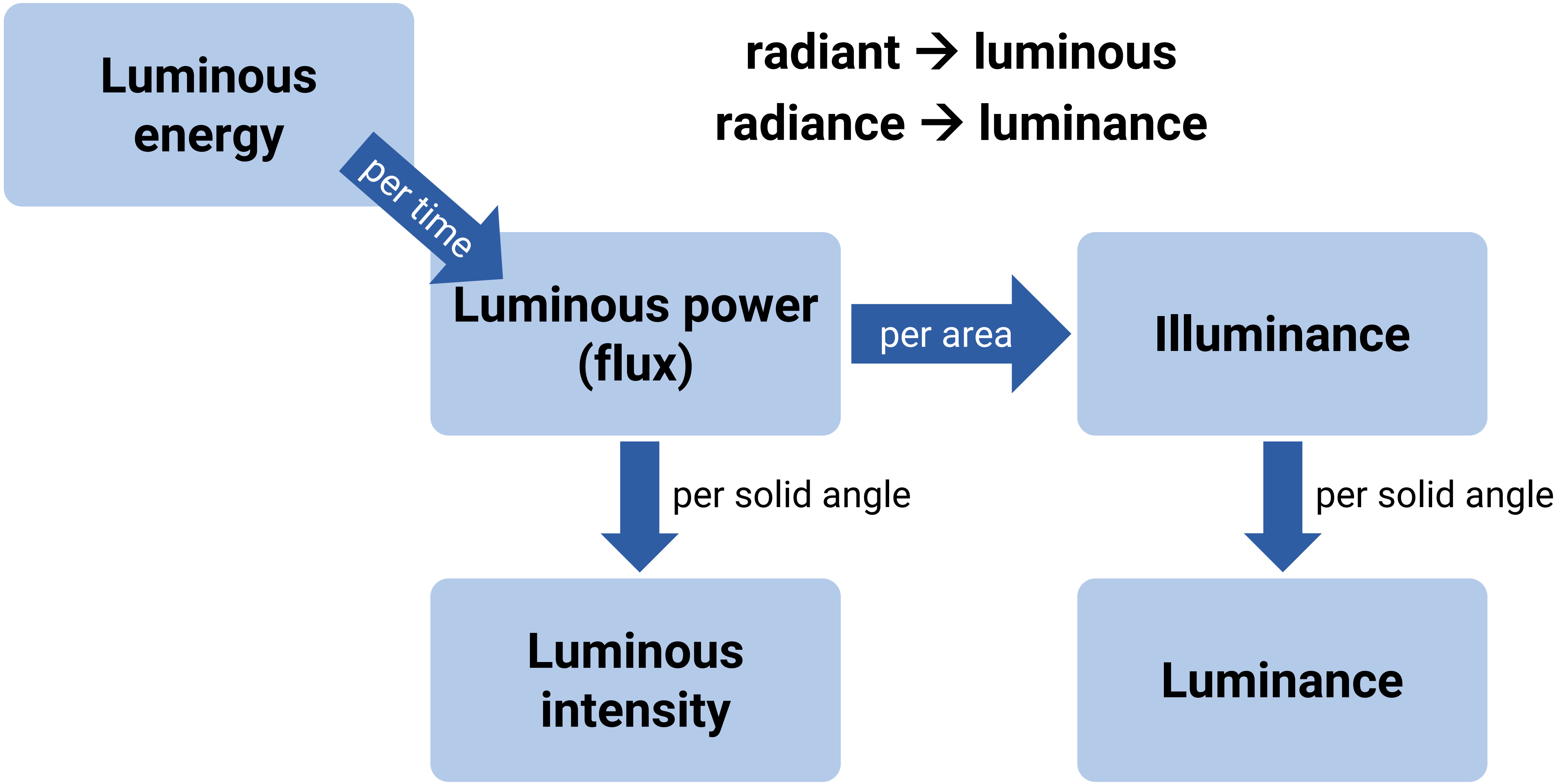
radiance → luminance



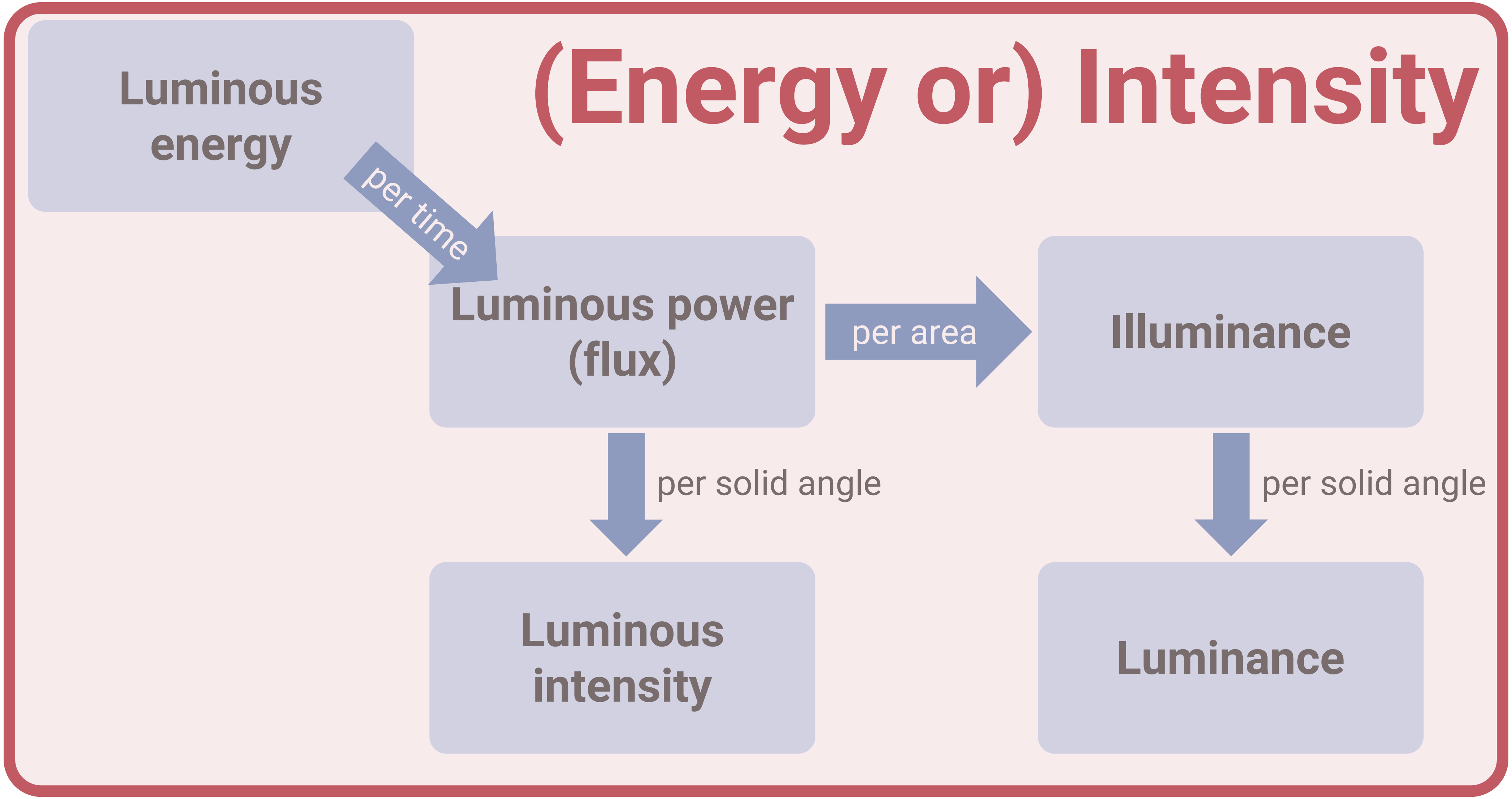
Photometry

radiant → luminous

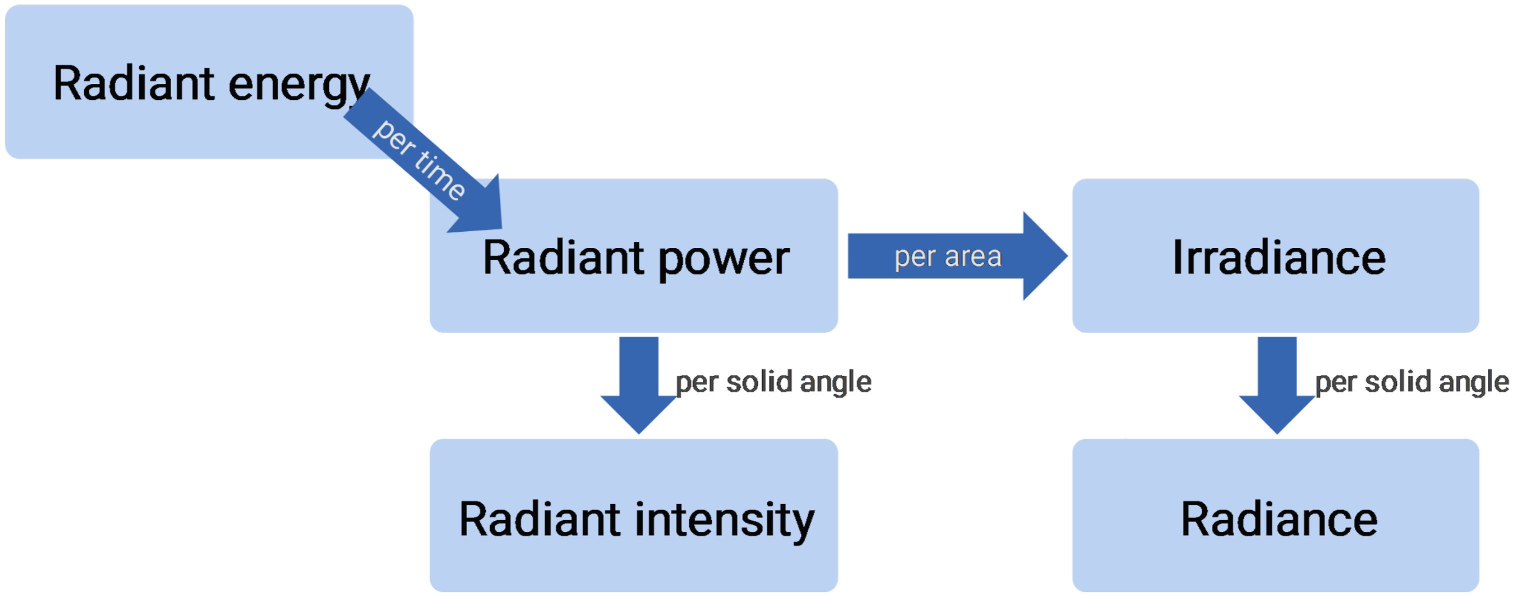
radiance → luminance



Slight abuse of terminology



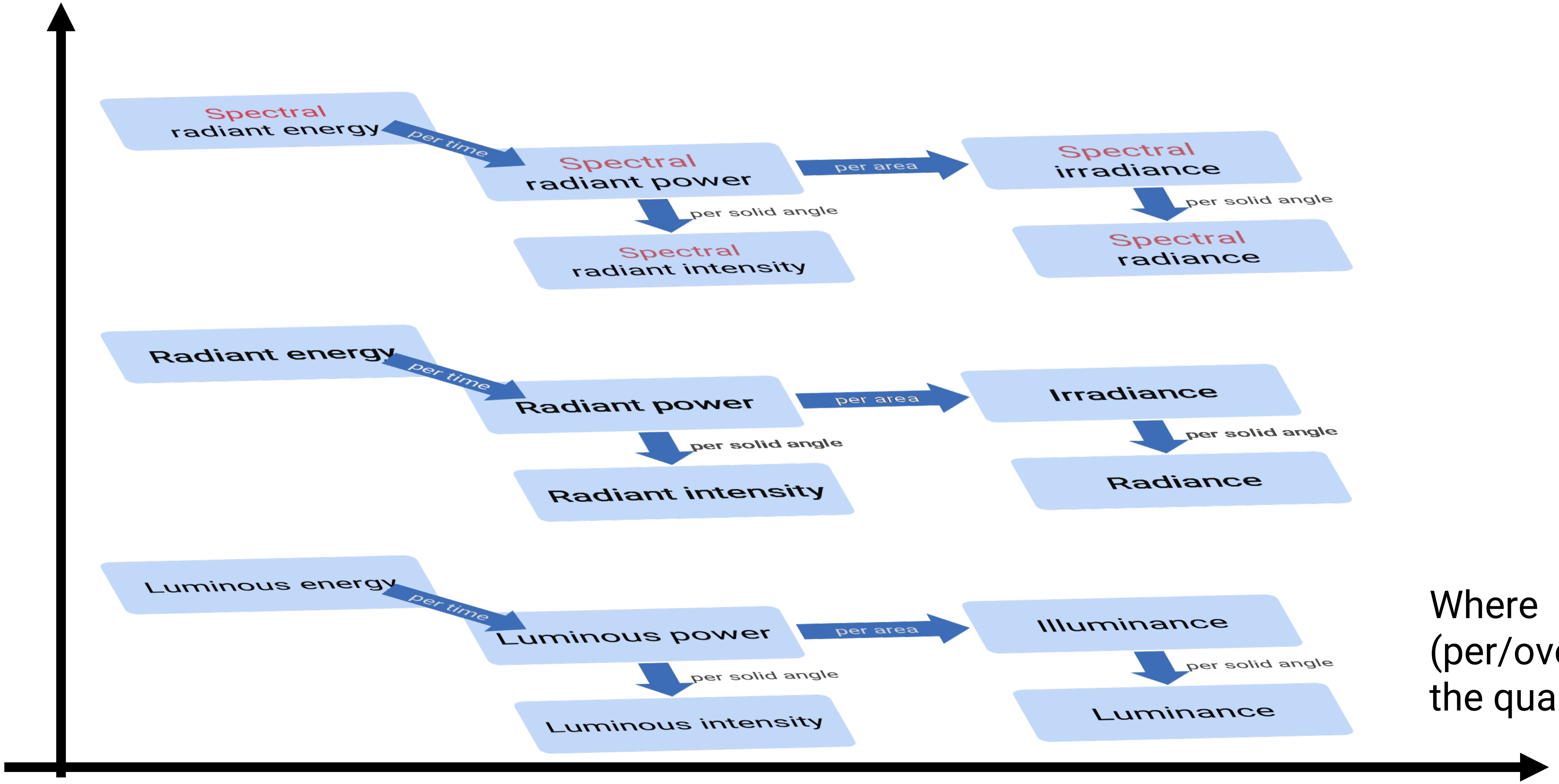
Radiometric quantities



Radiometric quantities

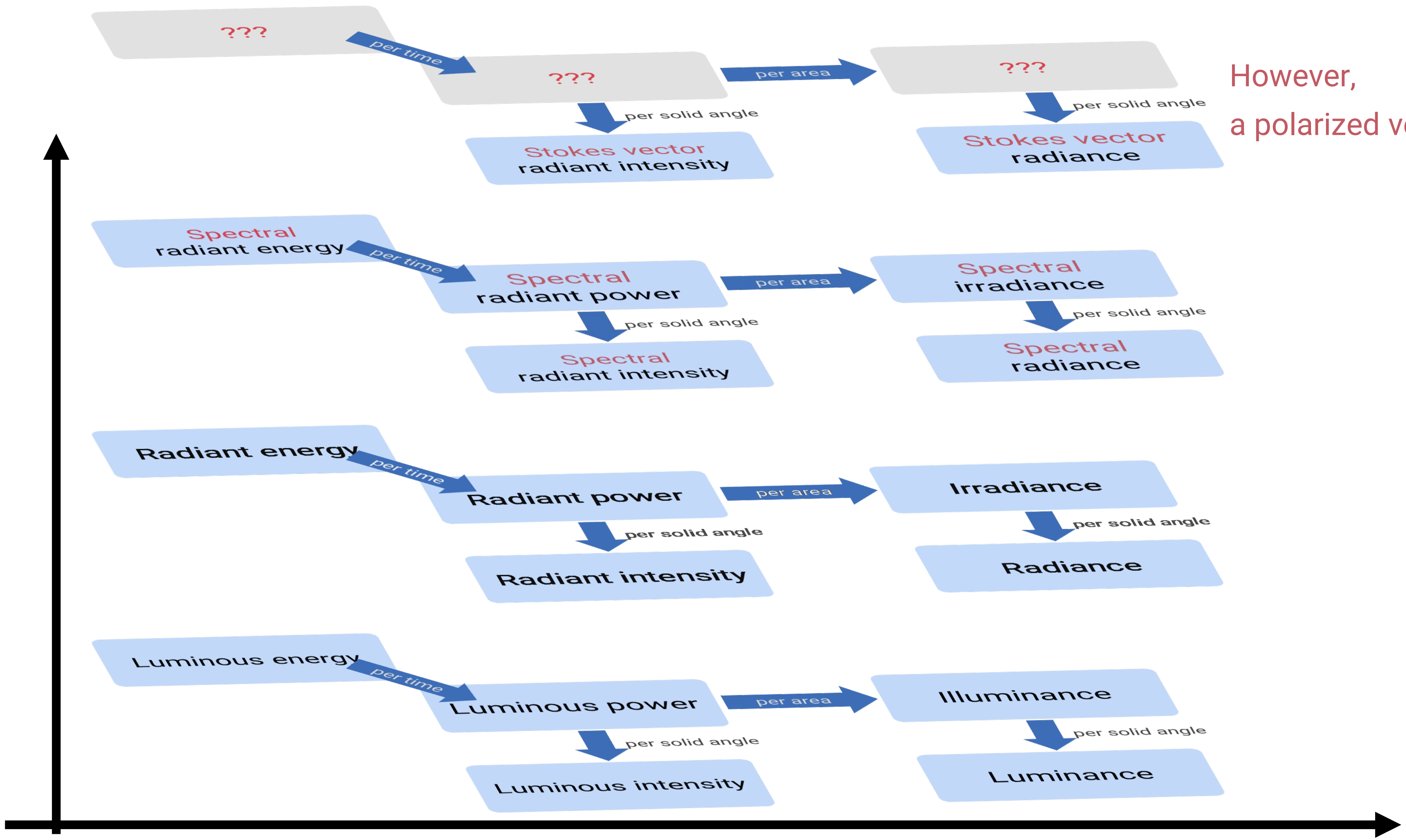
Physical energy? (per wavelength?)

Visible brightness for human?



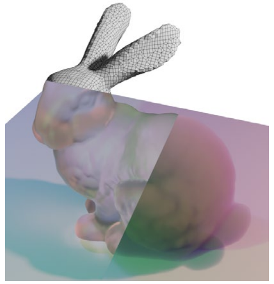
Where
(per/over position? direction?)
the quantity is defined

(Advanced) Radiometric quantities



