Laplace transform

If $f(t)$ is a function of t satisfying certain conditions, then the define integral

$$\phi(s) = \int_0^\infty e^{-st} \cdot f(t) \cdot dt$$

when it exists, is called Laplace Transform of $f(t)$ & it is written as $L[f(t)]$.
Thus,

$$L[f(t)] = \int_0^\infty e^{-st} \cdot f(t) \cdot dt \cdot$$

Q.1. Find $L[f(t)]$ where $f(t) = \cos(t-\alpha)$, $t > \alpha$
$= 0$, $t < \alpha$.

Ans.

$$L[f(t)] = \int_0^\infty e^{-st} \cdot f(t) \cdot dt \cdot$$

$$= \int_0^\alpha e^{-st} \cdot 0 \cdot dt + \int_\alpha^\infty e^{-st} \cdot \cos(t-\alpha) \cdot dt \cdot$$

$$= \cancel{0 + \cos(t-\alpha) \int_\alpha^\infty e^{-st} \cdot dt - \int \int_\alpha^\infty e^{-st} \cdot dt \cdot \frac{d(\cos(t-\alpha))}{dt}}$$

$$= \cancel{0 + \cos(t-\alpha) \left[\frac{e^{-st}}{-s}\right]_\alpha^\infty + \int \left[\frac{e^{-st}}{-s}\right]_\alpha^\infty \cdot \sin(t-\alpha)}$$

$$= \cancel{\cos(t-\alpha) \left[\frac{e^{-\alpha s}}{s}\right] + \int_\alpha^\infty \frac{e^{-\alpha s}}{s} \cdot \sin(t-\alpha)}.$$

$$= 0 + \left[\frac{e^{-st}}{(s)^2 + (1)^2} [-s\cos(t-\alpha) + \sin(t-\alpha)]\right]_\alpha^\infty \qquad \left[\int e^{at} \cos bt\right.$$

$$= 0 - \frac{e^{-as}}{s^2+1} [-s\cos(\alpha-\alpha) + \sin(\alpha-\alpha)] \qquad = \frac{e^{at}}{a^2+b^2} [a\cos bt + b\sin bt]$$

$$= -\frac{e^{-as}}{s^2+1} [-s(1)]$$

$$= \frac{se^{-as}}{s^2+1}$$

Linearity Property

If k_1 & K_2 are constant then $L[k_1 f_1(t) + k_2 f_2(t)]$
$= k\, L[f_1(t)] + k_2 L[f_2(t)]$

Laplace transform of some standard function

1) $L[k] = \frac{K}{s}$, $s>0$, where k is constant.

$$L[k] = \int_0^\infty e^{-st} \cdot K \cdot dt = \left[\frac{e^{-st}}{-s} \cdot k\right]_0^\infty = \left[0 + \frac{e^0}{s} k\right]$$

$$= \frac{k}{s}$$

2.) $L[e^{at}] = \frac{1}{s-a}$, $s>a$

$$L[e^{at}] = \int_0^\infty e^{-st} \cdot e^{at} \cdot dt = \int_0^\infty e^{-(s-a)t} dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= 0 + \frac{e^0}{s-a}$$

$$= \frac{1}{s-a}$$

~~Coll~~ Corollary of $L[e^{at}]$,

$$L[e^{-at}] = \frac{1}{s+a}$$

3) $L[c^{at}] = \frac{1}{s - a\log c}$, where c is constant

$\therefore$ Since,

$c^a = e^{a\log c}$.

$\therefore c^{at} = e^{(a\log c)t}$

4) $L[\sin at] =$

$$L[f(t)] = \int_0^\infty e^{-st} \cdot f(t) \cdot dt \cdot$$

$$\therefore L[\sin(at)] = \int_0^\infty e^{-st} \cdot \sin at \cdot dt \cdot$$

$$= \cancel{\frac{e^{-at}}{s^2+a^2}[-a\cos at]}$$

$$= \left[\frac{e^{-st}}{s^2+a^2}[-s\sin at + a\cos(-st)]\right]_0^\infty$$

$$= 0 - \frac{1}{s^2+a^2}[-s\cos 0 + a\cos(-0)]$$

$$= 0 + \frac{a}{s^2+a^2}$$

$$= \frac{a}{s^2+a^2}.$$

5) Also, find by e^{iat}:

6.) $\sinh at = \dfrac{e^{at} - e^{-at}}{2}$.

