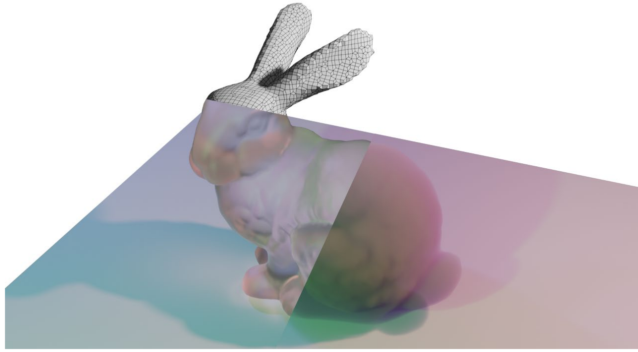
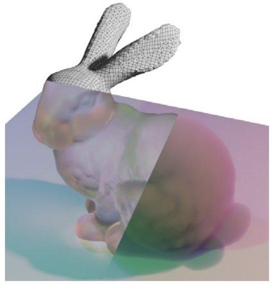


2. Probability and Statistical Inference

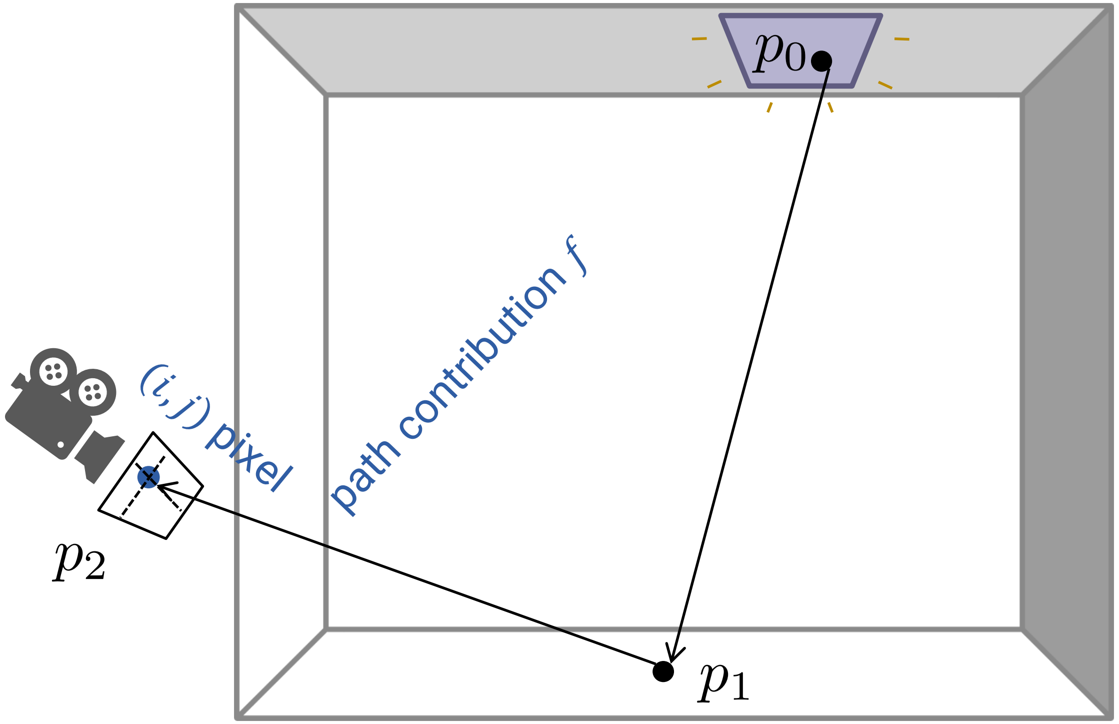
Physically Based Rendering

Shinyoung Yi (이신영)



Preview: a simple path tracing

Rendering equation for direct illumination



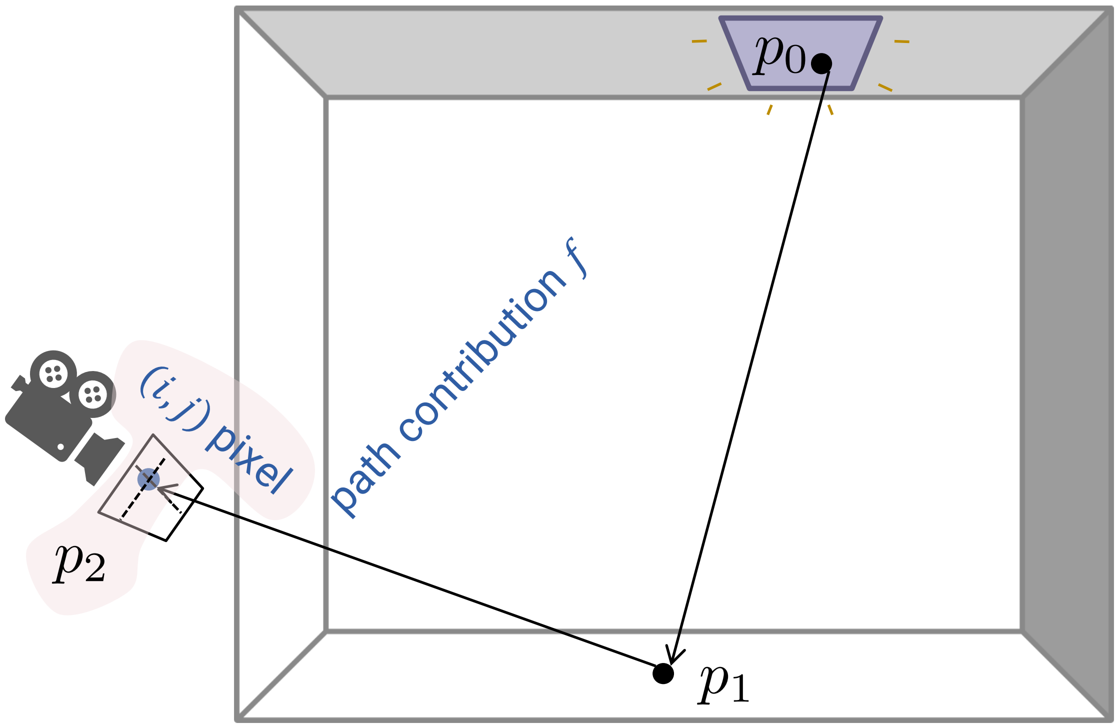
Terminology and convention:

- Notation / actual light: $p_0 \rightarrow p_1 \rightarrow p_2$
- Computation: $p_2 \rightarrow p_1 \rightarrow p_0$

- # of bounces < depth < # of vertices

1 **2** **3**

Rendering equation for direct illumination



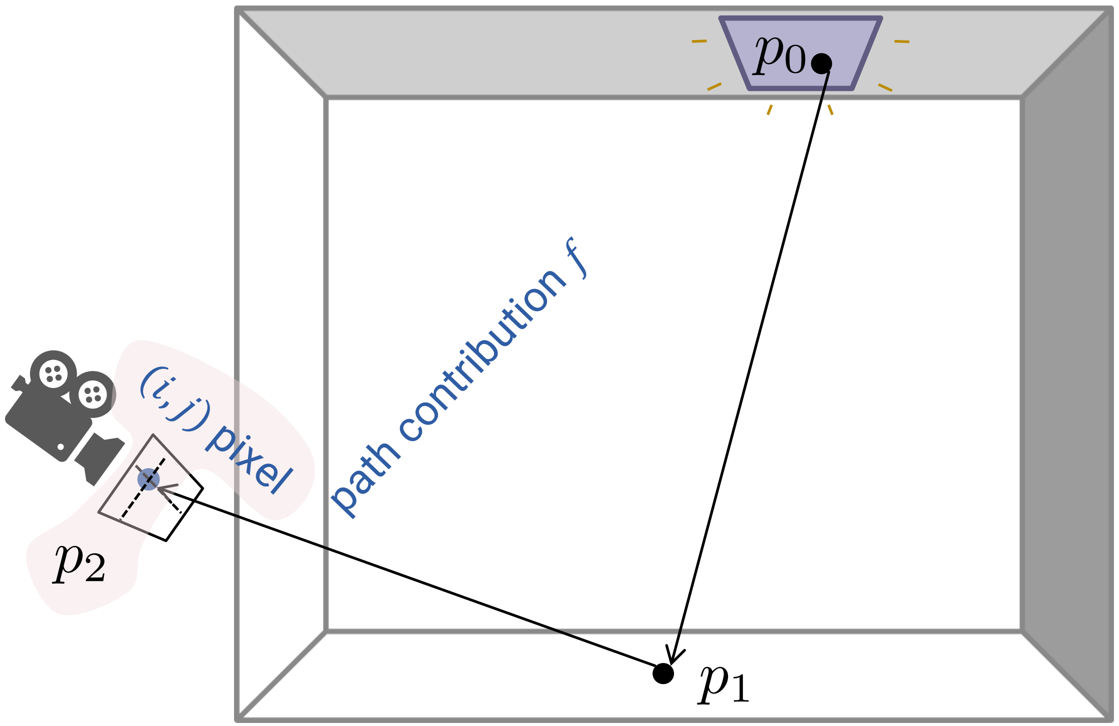
Notation:

- $\widehat{\omega}_{p_0 p_1} \coloneqq \frac{p_1 - p_0}{\|p_1 - p_0\|}$

Final measurement:

$$I^{(i,j)} = \int_{\mathbb{S}} W_e^{(i,j)}(\hat{\omega})\, L(p_2, \hat{\omega})\, \mathrm{d}\hat{\omega}$$

Rendering equation for direct illumination



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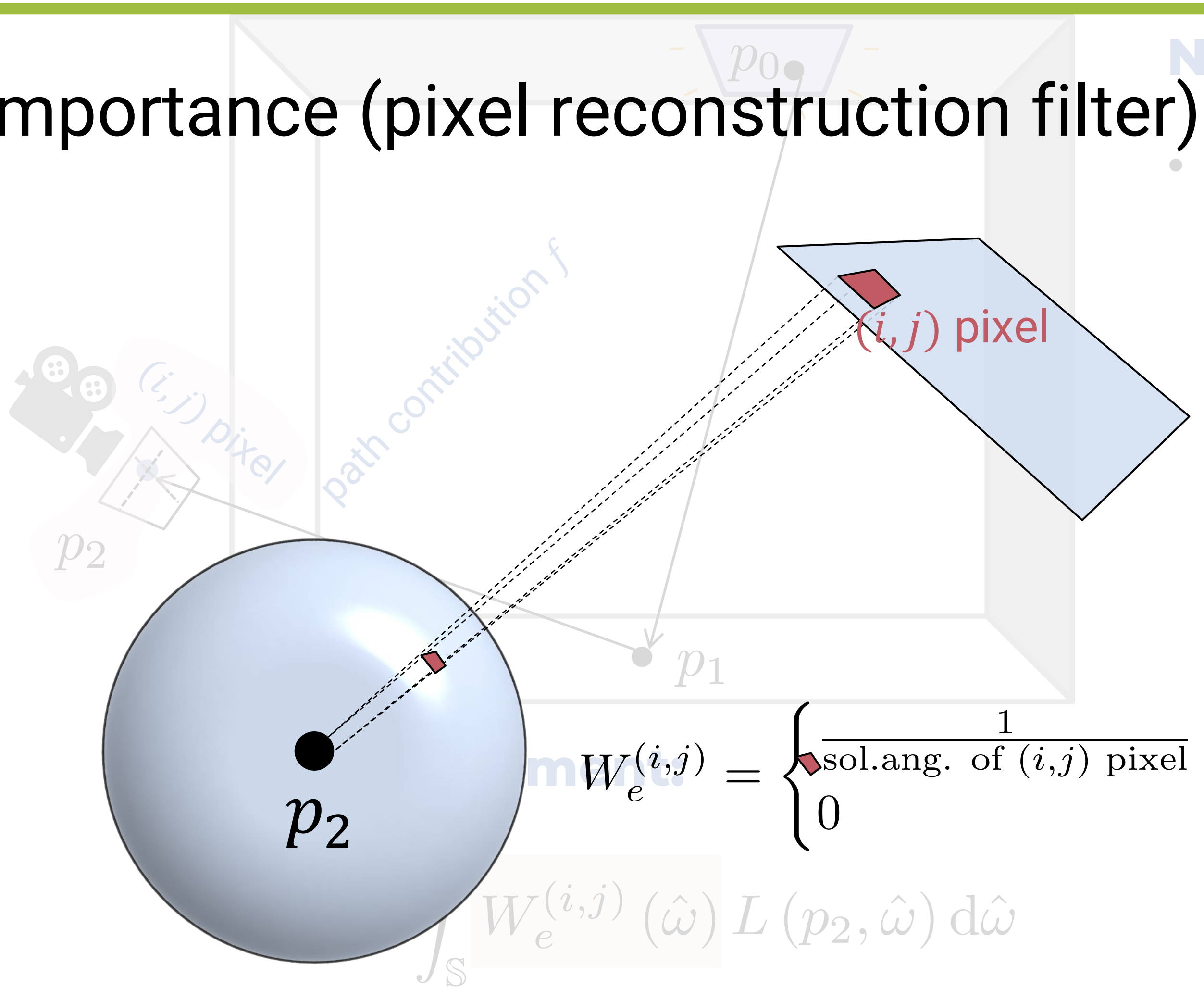
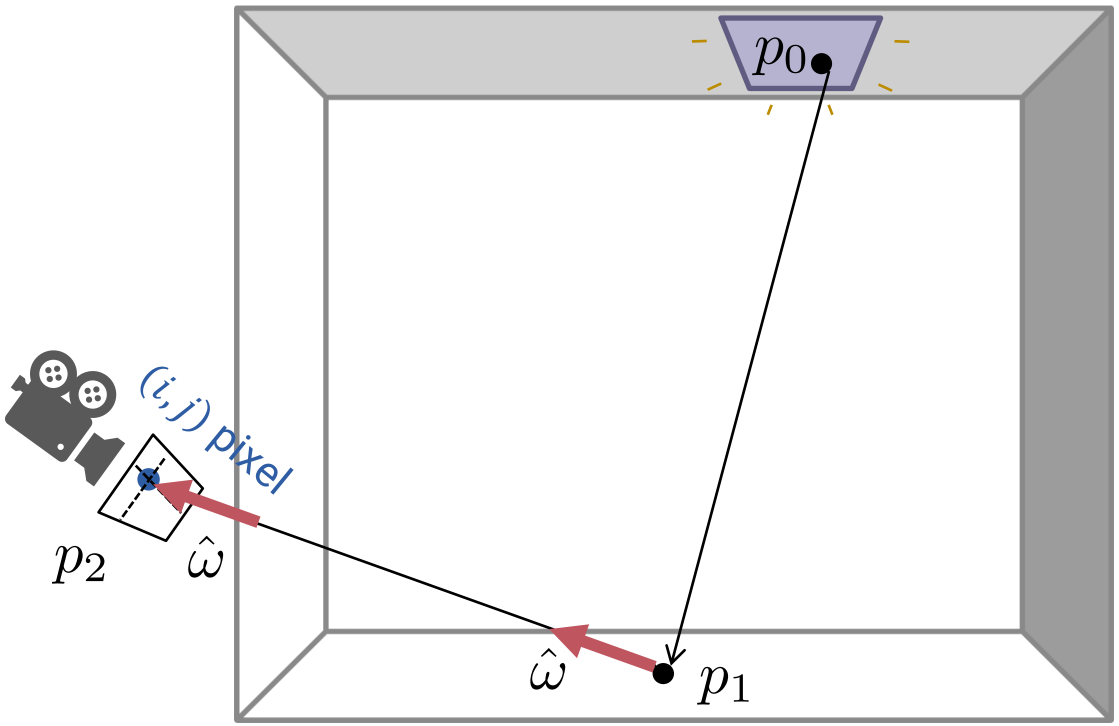
Rendering equation for direct illumination

Importance (pixel reconstruction filter) $W_e(p, \hat{\omega})$



$$W_e^{(i,j)} = \begin{cases} \dfrac{1}{\text{sol.ang. of } (i,j) \text{ pixel}} & \hat{\omega} \text{ in } (i,j) \text{ pixel} \\ 0 & \text{otherwise} \end{cases}$$

Rendering equation for direct illumination



Notation:

- $\widehat{\omega}_{p_0 p_1} \coloneqq \frac{p_1 - p_0}{\|p_1 - p_0\|}$
- r: $\mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}^3$
  - ray intersection

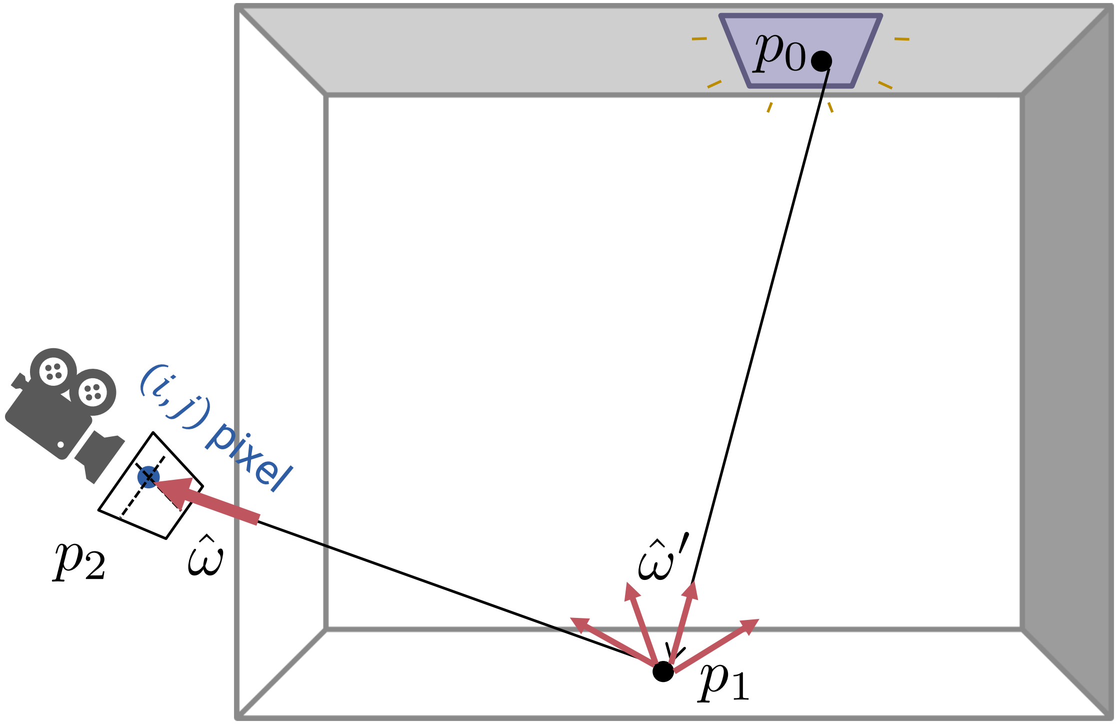
Final measurement:

$$I^{(i,j)} = \int_{\mathbb{S}} W_e^{(i,j)}(\hat{\omega})\, L(p_2, \hat{\omega})\, \mathrm{d}\hat{\omega}$$

$$L(p_2, \hat{\omega}) = L(p_1, \hat{\omega})$$

$p_1 = \mathrm{r}(p_2, -\hat{\omega})$

Rendering equation for direct illumination



Notation:

