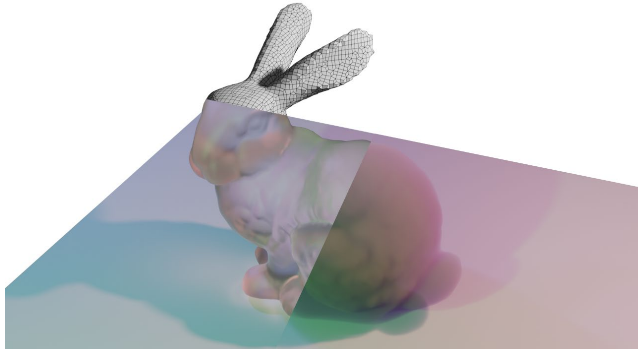
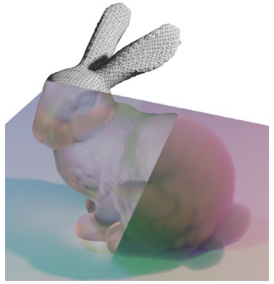


2. Probability and Statistical Inference

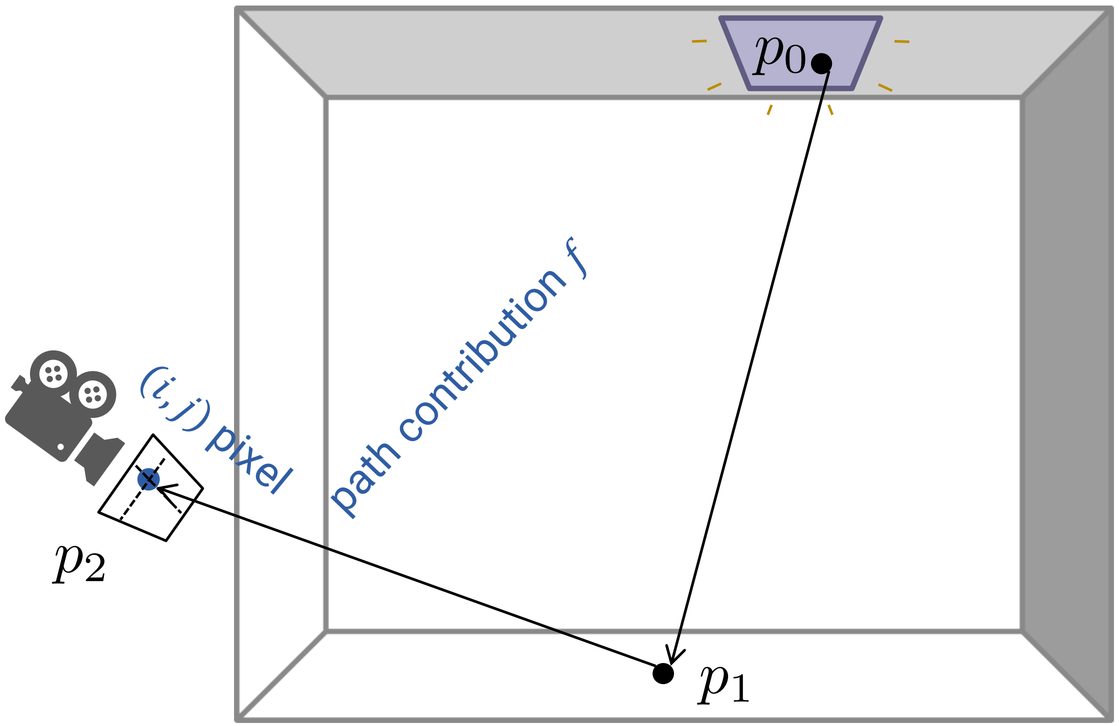
Physically Based Rendering

Shinyoung Yi (이신영)



Preview: a simple path tracing

Rendering equation for direct illumination



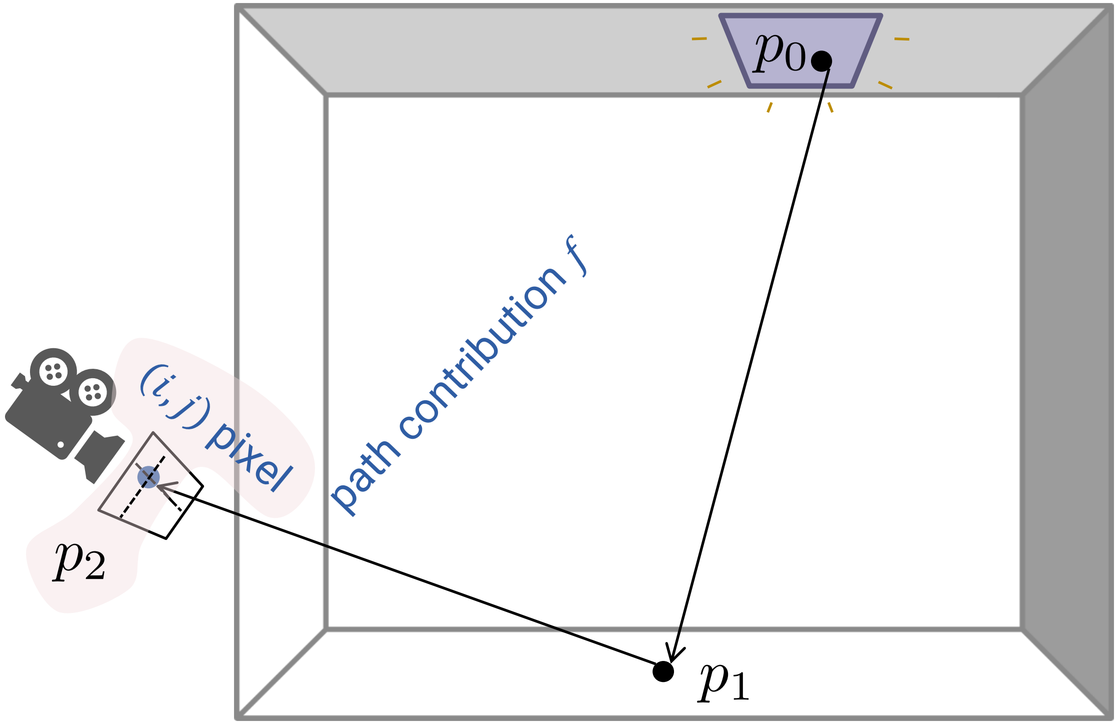
Terminology and convention:

- Notation / actual light: $p_0 \rightarrow p_1 \rightarrow p_2$
- Computation: $p_2 \rightarrow p_1 \rightarrow p_0$

- # of bounces < depth < # of vertices

1 **2** **3**

Rendering equation for direct illumination



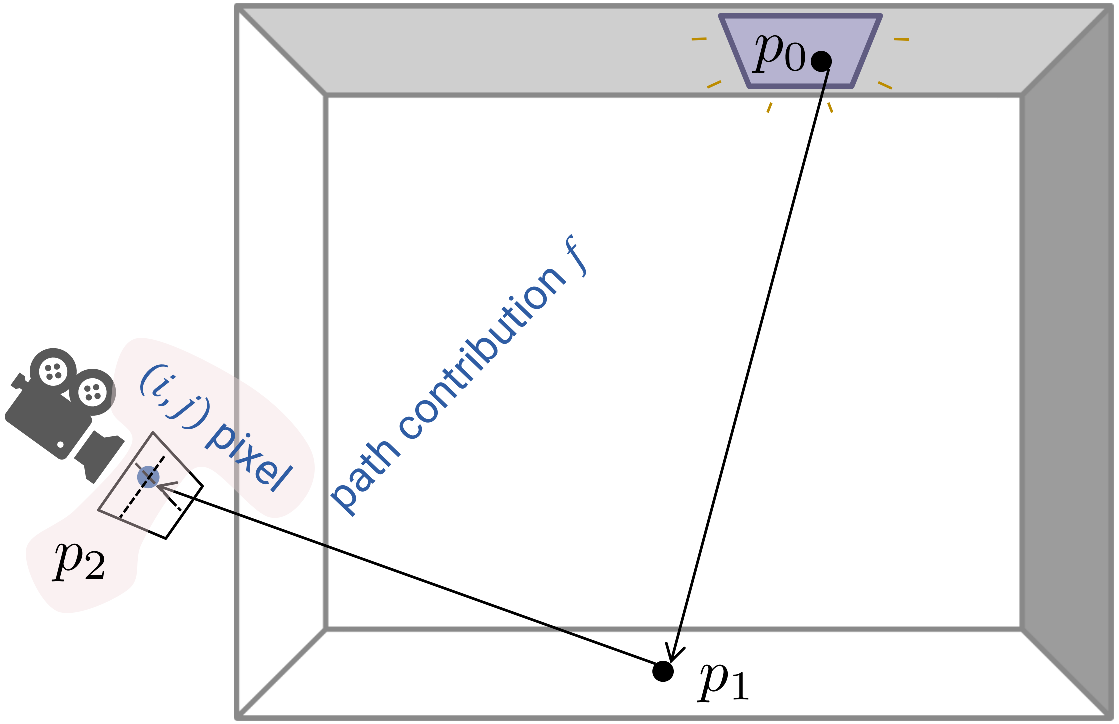
Notation:

- $\widehat{\omega}_{p_0 p_1} := \frac{p_1 - p_0}{\|p_1 - p_0\|}$

Final measurement:

$$I^{(i,j)} = \int_{\mathbb{S}} W_e^{(i,j)}(\hat{\omega})\, L(p_2, \hat{\omega})\, \mathrm{d}\hat{\omega}$$

Rendering equation for direct illumination



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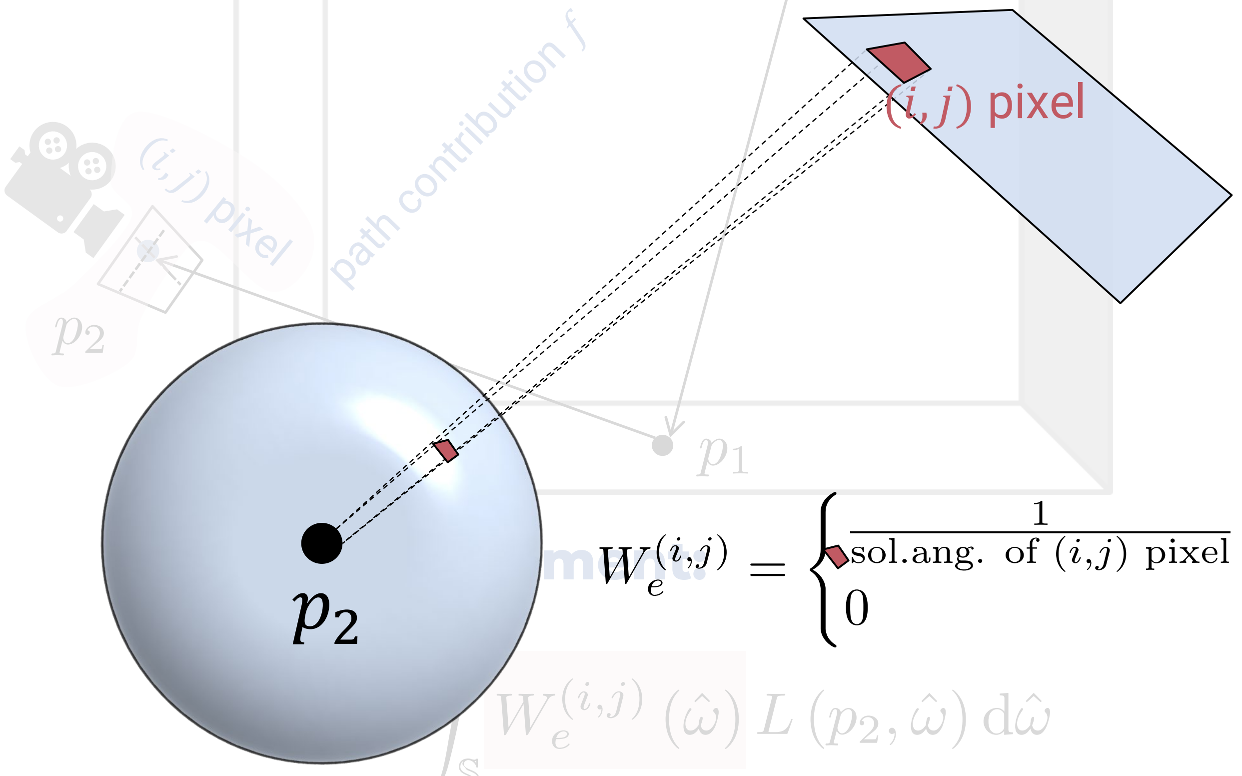
Final measurement:

$$I^{(i,j)} = \int_{\mathbb{S}} W_e^{(i,j)}(\hat{\omega})\, L(p_2, \hat{\omega})\, \mathrm{d}\hat{\omega}$$

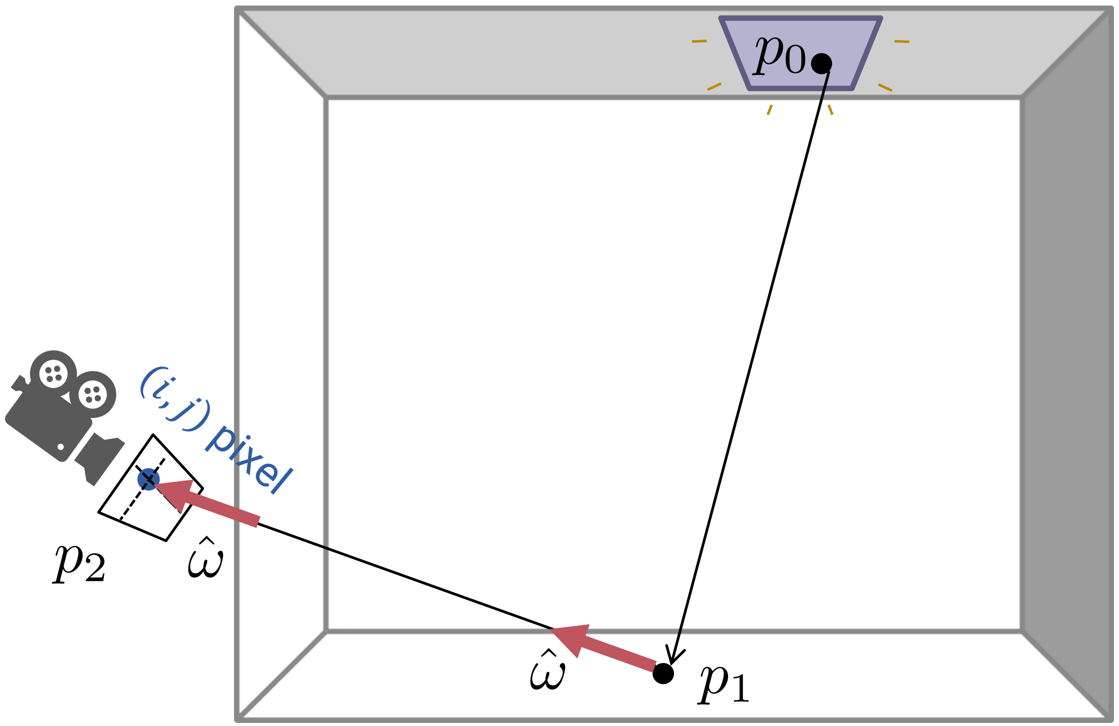
Rendering equation for direct illumination

Importance (pixel reconstruction filter) $W_e(p, \widehat{\omega})$



$$W_e^{(i,j)} = \begin{cases} \frac{1}{\text{sol.ang. of } (i,j) \text{ pixel}} & \hat{\omega} \text{ in } (i,j) \text{ pixel} \\ 0 & \text{otherwise} \end{cases}$$

Rendering equation for direct illumination



Notation:

- $\widehat{\omega}_{p_0 p_1} := \frac{p_1 - p_0}{\|p_1 - p_0\|}$
- r: $\mathbb{R}^3 \times \mathbb{S}^2 \rightarrow \mathbb{R}^3$
  - ray intersection

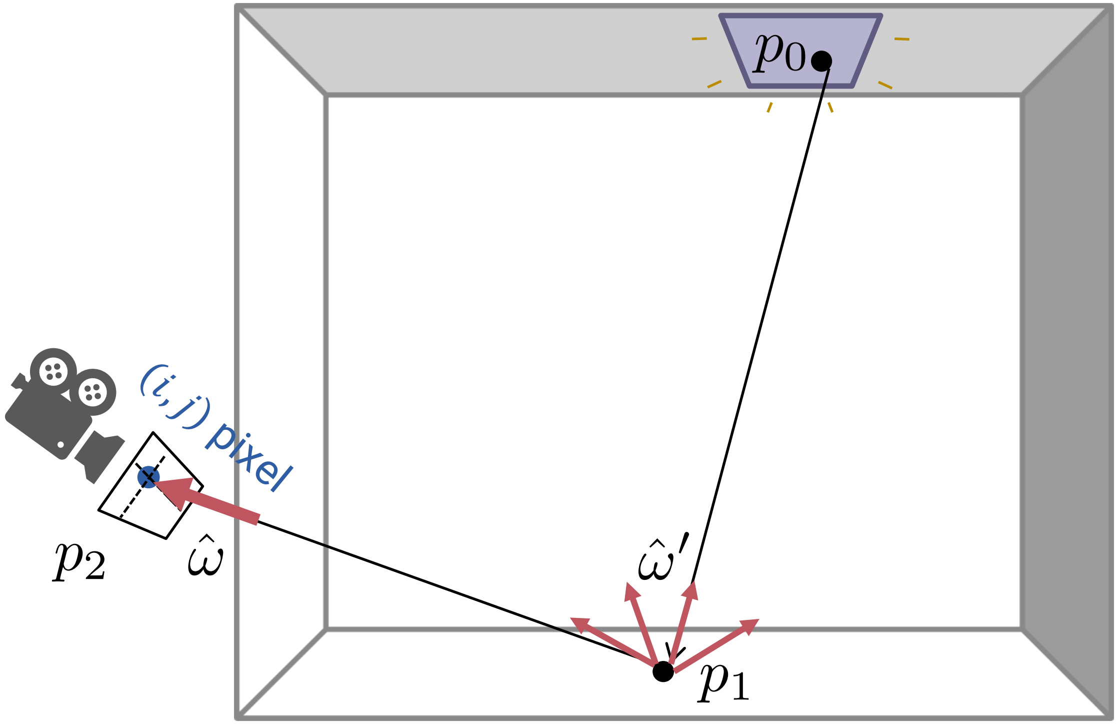
Final measurement:

$$I^{(i,j)} = \int_{\mathbb{S}} W_e^{(i,j)}(\hat{\omega}) \, L(p_2, \hat{\omega}) \, \mathrm{d}\hat{\omega}$$

$$L(p_2, \hat{\omega}) = L(p_1, \hat{\omega})$$

$p_1 = \mathrm{r}(p_2, -\hat{\omega})$

Rendering equation for direct illumination



Notation:

- $\widehat{\omega}_{p_0 p_1} := \frac{p_1 - p_0}{\|p_1 - p_0\|}$
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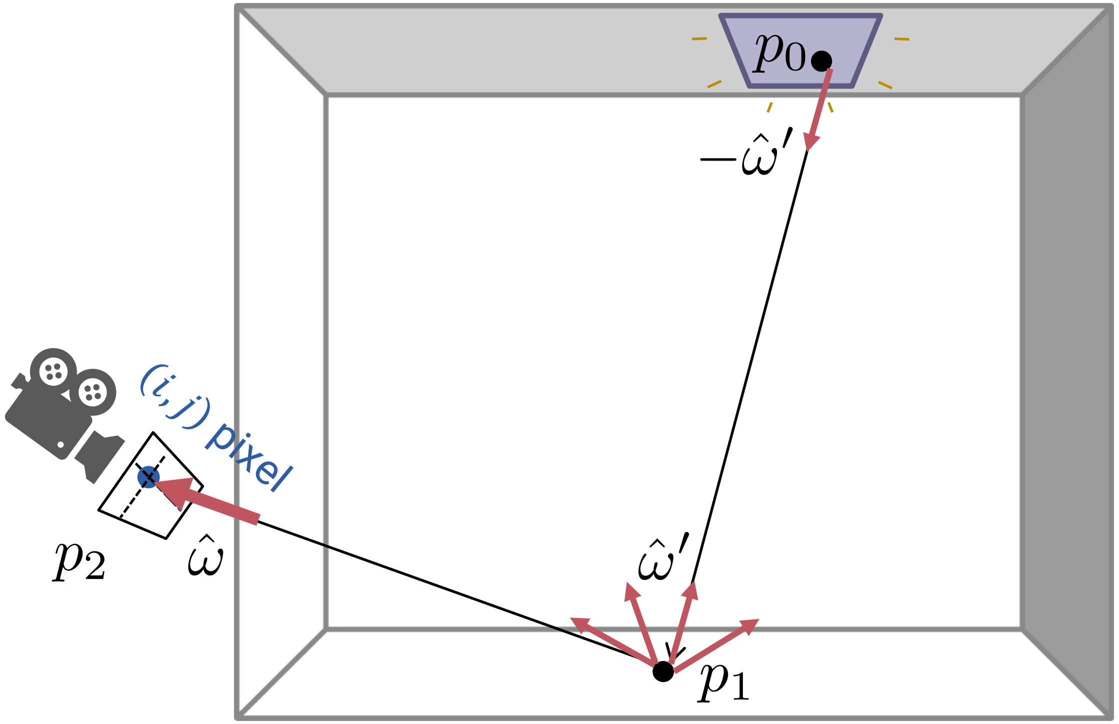
Final measurement:

$$I^{(i,j)} = \int_{\mathbb{S}} W_e^{(i,j)}(\hat{\omega})\, L(p_1, \hat{\omega})\, \mathrm{d}\hat{\omega}$$

$p_1 = \mathrm{r}(p_2, -\hat{\omega})$

$$L(p_1, \hat{\omega}) = L_e(p_1, \hat{\omega}) + \int_{\mathbb{S}} L^{(\mathrm{in})}(p_1, \hat{\omega}')\, \rho(p_1, \hat{\omega}', \hat{\omega})\, |\hat{n} \cdot \hat{\omega}'|\, \mathrm{d}\hat{\omega}'$$

Rendering equation for direct illumination



Notation:

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- r: $\mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}^3$
  - ray intersection

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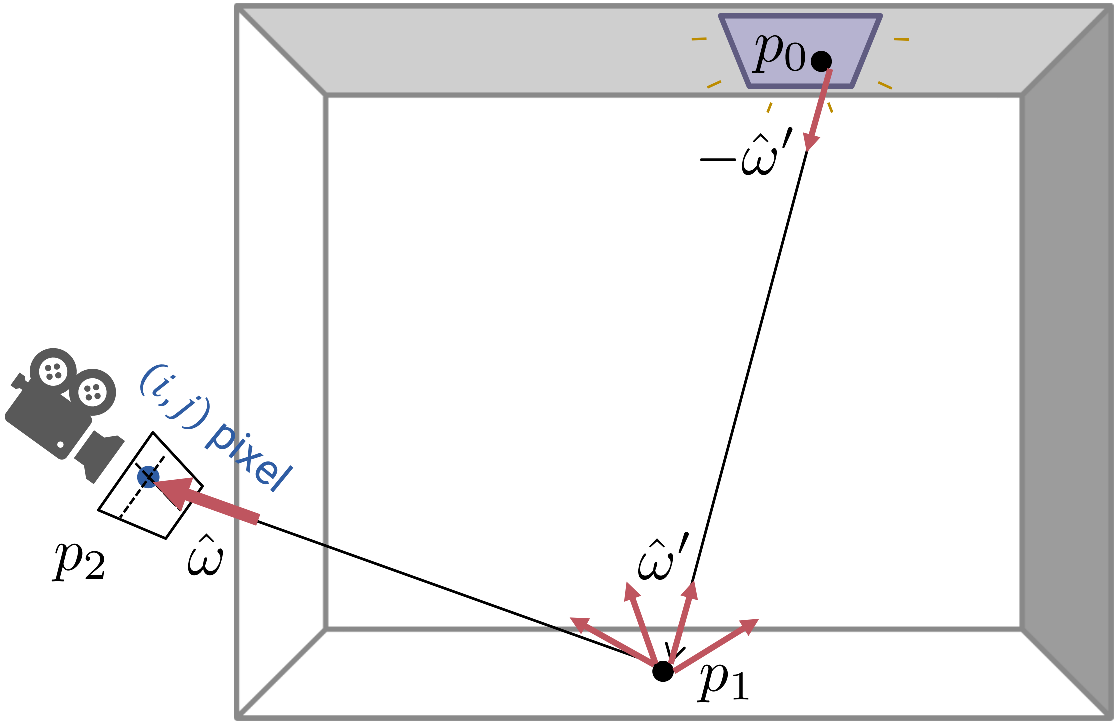
$$L(p_1, \hat{\omega}) = L_e(p_1, \hat{\omega}) + \int_{\mathbb{S}} L^{(\mathrm{in})}(p_1, \hat{\omega}')\, \rho(p_1, \hat{\omega}', \hat{\omega})\, |\hat{n} \cdot \hat{\omega}'|\, \mathrm{d}\hat{\omega}'$$

$p_0 = \mathrm{r}(p_1, \hat{\omega}')$

$$L^{(\mathrm{in})}(p_1, \hat{\omega}') = L(p_0 \ \ -\hat{\omega}')$$

Rendering equation for direct illumination



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Final measurement:

$$I^{(i,j)} = \int_{\mathbb{S}} W_e^{(i,j)}(\hat{\omega})\, L(p_1, \hat{\omega})\, \mathrm{d}\hat{\omega}$$