$\therefore L[\sinh at] = \frac{1}{2} L[e^{at}] - \frac{1}{2} [e^{-at}]$.

$= \frac{1}{2} \cdot \frac{1}{s-a} - \frac{1}{2} \cdot \frac{1}{s+a}$

$= \frac{1}{2} \left[\frac{s+a-(s-a)}{s^2-a^2} \right]$

$= \frac{1}{2} \left[\frac{2a}{s^2-a^2} \right]$

$= \underline{\underline{\frac{a}{s^2-a^2}}}, \ s > 0.$

7.) $\cosh at = \dfrac{e^{at} + e^{-at}}{2}$

$\therefore L[\cosh at] = \frac{1}{2} L[e^{at}] + \frac{1}{2} L[e^{-at}]$.

$= \frac{1}{2} \cdot \frac{1}{s-a} + \frac{1}{2} \cdot \frac{1}{s+a}$.

$= \frac{1}{2} \left[\frac{s+a+s-a}{s^2-a^2} \right]$

$= \frac{1}{2} \left[\frac{2s}{s^2-a^2} \right], \ s > 0.$

$= \underline{\underline{\frac{s}{s^2-a^2}}}$

Using Gamma function,

$$\int_0^\infty e^{-t} \cdot t^n \cdot dt = \Gamma(n+1)$$

8) $L[t^n] = \int_0^\infty e^{-st} \cdot t^n \cdot dt$.

Put $st = z$.

$\therefore t = \frac{z}{s}$

$\therefore dt = \frac{dz}{s}$ | when $t = 0$, $z = 0$
$t = \infty$, $z = \infty$.

$\therefore = \int_0^\infty e^{-z} \cdot \left(\frac{z}{s}\right)^n \cdot \frac{dz}{s}$.

$= \frac{1}{s^{n+1}} \int_0^\infty e^{-z} \cdot (z)^n \cdot dz$.

$= \frac{1}{s^{n+1}} \Gamma(n+1)$.

$= \frac{\Gamma(n+1)}{s^{n+1}}$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1.

Q1. Solve, find Laplace $f(t) = 4t^2 + \sin 3t + e^{2t}$.

Ans $L[f(t)] = L[4t^2 + \sin 3t + e^{2t}]$

$= L[4t^2] + L[\sin 3t] + L[e^{2t}]$

$= 4 \times \frac{\lfloor 3}{s^3} + \frac{3}{s^2+9} + \frac{1}{s-2}$

Q2. Find $L[\sin^5 t]$.

* One more way.

Ans. $\therefore$ ~~$L[\sin^5 t] =$~~

$f(t) = \sin^5 t$.

$\therefore e^{it} = \cos t + i \sin t$.

$\therefore e^{-it} = \cos t - i \sin t$.

$\therefore \sin t = \frac{e^{it} - e^{-it}}{2i}$

$\therefore \sin^5 t = \frac{1}{32i} (e^{it} - e^{-it})^5$

$= \frac{1}{32i} (e^{5it} - 5e^{4it} \cdot e^{-it} + 10 e^{3it} \cdot e^{-2it} - 10 e^{2it} \cdot e^{-3it} + 5 e^{it} \cdot e^{-4it} - e^{-5it})$.

