Solving Sustained Oscillation Induced by time delay equations with Runge-Kutta[*]

Edward Weisberg

2024/12/05

1. Introduction

The Time Delay graph explores oscillation in various parameters alongside the time delay (tau). Such a phenomenon in the graphs' dampening from sharp oscillations can be explained rather agreeably by the implication of Hopf bifurcation which is outside of this write-up's scope. The applications of such a graph are not merely a curiosity; implications across commodity markets, monetary policy and quantitative easing are all widespread and integral to the general workings of our economic climate.

2. Method

2.1 The Equations Explored:

Let $P(t)$ be the price at time t, $Q_d(t)$ the quantity demanded at time t, and $Q_s(t)$ the quantity supplied at time t. The model includes a time delay in the supply response.

The price change is described by the following differential equation:

$$\frac{dP(t)}{dt} = -aQ_d(t) + bQ_s(t) + cP(t),$$

where a, b, and c are parameters.

The rate of change of the quantity demanded is given by:

$$\frac{dQ_d(t)}{dt} = d - eP(t),$$

where d is the baseline demand and e is a multiplier that measures how sensitive demand is to price changes (price elasticity for demand).

The rate of change of the quantity supplied is modeled with a time delay:

$$\frac{dQ_s(t)}{dt} = fP(t-\tau) - gQ_s(t),$$

where f and g are parameters, and τ represents the time delay.

These Functions were solved using the Runge-Kutta approach.

2.2 The Solver:

We solve the following system of equations:

$$\frac{dP}{dt} = a(1 - P - bP_{\text{delayed}}) + 1,$$

$$\frac{dQ_s}{dt} = fP_{\text{delayed}} - gQ_s.$$

Using the 4th-order Runge-Kutta method:

Equation for $P(t)$:

$$P_{n+1} = P_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where:

$$k_1 = h \cdot \frac{dP}{dt}(P_n, P_{\text{delayed}}),$$

$$k_2 = h \cdot \frac{dP}{dt}\left(P_n + \frac{k_1}{2}, P_{\text{delayed}}\right),$$

$$k_3 = h \cdot \frac{dP}{dt}\left(P_n + \frac{k_2}{2}, P_{\text{delayed}}\right),$$

$$k_4 = h \cdot \frac{dP}{dt}(P_n + k_3, P_{\text{delayed}}).$$

Equation for $Q_s(t)$:

$$Q_{s,n+1} = Q_{s,n} + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where:

$$k_1 = h \cdot \frac{dQ_s}{dt}(P_{\text{delayed}}, Q_{s,n}),$$

$$k_2 = h \cdot \frac{dQ_s}{dt}\left(P_{\text{delayed}}, Q_{s,n} + \frac{k_1}{2}\right),$$

$$k_3 = h \cdot \frac{dQ_s}{dt}\left(P_{\text{delayed}}, Q_{s,n} + \frac{k_2}{2}\right),$$

$$k_4 = h \cdot \frac{dQ_s}{dt}(P_{\text{delayed}}, Q_{s,n} + k_3).$$

Time and Delay Handling:

At each step i:

$$t_i = i \cdot h,$$

and the delayed state is:

$$P_{\text{delayed}} = \text{value stored in the delay buffer at time } t_i.$$

[1]

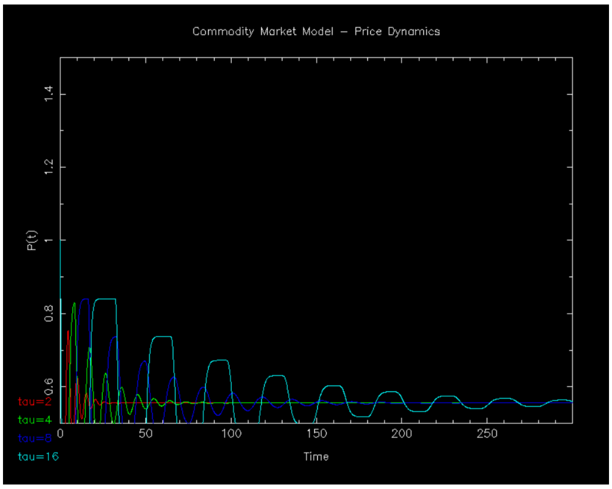

Figure 1: Demo Code (all general Taus graphed)