$p_1 = \mathrm{r}(p_2, -\hat{\omega})$

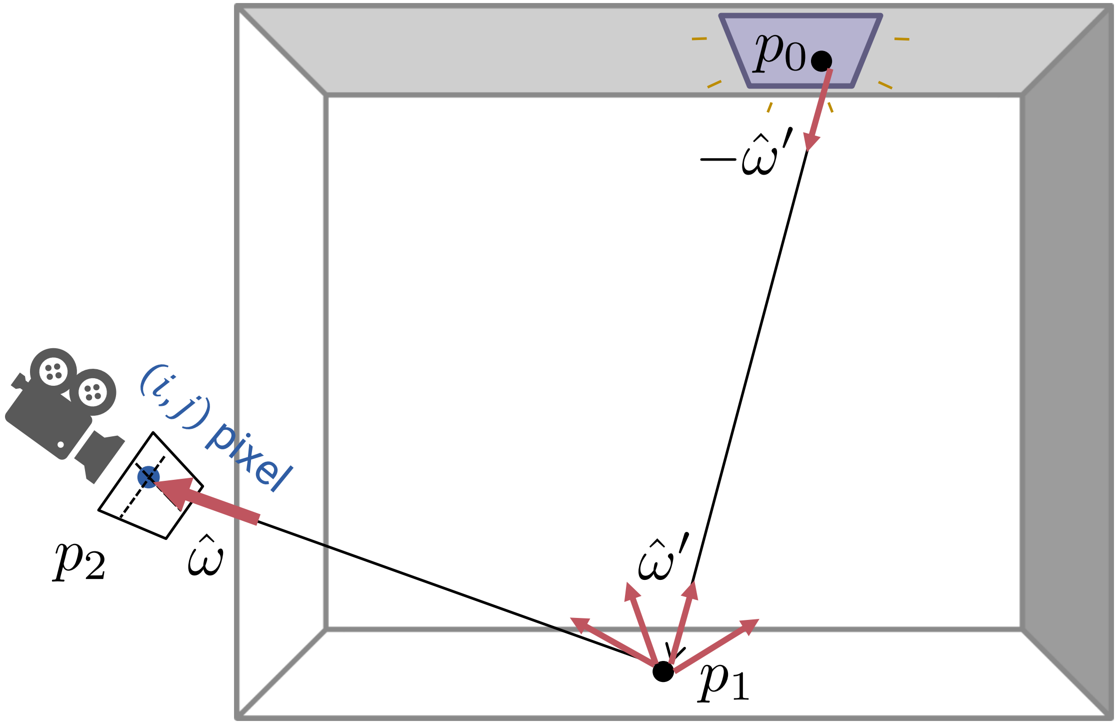
$$L(p_1, \hat{\omega}) = L_e(p_1, \hat{\omega}) + \int_{\mathbb{S}} L^{(\mathrm{in})}(p_1, \hat{\omega}')\, \rho(p_1, \hat{\omega}', \hat{\omega})\, |\hat{n} \cdot \hat{\omega}'|\, \mathrm{d}\hat{\omega}'$$

$p_0 = \mathrm{r}(p_1, \hat{\omega}')$

Direct illumination only

$$L(p_0, -\hat{\omega}') = L_e(p_0, -\hat{\omega}') + \cancel{\int_{\mathbb{S}} L^{(\mathrm{in})}(p_0, \hat{\omega}'')\, \rho(p_0, \hat{\omega}'', -\hat{\omega}')\, |\hat{n} \cdot \hat{\omega}''|\, \mathrm{d}\hat{\omega}''}$$

Rendering equation for direct illumination



Notation:

- $\widehat{\omega}_{p_0 p_1} \coloneqq \frac{p_1 - p_0}{\|p_1 - p_0\|}$
- r: $\mathbb{R}^3 \times \mathbb{S}^2 \rightarrow \mathbb{R}^3$
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$$I^{(i,j)} = \int_{\mathbb{S}} W_e^{(i,j)}(\hat{\omega})\, L(p_1, \hat{\omega})\, \mathrm{d}\hat{\omega}$$

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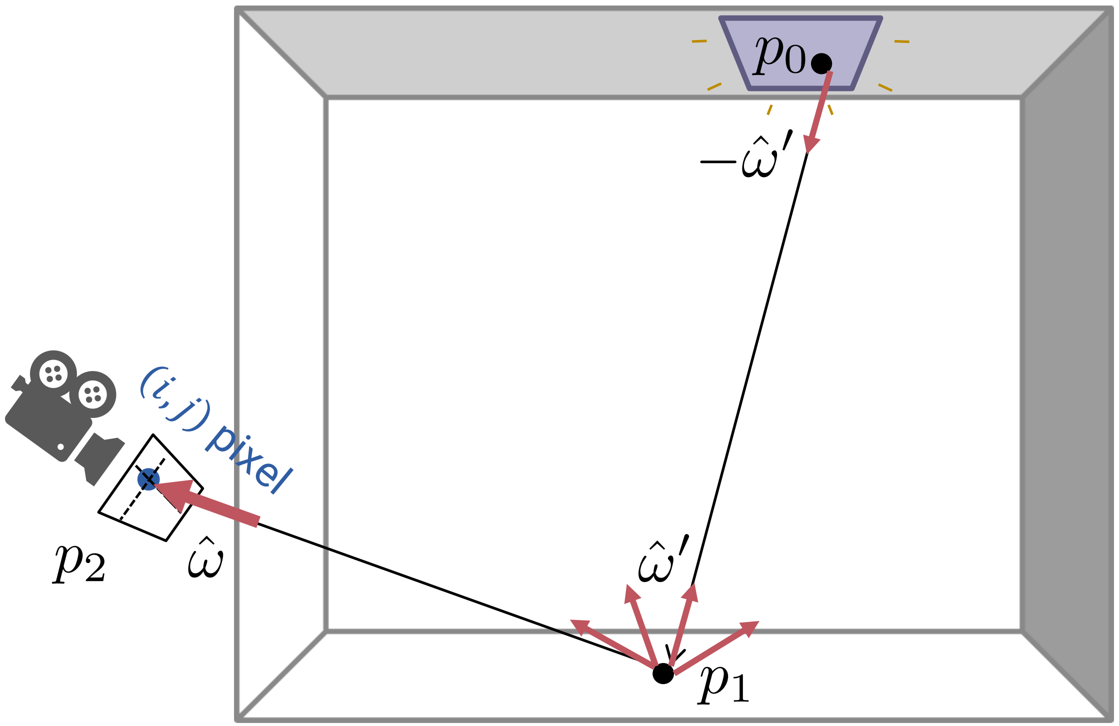
$$L(p_1, \hat{\omega}) = L_e(p_1, \hat{\omega}) + \int_{\mathbb{S}} L^{(\mathrm{in})}(p_1, \hat{\omega}')\, \rho(p_1, \hat{\omega}', \hat{\omega})\, |\hat{n} \cdot \hat{\omega}'|\, \mathrm{d}\hat{\omega}'$$

$p_0 = \mathrm{r}(p_1, \hat{\omega}')$

$$L(p_0, -\hat{\omega}') = L_e(p_0, -\hat{\omega}')$$

Rendering equation for direct illumination



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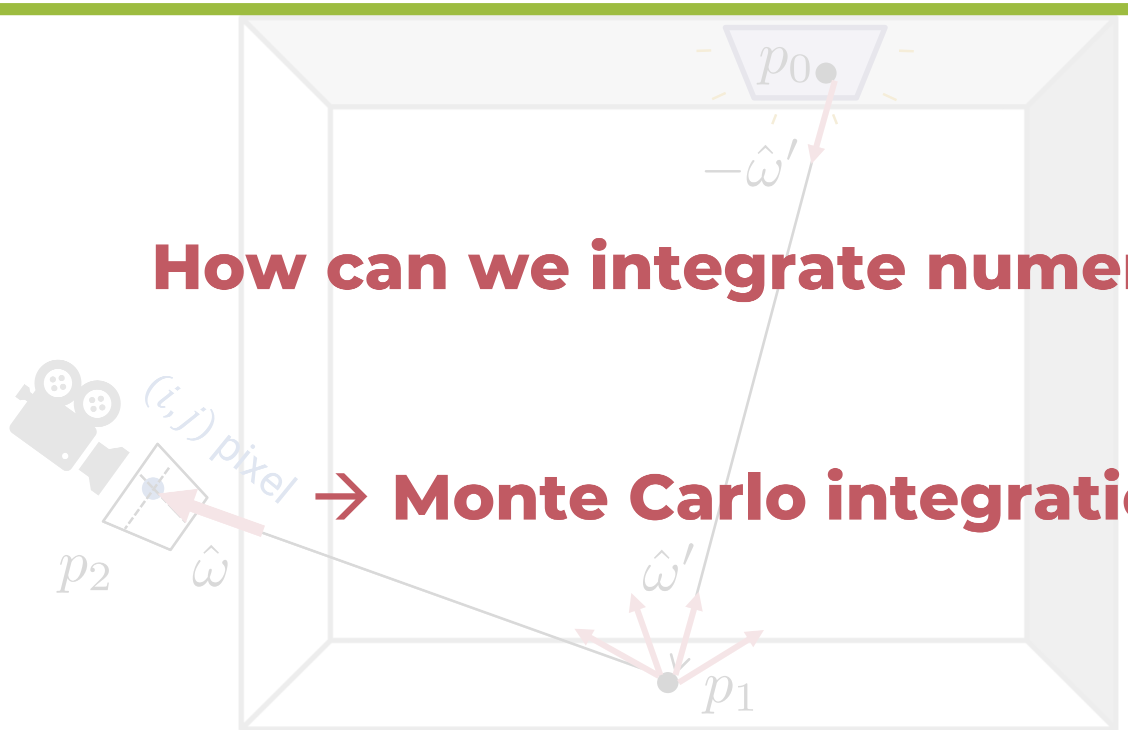
$p_1 = \mathrm{r}(p_2, -\hat{\omega})$
$p_0 = \mathrm{r}(p_1, \hat{\omega}')$

Rendering equation for direct illumination



Notation:

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- r : $\mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}^3$
  - ray intersection

How can we integrate numerically?