$= \frac{1}{32i} [e^{5it} - 5e^{3it} + 10 e^{it} - 10 e^{-it} + 5e^{-3it} - e^{-5it}]$

$= \frac{1}{32i} [e^{5it} - e^{-5it} - 5(e^{3it} - e^{-3it}) + 10(e^{it} - e^{-it})]$

$= \frac{\cancel{2}}{\cancel{32i}} = \frac{1}{16i} \left[\frac{e^{5it} - e^{-5it}}{2i} - 5\left(\frac{e^{3it} - e^{-3it}}{2i}\right) + 10\left(\frac{e^{it} - e^{-it}}{2i}\right) \right]$

$= \frac{1}{16} [\sin 5t - 5 \sin 3t + 10 \sin t]$.

$\therefore L[\sin^5 t] = \frac{1}{16} L[\sin 5t] - \frac{5}{16} L[\sin 3t] + \frac{10}{16} L[\sin t]$

$= \frac{1}{16} \cdot \left[\frac{5}{s^2+25}\right] - \frac{5}{16} \left[\frac{3}{s^2+9}\right] + \frac{10}{16} \left[\frac{1}{s^2+1}\right]$.

$$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$2\cos^2 A = 1 + \cos 2A$$

Q.3. Find $L[\cos t \cos 2t \cdot \cos 3t]$.

Ans.

$\therefore \cos t \cdot \cos 2t \cdot \cos 3t = \cos 3t\left[\frac{1}{2}(\cos(3t) + \cos(-t))\right]$

$= \cos 3t\left[\frac{1}{2}(\cos 3t + \cos t)\right]$

$= \frac{1}{2}\left[\cos^2 3t + \cos t \cdot \cos 3t\right]$

$= \frac{1}{2}\left[\cos^2 3t + \frac{1}{2}(\cos(4t) + \cos(2t))\right]$

$= \frac{1}{2}\left[\frac{1}{2}(1 + \cos 6t) + \frac{1}{2}(\cos(4t) + \cos(2t))\right]$

$= \frac{1}{4}(1 + \cos 6t) + \frac{1}{4}(\cos 4t + \cos 2t)$

$= \frac{1}{4}\left[1 + \cos 6t + \cos 4t + \cos 2t\right]$

$\therefore L[\cos t \cos 2t \cdot \cos 3t] = \frac{1}{4} L[1 + \cos 6t + \cos 4t + \cos 2t]$

$= \left[\frac{1}{4s} + \frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{s}{s^2+4}\right]$

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Change of Scale Property

If $L[f(t)] = F(s)$, then
$L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$

For E.g., If $L[f(t)] = \frac{2s}{s^2+4} = F(s)$

then $L[f(2t)] = \frac{1}{2} \times F\left(\frac{s}{a}\right)$

$= \frac{1}{2} \times \frac{2\cdot(s/2)}{(s/2)^2+4}$

$= \frac{1}{2} \times \frac{4s}{s^2+16}$

$= \frac{2s}{s^2+16}$

First shifting theorem.

If $L[f(t)] = F(s)$, then
$L[e^{-at}.f(t)] = F(s+a)$.

Corollary, $L[e^{at}.f(t)] = F(s-a)$.

E.g; If $L[\sin at] = \frac{a}{s^2+a^2}$

$L[e^{-bt}.\sin at] = \frac{a}{(s+b)^2+a^2}$

Q1. $f(t) = e^{-3t} \cdot \cosh 5t \sin 4t$

Find $L[f(t)]$

Ans. We know,

$$\cosh 5t = \frac{e^{5t} + e^{-5t}}{2}$$

$$\therefore e^{-3t} \cdot \cosh 5t = \frac{1}{2}(e^{2t} + e^{-8t})$$

$$\therefore e^{-3t} \cdot \cosh 5t \sin 4t = \frac{1}{2}(e^{2t} \cdot \sin 4t + e^{-8t} \cdot \sin 4t)$$