However,

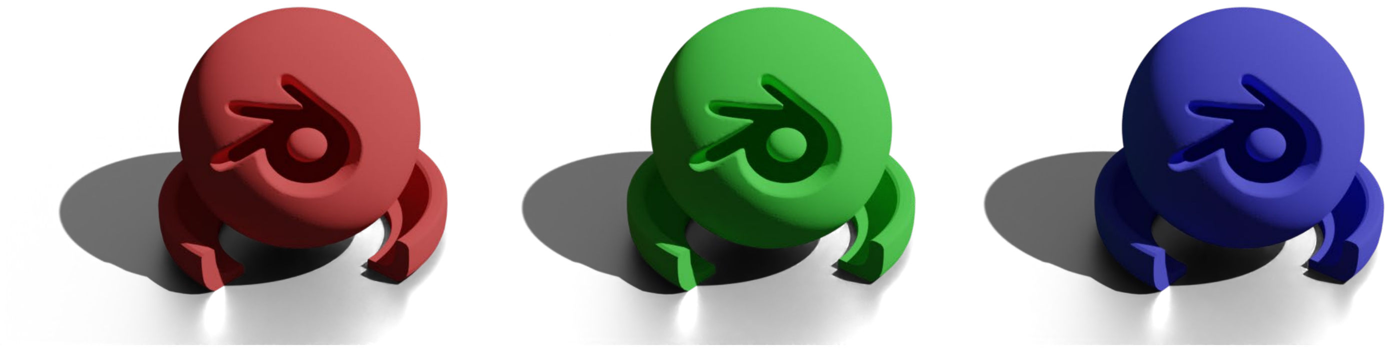
a polarized version is not straightforward



Material Appearance: BRDFs

Material Appearance

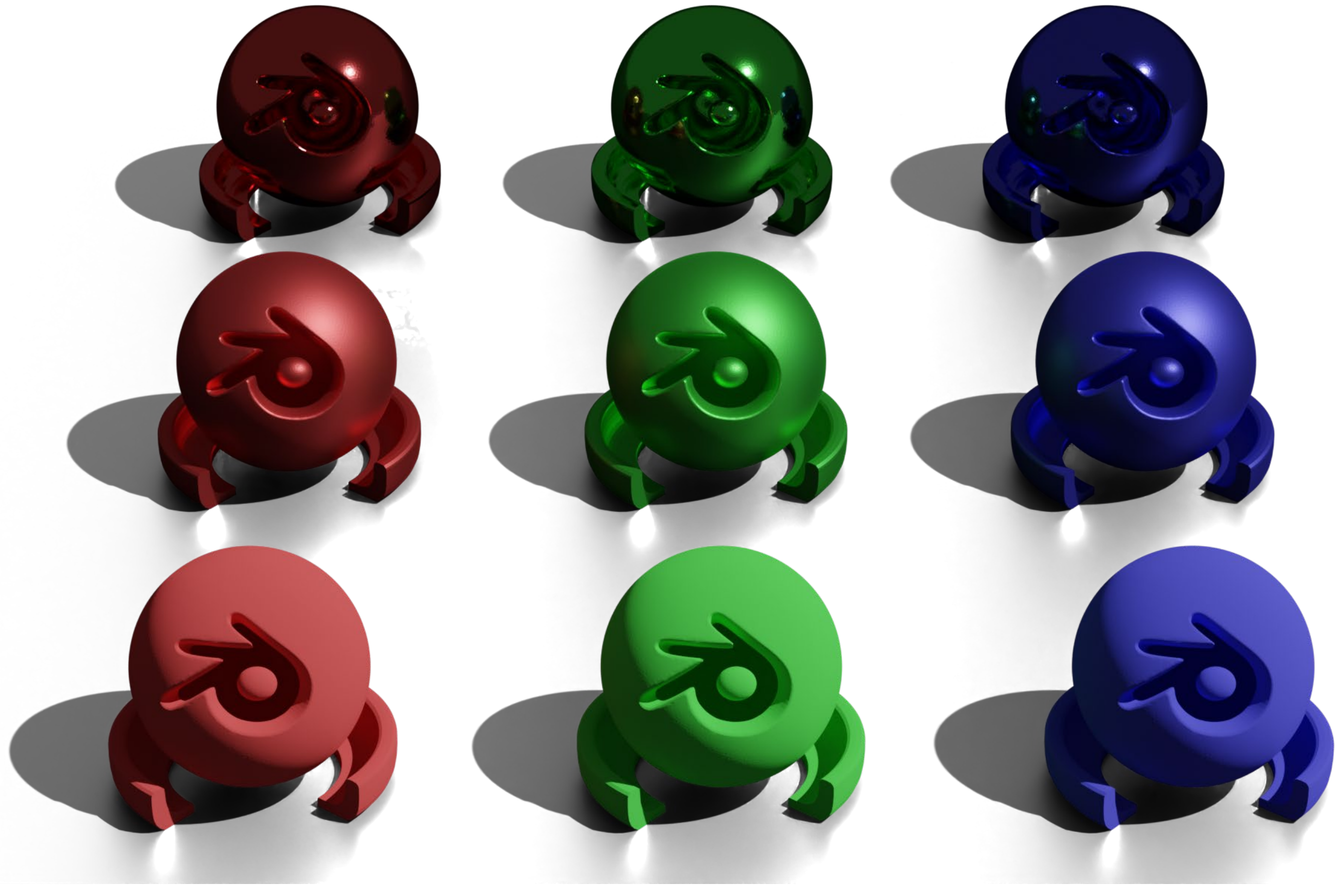
How can we characterize material appearance (refelctance)?



Is single numbers of reflectance per RGB channel (or wavelength) enough?

Material Appearance

How can we characterize material appearance (refelctance)?



Is single numbers of reflectance per RGB channel (or wavelength) enough?

Bidirectional reflectance distribution function

We roughly say….