3. Tests

Although the Demo code is not fully realized and has inconsistencies with time, dampening effects, oscillation frequency, and converging value, the general behavior is intact and the main takeaway is apparent - no matter the size of the time delay, or oscillations apparent, a converging price value will be reached given enough time

For the tests below, it should be noted that the source does not provide the parameters in each test case. As such generalized parameters that best fit the source graphs will be used:

The parameters for the model are defined as follows:

$$\begin{aligned}
a &= 1.5 \quad \text{(Adjusted price adjustment speed)} \\
b &= 0.8 \quad \text{(Adjusted demand elasticity)} \\
d &= 0.5 \quad \text{(Baseline demand)} \\
e &= 0.3 \quad \text{(Price elasticity of demand)} \\
f &= 0.7 \quad \text{(Supply response to price)} \\
g &= 0.6 \quad \text{(Decay of supply)}
\end{aligned}$$

[1] The equations chosen are simpler version from [1] for coding simplicity

3.1 Test 1:

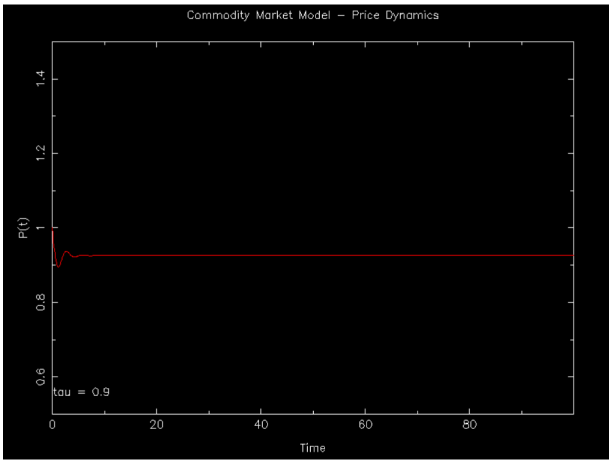

Figure 2: Demo Tau = 0.9

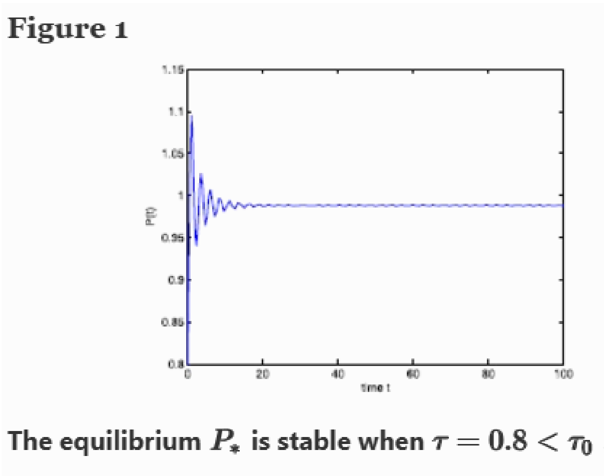

Figure 3: Source at Tau = 0.8

Using the extension presently which is the time delay and setting it to 0.9 creates a very similar plot to the source. As referenced before, the functions are simplified from the ones used in the source for feasibility reasons in terms of the functions and parameters utilized. With all this being said, however, the general oscillations and converging value are more or less identical, just that the beginning inaccurary is greater in the source