[By linearity property]

$$\therefore L[e^{-3t} \cdot \cosh 5t \sin 4t] = \frac{1}{2}\left(L[e^{2t} \sin 4t] + L[e^{-8t} \cdot \sin 4t]\right)$$

$$\therefore \text{Since, } L[\sin 4t] = \frac{4^2}{s^2 + 4^2}$$

$$\therefore L[e^{-3t} \cosh 5t \sin 4t] = \frac{1}{2}\left(\frac{4^2}{(s-2)^2 + 4^2} + \frac{4^2}{(s+8)^2 + 4^2}\right)$$

[By first shifting theorem]

Q.2 $f(t) = e^{4t} \cdot \sin^3 t$. Find $L(t)$

Ans.

$$\left[\because \sin^3 t = \frac{1}{4}[3 \sin t - \sin 3t]\right]$$

$$\therefore f(t) = e^{4t} \cdot \left[\frac{1}{4}[3 \sin t - \sin 3t]\right]$$

$$\therefore L(t) = \frac{1}{4}\left[e^{4t} \cdot 3 \sin t - e^{4t} \cdot \sin 3t\right]$$

$$\because L[\sin t] = \frac{1}{s^2+1}$$

$$\therefore L[t] = \frac{1}{4}\left[3 \cdot \left(\frac{1}{(s-4)^2 + 1^2}\right) - \left(\frac{3}{(s-4)^2 + 3^2}\right)\right]$$

$$\therefore L[\sin 3t] = \frac{3}{s^2+9}$$

$\because$ [By first shifting theorem].

Effect of multiplication by t.

If $L[f(t)] = F(s)$.

then, $L[t^n . f(t)] = (-1)^n . \frac{d^n}{ds^n} [F(s)]$.

Eg. $f(t) = t \cdot e^{at}$.

Since, $L[e^{at}] = \frac{1}{s-a}$

$\therefore L[t \cdot e^{at}] = (-1) \cdot \frac{d}{ds} \left[\frac{1}{s-a}\right]$

$= (-1) \cdot -\frac{1}{(s-a)^2}$.

$= \frac{1}{(s-a)^2}$.

Q.1. $f(t) = [1 + te^{-t}]^3$. Find $L[f(t)]$

Ans. $f(t) = 1^3 + 3(1)^2(te^{-t}) + 3(1)(te^{-t})^2 + (te^{-t})^3$

$= 1 + 3 \cdot (te^{-t}) + 3(t^2 e^{-2t}) + (t^3 e^{-3t})$.

$= ~~ep~~$

$\therefore L[t] = L[1 + 3 \cdot (te^{-t}) + 3(t^2 e^{-2t}) + (t^3 e^{-3t})]$

$= L[1] + 3 L[te^{-t}] + 3L[t^2 e^{-2t}] + L[t^3 e^{-3t}]$

$= \frac{1}{s} + 3 \cdot (-1) \cdot \frac{d}{ds} \left[\frac{1}{s+1}\right] + 3 \cdot (-1)^2 \cdot \frac{d^2}{ds^2} \left[\frac{1}{s+2}\right]$

$+ (-1)^3 \cdot \frac{d^3}{ds^3} \left[\frac{1}{s+3}\right]$.

$\left[\begin{array}{l} \because L[e^{-t}] = \frac{1}{s+1} \\ \because L[e^{-2t}] = \frac{1}{s+2} \\ \because L[e^{-3t}] = \frac{1}{s+3}. \end{array}\right.$

$= \frac{1}{s} + 3 \cdot \frac{1}{(s+1)^2} + \frac{3 \cdot 2}{(s+2)^3} + \frac{6}{(s+3)^4}$.

$= \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3} + \frac{6}{(s+3)^4}$

$\frac{-1}{(s+2)^2}$

$-(s+2)^{-2}$

$+2(s+2)^{-3}$

1 1
1 2 1
1 3 3 1

Q.2. $f(t) = t \cdot e^{3t} \cdot \sin 4t$. Find $L[f(t)]$.

Ans. Since,

$\therefore L[\sin 4t] = \dfrac{4}{s^2+4^2}$.

$\therefore L[e^{3t} \sin 4t] = \dfrac{4}{(s-3)^2+4^2}$ [By first shifting theorem].