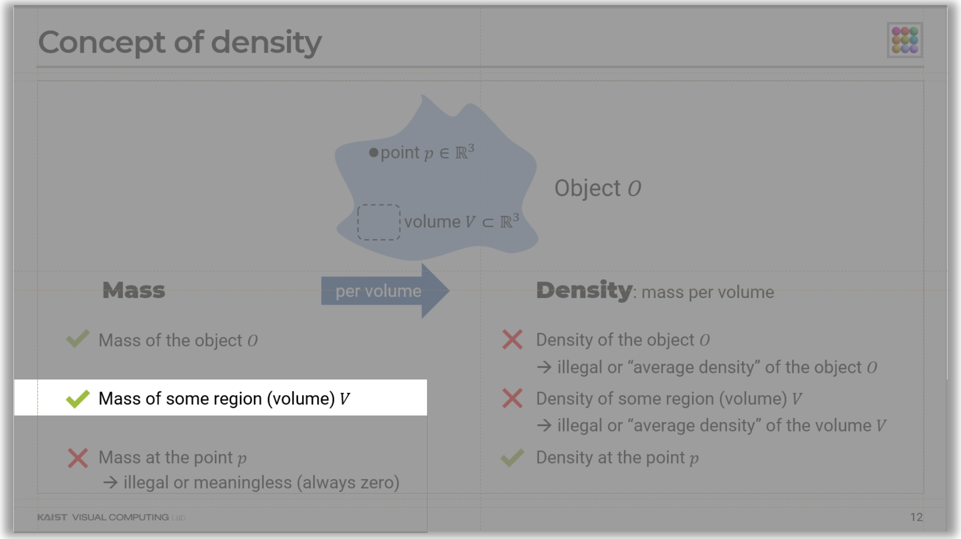
$$\text{BRDF } f_s\text{:} \frac{\text{outgoing radiance along } \omega_o}{\text{incident irradiance at } \omega_i}$$

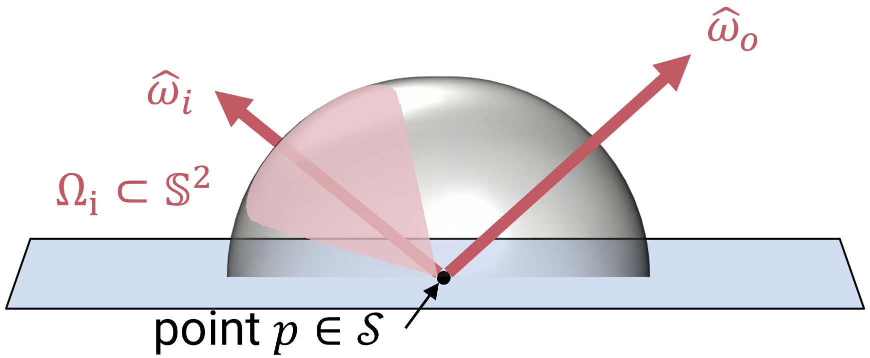
Previous slide

Irradiance ***at a point on a surface (dosen't depend on direction)***

????

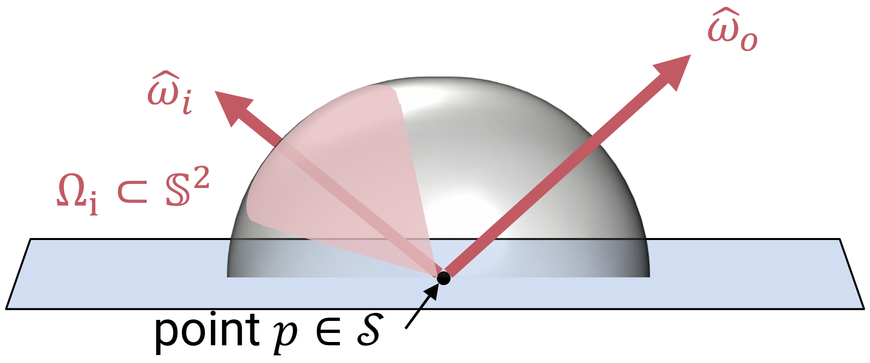
Bidirectional reflectance distribution function





$$\text{BRDF } f_s(p, \hat{\omega}_i, \hat{\omega}_o) = \lim_{\substack{\text{sol.ang.}(\Omega_i) \to 0 \\ \omega_i \in \Omega_i}} \frac{L(p, \hat{\omega}_o)}{E_{\mathcal{S}}^{(\text{in})}(p, \Omega_i)}$$

Bidirectional reflectance distribution function



$$\text{BRDF } f_s(p, \widehat{\omega}_i, \widehat{\omega}_o) = \lim_{\substack{\text{sol.ang.}(\Omega_i)\to 0 \\ \omega_i \in \Omega_i}} \frac{L(p, \widehat{\omega}_o\ )}{E_{\mathcal{S}}^{(\text{in})}(p, \Omega_i)}$$

$$= \lim_{\substack{\text{sol.ang.}(\Omega_i)\to 0 \\ \omega_i \in \Omega_i}} \frac{L(p, \widehat{\omega}_o\ )}{\int_{\Omega_i} L^{(\text{in})}(p, \widehat{\omega}_i)\, |\hat{n} \cdot \widehat{\omega}_i| \mathrm{d}\widehat{\omega}_i} = \lim_{\substack{\text{sol.ang.}(\Omega_i)\to 0 \\ \omega_i \in \Omega_i}} \frac{L(p, \widehat{\omega}_o\ )}{L^{(\text{in})}(p, \widehat{\omega}_i) |\hat{n} \cdot \widehat{\omega}_i| \text{sol. ang. } (\Omega_i)}$$

Rendering equation

$+L_e(p, \widehat{\omega}_o) \Rightarrow$**, then we get the rendering equation**

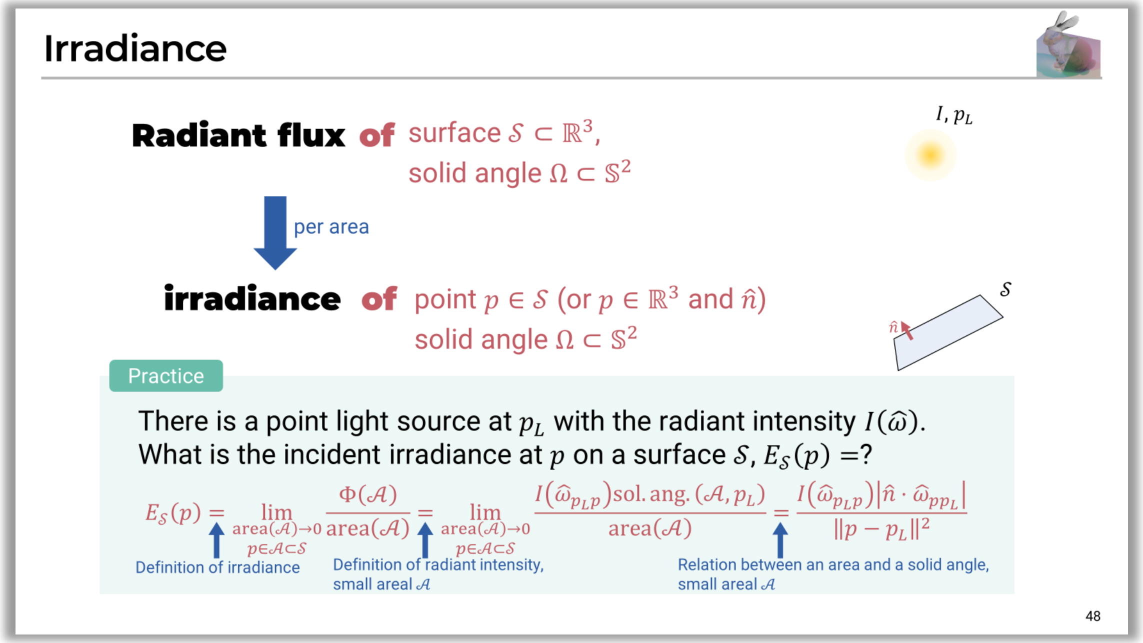
$$L^{(\text{out})}(p, \widehat{\omega}_o) = \int_{\mathbb{S}^2} L^{(\text{in})}(p, \widehat{\omega}_i) f_s(p, \widehat{\omega}_i, \widehat{\omega}_o) |\hat{n} \cdot \widehat{\omega}_i| \mathrm{d}\widehat{\omega}_i$$

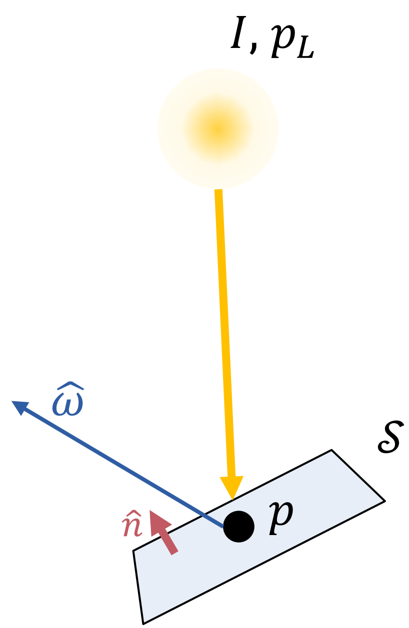
per solid angle

over solid angle

$$f_s(p, \widehat{\omega}_i, \widehat{\omega}_o) = \lim_{\substack{\text{sol.ang.}(\Omega_i) \to 0 \\ \omega_i \in \Omega_i}} \frac{L(p, \widehat{\omega}_o)}{L^{(\text{in})}(p, \widehat{\omega}_i) |\hat{n} \cdot \widehat{\omega}_i| \text{sol. ang. } (\Omega_i)}$$

Rendering equation





Practice

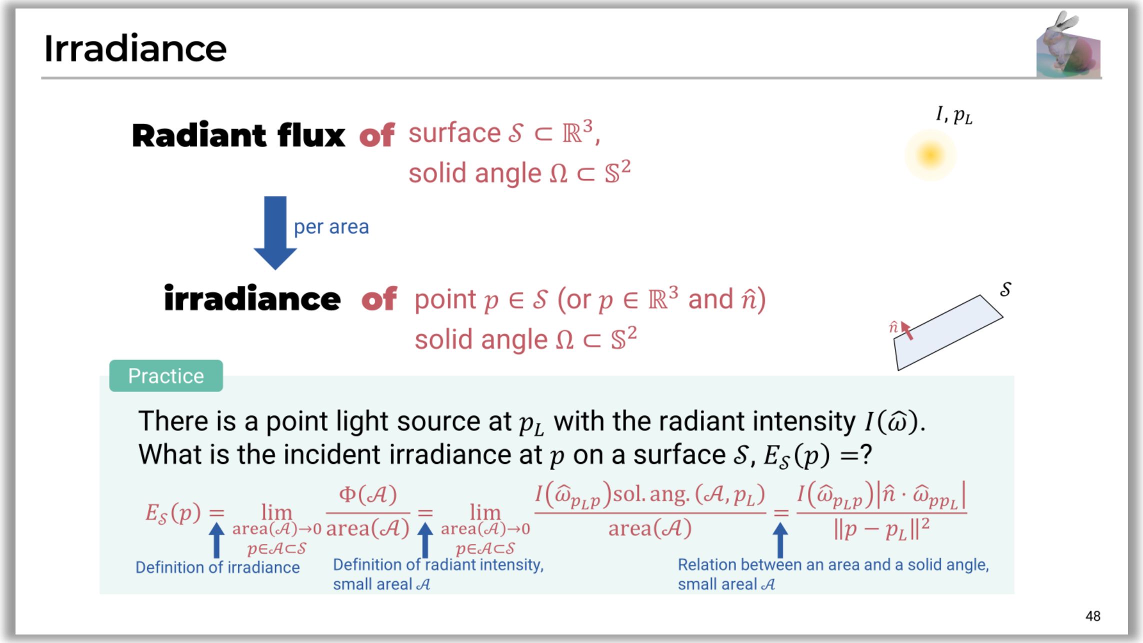
There is a point light source at p_L with the radiant intensity $I(\hat{\omega})$.
What is the reflected radiance at p on a surface $\mathcal{S}$, along $\hat{\omega}_o$?