- $\widehat{\omega}_{p_0 p_1} := \frac{p_1 - p_0}{\|p_1 - p_0\|}$
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  - ray intersection

Final measurement:

$$I^{(i,j)} = \int_{\mathbb{S}} W_e^{(i,j)}(\hat{\omega})\, L(p_1, \hat{\omega})\, \mathrm{d}\hat{\omega}$$

$$L(p_1, \hat{\omega}) = L_e(p_1, \hat{\omega}) + \int_{\mathbb{S}} L^{(\text{in})}(p_1, \hat{\omega}')\, \rho(p_1, \hat{\omega}', \hat{\omega})\, |\hat{n} \cdot \hat{\omega}'|\, \mathrm{d}\hat{\omega}'$$

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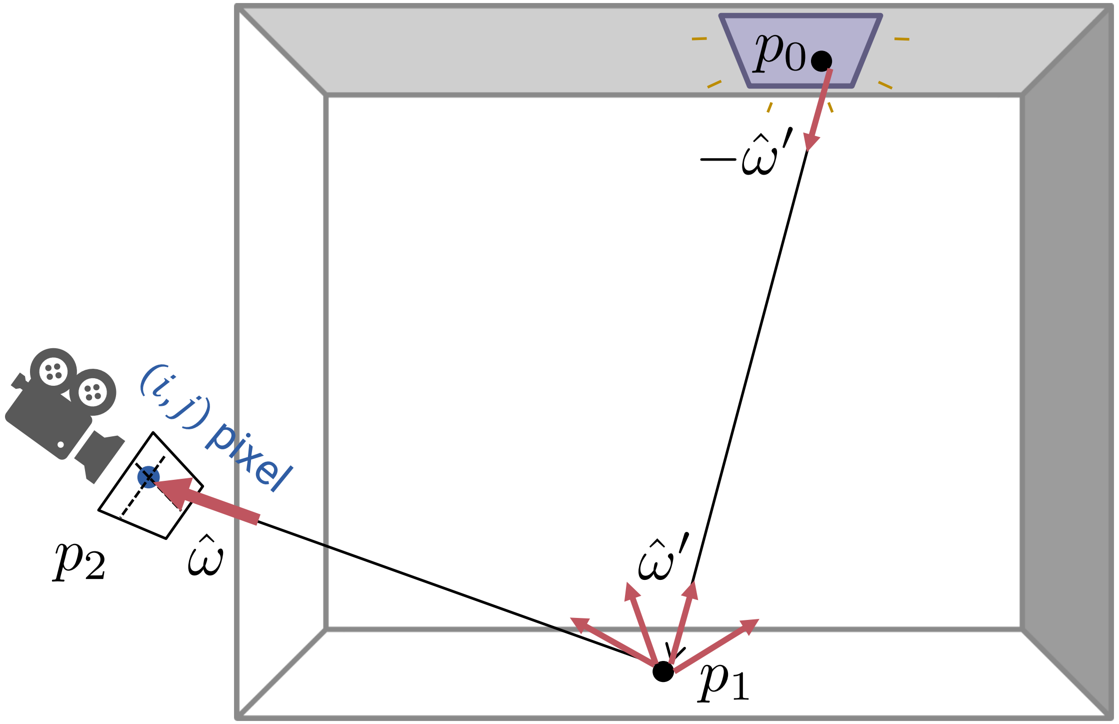
Rendering equation for direct illumination



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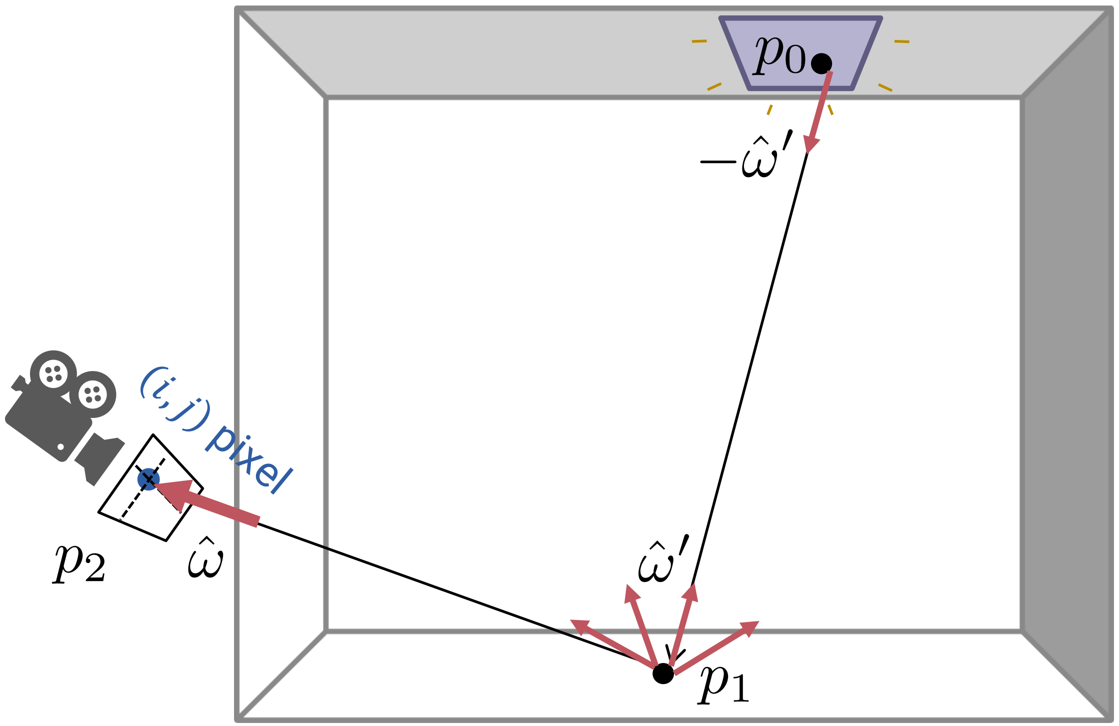
$p_0 = \mathrm{r}(p_1, \hat{\omega}')$

$$L^{(\mathrm{in})}(p_1, \hat{\omega}') = L(p_0\ \ -\hat{\omega}')$$

Rendering equation for direct illumination



Notation:

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- r: $\mathbb{R}^3 \times \mathbb{S}^2 \rightarrow \mathbb{R}^3$
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Final measurement:

$$I^{(i,j)} = \int_{\mathbb{S}} W_e^{(i,j)}(\hat{\omega})\, L(p_1, \hat{\omega})\, \mathrm{d}\hat{\omega}$$

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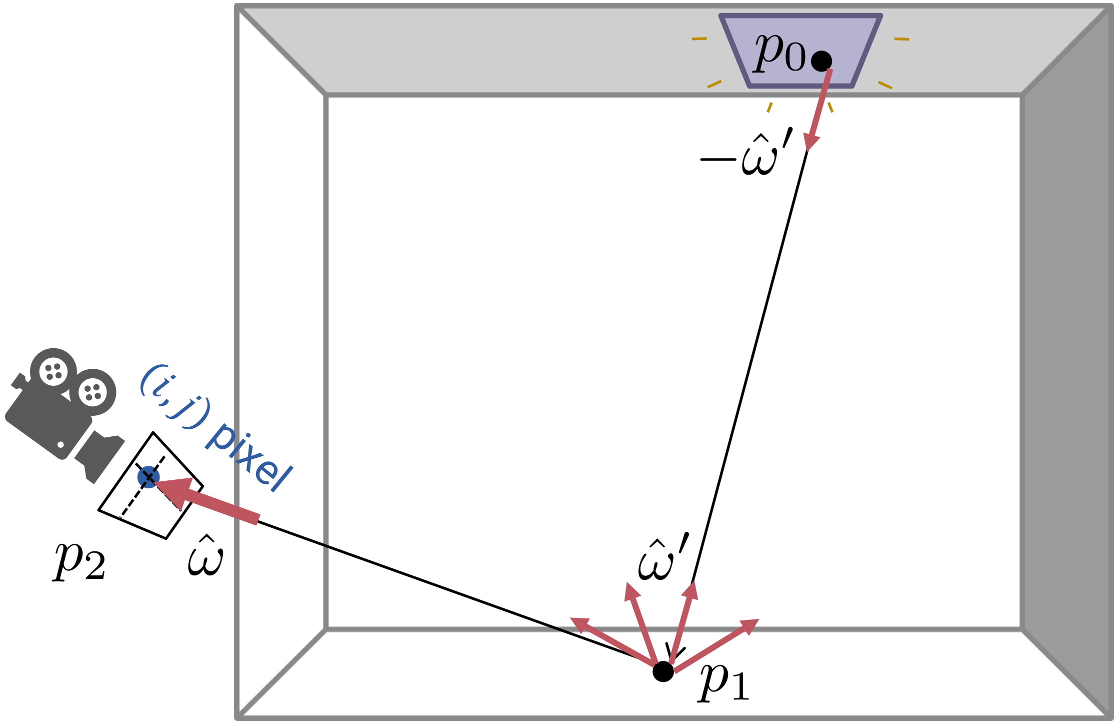
$$L(p_1, \hat{\omega}) = L_e(p_1, \hat{\omega}) + \int_{\mathbb{S}} L^{(\mathrm{in})}(p_1, \hat{\omega}')\, \rho(p_1, \hat{\omega}', \hat{\omega})\, |\hat{n} \cdot \hat{\omega}'|\, \mathrm{d}\hat{\omega}'$$

$p_0 = \mathrm{r}(p_1, \hat{\omega}')$

Direct illumination only

$$L(p_0, -\hat{\omega}') = L_e(p_0, -\hat{\omega}') + \cancel{\int_{\mathbb{S}} L^{(\mathrm{in})}(p_0, \hat{\omega}'')\, \rho(p_0, \hat{\omega}'', -\hat{\omega}')\, |\hat{n} \cdot \hat{\omega}''|\, \mathrm{d}\hat{\omega}''}$$

Rendering equation for direct illumination



Notation:

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- r: $\mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}^3$
  - ray intersection

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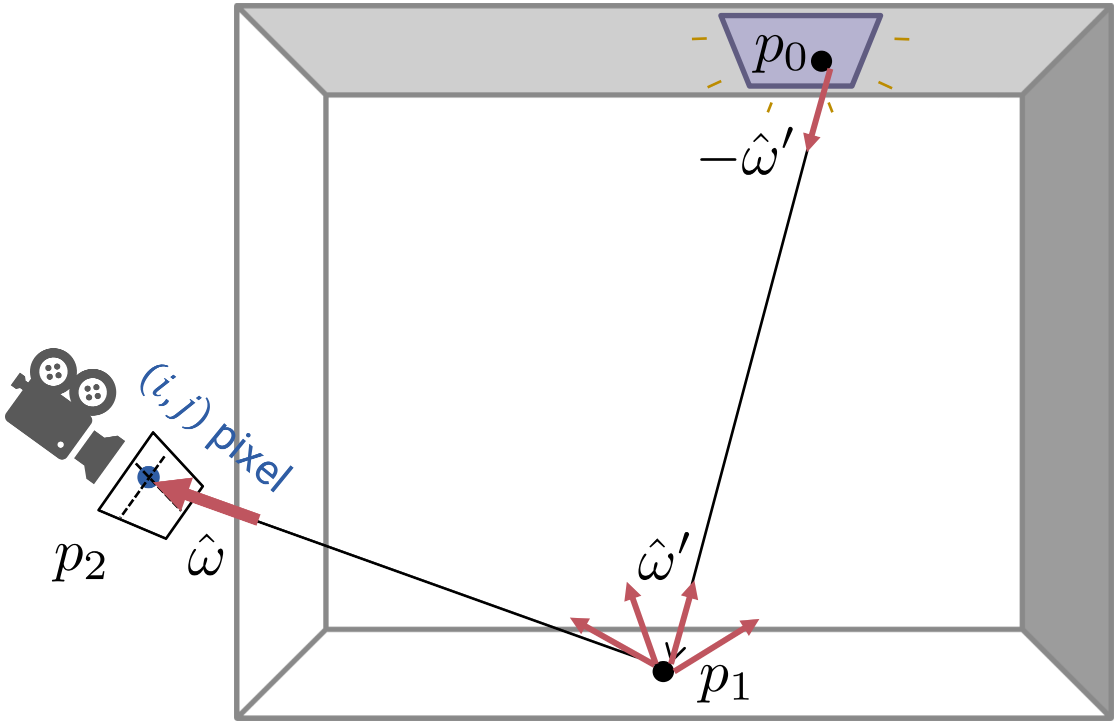
$p_1 = \mathrm{r}(p_2, -\hat{\omega})$

$p_0 = \mathrm{r}(p_1, \hat{\omega}')$

$$L(p_0, -\hat{\omega}') = L_e(p_0, -\hat{\omega}')$$

Rendering equation for direct illumination



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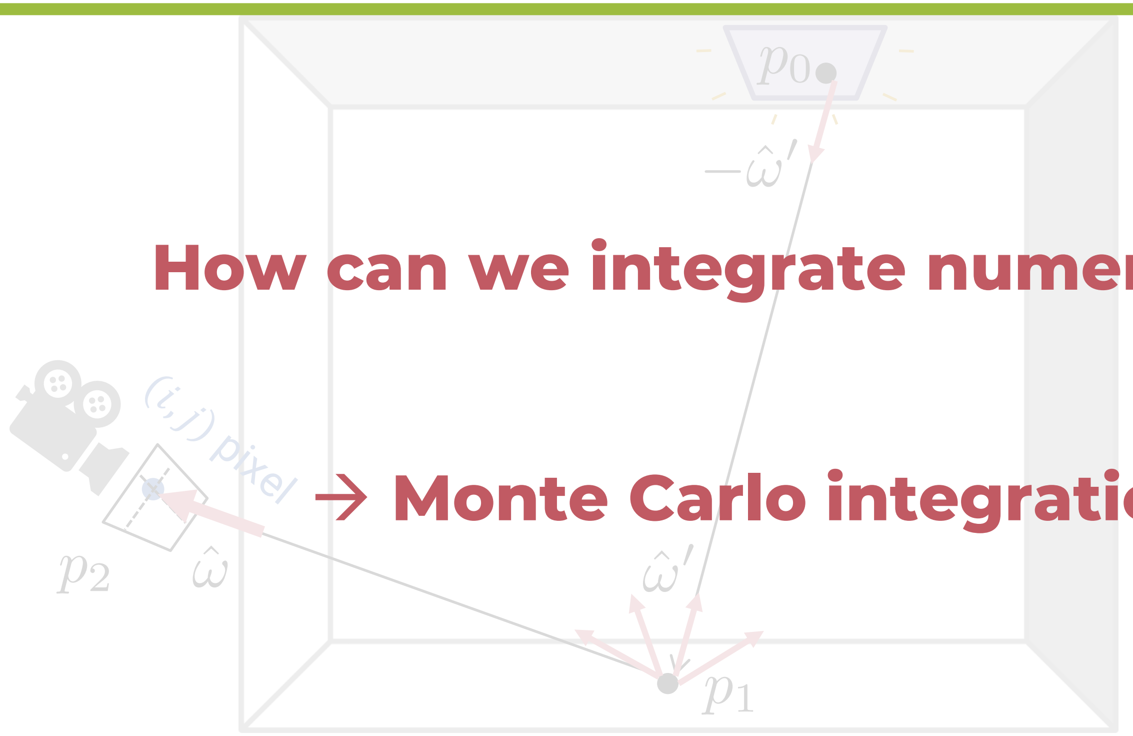
$p_1 = \mathrm{r}(p_2, -\hat{\omega})$
$p_0 = \mathrm{r}(p_1, \hat{\omega}')$

Rendering equation for direct illumination



How can we integrate numerically?

→ Monte Carlo integration

Notation:

- $\hat{\omega}_{p_0 p_1} := \frac{p_1 - p_0}{\|p_1 - p_0\|}$
- $\mathrm{r}: \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}^3$
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Final measurement:

$$I^{(i,j)} = \int_{\mathbb{S}} W_e^{(i,j)}(\hat{\omega})\, L(p_1, \hat{\omega})\, \mathrm{d}\hat{\omega}$$

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$p_1 = \mathrm{r}(p_2, -\hat{\omega})$
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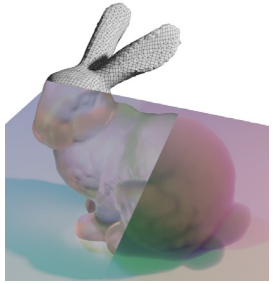
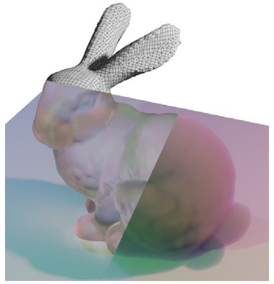


Basic Math: Sets and Functions

Sets and functions

Suppose $A, B, X, \cdots$ are sets and $f, g, \cdots$ are functions

- $f: X \to Y$, $A \subset X$ then $f(A) := \{f(x) | x \in A\}$ is the ***image of*** A ***under*** f
- $B \subset Y$ then $f^{-1}(B) := \{x | f(x) \in B\}$ is the ***preimage of*** B ***under*** f
  - Even if an inverse function does not exists, preimages are well defined.