$= \dfrac{4}{(s-3)^2+16}$.

$\therefore L[t \cdot e^{3t} \cdot \sin 4t] = (-1)^1 \cdot \dfrac{d}{ds}\left(\dfrac{4}{(s-3)^2+16}\right)$

$= (-1) \cdot \dfrac{d}{ds}\left(\dfrac{4}{s^2-6s+9+16}\right)$

$= (-1) \cdot \dfrac{d}{ds}\left(\dfrac{4}{s^2-6s+25}\right)$

$= (-1) \cdot \left[0 - \dfrac{4(2s-6)}{(s^2-6s+25)^2}\right]$

$= \dfrac{8s-24}{(s^2-6s+25)^2}$

Effect of division by t

If $L[f(t)] = \phi(s)$

then $L\left[\frac{1}{t} \cdot f(t)\right] = \int_s^{\infty} \phi(s) \cdot ds$

Eg. $f(t) = \frac{1}{t}[1-\cos t]$.

Ans $L[1-\cos t] = \frac{1}{s} - \frac{s}{s^2+1}$

$$\therefore L\left[\frac{1-\cos t}{t}\right] = \int_s^{\infty}\left[\frac{1}{s} - \frac{s}{s^2+1}\right] ds$$

$$= [\log s]_s^{\infty} - \left[\frac{1}{2}\log(s^2+1)\right]_s^{\infty}$$

Multiply & divide by 2.

$$= \frac{1}{2}\left[\log s^2 - \log(s^2+1)\right]_s^{\infty}$$

$$= -\frac{1}{2}\left[\log\left(\frac{s^2+1}{s^2}\right)\right]_s^{\infty}$$

$$= -\frac{1}{2}\left[\log\left[1+\frac{1}{s^2}\right]\right]_s^{\infty}$$

$$= -\frac{1}{2}\left[\log 1 - \log\left(1+\frac{1}{s^2}\right)\right]$$

$$= \frac{1}{2}\log\left(1+\frac{1}{s^2}\right)$$

Q.2. $f(t) = \dfrac{e^{-2t} \cdot \sin 2t \cosh t}{t}$

Ans: ~~f(t)~~

~~$L[\cosh t] = \dfrac{s^2}{s}$.~~

$\therefore \cosh t = \dfrac{e^{t} + e^{-t}}{2}$.

$\therefore e^{-2t} \cdot \cosh t = \dfrac{1}{2}(e^{-t} + e^{-3t})$.

$\therefore e^{-2t} \cdot \sin 2t \cdot \cosh t = \dfrac{1}{2}(e^{-t} \cdot \sin 2t + e^{-3t} \cdot \sin 2t)$.

$L[e^{-2t} \cdot \sin 2t \cosh t] = \dfrac{1}{2}\left[L[e^{-t} \cdot \sin 2t] + L[e^{-3t} \cdot \sin 2t]\right]$

$= \dfrac{1}{2}\left[\dfrac{2}{(s+1)^2 + 2^2} + \dfrac{2}{(s+3)^2 + 2^2}\right]$

$= \dfrac{1}{(s+1)^2 + 2^2} + \dfrac{1}{(s+3)^2 + 2^2}$

$\therefore L\left[\dfrac{e^{-2t} \cdot \sin 2t \cdot \cosh t}{t}\right] = \int_s^\infty \dfrac{1}{(s+1)^2 + 2^2} + \dfrac{1}{(s+3)^2 + 2^2}$

$= \left[\dfrac{1}{2} \tan^{-1}\left(\dfrac{s+1}{2}\right) + \dfrac{1}{2} \tan^{-1}\left(\dfrac{s+3}{2}\right)\right]_s^\infty$

$= \left[\dfrac{1}{2} \cdot \dfrac{\pi}{2} + \dfrac{1}{2} \cdot \dfrac{\pi}{2} - \tan^{-1}\left(\dfrac{s+1}{2}\right) - \tan^{-1}\left(\dfrac{s+3}{2}\right)\right.$

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Laplace transform of derivatives.