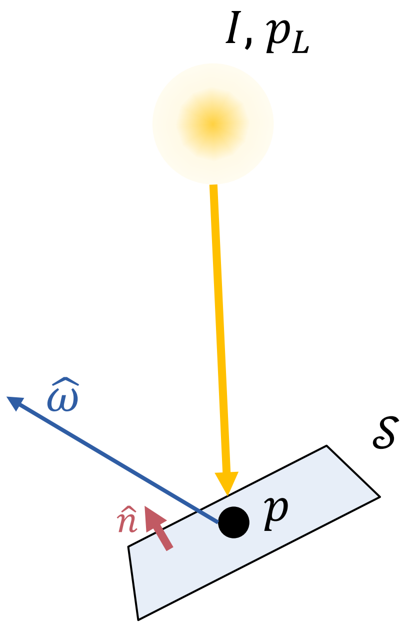
$$f_s(p, \hat{\omega}_i, \hat{\omega}_o) = \lim_{\substack{\mathrm{sol.ang.}(\Omega_i)\to 0 \\ \omega_i\in\Omega_i}} \frac{L(p, \hat{\omega}_o\ )}{E_{\mathcal{S}}^{(\mathrm{in})}(p, \Omega_i)} = \frac{\|p - p_L\|^2}{I(\hat{\omega}_{p_L p})|\hat{n}\cdot\hat{\omega}_{p p_L}|} L^{(\mathrm{out})}(p, \hat{\omega}_o)$$

Definition of BRDF

Previous slide: irradiance due to point source

Rendering equation



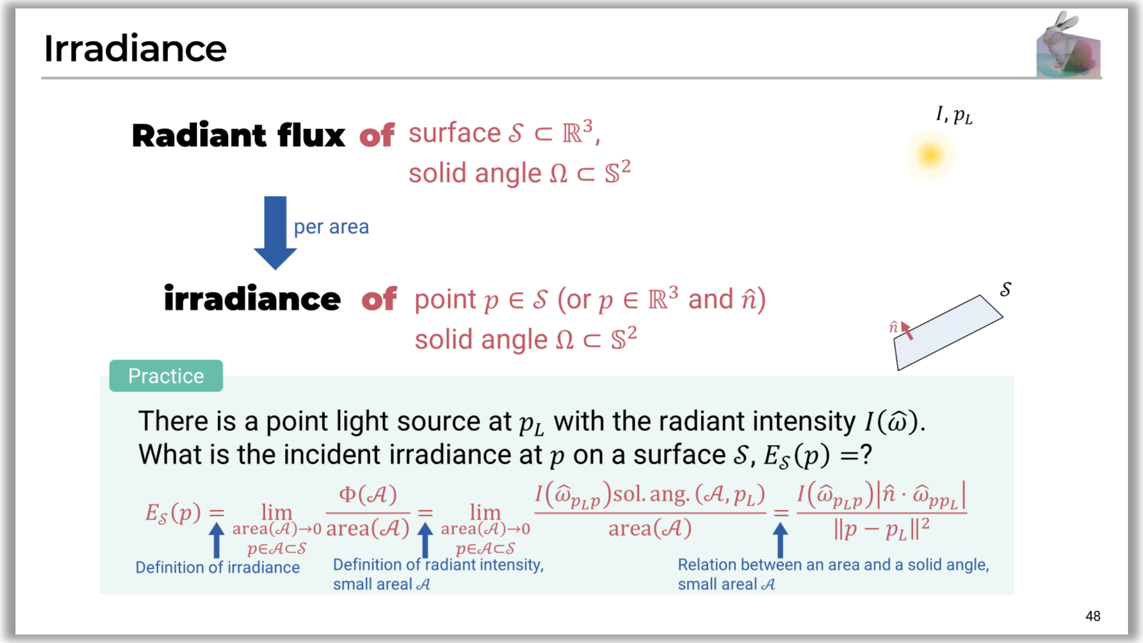


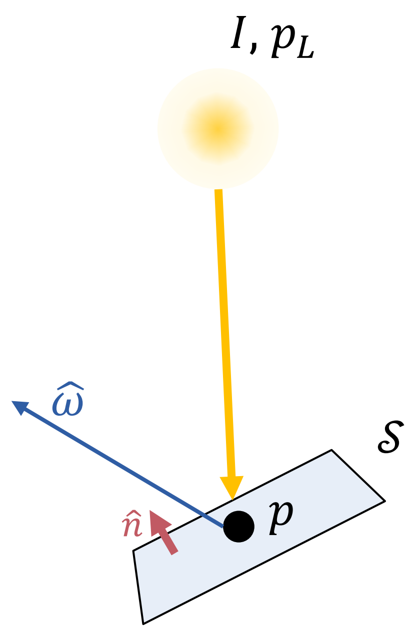
Practice

There is a point light source at p_L with the radiant intensity $I(\widehat{\omega})$.
What is the reflected radiance at p on a surface $\mathcal{S}$, along $\widehat{\omega}_o$?

$$f_{\mathcal{S}}(p, \widehat{\omega}_i, \widehat{\omega}_o) = \frac{\|p - p_L\|^2}{I(\widehat{\omega}_{p_L p})|\hat{n} \cdot \widehat{\omega}_{pp_L}|} L^{(\text{out})}(p, \widehat{\omega}_o)$$

Rendering equation



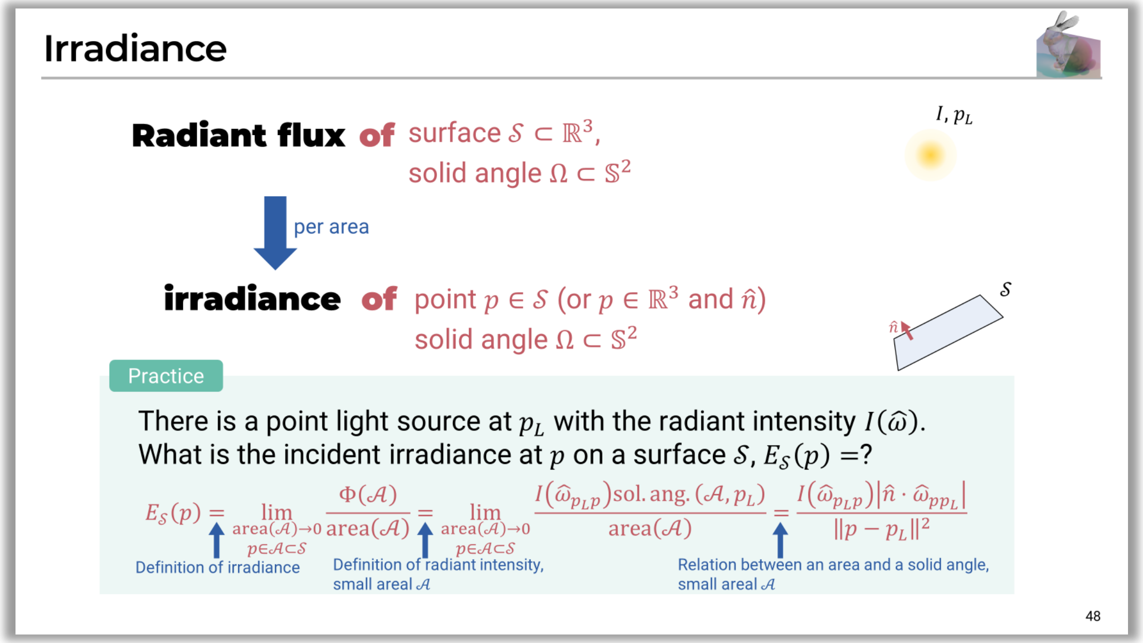


Practice

There is a point light source at p_L with the radiant intensity $I(\hat{\omega})$.
What is the reflected radiance at p on a surface $\mathcal{S}$, along $\hat{\omega}_o$?

$$\frac{\|p - p_L\|^2}{I(\hat{\omega}_{p_L p})|\hat{n}\cdot\hat{\omega}_{pP_L}|} L^{(\text{out})}(p, \hat{\omega}_o) = f_s(p, \hat{\omega}_i, \hat{\omega}_o)$$

Rendering equation


Irradiance

Radiant flux of surface $\mathcal{S} \subset \mathbb{R}^3$, solid angle $\Omega \subset \mathbb{S}^2$

per area

irradiance of point $p \in \mathcal{S}$ (or $p \in \mathbb{R}^3$ and $\hat{n}$) solid angle $\Omega \subset \mathbb{S}^2$

Practice

There is a point light source at p_L with the radiant intensity $I(\hat{\omega})$.
What is the incident irradiance at p on a surface $\mathcal{S}$, $E_{\mathcal{S}}(p) =$?

$$E_{\mathcal{S}}(p) = \lim_{\substack{\mathrm{area}(\mathcal{A})\to 0 \\ p\in\mathcal{A}\subset\mathcal{S}}} \frac{\Phi(\mathcal{A})}{\mathrm{area}(\mathcal{A})} = \lim_{\substack{\mathrm{area}(\mathcal{A})\to 0 \\ p\in\mathcal{A}\subset\mathcal{S}}} \frac{I(\hat{\omega}_{p_L p})\,\mathrm{sol.ang.}(\mathcal{A}, p_L)}{\mathrm{area}(\mathcal{A})} = \frac{I(\hat{\omega}_{p_L p})|\hat{n}\cdot\hat{\omega}_{pp_L}|}{\|p-p_L\|^2}$$

Definition of irradiance

Definition of radiant intensity, small areal $\mathcal{A}$

Relation between an area and a solid angle, small areal $\mathcal{A}$

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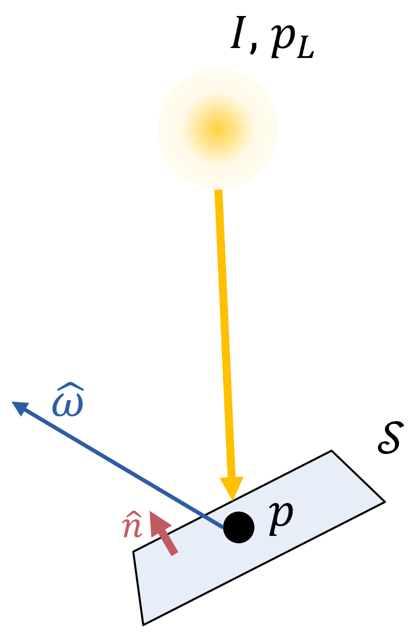
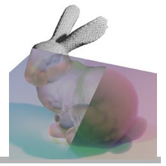


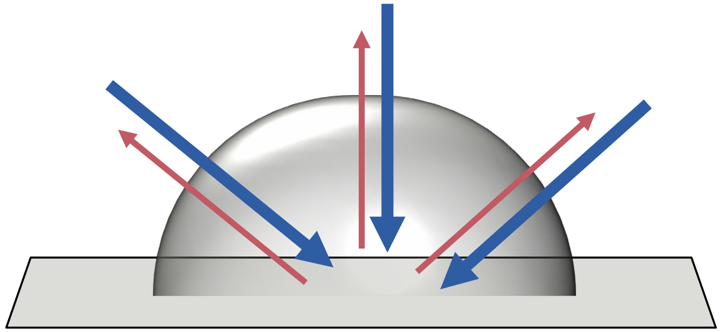
Practice

There is a point light source at p_L with the radiant intensity $I(\hat{\omega})$.
What is the reflected radiance at p on a surface $\mathcal{S}$, along $\hat{\omega}_o$?

$$L^{(\mathrm{out})}(p, \hat{\omega}_o) = f_s(p, \hat{\omega}_i, \hat{\omega}_o)\,\frac{I(\hat{\omega}_{p_L p})|\hat{n}\cdot\hat{\omega}_{pp_L}|}{\|p-p_L\|^2}$$

Properties of BRDF: energy conservation





$$\frac{\text{outgoing irradiance}}{\text{incident irradiance}} \leq 1, \text{ for any illumination condition}$$

$$\frac{\int_{\mathbb{S}^2} L^{(\text{out})}(p, \widehat{\omega})|\hat{n} \cdot \widehat{\omega}|\mathrm{d}\widehat{\omega}}{\int_{\mathbb{S}^2} L^{(\text{in})}(p, \widehat{\omega})|\hat{n} \cdot \widehat{\omega}|\mathrm{d}\widehat{\omega}} =$$

rendering equation

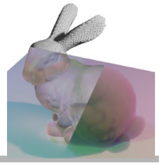
$$\frac{\int_{\mathbb{S}^2} \int_{\mathbb{S}^2} L^{(\text{in})}(p, \widehat{\omega}_i) f_s(p, \widehat{\omega}_i, \widehat{\omega}_o)|\hat{n} \cdot \widehat{\omega}_i|\mathrm{d}\widehat{\omega} \, |\hat{n} \cdot \widehat{\omega}_o|\mathrm{d}\widehat{\omega}_o}{\int_{\mathbb{S}^2} L^{(\text{in})}(p, \widehat{\omega})|\hat{n} \cdot \widehat{\omega}|\mathrm{d}\widehat{\omega}} \leq 1,$$

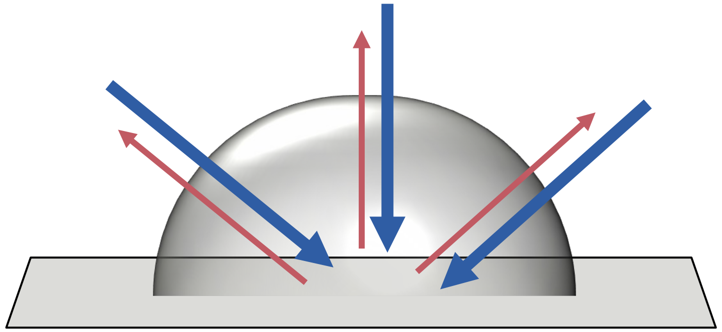
for **any positive function** $L^{(\text{in})}(p, \cdot)$.