3.2 Test 2:

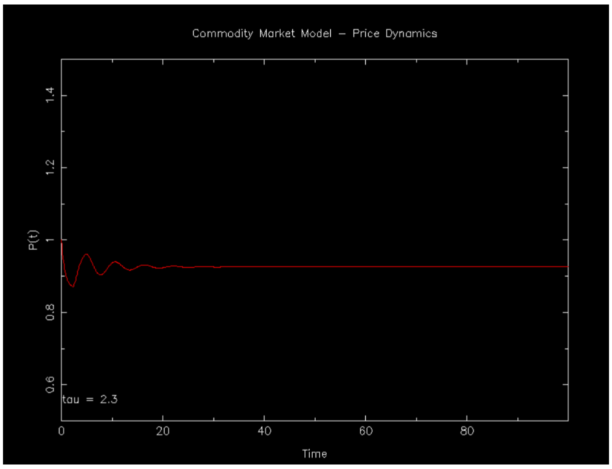

Figure 4: Tau = 2.3

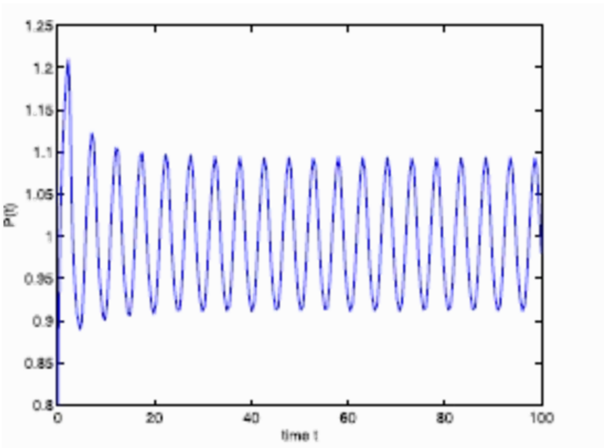

Figure 5: Source at Tau = 2

Using the extension presently - which is the time delay - and setting it to 2.3 creates a very similar plot to the source. As referenced before, the functions are simplified from the ones used in the source for feasibility reasons in terms of the functions and parameters utilized. With all this being said, however, the general oscillations and converging value are more or less identical, just that the beginning inaccurary is greater in the source

3.3 Test 3:

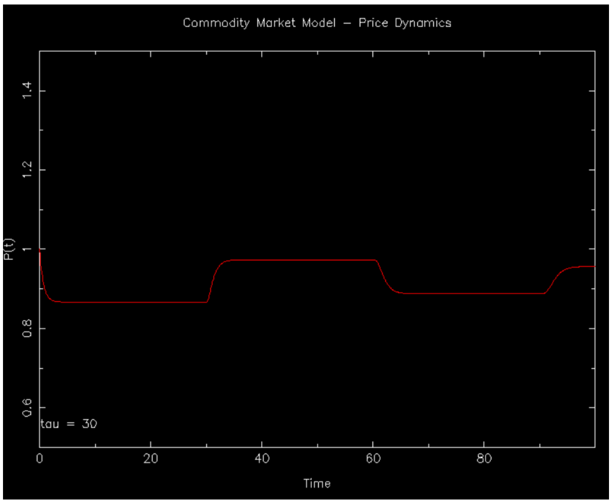

Figure 6: Tau = 30

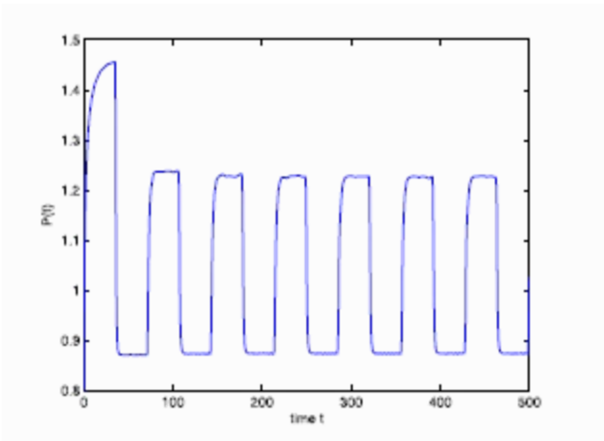

Figure 7: Source at Tau = 30

The simplifications in the equations utilized have caused the accuracy of the graphs plotted for greater values of Tau to be inhibited. The hastening of the dampening effects in the code has caused it to overshoot the actual value by a large margin. It should be noted that the general shape of the oscillations are relatively similar just that they are drawn out for longer periods of time.