$L[f'(t)] = -f(0) + sL[f(t)]$.

$L[f''(t)] = -f'(0) - sf(0) + s^2 . L[f(t)]$.

$L[f'''(t)] = -f''(0) - sf'(0) - s^2 f(0) + s^3 L[f(t)]$.

$\vdots$

$L[f^n(t)] = -f^{n-1}(0) - sf^{n-2}(0) - s^2 f^{n-3}(0) \dots \dots + s^n L[f(t)]$

Q1. Given, $f(t) = t+1$, $0 \le t \le 2$

$f(t) = 3$, $t > 2$.

Find $L[f(t)]$, $L[f'(t)]$ & $L[f''(t)]$.

Ans.

$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$.

$= \int_0^2 e^{-st}.(t+1)dt + \int_2^\infty e^{-st}. 3 .dt$.

$= \int_0^2 e^{-st}.t\, dt + \int_0^2 e^{-st} dt + \int_2^\infty e^{-st}. 3 .dt$.

U.V. $= u\int v.\,dt - \int \frac{du}{dt} \int v\,dt . dt$

$= \left\{ t\left[\frac{e^{-st}}{-s}\right]_0^2 - \left[\frac{e^{-st}}{s^2}\right]_0^2 \right\}_0^2 + \left[\frac{e^{-st}}{-s}\right]_0^2 + 3\left[\frac{e^{-st}}{-s}\right]_2^\infty$

$= 2\left[\frac{e^{-2s}}{-s}\right] - \left[\frac{e^{-2s}}{s^2}\right] - 0 + \frac{1}{s^2} + \left[\frac{e^{-2s}}{-s} + \frac{1}{s}\right]$

$+ \left[0 + \frac{3e^{-2s}}{s}\right]$.

$= \frac{\cancel{e^{-2s}}}{\cancel{-s}} - \frac{e^{-2s}}{s^2} + \frac{1}{s^2} + \frac{1}{s}$

$= \frac{1}{s} + \frac{1}{s^2}(1 - e^{-2s})$.

$f(0) = 0 + 1 = 1$

$L[f'(t)] = -f(0) + s\,L[f(t)]$

$= -1 + s\left[\frac{1}{s} + \frac{1}{s^2}(1-e^{-2s})\right]$

$f'(t) = 1 + 0 \therefore f'(0) = 1.$

$\therefore L[f''(t)] = -f(0) - s f'(0) + s^2 L[f(t)]$

$= -1 - s(1) + s^2\left[\frac{1}{s} + \frac{1}{s^2}(1-e^{-2s})\right].$

Q.2. Find out laplace transform of

$L\left[\frac{d}{dt}\left[\frac{\sin 3t}{t}\right]\right]$

Ans. $\therefore L[\sin 3t] = \frac{3}{s^2+9}.$

$\therefore L\left[\frac{\sin 3t}{t}\right] = \int_s^{\infty} \frac{3}{s^2+9} \cdot ds$

$= 3\left[\frac{1}{3}\tan^{-1}\left(\frac{s}{3}\right)\right]_s^{\infty}.$

$= 3\left[\frac{1}{3}\tan^{-1}\left(\frac{\infty}{3}\right) - \frac{1}{3}\tan^{-1}\left(\frac{s}{3}\right)\right]$

$= 3\left[\frac{1}{3}\times\frac{\pi}{2} - \frac{1}{3}\tan^{-1}\left(\frac{s}{3}\right)\right]$

$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{3}\right)$

$= \cot^{-1}\left(\frac{s}{3}\right).$

$\therefore L[f'(t)] = -f(0) + s\,L[f(t)]$, where $f(t) = \frac{\sin 3t}{t}$.