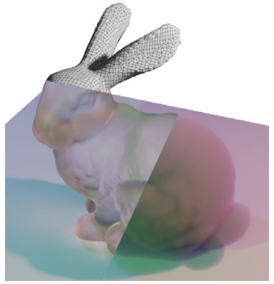
→ Monte Carlo integration

Final measurement:

$$I^{(i,j)} = \int_{\mathbb{S}} W_e^{(i,j)}(\hat{\omega})\, L(p_1, \hat{\omega})\, \mathrm{d}\hat{\omega}$$

$$L(p_1, \hat{\omega}) = L_e(p_1, \hat{\omega}) + \int_{\mathbb{S}} L_e(p_0, -\hat{\omega}')\, \rho(p_1, \hat{\omega}', \hat{\omega})\, |\hat{n} \cdot \hat{\omega}'|\, \mathrm{d}\hat{\omega}'$$

$p_1 = \mathrm{r}(p_2, -\hat{\omega})$
$p_0 = \mathrm{r}(p_1, \hat{\omega}')$

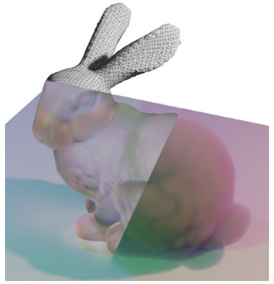


Basic Math: Sets and Functions

Sets and functions

Suppose $A, B, X, \cdots$ are sets and $f, g, \cdots$ are functions

- $f: X \to Y$, $A \subset X$ then $f(A) \coloneqq \{f(x) | x \in A\}$ is the ***image of*** A ***under*** f
- $B \subset Y$ then $f^{-1}(B) \coloneqq \{x | f(x) \in B\}$ is the ***preimage of*** B ***under*** f
  - Even if an inverse function does not exists, preimages are well defined.



Probability

What is probability?

random sample

probability

observation

distribution

random variable

Probability overview

- Sample space (probability space) Ω
- Event E
- Probability P

- Random variable X
- CDF of X
- PDF of X

Probability overview

- Sample space (probability space) Ω: a set of all things that the outcome can be.
- Event $E \subset \Omega$
- Probability $P(E) \in \mathbb{R}_{\geq 0}$
  - $P(E_1 \cup E_2) = P(E_1) + P(E_2)$ if $E_1 \cap E_2 = \phi$
  - $P(\Omega) = 1$
- Random variable $X: \Omega \rightarrow \mathbb{R}$ (codomain $\mathbb{R}^n$ or $\mathbb{S}^2$ is also okay)
- CDF of X: $F_X(x) = P(X \leq x) \coloneqq P(X^{-1}(\{y \in \mathbb{R} | y \leq x\})$
- PDF of X: some function $p_X: \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ s.t. $P(a \leq x \leq b) = \int_a^b p_X(x)\mathrm{d}x$
  - $p_X(x) = \frac{\mathrm{d}F_X(x)}{\mathrm{d}x}$ for continuous random variable

Why is probability **so confusing**?

Choice of sample space

Event: H, T

Probability: $P[H] = \frac{1}{2}, P[T] = \frac{1}{2}$

Choice 1

Sample space: $\{h, t\}$

$H = \{h\}, T = \{t\}$

$P[\{h\}] = \frac{1}{2}, P[\{t\}] = \frac{1}{2}$

Choice 2

Sample space: $\{\theta | 0 \leq \theta \leq 2\pi\}$

$H = \{\theta | 0 \leq \theta \leq \pi\}, T = \{\theta | \pi < \theta \leq 2\pi\}$

$P[\{\theta | 0 \leq \theta_1 < \theta \leq \theta_2 \leq 2\pi\}] = \frac{\theta_2 - \theta_1}{2\pi}$

Choice 3

Sample space: $\{\mathbf{R} | \mathbf{R} \in \mathrm{SO}(3)\}$

$H = \left\{\mathbf{R} \middle| 0 \leq \mathbf{R}\hat{z} \leq \frac{\pi}{2}\right\}, T = \left\{\mathbf{R}\hat{z} \middle| \frac{\pi}{2} < \mathbf{R}\hat{z} \leq 1\right\}$

$$P[A \subset \mathrm{SO}(3)] = \frac{1}{8\pi^2} \int_A \sin\beta \, \mathrm{d}\alpha \mathrm{d}\beta \mathrm{d}\gamma$$

Choice of sample space

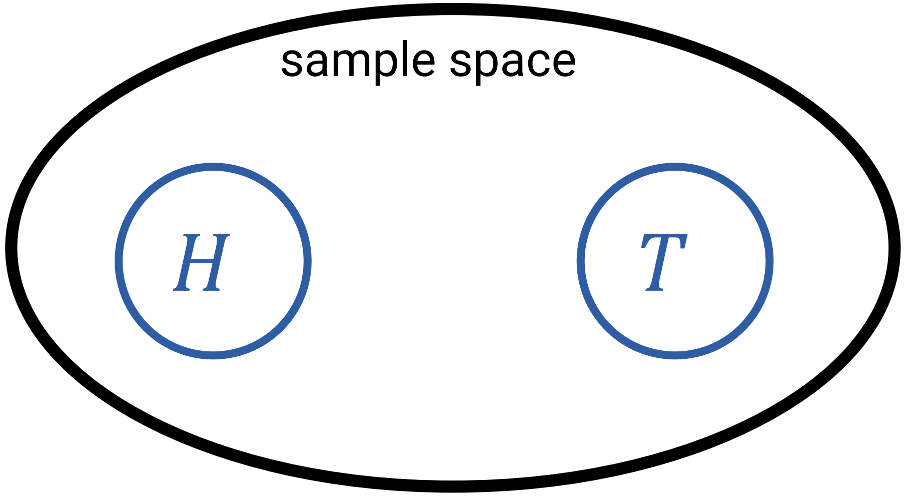
https://ko.wikipedia.org/wiki/%EB%8F%99%EC%A0%84_%EB%8D%98%EC%A7%80%EA%B8%B0#/media/%ED%8C%8C%EC%9D%BC:Coin_Toss_(3635981474).jpg

Choice 1

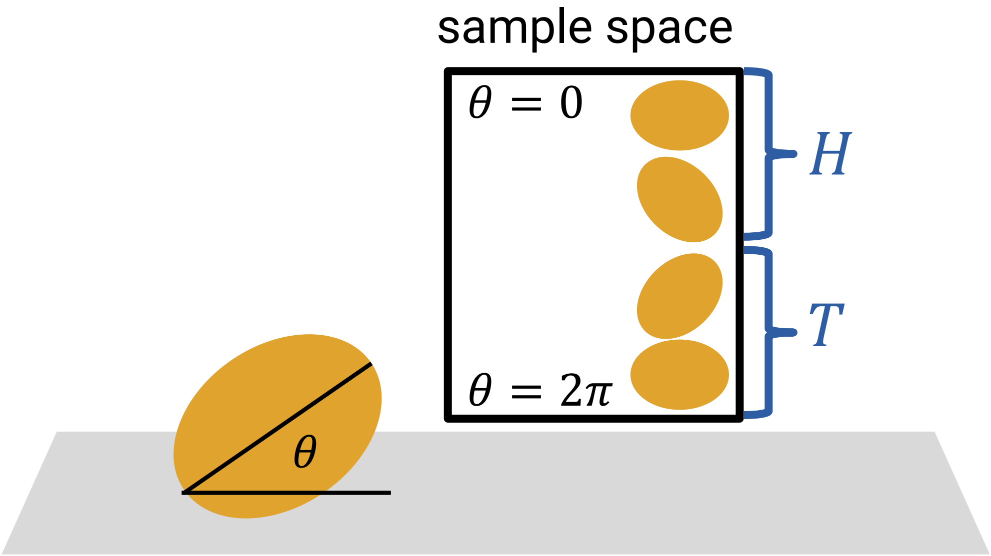
Sample space: $\{h, t\}$

$H = \{h\}, T = \{t\}$

$P[\{h\}] = \frac{1}{2}, P[\{t\}] = \frac{1}{2}$



Choice of sample space



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Choice 2

Sample space: $\{\theta | 0 \leq \theta \leq \pi\}$

$H = \{\theta | 0 \leq \theta \leq \pi\}, T = \{\theta | \pi < \theta \leq 2\pi\}$