Probability

What is probability?

random sample

probability

observation

distribution

random variable

Probability overview

- Sample space (probability space) Ω
- Event E
- Probability P

- Random variable X
- CDF of X
- PDF of X

Probability overview

- Sample space (probability space) Ω: a set of all things that the outcome can be.
- Event $E \subset \Omega$
- Probability $P(E) \in \mathbb{R}_{\geq 0}$
  - $P(E_1 \cup E_2) = P(E_1) + P(E_2)$ if $E_1 \cap E_2 = \phi$
  - $P(\Omega) = 1$
- Random variable $X: \Omega \to \mathbb{R}$ (codomain $\mathbb{R}^n$ or $\mathbb{S}^2$ is also okay)
  - We can write an event in a way as $\{X \leq x\} \coloneqq X^{-1}(\{y \in \mathbb{R} | y \leq x\})$
- CDF of X: $F_X(x) = P(\{X \leq x\})$
- PDF of X: some function $p_X: \mathbb{R} \to \mathbb{R}_{\geq 0}$ s.t. $P(a \leq x \leq b) = \int_a^b p_X(x) \mathrm{d}x$
  - $p_X(x) = \frac{\mathrm{d}F_X(x)}{\mathrm{d}x}$ for continuous random variable

Probability overview: summary

Event $\subset$ Ω sample space

Probability: $0 \leq P(E) \leq 1$

Random variable $X: \Omega \to \mathbb{R}$

Probability properties

- For a random variable $X\colon \Omega \to \mathbb{R}$
  - Expectation: $\mathbb{E}[X] \coloneqq \int_{\mathbb{R}} x p_X(x)\mathrm{d}x$
  - Variance: $Var(X) \coloneqq \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

- For a function $f\colon \mathbb{R} \to \mathbb{R}$
  - $f(X) = f \circ X\colon \Omega \to \mathbb{R}$ is also a random variable
  - $p_{f(X)}(y) = \sum_{x \in f^{-1}(y)} \frac{p_{X(x)}}{|f'(x)|}$

Probability properties

- Independence: definition
  - Events $A, B \subset \Omega$ are called independent if $P(A \cap B) = P(A)P(B)$
  - Random variables $X, Y: \Omega \to \mathbb{R}$ are call independent
    if for any $x, y \in \mathbb{R}$, $\{X \leq x\}$ and $\{Y \leq y\}$ are independent events

- Independence: properties
  - For independent random variables X and Y,

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$
$$p_{X,Y}(x, y) = p_X(x)p_Y(y)$$
$$\mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)]\mathbb{E}[g(Y)]$$

Why is probability **so confusing**?

Choice of sample space

https://ko.wikipedia.org/wiki/%EB%8F%99%EC%A0%84_%EB%8D%98%EC%A7%80%EA%B8%B0#/media/%ED%8C%8C%EC%9D%BC:Coin_Toss_(3635981474).jpg

Choice 1

Sample space: $\{h, t\}$

$H = \{h\}, T = \{t\}$

$P[\{h\}] = \frac{1}{2}, P[\{t\}] = \frac{1}{2}$

Choice 2

Sample space: $\{\theta | 0 \leq \theta \leq 2\pi\}$

$H = \{\theta | 0 \leq \theta \leq \pi\}, T = \{\theta | \pi < \theta \leq 2\pi\}$

$P[\{\theta | 0 \leq \theta_1 < \theta \leq \theta_2 \leq 2\pi\}] = \frac{\theta_2 - \theta_1}{2\pi}$

Choice 3

Sample space: $\{\mathbf{R} | \mathbf{R} \in \mathrm{SO}(3)\}$

$H = \left\{\mathbf{R} \middle| 0 \leq \mathbf{R}\hat{z} \leq \frac{\pi}{2}\right\}, T = \left\{\mathbf{R}\hat{z} \middle| \frac{\pi}{2} < \mathbf{R}\hat{z} \leq 1\right\}$

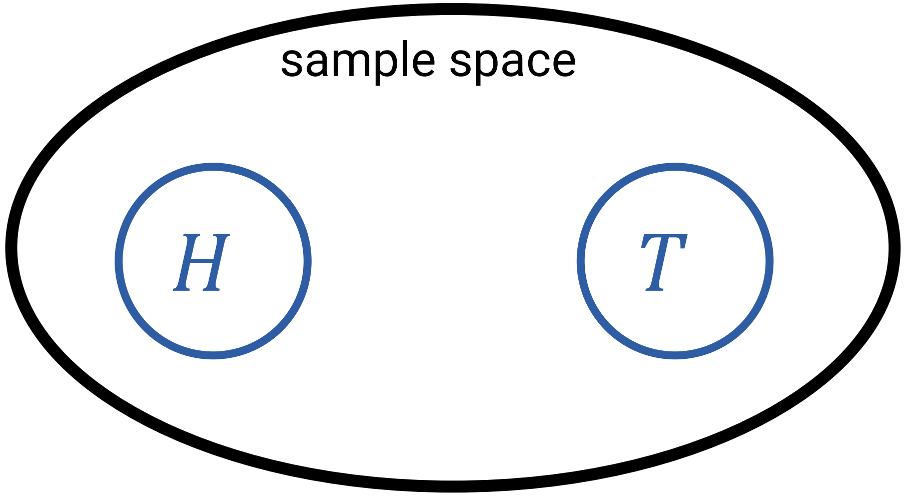
$$P[A \subset \mathrm{SO}(3)] = \frac{1}{8\pi^2} \int_A \sin\beta \, \mathrm{d}\alpha \mathrm{d}\beta \mathrm{d}\gamma$$

Event: H, T

Probability: $P[H] = \frac{1}{2}, P[T] = \frac{1}{2}$

Choice of sample space

https://ko.wikipedia.org/wiki/%EB%8F%99%EC%A0%84_%EB%8D%98%EC%A7%80%EA%B8%B0#/media/%ED%8C%8C%EC%9D%BC:Coin_Toss_(3635981474).jpg



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Choice of sample space

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Sample space: $\{\theta | 0 \leq \theta \leq \pi\}$

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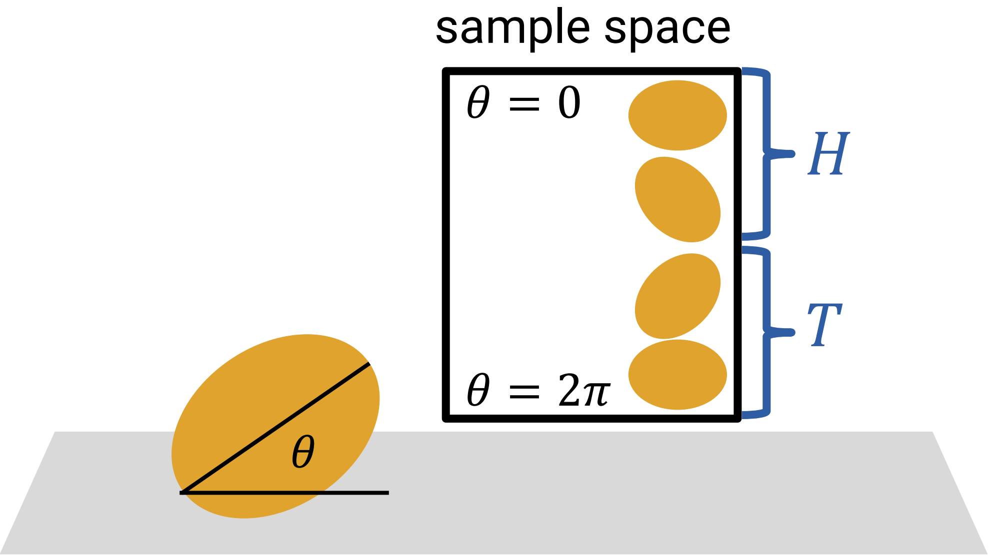
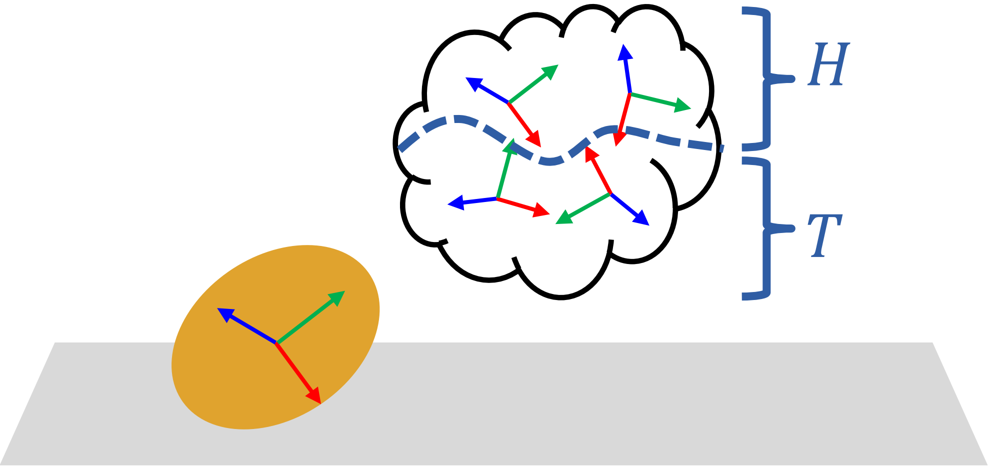


Choice of sample space

sample space = $\mathrm{SO}(3)$



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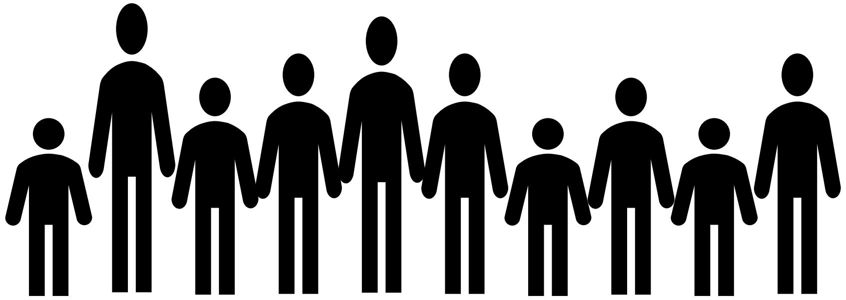
Choice 3

Sample space: $\{\mathbf{R} | \mathbf{R} \in \mathrm{SO}(3)\}$

$H = \left\{\mathbf{R} \middle| 0 \leq \mathbf{R}\hat{z} \leq \frac{\pi}{2}\right\}, T = \left\{\mathbf{R}\hat{z} \middle| \frac{\pi}{2} < \mathbf{R}\hat{z} \leq 1\right\}$

$$P[A \subset \mathrm{SO}(3)] = \frac{1}{8\pi^2} \int_A \sin\beta \, \mathrm{d}\alpha \mathrm{d}\beta \mathrm{d}\gamma$$