$= -3 + s.\cot^{-1}\left(\frac{s}{3}\right).$

$\therefore f(0) = \lim_{t\to 0} \frac{\sin 3t}{t}$

$= 3\lim_{t\to 0} \frac{\sin 3t}{3t}$

$= 3(1)$

$$\frac{1+\cos 2t}{2}.$$

Laplace transform of integrals.

If $L[f(t)] = F(s)$
then $L\left[\int_0^t f(u).du\right] = \frac{1}{s}.F(s)$.

Q.1. $L\left[\int_0^t \int_0^t \cdots \int_0^t f(u).(du)^n\right] = \frac{1}{s^n} F(s)$

Q.1 Find Laplace transform of $\int_0^t u.\cos^2 u.\, du$.

Ans. $L[t\cdot\cos^2 t] = F(s)$ $\qquad \cos^2 t = \frac{1+\cos 2t}{2}$.

$$\therefore L\left[\frac{1+\cos 2t}{2}\right] = L\left[\frac{1}{2}\right] + L\left[\frac{\cos 2t}{2}\right].$$

$$\therefore L[\cos^2 t] = \frac{1}{2s} + \frac{1}{2}\cdot\left[\frac{s}{s^2+4}\right]$$

$$\therefore L[t\cdot\cos^2 t] = (-1)^n \cdot \frac{d}{ds}\left[\frac{1}{2s} + \frac{1}{2}\left(\frac{s}{s^2+4}\right)\right].$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2} + \frac{s^2+4(1)-2s^2}{(s^2+4)^2}\right].$$

$$= \left[+\frac{1}{2s^2} \; \frac{-1}{2}\frac{(-s^2+4)}{(s^2+4)^2}\right]$$

$$= \frac{1}{2s^2} + \frac{1}{2}\frac{(s^2-4)}{(s^2+4)^2}$$

$$\therefore L\left[\int_0^t u\cdot\cos^2 u\right] = \frac{1}{s}\left[\frac{1}{2s^2} + \frac{1}{2}\frac{(s^2-4)}{(s^2+4)^2}\right]$$

Q.2. Find Laplace transform of $\cosh t \int_0^t e^u \sinh u \cdot du$

Ans $L[e^t \sinh t] = \cancel{\phi}$.

$\therefore \cancel{e^t} \cdot \sinh t = e^t$

$$L[\sinh t] = \frac{1}{s^2-1}$$

$$\therefore L[e^t \cdot \sinh t] = \cancel{\phi} \cdot \cancel{\frac{b}{s^2-a}} \frac{1}{(s-1)^2-1}$$

$$\therefore L\left[\int_0^t e^u \cdot \sinh u\right] = \frac{1}{s}\left[\frac{1}{(s-1)^2-1}\right]$$

$$\therefore L\left[\cosh t \int_0^t e^u \cdot \sinh u \cdot du\right] = L\left[\left(\frac{e^t+e^{-t}}{2}\right)\int_0^t e^u \cdot \sinh u\, du\right]$$

$$= \frac{1}{2}\left[L\left[e^t \cdot \int_0^t e^u \sinh u du\right]\right.$$

$$\left. + L\left[e^{-t} \cdot \int_0^t e^u \cdot \sinh u \cdot du\right]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-1}\left[\frac{1}{(s-1-1)^2-1}\right]\right.$$

$$\left. + \frac{1}{(s+1)}\left[\frac{1}{(s-1+1)^2-1}\right]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-1}\left[\frac{1}{(s-2)^2-1}\right] + \frac{1}{s+1}\left[\frac{1}{s^2-1}\right]\right]$$

Second shifting theorem:-

If $L[f(t)] = F(s)$.

& $g(t) = f(t-a)$, when $t > a$.

$\quad = 0$, when $t < a$.

then $L[g(t)] = e^{-as} F(s)$.

Q.1. Find $L[g(t)]$, $g(t) = \cos(t-\alpha)$, $t > \alpha$

$\quad = 0$, $t < \alpha$.