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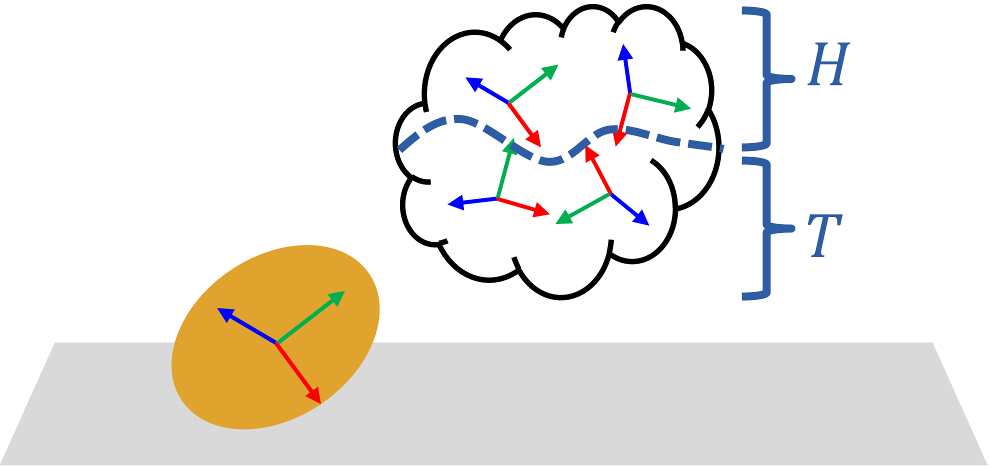
Choice of sample space

https://ko.wikipedia.org/wiki/%EB%8F%99%EC%A0%84_%EB%8D%98%EC%A7%80%EA%B8%B0#/media/%ED%8C%8C%EC%9D%BC:Coin_Toss_(3635981474).jpg

sample space = $\mathrm{SO}(3)$



Choice 1

Sample space: $\{h, t\}$

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$P[\{h\}] = \frac{1}{2}, P[\{t\}] = \frac{1}{2}$

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Sample space: $\{\theta | 0 \leq \theta \leq \pi\}$

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$P[\{\theta | 0 \leq \theta_1 < \theta \leq \theta_2 \leq 2\pi\}] = \frac{\theta_2 - \theta_1}{2\pi}$

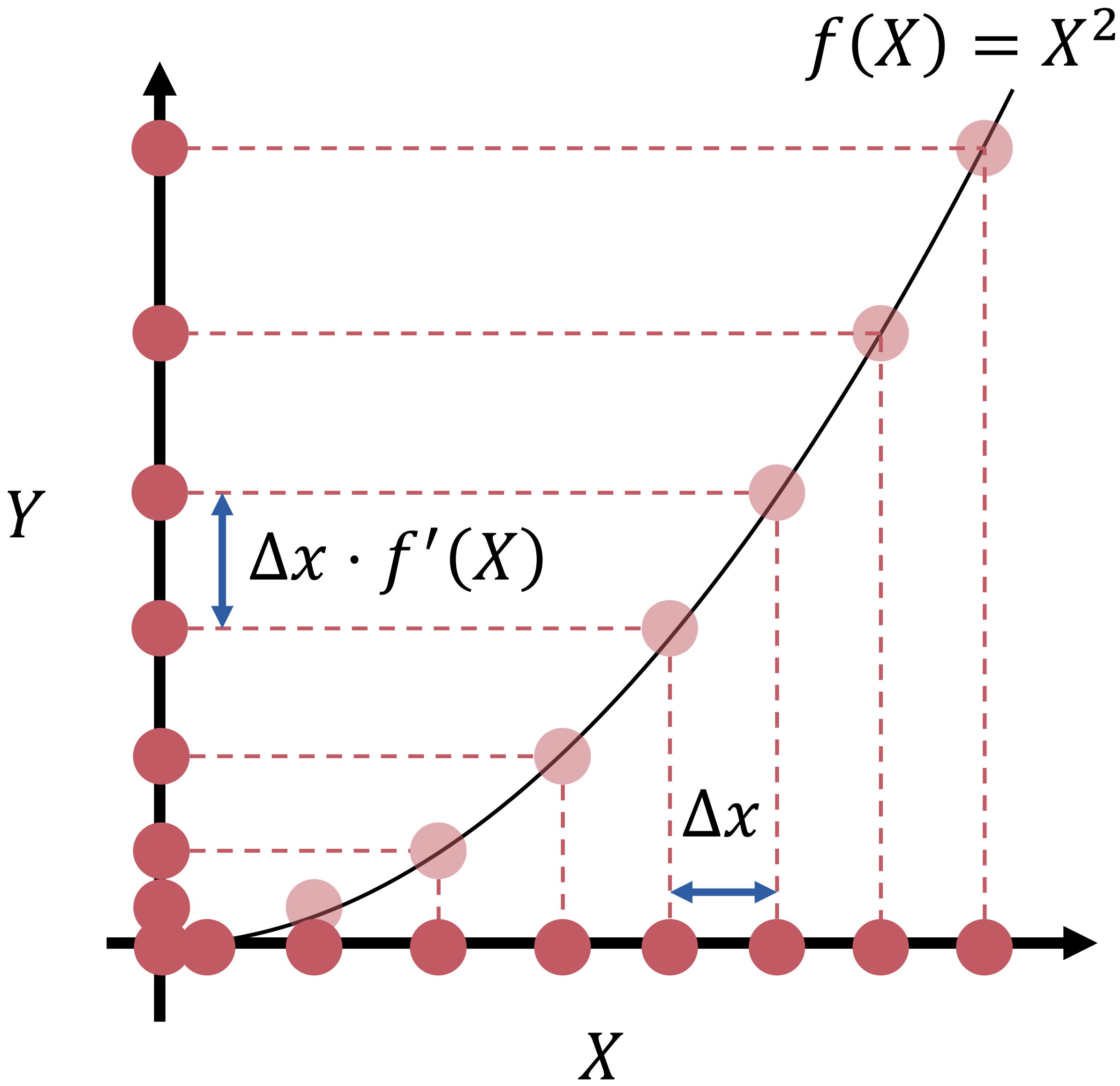
Choice 3

Sample space: $\{\mathbf{R} | \mathbf{R} \in \mathrm{SO}(3)\}$

$H = \left\{\mathbf{R} \middle| 0 \leq \mathbf{R}\hat{z} \leq \frac{\pi}{2}\right\}, T = \left\{\mathbf{R}\hat{z} \middle| \frac{\pi}{2} < \mathbf{R}\hat{z} \leq 1\right\}$

$$P[A \subset \mathrm{SO}(3)] = \frac{1}{8\pi^2} \int_A \sin\beta \, \mathrm{d}\alpha \mathrm{d}\beta \mathrm{d}\gamma$$