Choice of sample space

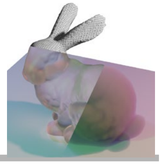


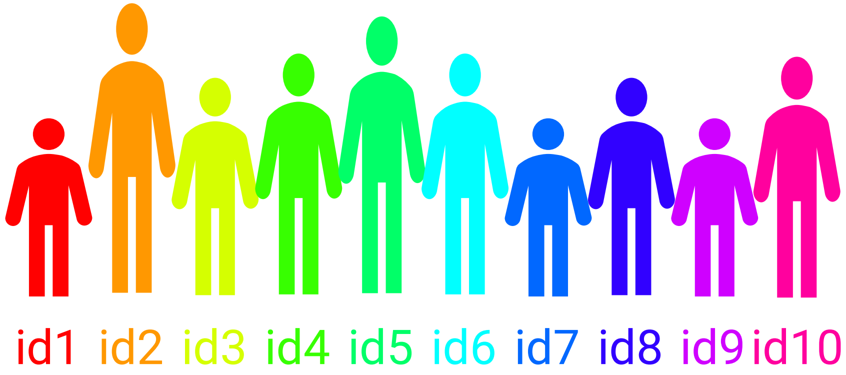
Event: each person

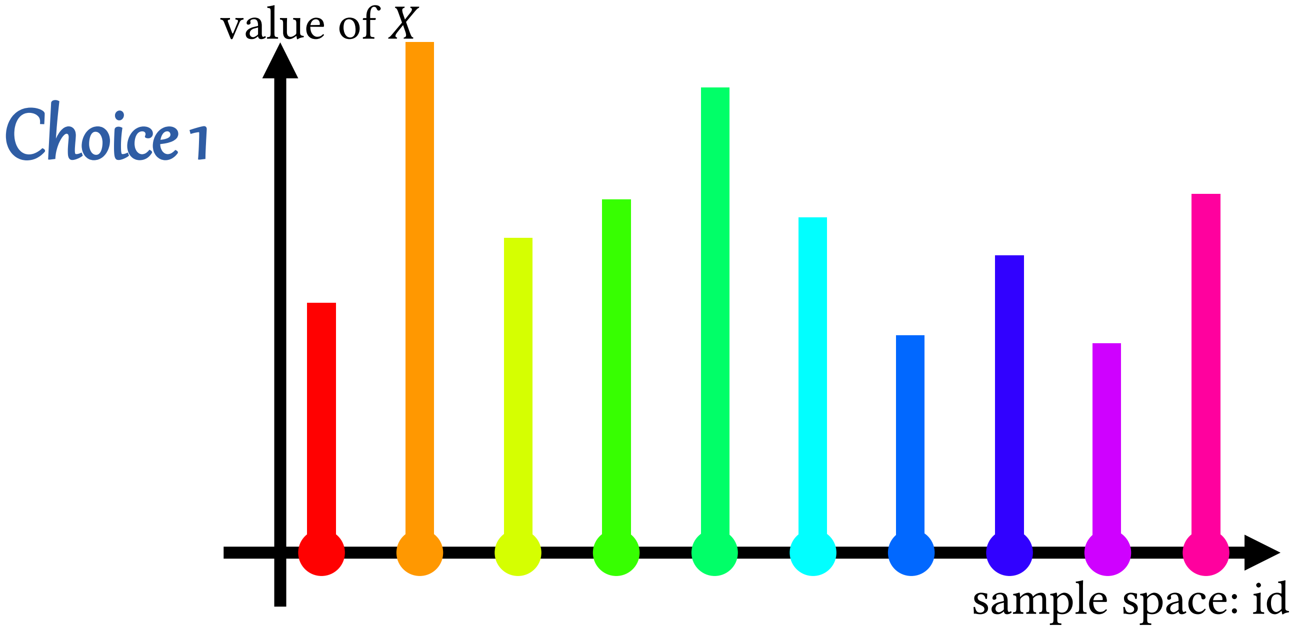
Probability: $\frac{1}{N}$

Random variable: height X

Choice of sample space





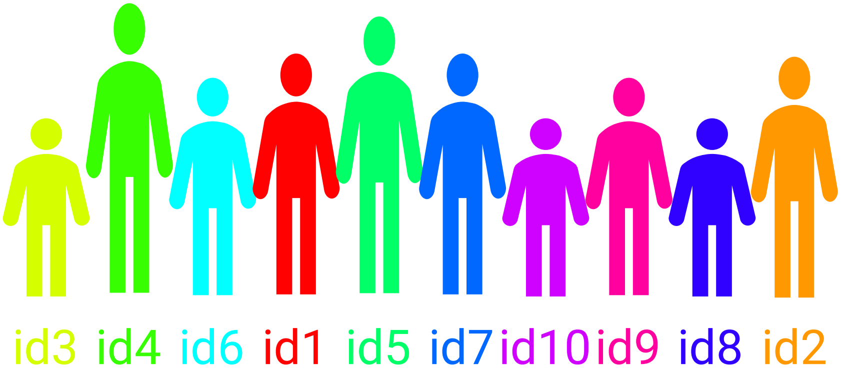


Event: each person

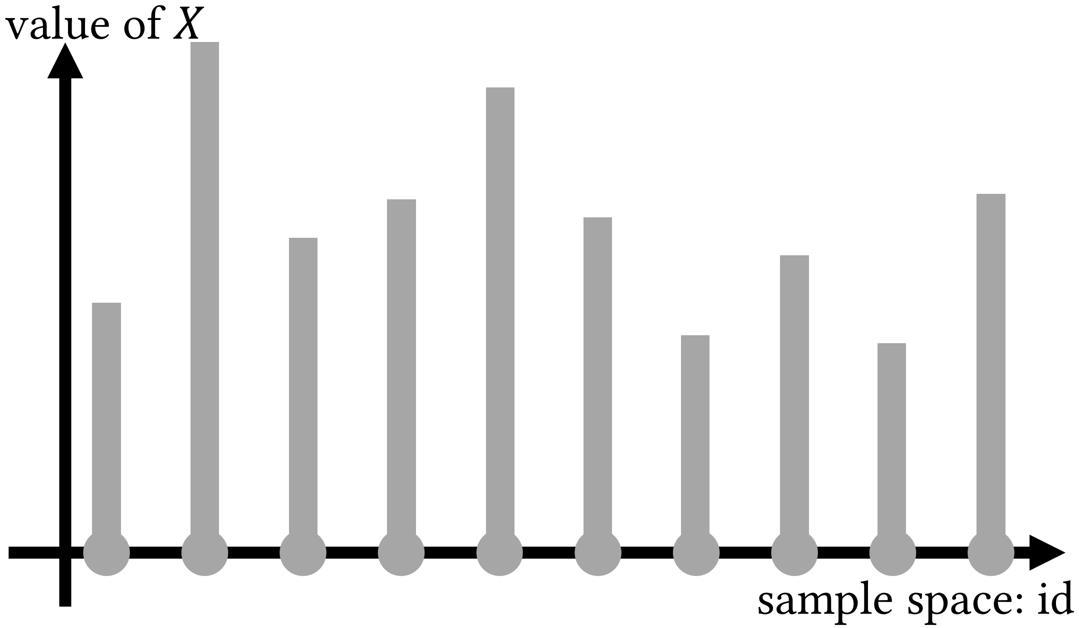
Probability: $\frac{1}{N}$

Random variable: height X

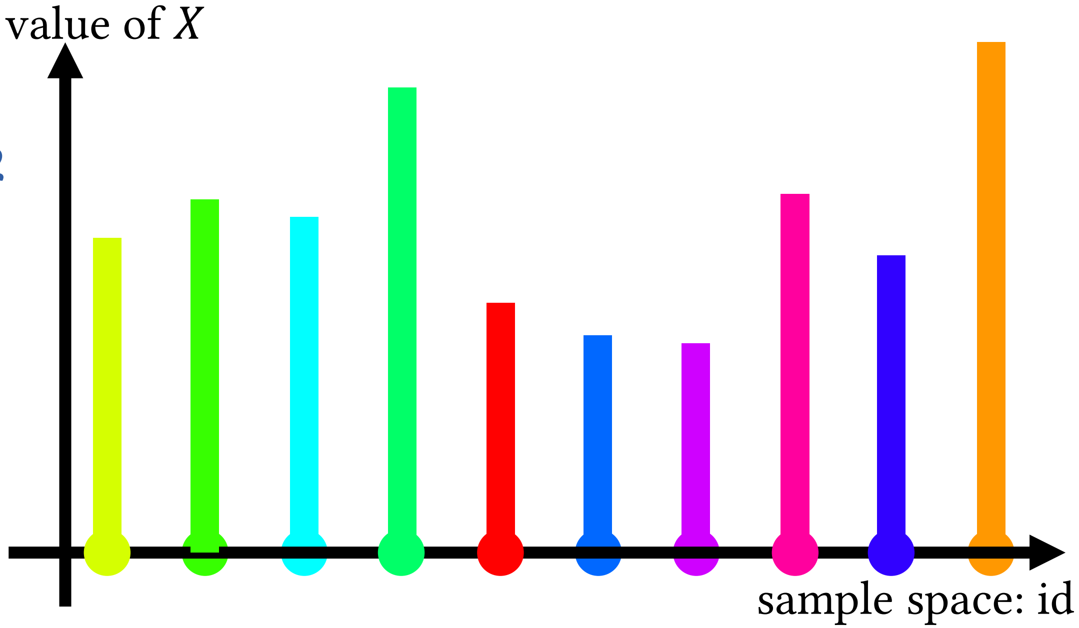
Choice of sample space



Choice 1



Choice 2

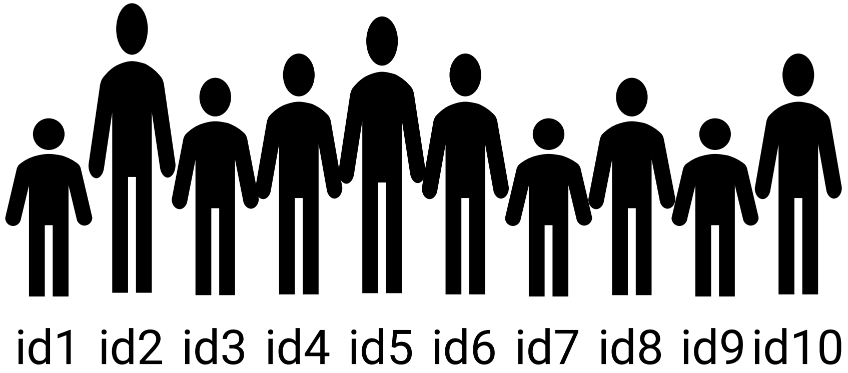


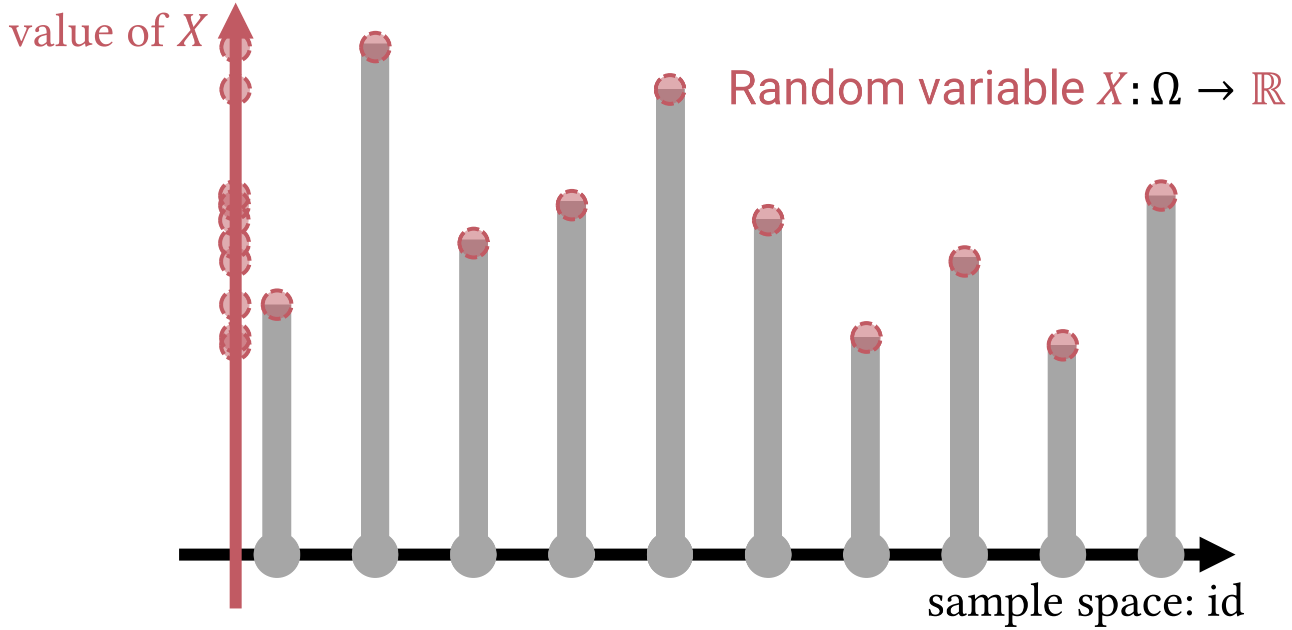
Event: each person

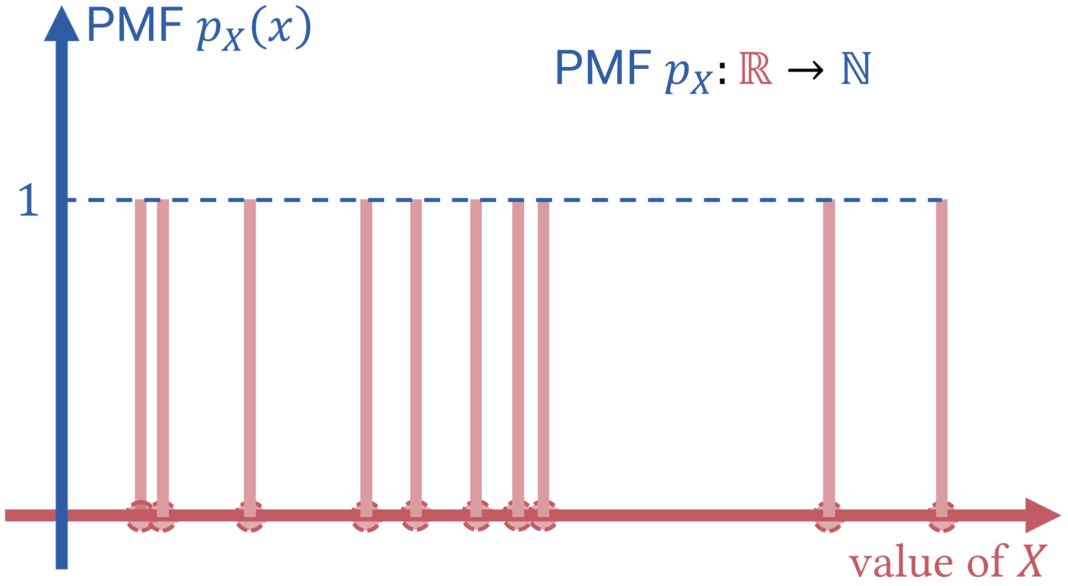
Probability: $\frac{1}{N}$

Random variable: height X

Two ways for viewing r.v. as a "function"







Event: each person

Probability: $\frac{1}{N}$

Random variable: height X

Underlying sample space is often implicit

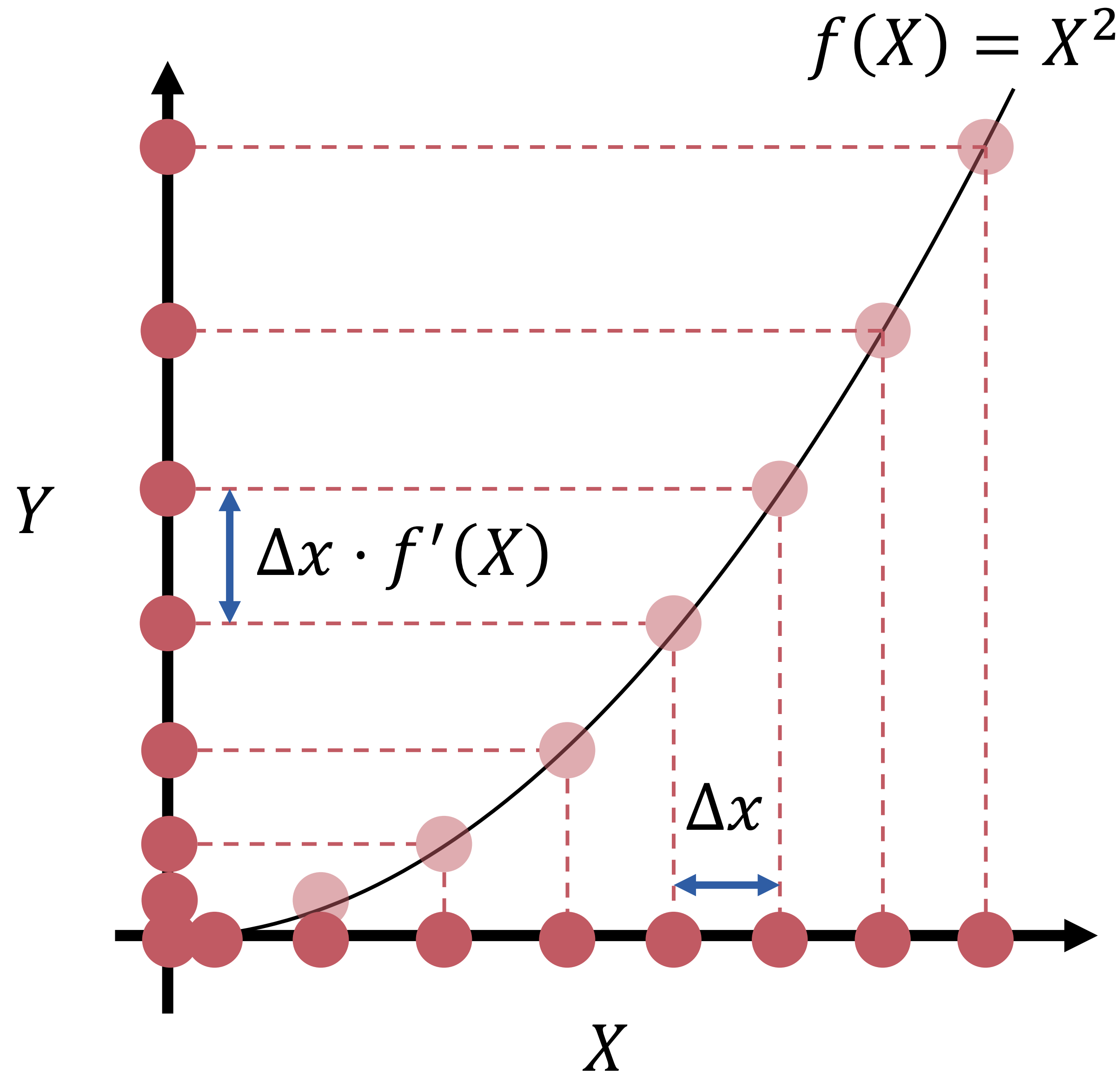
Suppose there is a random variable $X \sim U([0,1])$.

…

We can consider the underlying sample space is

We have a r.v. X:

PDF change of variables

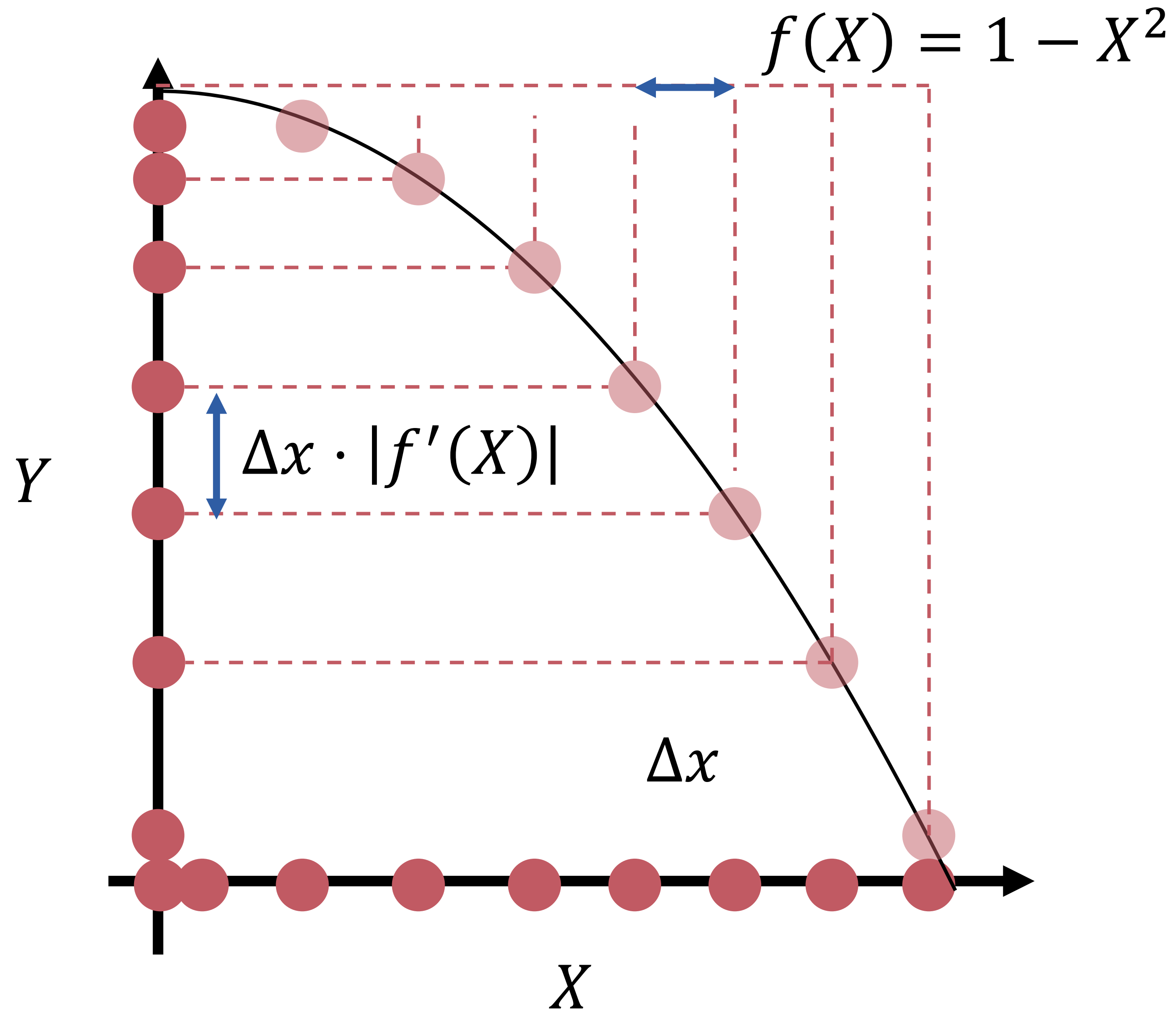


$$Y = f(X)$$

$$p_Y(y) = \frac{p_X\left(f^{-1}(y)\right)}{f'\left(f^{-1}(y)\right)} \; *$$

*: incomplete

PDF change of variables



$$Y = f(X)$$

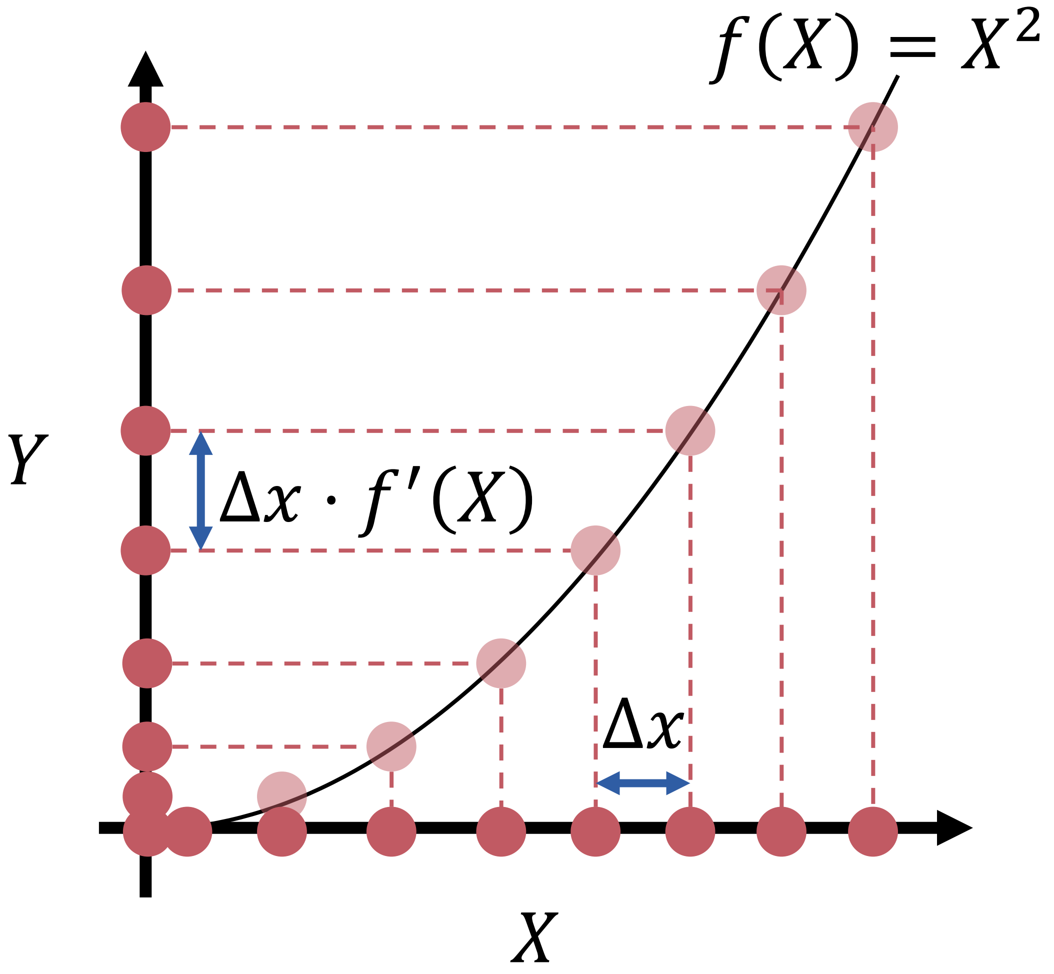
$$p_Y(y) = \frac{p_X\left(f^{-1}(y)\right)}{\left|f'\left(f^{-1}(y)\right)\right|} \; *$$