Taking $L^{(\text{in})}(p, \cdot)$ as a Dirac delta function centered at $\widehat{\omega}_i$,

$$\therefore \int_{\mathbb{S}^2} f_s(p, \widehat{\omega}_i, \widehat{\omega}_o)|\hat{n} \cdot \omega_o|\mathrm{d}\widehat{\omega}_o \leq 1, \forall \widehat{\omega}_i$$

Properties of BRDF: energy conservation





$$\frac{\text{outgoing irradiance}}{\text{incident irradiance}} \leq 1\text{, for any illumination condition}$$

< 1: energy losses

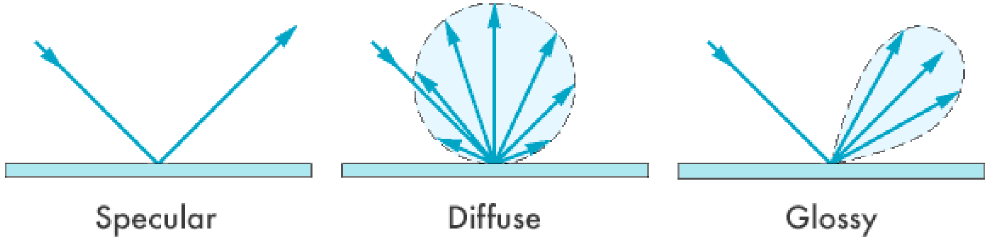
$= 1$: energy conserves

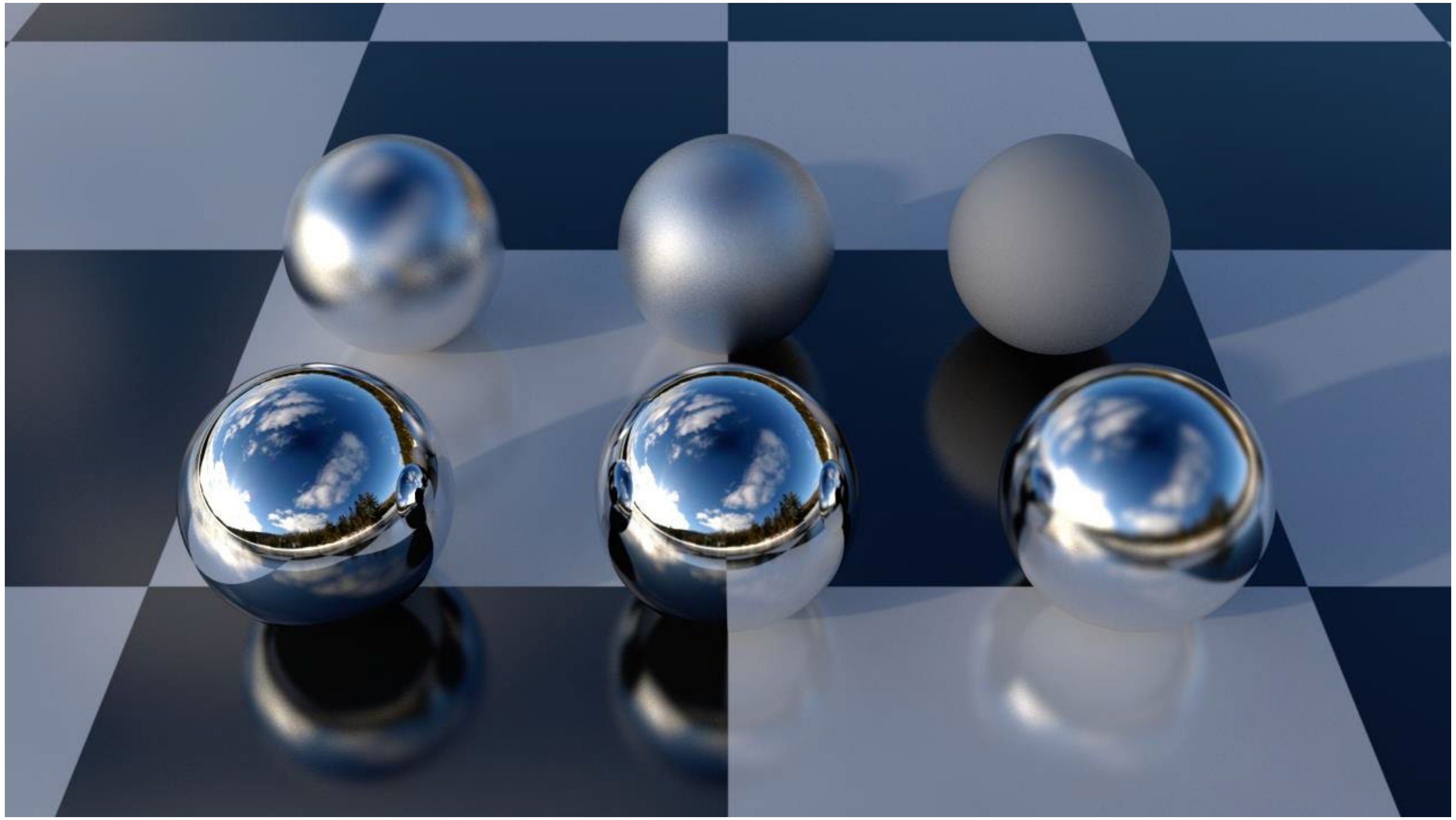
> 1: impossible!

Energy Conservation

$$\int_{\mathbb{S}^2} f_s(p, \hat{\omega}_i, \hat{\omega}_o) |\hat{n} \cdot \omega_o| \mathrm{d}\hat{\omega}_o \leq 1, \forall \hat{\omega}_i$$

Example BRDFs





source: Keenan Crane []

Example BRDFs

- Pure diffuse (Lambertian reflection)
  - Albedo ρ_d: ratio of energy conservation
  - f_s is a constant function on $\mathbb{S}^2_{\hat{z}}$

$$\int_{\mathbb{S}^2_{\hat{z}}} f_s |\hat{n} \cdot \widehat{\omega}_o| \mathrm{d}\omega_o = \pi f_s = \rho_d$$

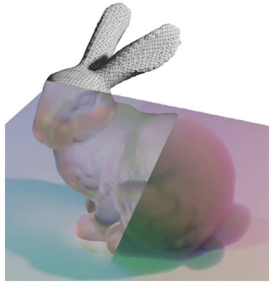
$$\therefore f_s = \frac{\rho_d}{\pi}$$

Example BRDFs

- Pure specular
  - A Dirac delta function centered at $\mathrm{refl}_{\hat{n}}(\hat{\omega}_i)$….
  - Be careful when you treat Dirac delta functions

$$f_s(\widehat{\omega}_i, \widehat{\omega}_o) = a \cdot \delta_{\mathbb{S}^2}\big(\widehat{\omega}_o, \mathrm{refl}_{\hat{n}}(\widehat{\omega}_i)\big), \int_{\mathbb{S}^2_{\hat{n}}} f_s(\widehat{\omega}_i, \widehat{\omega}_o) |\hat{n} \cdot \widehat{\omega}_o| \mathrm{d}\widehat{\omega}_o = 1$$

$$\therefore\ f_s(\widehat{\omega}_i, \widehat{\omega}_o) = \frac{\delta_{\mathbb{S}^2}\big(\widehat{\omega}_o, \mathrm{refl}_{\hat{n}}(\widehat{\omega}_i)\big)}{\hat{n} \cdot \widehat{\omega}_o}$$



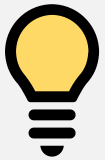
The Rendering Equation

Review: Rendering

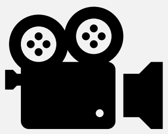
Scene

geometry

BRDF



emitter



sensor

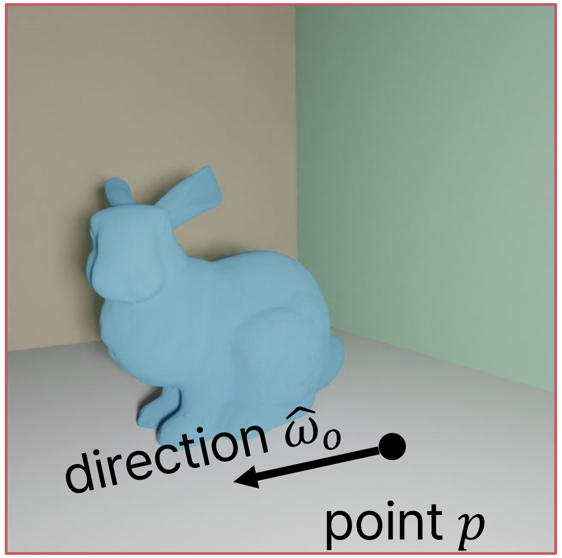
known

Rendering

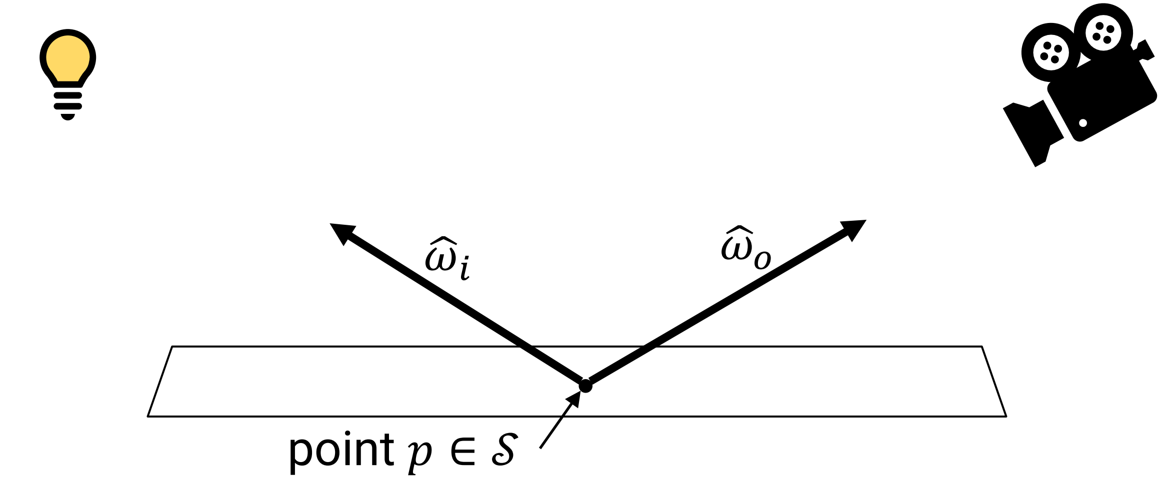
Rendering equation unknown

Find scalar $\boxed{\text{radiance } L(p, \hat{\omega})}$
at any point p, along any direction $\hat{\omega}$