PDF change of variables

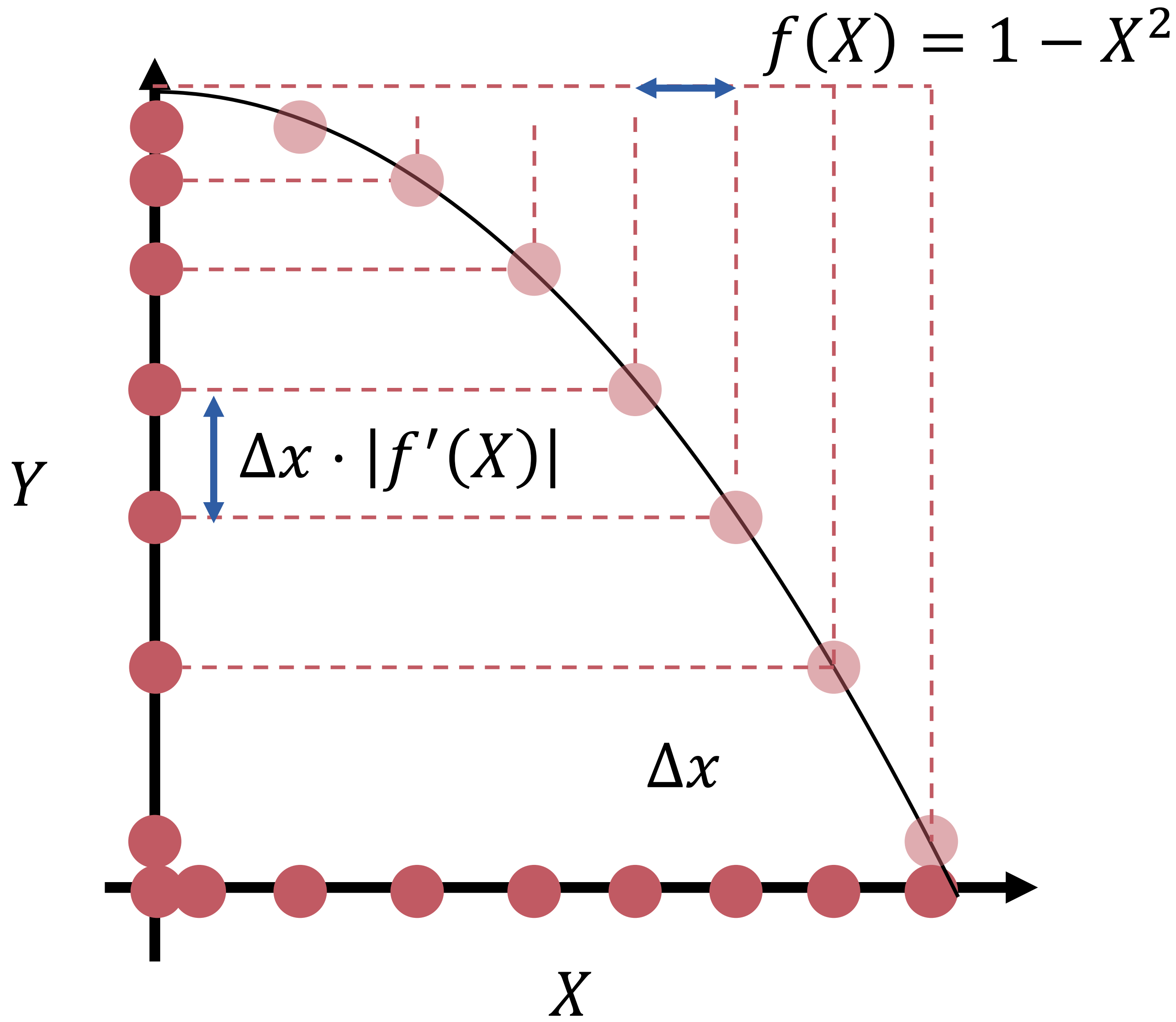


$$Y = f(X)$$

$$p_Y(y) = \frac{p_X\left(f^{-1}(y)\right)}{f'\left(f^{-1}(y)\right)} *$$

*: incomplete

PDF change of variables



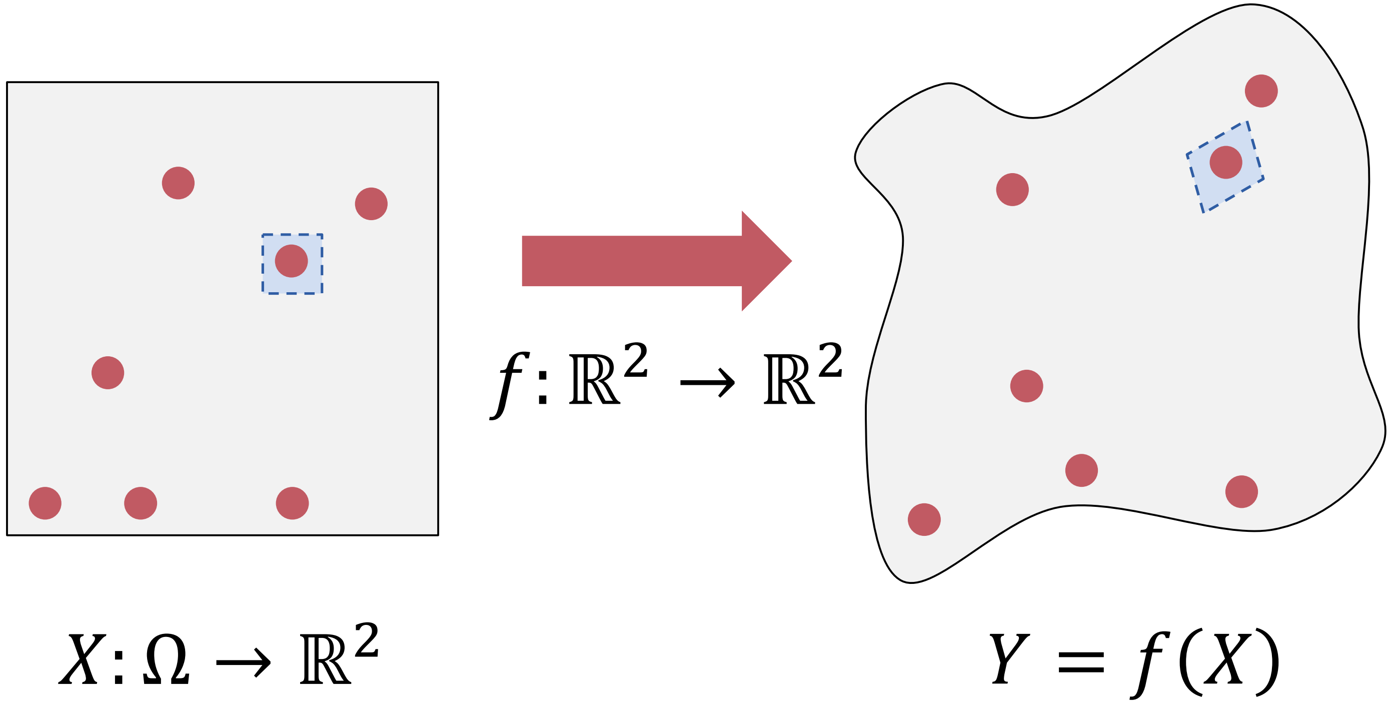
$$Y = f(X)$$

$$p_Y(y) = \frac{p_X\left(f^{-1}(y)\right)}{\left|f'\left(f^{-1}(y)\right)\right|} *$$

PDFs should be nonnegative!