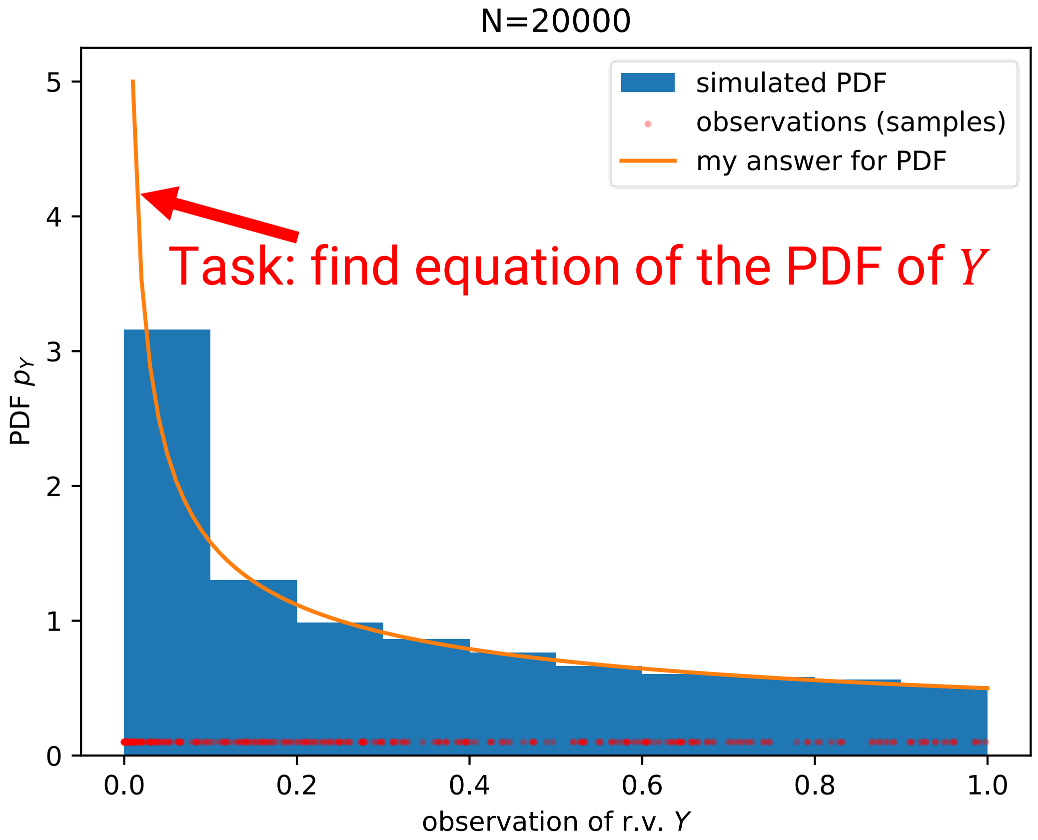
PDFs should be nonnegative!

*: incomplete

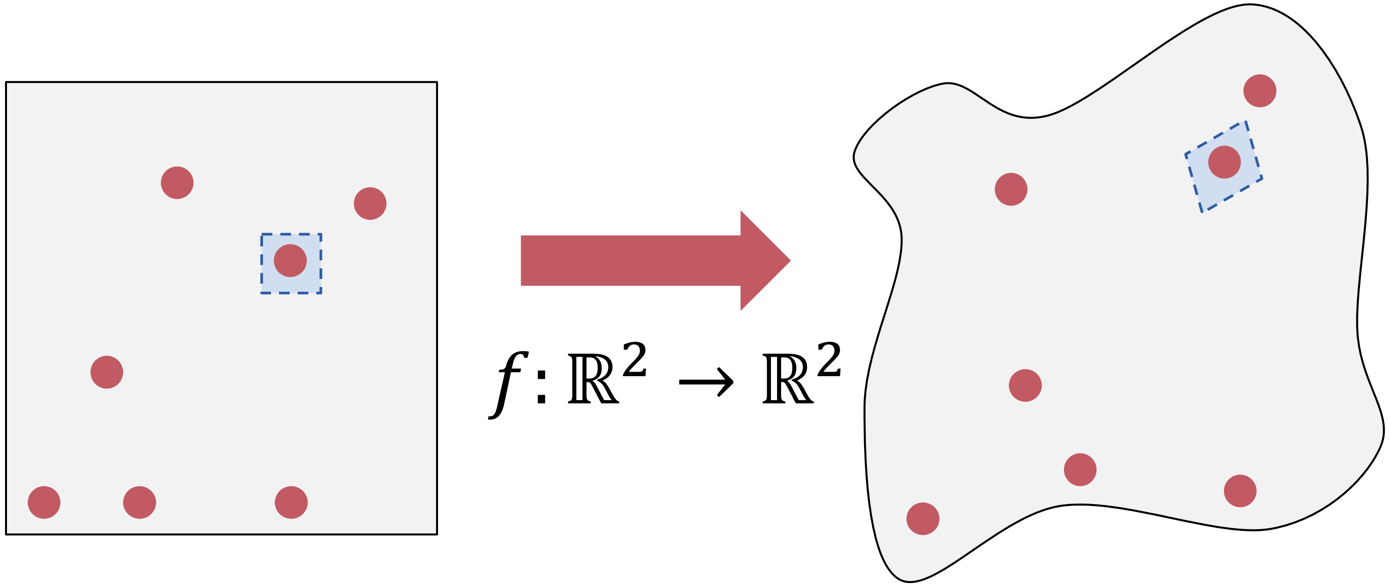
HW2 – Problem 1(a): PDF change of variable on 1D

- $X \sim \mathcal{U}([0,1])$
- $Y = X^2$





PDF change of variables



$X: \Omega \to \mathbb{R}^2$

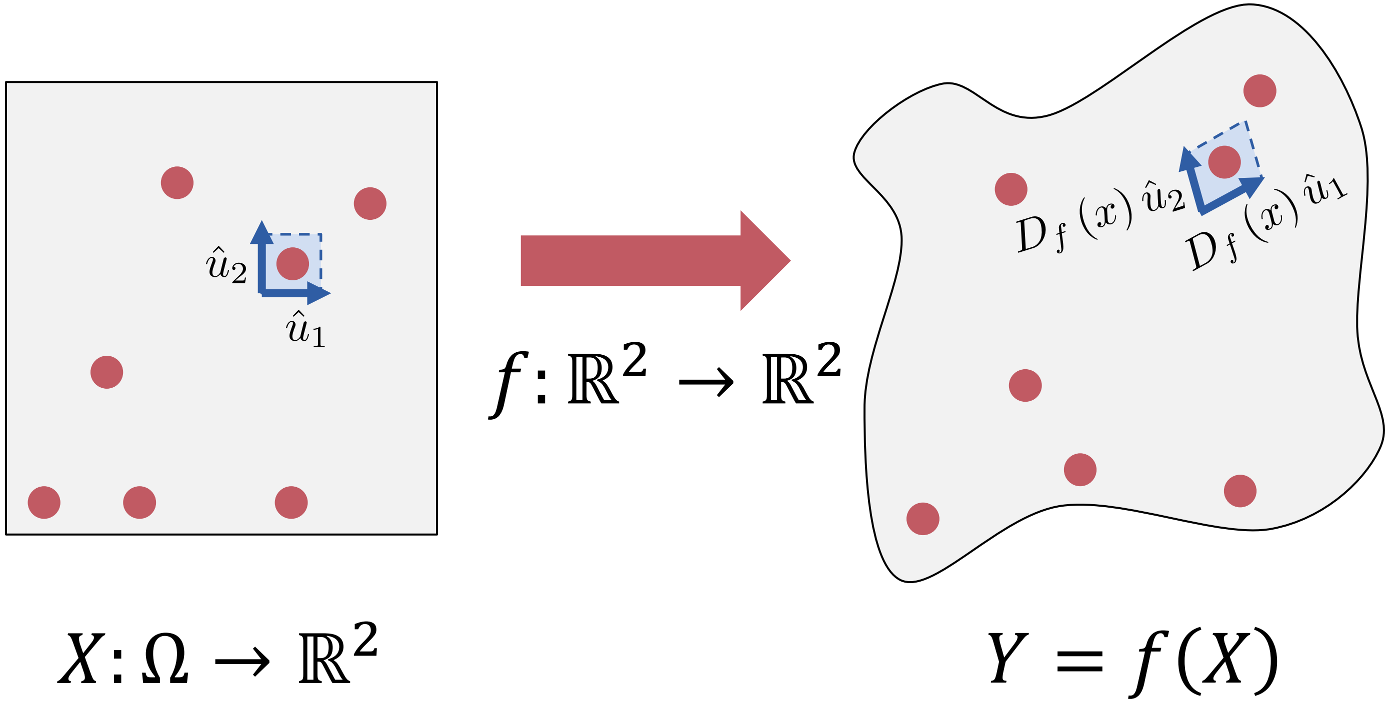
$Y = f(X)$

$$Y = f(X)$$

$$p_Y(y) = \frac{p_X\left(f^{-1}(y)\right)}{|\square| \,/\, |\square|} \; *$$

*: incomplete

PDF change of variables

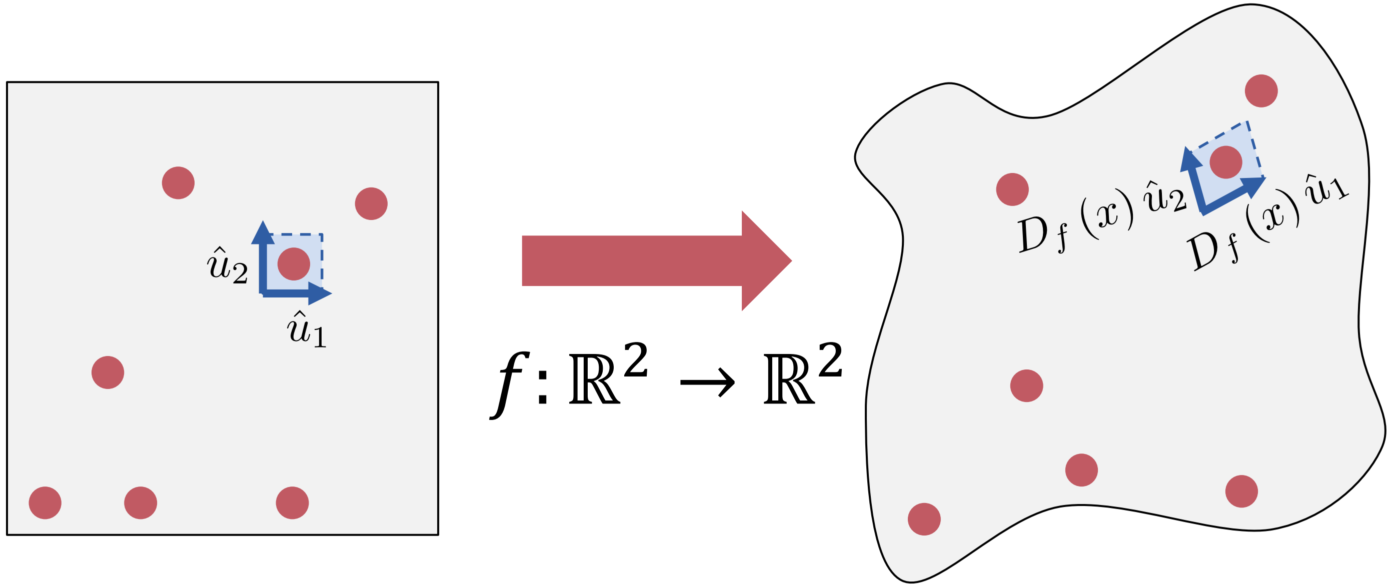


$$Y = f(X)$$

$$p_Y(y) = \frac{p_X\left(f^{-1}(y)\right)}{|\lozenge| \,/\, |\square|} \; *$$

$$\begin{aligned} |\lozenge| &= \left\| D_f(x)\,\hat{u}_1 \times D_f(x)\,\hat{u}_2 \right\| \\ &= \left|\det D_f(x)\right| \left\|\hat{u}_1\right\| \left\|\hat{u}_2\right\| \end{aligned}$$

PDF change of variables



$X\colon \Omega \to \mathbb{R}^2$

$Y = f(X)$

$$Y = f(X)$$

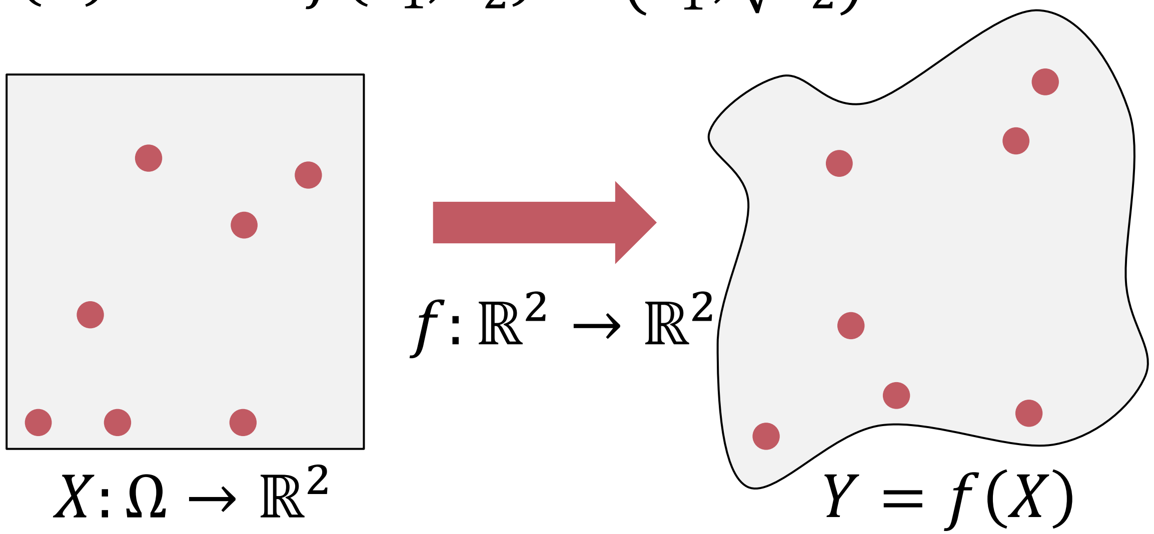
$$p_Y(y) = \frac{p_X\left(f^{-1}(y)\right)}{\left|\det D_f\left(f^{-1}(y)\right)\right|} \quad *$$

$$\begin{aligned} \left|\square\right| &= \left\| D_f(x)\,\hat{u}_1 \times D_f(x)\,\hat{u}_2 \right\| \\ &= \left|\det D_f(x)\right| \left\|\hat{u}_1\right\| \left\|\hat{u}_2\right\| \end{aligned}$$

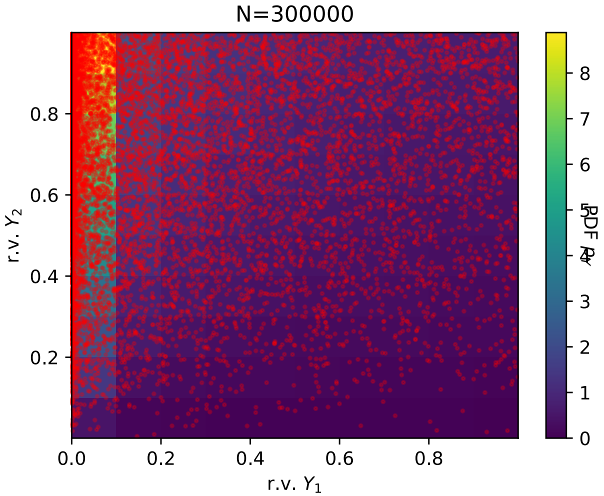
*: incomplete

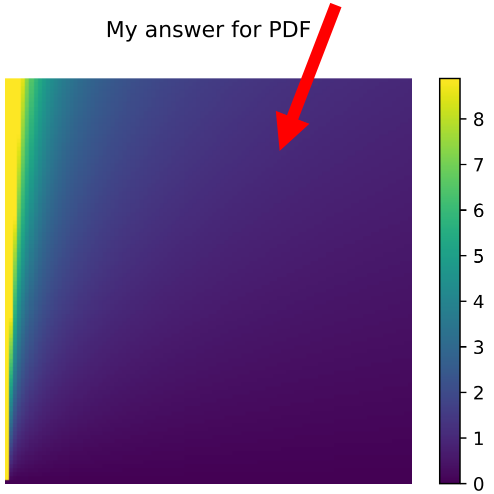
HW2 – Problem 1(c): PDF change of variable on 2D