Image $\mathbf{I} \in \mathbb{R}^{H \times W}$



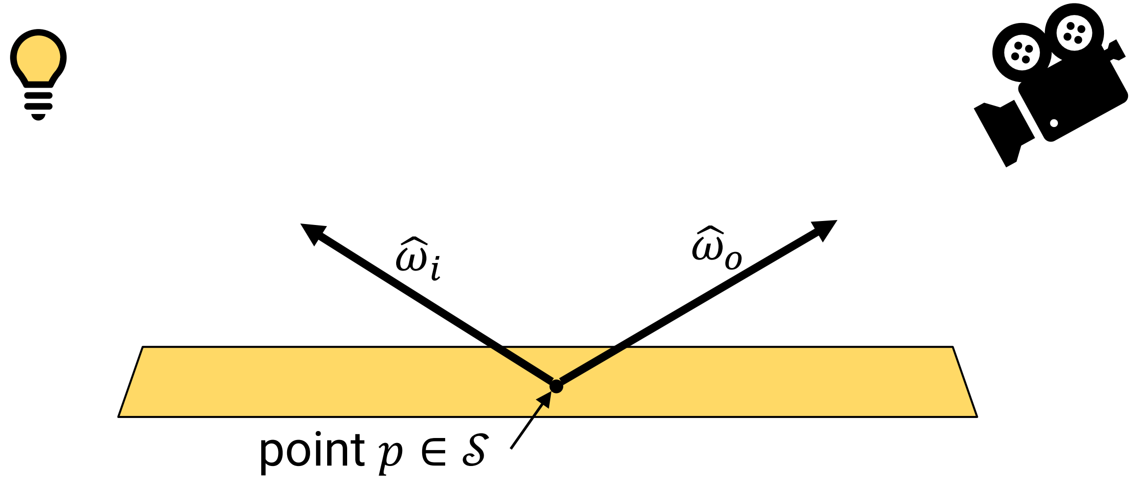
Rendering equation



From definition of BRDFs…

$$L^{(\text{out})}(p, \hat{\omega}_o) = \int_{\mathbb{S}^2} L^{(\text{in})}(p, \hat{\omega}_i) f_s(p, \hat{\omega}_i, \hat{\omega}_o) |\hat{n} \cdot \hat{\omega}_i| \mathrm{d}\hat{\omega}_i$$

Rendering equation



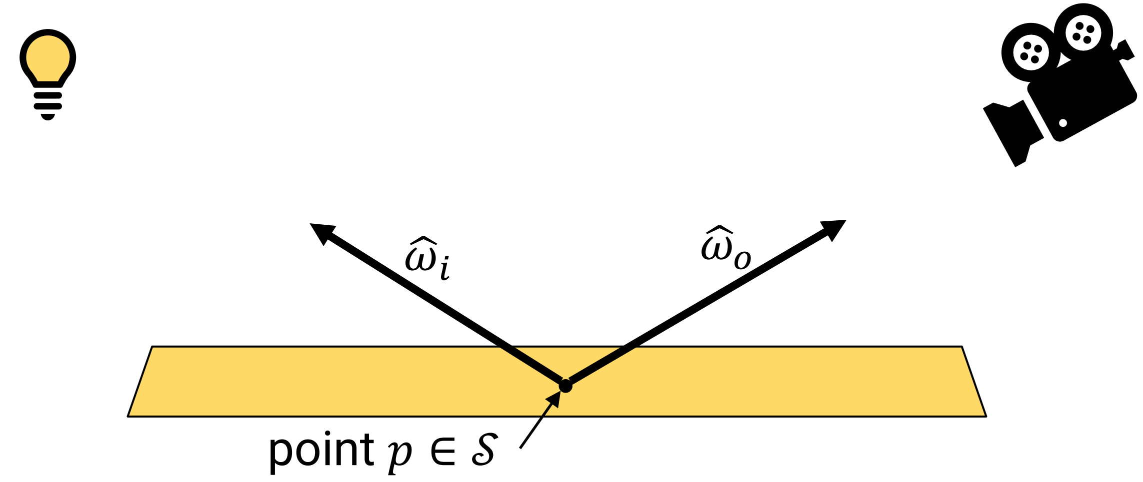
From definition of BRDFs…

+ Emission

Rendering equation
(light transport equation)

$$L^{(\mathrm{out})}(p, \widehat{\omega}_o) = L_e(p, \omega_{\hat{o}}) + \int_{\mathbb{S}^2} L^{(\mathrm{in})}(p, \widehat{\omega}_i) f_s(p, \widehat{\omega}_i, \widehat{\omega}_o) |\hat{n} \cdot \widehat{\omega}_i| \mathrm{d}\widehat{\omega}_i$$

Rendering equation

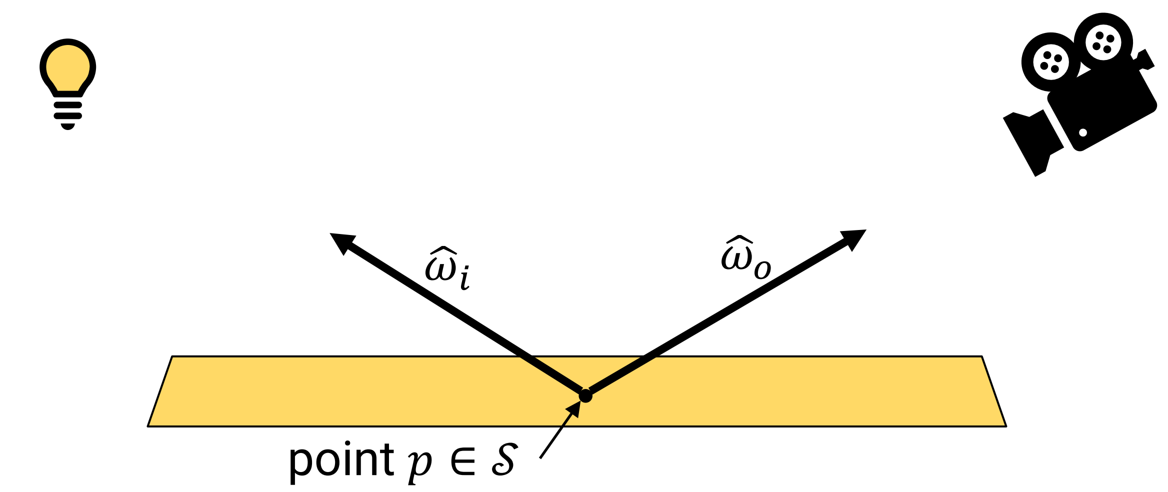


Q. What are knowns and unknowns?

Rendering equation
(light transport equation)

$$L^{(\mathrm{out})}(p, \widehat{\omega}_o) = L_e(p, \omega_{\hat{o}}) + \int_{\mathbb{S}^2} L^{(\mathrm{in})}(p, \widehat{\omega}_i) f_s(p, \widehat{\omega}_i, \widehat{\omega}_o) |\hat{n} \cdot \widehat{\omega}_i| \mathrm{d}\widehat{\omega}_i$$

Rendering equation



Q. What are knowns and unknowns?

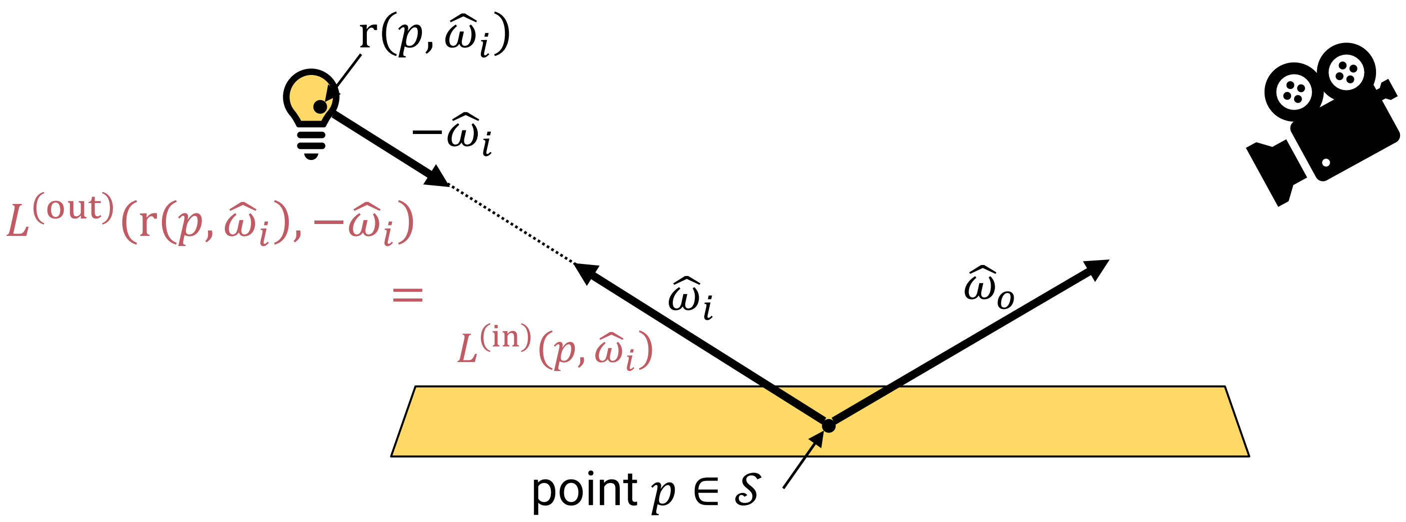
Rendering equation
(light transport equation)

known

$$L^{(\mathrm{out})}(p, \widehat{\omega}_o) = L_e(p, \omega_{\hat{o}}) + \int_{\mathbb{S}^2} L^{(\mathrm{in})}(p, \widehat{\omega}_i) f_s(p, \widehat{\omega}_i, \widehat{\omega}_o) |\hat{n} \cdot \widehat{\omega}_i| \mathrm{d}\widehat{\omega}_i$$

unknown

Rendering equation

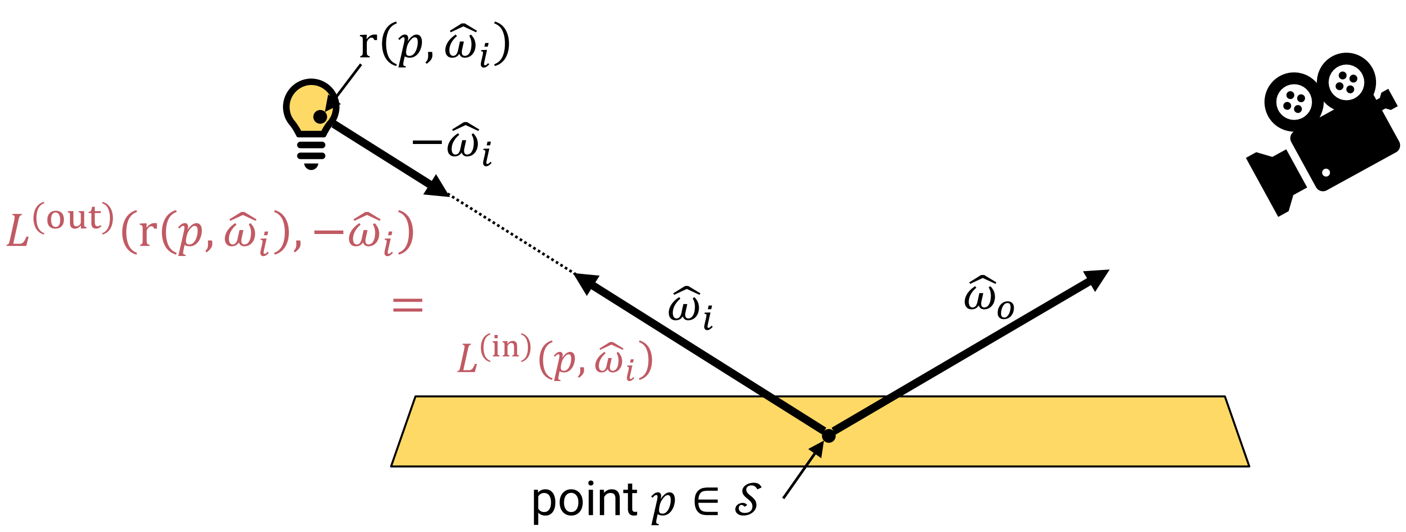


Q. Any relationship between $L^{(\mathrm{out})}$ and $L^{(\mathrm{in})}$?

Rendering equation
(light transport equation)

$$L^{(\mathrm{out})}(p, \widehat{\omega}_o) = L_e(p, \omega_{\hat{o}}) + \int_{\mathbb{S}^2} L^{(\mathrm{in})}(p, \widehat{\omega}_i) f_s(p, \widehat{\omega}_i, \widehat{\omega}_o) |\hat{n} \cdot \widehat{\omega}_i| \mathrm{d}\widehat{\omega}_i$$