3.4 Figures and Tables

Time	$P(t)$
0.0	1.000000
0.2	0.940450
0.4	0.882100
0.6	0.826090
0.8	0.773290
1.0	0.724230
1.2	0.679250
1.4	0.638620
1.6	0.602500
1.8	0.570990
2.0	0.544060
2.2	0.521600
2.4	0.503420
2.6	0.489310
2.8	0.479050
3.0	0.472440
3.2	0.469270
3.4	0.469340
3.6	0.472480
3.8	0.478490
4.0	0.487140
4.2	0.498200
4.4	0.511490
4.6	0.526820
4.8	0.544030
5.0	0.562950
5.2	0.583380
5.4	0.605100
5.6	0.627870
5.8	0.651460
6.0	0.675620
6.2	0.700110
6.4	0.724680
6.6	0.749090
6.8	0.773100
7.0	0.796470

Table 1: Table of $P(t)$ values for time up to $t = 10.0$ For estimated value Tau $= 0.9$.

Results

Test 1 (Comparing against demo code)

3.5 Test 1 (Comparing against demo code): 72

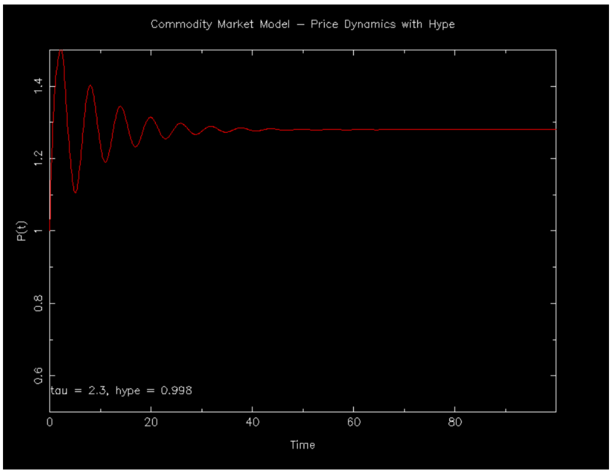

Figure 8: Extensionality (Hype) with Tau = 2.3

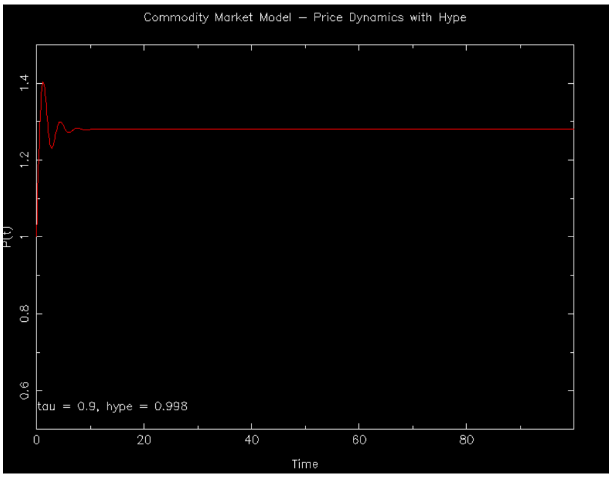

Figure 9: Extensionality (Hype) with Tau = 0.9

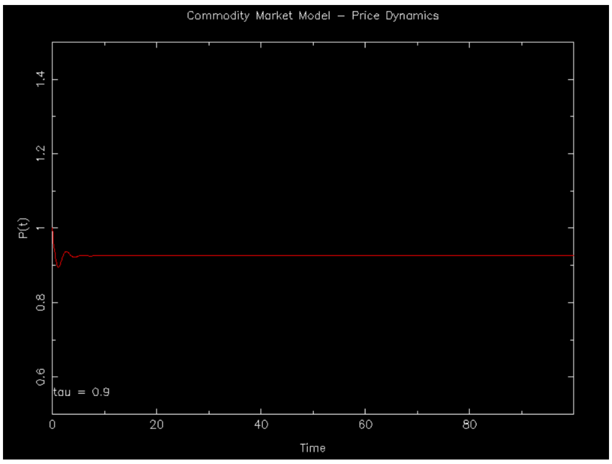

Figure 10: Demo Tau = 0.9

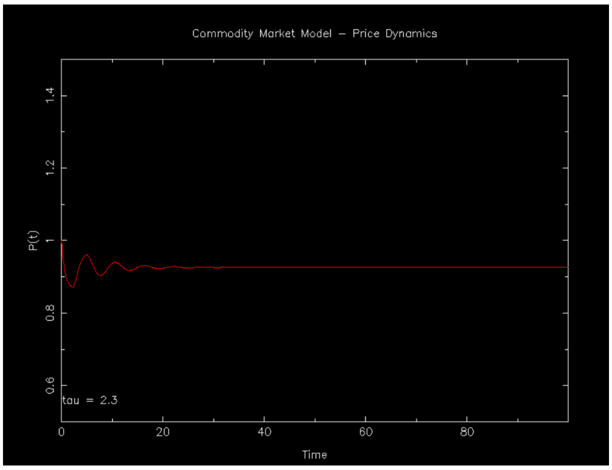

Figure 11: Demo Tau = 2.3

Figure 12: Playstation price [2]

[2]

Analyzing the figures, it's evident to see that hype variable is working effectively compared to the original demos. Hype pushes the overall price up (as increase demand would do) generally. It also causes the oscillations to be more pronounced throughout the graph. This makes sense as "hype" would cause more