- $X \sim \mathcal{U}([0,1]^2)$
- $Y = f(X)$ where $f(x_1, x_2) = \left(x_1^3, \sqrt{x_2}\right)$



Task: find equation of the PDF of Y



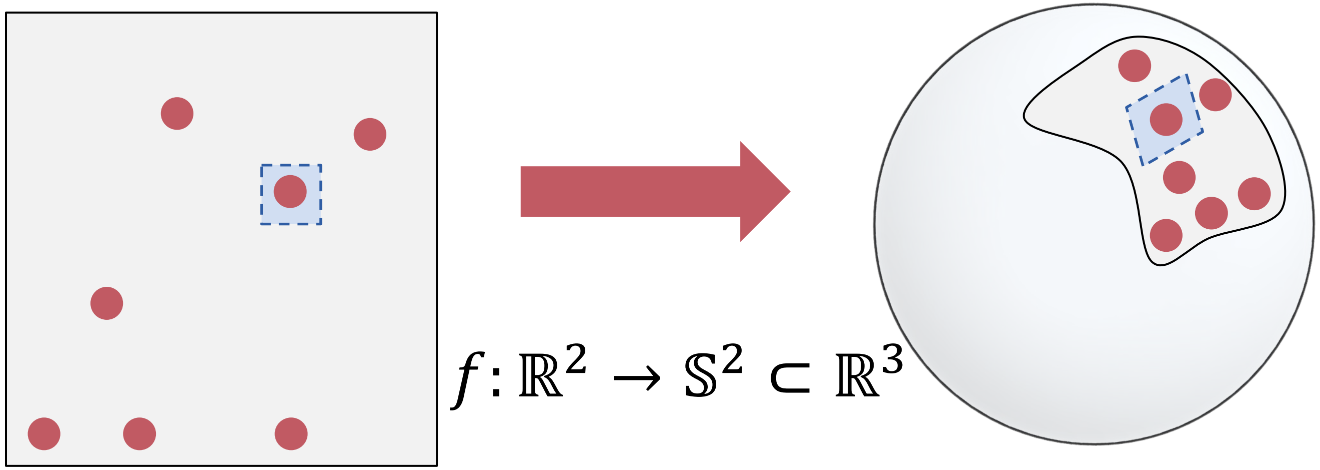


PDF change of variables

$$Y = f(X)$$

$$p_Y(y) = \frac{p_X\left(f^{-1}(y)\right)}{\left|\cancel{\det D_f}\left(f^{-1}(y)\right)\right|} \quad *$$

No determinant for 3×2 matrices



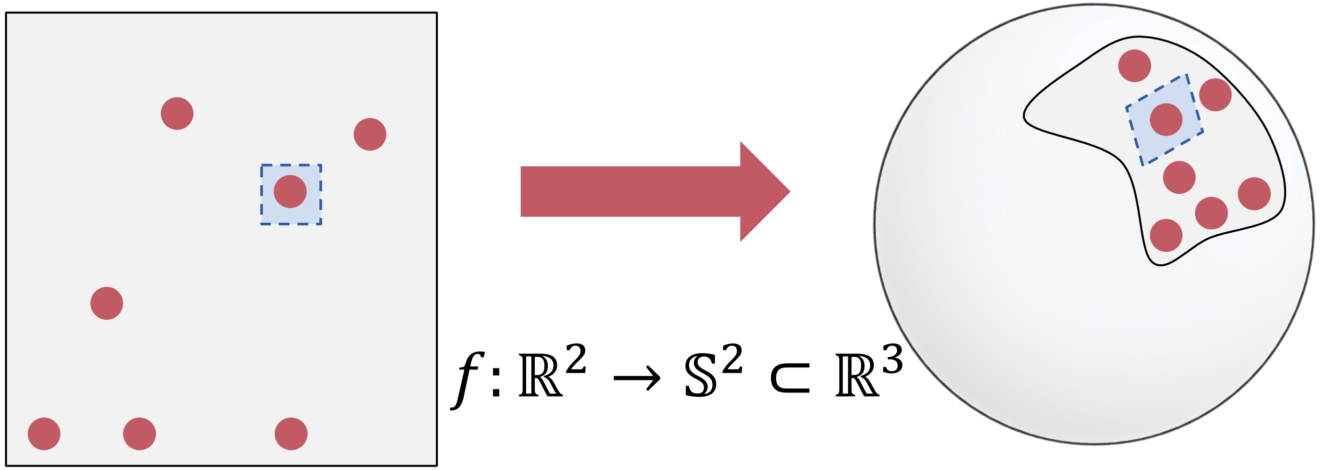
$f\colon \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$

$X\colon \Omega \to \mathbb{R}^2$

$Y = f(X)$

*: incomplete

PDF change of variables



$f : \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$

$X : \Omega \to \mathbb{R}^2$

$Y = f(X)$

Proposition

$$Y = f(X)$$

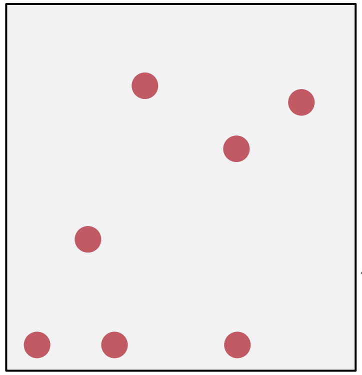
$$p_Y(y) = \frac{p_X\left(f^{-1}(y)\right)}{J_f\left(f^{-1}(y)\right)}$$

$$^{*}\, J_f(x) = \sqrt{\det\left(D_f(x)^T D_f(x)\right)}$$

*: not a very common notation

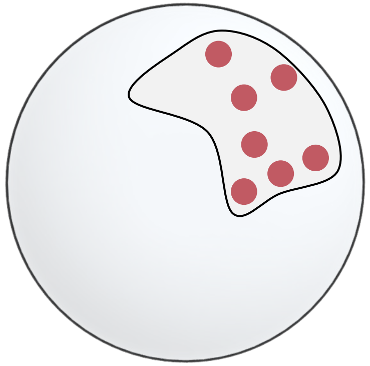
HW2 – Problem 1(c): PDF change of variable into $\mathbb{S}^2$

- $X \sim \mathcal{U}([0,1]^2)$
- Find f such that $Y = f(X)$ follows the uniform distribution on $\mathbb{S}^2$



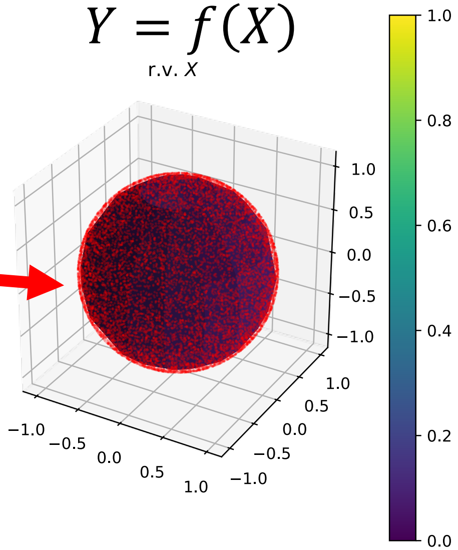
$X: \Omega \to \mathbb{R}^2$

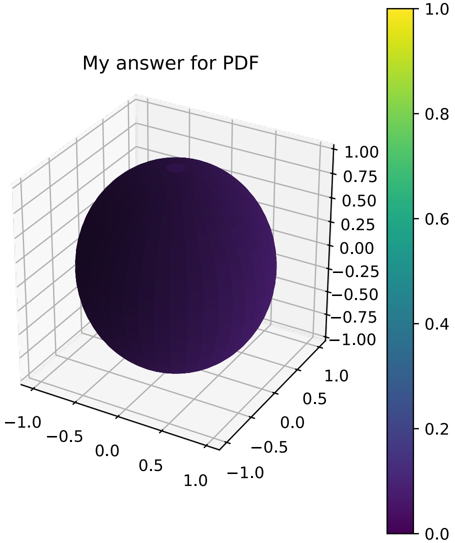
$f: \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$

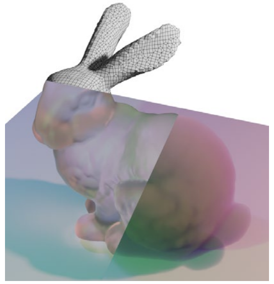


$Y = f(X)$

Task: find such f to make Y uniform





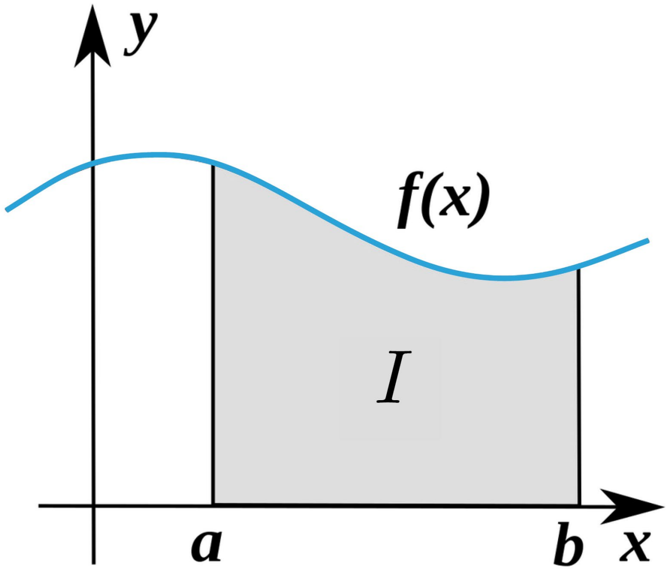


Numerical Integration

Integration

We have a function f on a domain $\mathcal{X}$.

Q. How can we compute $I = \int_{\mathcal{X}} f(x)\,\mathrm{d}x$?



$\mathcal{X} = [a, b] \subset \mathbb{R}$

source: User 4C / Wikipedia

[Adam Celarek]

1. Analytic (symbolic) integration

- We can integrate only few functions analytically.
- Hardly predict difficulty of given problem

Try analytic integration for:

$$\int e^{-x^2}\,\mathrm{d}x$$

$$\int x e^{-x^2}\,\mathrm{d}x$$

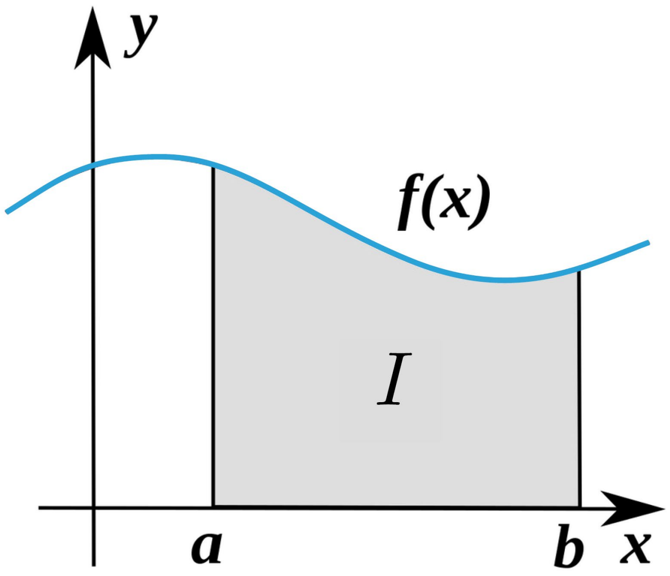
$$\int x^2 e^{-x^2}\,\mathrm{d}x$$

$$\int x^3 e^{-x^2}\,\mathrm{d}x$$

Integration

We have a function f on a domain $\mathcal{X}$.

Q. How can we compute $I = \int_{\mathcal{X}} f(x)\,\mathrm{d}x$?

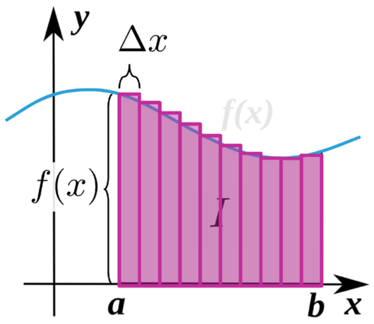


$\mathcal{X} = [a, b] \subset \mathbb{R}$

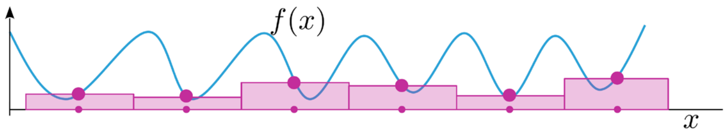
[Adam Celarek]

1. **Analytic (symbolic) integration**
- We can integrate only few functions analytically.
- Hardly predict difficulty of given problem

2. **Numerical approximation by uniform grids**
- For a fixed error level, we need $N^{\dim(\mathcal{X})}$ samples!
- Aliasing!

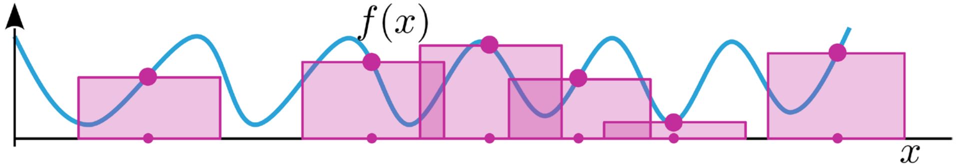


$$I \approx \sum_{n=1}^{N} f(x_n)\,\Delta x$$



3. **Monte Carlo estimation**: for a random variable $X \sim \mathcal{U}(\mathcal{X})$

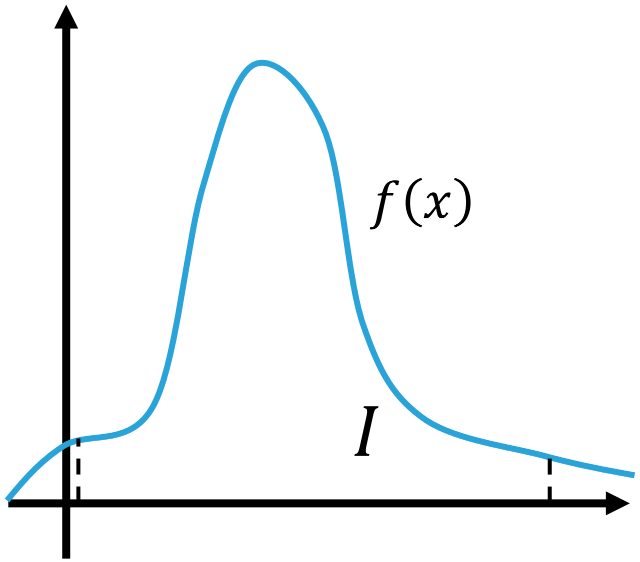
$$I \approx \hat{I} = \frac{|\mathcal{X}|}{N} \sum_{n=1}^{N} f(X_n)$$



Monte Carlo Integration

We have a function f on a domain $\mathcal{X}$.

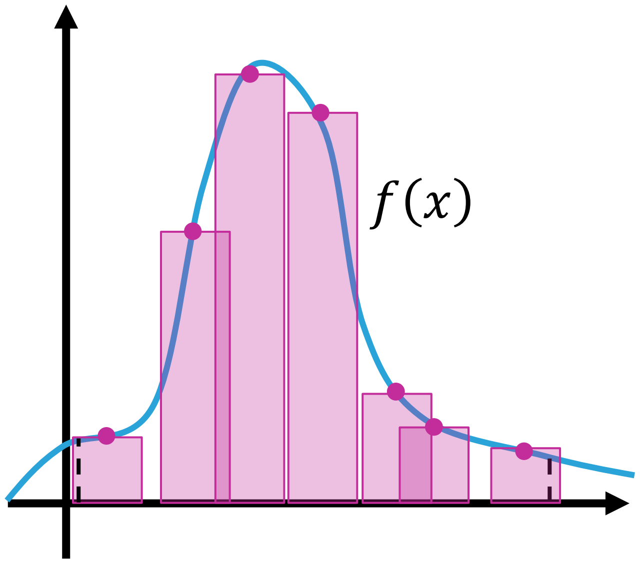
Q. How can we compute $I = \int_{\mathcal{X}} f(x)\,\mathrm{d}x$?



$\mathcal{X} = [a, b] \subset \mathbb{R}$

for a random variable $X \sim \mathcal{U}(\mathcal{X})$

$$I \approx \hat{I} = \frac{|\mathcal{X}|}{N} \sum_{n=1}^{N} f(X_n)$$



$$Var(\hat{I}) \propto Var(\square)$$

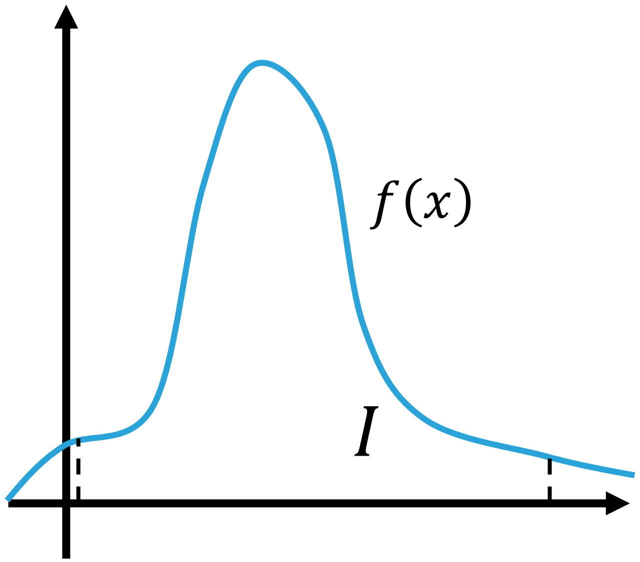
Monte Carlo Integration: importance sampling

for a random variable X with PDF hopefully $\propto f$

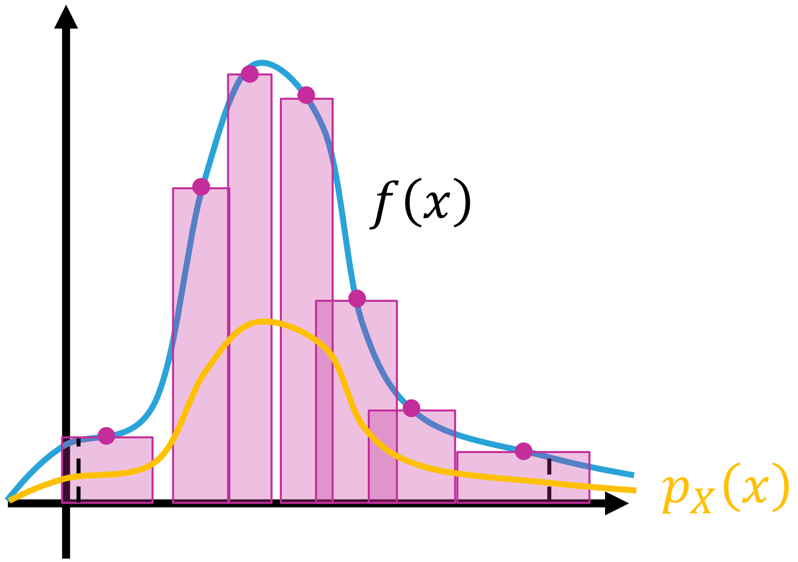
We have a function f on a domain $\mathcal{X}$.