Rendering equation

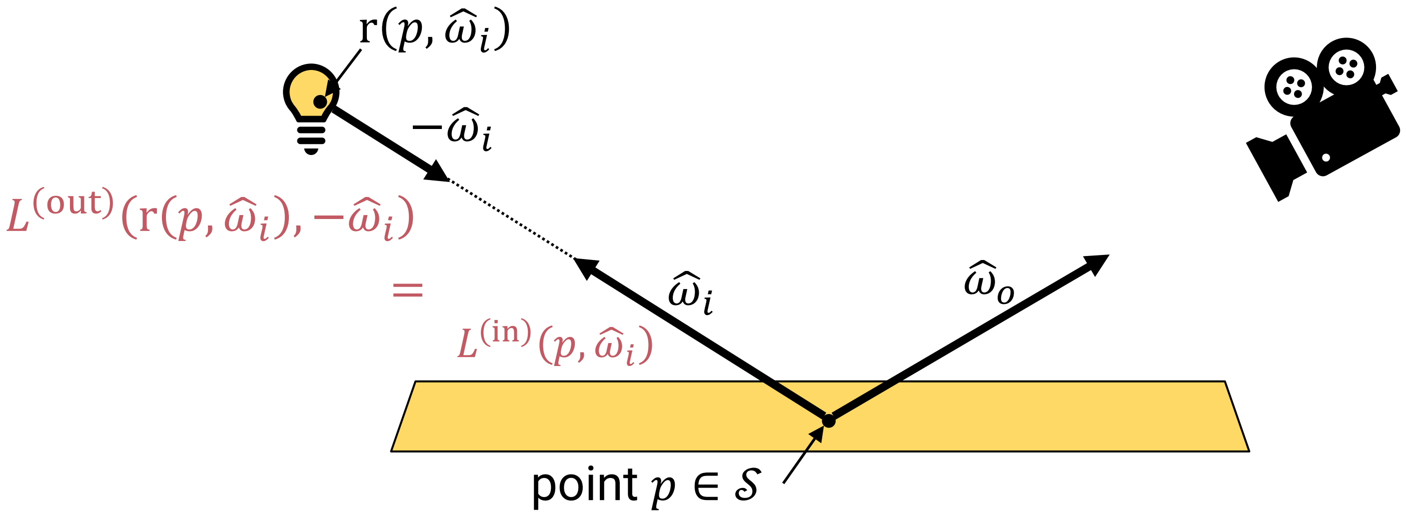


Q. Any relationship between $L^{(\mathrm{out})}$ and $L^{(\mathrm{in})}$?

Rendering equation
(light transport equation)

$$L^{(\mathrm{out})}(p, \hat{\omega}_o) = L_e(p, \omega_{\hat{o}}) + \int_{\mathbb{S}^2} L^{(\mathrm{out})}(\mathrm{r}(p, \hat{\omega}_i), -\hat{\omega}_i) f_s(p, \hat{\omega}_i, \hat{\omega}_o) |\hat{n} \cdot \hat{\omega}_i| \mathrm{d}\hat{\omega}_i$$

Rendering equation



Q. Any relationship between $L^{(\text{out})}$ and $L^{(\text{in})}$?

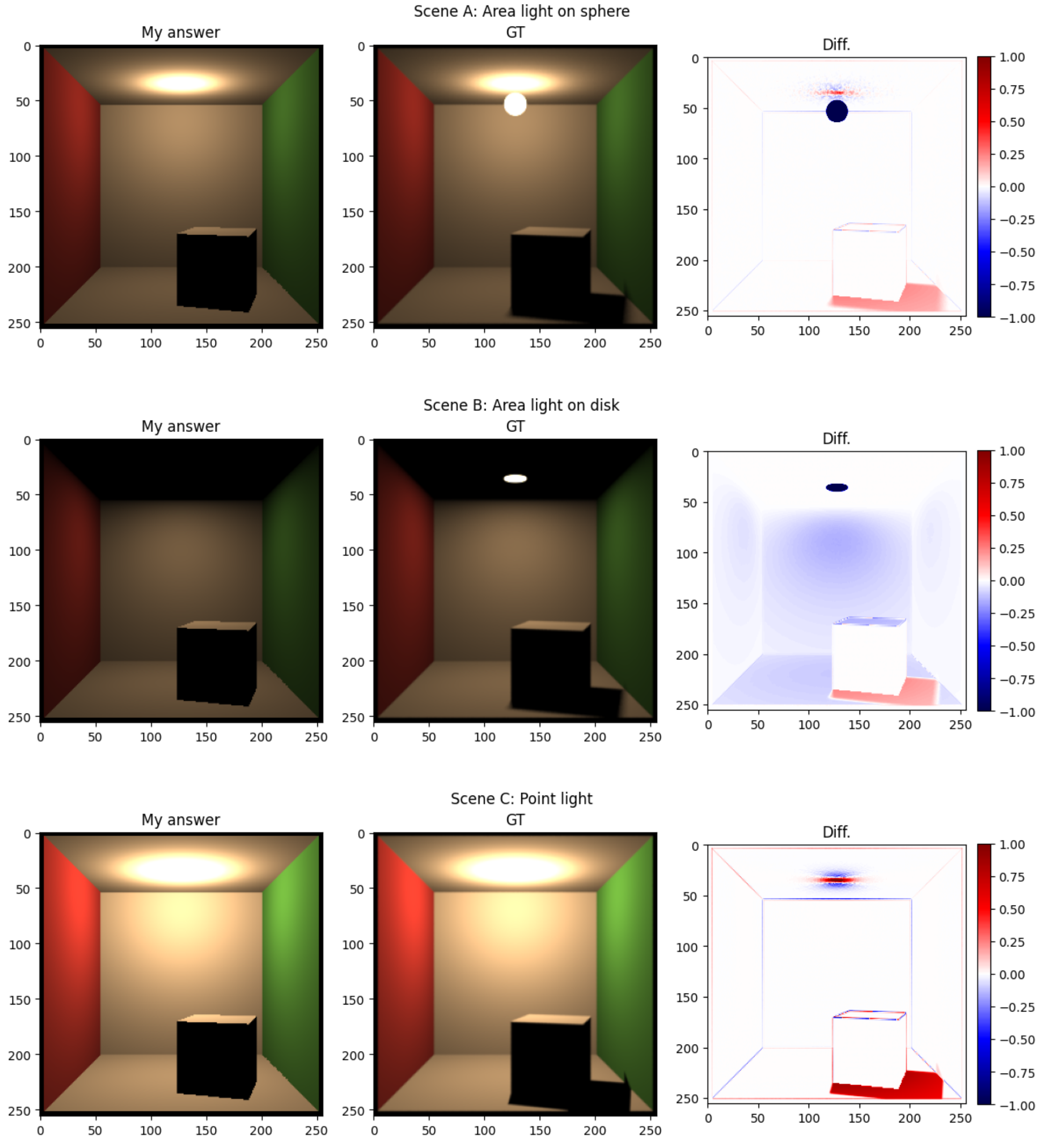
Rendering equation
(light transport equation)

$$L^{(\text{out})}(p, \hat{\omega}_o) = L_e(p, \omega_{\hat{o}}) + \int_{\mathbb{S}^2} L^{(\text{out})}(\mathrm{r}(p, \hat{\omega}_i), -\hat{\omega}_i) f_s(p, \hat{\omega}_i, \hat{\omega}_o) |\hat{n} \cdot \hat{\omega}_i| \mathrm{d}\hat{\omega}_i$$

- TODO: sphere light -> point light converge HW problem

HW 1 - Problem 3: Direct illuminations

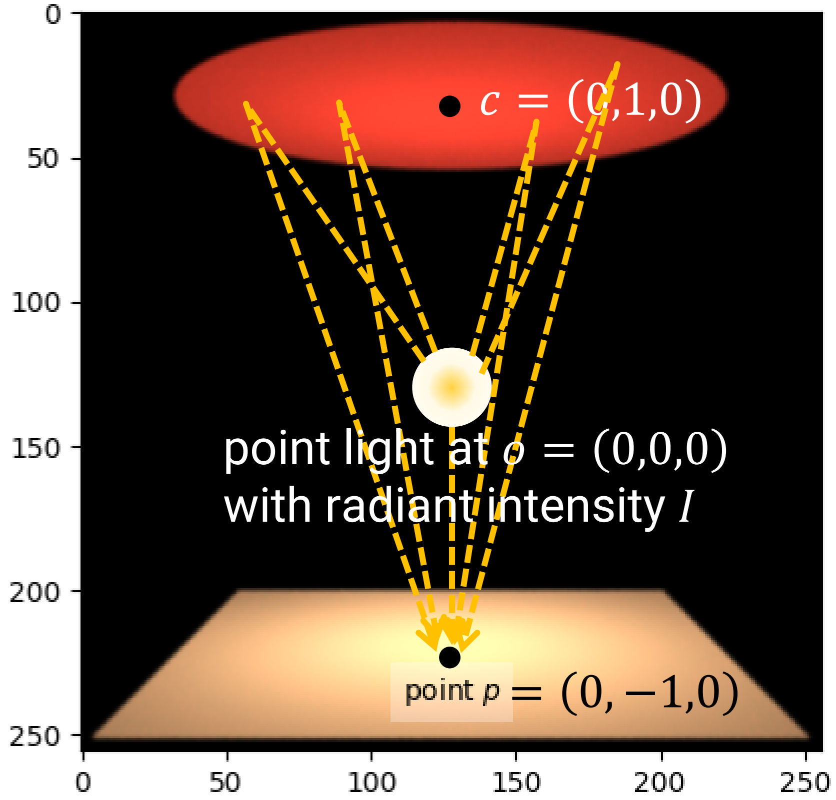
- Evaluate reflected radiance from surfaces with diffuse BRDFs in the scene, where the scene has the following emitter:
  A. Area source on a sphere
  B. Area source on a disk
  C. Point source



- Recall the rendering equation.
- Assume the source is sufficiently small.

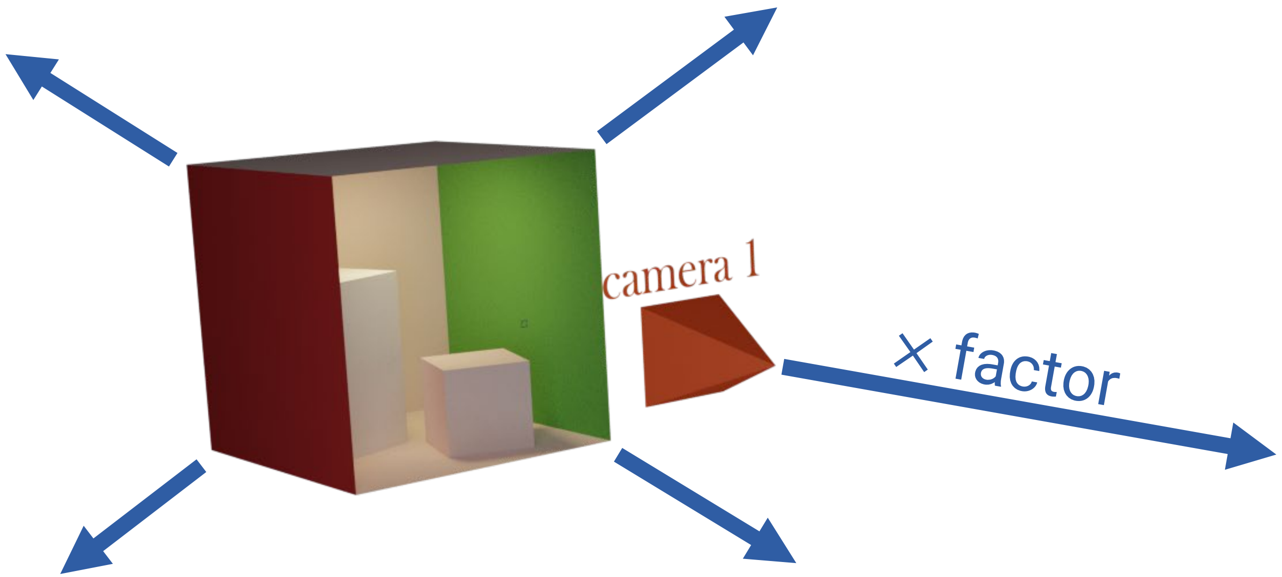
HW 1 - Problem 4: Indirect illuminations