volatility as uncertainty would clash with irrational expectations and prospects causing the price to rise and dip dramatically and quickly. Furthermore, the price hitting an equilibrium in a longer period of time would also make sense as hype slowly undermines itself over time. Observing real life too - via the PlayStation price graph - it is apparent that such a relationship is true practically. Due to many more external factor, the curve is more complex than the simulated one. Still though, it takes longer for the PlayStation curve to reach equilibrium just like the extension of hype.

4. Test 2 (Comparing with different parameters)

Default Parameter Values For Test:

[2] Yahoo Finance

$$a = 1.5 \quad \text{(Adjusted price adjustment speed)}$$
$$b = 0.8 \quad \text{(Adjusted demand elasticity)}$$
$$d = 0.5 \quad \text{(Baseline demand)}$$
$$e = 0.3 \quad \text{(Price elasticity of demand)}$$
$$f = 0.7 \quad \text{(Supply response to price)}$$
$$g = 0.6 \quad \text{(Decay of supply)}$$

New Parameter Values For Test: 87

$$a = 1.5 \qquad // \text{ Adjusted price adjustment speed}$$
$$b = 1.0 \qquad // \text{ Adjusted demand elasticity}$$
$$d = 0.8 \qquad // \text{ Baseline demand}$$
$$e = 0.5 \qquad // \text{ Price elasticity of demand}$$
$$f = 0.9 \qquad // \text{ Supply response to price}$$
$$g = 1.2 \qquad // \text{ Decay of supply}$$