Ans Let,

$f(t) = \cos t$, ~~t>α~~ ~~=0, t<α~~

$L[f(t)] = \dfrac{s}{s^2+1}$

By Second shifting theorem,

$L[g(t)] = e^{-\alpha s} \left(\dfrac{s}{s^2+1}\right)$.

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Evaluation of integrals using Laplace transform.

$$\int_0^\infty e^{-at}.f(t).dt.$$

$$L[f(t)] = F(s).$$

i.e. $\int_0^\infty e^{-st}.f(t).dt = F(s).$

$\therefore$ Put $s = a$

Q1. Evaluate $\int_0^\infty e^{-3t} \sin t .dt.$

$f(t) = \sin t$

Ans. $L[\sin t] = \frac{1}{s^2+1}$

$\therefore \int_0^\infty e^{-st}. \sin t.dt = \frac{1}{s^2+1}.$

Put. $s=3$

$\therefore \int_0^\infty e^{-3t}.\sin t.dt = \frac{1}{10}.$

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Q2. Evaluate $\int_0^\infty e^{-2t} t^5 \cosh t \, dt$

Ans. Let, $f(t) = t^5 \cdot \cosh t$.

Step 1: $L[t^5 \cdot \cosh t] = F(s)$

$\therefore$ Since $\cosh t = \frac{e^t + e^{-t}}{2}$.

$$t^5 \cosh t = \frac{1}{2}[e^t \cdot t^5 + e^{-t} \cdot t^5].$$

$$L[t^5 \cosh t] = \frac{1}{2}\left[L[e^t \cdot t^5] + L[e^{-t} \cdot t^5]\right]$$

$$= \frac{1}{2}\left[\frac{\lfloor 6}{(s-1)^6} + \frac{\lfloor 6}{(s+1)^6}\right]$$

$$= \frac{1}{2}\left[\frac{5!}{(s-1)^6} + \frac{5!}{(s+1)^6}\right].$$

Put $s = 2$

$$\therefore \int_0^\infty e^{-2t} \cdot t^5 \cosh t \cdot dt = \frac{1}{2}\left[\frac{5!}{(2-1)^6} + \frac{5!}{(2+1)^6}\right]$$

$$= \frac{5!}{2}\left[1 + \frac{1}{3^6}\right]$$

Q3. Evaluate $\int_0^\infty e^{-t}\, t\sqrt{1+\sin t}\; dt$.

Ans.

Let, $f(t) = t\sqrt{1+\sin t}$

$\therefore f(t) = t\sqrt{\cos^2\frac{t}{2} + \sin^2\frac{t}{2} + 2\sin\left(\frac{t}{2}\right)\cdot\cos\left(\frac{t}{2}\right)}$

$= t\cdot\sqrt{\left(\cos\left(\frac{t}{2}\right)+\sin\left(\frac{t}{2}\right)\right)^2}$

$= t\cdot\left[\sin\left(\frac{t}{2}\right) + \cos\left(\frac{t}{2}\right)\right]$

$\therefore L\left[\sin\frac{t}{2}\right] = \frac{1/2}{s^2+1/4}$

$\therefore L\left[\cos\frac{t}{2}\right] = \frac{s}{s^2+1/4}$

$\therefore L[\sqrt{1+\sin t}] = \frac{1/2}{s^2+1/4} + \frac{s}{s^2+1/4}$

$= \frac{1/2+s}{s^2+1/4}$

$= \frac{2(1+2s)}{4s^2+1}$

$= \frac{2+4s}{4s^2+1}.$

$\therefore L[t\cdot\sqrt{1+\sin t}] = (-1)^1\frac{d}{ds}\left(\frac{2+4s}{4s^2+1}\right)$

$= (-1)\left[\frac{(4s^2+1)4 - (2+4s)8s}{(4s^2+1)^2}\right]$

$\therefore$ Put $s = 1$

$\therefore \int_0^\infty e^{-t}\cdot t\sqrt{1+\sin t}\; dt = (-1)\left[\frac{20-48}{(4(1)+1)^2}\right]$

$= \frac{28}{25}$