Q. How can we compute $I = \int_{\mathcal{X}} f(x)\,\mathrm{d}x$?

$$I \approx \hat{I} = \frac{1}{N}\sum_{n=1}^{N} \frac{f(X_n)}{p_X(X_n)}$$



$\mathcal{X} = [a, b] \subset \mathbb{R}$



$$Var(\hat{I}) \propto Var(\boxed{})$$

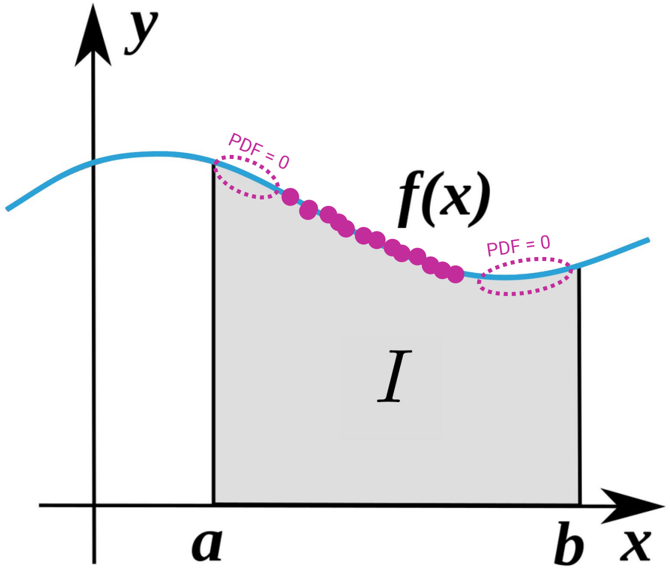
Monte Carlo Integration

When is the MC integration correct (unbiased)?

$$\int_{\mathcal{X}} f(x)\mathrm{d}x = I = \mathbb{E}\left[\hat{I} = \frac{1}{N}\sum_{n=1}^{N}\frac{f(X_n)}{p_X(X_n)}\right]$$

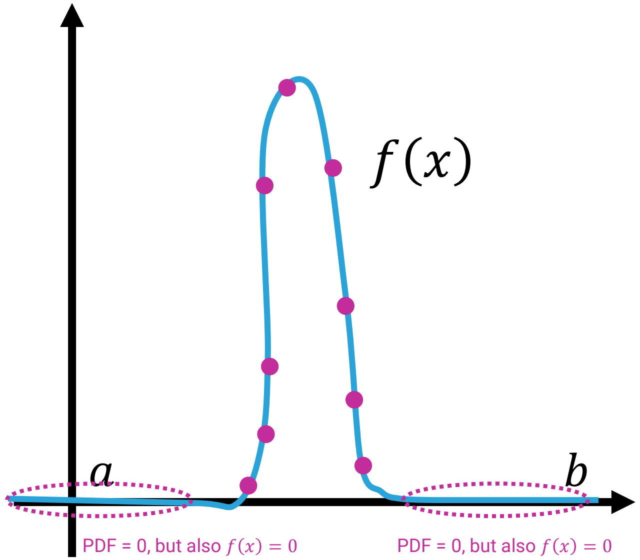
For any $x \in \mathcal{X}$, if $f(x) \neq 0$ then $p_X(x) > 0$

Failure case



$\mathcal{X} = [a, b] \subset \mathbb{R}$

Okay



$\mathcal{X} = [a, b] \subset \mathbb{R}$

Language of Monte Carlo Integration

Sampling: generate a realization of a random variable

Which PDF do we choose?
How draw a sample of a r.v. with such PDF?

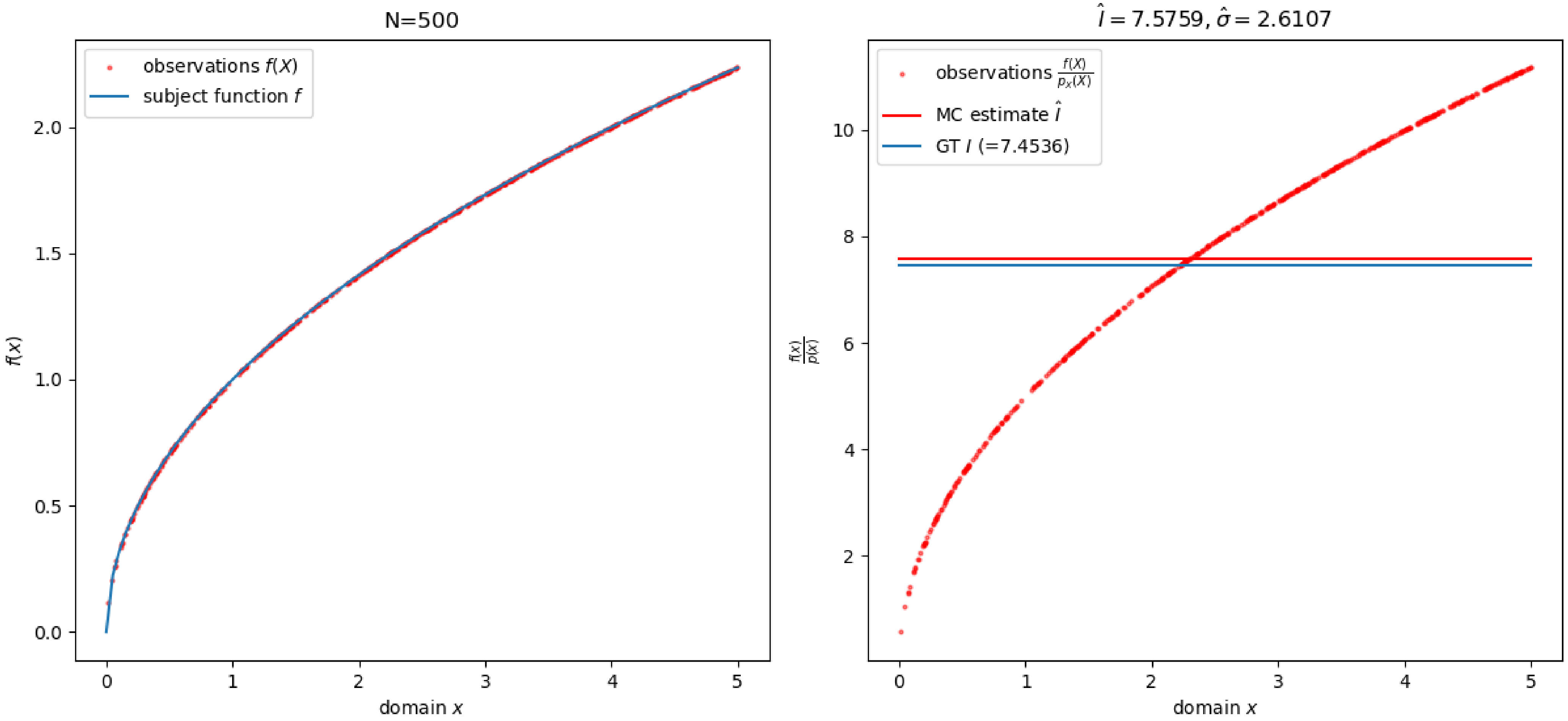
- *"This sampling strategy is good for given integrand f."*
- *"In path tracing, BSDF sampling and emitter samplings are used."*

Correct → **unbiased** (or **consistent**, at least)

Efficient → **low variance**

HW2 - Problem 2.(a): Monte Carlo on 1D

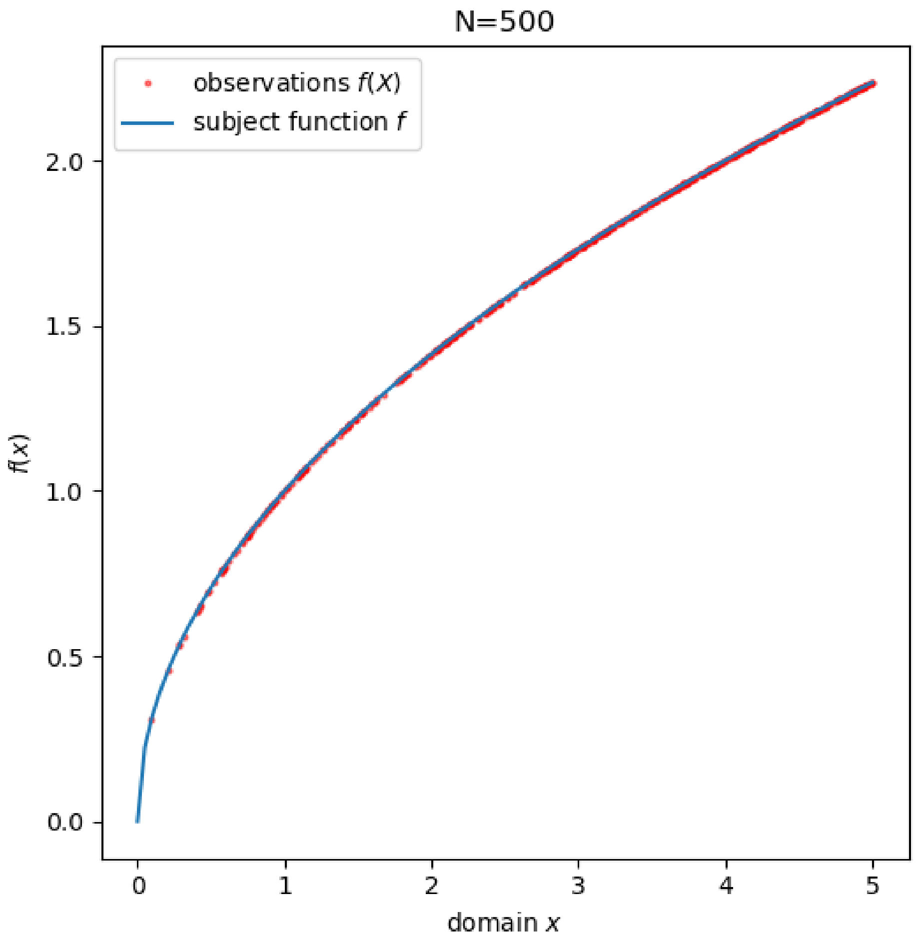
random variable $X \sim \mathcal{U}(\mathcal{X})$

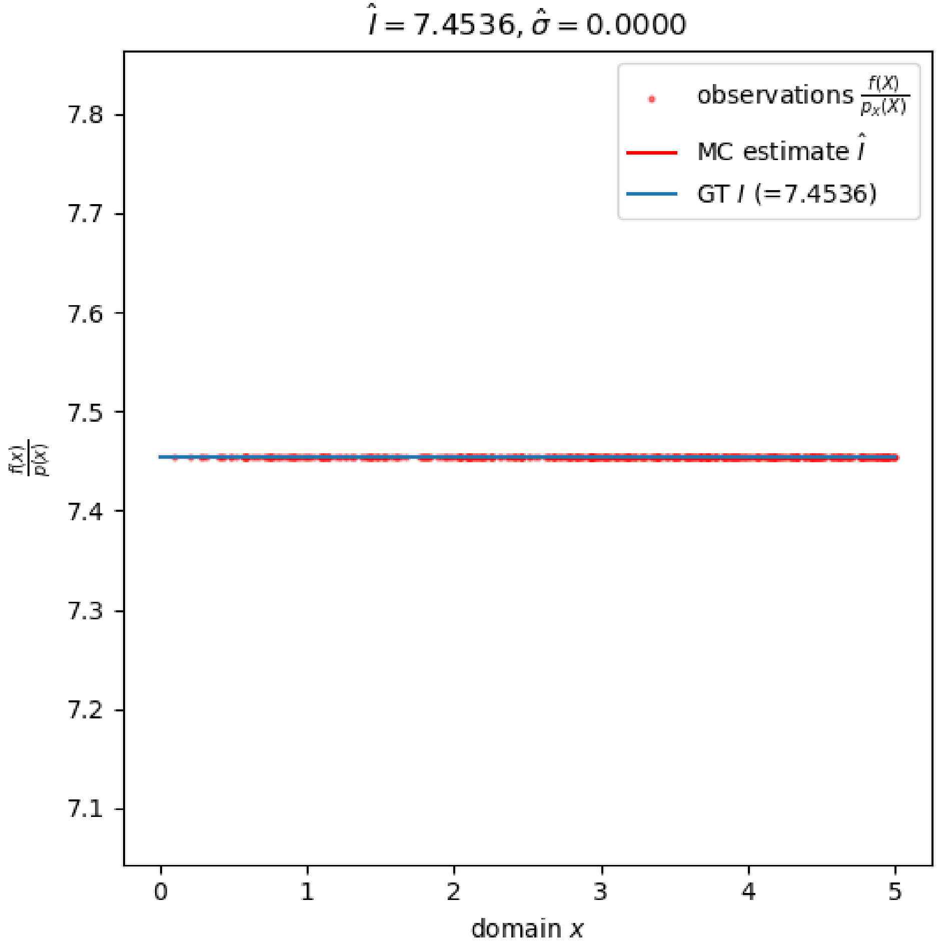


HW2 - Problem 2.(a): Monte Carlo on 1D

random variable X with PDF $\propto f$

Task: implement to draw X following such distribution

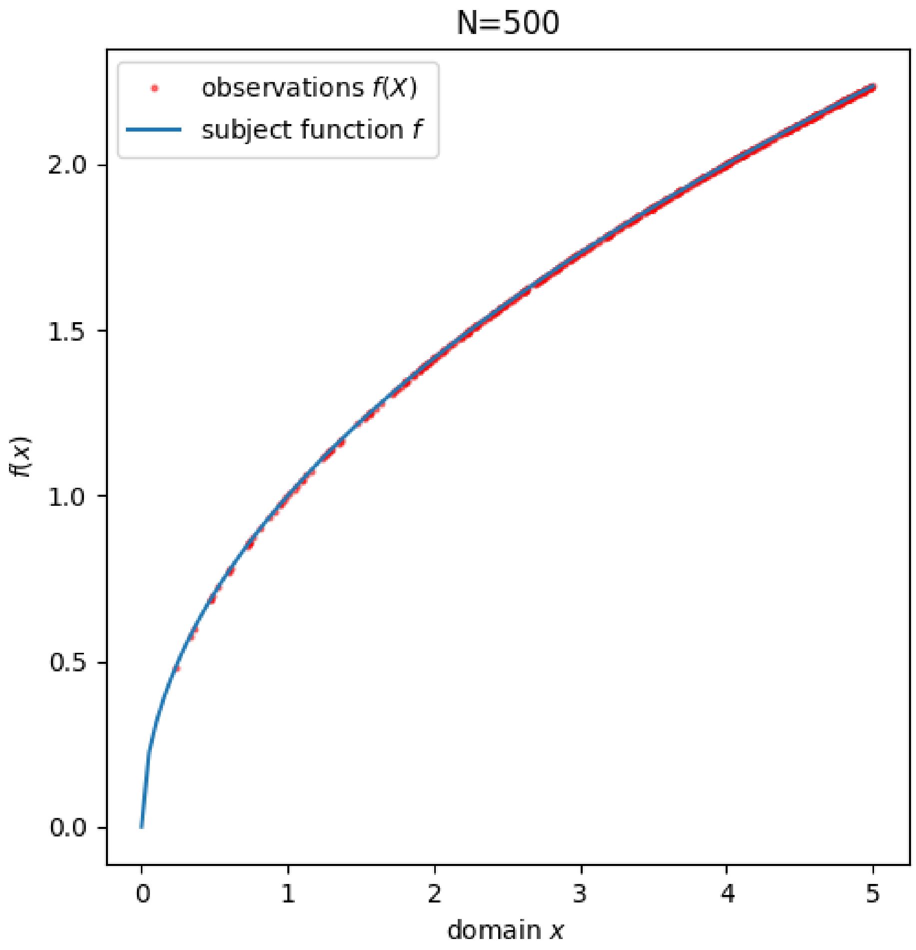


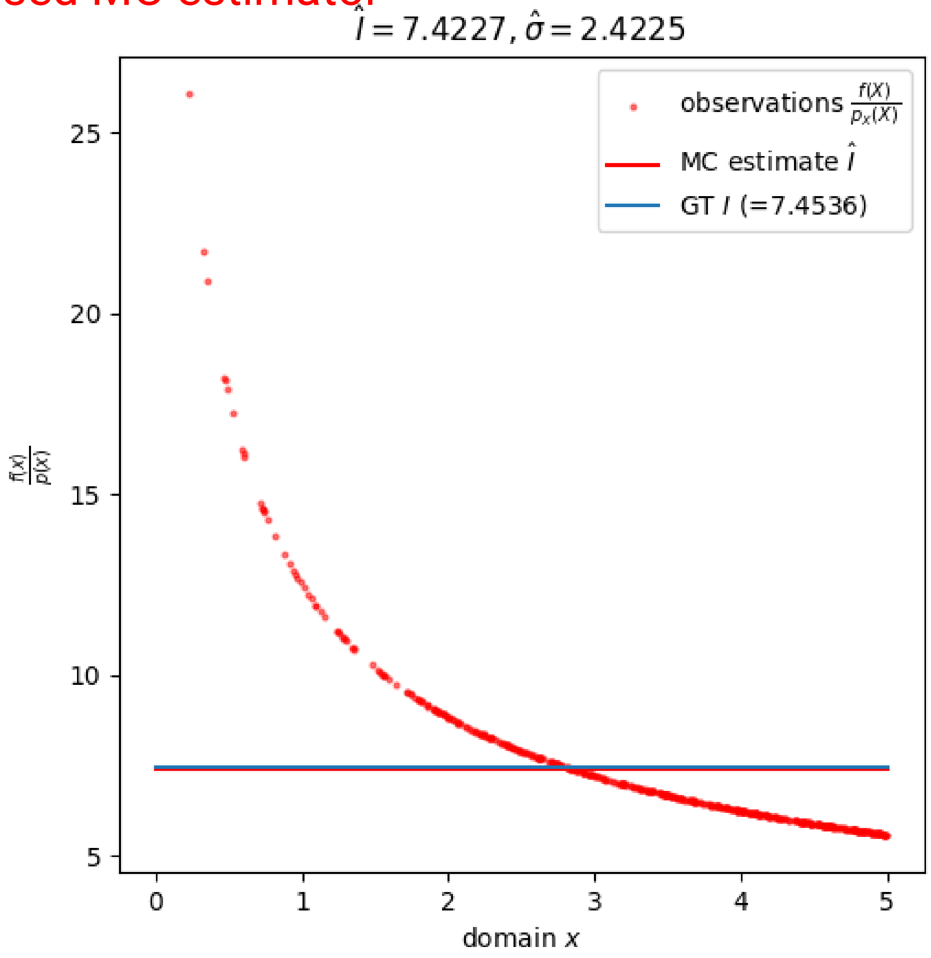


HW2 - Problem 2.(c): Not an optimal importance

random variable X a PDF having more similar trend to f than the uniform distribution