- Find reflected radiance at the point p, due to one-bound indirect illumination



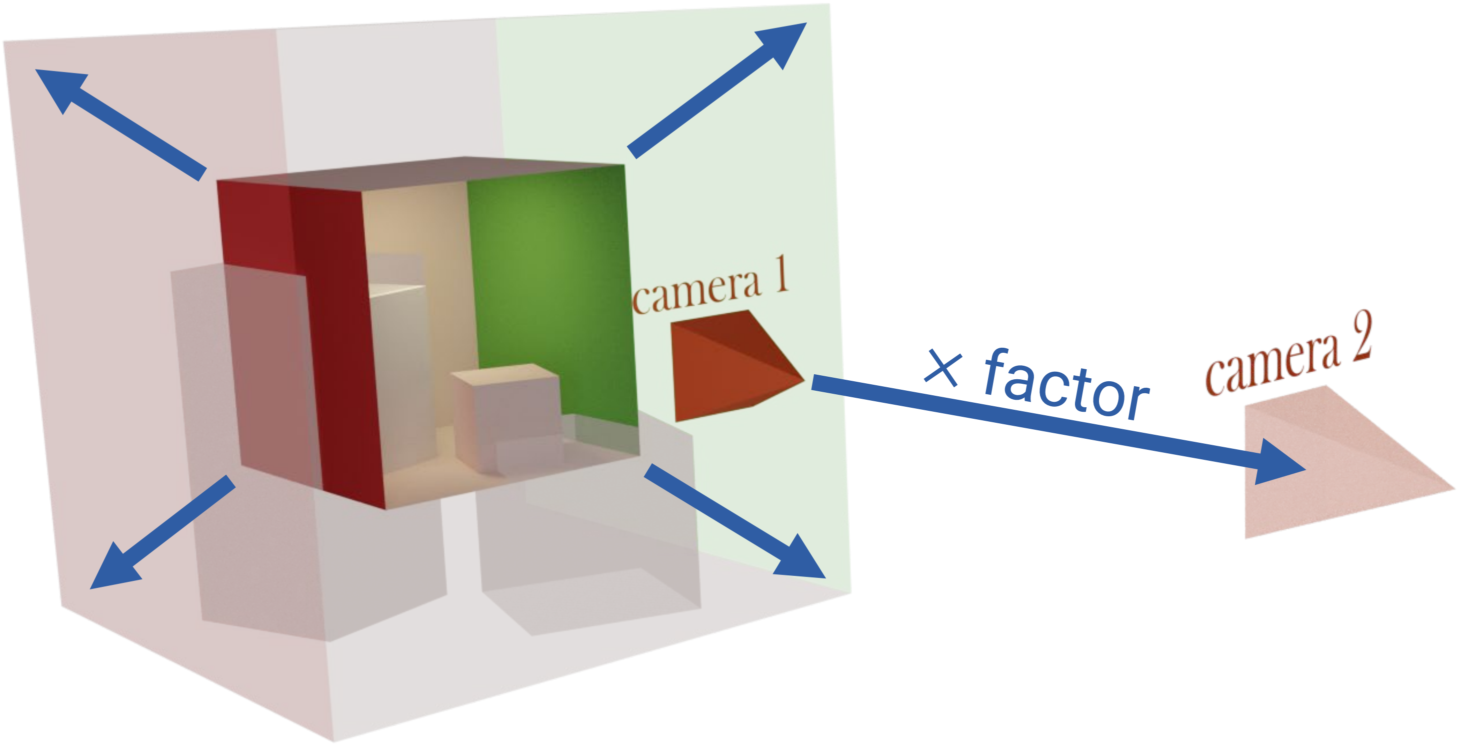
HW1 – Problem 5: Scale and physical dimensions

- Scale the entire scene geometry.



HW1 – Problem 5: Scale and physical dimensions

- Scale the entire scene geometry.
- To get the identical image from Camera 2, how should we change:
  A. Emitting radiance at area sources L_e?
  B. Emitting radiant intensity of point sources I?



HW1 – Problem 5: Scale and physical dimensions

- Recall rendering equation (+ point source)

Rendering equation
(light transport equation)

$$L^{(\text{out})}(p,\hat{\omega}_o) = L_e(p,\omega_{\hat{o}}) + \int_{\mathbb{S}^2} L^{(\text{out})}(\mathrm{r}(p,\hat{\omega}_i),-\hat{\omega}_i) f_s(p,\hat{\omega}_i,\hat{\omega}_o)|\hat{n}\cdot\hat{\omega}_i|\mathrm{d}\hat{\omega}_i$$

Practice

There is a point light source at p_L with the radiant intensity $I(\hat{\omega})$.
What is the reflected radiance at p on a surface $\mathcal{S}$, along $\hat{\omega}_o$?

$$L^{(\text{out})}(p,\hat{\omega}_o) = f_s(p,\hat{\omega}_i,\hat{\omega}_o)\frac{I(\hat{\omega}_{p_L p})|\hat{n}\cdot\hat{\omega}_{pp_L}|}{\|p-p_L\|^2}$$

HW1 – Problem 5: Scale and physical dimensions

- Recall rendering equation (+ point source)

Rendering equation
(light transport equation)

$$L^{(\mathrm{out})}(p,\widehat{\omega}_o) = L_e(p,\omega_{\hat{o}}) + \int_{\mathbb{S}^2} L^{(\mathrm{out})}(\mathrm{r}(p,\widehat{\omega}_i),-\widehat{\omega}_i) f_s(p,\widehat{\omega}_i,\widehat{\omega}_o)|\hat{n}\cdot\widehat{\omega}_i|\mathrm{d}\widehat{\omega}_i$$

$$+f_s(p,\widehat{\omega}_i,\widehat{\omega}_o)\frac{I\left(\widehat{\omega}_{p_L p}\right)\left|\hat{n}\cdot\widehat{\omega}_{PP_L}\right|}{\|p-p_L\|^2}$$

HW1 – Problem 5: Scale and physical dimensions

- Recall rendering equation (+ point source)

Rendering equation
(light transport equation)

$$L^{(\mathrm{out})}(p,\widehat{\omega}_o) = L_e(p,\omega_{\hat{o}}) + \int_{\mathbb{S}^2} L^{(\mathrm{out})}(\mathrm{r}(p,\widehat{\omega}_i),-\widehat{\omega}_i) f_s(p,\widehat{\omega}_i,\widehat{\omega}_o)|\hat{n}\cdot\widehat{\omega}_i|\mathrm{d}\widehat{\omega}_i$$

$$+f_s(p,\widehat{\omega}_i,\widehat{\omega}_o)\frac{I(\widehat{\omega}_{p_L p})|\hat{n}\cdot\widehat{\omega}_{PP_L}|}{\|p-p_L\|^2}$$

■ : want to make consistent

HW1 – Problem 5: Scale and physical dimensions

- Recall rendering equation (+ point source)

Rendering equation
(light transport equation)

$$L^{(\text{out})}(p,\hat{\omega}_o) = L_e(p,\omega_{\hat{o}}) + \int_{\mathbb{S}^2} L^{(\text{out})}(\mathrm{r}(p,\hat{\omega}_i),-\hat{\omega}_i) f_s(p,\hat{\omega}_i,\hat{\omega}_o) |\hat{n}\cdot\hat{\omega}_i| \mathrm{d}\hat{\omega}_i$$

$$+ f_s(p,\hat{\omega}_i,\hat{\omega}_o) \frac{I(\hat{\omega}_{p_L p}) |\hat{n}\cdot\hat{\omega}_{p p_L}|}{\|p - p_L\|^2}$$

■ : want to make consistent
or already consistent

HW1 – Problem 5: Scale and physical dimensions

- Recall rendering equation (+ point source)

Rendering equation
(light transport equation)

$$L^{(\mathrm{out})}(p, \widehat{\omega}_o) = L_e(p, \omega_{\hat{o}}) + \int_{\mathbb{S}^2} L^{(\mathrm{out})}(\mathrm{r}(p, \widehat{\omega}_i), -\widehat{\omega}_i) f_s(p, \widehat{\omega}_i, \widehat{\omega}_o) |\hat{n} \cdot \widehat{\omega}_i| \mathrm{d}\widehat{\omega}_i$$

$$+ f_s(p, \widehat{\omega}_i, \widehat{\omega}_o) \frac{I(\widehat{\omega}_{p_L p}) |\hat{n} \cdot \widehat{\omega}_{p p_L}|}{\|p - p_L\|^2}$$

(blue): want to make consistent or already consistent

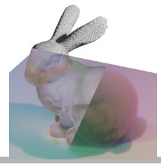
(red): becomes × factor^2

Q. Then how much factor do we need for L_e, I, and f_s?

Q. Observation with units?

- Radiance: $\mathrm{W/sr \cdot m^2}$, radiant intensity: cd, BRDF: sr^{-1}

(Advanced) … for volume rendering



- Recall rendering equation (+ point source)

Radiative transfer equation

$$\boxed{\hat{\omega}\cdot\nabla L(p,\hat{\omega})} = l_e(p,\omega_{\hat{o}}) - \mu_t L(p,\hat{\omega}) + \mu_s\int_{\mathbb{S}^2} L(p,\hat{\omega}')p(p,\hat{\omega}',\hat{\omega})\mathrm{d}\hat{\omega}_i$$

: want to make consistent
or already consistent

: becomes $\times\ \mathrm{factor}^{-1}$

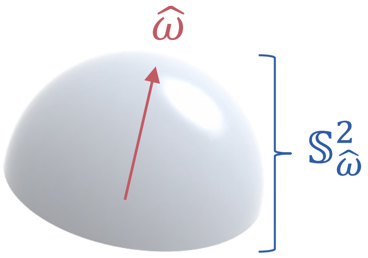
Q. Then how much factor do we need for $L_{e,\nu}$, μ_t, μ_s, and p?

Q. Observation with units?

Notation table

Sets

$\mathbb{R}^n$	Euclidean space
$\mathbb{S}^2$	the unit sphere (the set of all unit vectors)
$\mathbb{S}^2_{\hat{\omega}}$	the hemisphere facing a direction $\hat{\omega} \in \mathbb{S}^2$



Convention of variables

$p \in \mathbb{R}^3$	point in the space (or a surface)
$\hat{\omega} \in \mathbb{S}^2$	direction (unit vector)
• $\hat{\omega}_{p_1 p_2} := \frac{p_2 - p_1}{\lVert p_2 - p_1 \rVert}$ for any $p_1, p_2 \in \mathbb{R}^3$	
$\hat{n} \in \mathbb{S}^2$	surface normal, where a point and a surface are given in context
$\mathcal{S} \subset \mathbb{R}^3$	surface
$\mathcal{V} \subset \mathbb{R}^3$	volume
$\Omega \subset \mathbb{S}^2$	solid angle (region on the unit sphere $\mathbb{S}^2$)

Radiometric quantities

* time dependency is omitted for simplicity
* (· ", Ω") is usually omitted and assumed as an entire $\mathbb{S}^2$ or hemisphere

$\Phi(\mathcal{S}, \Omega)$ [W]	radiant power (flux) at a surface $\mathcal{S} \subset \mathbb{R}^3$ and a solid angle $\Omega \subset \mathbb{S}^2$
$I(\hat{\omega})$ [W/sr]	radiant intensity at a direction $\hat{\omega} \in \mathbb{S}^2$, where a point source is given in context
$E(p, \Omega)$ [W/m^2]	irradiance at $p \in \mathcal{S}$ and $\Omega \subset \mathbb{S}^2$, where the surface $\mathcal{S} \subset \mathbb{R}^3$ is given in context
$L(p, \omega)$ [W/m^{2}sr]	radiance at $p \in \mathbb{R}^3$ and $\hat{\omega} \in \mathbb{S}^2$
$f_s(p, \omega_i, \omega_o)$ [sr^{-1}]	BSDF at $p \in \mathcal{S}$ from $\hat{\omega}_i \in \mathbb{S}^2$ to $\hat{\omega}_o \in \mathbb{S}^2$, where the surface $\mathcal{S}$ is given in context