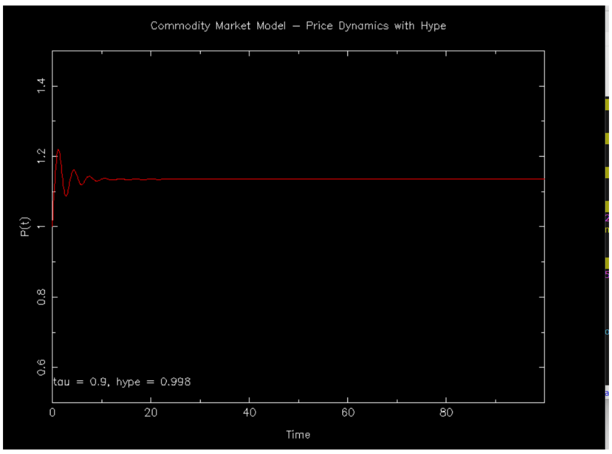

Figure 13: New Parameters with Tau = 0.9

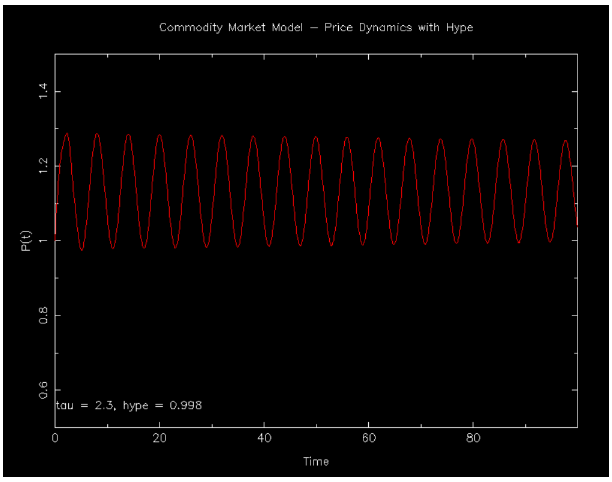

Figure 14: New Parameters with Tau = 2.3

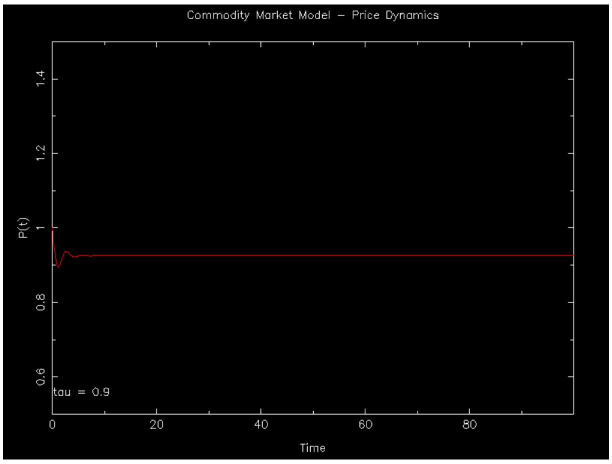

Figure 15: Demo Tau = 0.9

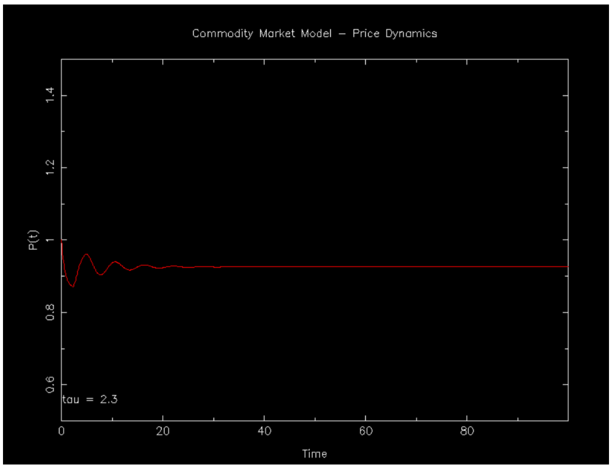

Figure 16: Demo Tau = 2.3

It is apparent that changing around the parameters and increasing the number from the demo proves the effects that would hype would have on such a graph. Prices take longer to reach an equilibrium in the case where hype is applied and the dampening effects are significantly hindered throughout. In the case of large time delays like Tau being equivalent to 2.3, there is no dampening effect. This would make sense because in real world scenarios it is one's intuition that delaying the advent of a highly coveted product or event would create a large amount of uncertainty coupled with hype driving prices up and down very quickly.

Conclusion

The problem the model solves is supply adjustment and time particularly when supply lags causes prices to overshoot with little explanation or justification. The price changes can also lead to cyclical price behavior. Such a model is imperative for policymakers to understand the relationship between time and supply's effects on price in a short period during averse economic scenarios and events. Specifically, it is used to gauge and think of different monetary policies to remedy such problems. Such models are integral in determining solutions for equilibrium prices in events of time delays. Although the required simplifications to the general model quickened the dampening and converging of the value, the usefulness of such conclusions (in less involved scenarios) cannot be understated.

References

[1] Y. Zhao, Y. Wei, and Z. Zhang, "Stability analysis of Caputo fractional differential equations with variable delays," *Advances in Continuous and Discrete Models*, vol. 2017, no. 13, pp. 1–16, 2017.

[2] Yahoo Finance. Graph showing Sony PlayStation 5 sales trends and consensus estimates. Retrieved from https://finance.yahoo.com/news/sonys-playstation-5-surpasses-50m-130800801.html.