*: incomplete

PDF change of variables

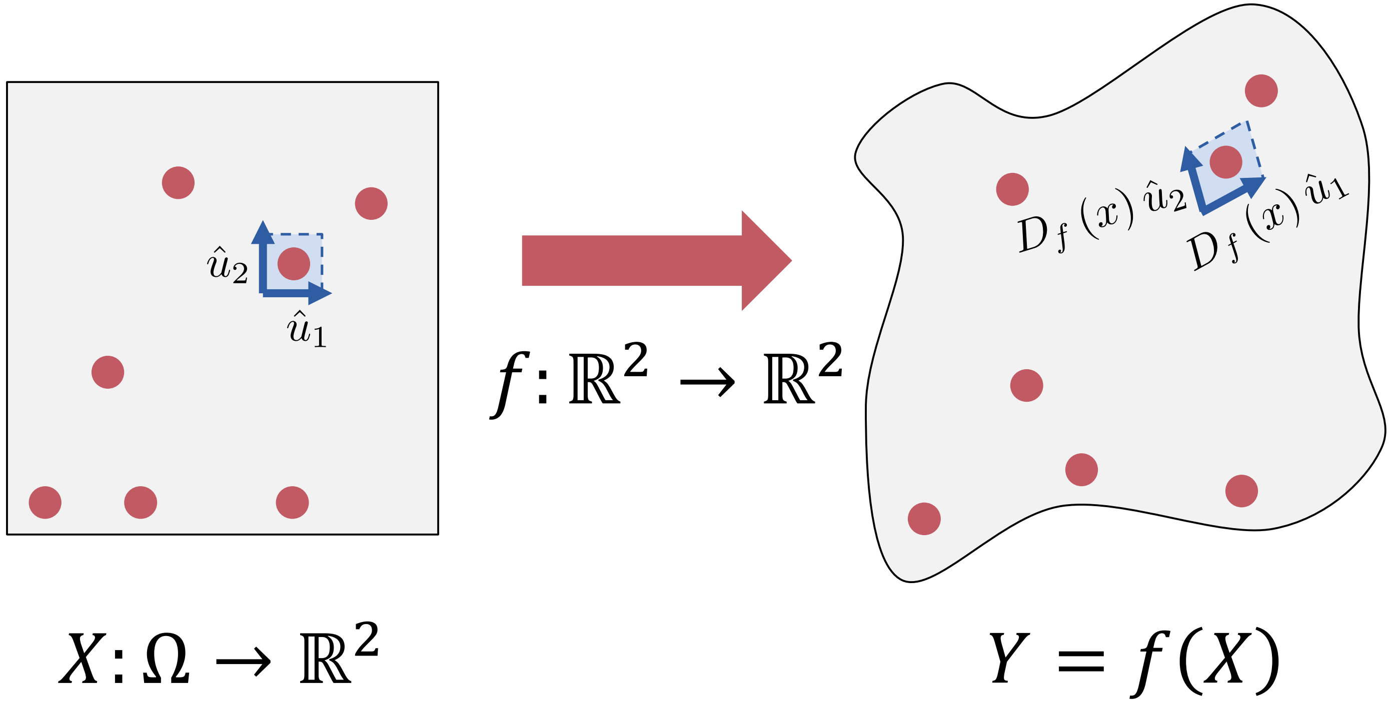


$$Y = f(X)$$

$$p_Y(y) = \frac{p_X\left(f^{-1}(y)\right)}{|\square| \,/\, |\square|} \; *$$

*: incomplete

PDF change of variables



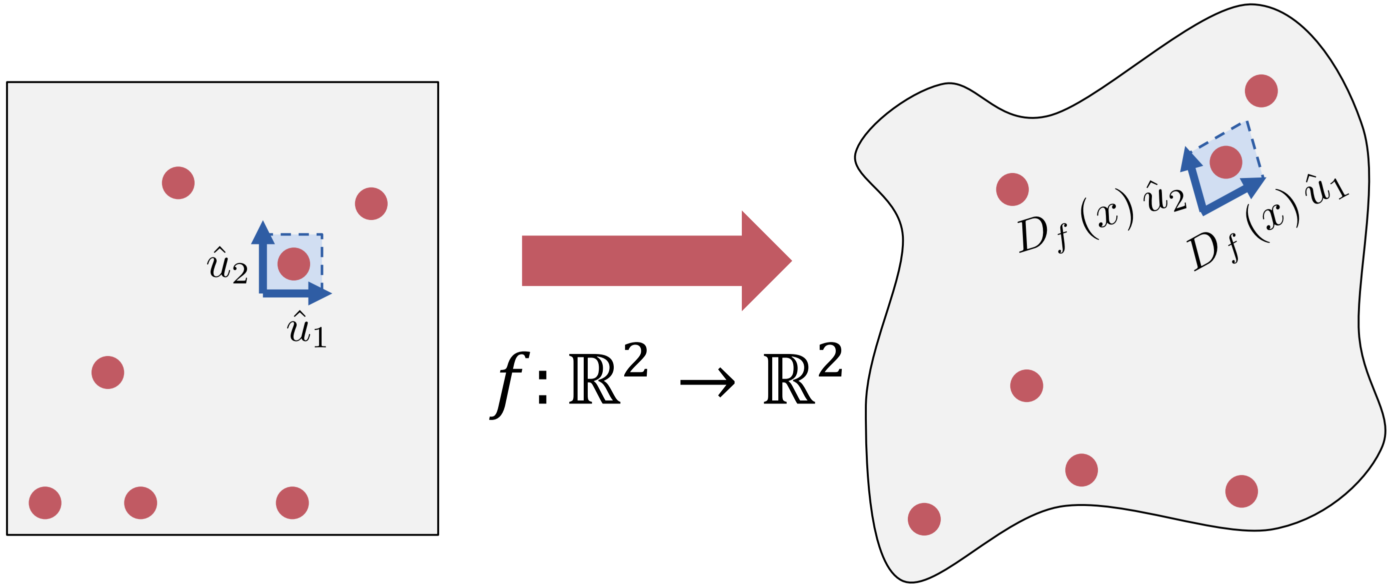
$$Y = f(X)$$

$$p_Y(y) = \frac{p_X\left(f^{-1}(y)\right)}{|\lozenge| \,/\, |\square|} \; *$$

$$\begin{aligned} |\lozenge| &= \|D_f(x)\,\hat{u}_1 \times D_f(x)\,\hat{u}_2\| \\ &= |\det D_f(x)|\,\|\hat{u}_1\|\,\|\hat{u}_2\| \end{aligned}$$

*: incomplete

PDF change of variables



$X\colon \Omega \to \mathbb{R}^2$

$Y = f(X)$

$$Y = f(X)$$

$$p_Y(y) = \frac{p_X\left(f^{-1}(y)\right)}{\left|\det D_f\left(f^{-1}(y)\right)\right|} \; *$$

$$\begin{aligned} |\quad| &= \left\|D_f(x)\,\hat{u}_1 \times D_f(x)\,\hat{u}_2\right\| \\ &= \left|\det D_f(x)\right| \left\|\hat{u}_1\right\| \left\|\hat{u}_2\right\| \end{aligned}$$

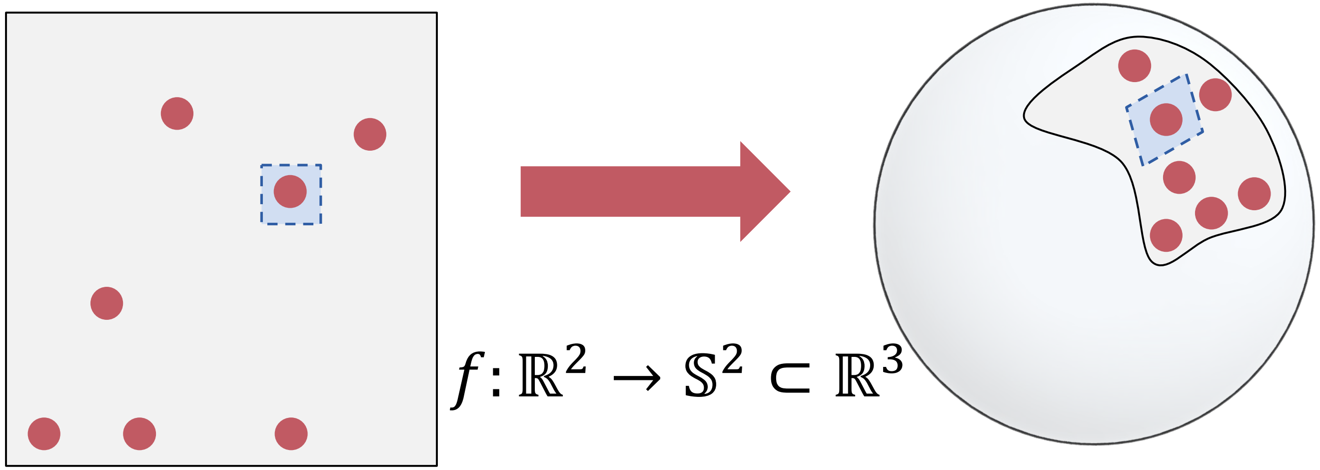
*: incomplete

PDF change of variables

$$Y = f(X)$$

$$p_Y(y) = \frac{p_X\left(f^{-1}(y)\right)}{\left|\cancel{\det D_f}\left(f^{-1}(y)\right)\right|} \quad *$$

No determinant for 3×2 matrices



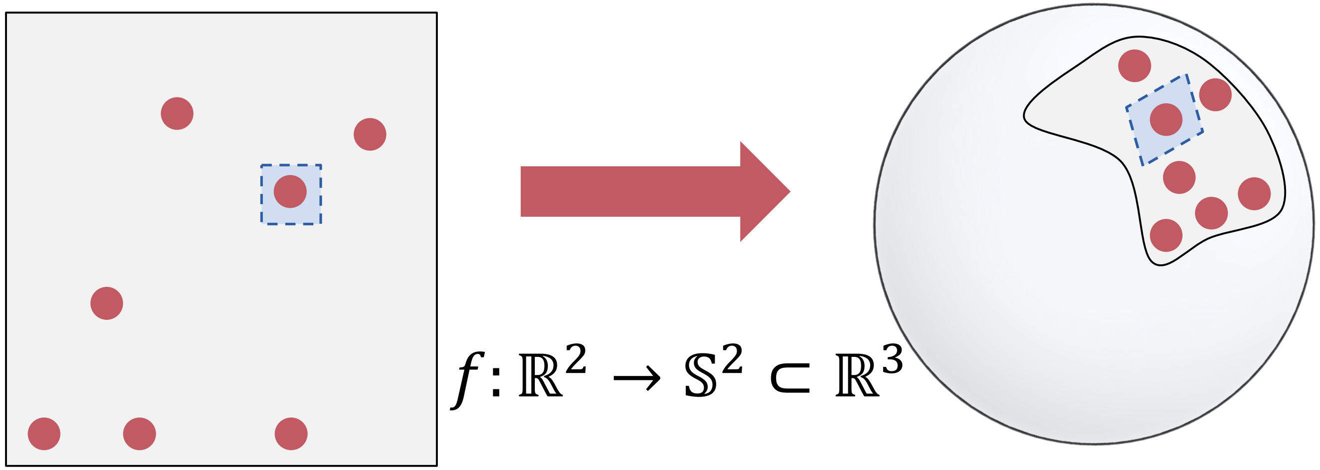
$f\colon \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$

$X\colon \Omega \to \mathbb{R}^2$

$Y = f(X)$

*: incomplete

PDF change of variables



$f: \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$

$X: \Omega \to \mathbb{R}^2$

$Y = f(X)$

Proposition

$$Y = f(X)$$

$$p_Y(y) = \frac{p_X\left(f^{-1}(y)\right)}{J_f\left(f^{-1}(y)\right)}$$

$$^{*}J_f(x) = \sqrt{\det\left(D_f(x)^T D_f(x)\right)}$$

*: not a very common notation

PDF change of variables: same object, different coordinates