Task: find the PDF of such X to build an unbiased MC estimator





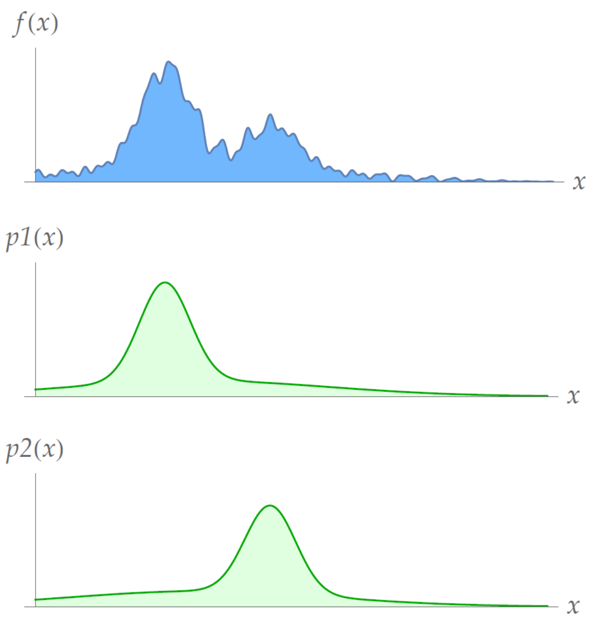
How is importance sampling useful in practice?

- Performing importance sampling with a PDF exactly $\propto f$ is as hard as the integration itself
  - However, using a PDF which roughly $\propto f$ and analytically integrable is still useful

- We have good importance sampling for f and g. How about $\int_{\chi} f(x)g(x)\mathrm{d}x$?

Multiple Importance Sampling (MIS)

- Many possible importance sampling strategies
- Which one should we use for a given integrand?
- MIS: *combine* strategies to preserve strengths of all of them
- *Balance heuristic* is (provably!) about as good as anything:



sum over strategies

sum over samples

jth sample taken with ith strategy

$$\frac{1}{N}\sum_{i=1}^{n}\sum_{j=1}^{n_i}\frac{f(x_{ij})}{\sum_k c_k p_k(x_{ij})}$$

total # of samples

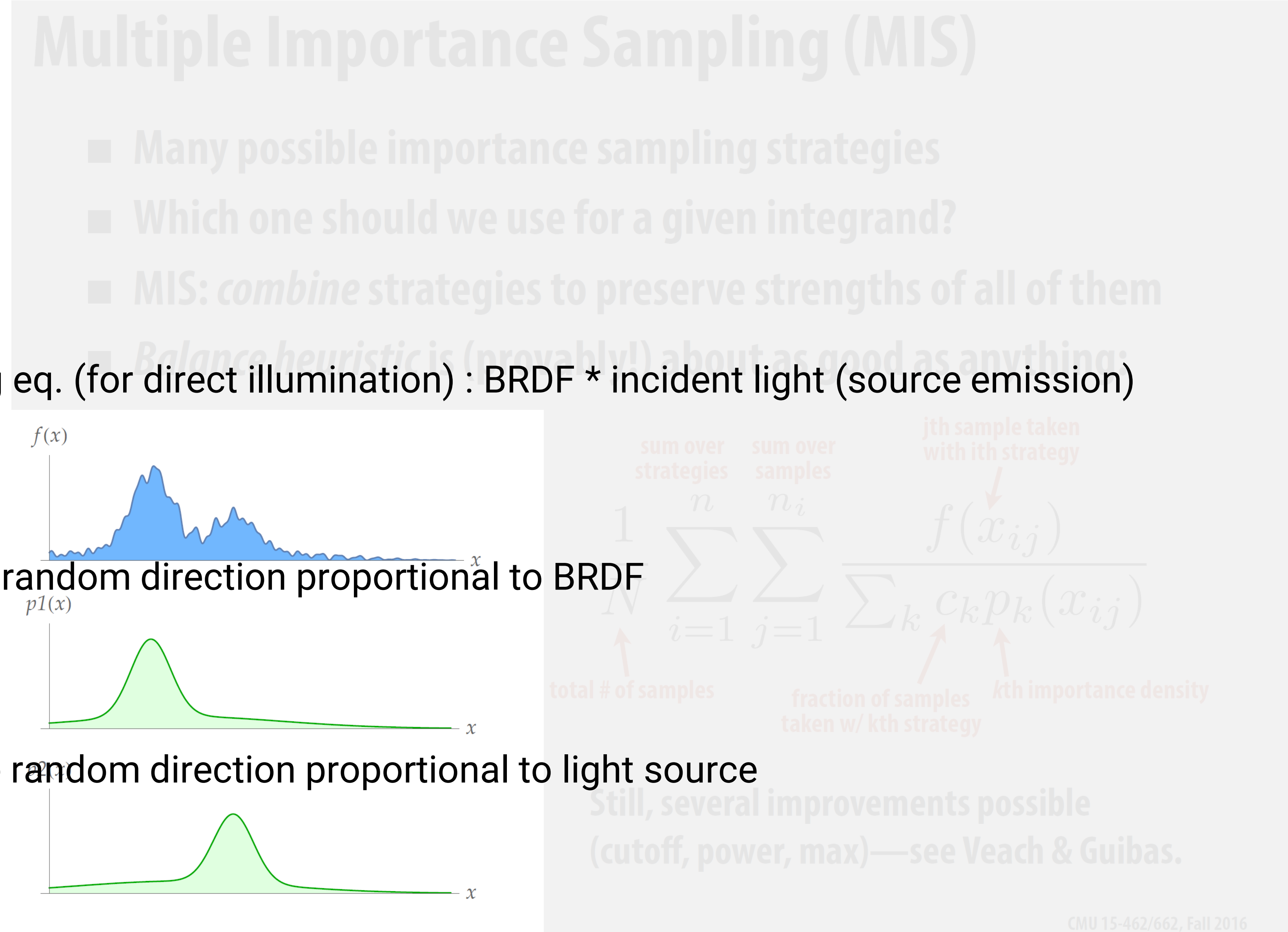
fraction of samples taken w/ kth strategy

*k*th importance density

Still, several improvements possible (cutoff, power, max)—see Veach & Guibas.

[Keenan Crane]

Multiple Importance Sampling: in context of rendering



rendering eq. (for direct illumination) : BRDF * incident light (source emission)

generate random direction proportional to BRDF

generate random direction proportional to light source

[Keenan Crane]

Multiple Importance Sampling: statement

For integration problem

$$I = \int_{\mathcal{X}} f(x)\mathrm{d}x$$

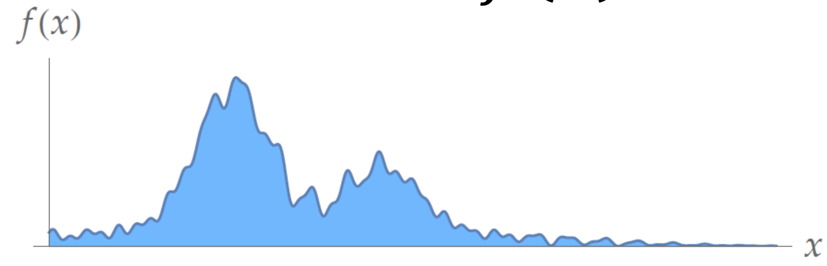
We have $2N$ samples $X_{A,i} \sim p_A$ and $X_{B,i} \sim p_B$ with $i = 1, \cdots, N$

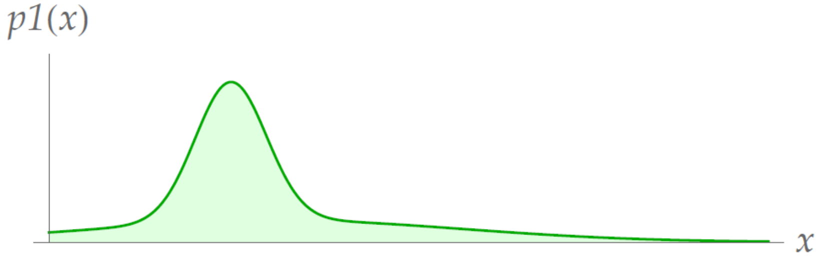
$$\hat{I}_{\mathrm{MIS}(A,B)} = \frac{1}{N} \sum_{s\in\{A,B\}} \sum_{i=1}^{N} w_s\left(X_{s,i}\right) \frac{f\left(X_{s,i}\right)}{p_s\left(X_{s,i}\right)}, \text{where}$$

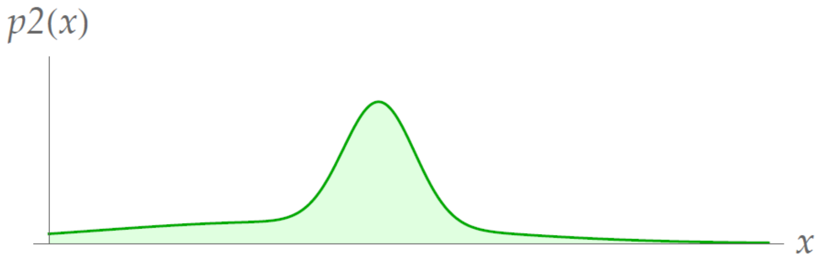
- $\sum_{s\in\{A,B\}} w_s(x) = 1$ whenever $f(x) \neq 0$,
- $w_s(x) = 0$ whenever $p_s(x) = 0$

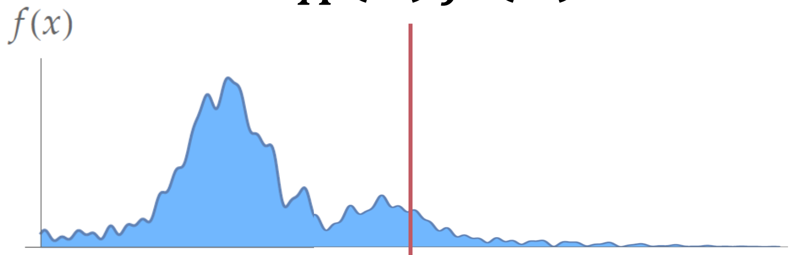
A way to understand MIS

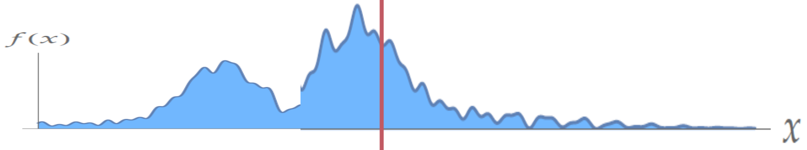
$$f(x) \quad = \quad w_A(x) f(x) \quad + \quad w_B(x) f(x)$$











MC using $X \sim p_A$

MC using $X \sim p_B$

$$\frac{1}{N}\sum_{i=1}^{N}\frac{w_A(X_{A,i})f(X_{A,i})}{p_A(X_{A,i})} + \frac{1}{N}\sum_{i=1}^{N}\frac{w_B(X_{B,i})f(X_{B,i})}{p_A(X_{B,i})}$$

[Keenan Crane]

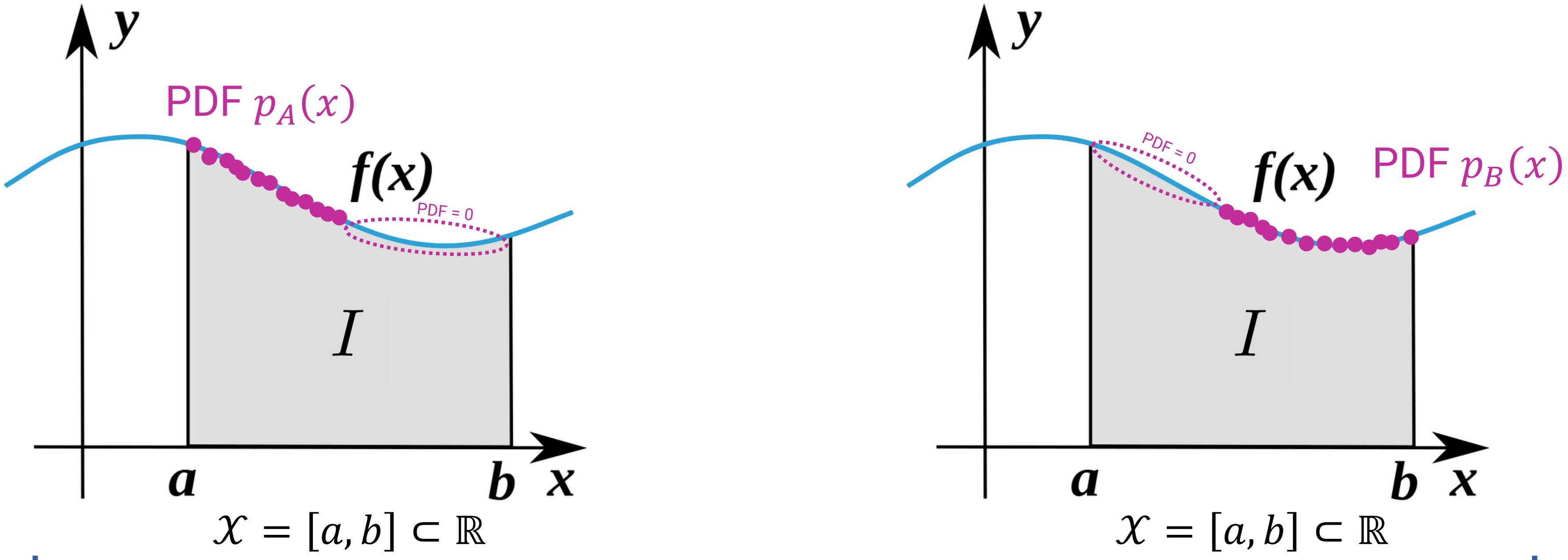
Multiple Importance Sampling: unbiasedness

When is the MIS correct (unbiased)?

For any $x \in \mathcal{X}$, if $f(x) \neq 0$ then either $p_A(x) > 0$ or $p_B(x) > 0$

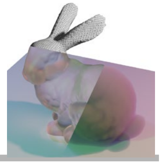
Fail, if use it alone

Fail, if use it alone



But $\hat{I}_{MIS(A,B)}$ is a correct estimator

MIS strategies



Q. Then how can we compute such MIS weightes $w_A(x)$ and $w_B(x)$?

1. Very naively, just $w_A = w_B = \frac{1}{2}$
   - only works if both of each strategy A and B are unbiased.
   - Do not use in practice

2. Balance heuristic: $w_A(x) = \frac{p_A(x)}{p_A(x)+p_B(x)}$, $w_B(x) = \frac{p_B(x)}{p_A(x)+p_B(x)}$
   - A certain upper bound of difference between it and an optimum is known
     - [Veach 1997] Theotem 9.2

3. Power heuristic: $w_A(x) = \frac{p_A(x)^\alpha}{p_A(x)^\alpha+p_B(x)^\alpha}$, $w_B(x) = \frac{p_B(x)^\alpha}{p_A(x)^\alpha+p_B(x)^\alpha}$
   - Usually $\alpha = 2$

MIS strategies

Q. Then how can we compute such MIS weightes $w_A(x)$ and $w_B(x)$?

1. Very naively, just $w_A = w_B = \frac{1}{2}$
   - only works if both of each strategy A and B are unbiased.
   - Do not use in practice

2. Balance heuristic: $w_A(x) = \frac{p_A(x)}{p_A(x)+p_B(x)}$, $w_B(x) = \frac{p_B(x)}{p_A(x)+p_B(x)}$
   - A certain upper bound of difference between it and an optimum is known
     - [Veach 1997] Theorem 9.2

Theorem 9.2. *Let f, n_i, and p_i be given, for $i = 1, \ldots, n$. Let F be any unbiased estimator of the form (9.4), and let $\hat{F}$ be the estimator that uses the weighting functions $\hat{w}_i$ (the balance heuristic). Then*

$$V[\hat{F}] - V[F] \leq \left(\frac{1}{\min_i n_i} - \frac{1}{\sum_i n_i} \right) \mu^2 , \qquad (9.9)$$

where $\mu = E[F] = E[\hat{F}]$ is the quantity to be estimated. (A proof is given in Appendix 9.A.)